EE110B Lab 7

There are several forms of Fourier transform or Fourier expansion, such as continuous-time Fourier transform (CTFT), discrete-time Fourier transform (DTFT), continuous-time Fourier series (CTFS) and discrete-time Fourier series (DTFS). CTFS and DTFS are for periodic signals in continuous-time and discrete-time respectively.

Let $\tilde{x}(n)$ be a periodic discrete-time signal with period N. Then $\tilde{x}(n) = \tilde{x}(n+N)$, and its DTFS is

$$\tilde{x}(n) = \sum_{k=0}^{N-1} \tilde{X}(k) \exp\left(j2\pi \frac{k}{N} n\right) \tag{1}$$

and the corresponding DTFS coefficients are

$$\tilde{X}(k) = \frac{1}{N} \sum_{n=0}^{N-1} \tilde{x}(n) \exp\left(-j2\pi \frac{k}{N} n\right). \tag{2}$$

One can verify that $\tilde{X}(k)$ is also periodic with the period N.

Let $x(n) = \tilde{x}(n)$ for $n = 0, 1, \cdots, N-1$ and $X(k) = N\tilde{X}(k)$ for $k = 0, 1, \cdots, N-1$. It follows that

$$X(k) = \sum_{n=0}^{N-1} x(n) \exp\left(-j2\pi \frac{k}{N} n\right) \tag{3}$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \exp\left(j2\pi \frac{k}{N} n\right) \tag{4}$$

where the finite sequence $\{X(k), k = 0, 1, \cdots, N-1\}$ is called the discrete Fourier transform (DFT) of the finite sequence $\{x(n), n = 0, 1, \cdots, N-1\}$, and $\{x(n), n = 0, 1, \cdots, N-1\}$ is called the inverse discrete Fourier transform (IDFT) of $\{X(k), k = 0, 1, \cdots, N-1\}$.

One can also verify that $\tilde{x}(n)$ is a periodic extension of $\{x(n), n = 0, 1, \cdots, N-1\}$, and $\tilde{X}(k)$ is a periodic extension of $\{\frac{1}{N}X(k), k = 0, 1, \cdots, N-1\}$.

In MATLAB, there are existing functions for DFT and IDFT, which are implemented via fast Fourier transform (FFT) and inverse fast Fourier transform (IFFT) respectively.

One of the powerful properties of FFT and IFFT is the following. Let $h(n), n = 0, 1, \cdots, N-1$ be a sequence with its last $\frac{N}{2}$ elements being zero, and $g(n), n = 0, 1, \cdots, N-1$ be another sequence with its last $\frac{N}{2}$ elements being zero. Then, for $n = 0, 1, \cdots, N-1$,

$$h(n) * g(n) = \mathtt{IFFT}\left[H(k)G(k)\right] \tag{5}$$

with $H(k) = \mathtt{FFT}[h(n)]$ and $G(k) = \mathtt{FFT}[g(n)]$.

Perform the following tasks:

1) Choose a large N that is an integer power of 2, i.e., $N = 2^p$ with p being an integer. For example, $N = 1024$. And generate two random sequences $h(n)$ and $g(n)$ meeting the above "zero tail" properties.
2) Compute the convolution $x(n) = h(n) * g(n)$ for $n = 0, 1, \cdots, N-1$ using the conventional method, i.e., $h(n) * g(n) = \sum_{l=0}^{N-1} h(l)g(n-l)$, and record the execution CPU time T_1.
3) Then compute $H(k) = \texttt{FFT}[h(n)]$ and $G(k) = \texttt{FFT}[g(n)]$ (for all $k = 0, 1, \cdots, N-1$) and $y(n) = \texttt{IFFT}\,[H(k)G(k)]$ (for all $n = 0, 1, \cdots, N-1$), and record the total execution time T_2.
4) Is T_2 much smaller than T_1?
5) Is $x(n)$ identical to $y(n)$ for $n = 0, 1, \cdots, N-1$? (Hint: compute the mean squared error $\texttt{MSE} = \frac{1}{N} \sum_{n=0}^{N-1} |x(n) - y(n)|^2$.)
6) Try an even larger N and repeat the above.