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CSC3001 Assignment 2

Q1. let p denotes the first integer and q denotes the second integer ($p \geq q$)

- case p is odd and q is odd:

$$\begin{aligned} p^2 - q^2 &= (2m+1)^2 - (2n+1)^2 \\ &= 4m^2 + 4m + 1 - 4n^2 - 4n - 1 \\ &= 4(m^2 + m - n^2 - n) \\ &\equiv 0 \pmod 4 \end{aligned}$$

- case p is odd and q is even:

$$\begin{aligned} p^2 - q^2 &= (2m+1)^2 - (2n)^2 \\ &= 4m^2 + 4m + 1 - 4n^2 \\ &= 4(m^2 + m - n^2) + 1 \\ &\equiv 1 \pmod 4 \end{aligned}$$

- case p is even and q is odd:

$$\begin{aligned} p^2 - q^2 &= (2m)^2 - (2n+1)^2 \\ &= 4m^2 - 4n^2 - 4n - 1 \\ &= 4(m^2 - n^2 - n) - 1 \\ &\equiv 3 \pmod 4 \end{aligned}$$

- case p is even and q is even:

$$\begin{aligned} p^2 - q^2 &= (2m)^2 - (2n)^2 \\ &= 4m^2 - 4n^2 \\ &= 4(m^2 - n^2) \\ &\equiv 0 \pmod 4 \end{aligned}$$

Since $4k + 2 \equiv 2 \pmod 4$, there doesn't exist any integer k s.t. $4k + 2 = p^2 - q^2$ (disproved)

Q2. - case x is odd and y is even:

$$\begin{aligned} x^2 + y^2 &= (2m+1)^2 + (2n)^2 \\ &= 4m^2 + 4m + 1 + 4n^2 \\ &= 4(m^2 + m + n^2) + 1 \\ &\equiv 1 \pmod 2 \end{aligned}$$

is odd

(similar for y is odd and x is even)

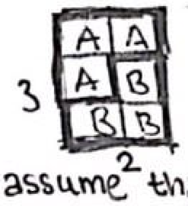
- case x is odd and y is odd:

$$\begin{aligned} 3xy &= 3(2m+1)(2n+1) \\ &= 12mn + 6m + 6n + 3 \\ &= 6(mn + m + n) + 3 \\ &\equiv 1 \pmod 2 \end{aligned}$$

is odd

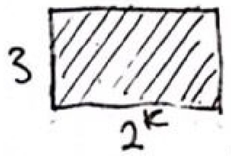
then, by contraposition, both x and y must be even numbers

Q3. 1) $n = 1$,



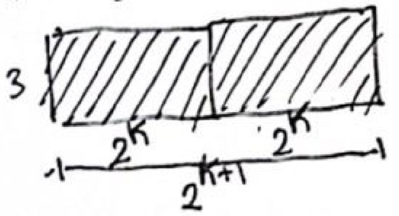
(covered)

assume that



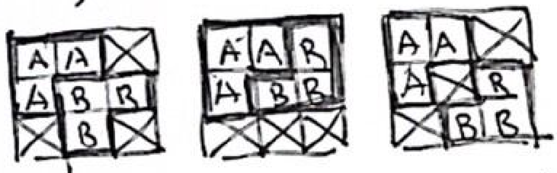
can be covered for $n = k$, k is integer

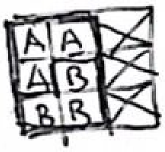
$n = k+1$,



$\therefore$ all 3×2^n checkerboard can be covered (proved)

2) $n = 1$,





(and 12 other similar cases by rotating the board 90°, 180°, and 270°)

there doesn't exist any possible configuration for $3^n \times 3^n$ checkerboard (disproved)

Q4. the possible configurations might not have 1 three-cent stamp and 2 four-cent stamps, which leads to an invalid proof.

example: 4 (0 three-cent stamp and 1 four-cent stamp)

Q5. let p be a non-zero rational number and q be a irrational number.

we assume that $p \cdot q$ is rational, by definition, there exists a and b such as $p \cdot q = \frac{a}{b}$.

$\Rightarrow q = \frac{a}{p \cdot b}$

since a, b, and p are rational numbers, q must be also a rational number (contradict)

therefore, $p \cdot q$ must be irrational

Q6. we have $a, b, c \in \mathbb{R}$.

assume there exist a triple (p, q, r) such that pq, pr, and qr are negative.

$\Rightarrow pq \cdot pr \cdot qr$

$= p^2 \cdot q^2 \cdot r^2 \geq 0$

this contradicts with the idea that $pq \cdot pr \cdot qr < 0$. therefore, we can select 2 out of 3 arbitrary real numbers such that their product is non-negative.

Q7. $\Rightarrow n = 4$,

$a_4 = a_1 + a_2 + a_3 = 3 < 16$ (true)

$\Rightarrow n = 5$,

$a_5 = a_2 + a_3 + a_4 = 5 < 32$ (true)

$\Rightarrow n = 6$,

$a_6 = a_3 + a_4 + a_5 = 9 < 64$ (true)

$\Rightarrow$ assume

$a_k < 2^k$

is true for $n = k$, $k \geq 7$

$\Rightarrow n = k+1$,

$$\begin{aligned} LHS: a_{k+1} &= a_k + a_{k-1} + a_{k-2} \\ &< 2^k + 2^{k-1} + 2^{k-2} \\ &< 2^k + 2^{k-1} + 2 \cdot 2^{k-2} \\ &= 2^k + 2 \cdot 2^{k-1} \\ &= 2 \cdot 2^k \\ &= 2^{k+1} \end{aligned}$$

therefore, for $n \geq 4$, $a_n < 2^n$

Q8. for $n = 0$,

we have $S = \{\}$ ($2^0 = 1$ element).

thus, it is true for $n = 0$

assume

a set S with k elements there are 2^k subsets of elements

is true for $n = k$.

for $n = k+1$,

$S_{k+1} = S_k \cup \{e_{k+1}\}$

case subset of S_{k+1} contains e_{k+1}:

subset S_k is formed with e_{k+1} which implies S_{k+1} has 2^k subsets containing e_{k+1}

case subset of S_{k+1} doesn't contain e_{k+1}:

subset S_{k+1} that doesn't contain e_{k+1} = subset S_k (2^k subsets)

$\therefore 2^k + 2^k = 2^{k+1}$ subsets (proved)

Q9. proof by Well-Ordering,

$\Rightarrow$ case 1: there is a person who beats everyone (proved)

$\Rightarrow$ case 2: every person is beaten by at least one other

suppose we have P_i that denotes the i-th person ($1 \leq i \leq n$).

we order the sequence such that P_{i+1} beats P_i (P_1 beats P_n)

we want to prove that $3 \in S$ where $S = \{k \mid \exists \text{ a cycle with length } k\}$.

suppose the length is denoted by $l > 3$,

case P_1 beats P_3:

(proved)

case P_3 beat P_1:

cycle with length $l-1$ (w/o P_2)

then l is not the least value (contradict)

$\therefore$ l must be 3. hence, proved

Q12. well-ordered set <=> totally ordered.

for every subset, it should have a minimum element.

A. not well-ordered as $(0,1) = \{x \mid 0 < x < 1\}$ is a non-empty but doesn't contain a least number

B. well-ordered as any finite total order is a well-order

C. not well-ordered as not positive integers