Computing the power set.

For a set S, define the power set of S

as $\mathcal{P}(S) = \{X \mid X \subseteq S\}$. (the set of all subsets)

$|\mathcal{P}(S)| = 2^{|S|}$ (if $|S| < \infty$)

Another way to describe $X \subseteq S$ is a function $f_X : |S| \longrightarrow \{0,1\}$

$$f_X(y) = \begin{cases} 1 & \text{if } y \in X \\ 0 & \text{else} \end{cases}$$

Alternatively, view $S = \{s_1, s_2, \dots s_n\}$.

Then $X \subseteq S$ is described by a length n vector where entry $i = 1 \iff s_i \in X$.

Ex: $\mathcal{P}(\{1,2,3\}) = \{ \{\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\} \}$

Note $|\mathcal{P}(\{1,2,3\})| = 8 = 2^3$.

Let's try to find a recursive algorithm.

Note: if $T \subseteq S$, $\mathcal{P}(T) \subseteq \mathcal{P}(S)$.

Say $S = \{1,2,3\}$. Let $T = \{1,2\} \subseteq S$.

$$\mathcal{P}(T) = \{\{\}, \{1\}, \{2\}, \{1,2\}\}.$$

$$\mathcal{P}(S) = \mathcal{P}(T) \cup \underbrace{\{\{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}}_{\approx \mathcal{P}(T).}$$

Base case: if $S = \{\}$, return $\{\{\}\}$.
$= \{S\}$.

Otherwise let $x \in S$, and set
$T = S \setminus \{x\}$.

Now compute $\mathcal{P}(T)$ (We can get this for free via recursive magic since $|T| \lneq |S|$)

Define $\mathcal{P}(T)^x$ = Same as $\mathcal{P}(T)$, but with x added to each element.

$$= \bigcup_{H \in \mathcal{P}(T)} (H \cup \{x\})$$

Then $\underline{\mathcal{P}(S) = \mathcal{P}(T) \cup \mathcal{P}(T)^x}$. ✓