Assignment 5

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Please note that

- **Released date: 8th Nov, Monday.**
- **Due date: 22nd Nov, Monday, by 11:59pm.**
- Late submission is **NOT** accepted.
- Please submit your answers as a PDF file with a name like "120010XXX ASS1.pdf" (Your student ID + ASS No.). You may either typeset you answers directly using computers, or scan your handwritten answers. (We recommend you use the printers on campus to scan. If you use your smartphone to scan, please limit the file size 10MB.)

Question 1.(Slides 15) For each of the following, compute (i) $\det(\boldsymbol{A})$, (ii) adj $\boldsymbol{A}$, and (iii) $\boldsymbol{A}^{-1}$:

(a). $\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$ (b). $\boldsymbol{A} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$

(c). $\boldsymbol{A} = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ -2 & 2 & -1 \end{bmatrix}$ (d). $\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Question 2.(Slides 16) Use Cramer's rule to solve each of the following systems:

(a). $\begin{cases} x_1 + 2x_2 = 3 \\ 3x_1 - x_2 = 1 \end{cases}$. (b). $\begin{cases} 2x_1 + 3x_2 = 2 \\ 3x_1 + 2x_2 = 5 \end{cases}$.

(c). $\begin{cases} 2x_1 + x_2 - 3x_3 = 0 \\ 4x_1 + 5x_2 + x_3 = 8 \\ -2x_1 - x_2 + 4x_3 = 2 \end{cases}$. (d). $\begin{cases} x_1 + 3x_2 + x_3 = 1 \\ 2x_1 + x_2 + x_3 = 5 \\ -2x_1 + 2x_2 - x_3 = -8 \end{cases}$.

Question 3.(Slides 15 and 16) Let

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \\ 3 & 4 & 3 \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}, \qquad \boldsymbol{C} = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 3 & 1 \end{bmatrix}$$

If $\boldsymbol{AXB} = \boldsymbol{C}$, find matrix $\boldsymbol{X}$.

Question 4.(Slides 16) Let

$$\boldsymbol{P} = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}, \quad \boldsymbol{\Lambda} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \boldsymbol{AP} = \boldsymbol{P\Lambda}.$$

Find $\boldsymbol{A}^n$.

Question 5.(Slides 16) Show that if $\boldsymbol{A}$ is nonsingular, then $\operatorname{adj}(\boldsymbol{A})$ is nonsingular and

$$(\operatorname{adj} \boldsymbol{A})^{-1} = \det(\boldsymbol{A}^{-1})\boldsymbol{A} = \operatorname{adj}(\boldsymbol{A}^{-1}).$$

Question 6.(Slides 15 and 16) Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be 3×3 matrices, with $\det(\boldsymbol{A}) = -3$ and $\det(\boldsymbol{B}) = -4$. Use the properties of determinants to compute:
(a). $\det(\boldsymbol{AB})$; (b). $\det(5\boldsymbol{A})$; (c). $\det(\boldsymbol{B}^T)$; (d). $\det(\boldsymbol{A}^{-1})$; (e). $\det(\boldsymbol{A}^3)$.

Question 7.(Slides 16) Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be 2×2 matrices and let

$$\boldsymbol{C} = \begin{bmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{bmatrix}, \qquad \boldsymbol{D} = \begin{bmatrix} b_{11} & b_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Show that

$$\det(\boldsymbol{A} + \boldsymbol{B}) = \det(\boldsymbol{A}) + \det(\boldsymbol{B}) + \det(\boldsymbol{C}) + \det(\boldsymbol{D}).$$

Question 8.(Slides 16) Compute the determinants of the following elementary matrices.

(a).

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{bmatrix}$$

(b).

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(c).

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Question 9.(Slides 15) Use a determinant to decide whether $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$ are linearly dependent or not, when

(a). $\boldsymbol{v}_1 = \begin{bmatrix} 5 \\ -7 \\ 9 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} -3 \\ 3 \\ -5 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 2 \\ -7 \\ 5 \end{bmatrix}$

Question 10.(Slides 15) Let U be a square matrix such that $U^T U = I$. Show that $\det(U) = \pm 1$

Question 11.(Slides 15 and 16) Assume matrix A is square, make each statement True or False

(a) If the columns of A are linearly dependent, then $det(A) = 0$

(b) $det(A^{-1}) = (-1)det(A)$

(c) The determinant of A is the product of the diagonal entries in A

(d) If $det(A)$ is zero, then two rows or two columns are the same, or a row or a column is zero

(e) $det(A + B) = det(A) + det(B)$

Question 12.(Slides 15) Let

$$A = \begin{bmatrix} 1 & 5 & 2 \\ 2 & 4 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

(a) Find the values of $det(M_{21})$, $det(M_{22})$ and $det(M_{23})$

(b) Find the values of A_{21}, A_{22} and A_{23}

(c) Use your answers from part (b) to compute $det(A)$

Question 13.(Slides 16) Let

$$\boldsymbol{A} = \begin{bmatrix} 3 & 1 & -2 \\ -1 & 0 & 0 \\ 5 & 0 & 6 \end{bmatrix}$$

(a) Find $det(\boldsymbol{A})$

(b) Compute $det(\boldsymbol{A}^4)$

(c) Find the solution to $\boldsymbol{A}x = \begin{bmatrix} 16 \\ -2 \\ -8 \end{bmatrix}$

Question 14.(Slides 15) Compute determinants using the cofactor formula

(a) $\boldsymbol{A} = \begin{bmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{bmatrix}$ (b) $\boldsymbol{B} = \begin{bmatrix} 6 & 3 & 2 & 4 & 0 \\ 9 & 0 & -4 & 1 & 0 \\ 8 & -5 & 6 & 7 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{bmatrix}$

Question 15.(Slides 16) Let A be a $k \times k$ matrix and let B be an $(n-k) \times (n-k)$ matrix. Let $E = \begin{bmatrix} I_k & 0 \\ 0 & B \end{bmatrix}, F = \begin{bmatrix} A & 0 \\ 0 & I_{k-1} \end{bmatrix}, C = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ where I_k and I_{k-1} are $k \times k$ and $(n-k) \times (n-k)$ identity matrices. Show that $det(C) = det(A)det(B)$.

Question 16.(Slides 17) Let $L_1 : \mathcal{U} \to \mathcal{V}$ and $L_2 : \mathcal{V} \to \mathcal{W}$ be linear transformations, and let $L = L_2 \circ L_2$ be the mapping defined by

$$L(\boldsymbol{u}) = L_2(L_1(\boldsymbol{u}))$$

for each $\boldsymbol{u} \in \mathcal{U}$. Show that L is a linear transformation mapping $\mathcal{U}$ into $\mathcal{W}$.

Question 17.(Slides 17) Determine the kernel of each of the following linear operators on $\mathbb{R}^3$:

(a).

$$L(\boldsymbol{x}) = \begin{bmatrix} x_3 & x_2 & x_1 \end{bmatrix}^T.$$

(b).

$$L(\boldsymbol{x}) = \begin{bmatrix} x_1 & x_2 & 0 \end{bmatrix}^T.$$

(c).

$$L(\boldsymbol{x}) = \begin{bmatrix} x_1 & x_1 & x_1 \end{bmatrix}^T.$$

Question 18.(Slides 16) Consider the 3×3 Vandermonde matrix

$$V = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix}$$

(a) show that $det(V) = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2)$.

(b) What conditions must the scalars x_1, x_2 and x_3 satisfy in order for V to be nonsingular

Question 19.(Slides 17) Let

$$\boldsymbol{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{b}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad \boldsymbol{b}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

and let L be the linear transformation from $\mathbb{R}_2$ into $\mathbb{R}_3$ defined by

$$L(\boldsymbol{x}) = x_1\boldsymbol{b}_1 + x_2\boldsymbol{b}_2 + (x_1 + x_2)\boldsymbol{b}_3.$$

Find the matrix $\boldsymbol{A}$ representing L with respect to the standard basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ of $\mathbb{R}_2$ and the basis $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$ of $\mathbb{R}_3$.

Question 20.(Slides 16) Let A be a nonsingular $n \times n$ matrix with $n > 1$. Show that $det(adjA) = (det(A))^{n-1}$