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MAT2040 Assignment 6

1(a) we can choose $S = I$, we have:

$$A = I^{-1} A I$$

which is always true

(b) $B = S^{-1} A S$

$\Rightarrow SB = AS$

$\cancel{\Rightarrow B = S}$

$\Rightarrow SBS^{-1} = A$

we take $S = S_1^{-1}$

$\Rightarrow S_1^{-1} B S_1 = A$

which implies A is similar to B

(c) $A = S_0^{-1} B S_0$

$B = S_1^{-1} C S_1$

$\Rightarrow A = S_0^{-1} S_1^{-1} C S_1 S_0$

$= (S_1 S_0)^{-1} C (S_1 S_0)$

we take $S = S_1 S_0$

$A = S^{-1} C S$

which implies A is similar to C

2. we have:

$V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $W = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$A = [a_1, a_2, a_3]$

$= [[L(v_1)]_W, [L(v_2)]_W, [L(v_3)]_W]$

$= \left[\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

3. The basis of P_2 is $[1, x, x^2]$

$\Rightarrow T(1) = 1 + 0 \times t + 0 \times t^2$

$\Rightarrow T(t) = -2 + 3t + 0 \times t^2$

$\Rightarrow T(t^2) = 4 - 12t + 9 \times t^2$

$\Rightarrow T$ w.r.t. P_2 is $\begin{bmatrix} 1 & -2 & 4 \\ 0 & 3 & -12 \\ 0 & 0 & 9 \end{bmatrix}$

4. $A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}$

$V = [v_1, v_2, v_3]$

$\cancel{[M]} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -2 \\ 1 & 0 & 0 \end{bmatrix}$

$B = V^{-1} A V = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

5. $L(x) = Ax$

$$\underbrace{\begin{bmatrix} 2x_1 - x_2 - x_3 \\ 2x_2 - x_1 - x_3 \\ 2x_3 - x_1 - x_2 \end{bmatrix}}_{L(x)} = \underbrace{\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x}$$

(a) $L\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(b) $L\left(\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$

(c) $L\left(\begin{bmatrix} -5 \\ 3 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} -5 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -15 \\ 9 \\ 6 \end{bmatrix}$

6. L is one-to-one implies that ~~the~~ there is only one vector that maps 0_{W} by linear tranformation, 0_V maps to 0_W.

$\Rightarrow \ker(L) = \{0_V\}$

we take v_1 & v_2 that map elements of W s.t. $L(v_1) = L(v_2) \Rightarrow L(v_1) - L(v_2) = 0_V$

$\Rightarrow L(v_1 - v_2) = 0_V \Rightarrow v_1 - v_2 = 0_V \Rightarrow v_1 = v_2$

because of these reasons, L is one-to-one if and only if $\ker(L) = \{0_V\}$

7a. $A = [T(e_1), T(e_2)]$

$= \begin{bmatrix} 1 & -1 \\ 0 & 3 \\ 4 & 5 \end{bmatrix}$

b. $A = [T(e_1), T(e_2)]$

$= \left[\begin{bmatrix} 3 \\ -6 \end{bmatrix} - \begin{bmatrix} -2 \\ 5 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \end{bmatrix} - \begin{bmatrix} 2 \\ -4 \end{bmatrix} \right]$

$= \begin{bmatrix} 5 & -4 \\ -11 & 9 \end{bmatrix}$

8. $A = [T(e_1), T(e_2)]$

$= [v_1, v_2]$

$= \begin{bmatrix} -3 & 7 \\ 5 & -2 \end{bmatrix}$

9. let $S: \mathbb{R}^p \to \mathbb{R}^n$, $T_0: \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformations, we want to show

$T_1: \mathbb{R}^p \to \mathbb{R}^m$

$x \to T_0(S(x))$ is a linear transformation

let $u, v \in \mathbb{R}^p$ and $\alpha, \beta \in \mathbb{R}$

consider $T_1(\alpha u + \beta v) = T_0(S(\alpha u + \beta v))$

$\Rightarrow T_1(\alpha u + \beta v) = T_0(\alpha S(u) + \beta S(v))$

$= \alpha T_0(S(u)) + \beta T_0(S(v))$

$= \alpha T_1(u) + \beta T_1(v)$

this shows that T_1 is a linear transformation, which implies

$x \to T_0(S(x)): \mathbb{R}^p \to \mathbb{R}^m$ is a linear transformation

10. assume $T(\alpha x) = \alpha T(x)$, let's consider the value of α, which is negative,

$\alpha < 0$

$\Rightarrow T(\alpha x) = \begin{bmatrix} \alpha x_1 + 2\alpha x_2 \\ \alpha x_1 - 4\alpha x_2 \end{bmatrix}$

$\neq \begin{bmatrix} \alpha x_1 - 2\alpha x_2 \\ \alpha x_1 - 4\alpha x_2 \end{bmatrix}$

which is a contradiction. hence, proved

11. $x = \begin{bmatrix} \sqrt{2} \\ 0 \\ 1 \end{bmatrix}$ $y = \begin{bmatrix} 0 \\ \sqrt{5} \\ 0 \end{bmatrix}$

$\|x\| = \sqrt{(\sqrt{2})^2 + 0^2 + 1^2} = \sqrt{3}$

$\|y\| = \sqrt{0^2 + (\sqrt{5})^2 + 0^2} = \sqrt{5}$

$\|x+y\| = \sqrt{(\sqrt{2})^2 + (\sqrt{5})^2 + 1^2} = \sqrt{8}$

$\left.\begin{array}{l} \|x\|^2 + \|y\|^2 = 3 + 5 = 8 \\ \|x+y\|^2 = 8 \end{array}\right\}$ same

12. note that $\alpha = \dfrac{a_1^T a_2}{\|a_2\|}$

by Pythagorers, $\|a_1\|^2 = h^2 + \alpha^2$

$h^2 = \|a_1\|^2 - \alpha^2$

$= \|a_1\|^2 - \left(\dfrac{a_1^T a_2}{\|a_2\|} \right)^2$

$\Rightarrow h^2 \|a_2\|^2 = \|a_1\|^2 \|a_2\|^2 - (a_1^T a_2)^2$

13. $\|a-b\| = \sqrt{\|a-b\|^2}$

$= \sqrt{(a^T - b^T)(a-b)}$

$= \sqrt{\|a\|^2 - 2(a \cdot b) + \|b\|^2}$

$= \sqrt{3}$

14a) $x^T y = [-2\ 3\ 0] \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} = 0$, which shows that x and y are orthogonal

b) $x^T z = y^T z = 0 \Rightarrow -2a + 3b = 0$ ...(1)

$3a + 2b + 16 = 0$ ...(2)

by (1) & (2) $\Rightarrow a = -\frac{48}{13}$ $b = -\frac{32}{13}$

15. $u^T v = a^2 + 4 - b^2 = 0$ & $a = b + 1$

$\Rightarrow (b+1)^2 + 4 - b^2 = 0$

$\Rightarrow b = -5/2$

$\Rightarrow a = -3/2$

16. $u^T v = \cos(\theta) - \sin(\theta) = 0$

$\Rightarrow \theta = \left\{ \frac{\pi}{4}, \frac{5\pi}{4} \right\}$

17. $W = \text{Span}\left\{ \underbrace{\begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}}_{w_1}, \underbrace{\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}}_{w_2} \right\}$

$$\begin{aligned} w^\perp \cdot w_1 &= 0 \\ w^\perp \cdot w_2 &= 0 \end{aligned} \Rightarrow w^\perp = \text{Null}\begin{pmatrix} 1 & 7 & 2 \\ -2 & 3 & 1 \end{pmatrix}$$

$$\Rightarrow x = \alpha \begin{bmatrix} 1/17 \\ -5/17 \\ 1 \end{bmatrix}$$

$$W^\perp = \begin{bmatrix} 1 \\ -5 \\ 17 \end{bmatrix}$$

18 a) True, V & W orthogonal $\Rightarrow V = W^\perp$

$\Rightarrow V^\perp$ & $W^\perp$ orthogonal

b) False

19. $S = \text{Span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$S^\perp = \text{Null}(S) = \mathbb{R}^4$

$= \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

20. $\vec{MN} = \begin{bmatrix} -2.25 \\ 0 \end{bmatrix}$

$\vec{KL} = \begin{bmatrix} 1.5 \\ 2 \end{bmatrix}$

$\text{proj}_{\vec{KL}} \vec{MN} = \left(\frac{\vec{MN} \cdot \vec{KL}}{\|\vec{KL}\|^2} \right) \cdot \vec{KL}$

$= \begin{bmatrix} -0.81 \\ -1.08 \end{bmatrix}$