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PHY1002 Homework on Error Analysis

1 Significant figures (2 pts)

1.1 Determine the significant figures

Write down the number of significant figures for the following numbers

(a) 1.00101 ( )

(b) 1.0110×10^{-3} ( )

(c) 0.0010011 ( )

(d) 3.140 ( )

(e) 1670. ( )

(f) 1.68×10^4 ( )

1.2 Significant figures in calculations

Express the results of the following calculations with the correct significant figures.

(a) $3.1416 \times 0.28/2.34$

(b) $123.62 + 7.1 - 5.33$

2 Propagation of Uncertainty (Error)

Let us first define the standard deviation s. Suppose we perform N measurements $x_1, x_2, \cdots, x_N$ with the average $\bar{x}$. Then the deviation of each measurement is given by $\delta x_i = x_i - \bar{x}$ with $i = 1, 2, \cdots, N$. The standard deviation s is

$$s = \sqrt{\frac{\sum_{i=1}^{N} (\delta x_i)^2}{N-1}}$$

When we report the average value of N measurements, the uncertainty we should associate with this average value is the standard error.

$$\sigma = \frac{s}{\sqrt{N}} = \sqrt{\frac{\sum_{i=1}^{N} (\delta x_i)^2}{N(N-1)}}$$

The standard error is smaller than the standard deviation by a factor of $1/\sqrt{N}$, since the statistical uncertainty can be reduced by large number of measurements. Also it is useful to write $\sigma_u^2 = \delta u^2 \equiv \frac{1}{N(N-1)} \sum_{i=1}^{N} \delta u_i^2$.

Suppose we want to determine a quantity $x = f(u, v)$, which depends on u and v. We want to know the error in $x = f(u, v)$ if we measure u and v with errors σ_u and σ_v. Using the Taylor expansion, we can obtain the law of the error propagation as follows

$$(\delta x)^2 = \left(\frac{\partial f}{\partial u}\right)^2 (\delta u)^2 + \left(\frac{\partial f}{\partial v}\right)^2 (\delta v)^2 + 2\left(\frac{\partial f}{\partial u}\frac{\partial f}{\partial v}\right)(\delta u \delta v)$$

If the measurements of u and v are uncorrelated, then, on the average, we should expect to find equal distributions of positive and negative values for this term, and we should expect $\overline{(\delta u \delta v)} = 0$. At the end of the day, using the definition of the standard error σ, we can obtain

$$\sigma_x = \sqrt{\left(\frac{\partial f}{\partial u}\right)^2 \sigma_u^2 + \left(\frac{\partial f}{\partial v}\right)^2 \sigma_v^2}$$

Exercise problems: Now find the standard error σ_x in $x = f(u, v)$ as a function of the errors in σ_u and σ_v for the following functions:

(a) $x = u + v$ (0.5 pts)
You can find the absolute uncertainty of the sum (or difference) is the root square sum of the individual absolute uncertainties when adding (or subtracting).

(b) $x = u \times v$ (0.5 pts)

(c) $x = u/v$ (1 pt)
You can find that the relative uncertainty of the product (quotient) is the root square sum of the individual relative uncertainties.

(d) $x = uv^2$ (1 pt)

(e) $x = u \exp(cv)$ with c constant. (0.5 pts)

(f) $x = 1/u$ (0.5 pts)

3 Snell's Law (2 pts)

Through the equation $n_1 \sin\theta_1 = n_2 \sin\theta_2$, the Snell's law relates the incident angle θ_1 of a ray traveling in a medium of index n_1 to the refraction angle θ_2 of the same light ray in the medium of refraction index n_2. Find n_2 and its uncertainty from the following measurements

$$\theta_1 = 22.0° \pm 0.2°$$
$$\theta_2 = 16.3° \pm 0.2°$$
$$n_1 = 1.000 \quad \text{(assumed to be exact)}$$

Note that $\delta\theta$ must be converted into radians when you compute the uncertainty. In calculus, we only use radians in trigonometrical functions.

4 Simple Pendulum (2 pts)

Suppose you determine the acceleration of gravity $g = \frac{4\pi^2 L}{T^2}$ by measuring the oscillation period T of a pendulum with length L. Determine the value and uncertainty of g from the following measurement

$$T = 2.01 \pm 0.02\, s, \quad L = 1.000 \pm 0.002 m.$$

If you wish to improve the above measurement significantly, which part of the measurement (T or L) do you want to improve? Why?