MAT3007 - Assignment 2

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Problem 1. True or False

1. The set of all optimal solutions (assuming existence) must be bounded;

   False. Optimal solutions are not necessarily bounded. The counterexample is shown as follows:

$$\begin{aligned} \min \quad & 0 \\ \text{s.t.} \quad & x_1 - x_2 = 0 \\ & x_1, x_2 \geq 0 \end{aligned}$$

   This still satisfies the constraint set P and linear independence for A's rows.

2. At every optimal solution, no more than m variables can be positive;

   False. Consider the previously provided LP:

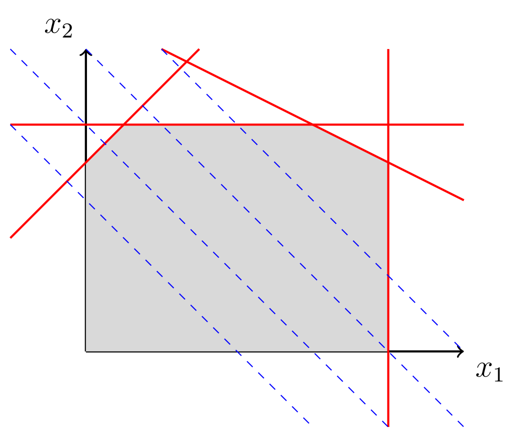
$$\begin{aligned} \min \quad & 0 \\ \text{s.t.} \quad & x_1 - x_2 = 0 \\ & x_1, x_2 \geq 0 \end{aligned}$$

The set of all solutions that satisfy the constraint $x_1 - x_2 = 0$ and $x_1, x_2 \geq 0$ is a line where x_1 equals x_2 in the first quadrant. Along this line, there are points (in fact, infinitely many) where both x_1 and x_2 are positive. For instance, $x_1 = 1, x_2 = 1$ or $x_1 = 0.5, x_2 = 0.5$. At these points, two variables are positive, which is more than m. Thus, the statement is false for this counterexample.

3. If there is more than one optimal solution, then there are uncountably many optimal solutions.

   True. If an LP in standard form has multiple optimal solutions, then the set of optimal solutions forms a convex combination. This implies that there exists a line segment (or a higher-dimensional equivalent) within the feasible set along which all the points yield the same optimal objective value. Since a line segment has uncountably many points, there are uncountably many optimal solutions. For example if x_1 and x_2 direct to optimal solution, so does for each $\alpha x_1 + (1-\alpha)x_2$ (uncountably many).

Problem 2. Graphical Method



The gray-shaded area is the feasible region defined by the constraints. We are going to use the concept of iso-profit lines to find the optimal solution. The iso-profit lines, shown as dashed lines in the above figure, are our objective function: $x_1 + x_2 = C$, where C is a constant that can be adjusted. Greedily, we want it to increase as much as it can as the objective is to maximize and we can find the last point where this line touches the feasible region, which will be the optimal solution.

The vertices of the feasible region are:

1. $(0, 2.5)$
2. $(0.5, 3)$
3. $(3, 3)$
4. $(4, 2.5)$
5. $(4, 0)$
6. $(0, 0)$

The maximum value of $x_1 + x_2$ will occur at the vertex $(4, 2.5)$, giving an optimal value of 6.5. Thus, the constraints active at the optimal solution are $x_1 + 2x_2 \leq 9$ and $0 \leq x_1 \leq 4$.

Problem 3. Basic solutions and basic feasible solutions

$$\begin{aligned} \min_{x} \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

$$, \text{ where } c = \begin{bmatrix} -1 \\ -2 \\ -4 \\ 0 \\ 0 \end{bmatrix}, A = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 \end{bmatrix}, b = \begin{bmatrix} 8 \\ 15 \end{bmatrix}, \text{ and } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ s_1 \\ s_2 \end{bmatrix}$$

In an n-dimensional space, it is required to have at least n planes to pinpoint a specific location. However, notice that in our scenario, we only have m planes (constraints) and $m < n$. Geometrically-wised, this means that we cannot fully enclose a region or determine a unique point in the space with the given constraints. Instead, these constraints will form a shape (in our case, a 2D surface in a 3D space) that extends infinitely in at least one dimension.

By the definition of a basic solution, for any subset of m constraints that are linearly independent, they determine a unique point (or vertex) on this surface. This vertex represents a basic solution. Given that we only have $m = 2$ constraints in our 3D space ($n = 3$), any such vertex determined by the intersection of two constraints implies that one of the three variables must be zero at this vertex. That is, only two out of the three variables are positive.

Given that optimal solutions to LPs are located at vertices of the feasible region, and each of the vertices in our case has no more than 2 positive variables, it's conclusive that there must exist an optimal solution with no more than 2 positive variables.

Definition: Basic Solution

We call $\mathbf{x}$ a basic solution of the LP if and only if

1. $\mathbf{Ax} = \mathbf{b}$.
2. There exist indices $B(1), \ldots, B(m)$ such that the columns of

$$\begin{bmatrix} | & | & & | \\ A_{B(1)} & A_{B(2)} & \cdots & A_{B(m)} \\ | & | & & | \end{bmatrix} = \mathbf{A}_B$$

   are linearly independent and $x_i = 0$ for $i \neq B(1), ..., B(m)$.

By definition, we can find B as:

1. $B = \{1, 2\}$

   , which implies: $x_B = (8, 15)$, $c^T x = -38$, and it is a BFS.

2. $B = \{1, 3\}$

   , which implies: $x_B = (0.5, 7.5)$, $c^T x = -30.5$, and it is a BFS.

3. $B = \{1, 4\}$

   , which is not linearly independent.

4. $B = \{1, 5\}$

   , which implies: $x_B = (8, 15)$, $c^T x = -8$, and it is a BFS.

5. $B = \{2, 3\}$

   , which implies: $x_B = (-1, 8)$, $c^T x = -30$, and it is **NOT** a BFS.

6. $B = \{2, 4\}$

   , which implies: $x_B = 15, 8$, $c^T x = -30$, and it is a BFS.

7. $B = \{2, 5\}$

   , which is not linearly independent.

8. $B = \{3, 4\}$

   , which implies: $x_B = (4, 11)$, $c^T x = -16$, and it is a BSF.

9. $B = \{3, 5\}$

   , which implies: $x_B = (8, -1)$, $c^T x = -24$, and it is **NOT** a BFS.

10. $B = \{4, 5\}$

    , which implies: $x_B = (8, 15)$, $c^T x = 0$, and it is a BFS.

For all obtained BFS, we may find that the optimal value is obtained when $c^T x = -38$, which is obtainable by having $(8, 15, 0, 0, 0)$.

Problem 4. Vertex Covering Problem

Python code for $x_i \in \{0, 1\}$ is as follows:

```
import cvxpy as cp

if __name__ == "__main__":
    n = 10

    # a is denoted by 0, b is denoted by 1, ...
    edges = [(0, 1), (1, 2), (2, 3), (3, 4), (4, 0),
             (0, 5), (1, 6), (2, 7), (3, 8), (4, 9),
             (5, 7), (6, 8), (7, 9), (8, 5), (9, 6)]
```

```
    X = cp.Variable(n, boolean = True)

    obj = cp.Minimize(cp.sum(X))
    consts = [
        X[u] + X[v] >= 1 for (u, v) in edges
    ]

    print("Minimum objective value:", cp.Problem(obj, consts).solve())
    print("Obtainable by having X =\n", X.value)
```

Console output:

```
Minimum objective value: 6.0
Obtainable by having X =
 [-0.  1.  1. -0.  1.  1. -0. -0.  1.  1.]
```

Python code for $x_i \in [0,1]$ is as follows:

```
import cvxpy as cp

if __name__ == "__main__":
    n = 10

    # a is denoted by 0, b is denoted by 1, ...
    edges = [(0, 1), (1, 2), (2, 3), (3, 4), (4, 0),
             (0, 5), (1, 6), (2, 7), (3, 8), (4, 9),
             (5, 7), (6, 8), (7, 9), (8, 5), (9, 6)]

    X = cp.Variable(n, nonneg = True)

    obj = cp.Minimize(cp.sum(X))
    consts = [
        X[u] + X[v] >= 1 for (u, v) in edges
    ] + [
        X[u] <= 1 for u in range(n)
    ]

    print("Minimum objective value:", cp.Problem(obj, consts).solve())
    print("Obtainable by having X =\n", X.value)
```

Console output:

```
Minimum objective value: 5.000000000061929
Obtainable by having X =
 [0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5]
```

When we set the constraints for each x_i to be $\in [0, 1]$ instead of $\in \{0, 1\}$, we obtain the LP solution as $X = [0.5\ 0.5\ \dots\ 0.5]$ that results the optimal value as 5 (0.5×10). While we have the optimal value of the true problem to be 6, we conclude that one cannot simply remove the integer constraint and expect to get a correct, integer solution to the vertex cover problem from the LP relaxation.

Problem 5. A Robust LP Formulation

The optimization problem rewritten as a linear problem is as follows:

$$\begin{aligned} \min_x \quad & c^T x \\ \text{s.t.} \quad & A_i x - b_i \leq \delta, \forall i = 1, \dots, m \\ & -A_i x + b_i \leq \delta, \forall i = 1, \dots, m \\ & x \geq 0 \end{aligned}$$

Denote x_1 and x_2 as the number of processed fruit salads A and B, respectively.

$$\begin{aligned} \max_x \quad & 10x_1 + 20x_2 \\ \text{s.t.} \quad & \frac{1}{4}x_1 + \frac{1}{2}x_2 = 25 \\ & \frac{1}{8}x_1 + \frac{1}{4}x_2 = 10 \\ & 3x_1 + x_2 = 120 \\ & x_1, x_2 \geq 0 \\ & x_1, x_2 \in Z \end{aligned}$$

The standard form is as follows:

$$\begin{aligned}\min_x \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0\end{aligned}$$

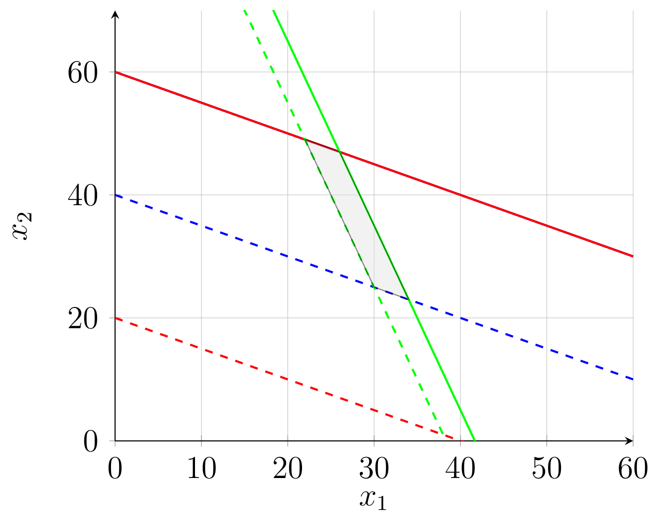
, where $c = \begin{bmatrix} -10 \\ -20 \end{bmatrix}$, $A = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{8} & \frac{1}{4} \\ 3 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 25 \\ 10 \\ 120 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $x_1, x_2 \in Z$.

Clearly both $\frac{1}{4}x_1 + \frac{1}{2}x_2 = 25$ and $\frac{1}{8}x_1 + \frac{1}{4}x_2 = 10$ can't be true together. Therefore, we cannot simultaneously satisfy the constraints for mangoes and pineapples. As a result, the LP is infeasible.

With a robust variant $\delta = 5$, our LP problem becomes:

$$\begin{aligned}\min_x \quad & -10x_1 - 20x_2 \\ \text{s.t.} \quad & \frac{1}{4}x_1 + \frac{1}{2}x_2 \leq 25 + 5 \\ & \frac{1}{4}x_1 + \frac{1}{2}x_2 \geq 25 - 5 \\ & \frac{1}{8}x_1 + \frac{1}{4}x_2 \leq 10 + 5 \\ & \frac{1}{8}x_1 + \frac{1}{4}x_2 \geq 10 - 5 \\ & 3x_1 + x_2 \leq 120 + 5 \\ & 3x_1 + x_2 \geq 120 - 5 \\ & x_1, x_2 \geq 0 \\ & x_1, x_2 \in Z\end{aligned}$$

The feasible set (region) is sketched as follows.



We have 4 points to be considered: $(22, 49)$, $(26, 47)$, $(34, 23)$, $(30, 25)$. Put them into our objective function, we have -1200, -1200, -800, -800, respectively. Then, the optimal value is 1200 and obtained by having x anywhere in the line segment between $(22, 49)$ and $(26, 47)$ (inclusive). The active constraints include both $\frac{1}{4}x_1 + \frac{1}{2}x_2 \leq 30$ and $\frac{1}{8}x_1 + \frac{1}{4}x_2 \leq 15$. Integer solutions to the problem are: $(22, 49)$, $(24, 48)$, $(26, 47)$.