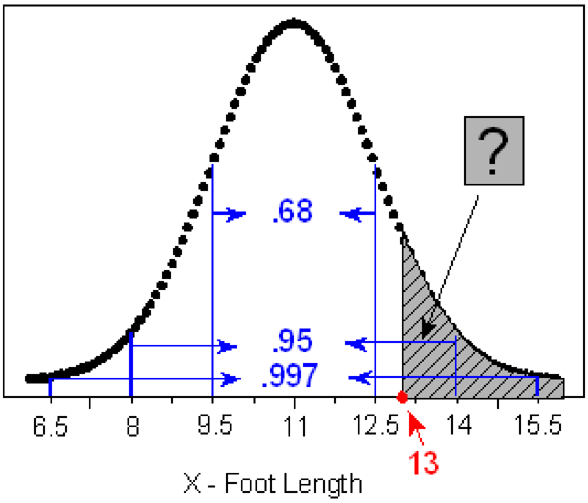
Introduction to Normal Random Variables: Standardizing Values

Learning Objective: Find probabilities associated with the normal distribution.

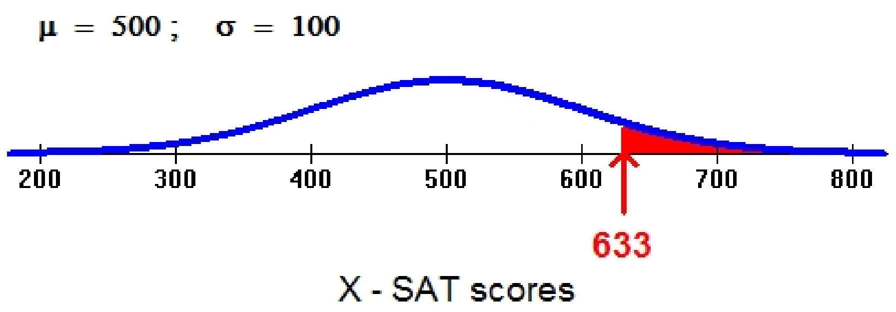
Let's go back to our example of foot length:

How likely or unlikely is it for a male's foot length to be more than 13 inches?



Since 13 inches doesn't happen to be exactly 1, 2, or 3 standard deviations away from the mean, we would only be able to give a very rough estimate of the probability at this point. Clearly, the Standard Deviation Rule only describes the tip of the iceberg, and while it serves well as an introduction to the normal curve, and gives us a good sense of what would be considered likely and unlikely values, it is very limited in the probability questions it can help us answer.

Here is another familiar normal distribution:



Suppose we are interested in knowing the probability that a randomly selected student will score 633 or more on the math portion of his or her SAT (this is represented by the red area). Again, 633 does not fall exactly 1, 2, or 3 standard deviations above the mean. Notice, however, that an SAT score of 633 and a foot length of 13 are both about 1/3 of the way between 1 and 2 standard deviations. As you continue to read this page, you'll realize that this positioning relative to the mean is the key to finding probabilities.

Finding Probabilities for a Normal Random Variable

As we saw, the Standard Deviation Rule is very limited in helping us answer probability questions, and basically limited to questions involving values that fall exactly 1, 2, and 3 standard deviations away from the mean. How do we answer probability questions in general? The key is the position of the value relative to the mean, measured in standard deviations.

We can approach the answering of probability questions two possible ways: a table and technology. In the next sections, you will learn how to use the "standard normal table," and then how the same calculations can be done with technology.

Standardizing Values

The first step to assessing a probability associated with a normal value is to determine the **relative** value with respect to all the other values taken by that normal variable. This is accomplished by determining how many standard deviations below or above the mean that value is.

Example: Foot Length

How many standard deviations below or above the mean male foot length is 13 inches? Since the mean is 11 inches, 13 inches is 2 inches above the mean. Since a standard deviation is 1.5 inches, this would be 2 / 1.5 = 1.33 standard deviations above the mean. Combining these two steps, we could write:

(13 in. - 11 in.) / (1.5 inches per standard deviation) = (13 - 11) / 1.5 standard deviations = +1.33 standard deviations.

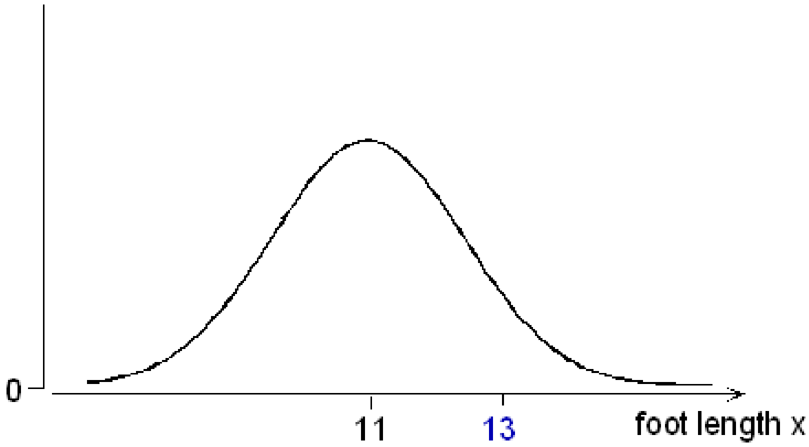
In the language of statistics, we have just found the **z-score** for a male foot length of 13 inches to be z = +1.33. Or, to put it another way, we have **standardized** the value of 13. In general, the standardized value z tells how many standard deviations below or above the mean the original value is, and is calculated as follows:

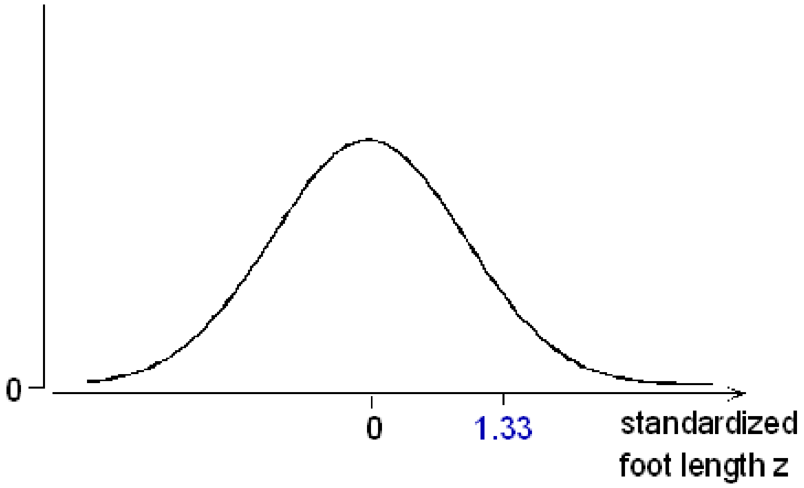
z-score = (value - mean)/standard deviation

The convention is to denote a value of our normal random variable X with the letter "x." Since the mean is written μ and the standard deviation σ, we may write the standardized value as

$$z = \frac{x - \mu}{\sigma}$$

Notice that since σ is always positive, for values of x above the mean (μ), z will be positive; for values of x below μ, z will be negative.





Example: Standardizing Foot Measurements

Let's go back to our foot length example, and answer some more questions.

(a) What is the standardized value for a male foot length of 8.5 inches? How does this foot length relate to the mean?

z = (8.5 - 11) / 1.5 = -1.67. This foot length is 1.67 standard deviations **below** the mean.

(b) A man's standardized foot length is +2.5. What is his actual foot length in inches? If z = +2.5, then his foot length is 2.5 standard deviations above the mean. Since the mean is 11, and each standard deviation is 1.5, we get that the man's foot length is: 11 + 2.5(1.5) = 14.75 inches.

z-scores also allow us to compare values of different normal random variables. Here is an example:

(c) In general, women's foot length is shorter than men's. Assume that women's foot length follows a normal distribution with a mean of 9.5 inches and standard deviation of 1.2. Ross' foot length is 13.25 inches, and Candace's foot length is only 11.6 inches. Which of the two has a longer foot relative to his or her gender group?

To answer this question, let's find the z-score of each of these two normal values, bearing in mind that each of the values comes from a different normal distribution.

Ross: z-score = (13.25 - 11) / 1.5 = 1.5 (Ross' foot length is 1.5 standard deviations above the mean foot length for men).

Candace: z-score = (11.6 - 9.5) / 1.2 = 1.75 (Candace's foot length is 1.75 standard deviations above the mean foot length for women).

Note that even though Ross' foot is longer than Candace's, Candace's foot is longer relative to their respective genders.

TO SUM UP...

Part (c) illustrates how z-scores become crucial when you want to **compare distributions**.

Learn By Doing

1/1 point (graded)

The hourly salary rate for accountants at the “We are the Best Accounting Firm” follow a normal distribution, with a mean of \$27 and a standard deviation of \$4.

Tom earns \$22.50 an hour. What is his z score?

- ○ 22.50
- ● -1.125 ✔
- ○ -4.50
- ○ 1.125

Answer
Correct: Indeed, Tom's z-score = (22.50-27)/4 = -1.125

Submit

Learn By Doing

1/1 point (graded)
This means that Tom's hourly rate is how far from the mean?

- ○ 1.125 dollars below the mean
- ● 1.125 standard deviations below the mean ✔
- ○ 1.125 dollars above the mean
- ○ 1.125 standard deviations above the mean

Answer
Correct:
Indeed, a z-score of 1.125 indicates that Tom's hourly rate of $22.50 is 1.125 standard deviations below the mean score.

Submit

Learn By Doing

1/1 point (graded)
John also works for the "We are the Best Accounting Firm" and the z-score for his hourly salary is 2.6. This means that his hourly rate is:

- Much lower than average
- Slightly lower than average
- Slightly above average
- Much higher than average ✔

Answer
Correct:
Indeed, John's z-score is 2.6 standard deviations above the mean, which is much above average. In particular, from the standard deviation rule, we know that,since John's z-score is more than 2 standard deviations above the mean, his hourly rate is in the highest 2.5% of salaries.

Submit

Learn By Doing

1/1 point (graded)
What is John's actual hourly salary?

- $37.40 ✔
- $29.60
- $16.60
- $24.40

Answer
Correct:
Indeed, John's z-score of 2.6 indicates that his salary is 2.6 standard deviations above the mean. Since the mean is $27 and each standard deviation is $4, we get that John's salary = 27 + 2.6(4) = $37.40.

Submit

Learn By Doing

1/1 point (graded)

Cindy works for a competing accounting firm called “Ace Pro Accountants.” The hourly salary rate for accountants follow a normal distribution, with a mean of $24 and a standard deviation of $3. Cindy earns $18 an hour.

Whose salary is lower relative to his/her company’s mean salary: Tom or Cindy’s?

- Tom
- Cindy ✔
- They are both equally low.

Answer
Correct:
Indeed, Cindy's z-score is -2, while Tom’s z-score is (22.50-27) / 4 = -1.125. Cindy’s hourly salary is lower, since she scored 2 standard deviations below the mean, while Tom scored only 1.125 standard deviations below the mean.

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Scenario: Final Exam Scores

Scores on the final exam in Professor Meyer's statistics class follow a normal distribution, with a mean of 82 and a standard deviation of 5.

Did I Get This

1/1 point (graded)
Ron, who took Professor Meyer's class, scored 88 on the final. What is Ron's z-score?

- 88
- 6
- -1.2
- 1.2 ✔

Answer

Correct: Indeed, Ron's z-score = (88 - 82) / 5 = 1.2.

Submit

Did I Get This

1/1 point (graded)

This means that Ron's final score is how far from the mean?

- 1.2 points above the mean
- 1.2 standard deviations above the mean ✔
- 1.2 points below the mean
- 1.2 standard deviations below the mean

Answer

Correct:

Indeed, a z-score of 1.2 indicates that Ron's score of 88 is 1.2 standard deviations above the mean score.

Submit

Did I Get This

1/1 point (graded)

Dan also took Professor Meyer's class, and the z-score of his final is -2.4. This means that Dan's score is:

- much lower than average. ✔
- slightly lower than average.
- slightly above average.
- much higher than average.

Answer

Correct:
Indeed, Dan's z-score indicates that his score is 2.4 standard deviations below the mean, which is much below average. In particular, from the standard deviation rule, we know that, since Dan's score is less than 2 standard deviations below the mean, his score is in the lowest 2.5% of scores.

Submit

Did I Get This

1/1 point (graded)
What is Dan's actual score on the final?

- 94
- 84.4
- 70 ✔
- 79.6

Answer
Correct:
Indeed, Dan's z-score of -2.4 indicates that his score is 2.4 standard deviations below the mean. Since the mean is 82 and each standard deviation is 5, we get that Dan's score = 82 - 2.4(5) = 70.

Submit

Did I Get This

1/1 point (graded)
Julie was assigned to take her statistics class with Professor Fisher, whose final scores follow a normal distribution with mean 75 and standard deviation of 6. Her score on the final was 84. Which of the two scores, Ron's or Julie's, is more impressive?

- Ron's score is more impressive.
- Julie's score is more impressive. ✔
- Both scores are equally impressive.

Answer

Correct:
Indeed, Ron's z-score is 1.2, and Julie's z-score is (84 - 75) / 6 = 1.5. Julie's score is more impressive, since she scored 1.5 standard deviations above the mean, while Ron scored only 1.2 standard deviations above the mean.

Submit