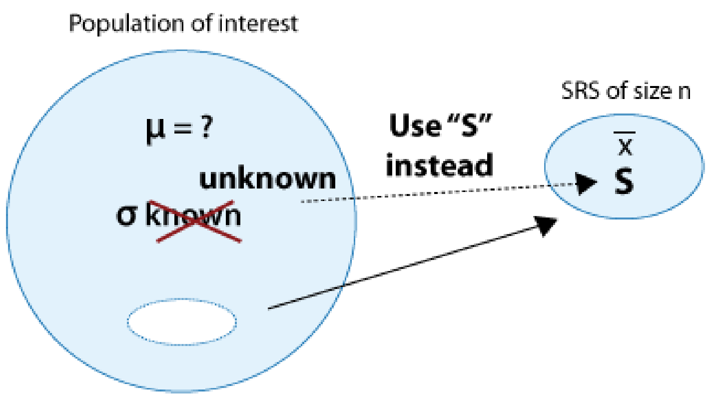
Hypothesis Testing for the Population Mean: the t Distribution

Learning Objective: Carry out hypothesis testing for the population proportion and mean (when appropriate), and draw conclusions in context.

Tests About μ When σ is Unknown—The t-test for the Population Mean

As we mentioned earlier, only in a few cases is it reasonable to assume that the population standard deviation, σ, is known. The case where σ is unknown is much more common in practice. What can we use to replace σ? If you don't know the population standard deviation, the best you can do is find the sample standard deviation, S, and use it instead of σ. (Note that this is exactly what we did when we discussed confidence intervals).



Is that it? Can we just use S instead of σ, and the rest is the same as the previous case? Unfortunately, it's not that simple, but not very complicated either.

We will first go through the four steps of the t-test for the population mean and explain in what way this test is different from the z-test in the previous case. For comparison purposes, we will then apply the t-test to a variation of the two examples we used in the previous case, and end with an activity where you'll get to carry out the t-test yourself.

Let's start by describing the four steps for the t-test:

I. Stating the hypotheses.

In this step there are no changes:

* The null hypothesis has the form:

$$H_0 : \mu = \mu_0$$

(where μ_0 is the null value).

* The alternative hypothesis takes one of the following three forms (depending on the context):

$H_a : \mu < \mu_0$(one-sided)

$H_a : \mu > \mu_0$ (one-sided)

$H_a : \mu \neq \mu_0$ (two-sided)

II. Checking the conditions under which the t-test can be safely used and summarizing the data.

Technically, this step only changes slightly compared to what we do in the z-test. However, as you'll see, this small change has important implications. The conditions under which the t-test can be safely carried out are exactly the same as those for the z-test:

(i) The sample is random (or at least can be considered random in context).

(ii) We are in one of the three situations marked with a green check mark in the following table (which ensure that $\overline{X}$ is at least approximately normal):

Conditions: z-test for a population mean	Small sample size	Large sample size
Variable varies normally in the population	✓	✓
Variable doesn't vary normally in the population	✗	✓

Assuming that the conditions are met, we calculate the sample mean $\overline{x}$ and the sample standard deviation, S (which replaces σ), and summarize the data with a test statistic. As in the z-test, our test statistic will be the standardized score of $\overline{x}$ assuming that $\mu = \mu_0$ (H_o is true). The difference here is that we don't know σ, so we use S instead. The test statistic for the t-test for the population mean is therefore:

$$t = \frac{\overline{x} - \mu_0}{\frac{s}{\sqrt{n}}}$$

The change is in the denominator: while in the z-test we divided by the standard **deviation** of $\overline{X}$, namely $\frac{\sigma}{\sqrt{n}}$, here we divide by the standard **error** of $\overline{X}$, namely $\frac{s}{\sqrt{n}}$. Does this have an effect on the rest of the test? Yes. The t-test statistic in the test for the mean does not follow a standard normal distribution. Rather, it follows another bell-shaped distribution called the t distribution. So we first need to introduce you to this new distribution as a general object. Then, we'll come back to our discussion of the t-test for the mean and how the t-distribution arises in that context.

The t Distribution

We have seen that variables can be visually modeled by many different sorts of shapes, and we call these shapes distributions. Several distributions arise so frequently that they have been given special names, and they have been studied mathematically. So far in the course, the only one we've named is the normal distribution, but there are others. One of them is called the t distribution.

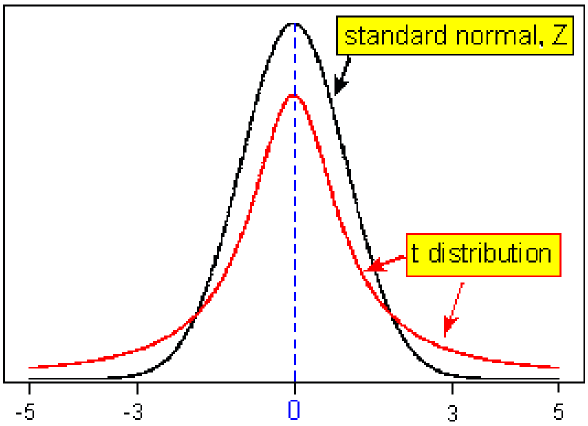
The t distribution is another bell-shaped (unimodal and symmetric) distribution, like the normal distribution; and the center of the t distribution is standardized at zero, like the center of the normal distribution.

Like all distributions that are used as probability models, the normal and the t distribution are both scaled, so the total area under each of them is 1.

So how is the t distribution fundamentally **different** from the normal distribution?

The **spread**.

The following picture illustrates the fundamental difference between the normal distribution and the t distribution:



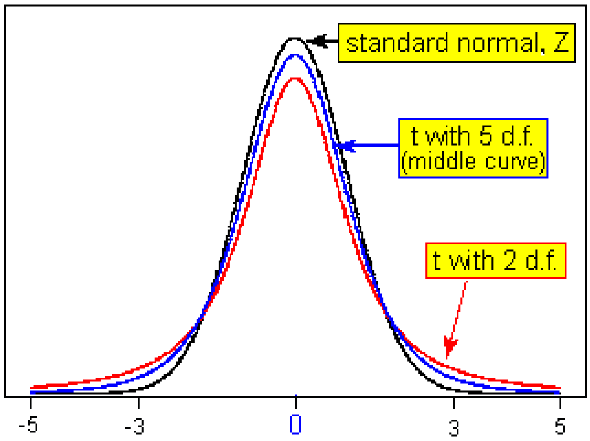
You can see in the picture that the t distribution has **slightly less area near the expected central value** than the normal distribution does, and you can see that the t distribution has correspondingly **more area in the "tails"** than the normal distribution does. (It's often said that the t distribution has "fatter tails" or "heavier tails" than the normal distribution.)

This reflects the fact that the t distribution **has a larger spread** than the normal distribution. The same total area of 1 is spread out over a slightly wider range on the t distribution, making it a bit lower near the center compared to the normal distribution, and giving the t distribution slightly more probability in the 'tails' compared to the normal distribution.

Therefore, the t distribution ends up being the appropriate model in certain cases where there is **more variability** than would be predicted by the normal distribution. One of these cases is stock values, which have more variability (or "volatility," to use the economic term) than would be predicted by the normal distribution.

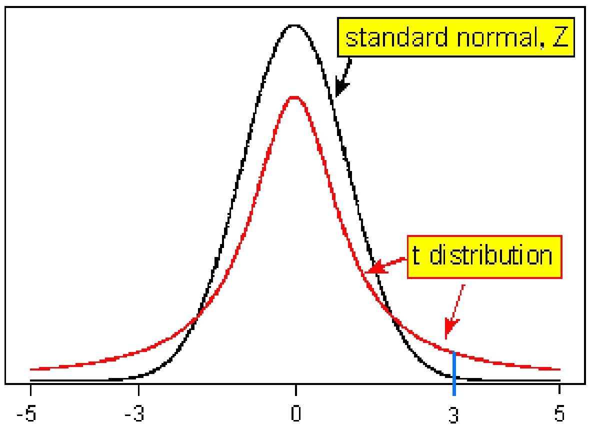
There's actually an entire family of t distributions. They all have similar formulas (but the math is beyond the scope of this introductory course in statistics), and they all have slightly "fatter tails" than the normal distribution. But some are closer to normal than others. The t distributions that are closer to normal are said to have higher "degrees of freedom" (that's a mathematical concept that we won't use in this course, beyond merely mentioning it here). So, there's a t distribution "with one degree of freedom," another t distribution "with 2 degrees of freedom" which is slightly closer to normal, another t distribution "with 3 degrees of freedom." which is a bit closer to normal than the previous ones, and so on.

The following picture illustrates this idea with just a couple of t distributions (note that "degrees of freedom" is abbreviated "d.f." on the picture):



Learn By Doing

The following figure of the standard normal distribution together with a t distribution will visually help you answer the following questions.



Learn By Doing

1/1 point (graded)

We know that the "upper tail probability" of 3 under the Z distribution, P(Z > 3), is roughly 0.0015 (using the 99.7% part of the Standard Deviation Rule for the normal distributions). Which of the following do you think is the upper tail probability of 3 under the t distribution in the figure above?

- ◯ 0.0015, same as for Z
- ◉ larger than 0.0015 ✔
- ◯ smaller than 0.0015

Answer

Correct:

Indeed, the t distribution is more spread out than the Z distribution and, therefore, values that are further away from the mean (0) are more likely. In particular, under the t distribution it is more likely to get values that are above 3 than under the Z distribution. Visually, you can see that the t distribution has heavier tails and, therefore, the area under the t distribution to the right of 3 is larger than the area to the right of 3 under the Z distribution.

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Learn By Doing

1/1 point (graded)

The 95th percentile of the Z distribution is 1.645, since $P(Z < 1.645) = 0.95$, or $P(Z > 1.645) = 0.05$. Which of the following do you think is the 95th percentile of the t distribution in the figure above?

- (x) larger than 1.645 ✔
- ( ) smaller than 1.645
- ( ) 1.645

Answer

Correct:

Indeed, since the t distribution has heavier tails, the area under the t distribution to the right of 1.645 would be larger than 0.05. The 95th percentile of the t distribution, therefore, must be to the right of, or larger than, 1.645.

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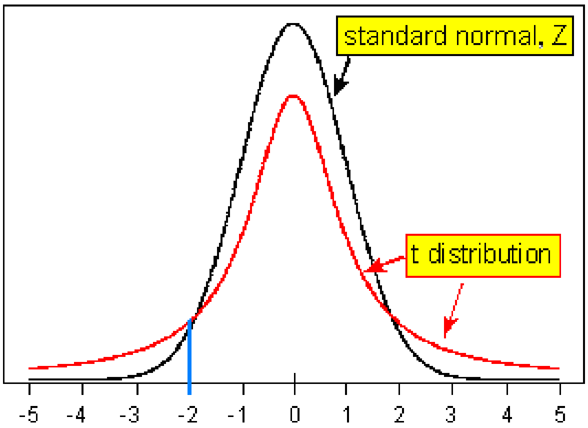
Did I Get This?

The following figure of the standard normal distribution together with a t distribution will visually help you answer the following questions.



Did I Get This

1/1 point (graded)

We know that the "lower tail probability" of less than a score of -2 under the normal distribution, P(Z < -2), is approximately 0.025 (from the 95% part of the approximation rule for normal distributions). Which of the following will be the lower tail probability of less than a score of -2 under the t distribution pictured above?

- smaller than 0.025
- 0.025, same as for Z
- larger than 0.025 ✔

Answer

Correct:

Indeed, the t distribution is more spread out than the Z distribution and, therefore, values that are further away from the mean (0) are more likely on the t distribution. In particular, getting values of less than a score of -2 is more likely with the t distribution than with the Z distribution. Visually, this is because the t distribution has "fatter tails" and, thus, there is more area to the left of -2 under the t distribution than under the Z distribution.

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Did I Get This

1/1 point (graded)

We know from the standard deviation rule that there is approximately a 68% chance that any normal random variable will get values within one standard deviation of the mean. In particular, P(-1 < Z < 1) = 0.68. Which of the following is true regarding the probability of getting values between -1 and 1 under the t distribution?

- (selected) The probability is less than 0.68. ✔
- The probability is more than 0.68.
- The probability is 0.68, the same as it is for Z.

Answer
Correct:
The t distribution is more spread out than the Z distribution and therefore the tail probabilities above 1 and below -1 are larger under the t distribution. As a result, the probability of getting values between -1 and 1 is **smaller** under the t distribution than it is under the Z distribution. Visually, the figure above shows that the area between the two blue lines is smaller under the t distribution compared to the area under the Z distribution.

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Now let's return to our discussion of the test for the mean, and let's see how and why the t distribution arises in that context.
