Homework 11

Due Tuesday, December 8

Please either give the assignment to Loraine at the CDS or send it via email to the graders **before noon**.

1. *Convexity (10 points).* Let $f_i : \mathbb{R} \to \mathbb{R}$, $1 \leq i \leq m$ be m convex functions.
   a. Prove that $f := \sum_{i=1}^{n} a_i f_i$ is a convex function if the constants $a_1, \ldots, a_m$ are nonnegative.
   b. Prove that the function
   $$g(x) = \max_{1 \leq i \leq m} f_i(x) \tag{1}$$
   is convex.
   c. Prove that the function $h := \prod_{i=1}^{m} f_i$ is not necessarily convex.

2. *1D optimization (10 points).* In this problem you will implement the 1D optimization algorithms described in Lecture Notes 11.
   a. Complete the script *hw11_pb2.py*, implementing derivative descent and Newton's method. Submit the code that you add (*not* the whole script, just the code you write).

3. *Projected gradient descent (10 points).* We have focused on methods for unconstrained optimization. However, it is often of interest to solve problems of the form
   $$\min_{\boldsymbol{x} \in \mathcal{S}} f(\boldsymbol{x}) \tag{2}$$
   where $\mathcal{S}$ is a set.

   Projected gradient descent is an algorithm that solves this type of problem by projecting onto $\mathcal{S}$ after each iteration of gradient descent. This way the points always stay in $\mathcal{S}$.
   a. What is the projection of a vector $\boldsymbol{x} \in \mathbb{R}^n$ on the positive orthant $\mathbb{R}^n_+$?
   $$\mathbb{R}^n_+ = \{\boldsymbol{x} \quad | \quad x_i \geq 0,\ 1 \leq i \leq n\}. \tag{3}$$
   b. What is the projection of a vector $\boldsymbol{x} \in \mathbb{R}^n$ on the unit ℓ_2 ball $\mathcal{B}_{\ell_2}$ in n dimensions?
   $$\mathcal{B}_{\ell_2} = \{\boldsymbol{x} \quad | \quad ||\boldsymbol{x}||_2 \leq 1\}. \tag{4}$$
   c. Complete the script *hw11_pb3.py*. Limit the projected gradient descent iterations to 10. Submit your code and the figures that you obtain.
   d. Justify intuitively using the concept of contour line that the algorithm reaches the optimal point for both sets of interest.