# DDA6050 Assignment 3

Zihan, Zhou  119010484@link.cuhk.edu.cn

Due by: 23:59, Nov 17th

If you have any questions about grading, please feel free to reach out to me via email at 119010484@link.cuhk.edu.cn or visit me during my office hours, which are held every Wednesday from 16:00 to 17:00 at the SDS Research Lab, Seat 88, Zhi Xin Building.

## 1 Amortized Analysis. (20 marks)

(a). Assuming we possess two stacks capable of performing push and pop operations at a constant $O(1)$ cost per operation, construct a queue with enqueue and dequeue operations in such a way that the amortized time for each queue operation remains within $O(1)$. Also prove that the implemented queue indeed has the $O(1)$ amortized time per operation. (10 marks)

(b). Suppose that we run a program over a sequence of $n$ days. On the $i$-th day, if $\log_2(i)$ is an integer, then this program costs $i$ units of computation resources to examine the outputs obtained so far. Otherwise, it only costs 1 unit of computation resource this day. Compute the amortized computation cost per day. (10 marks)

## 2 Governing Set Problem. (35 marks)

In this question, we only consider the connected graph. A **governing set** in a graph $G = (V, E)$ is a subset of vertices $S \subseteq V$ such that every vertex $v \in V \backslash S$ is adjacent to a vertex $s \in S$. Now, the governing set problem is that, given a graph $G = (V, E)$ and an integer $k$, we want to determine the existence of a **governing set** $S \subseteq V$ such that $|S| \leq k$. Show that the vertex-cover problem $\leq_p$ governing set problem. (Hint: You may consider the following algorithm. Try to prove the correctness of the following algorithm. This algorithm claims a necessary and sufficient condition, so you must demonstrate both aspects.)

**Algorithm 1** Vertex cover problem to governing set reduction

**Input**: graph $G = (V, E)$ and and integer $k$.

**Output**: Graph $H = (V', E')$ such that $G$ has a vertex cover of size $k$ if and only if $H$ has a governing set of size $k$.

Initialize $V' \leftarrow V, E' \leftarrow E$.
**for** every edge $\{u, v\} \in E$ **do**
  Add vertex $w_{uv}$ to $V'$.
  Add edges $\{u, w_{uv}\}, \{v, w_{uv}\}$ to $E'$.
**end for**
**return** $H = (V', E')$

# 3 NP-complete. (35 marks)

Prove that the following problem is NP-complete. Given a set $X$ and a family $\mathcal{F}$ of subsets of $X$, whether there are two disjoint sets $S_1, S_2$ such that $S_1 \cup S_2 = X$ and for any set $A$ in $\mathcal{F}$, $A$ is not a subset of $S_1$ and $S_2$. The following is an example. Let $X = \{1, 2, 3, 4, 5\}, \mathcal{F} = \{\{1, 2\}, \{3, 4\}, \{1, 2, 4\}\}$. A possible assignment is $S_1 = \{1, 4\}, S_2 = \{2, 3, 5\}$. The assignment of $S_1 = \{1, 3, 4\}, S_2 = \{2, 5\}$ is infeasible since a subset $\{3, 4\}$ of $\mathcal{F}$ is a subset of $S_1$. (Hint: consider not-all-equal 3-satisfiability (NAE3SAT, see Wikipedia) which is NP-complete)

# 4 Randomized Algorithm. (10 marks)

Suppose we have $n$ servers and $m$ tasks. Each task is independently and uniformly randomly assigned to a server among $n$ of them. In this question, you do not need to rigorously consider if a value is integer. Suppose that $m = 2n \log n$.

(a) We focus on the first server. Show that the probability of it receiving at least $2e \cdot \log n$ tasks is no larger than $1/n^2$. (5 marks)

(b) Following (a), show that when $n$ is large enough, then with high probability, no server receives at least $2e \log n$ tasks. (5 marks)