yohandi – assignment 10

1a. $M_{X_2}(t) = E(e^{tX_2})$

$\quad = E(e^{t(Y-X_1)})$

$\quad = E(e^{tY}) E(e^{-tX_1})$

$\quad = (1-2t)^{-r/2} (1-2t)^{r_1/2}$

$\quad = (1-2t)^{(r_1-r)/2}$

b. $X_2 \sim \chi^2(r-r_1)$

2. $M_Y(t) = E(e^{tX^2})$

$\quad = \int_{-\infty}^{\infty} e^{tx^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$

$\quad = \int_{-\infty}^{\infty} e^{x^2(t-\frac{1}{2})} \frac{1}{\sqrt{2\pi}} dx$

let $w = x\sqrt{1-2t}$

$\frac{dw}{dx} = \sqrt{1-2t}$

$\Rightarrow M_y(t) = \int_{-\infty}^{\infty} e^{-\frac{w^2}{2}} \frac{1}{\sqrt{2\pi}} \frac{dw}{\sqrt{1-2t}}$

$\quad = \frac{1}{\sqrt{1-2t}}$

this implies that $Y \sim \chi^2(1)$

3.

$f(x)$

(c) $n=36$

(b) $n=9$

(a) $n=1$

x
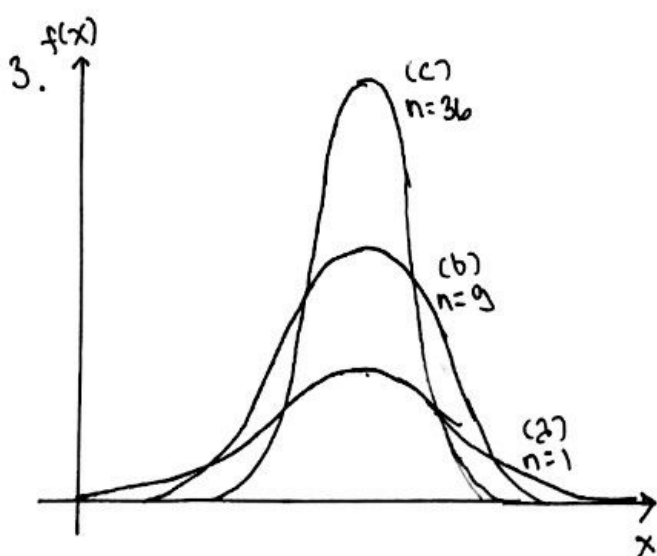
4. $X \sim N(6.05, 0.0004)$

a. $P(X < 6.0171) = \phi\left(\frac{6.0171 - 6.05}{\sqrt{0.0004}}\right) \approx 0.050$

$Y \sim B(9, 0.050)$

b. $P(Y \leq 2) = \sum_{i=0}^{2} \binom{9}{i}(0.050)^i (1-0.050)^{9-i} \approx 0.992$

$\bar{X} \sim N(6.05, \frac{0.0004}{9})$

c. $P(\bar{X} \leq 6.035) = \phi\left(\frac{6.035 - 6.05}{\sqrt{\frac{0.0004}{9}}}\right) \approx 0.012$

6a. $W = \frac{z_1}{\sqrt{\frac{z_2^2 + z_3^2}{2}}} = \frac{z}{\sqrt{\frac{u}{r}}}$ where $\begin{array}{l} z \sim N(0,1) \\ u \sim \chi^2(2) \\ r=2 \end{array}$

this implies that $W \sim t(2)$

b. $V = \frac{z_1}{\sqrt{\frac{z_1^2 + z_2^2}{2}}}$

$F(v) = P(V \leq v)$

$\quad = P\left(\frac{z_1}{\sqrt{\frac{z_1^2+z_2^2}{2}}} \leq v\right)$

$\quad = P\left(-\frac{v}{\sqrt{2-v^2}}\sqrt{z_2^2} \leq z_1 \leq \frac{v}{\sqrt{2-v^2}}\sqrt{z_2^2}\right)$

$\quad = \int_0^{\infty}\int_{-\infty}^{\frac{v|z_2|}{\sqrt{2-v^2}}} \frac{1}{2\pi} e^{-\frac{(z_1^2+z_2^2)}{2}} dz_1 dz_2 - \int_0^{\infty}\int_{-\infty}^{\frac{-v|z_2|}{\sqrt{2-v^2}}} \frac{1}{2\pi} e^{-\frac{(z_1^2+z_2^2)}{2}} dz_1 dz_2$

$f(v) = \frac{d}{dv} F(v)$

$\quad = \frac{1}{\pi(2-v^2)^{3/2}} \left(\int_0^{\infty} z_2 \cdot e^{(\frac{-2}{\sqrt{2}-2})z_2^2(\frac{1}{2})} dz_2\right)$

$\quad = \frac{2}{\pi(2-v^2)^{3/2}}\left(\frac{4}{v^2-2}\right)\int_0^{\infty} e^{-w/2} dw$

$\quad = \frac{1}{\pi\sqrt{2-v^2}}$

c. $E(v) = \int_{-\infty}^{\infty} v \, f(v)$

$= \int_{-\sqrt{2}}^{\sqrt{2}} v \cdot \underbrace{\frac{1}{\pi\sqrt{2-v^2}}}_{\text{odd function}} dv$

$= 0$

d. $Var(v) = E(v^2) - E(v)^2$

$= \int_{-\infty}^{\infty} v^2 f(v)$

$= \int_{-\sqrt{2}}^{\sqrt{2}} \frac{v^2}{\pi\sqrt{2-v^2}} dv$

$= \frac{\arcsin(\frac{v}{\sqrt{2}})}{\pi} - \frac{v\sqrt{2-v^2}}{2\pi} \Big|_{v=-\sqrt{2}}^{\sqrt{2}}$

$= 1$

$\sigma = \sqrt{\sigma_v^2} = \sqrt{Var(v)} = 1$

e. In succinct way, both (a) & (b) are mainly different on their dependencies. Since $z_1, z_2,$ and z_3 have independent distribution,

z_1 is independent with $\sqrt{\frac{z_2^2 + z_3^2}{2}}$.

however,

z_1 might not be independent

with $\sqrt{\frac{z_1^2 + z_2^2}{2}}$

6. $T = \frac{z}{\sqrt{\frac{u}{r}}}$ (here, $r=2$)

$\rightarrow E(T) = \underbrace{E(z)}_{0 \ (\text{since } z \sim N(0,1))} \cdot E\left(\frac{\sqrt{r}}{\sqrt{u}}\right)$

$= 0$

$\rightarrow Var(T) = E\left(\frac{rz^2}{u}\right) - E(T)^2$

$= r \cdot E(z^2) \, E(\frac{1}{u})$

$= r \, (E(z)^2 + Var(z)) \, E(\frac{1}{u})$

$= r \cdot (0^2 + 1) \cdot \int_0^\infty \frac{1}{u}\left(\frac{u^{r/2-1}e^{-u/2}}{2^{r/2}\Gamma(\frac{r}{2})}\right) du$

$= r \cdot \frac{1}{r-2}$

$= \frac{r}{r-2}$

7a. $t_{0.025}(9-1) = 2.306$

b. $-t_{0.025} \leq T \leq t_{0.025}$

$= -t_{0.025} \leq \frac{\bar{x}-\mu}{\frac{s}{\sqrt{n}}} \leq t_{0.025}$

$= -2.306 \leq \frac{\bar{x}-\mu}{\frac{s}{\sqrt{9}}} \leq 2.306$

$= \bar{x} - \frac{2.306}{3}s \leq \mu \leq \bar{x} + \frac{2.306}{3}s$

8a. $Y \sim \chi^2_{(18)}$

b. $Y \sim N(18, 36)$

$P(Y \leq 9.390) \approx 0.07565$ & $P(Y \leq 9.390) \approx 0.05$

(CLT)   (table)

$P(Y \leq 34.80) \approx 0.9974$ & $P(Y \leq 34.80) \approx 0.99$

(CLT)   (table)

9a. $E(\bar{x}) = E(x) = 24.43$

b. $Var(\bar{x}) = \frac{Var(x)}{n} = 0.0\overline{73}$

c. $\phi\left(\frac{24.82-24.43}{\sqrt{0.073}}\right) - \phi\left(\frac{24.19-24.43}{\sqrt{0.073}}\right) = 0.7566$

10. $Y \sim N(10 \cdot 20, 4 \cdot 20)$

$P\left(z \geq \frac{Y-200}{\sqrt{80}}\right) = 1 - P\left(z \leq \frac{Y-200}{\sqrt{80}}\right) \approx 20\%$

$\Rightarrow \phi\left(\frac{Y-200}{\sqrt{80}}\right) = \phi(0.84)$

$\Rightarrow Y \approx 207.513$

$\Rightarrow \lceil Y \rceil = 208$ days for budget

yohandi - assignment 10 (computer-based)

S	-1	1
f(S)	0.5	0.5

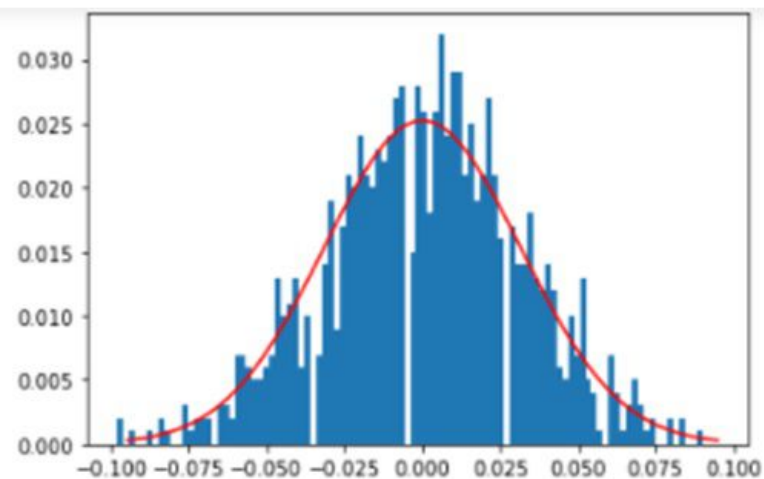$$E(X) = E\left(\frac{1}{N}\sum_{i=1}^{N} S_i\right) = \frac{1}{N}\sum_{i=1}^{N} \underbrace{E(S_i)}_{0} = 0$$

$$Var(x) = Var\left(\frac{1}{N}\sum_{i=1}^{N} S_i\right) = \frac{1}{N^2}\underbrace{\sum_{i=1}^{N} Var(S_i)}_{N} = \frac{1}{N}$$
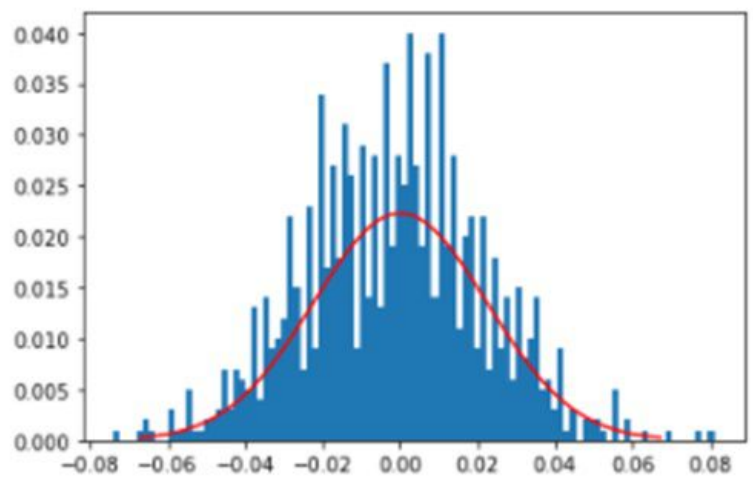
from the obtained $Var(x)$,

the bigger N, the smaller $Var(x)$
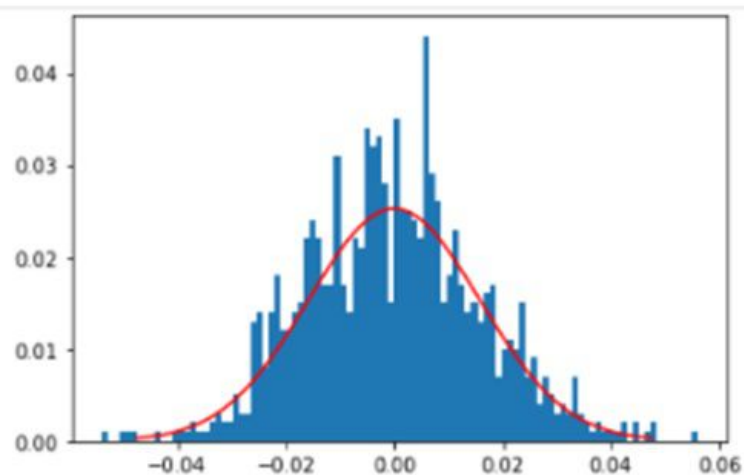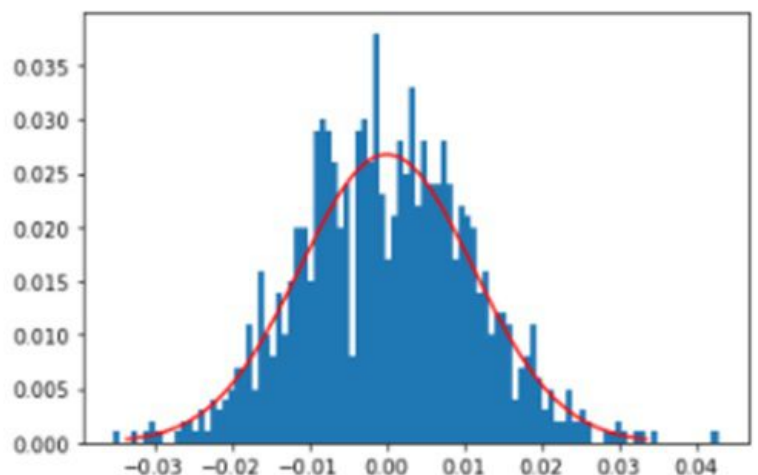$\Rightarrow$ the graph tends to be more compact

N=1000

N=2000

N=4000

N=8000

```python
import math
import matplotlib.pyplot as plt
import numpy as np
import random
import scipy.stats

total_experiment = 4
N_list = [(1000 * 2 ** i) for i in range(total_experiment)]

def random_walk(N):
    return sum([(random.randint(0, 1) * 2 - 1) for i in range(N)]) / N

def experiment(ith, N, M = 1000, K = 100):
    ith_list = [0.002, 0.00125, 0.001, 0.00075]
    sample_X = np.array([random_walk(N) for i in range(M)])
    ax = plt.figure().add_subplot(111)
    n, bins, patches = ax.hist(sample_X, bins = K, cumulative = 0, weights = np.zeros_like(sample_X) + 1. / len(sample_X))
    ax.legend
    mean = 0
    variance = 1 / N
    sigma = math.sqrt(variance)
    X = np.linspace(mean - 3 * sigma, mean + 3 * sigma, 100)
    Y = [i * ith_list[ith] for i in scipy.stats.norm.pdf(X, mean, sigma)]
    plt.plot(X, Y, color = "r")
    plt.show()

for i in range(total_experiment):
    experiment(i, N_list[i])
```