yohandi - assignment 7

1. $f(x,y) = \frac{x+y}{32}$

a. $f_x(x,y) = \frac{1}{32}$

$f(x) = \sum_{i=1}^{4} f(x, y_i) = \frac{4x+10}{32} = \frac{2x+5}{16}$

b. $f_y(x,y) = \frac{1}{32}$

$f(y) = \sum_{i=1}^{2} f(x_i, y) = \frac{2y+3}{32}$

c. $P(X>Y) = f(2,1) = \frac{3}{32}$

d. $P(Y=2X) = f(1,2) + f(2,4) = \frac{9}{32}$

e. $P(X+Y=3) = f(1,2) + f(2,1) = \frac{3}{16}$

f. $P(X \leq 3-Y) = P(X+Y \leq 3)$

$= f(1,1) + f(1,2) + f(2,1)$

$= \frac{1}{4}$

g. Since $f(x,y) \neq f(x) f(y)$,

$\therefore$ X and Y are dependent

h. $\mu_X = \sum_{x=1}^{2} x f(x) = \frac{7}{16} + 2 \cdot \frac{9}{16} = \frac{25}{16}$

$\sigma_X^2 = \sum_{x=1}^{2} x^2 f(x) - \mu_X^2 = \frac{7}{16} + 4 \cdot \frac{9}{16} - \left(\frac{25}{16}\right)^2$

$= \frac{63}{256}$

$\mu_Y = \sum_{y=1}^{4} y f(y) = \frac{5}{32} + 2 \cdot \frac{7}{32} + 3 \cdot \frac{9}{32} + 4 \cdot \frac{11}{32}$

$= \frac{45}{16}$

$\sigma_Y^2 = \sum_{y=1}^{4} y^2 f(y) - \mu_Y^2 = \frac{5}{32} + 4 \cdot \frac{7}{32} + 9 \cdot \frac{9}{32} + 16 \cdot \frac{11}{32} - \left(\frac{45}{16}\right)^2$

$= \frac{295}{256}$

2a.

y \ x	12	14	16	18	20	22
24	$\frac{1}{36}$	0	0	0	0	0
25	$\frac{1}{36}$	0	0	0	0	0
26	$\frac{1}{36}$	$\frac{1}{36}$	0	0	0	0
27	$\frac{1}{36}$	$\frac{1}{36}$	0	0	0	0
28	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	0	0	0
29	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	0	0	0
30	0	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	0	0
31	0	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	0	0
32	0	0	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	0
33	0	0	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	0
34	0	0	0	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
35	0	0	0	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
36	0	0	0	0	$\frac{1}{36}$	$\frac{1}{36}$
37	0	0	0	0	$\frac{1}{36}$	$\frac{1}{36}$
38	0	0	0	0	0	$\frac{1}{36}$
39	0	0	0	0	0	$\frac{1}{36}$

b. $f(x) = \sum_{y_i \in S_y} f(x, y_i) = 6 \cdot \frac{1}{36} = \frac{1}{6}$

$$f(y) = \begin{cases} \sum_{x_i \in S_y} f(x_i, y) = \frac{1}{36}, & y \in \{24, 25, 38, 39\} \\ \sum_{x_i \in S_y} f(x_i, y) = 2 \cdot \frac{1}{36}, & y \in \{26, 27, 36, 37\} \\ \sum_{x_i \in S_y} f(x_i, y) = 3 \cdot \frac{1}{36}, & \text{otherwise} \end{cases}$$

c. $y \in \{24, 25, 38, 39\}$,

$\frac{1}{36} \neq \frac{1}{6} \cdot \frac{1}{36}$

$y \in \{26, 27, 36, 37\}$,

$2 \cdot \frac{1}{36} \neq \frac{1}{6} \cdot 2 \cdot \frac{1}{36}$

$y \in$ otherwise,

$3 \cdot \frac{1}{36} \neq \frac{1}{6} \cdot 3 \cdot \frac{1}{36}$

$\therefore$ X and Y are dependent

3 a. $X \in \{1,2,3,4\}$

$Y \in \{2,3,4,5,6,7,8\}$

b.

		X			
		1	2	3	4
	2	$\frac{1}{16}$	0	0	0
	3	$\frac{1}{16}$	$\frac{1}{16}$	0	0
	4	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	0
Y	5	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$
	6	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$
	7	0	0	$\frac{1}{16}$	$\frac{1}{16}$
	8	0	0	0	$\frac{1}{16}$

c. $f(x) = \sum_{i=1}^{7} f(x, y_i) = 4 \cdot \frac{1}{16} = \frac{1}{4}$

d.

$$f(y) = \begin{cases} \sum_{i=1}^{4} f(x_i, y) = \frac{1}{16}, & y \in \{2,8\} \\ \sum_{i=1}^{4} f(x_i, y) = 2 \cdot \frac{1}{16}, & y \in \{3,7\} \\ \sum_{i=1}^{4} f(x_i, y) = 3 \cdot \frac{1}{16}, & y \in \{4,6\} \\ \sum_{i=1}^{4} f(x_i, y) = 4 \cdot \frac{1}{16}, & \text{otherwise} \end{cases}$$

e. $y \in \{2,8\}$,

$\frac{1}{16} \neq \frac{1}{4} \cdot \frac{1}{16}$

$y \in \{3,7\}$,

$2 \cdot \frac{1}{16} \neq \frac{1}{4} \cdot 2 \cdot \frac{1}{16}$

$y \in \{4,6\}$,

$3 \cdot \frac{1}{16} \neq \frac{1}{4} \cdot 3 \cdot \frac{1}{16}$

$y \in$ otherwise,

$4 \cdot \frac{1}{16} \neq \frac{1}{4} \cdot 4 \cdot \frac{1}{16}$

$\therefore$ X and Y are dependent

4. $P_x = 0.63$ $\quad P_y = 0.13$ $\quad q = 1 - P_x - P_y = 0.24$

a. using trinomial pmf:

$$f(x,y) = \frac{9!}{x!\,y!\,(9-x-y)!} 0.63^x \, 0.13^y \, 0.24^{9-x-y}$$

b. marginal pmf of $X \sim B(9, 0.63)$

$$f(x) = \binom{9}{x} (0.63)^x (0.37)^{9-x}$$

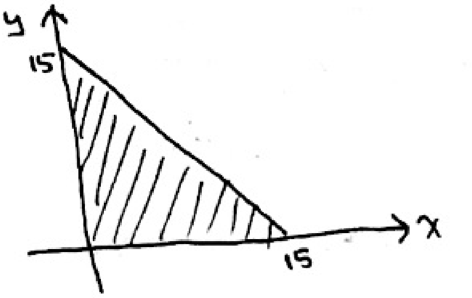
5. $P_x = \frac{6}{10}$ $\quad P_y = \frac{3}{10}$ $\quad q = 1 - P_x - P_y = \frac{1}{10}$

a. using trinomial pmf:

$$f(x,y) = \frac{15!}{x!\,y!\,(15-x-y)!} \left(\frac{6}{10}\right)^x \left(\frac{3}{10}\right)^y \left(\frac{1}{10}\right)^{15-x-y}$$

b. in the function $f(x,y)$, both x and y are defined when $x + y \leq 15$.

sketch:



from the sketch we can conclude that X and Y are dependent

c. $f(10,4) = \frac{15!}{10!\,4!\,1!} \left(\frac{6}{10}\right)^{10} \left(\frac{3}{10}\right)^4 \left(\frac{1}{10}\right)$

≈ 0.0735399

d. marginal pmf of $X \sim B(15, \frac{6}{10})$

$$f(x) = \binom{15}{x} \left(\frac{6}{10}\right)^x \left(\frac{4}{10}\right)^{15-x}$$

e. $P(X \leq 11) = 1 - \sum_{x=12}^{15} f(x) \approx 1 - 0.0905019$

$= 0.9094981$

6a.

		X = 0	1
Y	0	0.2	0
	1	0.3	0.3
	2	0	0.2

b. $f(x) = \sum_{y=0}^{2} f(x,y) = 0.5$

$$f(y) = \begin{cases} \sum_{x=0}^{1} f(x,y) = 0.2, & y \in \{0,2\} \\ \sum_{x=0}^{1} f(x,y) = 0.6, & \text{otherwise} \end{cases}$$

c. $\mu_X = \sum_{x=0}^{1} x f(x) = 0.5$

$\sigma_X^2 = \sum_{x=0}^{1} x^2 f(x) - \mu_X^2 = 0.25$

$\mu_Y = \sum_{y=0}^{2} y f(y) = 1$

$\sigma_Y^2 = \sum_{y=0}^{2} y^2 f(y) - \mu_Y^2 = 0.4$

$E(XY) = \sum_{x,y \in S} xy f(x,y) = 0.7$

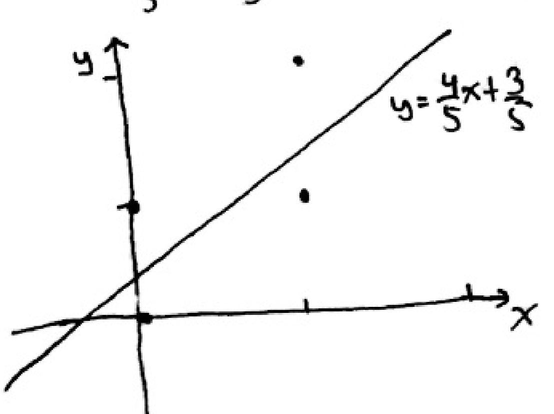
$Cov(X,Y) = E(XY) - \mu_X \mu_Y = 0.2$

$\rho = \frac{Cov(X,Y)}{\sigma_X \cdot \sigma_Y} = \frac{1}{5}\sqrt{10}$

d. $y = \mu_Y + \rho \frac{\sigma_Y}{\sigma_X}(x - \mu_X)$

$y = 1 + \frac{1}{5}\sqrt{10} \cdot \frac{\sqrt{0.4}}{\sqrt{0.25}}(x - 0.5)$

$y = \frac{4}{5}x + \frac{3}{5}$



the regession line looks accurate as every points have different weight

7. $f(x) = \frac{1}{4}, x \in \{1,2,3,4\}$

$$f(y) = \begin{cases} \frac{1}{16}, & y \in \{2,8\} \\ \frac{2}{16}, & y \in \{3,7\} \\ \frac{3}{16}, & y \in \{4,6\} \\ \frac{4}{16}, & \text{otherwise} \end{cases}$$

a. $\mu_X = \sum_{x=1}^{4} x f(x) = \frac{5}{2}$

$\sigma_X^2 = \sum_{x=1}^{4} x^2 f(x) - \mu_X^2 = \frac{5}{4}$

$\mu_Y = \sum_{y=2}^{8} y f(y) = 5$

$\sigma_Y^2 = \sum_{y=2}^{8} y^2 f(y) - \mu_Y^2 = \frac{5}{2}$

$E(XY) = \sum_{x,y \in S} xy f(x,y) = \frac{55}{4}$

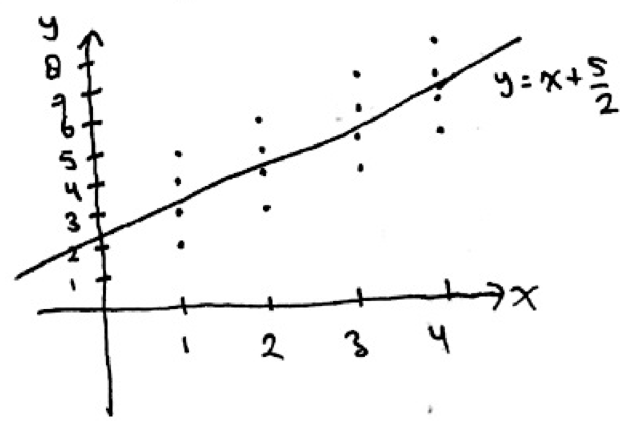
$Cov(X,Y) = E(XY) - \mu_X \mu_Y = 1.25$

$\rho = \frac{Cov(X,Y)}{\sigma_X \cdot \sigma_Y} = \frac{1}{2}\sqrt{2}$

b. $y = \mu_Y + \rho \cdot \frac{\sigma_Y}{\sigma_X}(x - \mu_X)$

$y = 5 + \frac{1}{2}\sqrt{2} \cdot \frac{\sqrt{2.5}}{\sqrt{1.25}}(x - 2.5)$

$y = x + \frac{5}{2}$



8. $k(a,b) = E[(Y - a - bX)^2]$

$= E[Y^2] - 2E[Y(a+bX)] + E[(a+bX)^2]$

$\frac{\partial k}{\partial a} = -2E(Y) + 2a + 2b\,E(X) = 2a - 2\mu_Y + 2b\mu_X$

$\frac{\partial k}{\partial b} = -2E(XY) + 2aE(X) + 2bE(X^2) = -2E(XY) + 2a\mu_x + 2bE(X^2)$

$\frac{\partial k}{\partial a} = 0 \qquad 2a - 2\mu_y + 2b\mu_x = 0$

$\Rightarrow -2E(XY) + 2a\mu_x + 2bE(X^2) = 0$

$\frac{\partial k}{\partial b} = 0$

$a = \mu_y - b\mu_x \qquad = 0$

$\Rightarrow -E(X,Y) - b\mu_x^2 + bE(X^2) = 0$

$\Rightarrow a = \mu_y - \frac{Cov(X,Y)}{\sigma^2_x}\mu_x = \mu_y - p\frac{\sigma_y}{\sigma_x}\mu_x$

$b = \frac{Cov(X,Y)}{\sigma_x^2} = p \cdot \frac{\sigma_y}{\sigma_x}$

9a. $\sum_{x,y \in S} f(x,y) = 1$

$\left(\begin{matrix} 12c + 18c + 18c + 12c + \\ 24c + 24c + 16c + \\ 18c + 12c \end{matrix}\right) = 1$

$c = \frac{1}{154}$

b.

		x = 0	1	2	3
y	0	$\frac{12}{154}$	$\frac{18}{154}$	$\frac{18}{154}$	$\frac{12}{154}$
	1	0	$\frac{24}{154}$	$\frac{24}{154}$	$\frac{16}{154}$
	2	0	0	$\frac{18}{154}$	$\frac{12}{154}$

c. $f(x) = \begin{cases} \frac{12}{154}, & x=0 \\ \frac{42}{154}, & x=1 \\ \frac{60}{154}, & x=2 \\ \frac{40}{154}, & \text{otherwise} \end{cases}$ $\qquad f(y) = \begin{cases} \frac{60}{154}, & y=0 \\ \frac{64}{154}, & y=1 \\ \frac{30}{154}, & \text{otherwise} \end{cases}$

d. X and Y are dependent

e. $\mu_x = \sum_{x \in S} x f(x) = \frac{282}{154}$

$\sigma_x^2 = \sum_{x \in S} x^2 f(x) - \mu_x^2 = \frac{4836}{5929}$

f. $\mu_y = \sum_{y \in S} y f(y) = \frac{124}{154}$

$\sigma_y^2 = \sum_{y \in S} y^2 f(y) - \mu_y^2 = \frac{3240}{5929}$

g. $Cov(X,Y) = E(XY) - \mu_x\mu_y = \frac{1422}{5929}$

h. $p = \frac{Cov(X,Y)}{\sigma_x \sigma_y} \approx 0.3592$

i. $y = \mu_y + p \cdot \frac{\sigma_y}{\sigma_x}(x - \mu_x)$

$y = \frac{237}{806}x + \frac{215}{806}$

10. consider,

$\Rightarrow h(v) = E([(X-\mu_x) + v(Y-\mu_Y)]^2)$

$\Rightarrow h(v) \geq 0$

$\Rightarrow h(v) = E((X-\mu_x)^2) + E(2v(X-\mu_x)(Y-\mu_Y)) + E(v^2(Y-\mu_Y)^2)$

$= \sigma^2_x + 2vE((X-\mu_x)(Y-\mu_Y)) + v^2\sigma^2_y$

$= \sigma^2_y v^2 + 2\sigma_x\sigma_y p v + \sigma_x^2$

recall that $h(v) \geq 0$

$\Rightarrow \sigma_y^2 v^2 + 2vp\sigma_x\sigma_y + \sigma_x^2 \geq 0$

since $h(v)$ follows a quadratic equation and $h(v)$ can't be negative,

$D \leq 0$

$4p^2\sigma_x^2\sigma_y^2 - 4\sigma_x^2\sigma_y^2 \leq 0$

$4\sigma_x^2\sigma_y^2(p^2-1) \leq 0$

$p^2 \leq 1$

$-1 \leq p \leq 1$

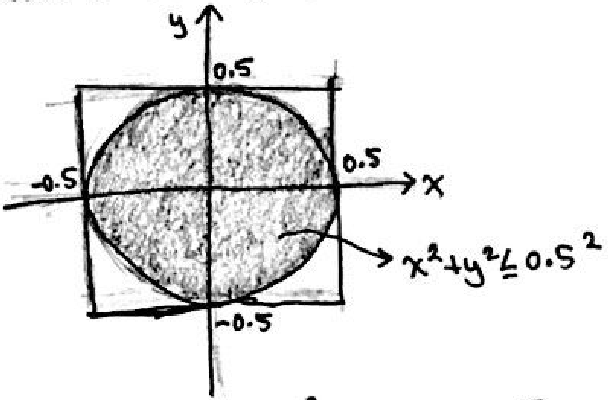
yohandi - assignment 7 (computer-based)

$X \sim U(-0.5, 0.5)$ and $Y \sim U(-0.5, 0.5)$,

consider $A \sim X^2 + Y^2$

$P(A \leq (0.5)^2) = P(x^2 + y^2 \leq (0.5)^2)$

→ to compute the given probability let's consider the circular region with radius 0.5 centered at origin,

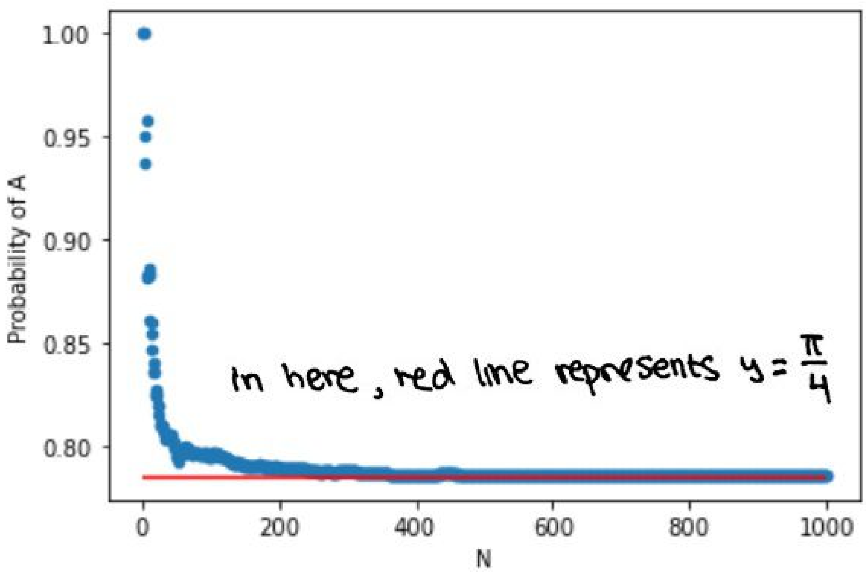


$$\Rightarrow P(A \leq (0.5)^2) = \frac{\pi(0.5)^2}{(0.5-(-0.5))^2} = 0.25\pi = \frac{\pi}{4}$$

the probability can also be estimated using the relative frequency interpretation of $P(A)$. for every values of N, we count the number of pairs (x_i, y_i) $i = 1, \dots, N$ where $x_i^2 + y_i^2 \leq (0.5)^2$.

$$\Rightarrow P(A) = \lim_{N \to \infty} \frac{N(A)}{N}$$

below is the graph obtained from computing:



we can see that the bigger N the more accurate the probability

```python
import math
import matplotlib.pyplot as plt
import numpy as np
import pandas as pd

def randomExperiment(N):
    ret = 0
    x = np.random.uniform(-0.5, 0.5, N)
    y = np.random.uniform(-0.5, 0.5, N)
    for i in range(N):
        if x[i] ** 2 + y[i] ** 2 <= (0.5) ** 2:
            ret += 1
    return ret

N = []
totalP = [0]
P = []
for i in range(1, 1001):
    N.append(i)
    totalP.append(totalP[-1])
    totalP[-1] += randomExperiment(i) / i
    P.append(totalP[-1] / N[-1])

data = {'N' : N, 'Probability of A' : P}
df = pd.DataFrame(data, columns = ['N', 'Probability of A'])
df.plot(x = 'N', y = 'Probability of A', kind = 'scatter')
plt.hlines(math.pi / 4, 0, 1000, colors = 'red')
plt.show()
```