Confidence Intervals for the Population Mean: When To Use

Learning Objective: Explain what a confidence interval represents and determine how changes in sample size and confidence level affect the precision of the confidence interval.

Learning Objective: Find confidence intervals for the population mean and the population proportion (when certain conditions are met), and perform sample size calculations.

We are almost done with this section. We need to discuss just a few more questions:

- Is it always okay to use the confidence interval we developed for μ when σ is known?
- What if σ is unknown?
- How can we use statistical software to calculate confidence intervals for us?

When Is It Safe to Use the Confidence Interval We Developed?

One of the most important things to learn with any inference method is the conditions under which it is safe to use it. It is very tempting to apply a certain method, but if the conditions under which this method was developed are not met, then using this method will lead to unreliable results, which can then lead to wrong and/or misleading conclusions. As you'll see throughout this section, we always discuss the conditions under which each method can be safely used.

In particular, the confidence interval for μ (when σ is known), $\overline{x} \pm z^* * \frac{\sigma}{\sqrt{n}}$, was developed assuming that the sampling distribution of $\overline{X}$ is normal; in other words, that the Central Limit Theorem applies. In particular, this allowed us to determine the values of z*, the confidence multiplier, for different levels of confidence.

First, **the sample must be random.** Assuming that the sample is random, recall from the Probability unit that the Central Limit Theorem works when the **sample size is large** (a common rule of thumb for "large" is n > 30), or, for **smaller sample sizes**, if it is known that the quantitative **variable** of interest is **distributed normally** in the population. The only situation in which we cannot use the confidence interval, then, is when the sample size is small and the variable of interest is not known to have a normal distribution. In that case, other methods, called nonparametric methods, which are beyond the scope of this course, need to be used. This can be summarized in the following table:

	Small sample size	Large sample size
Variable varies normally	✓	✓
Variable doesn't vary normally	✗	✓

Did I Get This

1/1 point (graded)

Below are four different situations in which a confidence interval for μ is called for.

Situation A: In order to estimate μ, the mean annual salary of high-school teachers in a certain state, a random sample of 150 teachers was chosen and their average salary was found to be $38,450. From past experience, it is known that teachers' salaries have a standard deviation of $5,000.

Situation B: A medical researcher wanted to estimate μ, the mean recovery time from open-heart surgery for males between the ages of 50 and 60. The researcher followed the next 15 male patients in this age group who underwent open-heart surgery in his medical institute through their recovery period. (Comment: Even though the sample was not strictly random, there is no reason to believe that the sample of "the next 15 patients" introduces any bias, so it is as good as a random sample). The mean recovery time of the 15 patients was 26 days. From the large body of research that was done in this area, it is assumed that recovery times from open-heart surgery have a standard deviation of 3 days.

Situation C: In order to estimate μ, the mean score on the quantitative reasoning part of the GRE (Graduate Record Examination) of all MBA students, a random sample of 1,200 MBA students was chosen, and their scores were recorded. The sample mean was found to be 590. It is known that the quantitative reasoning scores on the GRE vary normally with a standard deviation of 150.

Situation D: A psychologist wanted to estimate μ, the mean time it takes 6-year-old children diagnosed with Down's Syndrome to complete a certain cognitive task. A random sample of 12 children was chosen and their times were recorded. The average time it took the 12 children to complete the task was 7.5 minutes. From past experience with similar tasks, the time is known to vary normally with a standard deviation of 1.3 minutes.

In which situation can we **NOT** use the confidence interval that we developed?

- ○ Situation A
- ◉ Situation B ✔
- ○ Situation C
- ○ Situation D

Answer
Correct:
Even though the sample was selected in a manner that is not strictly random, no bias is expected. However, the sample size is small enough (less than 30) to require that the recovery times are normally distributed. Since we do not know this, we cannot use this confidence interval.

Submit

Did I Get This

1/1 point (graded)
Below are four different situations in which a confidence interval formula would be useful:

Situation A: A marketing executive wants to estimate the average time, in days, that a watch battery will last. She tests 50 randomly selected batteries and finds that the distribution is skewed to the left, since a couple of the batteries were defective. It is known from past experience that the standard deviation is 25 days.

Situation B: A college professor desires an estimate of the mean number of hours per week that full-time college students are employed. He randomly selected 250 college students and found that they worked a mean time of 18.6 hours per week. He uses previously known data for his standard deviation.

Situation C: A medical researcher at a sports medicine clinic uses 35 volunteers from the clinic to study the average number of hours the typical American exercises per week. It is known that hours of exercise are normally distributed and past data give him a standard deviation of 1.2 hours.

Situation D: A high-end auto manufacturer tests 5 randomly selected cars to find out the damage caused by a 5 mph crash. It is known that this distribution is normal. Assume that the standard deviation is known.

In which situation will we **NOT** be able to use the confidence interval we developed?

- Situation A
- Situation B
- Situation C ✔
- Situation D

Answer
Correct:
The first requirement of a random sample was not achieved. Not only did the researcher use volunteers, but it is quite possible that patients at a sports clinic exercise more than the typical person. Without a random sample, it does not matter what the sample size or the type of distribution are.

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Scenario: First-Time Mothers Age 35 or Older and Low Birth Weight

Background: Some studies suggest that women having their first baby at age 35 or older are at increased risk of having a baby with a low birth weight. A medical researcher wanted to estimate μ, the mean weight of newborns who are the first child for women over the age of 35. To this end, the researcher chose a random sample of 125 women ages 35 and older who were pregnant with their first child and followed them through the pregnancy. The datafile contains the birth weight (in grams) of the 125 newborns (women pregnant with more than one child were excluded from the study). From past research, it is assumed that the weight of newborns has a standard deviation of σ = 500 grams.

The confidence interval for μ when σ is known is, $\overline{x} \pm z^* * \frac{\sigma}{\sqrt{n}}$.

Learn By Doing

1/1 point (graded)
Given that the sample mean is 3111 grams, what is the lower bound estimate for the 99% confidence interval? Round your answer to the nearest whole number.

2996

2996

Answer
Correct: 3111 - 2.576 * (500 / √125) = 2996

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Learn By Doing

1/1 point (graded)
Given that the sample mean is 3111.36 grams, what is the upper bound estimate for the 99% confidence interval? Round your answer to the nearest whole number.

3227

3227

Answer
Correct: 3111 + 2.576 * (500 / √125) = 3227

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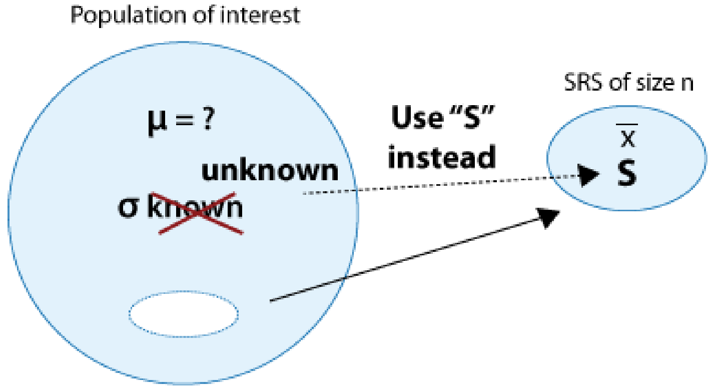
What if σ is unknown?

As we discussed earlier, when variables have been well-researched in different populations it is reasonable to assume that the population standard deviation (σ) is known. However, this is rarely the case. What if σ is unknown?

Well, there is some good news and some bad news.

The good news is that we can easily replace the population standard deviation, σ, with the **sample** standard deviation, s.



The bad news is that once σ has been replaced by s, we lose the Central Limit Theorem, together with the normality of $\overline{X}$, and therefore the confidence multipliers z* for the different levels of confidence (1.645, 2, 2.576) are (generally) not accurate any more. The new multipliers come from a different distribution called the "t distribution" and are therefore denoted by t* (instead of z*). We will discuss the t distribution in more detail when we talk about hypothesis testing.

The confidence interval for the population mean (μ) when (σ) is unknown is therefore:

$$\overline{x} \pm t^* * \frac{s}{\sqrt{n}}$$

(Note that this interval is very similar to the one when σ is known, with the obvious changes: s replaces σ, and t* replaces z* as discussed above.)

There is an important difference between the confidence multipliers we have used so far (z*) and those needed for the case when σ is unknown (t*). Unlike the confidence multipliers we have used so far (z*), which depend only on the level of confidence, the new multipliers (t*) have the **added complexity** that they **depend on both the level of confidence and on the sample size** (for example, the t* used in a 95% confidence when n = 10 is different from the t* used when n = 40). Due to this added complexity in determining the appropriate t*, we will rely heavily on software in this case.

Comments

1. Since it is quite rare that σ is known, this interval (sometimes called a *one-sample t confidence interval*) is more commonly used as the confidence interval for estimating μ. (Nevertheless, we could not have presented it without our extended discussion up to this point, which also provided you with a solid understanding of confidence intervals.)

2. The quantity $\frac{s}{\sqrt{n}}$ is called the **standard error** of $\overline{X}$. The central limit theorem tells us that $\frac{\sigma}{\sqrt{n}}$ is the **standard deviation** of $\overline{X}$ (and this is the quantity used in confidence interval when σ is known). In general, whenever we replace parameters with their sample counterparts in the standard deviation of a statistic, the resulting quantity is called the standard error of the statistic. In this case, we replaced σ with its sample counterpart (s), and thus $\frac{s}{\sqrt{n}}$ is the **standard error** of (the statistic) $\overline{X}$.

3. As before, to safely use this confidence interval, the sample **must be random**, and the only case when this interval cannot be used is when the sample size is small and the variable is not known to vary normally.

Learn By Doing

1/1 point (graded)

In the following situation determine whether or not it is safe to use the formula for the confidence interval for the population mean, μ.

In order to estimate μ, the mean weight of a typical passenger + his/her luggage on an early morning flight from New York City to Washington DC, an airline company chose a random sample of 16 passengers who took the flight. From past research on similar flights, it is known that weights of passengers + their luggage have a normal distribution with standard deviation σ = 20 pounds.

Can we safely estimate μ with the confidence interval that we developed?

- (selected) Yes ✔
- No

Answer

Correct:

Even though the sample size is only 16, we can safely use the confidence interval for the mean since we are given that the weights have a normal distribution.

Submit

Learn By Doing

1/1 point (graded)

In the following situation determine whether or not it is safe to use the formula for the confidence interval for the population mean, μ.

In order to estimate μ, the mean weight of a typical passenger + his/her luggage on an early morning flight from New York City to Washington DC, an airline company chose a random sample of 16 passengers who took the flight. From past research on similar flights, it is known that weights of passengers + their luggage have a standard deviation σ = 20 pounds.

Can we safely estimate μ with the confidence interval that we developed?

Yes

No ✔

Answer
Correct:
We cannot safely use the confidence interval for the mean since the sample is small (n = 16) and the variable of interest (weights) is not known to have a normal distribution.

Submit

Learn By Doing

1/1 point (graded)
IIn the following situation determine whether or not it is safe to use the formula for the confidence interval for the population mean, μ.

In order to estimate μ, the mean weight of a typical passenger + his/her luggage on an early morning flight from New York City to Washington DC, an airline company chose a random sample of 64 passengers who took the flight. From past research on similar flights, it is known that weights of passengers + their luggage have a standard deviation σ = 20 pounds.

Can we safely estimate μ with the confidence interval that we developed?

Yes ✔

No

Answer
Correct:
Even though we are not told that the variable of interest (weight) has a normal distribution, the sample size (n = 64) is large enough for us to be able to safely use the formula for the confidence interval for the mean.

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Scenario: Hours College Students Sleep

Background: As part of a large survey conducted at a large state university, a random sample of 142 students were asked: "How many hours do you sleep in a typical day?"

Even though σ is not known since the sample size is large researchers can use the sample standard deviation to estimate μ, the mean number of hours college students at this university sleep in a typical day.

The confidence interval for μ when σ is NOT known is, $\overline{x} \pm t^{*} * \frac{s}{\sqrt{n}}$.

Using this formula, the researchers calculate the 95% confidence interval to be: (7.09, 7.62).

Learn By Doing

1/1 point (graded)
True or false? Even though σ is unknown, we can safely use this confidence interval since the sample is random.

- (•) True ✔
- ( ) False

Answer
Correct:
To safely use this confidence interval, the sample must be random, and the only case when this interval cannot be used is when the sample size is small and the variable is not known to vary normally.

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Final Comment

It turns out that for large values of n, the t* multipliers are not that different from the z* multipliers, and therefore using the interval formula:

$$\overline{x} \pm z^{*} * \frac{s}{\sqrt{n}}$$

for μ when σ is unknown provides a pretty good approximation.
