Binomial Random Variables: Probability Distribution Examples

Learning Objective: Fit the binomial model when appropriate, and use it to perform simple calculations.

Let's look at another example:

Example: Blood Type A

The probability of having blood type A is 0.4. Choose 4 people at random and let X be the number with blood type A.

X is a binomial random variable with n = 4 and p = 0.4.

As a review, let's first find the probability distribution of X the long way: construct an interim table of all possible outcomes in S, the corresponding values of X, and probabilities. Then construct the probability distribution table for X.

S	X	Probability
NNNN	0	$.4^0 .6^4$
NNNA	1	$.4^1 .6^3$
NNAN	1	$.4^1 .6^3$
NANN	1	$.4^1 .6^3$
ANNN	1	$.4^1 .6^3$
NNAA	2	$.4^2 .6^2$
NANA	2	$.4^2 .6^2$
NAAN	2	$.4^2 .6^2$
ANNA	2	$.4^2 .6^2$
ANAN	2	$.4^2 .6^2$
AANN	2	$.4^2 .6^2$
NAAA	3	$.4^3 .6^1$
ANAA	3	$.4^3 .6^1$
AANA	3	$.4^3 .6^1$
AAAN	3	$.4^3 .6^1$
AAAA	4	$.4^4 .6^0$

As usual, the addition rule lets us combine probabilities for each possible value of X:

X	Probability
0	(1) $.4^0 .6^4$ =.1296
1	(4) $.4^1 .6^3$ =.3456
2	(6) $.4^2 .6^2$ =.3456
3	(4) $.4^3 .6^1$ =.1536
4	(1) $.4^4 .6^0$ =.0256

Now let's apply the formula for the probability distribution of a binomial random variable, and see that by using it, we get exactly what we got the long way.

Recall that the general formula for the probability distribution of a binomial random variable with n trials and probability of success p is:

$P(X = x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{(n-x)}$ for x = 0, 1, 2, 3, … , n

In our case, X is a binomial random variable with n = 4 and p = 0.4, so its probability distribution is:

$P(X = x) = \frac{4!}{x!(4-x)!} (0.4)^x (0.6)^{4-x}$ for x = 0, 1, 2, 3, 4

Let's use this formula to find P(X = 2) and see that we get exactly what we got before.

$$P(X = 2) = \frac{4!}{2!(4-2)!} (0.4)^2 (0.6)^{4-2} = \frac{1*2*3*4}{(1*2)(1*2)} (0.4)^2 (0.6)^2 = 0.3456$$

Scenario: Multiple-Choice Test

A multiple choice test has 10 questions, each with 5 possible answers, only one of which is correct. A student who did not study is absolutely clueless, and therefore uses an independent random guess to answer each of the 10 questions.

Let X be the number of questions the student gets right.

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1/1 point (graded)
X has a binomial distribution. What is the value of the parameter n?

10

10

Answer
Correct: Since the multiple-choice test has 10 questions, n = 10.

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X has a binomial distribution. What is the value of the parameter p? Report your answer to one decimal place.

0.2

0.2

Answer
Correct:
Since each multiple-choice question has 5 options, the probability of guessing the correct answer is 0.2.

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What is the probability that the student gets exactly 4 questions right, P(X = 4)? Round your answer to TWO decimal places.

0.09

0.09

Answer

Correct: Plugging in X = 4 in the probability distribution formula we get P(X = 4) = 0.0881 = 0.09.

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Here is another interesting example.

Example: Choosing Numbers at Random

Do people really choose numbers at random?

Each student in a group of 15 students is asked to each pick a number from 1 to 20 completely at random. 3 of the 15 happen to pick the number 7 (this is a probability of 0.20). Is this an improbably high proportion to choose a particular number?

If the selections are truly random, then each number from 1 to 20, including 7, has probability p = 1/20 = .05 of being selected. The number of trials is n = 15. The probability of at least 3 successes in 15 trials, when each trial has probability of success 0.05, can be found by applying the binomial formula.

To make the notation easier, we will use a shorthand notation for the number of possible outcomes with x successes out of n. $\frac{n!}{x!(n-x)!}$ will be written as: $\binom{\mathrm{n}}{\mathrm{x}}$.

$$P(X \geq 3) = P(X = 3) + P(X = 4) + \ \dots \ + P(X = 15)$$

$$= \binom{15}{3} \ (0.05)^3 \ (0.95)^{12} + \binom{15}{4} \ (0.05)^4 \ (0.95)^{11} + \ \dots \ + \binom{15}{15} \ (0.05)^{15} \ (0.95)^0$$

$$= \ .0307 \ + \ .0049 \ + \ .0006 \ + \ \dots \ = \ 0.0362$$

where all remaining terms after the first 3 are less than 0.0001. The probability of at least 3 out of 15 people picking 7, when choosing at random from the numbers 1 to 20, is only 0.0362. Thus, 3 out of 15 is rather improbably high. People may think they are choosing at random, but in fact they tend to favor certain numbers, like the number 7.

Now let's look at some truly practical applications of binomial random variables.

Example: Airline Flights

Past studies have shown that 90% of the booked passengers actually arrive for a flight. Suppose that a small shuttle plane has 45 seats. We will assume that passengers arrive independently of each other. (This assumption is not really accurate, since not all people travel alone, but we'll use it for the purposes of our experiment).

Many times airlines "**overbook**" flights. This means that the airline sells more tickets than there are seats on the plane. This is due to the fact that sometimes passengers don't show up, and the plane must be flown with empty seats. However, if they do overbook, they run the risk of having more passengers than seats. So, some passengers may be unhappy. They also have the extra expense of putting those passengers on another flight and possibly supplying lodging.

With these risks in mind, the airline decides to sell more than 45 tickets. If they wish to keep the probability of having more than 45 passengers show up to get on the flight to less than 0.05, how many tickets should they sell?

This is a binomial random variable that represents the number of passengers that show up for the flight. It has $p = 0.90$, and n to be determined.

Suppose the airline sells 50 tickets. Now we have $n = 50$ and $p = 0.90$. We want to know $P(X > 45)$, which is $1 - P(X \leq 45) = 1 - 0.57$ or 0.43. Obviously, all the details of this calculation were not shown, since a statistical technology package was used to calculate the answer. This is certainly more than 0.05, so the airline must sell fewer seats.

If we reduce the number of tickets sold, we should be able to reduce this probability. We have calculated the probabilities in the following table:

# tickets sold	P(X > 45)
50	0.43
49	0.26
48	0.13
47	0.04
46	0.008

From this table, we can see that by selling 47 tickets, the airline can reduce the probability that it will have more passengers show up than there are seats to less than 5%.

Note: For practice in finding binomial probabilities, you may wish to verify one or more of the results from the table above.

Scenario: In Vitro Fertilization

Purpose

The purpose of this activity is to gain experience making probabilistic decisions using binomial random variables.

Background

In vitro fertilization is becoming more and more common these days. Suppose each embryo that is implanted has a 20% chance of resulting in a pregnancy that results in delivering a baby. Also, assume that each embryo's chance of surviving and resulting in a baby is independent of the others.

It is an expensive procedure, so we want to do it only once. We wish to try to find the optimum number of embryos to implant so that the probability of at least 1 child being born is high, but the probability of more than 2 children being born is low. In other words, we want a baby, and we're willing to have twins, but we don't want triplets, quadruplets, etc.

Note that unlike the airline flight example, where we needed to control only one probability, in this case there are two probabilities that we wish to control.

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The two conditions we've outlined mean that we'll need two probabilities.

The first is the probability of having at least one child. How can we express this?

- ○ $P(X > 1)$
- ◉ $P(X \geq 1)$ ✔
- ○ $P(X < 1)$
- ○ $P(X \leq 1)$

Answer

Correct: At least one child means one or more.

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How will we calculate the above probability?

- ○ 1 - P(X > 1)
- ○ 1 - P(X ≥ 1)
- ◉ 1 - P(X = 0) ✔
- ○ 1 - P(X ≤ 1)

Answer

Correct:

To find P(X ≥ 1), we must use complements, since technology uses only cumulative P(X ≤ x) probabilities. The complement of **at least one**, is **less than** one. However, we need less than or equal to something. Less than one is none, so we really have: 1 - P(X = 0).

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The second probability we'll need is the probability of having more than two children. How can we express this one?

- ◉ P(X > 2) ✔
- ○ P(X ≥ 2)
- ○ P(X < 2)
- ○ P(X ≤ 2)

Answer

Correct: **More than two**, which is the same as greater than two.

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1/1 point (graded)

How will we calculate this second probability?

- ○ 1 - P(X > 2)
- ○ 1 - P(X ≥ 2)
- ○ 1 - P(X < 2)
- ● 1 - P(X ≤ 2) ✔

Answer

Correct:

To find P(X > 2) we need to use the complement, since technology finds only cumulative P(X ≤ x) probabilities. The complement of greater than two is less than or equal to two.

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We will let X represent the number of implanted embryos resulting in a baby. It is a binomial random variable with n = number of implanted embryos and p = 0.20 (the probability that an implanted embryo results in a baby).

It is customary to implant between n = 1 and n = 7 embryos. We have provided a table that contains the two probabilities mentioned in the previous question, for values of n ranging from 1 to 7.

# embryos	P(X ≥ 1)	P(X > 2)
1	0.20	0
2	0.36	0
3	0.488	0.008
4	0.590	0.027
5	0.672	0.058
6	0.738	0.099
7	0.790	0.148

Learn By Doing (1/1 point)

Using the table of probabilities provided and our conditions that we want $P(X \geq 1)$ to be high, while keeping $P(X > 2)$ low, decide how many embryos you'd implant and why.

Your Answer:

4 embryos, because that way, P(X>2) is still lower than 5% yet P(X>=1) is already more than 50%

Our Answer:

Even though the probability of having one child keeps increasing, so does the probability of having more than two children. We think the optimum number is five embryos. You have almost a 70% chance of a baby, while keeping the probability of triplets, quadruplets, etc. to well under 10%.

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