EE 105: First order systems in Simulink

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Abstract

The objective of this laboratory is to familiarize the student with the Simulink while exercising systems concepts such as transfer functions, time constants, pole locations, DC gain, and frequency response.

All of these concepts may be on exams.

1 First Order Systems

The majority of this lab will focus on the analysis of first-order, linear ordinary differential equations (ODE's) in state space form. In the context of first order systems we will demonstrate certain useful concepts that generalize to higher order systems. Those generalizations, which will be useful throughout your engineering career, are summarized in Section 2.

First-order, linear state-space models have the form

$$\frac{d}{dt}x(t) = -a\ x(t) + b\ u(t) \tag{1}$$

$$y(t) = c\ x(t) \tag{2}$$

where the signals $u(t)$ and $y(t)$ are the input and output and the signal $x(t)$ is the state of the system. The parameters a, b, and c are real constants with numeric values determined by the designer of the system that is of interest. One possible Simulink implementation of eqns. (1–2) is shown in Figure 1.

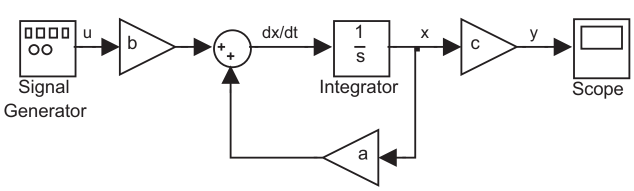

Figure 1: Simulink implementation of eqns. (1–2).

1.1 Solutions

To determine analytically the response of the system to an input $u(t)$ we use the Laplace transform of eqn. (1):

$$\begin{aligned} s\,X(s) - x(0) &= -a\,X(s) + b\,U(s) \\ (s-a)X(s) &= x(0) + b\,U(s) \\ X(s) &= \frac{x(0)}{s+a} + \frac{b}{s+a}U(s) \end{aligned}$$

and similarly

$$\begin{aligned} Y(s) &= c\,X(s) \\ Y(s) &= \frac{c\,x(0)}{s+a} + \frac{c\,b}{s+a}U(s). \end{aligned} \tag{3}$$

The inverse Laplace transform of eqn. (3) provides the general output response in the time-domain:

$$y(t) = ce^{-a\,t}x(0) + \int_0^t c\,e^{-a(t-\tau)}b\,u(\tau)d\tau, \text{ for } t \geq 0 \tag{4}$$

where τ is a dummy variable used only for integration. The two terms in the right hand side of eqn. (4) each have names related to their physical meaning. If the input $u(t) = 0$ for all $t \geq 0$, then $y(t) = ce^{-at}x(0)$; therefore, this term is referred to as the *zero input response*. If the initial value of the state is zero (i.e., $x(0) = 0$), then $y(t) = \int_0^t ce^{-a(t-\tau)}bu(\tau)d\tau$; therefore, this term is referred to as the *zero state response*. The general solution is the sum of the zero input and the zero state responses.

If $a > 0$, then the zero input response will converge to zero as $t \to \infty$. Such a system is *stable*. If $a < 0$, then the magnitude of the zero input response will diverge toward infinity as $t \to \infty$. Such a system is *unstable*. In the remainder of this document, we will only be interested in stable systems.

1.2 Terminology

The *transfer function* $H(s)$ from u to y is defined as the ratio of $Y(s)$ to $U(s)$ when the initial state is zero. From eqn. (3) we see that the transfer function for the system described by eqns. (1–2) is

$$H(s) = \frac{Y(s)}{U(s)} = \frac{c\,b}{s+a}. \tag{5}$$

The *impulse response* is the inverse Laplace transform of the transfer function. For the system described by eqns. (1–2) the impulse response is

$$h(t) = c\,e^{-at}\,b \tag{6}$$

which plays an important role in the convolution integral of the zero state response.

The polynomial in the denominator of the transfer function is called the *characteristic polynomial*. The roots (or zeros) of the characteristic polynomial are the *poles* of the system. The pole locations are critical to the determination of the stability of the system and affect the transient response. For the first-order system of this lab, the pole is at $s = -a$. The *time constant* of a first-order system is $\tau = \frac{1}{|a|}$.

1.3 First-Order Input Response

Certain types of inputs are common enough that they are often used to specify the performance of systems. This section analyzes the response of the first-order state-space system to some common input signals.

1.3.1 Zero Input: $u(t) = 0$

As stated above, the response of the first-order system when the input is zero, is completely determined by the initial condition: $y(t) = c\,e^{-a\,t}\,x(0)$. This can also be written in time constant form as

$$y(t) = c\,e^{\frac{-t}{\tau}}\,x(0)$$

where $\tau = \frac{1}{|a|}$ is the time constant.

For a stable system, the time constant characterizes the rate of decay of the initial condition. At $t = \tau$, the output will have deceased to approximately 37% of its initial value, since

$$y(\tau) = c\,e^{-1}\,x(0) \approx 0.37\,y(0).$$

When a person has data corresponding to a first-order system, this formula provides one method for estimating the time constant (i.e., estimating the parameter a). The person simply finds the value of time at which $y(t) = 0.37\,y(0)$.

A second method to estimate the value of the time constant is to use the slope of $y(t)$ at $t = 0$. Note that by differentiation of $y(t) = c\,e^{-a\,t}\,x(0)$ it is straightforward to show that $\frac{dy}{dt}(t) = -a\,c\,e^{-a\,t}\,x(0) = -a\,y(t)$. The line tangent to the graph of $y(t)$ at $t = 0$ is

$$v(t) = y(0) - a\,y(0)\,t.$$

This line intersects the t-axis at $t = \frac{1}{|a|} = \tau$. To use this method, on the plot of the $y(t)$ data, the person draws a line tangent to $y(t)$ at $t = 0$ and measures the time at which the line intersects the t-axis.

1.3.2 Step Input: $u(t) = A\ 1(t)$

The unit step function is represented as

$$1(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ 1 & \text{for} \quad t \geq 0. \end{cases}$$

The Laplace transform of this step function is $U(s) = \frac{A}{s}$. Therefore, using partial fractions on eqn. (3)

$$\begin{aligned} Y(s) &= \frac{c\,x(0)}{s+a} + \frac{c\,b\,A}{(s+a)\,s} \\ &= \left(c\,x(0) - \frac{c\,b\,A}{a}\right)\frac{1}{s+a} + \frac{c\,b\,A}{a\,s}. \end{aligned} \tag{7}$$

Using the inverse Laplace transform, the time response of the system is

$$y(t) = \left(c\,x(0) - \frac{c\,b\,A}{a}\right)e^{-a\,t} + \frac{c\,b\,A}{a}, \quad \text{for } t \geq 0. \tag{8}$$

The *DC Gain* of a stable system is the ratio of the magnitude of the steady state output to the magnitude of the applied step input. Steady state applies for values of t large enough that all transients are effectively finished. For first-order stable linear systems, steady state is often considered to apply for any $t \geq 4\,\tau$, since $e^{-4} < 2\%$.

By eqn. (8) as $t \to \infty$, $y(t) \to \frac{c\,b}{a}\,A = y_\infty$. Therefore, the DC gain is $\frac{y_\infty}{A} = \frac{c\,b}{a}$.

The DC gain can also be computed directly from the transfer function without using partial fractions. To derive the formula, we apply the Final Value Theorem to $Y(s)$:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} sH(s)U(s) = H(0)A. \tag{9}$$

Therefore, the DC gain is (again) $\frac{\lim_{t\to\infty} y(t)}{A} = H(0)$. This method for computing the DC gain easily extends to higher order systems. For the first order example of this lab, the DC gain is $\frac{cb}{a}$.

1.3.3 Sinusoidal Input: $u(t) = A\,\cos(\omega\,t)$ for $t \geq 0$

When the input is the sinusoid $u(t) = A\,\cos(\omega\,t)$ for $t \geq 0$, where ω is the radian frequency, the output response is computed using Laplace methods as

$$\begin{aligned} Y(s) &= \frac{c\,x(0)}{s+a} + \frac{c\,b}{(s+a)}\frac{A\,s}{(s^2+\omega^2)} \\ &= \frac{c\,x(0)}{s+a} + \frac{c\,b\,a\,A}{(\omega^2+a^2)}\frac{1}{(s+a)} + \frac{c\,b\,A}{(\omega^2+a^2)}\frac{-as+\omega^2}{(s^2+\omega^2)}. \end{aligned}$$

The time domain equivalent is

$$\begin{aligned} y(t) &= \left(c\,x(0) + \frac{c\,a\,b\,A}{w^2+a^2}\right)e^{-a\,t} + \\ &\quad \frac{c\,b\,A}{\sqrt{\omega^2+a^2}}\cos\left(\omega t + \operatorname{atan2}\left(-\omega, a\right)\right). \end{aligned}$$

For a stable system ($a < 0$), the exponential term in the first line decays toward zero as time increases. In steady state, the response is

$$y(t) \approx \frac{c\,b}{\sqrt{\omega^2+a^2}}A\cos\left(\omega t + \operatorname{atan2}\left(-\omega, a\right)\right). \tag{10}$$

Note that the steady state output is also a sinusoid with amplitude amplified by the gain $\left(\frac{cb}{\sqrt{\omega^2+a^2}}\right)$ and phase shifted by $\operatorname{atan2}\left(-\omega, a\right)$ relative to the applied input. In the following, we show that this gain and phase shift can be directly calculate from the transfer function. This result also generalizes to higher order systems. This method called frequency response is a very useful engineering tool.

The transfer function $H(s)$ is a complex function of the complex variable s. The *frequency response* corresponds to $H(s)$ evaluated for $s = j\omega$ where $j = \sqrt{-1}$ and $\omega \in [0, \infty)$. For our first order system,

$$\begin{aligned} H(j\omega) &= \frac{cb}{j\omega+a} = \frac{cb}{(j\omega+a)}\frac{(-j\omega+a)}{(-j\omega+a)} \\ &= \frac{cb(-j\omega+a)}{a^2+\omega^2} \\ &= \frac{cb}{\sqrt{a^2+\omega^2}}\exp\left(j\tan^{-1}\left(\frac{-\omega}{a}\right)\right) \\ &= M(w)e^{j\Phi(\omega)} \end{aligned}$$

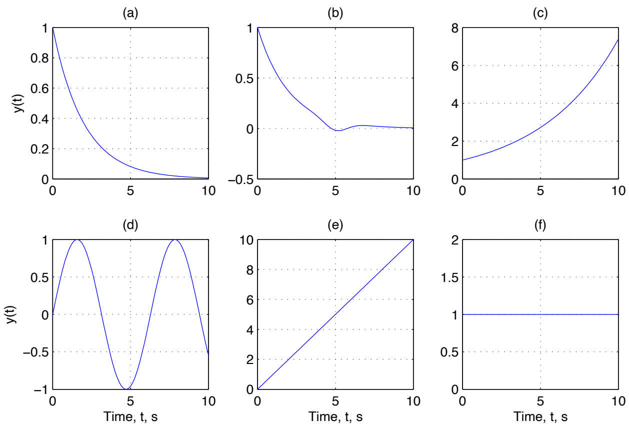

Figure 2: Figures for the Prelab Part 1.

where

$$M(w) = \frac{cb}{\sqrt{a^2+\omega^2}} \quad \text{and} \quad \Phi(\omega) = \tan^{-1}\left(\frac{-\omega}{a}\right). \tag{11}$$

Comparing these expressions with the steady state time response in eqn. (10) we see that as $t \to \infty$

$$y(t) \quad \approx \quad M(w)\, A \cos\left(\omega\, t + \Phi(\omega)\right).$$

The output has the same sinusoidal form as the input, with magnitude changed by the factor $M(\omega)$ and phase shifted by $\Phi(\omega)$.

2 Higher Order Systems

If we use the notation $d(s)$ to denote the denominator polynomial of the transfer function $H(s)$, then the zeros of the equation $d(s) = 0$ are the poles of the system. The location of the poles determines the stability properties of the system and greatly affect the transient behavior. If the real part of all poles are in the left half plane (LHP), then the system is stable. If the real part of any pole is not in the LHP, then the system is unstable.

The DC gain of a stable system with transfer function $H(s)$ is

$$DC\ Gain = \lim_{s\to 0} H(s).$$

For a sinusoidal input $u(t) = A\cos(\omega t)$ to a system with stable transfer function $H(s)$ as $t \to \infty$

$$y(t) \quad \to \quad M(w) A \cos\left(\omega t + \Phi(\omega)\right)$$

where

$$H(j\omega) \quad = \quad M(w) e^{j\Phi(\omega)}.$$

When $20log(M(w))$ and $\Phi(\omega)$ are plotted versus ω the resulting plots are called *Bode plots*. Matlab provides routines for plotting the Bode plots (type 'Help bode').

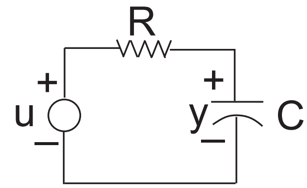

Figure 3: RC Circuit for Prelab Part 2 and Lab Part 2.

3 Prelab

1. Read the above text, which provides information to do the following.

2. Figure 2 plots the zero-input (i.e., $u(t) = 0$) time-response of various different systems. For each:

   (a) State whether of not the response could correspond to a first order linear system.

   (b) If not, state why not.

   (c) If so, state the sign and estimate the value of the time constant and the parameter a.

3. For the circuit shown in Figure 3:

   (a) Find the transfer function from u to y.

   (b) Using the capacitor voltage as the state variable, find the state space representation in the form of eqns. (1–2).

   (c) Using the values $R = 10\times 10^4$ ohms and $C = 1.0 \times 10^{-6}$ Farads, find an expression for the magnitude and phase of the transfer function at $\omega = 0.00$, 0.01, 0.10, 1.00, 10.00, and 100.00 rad/sec.

   (d) Use the Matlab 'Bode' function to plot the magnitude and phase versus ω. Make sure that these plots match your answers from Part 3c above.

4 Lab

This lab has two parts. In Part 1, you will be asked to compute the time constant applicable to a data set corresponding to a first order system. In Part 2, you will simulate a first order system to verify the DC gain and frequency response formulas.

Note that Parts 1 and 2 deal with two different systems. The time constant in Part 1 is not known and must be computed from the data. In Part 2, you know the value of the time constant from the Prelab.

4.1 Part 1 – Time Constant Estimation

1. Download from ILearn the data set listed for this lab. Plot y versus t. Make sure that the figure is properly labeled as indicated in Lab 1.

2. Section 1.3.1 describes two methods for estimation of time constants from data. Use each method to estimate the time constant for the data that you just plotted.[1]

Use the 'grid on' plot option. Note that you can zoom in on portions of the graph using the magnifying glass in the plot window menu. Ensure that your report clearly describes the method, computations, and data values that you use.

4.2 Part 2 – Simulation

1. Simulation setup.
   (a) Implement a Simulink all integrator block diagram (similar to that shown in Figure 1) for the RC circuit of Figure 3. Enter the values of a, b, and c by double clicking the gain icons.
   (b) Make sure to have a 'scope' connected to the signals u and y so that you can analyze their response. A good approach is to use the 'mux' block so that you can plot both u and y using a single scope.
   (c) In the Simulink window, under the 'Simulink' menu, edit the 'Model Configuration Parameters' to change the 'Max. step size' to about $0.01\,\tau$ and set the duration of the simulation to be about 10 time constants. The first change ensures that you have at least 100 time steps per time constant. The second change ensures that you will see both the transient and steady state response.
   (d) Double click on the integrator icon and set the initial condition of the integrator to 1.0. Ensure that your report states the units of this initial condition.
   (e) Simulate the system with the $u = 0$. Verify that the convergence rate of the zero input response matches that expected for the time constant. If it does not, then you need to check your derivations and Simulink implementation.
   (f) Set the initial condition of the integrator to 0.0.
2. Step input.
   (a) From the 'Simulink:Sources' library folder, select the 'Pulse Generator' as the input u. Once you have it connected, double click on it to open its user interface. Set the amplitude to 1 and the period to 20 time constants (longer than the simulation). It should change from 0 to 1 exactly one time. Close that icon's user interface. Leave the simulation duration at 10 time constants.
   (b) Simulate the system.
   (c) How long does the response take to get to steady state? How does this compare with the time constant?
   (d) What is the steady state value of y? Does this match the value predicted by the DC Gain analysis?
   (e) Remove the pulse generator from the block diagram.
3. Sinusoidal input.
   (a) From the 'Simulink:Sources' library folder, select and connect the 'Signal Generator' as the input u. Once you have it connected, double click on it to open its user interface.
   (b) Set the amplitude to 1 and the wave form to sine.
   (c) Set the simulation duration to 100s.
   (d) Simulate the system for each of the radian frequencies state in Prelab Part 3c.
   (e) For each frequncy, compare the magnitude and phase between y and u. They should match the frequency response predictions.

[1] For the second method (tangent line) you can either draw the tangent manually or use the first two points to determine the slope of the tangent line and draw it with Matlab.