120040025
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yohandi - assignment 5 (computer-based)

1. In theory, for every $i \in [1, n]$ such that $T_i \sim Exp(\lambda_i)$,

$$T = T_1 + T_2 + \ldots + T_n$$

$$f_T(x) = \left(\prod_{i=1}^{n} \lambda_i\right)\left(\sum_{i=1}^{n} \frac{e^{-\lambda_i x}}{\prod_{j=1, \lambda_j \neq \lambda_i}^{n} (\lambda_j - \lambda_i)}\right)$$

(proof will be provided on the last page)

in this case $n = 3$ and $\lambda_i = \lambda_j$, $i \neq j$

$$\Rightarrow f_T(x) = \frac{\lambda^3 . x^2 e^{-\lambda x}}{2!}$$

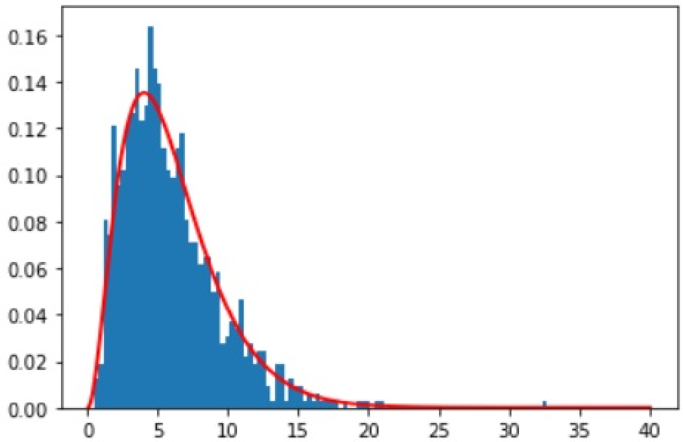
for $X \sim \Gamma(k, \theta) \equiv Gamma(k, \theta)$,

$$f(x; k, \theta) = \frac{x^{k-1} e^{-\frac{x}{\theta}}}{\theta^k \Gamma(k)}, x > 0 \text{ and } k, \theta > 0$$

$$f(x; 3, \tfrac{1}{\lambda}) = \frac{\lambda^3 . x^2 . e^{-\lambda x}}{2}$$

therefore, $T \sim Gamma(3, 2)$.

2 & 3 & 4.



here, red line represents the theoretical pdf of T

5. The Erlang distribution has two parameters "k" (denotes "shape") and "λ" (denotes "rate").

-> $k=1$ simplifies to the exponential distribution

-> sum of k independent exponential with mean $1/\lambda$ each simplifies to the gamma distribution

```
import numpy as np
import matplotlib.pyplot as plt
import scipy.special as sps

T1 = random.exponential(scale = 2, size = 1000)
T2 = random.exponential(scale = 2, size = 1000)
T3 = random.exponential(scale = 2, size = 1000)
T = T1 + T2 + T3
plt.hist(T, density = True, bins = 100)

count, bins = np.histogram(random.gamma(shape = 3, scale = 2, size = 100000), 100000, density = True)
y = bins ** (3 - 1) * (np.exp(-bins / 2) / (sps.gamma(3) * 2 ** 3))
plt.plot(bins, y, linewidth = 2, color = 'r')

plt.show()
```

proof :

$\Rightarrow$ for $n=2$,

$$f_{x_1+x_2}(x) = f_{x_1}(x) * f_{x_2}(x) = \int_0^x \lambda_1 \cdot e^{-\lambda_1(x-t)} \cdot \lambda_2 \cdot e^{-\lambda_2 \cdot t} dt = \lambda_1 \cdot \lambda_2 \cdot \frac{e^{-\lambda_2 x} - e^{-\lambda_1 x}}{\lambda_1 - \lambda_2}$$

(it is true)

$\Rightarrow$ assume that

$$f_{x_1+x_2+\dots x_n}(x) = f_{x_1+x_2+\dots+x_{n-1}}(x) * f_{x_n}(x) = \left(\prod_{i=1}^{n-1} \lambda_i\right)\left(\sum_{j=1}^{n-1} \frac{e^{-\lambda_j x}}{\prod_{k\neq j,\, k=1}^{n-1} (\lambda_k - \lambda_j)}\right) * f_{x_n}(x)$$

is true for $n \geq 3$ & "n" $= n-1$.

Since the coefficient of $e^{-\lambda_n x}$ fits the coefficients in our lemma,

$$-\sum_{j=1}^{n-1} \frac{1}{\prod_{k\neq j,\, k=1}^{n} (\lambda_k - \lambda_j)} = \frac{1}{\prod_{k=1}^{n-1} (\lambda_k - \lambda_n)},$$

equivalently,

$$\sum_{j=1}^{n} \frac{1}{\prod_{k\neq j,\, k=1}^{n} (\lambda_k - \lambda_j)} = 0$$

$$\Rightarrow \sum_{j=1}^{n} \prod_{\substack{k\neq l \neq j \\ k,l=1}}^{n} (\lambda_k - \lambda_l) = \sum_{j=1}^{n} \prod_{\substack{j\neq k \neq l \neq j \\ k,l=1}}^{n} (\lambda_k - \lambda_l) \prod_{\substack{k=j\neq l \\ k,l=1}}^{n} (\lambda_k - \lambda_l)$$

which equals to 0 if and only if :

$$\sum_{j=1}^{n} \prod_{\substack{j\neq k > l \neq j \\ k,l=1}}^{n} (\lambda_k - \lambda_l)(-1)^j = 0$$

therefore,

$$\begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^{n-2} \\ 1 & \lambda_2 & \lambda_2^2 & \dots & \lambda_2^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \dots & \lambda_n^{n-2} \end{vmatrix} = 0$$

It is true of $n \in \mathbb{N}$