yohandi - assignment 6

1. for $X$ follows the gamma distribution with parameter $\alpha$ and $\theta$,

$f(x)$ is given $\frac{1}{\Gamma(\alpha)\,\theta^{\alpha}} \cdot x^{\alpha-1} e^{-\frac{x}{\theta}}$

$$M(t) = E(e^{tX})$$
$$= \int_0^\infty e^{tx} \cdot \frac{x^{\alpha-1} \cdot e^{-\frac{x}{\theta}}}{\Gamma(\alpha)\cdot\theta^{\alpha}}\, dx$$
$$= \frac{1}{\Gamma(\alpha)\cdot\theta^{\alpha}} \int_0^\infty x^{\alpha-1} \cdot e^{x(t-\frac{1}{\theta})} dx$$

let $y = \frac{1-\theta t}{\theta} x$, $1-\theta t > 0$

$$\frac{dy}{dx} = \frac{1-\theta t}{\theta}$$

$$M(t) = \frac{1}{\Gamma(\alpha)\cdot\theta^{\alpha}} \int_0^\infty \left(\frac{\theta y}{1-\theta t}\right)^{\alpha-1} \cdot e^{-y} \cdot \left(\frac{\theta}{1-\theta t}\right) dy$$
$$= \frac{1}{\Gamma(\alpha)\cdot(1-\theta t)^{\alpha}} \int_0^\infty e^{-y} \cdot y^{\alpha-1}\, dy$$
$$= \frac{1}{(1-\theta t)^{\alpha}}$$

2. $M(t) = \frac{1}{(1-7t)^{20}}$,

W follows the gamma distribution with parameter $\alpha = 20$ and $\theta = 7$.

$$\Rightarrow f(w) = \frac{1}{\Gamma(20)\cdot 7^{20}} \cdot w^{19} e^{-\frac{w}{7}}$$
$$M'(t) = \frac{d}{dt} M(t) = \frac{140}{(1-7t)^{21}}$$
$$M''(t) = \frac{d}{dt} M'(t) = \frac{20580}{(1-7t)^{22}}$$
$$\Rightarrow E(w) = M'(0) = 140$$
$$\Rightarrow Var(w) = M''(0) - M'(0) = 980$$

3. $X \sim \chi^2(17)$

a. $P(X < 7.564) = 1 - P(X \geq 7.564) = 0.025$

b. $P(X > 27.59) \approx P(X > 27.587) = 0.050$

c. $P(6.408 < X < 27.59) = P(X > 6.408) - P(X > 27.59)$
$= 0.940$

d. $\chi^2_{0.95}(17) = 8.672$

e. $\chi^2_{0.025}(17) = 30.191$

4. $F(x) = \int_{-\infty}^{x} f(t)\, dt$
$$= \int_{-\infty}^{x} \frac{-d(1+e^{-t})}{(1+e^{-t})^2}\, dt$$
$$= \frac{1}{1+e^{-x}}$$

for $Y = \frac{1}{1+e^{-x}}$,

let $G$ be the cumulative distribution function of $Y$, we obtain:

$$G(y) = P(Y \leq y)$$
$$= P\left(\frac{1}{1+e^{-x}} \leq y\right)$$
$$= P\left(X \leq -\ln\left(\frac{1}{y} - 1\right)\right)$$
$$= \frac{1}{\left(\frac{1}{y}\right)}$$
$$= y$$

as the pdf function of $Y$, $g = G' = 1$ is a uniform distribution with $a = 0$ and $b = 1$ (i.e. $Y \sim U(0,1)$)

5a. $P(0 \leq Z \leq 0.87) = \phi\left(\frac{0.87-0}{\sqrt{1}}\right) - \phi\left(\frac{0-0}{\sqrt{1}}\right)$
$= 0.30785$

b. $P(-2.64 \leq Z \leq 0) = \phi\left(\frac{0-0}{\sqrt{1}}\right) - \phi\left(\frac{-2.64-0}{\sqrt{1}}\right)$
$= 0.49585$

c. $P(-2.13 \leq Z \leq -0.56) = \phi\left(\frac{-0.56-0}{\sqrt{1}}\right) - \phi\left(\frac{-2.13-0}{\sqrt{1}}\right)$
$= 0.27115$

d. $P(|Z| > 1.39) = 1 - \left(2\phi\left(\frac{1.39-0}{\sqrt{1}}\right) - 1\right)$
$= 0.16452$

e. $P(Z < -1.62) = \phi\left(\frac{-1.62-0}{\sqrt{1}}\right)$
$= 0.05262$

f. $P(|Z| > 1) = 1 - \left(2\phi\left(\frac{1-0}{\sqrt{1}}\right) - 1\right)$
$= 0.31732$

g. $P(|Z| > 2) = 1 - \left(2\phi\left(\frac{2-0}{\sqrt{1}}\right) - 1\right)$
$= 0.04550$

h. $P(|Z|>3) = 1-\left(2\phi\left(\frac{3-0}{\sqrt{1}}\right)-1\right)$

$= 0.00270$

6. a. $P(Z \geq c) = 0.025$

$c = \phi^{-1}(1-0.025) = 1.9600$

b. $P(|Z| \leq c) = 0.95$

$c = \phi^{-1}\left(\frac{0.95+1}{2}\right) = 1.9600$

c. $P(Z > c) = 0.05$

$c = \phi^{-1}(1-0.05) = 1.6449$

d. $P(|Z| \leq c) = 0.90$

$c = \phi^{-1}\left(\frac{0.90+1}{2}\right) = 1.6449$

7. a. $P(6 \leq X \leq 12) = \phi\left(\frac{12-6}{\sqrt{25}}\right) - \phi\left(\frac{6-6}{\sqrt{25}}\right)$

$= 0.38493$

b. $P(0 \leq X \leq 8) = \phi\left(\frac{8-6}{\sqrt{25}}\right) - \phi\left(\frac{0-6}{\sqrt{25}}\right)$

$= 0.54035$

c. $P(-2 < X \leq 0) = \phi\left(\frac{0-6}{\sqrt{25}}\right) - \phi\left(\frac{-2-6}{\sqrt{25}}\right)$

$= 0.06027$

d. $P(X > 21) = 1 - \phi\left(\frac{21-6}{\sqrt{25}}\right)$

$= 0.00135$

e. $P(|X-6|<5) = \phi\left(\frac{11-6}{\sqrt{25}}\right) - \phi\left(\frac{1-6}{\sqrt{25}}\right)$

$= 0.68268$

f. $P(|X-6|<10) = \phi\left(\frac{16-6}{\sqrt{25}}\right) - \phi\left(\frac{-4-6}{\sqrt{25}}\right)$

$= 0.95450$

g. $P(|X-6|<15) = \phi\left(\frac{21-6}{\sqrt{25}}\right) - \phi\left(\frac{-9-6}{\sqrt{25}}\right)$

$= 0.99730$

h. $P(|X-6|<12.41) = \phi\left(\frac{18.41-6}{\sqrt{25}}\right) - \phi\left(\frac{-6.41-6}{\sqrt{25}}\right)$

$= 0.98694$

8. $M_X(t) = e^{166t+200t^2}$

$\mu = 166$

$\frac{\sigma^2}{2} = 200 \Rightarrow \sigma^2 = 400$

$X \sim N(166, 400)$

a. $E(X) = \mu = 166$

b. $Var(X) = \sigma^2 = 400$

c. $P(170 < X < 200) = \phi\left(\frac{200-166}{\sqrt{400}}\right) - \phi\left(\frac{170-166}{\sqrt{400}}\right)$

$= 0.37617$

d. $P(148 \leq X \leq 172) = \phi\left(\frac{172-166}{\sqrt{400}}\right) - \phi\left(\frac{148-166}{\sqrt{400}}\right)$

$= 0.43385$

9. for $X \sim N(\mu, \sigma^2)$ and $Y = aX+b$,

$P(Y \leq y) = P(aX+b \leq y)$

$= P\left(X \leq \frac{y-b}{a}\right)$

$= \phi\left(\frac{\frac{y-b}{a}-\mu}{\sigma}\right)$

$= \phi\left(\frac{y-(b+a\mu)}{a\sigma}\right)$

$\therefore Y \sim N(b+a\mu, a^2\sigma^2)$

10. for $Y \sim N(10,1)$ and $X = e^Y$,

cdf: $F(x) = P(Y \leq \ln(x))$

$= \int_{-\infty}^{\ln(x)} \frac{1}{\sqrt{2\pi \cdot 1}} \cdot e^{-\frac{1}{2}\left(\frac{t-10}{1}\right)^2} dt$

pdf: $f(x) = \frac{d}{dx} F(x)$

$= \frac{d}{dx}\ln(x) \cdot \frac{1}{\sqrt{2\pi \cdot 1}} e^{-\frac{1}{2}(\ln(x)-10)^2}$

$= \frac{1}{x\sqrt{2\pi}} e^{-\frac{1}{2}(\ln(x)-10)^2}$

$P(10000 < X < 20000)$

$= \phi\left(\frac{\ln(20000)-10}{1}\right) - \phi\left(\frac{\ln(10000)-10}{1}\right)$

$= 0.24670$

yohandi - assignment 6 (computer-based)

1. Theoretically,

$$\text{mean } (\mu) = E(X) = np = 40\left(\tfrac{1}{2}\right) = 20$$

$$\text{variance } (\sigma^2) = Var(X) = npq = 40\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) = 10$$

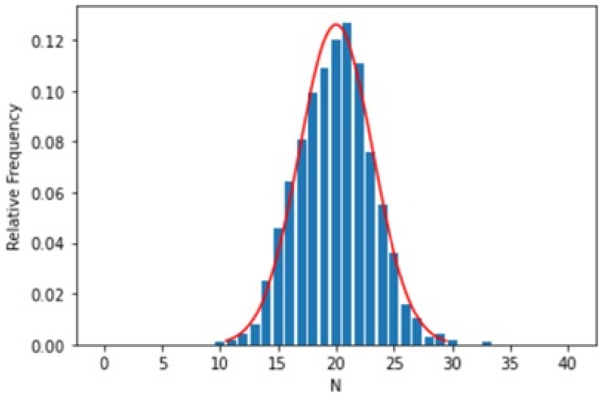
$X$ follows the normal distribution with mean $\mu = 20$ and variance $\sigma^2 = 10$

$\Rightarrow X \sim N(20, 10)$

2&3.

```python
import math
import matplotlib.pyplot as plt
import numpy as np
import pandas as pd
import random
import scipy.stats as stats

def experiment():
    ret = 0
    for i in range(40):
        if random.randint(1, 2) == 1:
            ret += 1
    return ret

simulation = [0 for i in range(41)]
relativeFrequency = []
for i in range(1000):
    simulation[experiment()] += 1
for i in range(41):
    relativeFrequency.append(simulation[i] / 1000)

mu = 20
variance = 10
sigma = math.sqrt(variance)
x = np.linspace(mu - 3 * sigma, mu + 3 * sigma, 100)
plt.plot(x, stats.norm.pdf(x, mu, sigma), color = 'r')

plt.bar(tuple([i for i in range(41)]), tuple(relativeFrequency), align = 'center')
plt.xlabel('N')
plt.ylabel('Relative Frequency')

plt.show()
```

4.




5. $P(19.5 < x < 20.5) = \phi\left(\frac{20.5-20}{\sqrt{10}}\right) - \phi\left(\frac{19.5-20}{\sqrt{10}}\right)$

$\approx 0.56282 - 0.43718$

$\approx 0.12564$

let $Y$ be the number of heads that occurs,

$Y \sim B(40, \frac{1}{2})$

$P(Y=20) = \binom{40}{20}\left(\frac{1}{2}\right)^{20}\left(\frac{1}{2}\right)^{40-20}$

$\approx 0.12537$

from the result,

$$\mathcal{E} = |P(19.5 < x < 20.5) - P(Y=20)| \approx 0.00027 < 10^{-3}$$

the error is way less than $10^{-3}$; therefore, the approximation using the normal distribution in this case is accurate to 3 decimal places.