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MAT2040 Assignment 1

Q1.
- False
- True
- False

  counter example:

  $A = [1 \ \ 1]$

  $B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

- True

  RHS: $A^2 - AI + IA - I^2 = A^2 - I^2$

- False

  LHS: $A^2 + AB + BA + B^2 \neq A^2 + 2AB + B^2$

Q2. define S_i as the solution set of i-th ~~l~~ linear system (i.e. $S_i = \{x_i, y_i, z_i\}$)

- $$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 1 & 1 & 8 \\ 1 & 1 & 0 & 5 \end{array}\right] \xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 3 & -3 & 6 \\ 0 & 2 & -2 & 4 \end{array}\right] \ldots$$

$$\xrightarrow{R_2 = \frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & -2 & 4 \end{array}\right] \xrightarrow{R_3 = R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$S_1 : \{3 - t, 2 + t, t \mid t \in \mathbb{R}\}$

- $$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & 2 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 0 & 1 & -1 & 2 \\ 1 & 0 & 1 & 3 \end{array}\right] \ldots$$

$$\xrightarrow{R_3 = R_3 - R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$S_2 : \{3 - t, 2 + t, t \mid t \in \mathbb{R}\}$

since $S_1 = S_2$, the linear systems are equivalent

Q3. 1)
$$\begin{bmatrix} 4 & 5 & 3 & 3 & 4 \\ 2 & 3 & 1 & 0 & 1 \\ 3 & 4 & 2 & 1 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccccc|c} 4 & 5 & 3 & 3 & 4 & -5 \\ 2 & 3 & 1 & 0 & 1 & -3 \\ 3 & 4 & 2 & 1 & 1 & -1 \end{array}\right]$$

2)
$$\left[\begin{array}{ccccc|c} 4 & 5 & 3 & 3 & 4 & -5 \\ 2 & 3 & 1 & 0 & 1 & -3 \\ 3 & 4 & 2 & 1 & 1 & -1 \end{array}\right] \ldots$$

$$\xrightarrow[R_3 = 4R_3]{R_2 = 2R_2} \left[\begin{array}{ccccc|c} 4 & 5 & 3 & 3 & 4 & -5 \\ 4 & 6 & 2 & 0 & 2 & -6 \\ 12 & 16 & 8 & 4 & 4 & -4 \end{array}\right] \ldots$$

$$\xrightarrow[R_3 = R_3 - 3R_1]{R_2 = R_2 - R_1} \left[\begin{array}{ccccc|c} 4 & 5 & 3 & 3 & 4 & -5 \\ 0 & 1 & -1 & -3 & -2 & -1 \\ 0 & 1 & -1 & -5 & -8 & 11 \end{array}\right] \ldots$$

$$\xrightarrow{R_3 = R_3 - R_2} \left[\begin{array}{ccccc|c} 4 & 5 & 3 & 3 & 4 & -5 \\ 0 & 1 & -1 & -3 & -2 & -1 \\ 0 & 0 & 0 & -2 & -6 & 12 \end{array}\right] \ldots$$

$$\xrightarrow{R_3 = -\frac{1}{2}R_3} \left[\begin{array}{ccccc|c} 4 & 5 & 3 & 3 & 4 & -5 \\ 0 & 1 & -1 & -3 & -2 & -1 \\ 0 & 0 & 0 & 1 & 3 & -6 \end{array}\right] \ldots$$

$$\xrightarrow[R_2 = R_2 + 3R_3]{R_1 = R_1 - 3R_3} \left[\begin{array}{ccccc|c} 4 & 5 & 3 & 0 & -5 & 13 \\ 0 & 1 & -1 & 0 & 7 & -19 \\ 0 & 0 & 0 & 1 & 3 & -6 \end{array}\right] \ldots$$

$$\xrightarrow{R_1 = R_1 - 5R_2} \left[\begin{array}{ccccc|c} 4 & 0 & 8 & 0 & -40 & 108 \\ 0 & 1 & -1 & 0 & 7 & -19 \\ 0 & 0 & 0 & 1 & 3 & -6 \end{array}\right]$$

$$\xrightarrow{R_1 = \frac{1}{4}R_1} \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -10 & 27 \\ 0 & 1 & -1 & 0 & 7 & -19 \\ 0 & 0 & 0 & 1 & 3 & -6 \end{array}\right]$$

$x_1 : 27 - 2x_3 + 10x_5$

$x_2 : -19 + x_3 - 7x_5$

$x_4 : -6 - 3x_5$

x_3 & x_5 are free

Q4.
$$\left[\begin{array}{ccc|c} 3 & k & 1 & 0 \\ 0 & 4 & 1 & 0 \\ k & -5 & -1 & 0 \end{array}\right] \xrightarrow{R_3 = 3R_3} \left[\begin{array}{ccc|c} 3 & k & 1 & 0 \\ 0 & 4 & 1 & 0 \\ 3k & -15 & -3 & 0 \end{array}\right] \ldots$$

$$\xrightarrow{R_3 = R_3 - kR_1} \left[\begin{array}{ccc|c} 3 & k & 1 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & -15-k^2 & -3-k & 0 \end{array}\right] \ldots$$

$$\xrightarrow{R_3 = -4R_3} \left[\begin{array}{ccc|c} 3 & k & 1 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 60+4k^2 & 12+4k & 0 \end{array}\right] \ldots$$

$$\xrightarrow{R_3 = R_3 - (15+k^2)R_2} \left[\begin{array}{ccc|c} 3 & k & 1 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & -3+4k-k^2 & 0 \end{array}\right]$$

the system will have infinite solutions (other than the non-zero solution) if $-3 + 4k - k^2 = 0$ (i.e. $k = 3$ or $k = 1$)

Q5. linear system:

$$\left[\begin{array}{ccc|c} 1 & 4 & -2 & 1 \\ 1 & 7 & -6 & 6 \\ 0 & 3 & p & 9 \end{array}\right] \xrightarrow{R_2=R_2-R_1} \left[\begin{array}{ccc|c} 1 & 4 & -2 & 1 \\ 0 & 3 & -4 & 5 \\ 0 & 3 & p & 9 \end{array}\right] \dots$$

$$\xrightarrow{R_3=R_3-R_2} \left[\begin{array}{ccc|c} 1 & 4 & -2 & 1 \\ 0 & 3 & -4 & 5 \\ 0 & 0 & p+4 & q-5 \end{array}\right]$$

1) $\begin{matrix} p+4=0 \\ q-5\neq 0 \end{matrix} \Rightarrow \begin{matrix} p=-4 \\ q\neq 5 \end{matrix}$

2) $p+4\neq 0 \Rightarrow p\neq -4$

3) $\begin{matrix} p+4=0 \\ q-5=0 \end{matrix} \Rightarrow \begin{matrix} p=-4 \\ q=5 \end{matrix}$

Q6. system:

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 1 & 2 & 4 & b & 0 \end{array}\right] \xrightarrow{R_4=R_4-R_1} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 2 & 5 & b-1 & 0 \end{array}\right] \dots$$

$$\xrightarrow{R_4=R_4-2R_2} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & b-3 & 0 \end{array}\right] \dots$$

$$\xrightarrow{R_4=R_4-R_3} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & b-2 & 0 \end{array}\right]$$

the system will have infinitely many solutions if and only if:

$\Rightarrow b-2=0$

$\Rightarrow b=2$

Q7. system:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & a & -1 \\ 2 & 3 & 0 & b \end{array}\right] \xrightarrow{R_3=R_3-R_2-R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & a & -1 \\ 0 & 0 & -a-1 & b-1 \end{array}\right] \dots$$

$$\xrightarrow{R_2=R_2-R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & a-1 & -3 \\ 0 & 0 & -a-1 & b-1 \end{array}\right]$$

1) $\begin{matrix} -a-1=0 \\ b-1\neq 0 \end{matrix} \Rightarrow \begin{matrix} a=-1 \\ b\neq 1 \end{matrix}$

2) $-a-1\neq 0 \Rightarrow a\neq -1$

3) $\begin{matrix} -a-1=0 \\ b-1=0 \end{matrix} \Rightarrow \begin{matrix} a=-1 \\ b=1 \end{matrix}$

Q8. system:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & a & 0 \\ 1 & 4 & a^2 & 0 \\ 1 & 2 & 1 & a-1 \end{array}\right] \xrightarrow{R_4=R_4-R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & a & 0 \\ 1 & 4 & a^2 & 0 \\ 0 & 0 & 1-a & a-1 \end{array}\right]$$

the system will have at least one solution if and only if:

$\Rightarrow 1-a=a-1$

$\Rightarrow a=1$

when $a=1$, system:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_3=R_3-R_1]{R_2=R_2-R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

the solution set: $x_1=-x_3$

$x_2=0$

x_3 is free

Q9. $A^k = \begin{cases} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, & k \text{ is even} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, & k \text{ is odd} \end{cases}$

proof: $A^2 = A\cdot A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow A^2 = I_3$

$\Rightarrow$ for $k=2n$ (n is integer),

$A^k = A^{2n} = (A^2)^n = I_3{}^n = I_3$

$\Rightarrow$ for $k=2n+1$ (n is integer),

$A^k = A^{2n+1} = (A^2)^n\cdot A = I_3{}^n \cdot A = A$

hence, proved.

Q10. (a) $(A^2-B^2)^T = (A^2)^T - (B^2)^T$
$= (A^T)^2 - (B^T)^2$
$= A^2 - B^2$ sym.

(b) $((A+B)(A-B))^T = (A-B)^T(A+B)^T$
$= (A^T - B^T)(A^T + B^T)$
$= (A-B)(A+B)$ non-sym.

(c) $(ABA)^T = A^T B^T A^T$
$= ABA$ sym.

(d). $(ABAB)^T = B^T A^T B^T A^T$
$= BABA$ non-sym.

Q11. $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 2 & 1 & 0 \\ 3 & \frac{3}{2} & 0 \end{bmatrix}$

$A^2 = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 2 & 0 \\ 6 & 3 & 0 \end{bmatrix} = 2A$

$A^3 = A^2 . A = 2A . A = 2A^2 = 2 . 2A = 2^2 A$

$\vdots$

$A^{11} = 2^{10} A = \begin{bmatrix} 1024 & 512 & 0 \\ 2048 & 1024 & 0 \\ 3072 & 1536 & 0 \end{bmatrix}$

Q12. $A^2 = A.A = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = 9 I_3$

$A^8 = (A^2)^4 = 9^4 I_3$

$\Rightarrow A^8 - 6400 I_3 = 161 I_3 = \begin{bmatrix} 161 & 0 & 0 \\ 0 & 161 & 0 \\ 0 & 0 & 161 \end{bmatrix}$

Q13. $(I-A)(I + A + A^2 + \dots + A^{k-1})$
$= (I + A + A^2 + \dots + A^{k-1}) - (A + A^2 + A^3 + \dots + A^k)$
$= I - A^k$
$= I$

Q14 (a) $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

$AB = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

we can see $A \neq 0$ and $B \neq 0$ but $AB = 0$

(b) $A = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$AB - BA = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= 0$

we can see $A \neq B$ but $AB - BA = 0$

Q15 (a) $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

we can see $A \neq 0$ but $A^2 = 0$

(b) $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = A$

we can see $A \neq 0$ and $A \neq I$ but $A^2 = A$

Q16. $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$(A-B)^2 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

we can see $A \neq B$ but $(A-B)^2 = 0$

Q17. $A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ & \vdots & & \\ A_{N1} & A_{N2} & \cdots & A_{NN} \end{bmatrix}$ $A^T = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{N1} \\ & \vdots & & \\ A_{1N} & A_{2N} & \cdots & A_{NN} \end{bmatrix}$

note that $A^TA = (0)_{N\times N}$

consider the main diagonal A^TA,

$$A^TA_{ii} = \sum_{j=1}^{N} A_{ij}^2 = 0$$

since $A_{ij}^2 \geq 0 \Rightarrow A_{ij} = 0$

for every $i = 1, \dots, N$

and every $j = 1, \dots, N$

$\therefore A = (A_{ij})_{N\times N} = (0)_{N\times N}$

$= O_{N\times N}$

Q18. $(A+B)^2 = A^2 + AB + BA + B^2 = A^2 + B^2$

$\Rightarrow AB + BA = 0$

$\Rightarrow BA = -AB$

$AB(A+I) = ABA + AB$

$= A(-AB) + AB$

$= -A^2B + AB$

$= -AB + AB$

$= 0$

Q19. $A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$A^3 = A^2 \cdot A = I_4 \cdot A$

$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

for $A^{2n} = (A^2)^n = (I_4)^n = I_4$

for $A^{2n+1} = A^{2n} \cdot A = I_4 \cdot A = A$

Q20. $\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$. $\alpha^T = [\alpha_1\ \alpha_2\ \alpha_3]$

$\alpha \cdot \alpha^T = \begin{bmatrix} \alpha_1^2 & \alpha_1\alpha_2 & \alpha_1\alpha_3 \\ \alpha_2\alpha_1 & \alpha_2^2 & \alpha_2\alpha_3 \\ \alpha_3\alpha_1 & \alpha_3\alpha_2 & \alpha_3^2 \end{bmatrix}$

$\alpha^T \cdot \alpha = [\alpha_1^2 + \alpha_2^2 + \alpha_3^2]$

$= [1^2 + 1^2 + 1^2]$

$= [3]$