The Chinese University of Hong Kong, Shenzhen
School of Data Science

MAT3007 · Homework 4

Due: 11:59pm, Oct. 29 (Sunday), 2023

Instructions:

- Homework problems must be carefully and clearly answered to receive full credit. Complete sentences that establish a clear logical progression are highly recommended.
- Please submit your assignment on Blackboard.
- The homework must be written in English.
- Late submission will not be graded.
- Each student must not copy homework solutions from another student or from any other source.

Problem 1 (10+10+10=30pts).
Consider the following linear program:

$$\begin{aligned} \text{maximize} \quad & 3x_1 + x_2 + 4x_3 \\ \text{subject to} \quad & x_1 + 3x_2 + x_3 \leq 5 \\ & x_1 + 2x_2 + 2x_3 \leq 7 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

(a). What is the corresponding dual problem?

(b). Solve the dual problem graphically.

(c). Use complementarity conditions for the primal-dual pair to solve the primal problem.

Problem 2 (5 + 10 + 10 + 10 = 35pts).
Consider the following table of food and corresponding nutritional values:

	Protein, g	Carbohydrates, g	Calories	Cost
Bread	4	7	130	3
Milk	6	10	120	4
Fish	20	0	150	9
Potato	1	30	70	1

The ideal intake for an adult is at least 30 grams of protein, 40 grams of carbohydrates, and 400 calories per day. The problem is to find the **least** costly way to achieve those amounts of nutrition by using the four types of food shown in the table.

(a). Formulate this problem as a linear optimization problem (specify the meaning of each decision variable and constraint).

(b). Solve it using MATLAB, find an optimal solution and the optimal value.

(c). Formulate the dual problem. Interpret the dual problem. (Hint: Suppose a pharmaceutical company produces each of the nutrients in pill form and sells them each for a certain price.)

(d). Use MATLAB to solve the dual problem. Find an optimal solution and the optimal value.

Problem 3 (15pts) Farkas's Lemma.
Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Then exactly one of the following two condition holds:

(1). $\exists x \in \mathbb{R}^n$ such that $Ax = b$ and $x \geq 0$.

(2). $\exists y \in \mathbb{R}^m$ such that $A^\top y \geq 0$ and $y^\top b < 0$.

Problem 4 (10 + 10 = 20pts). Special Dual problem.
Suppose M is a square matrix such that $M = -M^T$, for example,

$$M = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -4 \\ -2 & 4 & 0 \end{pmatrix}$$

Consider the following optimization problem:

$$\begin{array}{rc} \text{minimize}\, x & c^T x \\ \text{subject to} & Mx \geq -c \\ & x \geq 0 \end{array}$$

(a). Show that the dual problem of it is equivalent to the primal problem.

(b). Argue that the problem has optimal solution if and only if there is a feasible solution.