Hypothesis Testing for the Population Proportion p: z-score

Learning Objective: Carry out hypothesis testing for the population proportion and mean (when appropriate), and draw conclusions in context.

For the reason illustrated in the examples at the end of the previous page, the test statistic cannot simply be the difference $\hat{p} - p_0$, but must be some form of that formula that accounts for the sample size. In other words, we need to somehow standardize the difference $\hat{p} - p_0$ so that comparison between different situations will be possible. We are very close to revealing the test statistic, but before we construct it, let's be reminded of the following two facts from probability:

1. When we take a random sample of size n from a population with population proportion p, the possible values of the sample proportion $\hat{p}$ (when certain conditions are met) have approximately a normal distribution with:

* mean: p

* standard deviation: $\sqrt{\frac{p(1-p)}{n}}$

2. The z-score of a normal value (a value that comes from a normal distribution) is:

$$z = \frac{\text{value} - \text{mean}}{\text{standard deviation}}$$

and it represents how many standard deviations below or above the mean the value is.

We are finally ready to reveal the test statistic:

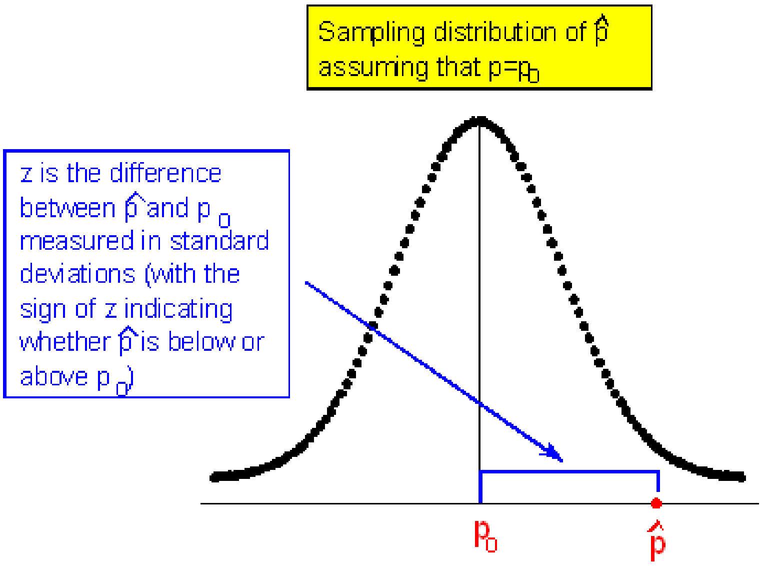
The test statistic for this test measures the difference between the sample proportion $\hat{p}$ and the null value p_0 by the z-score (standardized score) of the sample proportion $\hat{p}$, assuming that the null hypothesis is true (i.e., assuming that $p = p_0$).

From fact 1, we know that the values of the sample proportion ($\hat{p}$) are normal, and we are given the mean and standard deviation.

Using fact 2, we conclude that the z-score of $\hat{p}$ when $p = p_0$ is:

$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

This is the test statistic. It represents the difference between the sample proportion ($\hat{p}$) and the null value (p_0), measured in standard deviations.



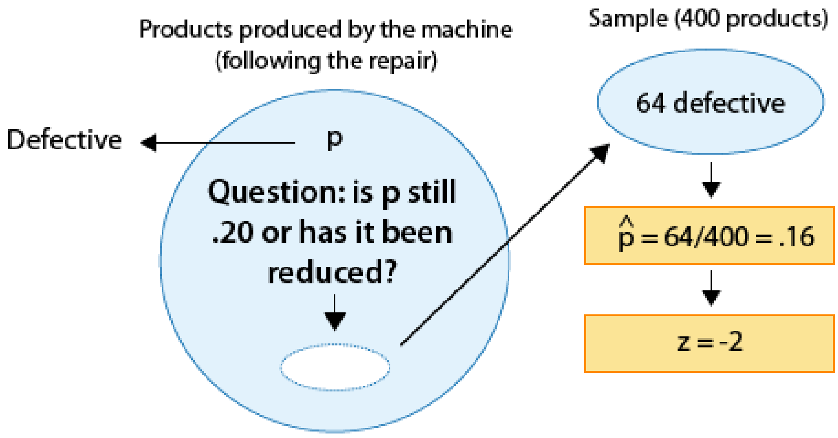
Here is a representation of the sampling distribution of $\hat{p}$, assuming p = p_o. In other words, this is a model of how $\hat{p}$'s behave if we are drawing random samples from a population for which H_o is true. Notice the center of the sampling distribution is at p_o, which is the hypothesized proportion given in the null hypothesis (H_o: p = p_o.) We could also mark the axis in standard deviation units, $\sqrt{\frac{p_0(1-p_0)}{n}}$. For example, if our null hypothesis claims that the proportion of U.S. adults supporting the death penalty is 0.64, then the sampling distribution is drawn as if the null is true. We draw a normal distribution centered at p = 0.64 with a standard deviation dependent on sample size, $\sqrt{\frac{0.64(1-0.64)}{n}}$.

Important Comment

Note that under the assumption that H_o is true (i.e., $p = p_0$), the test statistic, by the nature of the fact that it is a z-score, has N(0,1) (standard normal) distribution. Another way to say the same thing which is quite common is: "The null distribution of the test statistic is N(0,1)." By "null distribution," we mean the distribution under the assumption that H_o is true. As we'll see and stress again later, the null distribution of the test statistic is what the calculation of the p-value is based on.

Let's go back to our three examples and find the test statistic in each case:

Example: 1



Since the null hypothesis is H_o: p = 0.20, the standardized score of $\hat{p} = 0.16$ is: $z = \frac{0.16-0.20}{\sqrt{\frac{0.20(1-0.20)}{400}}} = -2$

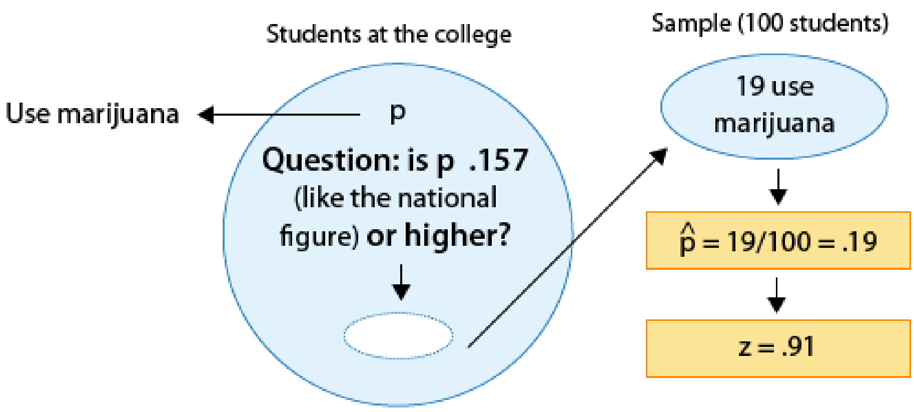
.

This is the value of the test statistic for this example.

What does this tell me?

This z-score of -2 tells me that (assuming that H_o is true) the sample proportion $\hat{p} = 0.16$ is 2 standard deviations below the null value (0.20).

Example: 2



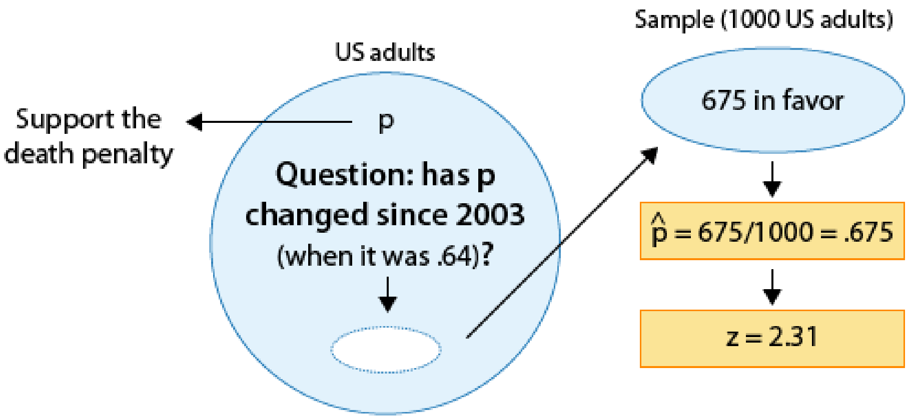
Since the null hypothesis is H_o: p = 0.157, the standardized score of $\hat{p} = 0.19$ is:

$$z = \frac{0.19-0.157}{\sqrt{\frac{0.157(1-0.157)}{100}}} \approx 0.91.$$

This is the value of the test statistic for this example.

We interpret this to mean that, assuming that H_o is true, the sample proportion $\hat{p} = 0.19$ is 0.91 standard deviations above the null value (0.157).

Example: 3



Since the null hypothesis is H_o: p = 0.64, the standardized score of $\hat{p} = 0.675$ is:

$$z = \frac{0.675-0.64}{\sqrt{\frac{0.64(1-0.64)}{1000}}} \approx 2.31.$$

This is the value of the test statistic for this example.

We interpret this to mean that, assuming that H_o is true, the sample proportion $\hat{p} = 0.675$ is 2.31 standard deviations above the null value (0.64).

Scenario: Automobile Color

We think that the most common color for automobiles is silver and that 24% of all automobiles sold are silver. We take a random sample of 225 cars and find that 63 of them are silver.

Learn By Doing

1/1 point (graded)

Calculate the value of the test statistic for this.

- 1.34
- -1.40
- 1.40 ✔
- 2.14
- 2.50

Answer
Correct:
$z = (\hat{p} - p_o) / \text{sqrt}((p_o*(1 - p_o))/n) = (0.28 - 0.24) / \text{sqrt}(((0.24*(1 - 0.24))/225) = 0.04 / \text{sqrt}(0.1824 / 225) = 0.04 / 0.02847 = 1.40$

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Learn By Doing

1/1 point (graded)
If we were to find that 72 (instead of 63) of the 225 randomly chosen cars are silver, would the test statistic increase, decrease, or stay the same? Try to answer this question without actually calculating the test statistic.

- Increase ✔
- Decrease
- Stay the same

Answer
Correct:
With the increased number of silver cars from 63 to 72 (out of 225), the sample proportion increases (from 0.28 to 0.30) and therefore will be further away from the assumed (null) value of 0.24. Since the test statistic measures the difference between the sample proportion and the null value in standard deviations, the test statistic will increase.

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Learn By Doing

1/1 point (graded)

If we take a different random sample of 225 cars and find that 20% are silver, is the test statistic positive, negative, or zero?

- Positive
- Negative ✔
- Zero

Answer

Correct: $\hat{p} = 0.20$. The test statistic is negative when $\hat{p}$ is less than p_o.

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Learn By Doing

1/1 point (graded)

If we take a different random sample and get a test statistic of zero, identify whether the following conclusion is true or false.

True or false? $\hat{p} = p_o$.

- True ✔
- False

Answer

Correct: When $\hat{p} = p_o$, we get z = 0 / standard deviation = 0.

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Learn By Doing

1/1 point (graded)

If we take a different random sample and get a test statistic of zero, identify whether the following conclusion is true or false.

True or false? The proportion for this sample is equal to the hypothesized value for the proportion in the population.

- (•) True ✔
- ( ) False

Answer

Correct: This is another way to say: $\hat{p} = p_o$.

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Learn By Doing

1/1 point (graded)

If we take a different random sample and get a test statistic of zero, identify whether the following conclusion is true or false.

True or false? $\hat{p}$ is 0 standard deviations away from p_o.

- (•) True ✔
- ( ) False

Answer

Correct: The test statistic z measures how many standard deviations $\hat{p}$ is from p_o.

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Learn By Doing

1/1 point (graded)

If we take a different random sample and get a test statistic of zero, identify whether the following conclusion is true or false.

The standard deviation is 0.

- ( ) True

False ✔

Answer
Correct:
If the standard deviation is 0, then there is no variability in sample proportions. $\hat{p} = p_o$ for every sample. In this strange situation we would not need to do a hypothesis test, because we would already know the value of p_o. Also the test statistic is not defined when the denominator is 0.

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Comments About the Test Statistic

1. We mentioned earlier that to some degree, the test statistic captures the essence of the test. In this case, the test statistic measures the difference between $\hat{p}$ and p_0 in standard deviations. This is exactly what this test is about. Get data, and look at the discrepancy between what the data estimates p to be (represented by $\hat{p}$) and what H_o claims about p (represented by p_0).

2. You can think about this test statistic as a measure of evidence in the data against H_o. The larger the test statistic, the "further the data are from H_o" and therefore the more evidence the data provide against H_o.

Did I Get This (1/1 point)

The UCLA Internet Report (February 2003) estimated that a proportion of roughly 0.75 of online homes are still using dial-up access, but claimed that the use of dial-up is declining. Is that really the case? To examine this, a follow-up study was conducted a year later in which, out of a random sample of 1,308 households that had Internet access, 804 were connecting using a dial-up modem. Let p be the proportion of all U.S. Internet-using households who have dial-up access. In the previous activity, we established that the appropriate hypotheses here are: Ho: $p = 0.75$ and Ha: $p < 0.75$Based on the data, what is the sample proportion of Internet households that use a dial-up connection?

Your Answer:

$p = 0.75$
$\hat{p} = 0.61$

0.61

Our Answer:
Out of a random sample of 1,308 households that had Internet connections, 804 used a dial-up connection, and so $\hat{p} = 804 / 1308 = 0.615$.

Resubmit Reset

Did I Get This

1/1 point (graded)

The test statistic in the case described in the previous question, therefore, is:

$$z = \frac{.615 - .75}{\sqrt{\frac{.75(1-.75)}{1308}}} \approx -11.3$$

This means that:

- (selected) If p is really still 0.75, the sample proportion we got is 11.3 standard deviations below it. ✔
- If p is really still 0.75, the sample proportion we got is 11.3 percentage points below it.
- If p is really still 0.75, the sample proportion we got is 11.3 standard deviations above it.
- If p is really still 0.75, the sample proportion we got is 11.3 percentage points above it.

Answer

Correct:

Indeed, the test statistic measures how many standard deviations away from p_o our sample result $\hat{p}$ is, assuming that p_o is the true value of p. The sign of z indicates whether the sample proportion is above (+) or below (-) the null value. In this case since z = -11.3, this indicates that the sample proportion is 11.3 standard deviations below the null value 0.75 (assuming that 0.75 is the true proportion).

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Scenario: M&M Colors

Ann and Sam are both testing the hypothesis that 40% of plain M&M's are orange, H_o: p = 0.40. Ann draws a sample of M&M's and 45% of her sample are orange. She calculates a test statistic of z = 1.25. Sam draws a sample of M&M's and 50% of his sample are orange. He calculates a test statistic of z = 1.

Did I Get This

1/1 point (graded)

Identify whether the following conclusion is true or false.

True or false? Sam's data provide stronger evidence against H_o because 0.50 is further from 0.40.

- True
- False ✔

Answer

Correct:

It is true that Sam's is further from p = 0.40, but the test statistic is smaller. So his sample gives weaker evidence against H_o.

Submit

Did I Get This

1/1 point (graded)

Identify whether the following conclusion is true or false.

True or false? Ann's data provides stronger evidence against H_o because her test statistic is larger.

- True ✔
- False

Answer

Correct: The further the test statistic is from zero, the stronger the evidence against H_o.

Submit

Did I Get This

1/1 point (graded)

Identify whether the following conclusion is true or false.

True or false? Someone made a mistake in calculating the test statistic. Sam should have the larger test statistic since 0.50 is further from 0.40.

True

False ✔

Answer
Correct:
The test statistic is the number of **standard deviations** $\hat{p}$ is from p_o, not the distance between $\hat{p}$ and p_o.

Submit

Did I Get This

1/1 point (graded)
Identify whether the following conclusion is true or false.

True or false? Sam must have drawn a smaller sample than Ann.

True ✔

False

Answer
Correct:
A smaller sample will have a sampling distribution with more variability. So Sam's $\hat{p}$ is 0.10 from p_o, but this distance is only 1 standard deviation. Ann must be drawing a larger sample with less variability. So her $\hat{p}$ is only 0.05 from p_o, but this distance is more than 1 standard deviation.

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