Yohandi - homework week 6

Exercises 13.4

3. $r(t) = (2t+3)i + (5-t^2)j$

$v(t) = \frac{d}{dt}r(t) = (2)i + (-2t)j$

$|v(t)| = \sqrt{(2)^2+(-2t)^2} = 2\sqrt{t^2+1}$

$T = \frac{v(t)}{|v(t)|} = \frac{i}{\sqrt{t^2+1}} - \frac{t\,j}{\sqrt{t^2+1}}$

$\frac{dT}{dt} = -\frac{t}{(t^2+1)\sqrt{t^2+1}}i - \frac{1}{(t^2+1)\sqrt{t^2+1}}j$

$\left|\frac{dT}{dt}\right| = \frac{1}{t^2+1}$

$N = \frac{\frac{dT}{dt}}{\left|\frac{dT}{dt}\right|} = -\frac{t}{\sqrt{t^2+1}}i - \frac{1}{\sqrt{t^2+1}}j$

$k = \frac{1}{2\sqrt{t^2+1}} \cdot \frac{1}{t^2+1} = \frac{1}{2(t^2+1)\sqrt{t^2+1}}$

5a. $r(x) = xi + f(x)j = xi + yj$

$T = \frac{v(t)}{|v(t)|} = \frac{dx\,i + dy\,j}{\sqrt{(dx)^2+(dy)^2}} = \frac{i+(\frac{dy}{dx})j}{\sqrt{1+(\frac{dy}{dx})^2}}$

$\frac{dT}{dt} = \frac{\frac{d^2y}{dx\,dt}\sqrt{1+(\frac{dy}{dx})^2}\,j - (i+(\frac{dy}{dx})j)\frac{2\frac{dy}{dx}\frac{d^2y}{dx\,dt}}{2\sqrt{1+(\frac{dy}{dx})^2}}}{1+(\frac{dy}{dx})^2}$

$= \frac{\frac{d^2y}{dx\,dt}(-\frac{dy}{dx}i + j)}{(1+(\frac{dy}{dx})^2)^{3/2}}$

$K = \left| \frac{dt}{\sqrt{(dx)^2+(dy)^2}} \frac{\frac{d^2y}{dx\,dt}}{(1+(\frac{dy}{dx})^2)^{3/2}} \cdot \sqrt{(\frac{dy}{dx})^2+1} \right|$

$= \left| \frac{\frac{d^2y}{dx^2}}{(1+(\frac{dy}{dx})^2)^{3/2}} \right|$

b. $y(x) = \ln(\cos x)$

$\frac{d}{dx}y(x) = -\tan x$

$\frac{d^2}{dx^2}y(x) = -\sec^2 x$

$K(x) = \frac{\sec^2(x)}{(\sec^2(x))^{3/2}}$

$= \cos(x)$

c. $k(x) = 0 = \left| \frac{\frac{d^2y}{dx^2}}{(1+(\frac{dy}{dx})^2)^{3/2}} \right|$

$\frac{d^2y}{dx^2} = 0 \Rightarrow$ inflection point requirement

13. $r(t) = (\frac{t^3}{3})i + (\frac{t^2}{2})j$

$v(t) = t^2 i + t\,j$

$|v(t)| = t\sqrt{t^2+1}$

$T = \frac{v(t)}{|v(t)|} = \frac{t}{\sqrt{t^2+1}}i + \frac{1}{\sqrt{t^2+1}}j$

$\frac{dT}{dt} = \frac{1}{(t^2+1)^{3/2}}i - \frac{t}{(t^2+1)^{3/2}}j$

$\left|\frac{dT}{dt}\right| = \frac{1}{t^2+1}$

$N = \frac{\frac{dT}{dt}}{\left|\frac{dT}{dt}\right|} = \frac{1}{\sqrt{t^2+1}}i - \frac{t}{\sqrt{t^2+1}}j$

$K = \frac{1}{t\sqrt{t^2+1}} \cdot \frac{1}{t^2+1} = \frac{1}{t(t^2+1)^{3/2}}$

18. $\frac{dx}{dt} = -a\sin t.\quad \frac{dy}{dt} = b\cos t$

$\frac{d^2x}{dt^2} = -a\cos t \quad \frac{d^2y}{dt^2} = -b\sin t$

$k = \frac{(-a\sin t)(-b\sin t) - (-a\cos t)(b\cos t)}{((-a\cos t)^2+(b\cos t)^2)^{3/2}}$

$= \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}, \quad a>0,\ b>0$

$\frac{dk}{dt} = \frac{-\frac{3}{2}ab(a^2-b^2)\sin 2t}{(a^2\sin^2 t + b^2\cos^2 t)^{5/2}}$

$\frac{dk}{dt} = 0 \Rightarrow t=0, \quad t=\frac{\pi}{2}$

(maximum) (minimum)

at $(a,0)$ at $(0,b)$

Exercises 14.1

1a. $f(0,0) = 0^2 + 0 \cdot 0^3 = 0$

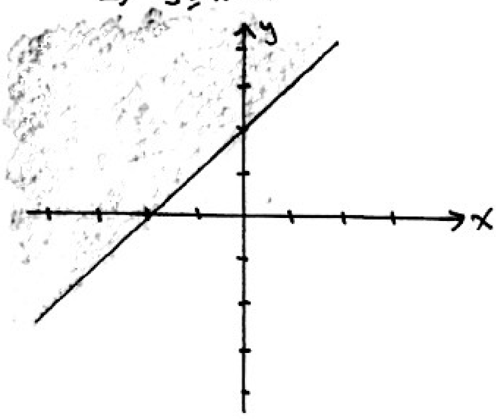
b. $f(-1,1) = (-1)^2 + (-1)1^3 = 0$

c. $f(2,3) = 2^2 + 2 \cdot 3^3 = 58$

d. $f(-3,-2) = (-3)^2 + (-3)(-2)^3 = 33$

5. $f(x,y)$ exists when $y - x - 2 \geq 0$,

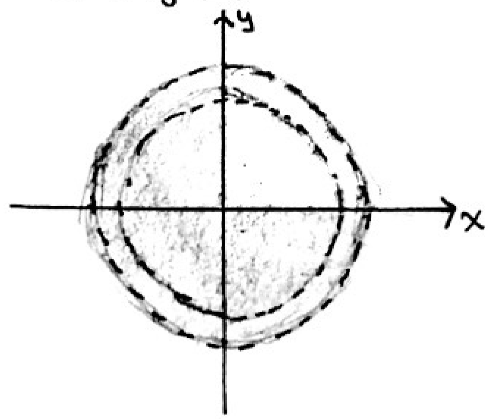
$\Rightarrow y \geq x + 2$



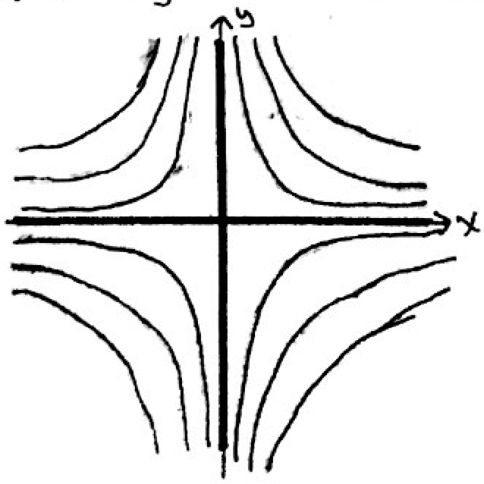
12. $f(x,y) = \dfrac{1}{\ln(4 - x^2 - y^2)}$ exists when $\ln(4 - x^2 - y^2) \neq 0$ and $4 - x^2 - y^2 > 0$,

$\Rightarrow x^2 + y^2 < 4$

$\Rightarrow x^2 + y^2 \neq 3$



15. $f(x,y) = xy \in [-9, -4, -1, 0, 1, 4, 9]$



19. $f(x,y) = 4x^2 + 9y^2$

a. $x \in \mathbb{R}$ and $y \in \mathbb{R}$

b. $f(x,y) > 0$ when ($x \neq 0$ and $y \neq 0$) or ($x \neq 0$ and $y = 0$) or ($x = 0$ and $y \neq 0$)

$f(x,y) = 0$ when $x = 0$ and $y = 0$

$f(x,y) < 0$ is impossible as $x^2 \geq 0$ and $y^2 \geq 0$

$\therefore f(x,y) \in [0, \infty)$

c. $f(x,y) = 4x^2 + 9y^2 = c$

$$\Rightarrow x^2 + \frac{y^2}{(\frac{2}{3})^2} = \frac{c}{4}$$

represents an ellipse

d. There is no boundary point as both doman include Real elements

e. The domain is both open and closed as every point in its domain is an interior point and the domain contains the boundary point

f. Bounded

27. $f(x,y) = \arcsin(y - x)$

a. $|y - x| \leq 1 \Rightarrow -1 \leq y - x \leq 1$

b. $f(x,y) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ as $f(x,y)_{min} = \sin^{-1}(-1) = -\frac{\pi}{2}$ and $f(x,y)_{max} = \sin^{-1}(1) = \frac{\pi}{2}$

c. $f(x,y) = \sin^{-1}(x - y) \in [-1,1]$

$\Rightarrow x - y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$y \in [x - \frac{\pi}{2}, x + \frac{\pi}{2}]$

a straight line w/ slope = 1 with y-intercepts $\in [-\frac{\pi}{2}, \frac{\pi}{2}]$

d. As $-1 \leq y - x \leq 1$, the points are bounded by line $y = x - 1$ and $y = x + 1$

e. As the points $\in [y = x + [-1, 1]]$, domain is closed

f. Unbounded

31. f

32. e

33. a

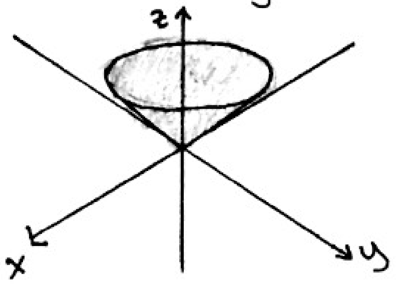
34. c

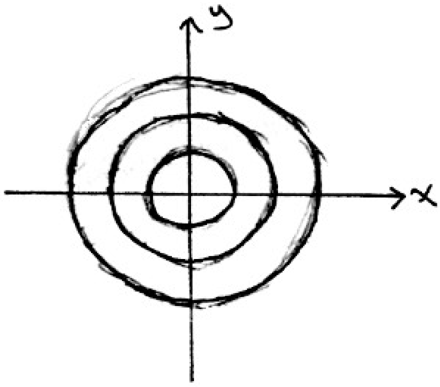
35. d

36. b

40. $f(x,y) = z = \sqrt{x^2+y^2}$

$\Rightarrow z^2 = x^2+y^2$

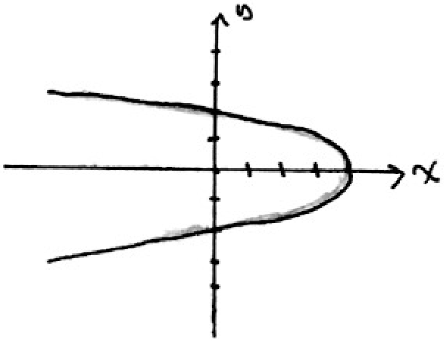




51. $f(x,y) = \sqrt{x+y^2-3}$

$f(3,-1) = 1 = \sqrt{x+y^2-3}$

$\Rightarrow x = 4-y^2$



Exercises 14.2

3. $\lim_{(x,y)\to(3,4)} \sqrt{x^2+y^2-1} = \sqrt{(3)^2+(4)^2-1} = 2\sqrt{6}$

11. $\lim_{(x,y)\to(1,\frac{\pi}{6})} \frac{x \sin y}{x^2+1} = \frac{1 \cdot \sin(\frac{\pi}{6})}{1^2+1} = \frac{1}{4}$

17. $\lim_{\substack{(x,y)\to(0,0)\\ x\neq y}} \frac{x-y+2\sqrt{x}-2\sqrt{y}}{\sqrt{x}-\sqrt{y}} = \lim_{\substack{(x,y)\to(0,0)\\ x\neq y}} \frac{(\sqrt{x}+1)^2-(\sqrt{y}+1)^2}{(\sqrt{x}+1)-(\sqrt{y}+1)} = \lim_{\substack{(x,y)\to(0,0)\\ x\neq y}} \sqrt{x}+\sqrt{y}+2 = 2$

23. $\lim_{(x,y)\to(1,-1)} \frac{x^3+y^3}{x+y} = \lim_{(x,y)\to(1,-1)} x^2-xy+y^2 = 3$

27. $\lim_{(x,y,z)\to(\pi,\pi,0)} \sin^2 x+\cos^2 y+\sec^2 z = \sin^2\pi+\cos^2\pi+\sec^2 0 = 2$

33a. $g(x,y) = \sin\frac{1}{xy}$

g is continuous when $xy \nrightarrow 0 \Rightarrow x\neq 0$ and $y\neq 0$

b. $g(x,y) = \frac{x+y}{2+\cos x}$

g is continuous when $2+\cos x \nrightarrow 0 \Rightarrow x\in\mathbb{R}$ and $y\in\mathbb{R}$

37a. $h(x,y,z) = xy \sin\frac{1}{z}$

h is continuous when $z \nrightarrow 0 \Rightarrow z\neq 0$

b. $h(x,y,z) = \frac{1}{x^2+z^2-1}$

h is continuous when $x^2+z^2-1 \nrightarrow 0 \Rightarrow x^2+z^2\neq 0$

43. $\lim_{(x,y)\to(0,0)} \left(f(x,y) = \frac{x^4-y^2}{x^4+y^2}\right)$

for $y=h(x)=x^2g(x)$,

$$\lim_{(x,y)\to(0,0)} \frac{x^4-x^4g^2(x)}{x^4+x^4g^2(x)} = \lim_{(x,y)\to(0,0)} \frac{1-g^2(x)}{1+g^2(x)} \in [-1,1]$$

since the mapped value for the limit is finitely countable; therefore, the limit doesn't exist

45. $\lim_{(x,y)\to(0,0)} \left(g(x,y) = \frac{x-y}{x+y}\right)$

for $y=h(x)=xf(x)$,

$$\lim_{(x,y)\to(0,0)} \frac{x-xf(x)}{x+xf(x)} = \lim_{(x,y)\to(0,0)} \frac{1-f(x)}{1+f(x)} \in (-\infty,-1)\cup(-1,\infty)$$

since the mapped value for the limit is infinitely uncountable; therefore, the limit doesn't exist

49. $\lim_{(x,y)\to(1,1)} \frac{xy^2-1}{y-1}$

for $y = h(x) = xg(x)$,

$$\lim_{(x,y)\to(1,1)} \frac{x^3 g^4(x) - 1}{x g(x) - 1} = \lim_{(x,y)\to(1,1)} g(x)+1$$

since $\lim_{x\to 1} g(x)$ could be not exist, so does the limit of $\frac{xy^2-1}{y-1}$

55. $g(x,y) = 1 - \frac{x^2y^2}{3}$

$$\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{(x,y)\to(0,0)} 1 - \frac{x^2-y^2}{3} = 1$$

$h(x,y) = 1$

$$\lim_{(x,y)\to(0,0)} h(x,y) = 1$$

since both limits are equal,

$$\lim_{(x,y)\to(0,0)} \frac{\tan^{-1} xy}{xy} = 1$$

58. $-1 \le \cos\left(\frac{1}{y}\right) \le 1$

$\Rightarrow -x \le x\cos\left(\frac{1}{y}\right) \le x$

$g(x,y) = -x$

$$\lim_{(x,y)\to(0,0)} g(x,y) = 0$$

$h(x,y) = x$

$$\lim_{(x,y)\to(0,0)} h(x,y) = 0$$

since both limits are equal,

$$\lim_{(x,y)\to(0,0)} x\cos\left(\frac{1}{y}\right) = 0$$

61. let $x = r\cos\theta$ and $y = r\sin\theta$.

$$f(r\cos\theta, r\sin\theta) = \frac{r\cos\theta\,(r^2\cos^2\theta - r^2\sin^2\theta)}{r^2\cos^2\theta + r^2\sin^2\theta}$$

$$= r\cos\theta\cos 2\theta$$

since both $\cos\theta$ and $\sin\theta$ can't be 0 at the same time

$\Rightarrow r = 0$

therefore,

$$\lim_{r\to 0} f(r\cos\theta, r\sin\theta) = \lim_{r\to 0} r\cos\theta\cos 2\theta$$

$$= 0$$

67. $$\lim_{(x,y)\to(0,0)} \ln\left(\frac{3x^2 - x^2y^2 + 3y^2}{x^2+y^2}\right)$$

$$= \lim_{r\to 0} \ln\left(\frac{3r^2\cos^2\theta - r^4\cos^2\theta\sin^2\theta + 3r^2\sin^2\theta}{r^2}\right)$$

$$= \lim_{r\to 0} \ln(3(\cos^2\theta + \sin^2\theta))$$

$$= \ln(3)$$

Exercises 14.3

5. $f(x,y) = (xy-1)^2$

$\frac{\partial f}{\partial x}(x,y) = 2y(xy-1)$

$\frac{\partial f}{\partial y}(x,y) = 2x(xy-1)$

17. $f(x,y) = \sin^2(x-3y)$

$\frac{\partial f}{\partial x}(x,y) = 2\sin(x-3y)\cos(x-3y)$

$= \sin(2x-6y)$

$\frac{\partial f}{\partial y}(x,y) = -2.3\sin(x-3y)\cos(x-3y)$

$= -3\sin(2x-6y)$

21. $f(x,y) = \int_x^y g(t)\,dt = G(y) - G(x)$

$\frac{\partial f}{\partial x}(x,y) = \frac{\partial G(y)}{\partial x}(x,y) - \frac{\partial G(x)}{\partial x}(x,y)$

$= -g(x)$

$\frac{\partial f}{\partial y}(x,y) = \frac{\partial G(y)}{\partial y}(x,y) - \frac{\partial G(x)}{\partial x}(x,y)$

$= g(y)$

25. $f(x,y,z) = x - \sqrt{y^2+z^2}$

$\frac{\partial f}{\partial x}(x,y,z) = 1$

$\frac{\partial f}{\partial y}(x,y,z) = -\frac{y}{\sqrt{y^2+z^2}}$

$\frac{\partial f}{\partial z}(x,y,z) = -\frac{z}{\sqrt{y^2+z^2}}$

31. $f(x,y,z) = e^{-(x^2+y^2+z^2)}$

$\frac{\partial f}{\partial x}(x,y,z) = -2x e^{-(x^2+y^2+z^2)}$

$\frac{\partial f}{\partial y}(x,y,z) = -2y e^{-(x^2+y^2+z^2)}$

$\frac{\partial f}{\partial z}(x,y,z) = -2z e^{-(x^2+y^2+z^2)}$

43. $g(x,y) = x^2y + \cos y + y\sin x$

$\frac{\partial g}{\partial x}(x,y) = 2yx + y\cos x$

$\frac{\partial^2 g}{\partial x^2}(x,y) = 2y - y\sin x$

$\frac{\partial g}{\partial y}(x,y) = x^2 - \sin y + \sin x$

$\frac{\partial^2 y}{\partial y^2}(x,y) = -\cos y$

53. $w = xy^2 + x^2y^3 + x^3y^4$

$\frac{\partial w}{\partial x} = y^2 + 2xy^3 + 3x^2y^4$

$\frac{\partial w}{\partial x\partial y} = 2y + 6xy^2 + 12x^2y^3$

$\frac{\partial w}{\partial y} = 2xy + 3x^2y^2 + 4x^3y^3$

$\frac{\partial^2 w}{\partial y.\partial x} = 2y + 6xy^2 + 12x^2y^3$

$\Rightarrow \frac{\partial^2 w}{\partial x.\partial y} = \frac{\partial^2 w}{\partial y.\partial x}$

60. $f(x,y) = \begin{cases} \frac{\sin(x^3+y^4)}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0 \end{cases}$

$\frac{\partial f}{\partial x}\Big|_{(0,0)} = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$

$= \lim_{h\to 0} \frac{\frac{\sin(h^3)}{h^2}}{h}$

$= 1$

$\frac{\partial f}{\partial y}\Big|_{(0,0)} = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h}$

$= \lim_{h\to 0} \frac{\frac{\sin(h^4)}{h^2}}{h}$

$= \lim_{h\to 0} h . \lim_{h\to 0} \frac{\sin(h^4)}{h^4}$

$= 0$

65. $xy+z^3x-2yz=0$

$$y+3z^2\cdot\frac{\partial z}{\partial x}\cdot x+z^3-2y\cdot\frac{\partial z}{\partial x}=0$$

when $x=1$ & $y=1$ & $z=1$,

$$1+3(1)^2\cdot\frac{\partial z}{\partial x}\cdot(1)+(1)^3-2(1)\cdot\frac{\partial z}{\partial x}=0$$

$$\frac{\partial z}{\partial x}=-2$$

72. $f(x,y)=\begin{cases} xy\cdot\dfrac{x^2-y^2}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0)\end{cases}$

a. $\frac{\partial f}{\partial y}(x,0)=x\cdot\frac{x^2-0^2}{x^2+0^2}+x(0)\cdot\frac{\partial}{\partial y}\left(\frac{x^2-y^2}{x^2+y^2}\right)$

$=x$

$\frac{\partial f}{\partial x}(0,y)=y\cdot\frac{0^2-y^2}{0^2+y^2}+(0)\cdot y\cdot\frac{\partial}{\partial x}\left(\frac{x^2-y^2}{x^2+y^2}\right)$

$=-y$

b. $\frac{\partial^2 f}{\partial y\partial x}(0,0)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(x,0)\right)=1$

$\frac{\partial^2 f}{\partial x\partial y}(0,0)=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(0,y)\right)=-1$

75. $f(x,y)=e^{-2y}\cos 2x$

$\frac{\partial f}{\partial x}(x,y)=-2e^{-2y}\sin 2x \quad \frac{\partial f}{\partial y}(x,y)=-2e^{-2y}\cos 2x$

$\frac{\partial^2 f}{\partial x^2}(x,y)=-4e^{-2y}\cos 2x \quad \frac{\partial^2 f}{\partial y^2}(x,y)=4e^{-2y}\cos 2x$

$\nabla^2 f=\frac{\partial^2 f}{\partial x^2}(x,y)+\frac{\partial^2 f}{\partial y^2}(x,y)$

$=-4e^{-2y}\cos 2x+4e^{-2y}\cos 2x$

$=0$ (It satisfies the laplace equation)

79. $f(x,y,z)=(x^2+y^2+z^2)^{-1/2}$

$\frac{\partial f}{\partial x}(x,y,z)=-x(x^2+y^2+z^2)^{-3/2}$

$\frac{\partial^2 f}{\partial x^2}(x,y,z)=-(x^2+y^2+z^2)^{-3/2}+3x^2(x^2+y^2+z^2)^{-5/2}$

$\frac{\partial f}{\partial y}(x,y,z)=-y(x^2+y^2+z^2)^{-3/2}$

$\frac{\partial^2 f}{\partial y^2}(x,y,z)=-(x^2+y^2+z^2)^{-3/2}+3y^2(x^2+y^2+z^2)^{-5/2}$

$\frac{\partial f}{\partial z}(x,y,z)=-z(x^2+y^2+z^2)^{-3/2}$

$\frac{\partial^2 f}{\partial z^2}(x,y,z)=-(x^2+y^2+z^2)^{-3/2}+3z^2(x^2+y^2+z^2)^{-5/2}$

$\nabla^2 f=\frac{\partial^2 f}{\partial x^2}(x,y,z)+\frac{\partial^2 f}{\partial y^2}(x,y,z)+\frac{\partial^2 f}{\partial z^2}(x,y,z)$

$=\frac{(-x^2-y^2+2z^2)+(-x^2+2y^2-z^2)+(2x^2-y^2-z^2)}{(x^2+y^2+z^2)^{5/2}}$

$=0$ (It satisfies the laplace equation)

83. $\frac{\partial}{\partial x}(\sin(x+ct)+\cos(2x+2ct))$

$=\cos(x+ct)-2\sin(2x+2ct)$

$\frac{\partial}{\partial t}(\sin(x+ct)+\cos(2x+2ct))$

$=c(\cos(x+ct))-2c\sin(2x+2ct)$

$\Rightarrow c^2\frac{\partial^2}{\partial x^2}(\sin(x+ct)+\cos(2x+2ct))$

$=c^2\cdot[\cos(x+ct)-4\sin(2x+2ct)]$

$\Rightarrow \frac{\partial^2}{\partial t^2}(\sin(x+ct)+\cos(2x+2ct))$

$=-c^2\sin(x+ct)-4c^2\cos(2x+2ct)$

$\Rightarrow \frac{\partial^2}{\partial t^2}(\sin(x+ct)+\cos(2x+2ct))$

$=c^2\cdot\frac{\partial^2}{\partial x^2}(\sin(x+ct)+\cos(2x+2ct))$

(satisfies the wave equation)

90. $f(x,t)=\sin(\alpha x)\cdot e^{-\beta t}$

$\frac{\partial f}{\partial t}=-\beta\sin(\alpha x)\cdot e^{-\beta t}$

$\frac{\partial f}{\partial x}=\alpha\cos(\alpha x)\cdot e^{-\beta t}$

$\frac{\partial^2 f}{\partial x^2}=-\alpha^2\sin(\alpha x)\cdot e^{-\beta t}$

$\Rightarrow \frac{\partial f}{\partial t}=\frac{\partial^2 f}{\partial x^2}$

$-\beta\sin(\alpha x)\cdot e^{-\beta t}=-\alpha^2\sin(\alpha x)\cdot e^{-\beta t}$

$\beta=\alpha^2$

91. let $f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & ,(x,y) \neq (0,0) \\ 0 & ,(x,y) = (0,0) \end{cases}$

$$\frac{\partial f}{\partial x}(x,y) = \frac{y^2(x^2+y^4) - xy^2(2x)}{(x^2+y^4)^2}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{2xy(x^2+y^4) - xy^2(4y^3)}{(x^2+y^4)^2}$$

if $f_x(0,0)$ & $f_y(0,0)$ don't exist, f is not differentiable at $(0,0)$.

$$\lim_{(y^2,y)\to(0,0)} f_x = \lim_{(y^2,y)\to(0,0)} \frac{y^6 - y^6 - y^8}{(y^4+y^4)^2} = -\frac{1}{4}$$

$$\lim_{(-y^2,y)\to(0,0)} f_x = \lim_{(-y^2,y)\to(0,0)} \frac{y^6 - y^6 + y^8}{(y^4+y^4)^2} = \frac{1}{4}$$

since f_x is not continuous at $(0,0)$, f_y is also not continuous at $(0,0)$

$\Rightarrow f$ is not differentiable at $(0,0)$