yohandi - math homework week 11

Exercises 7.8

9a. $\lim_{x\to\infty} \frac{x}{x} = 1 \neq 0$ (false)

b. $\lim_{x\to\infty} \frac{x}{x+5} = 1 \neq 0$ (false)

c. $\lim_{x\to\infty} \frac{x}{x+5} = 1 < \infty$ (true)

d. $\lim_{x\to\infty} \frac{x}{2x} = \frac{1}{2} < \infty$ (true)

e. $\lim_{x\to\infty} \frac{e^x}{e^{2x}} = \lim_{x\to\infty} e^{-x} = 0$ (true)

f. $\lim_{x\to\infty} \frac{x+\ln x}{x}$ "$\frac{\infty}{\infty}$" indeterminate form (apply L'Hôpital)

$\lim_{x\to\infty} \frac{1+\frac{1}{x}}{1} = 1 < \infty$ (true)

g. $\lim_{x\to\infty} \frac{\ln(x)}{\ln(2x)}$

"$\frac{\infty}{\infty}$" indeterminate form (apply L'Hôpital)

$\lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{2}{2x}} = 1 \neq 0$ (false)

h. $\lim_{x\to\infty} \frac{\sqrt{x^2+5}}{x} = 1 < \infty$ (true)

10a. $\lim_{x\to\infty} \frac{\frac{1}{x+3}}{\frac{1}{x}} = 1 < \infty$ (true)

b. $\lim_{x\to\infty} \frac{\frac{1}{x}+\frac{1}{x^2}}{\frac{1}{x}} = 1 < \infty$ (true)

c. $\lim_{x\to\infty} \frac{\frac{1}{x}-\frac{1}{x^2}}{\frac{1}{x}} = 1 \neq 0$ (false)

d. $\lim_{x\to\infty} \frac{2+\cos x}{2} = L$

where, $\frac{1}{2} \le L \le \frac{3}{2} < \infty$ (true)

e. $\lim_{x\to\infty} \frac{e^x+x}{e^x}$

"$\frac{\infty}{\infty}$" indeterminate form (apply L'Hôpital)

$\lim_{x\to\infty} \frac{e^x+1}{e^x}$

"$\frac{\infty}{\infty}$" indeterminate form (apply L'Hôpital)

$\lim_{x\to\infty} \frac{e^x}{e^x} = 1 < \infty$ (true)

f. $\lim_{x\to\infty} \frac{x\ln x}{x^2} = \lim_{x\to\infty} \frac{\ln x}{x}$.

"$\frac{\infty}{\infty}$" indeterminate form (apply L'Hôpital)

$\lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$ (true)

g. $\lim_{x\to\infty} \frac{\ln(\ln(x))}{\ln(x)}$

"$\frac{\infty}{\infty}$" indeterminate form (apply L'Hôpital)

$\lim_{x\to\infty} \frac{\frac{\frac{1}{x}}{\ln(x)}}{\frac{1}{x}} = 0 < \infty$ (true)

h. $\lim_{x\to\infty} \frac{\ln(x)}{\ln(x^2+1)}$

"$\frac{\infty}{\infty}$" indeterminate form (apply L'Hôpital)

$\lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{2x}{x^2+1}} = \lim_{x\to\infty} \frac{x^2+1}{2x^2} = \frac{1}{2} \neq 0$ (false)

11. since both $f(x)$ and $g(x)$ are growing at the same rate there exist $\lim_{x\to\infty} \frac{f(x)}{g(x)} = M$ where M is a positive constant.

this implies that $0 < \lim_{x\to\infty} \frac{f(x)}{g(x)} < \infty$ ($f(x) = O(g(x))$).

$\lim_{x\to\infty} \frac{g(x)}{f(x)} = \frac{1}{M}$, since $M > 0 \Rightarrow \frac{1}{M} > 0$.

and $\frac{1}{M} < \infty \Rightarrow g(x) = O(f(x))$

12. case order f < order g:

$$\lim_{x\to\infty} \frac{\sum_{i=1}^{n} a_i x^i}{\sum_{i=1}^{m} b_i x^i} = \lim_{x\to\infty} \frac{x^n \sum_{i=1}^{n} a_i x^{i-n}}{x^m \sum_{i=1}^{m} b_i x^{i-m}} = 0$$

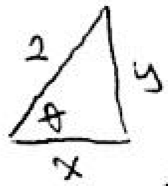
case order f = order g:

$$\lim_{x\to\infty} \frac{\sum_{i=1}^{n} a_i x^i}{\sum_{i=1}^{n} b_i x^i} = \frac{\sum_{i=1}^{n} a_i x^{i-n}}{\sum_{i=1}^{n} b_i x^{i-n}} = \frac{a_i}{b_i}$$

case order f > order g

$\lim_{x\to\infty} \frac{1}{\text{"case order } f < \text{order } g\text{"}} = \infty$ therefore, $f(x) = o(g(x))$

Exercises 7.6

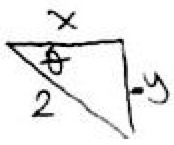
1a.


$\frac{y}{x} = 1$

$x^2+y^2=4$

$\Rightarrow x = y = \sqrt{2}$

$\theta = \frac{\pi}{4}$

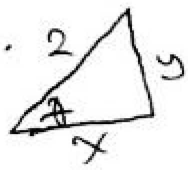
b.


$-\frac{y}{x} = -\sqrt{3}$

$x^2+y^2=4$

$\Rightarrow x=1,\ y=\sqrt{3}$

$\theta = -\frac{\pi}{3}$

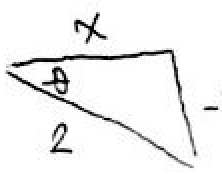
c.


$\frac{y}{x} = \frac{1}{\sqrt{3}}$

$x^2+y^2=4$

$\Rightarrow x=\sqrt{3},\ y=1$

$\theta = \frac{\pi}{6}$

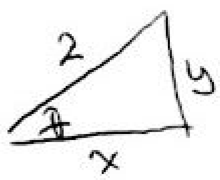
3a.


$\frac{-y}{2} = -\frac{1}{2}$

$x^2+y^2=4$

$\Rightarrow x=\sqrt{3},\ y=1$

$\theta = -\frac{\pi}{6}$

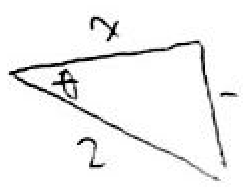
b.


$\frac{y}{2} = \frac{1}{\sqrt{2}}$

$x^2+y^2=4$

$\Rightarrow x=y=\sqrt{2}$

$\theta = \frac{\pi}{4}$

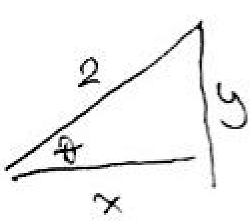
c.


$-\frac{y}{2} = -\frac{\sqrt{3}}{2}$

$x^2+y^2=4$

$\Rightarrow y=\sqrt{3},\ x=1$

$\theta = -\frac{\pi}{3}$

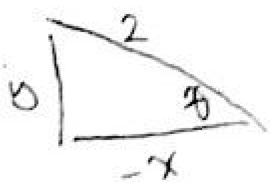
5a.


$\frac{x}{2} = \frac{1}{2}$

$x^2+y^2=4$

$\Rightarrow x=1,\ y=\sqrt{3}$

$\theta = \frac{\pi}{3}$

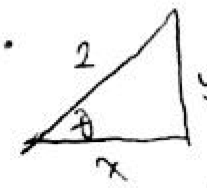
b


$-\frac{x}{2} = -\frac{1}{\sqrt{2}}$

$x^2+y^2=4$

$\Rightarrow x=\sqrt{2},\ y=\sqrt{2}$

$\theta = \frac{3\pi}{4}$

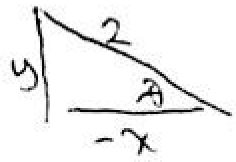
c.


$\frac{x}{2} = \frac{\sqrt{3}}{2}$

$x^2+y^2=4$

$\Rightarrow x=\sqrt{3},\ y=1$

$\theta = \frac{\pi}{6}$

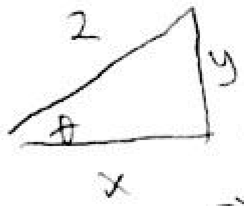
7a.


$-\frac{x}{2} = -\frac{1}{\sqrt{2}}$

$x^2+y^2=4$

$\Rightarrow x=\sqrt{2},\ y=\sqrt{2}$

$\theta = \frac{3\pi}{4}$

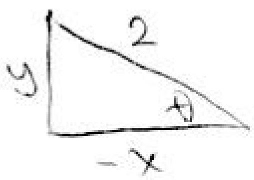
b


$\frac{x}{2} = \frac{\sqrt{3}}{2}$

$x^2+y^2=4$

$\Rightarrow x=\sqrt{3},\ y=1$

$\theta = \frac{\pi}{6}$

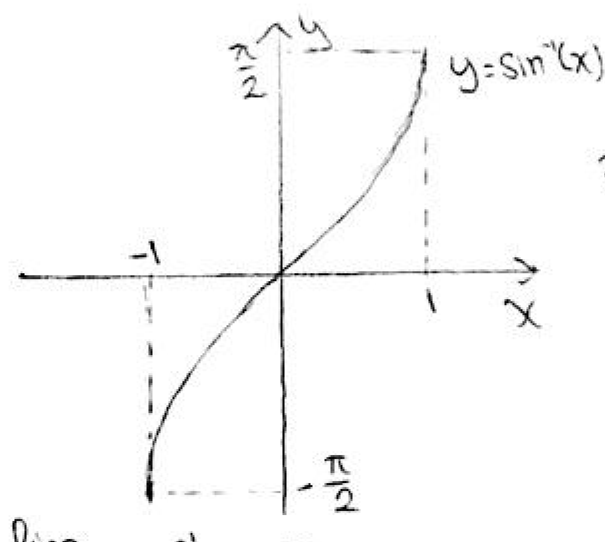
c.


$-\frac{x}{2} = -\frac{1}{2}$

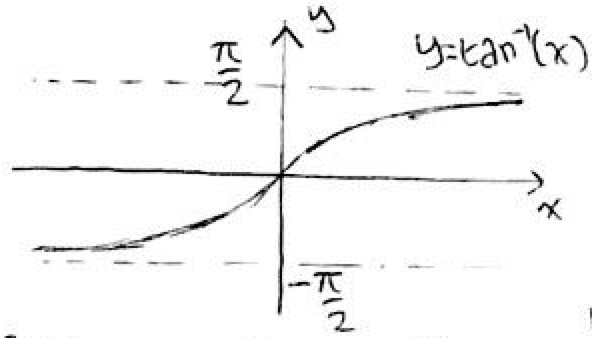
$x^2+y^2=4$

$\Rightarrow x=1,\ y=\sqrt{3}$

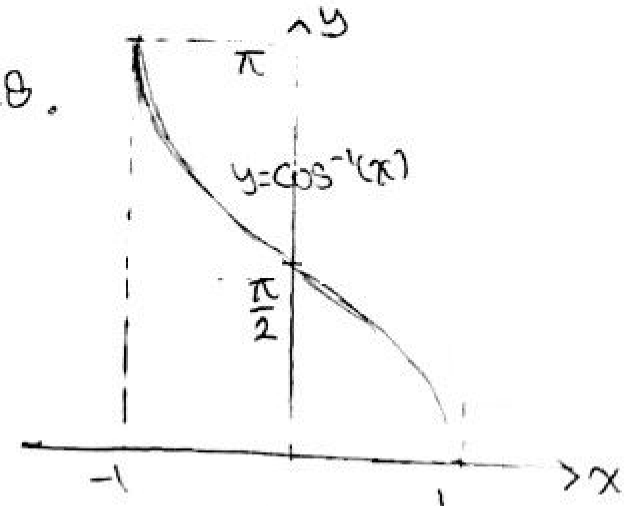
$\theta = \frac{2\pi}{3}$

13.


$\lim_{x\to 1^-} \sin^{-1}(x) = \frac{\pi}{2}$

16.


$\lim_{x\to -\infty} \tan^{-1}(x) = -\frac{\pi}{2}$

18.


$\lim_{x\to -\infty} \sec^{-1}(x) = \lim_{x\to 0^-} \cos^{-1}(x) = \frac{\pi}{2}$

22. $y = \cos^{-1}\left(\frac{1}{x}\right)$

$\frac{1}{x} = \cos(y)$ $\quad \frac{d(\dots)}{dx}$

$-\frac{1}{x^2} = \frac{dy}{dx}(-\sin(y))$

$\frac{dy}{dx} = \left|\frac{1}{x^2 \sin y}\right| = \frac{1}{\sqrt{x^4 - x^2}}$

25. $y = \sec^{-1}(2s+1)$

$2s+1 = \sec(y)$ $\quad \frac{d(\dots)}{ds}$

$2 = \frac{dy}{ds}\sec(y)\tan(y)$

$\frac{dy}{ds} = \left|\frac{1}{(2s+1)\sqrt{s^2+s}}\right|$

33. $y = \ln(\tan^{-1}(x))$

$\frac{dy}{dx} = \frac{1}{(x^2+1)\tan^{-1}(x)}$

41. $y = x\sin^{-1}x + \sqrt{1-x^2}$

$\frac{dy}{dx} = \sin^{-1}x + \frac{x}{\sqrt{1-x^2}} - \frac{2x}{2\sqrt{1-x^2}}$

$= \sin^{-1}(x)$

42. $y = \ln(x^2+4) - x\tan^{-1}\left(\frac{x}{2}\right)$

$\frac{dy}{dx} = \frac{2x}{x^2+4} - \tan^{-1}\left(\frac{x}{2}\right) - \frac{\frac{1}{2}x}{1+\frac{x^2}{4}}$

$= -\tan^{-1}\left(\frac{x}{2}\right)$

44. $\int \frac{dx}{\sqrt{1-4x^2}}$

let $2x = \sin\theta$

$2 = \cos\theta \cdot \frac{d\theta}{dx}$

$= \int \frac{1}{2} \cdot \frac{\cos\theta\, d\theta}{\cos\theta}$

$= \frac{1}{2}\arcsin(2x) + C$

47. $\int \frac{dx}{x\sqrt{25x^2-2}}$

let $5x = \sqrt{2}\sec\theta$

$5 = \sqrt{2}\sec\theta \frac{d\theta}{dx} \tan\theta$

$= \int \frac{\sqrt{2}}{5} \cdot \frac{\sec\theta\tan\theta\, d\theta}{\sqrt{2} \cdot \frac{\sqrt{2}}{5}\sec\theta\tan\theta}$

$= \frac{\operatorname{arcsec}(\frac{5}{2}\sqrt{2}|x|) + C}{\sqrt{2}}$

52. $\int_{-2}^{2} \frac{dt}{4+3t^2}$

$= 2\int_{0}^{2} \frac{dt}{4+3t^2}$

let $\sqrt{3}t = 2\tan\theta$

$\sqrt{3} = 2\sec^2\theta \cdot \frac{d\theta}{dt}$

$= \frac{4}{\sqrt{3}} \int_{0}^{\pi/3} \frac{\sec^2\theta}{4\sec^2\theta}\, d\theta$

$= \frac{1}{\sqrt{3}}\theta \Big|_{\theta=0}^{\pi/3}$

$= \frac{\pi}{3\sqrt{3}}$

63. $\int_{0}^{\ln\sqrt{3}} \frac{e^x dx}{1+e^{2x}}$

let $e^x = \tan\theta$

$e^x = \sec^2\theta \cdot \frac{d\theta}{dx}$

$= \int_{\pi/4}^{\pi/3} \frac{\sec^2\theta\, d\theta}{\sec^2\theta}$

$= \theta \Big|_{\theta=\pi/4}^{\pi/3}$

$= \frac{\pi}{12}$

65. $\int \frac{y\, dy}{\sqrt{1-y^4}}$

let $y^2 = \sin\theta$

$2y = \cos\theta \cdot \frac{d\theta}{dy}$

$= \frac{1}{2}\int \frac{\cos\theta\, d\theta}{\cos\theta}$

$= \frac{1}{2}\theta + C$

$= \frac{1}{2}\arcsin(y^2) + C$

69. $\int_{-1}^{0} \frac{6\, dt}{\sqrt{4-(t+1)^2}}$

let $t+1 = 2\sin\theta$

$1 = 2\cos\theta \frac{d\theta}{dt}$

$= \int_{0}^{\pi/6} \frac{12\cos\theta\, d\theta}{2\cos\theta}$

$= 6\theta \Big|_{\theta=0}^{\pi/6}$

$= \pi$

74. $\int_{2}^{4} \frac{2dx}{(x-3)^2+1}$

let $x-3 = \tan\theta$

$1 = \sec^2\theta \cdot \frac{d\theta}{dx}$

$= \int_{-\pi/4}^{\pi/4} \frac{2\sec^2\theta\, d\theta}{\sec^2\theta}$

$= 2\theta \Big|_{\theta=-\pi/4}^{\pi/4}$

$= \pi$

80. $\int \frac{dx}{(x-2)\sqrt{(x-2)^2-1}}$

let $x-2 = \sec\theta$

$1 = \sec\theta\tan\theta \cdot \frac{d\theta}{dx}$

$= \int \frac{\sec\theta\tan\theta}{\sec\theta\tan\theta}\, d\theta$

$= \operatorname{arcsec}|x-2| + C$

82. $\int \frac{e^{\cos^{-1}(x)} dx}{\sqrt{1-x^2}}$

let $x = \cos\theta$

$1 = \sin\theta \frac{d\theta}{dx}$

$= \int \frac{-\sin\theta \cdot e^{\theta}\, d\theta}{\sin\theta}$

$= -e^{\theta} + C$

$= -e^{\arccos(x)} + C$

87. $\int_{\sqrt{2}}^{2} \frac{\sec^2(\sec^{-1}(x))\, dx}{x\sqrt{x^2-1}}$

let $x = \sec\theta$

$1 = \sec\theta\tan\theta \frac{d\theta}{dx}$

$= \int_{\pi/4}^{\pi/3} \frac{\sec^2\theta\tan\theta}{\tan\theta}\, d\theta$

$= \tan\theta \Big|_{\theta=\pi/4}^{\pi/3}$

$= \sqrt{3} - 1$

90. $\int \frac{e^x \sin^{-1}(e^x)}{\sqrt{1-e^{2x}}}\, dx$

let $e^x = \sin\theta$

$e^x = \cos\theta \cdot \frac{d\theta}{dx}$

$= \int \frac{\theta\cos\theta\, d\theta}{\cos\theta}$

$= \frac{1}{2}(\arcsin(e^x))^2 + C$

91. $\lim_{x\to 0} \frac{\sin^{-1}5x}{x}$

"$\frac{0}{0}$" indeterminate form (apply L'Hôpital)

$= \lim_{x\to 0} \frac{\frac{5}{\sqrt{1-(5x)^2}}}{1} = 5$

92. $\lim_{x\to 1^+} \frac{\sqrt{x^2-1}}{\sec^{-1}(x)}$

"$\frac{0}{0}$" indeterminate form (apply L'Hôpital)

$= \lim_{x\to 1^+} \frac{\frac{x}{\sqrt{x^2-1}}}{\frac{1}{|x|(x^2-1)}}$

$= 1$

96. $\lim\limits_{x\to\infty} \dfrac{e^x \arctan(e^x)}{e^{2x}+x}$

"$\frac{\infty}{\infty}$" indeterminate form (apply L'Hôpital)

$= \lim\limits_{x\to\infty} \dfrac{e^x\left[\arctan(e^x) + \frac{e^x}{e^{2x}+1}\right]}{2e^{2x}+1}$

$= \lim\limits_{x\to\infty} \dfrac{\frac{\arctan(e^x)}{e^x} + \frac{1}{e^{2x}+1}}{2+e^{-2x}}$

$= 0$

97. $\lim\limits_{x\to 0^+} \dfrac{(\tan^{-1}\sqrt{x})^2}{x\sqrt{x+1}}$

"$\frac{0}{0}$" indeterminate form (apply L'Hôpital)

$= \lim\limits_{x\to 0^+} 2\tan^{-1}(\sqrt{x}) \cdot \dfrac{\frac{1}{2\sqrt{x}}}{x+1}$

$= \lim\limits_{x\to 0^+} \dfrac{\tan^{-1}(\sqrt{x})}{\sqrt{x}}$

"$\frac{0}{0}$" indeterminate form (apply L'Hôpital)

$= \lim\limits_{x\to 0^+} \dfrac{\frac{1}{2\sqrt{x}(x+1)}}{\frac{1}{2\sqrt{x}}} = 1$

109. $\sin\theta = \frac{r}{3}$

$\cos\theta = \frac{h}{3}$

$V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi(3\sin\theta)^2(3\cos\theta)$

$= 9\pi \sin^2\theta \cos\theta$.

$\frac{dV}{d\theta} = 9\pi\left[2\sin\theta\cos^2\theta - \sin^3\theta\right] = 0$

$\sin\theta = 0$, $V = 0$ (minimum)

$\tan\theta = \sqrt{2}$, $V = 9\pi\,\frac{2}{3}\cdot\frac{1}{\sqrt{3}}$

$= 2\pi\sqrt{3}$ (maximum)

$\boxed{\therefore\ \theta = \arctan(\sqrt{2})}$

123a. $\int_{-1}^{1} \pi\left(\frac{1}{\sqrt{1+x^2}}\right)^2 dx$

$= \pi \arctan(x)\Big|_{x=-1}^{1}$

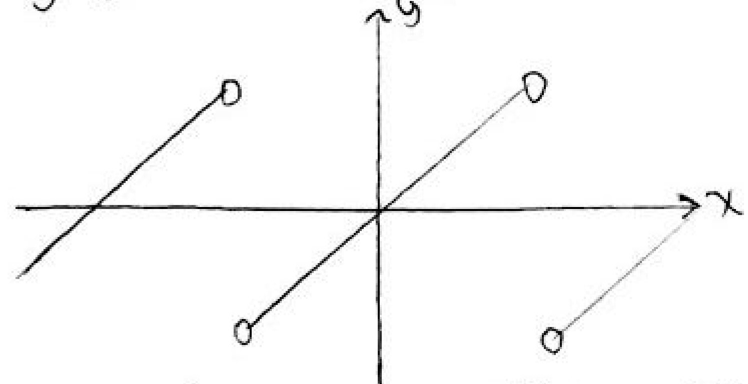
$= \pi\left(2\cdot\frac{\pi}{4}\right) = \frac{\pi^2}{2}$

b. $\int_{-1}^{1} \left(\frac{2}{\sqrt{1+x^2}}\right)^2 dx$

$= 4\arctan(x)\Big|_{x=-1}^{1}$

$= 4\cdot\left(2\cdot\frac{\pi}{4}\right) = 2\pi$

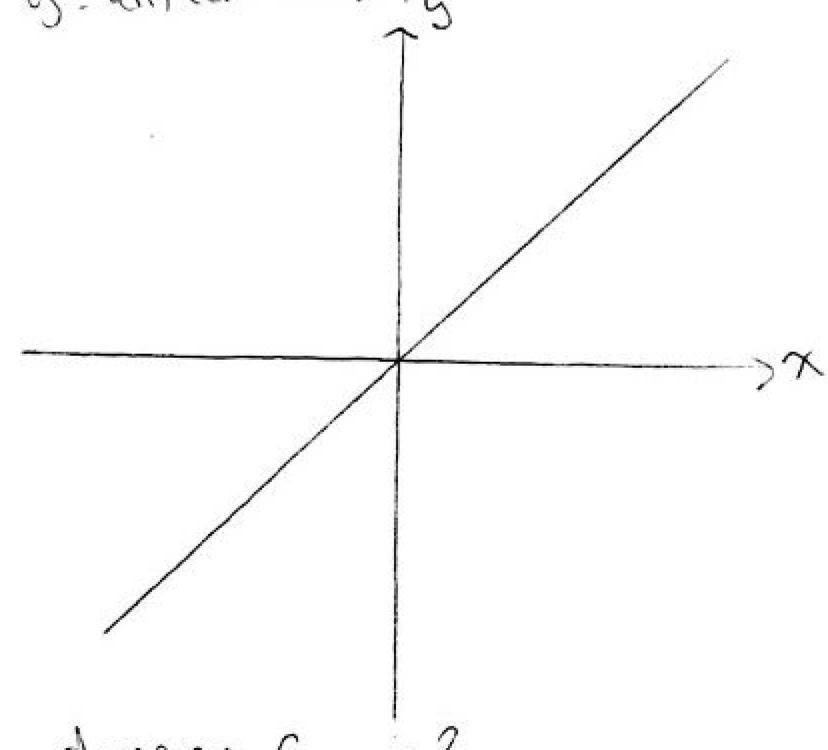
127a. $y = \arctan(\tan(x))$



domain: $\{x\in\mathbb{R} \cap (x \neq \frac{\pi}{2} + n\pi,\ \forall n \in \mathbb{N})\}$

range: $\{-\frac{\pi}{2} < y < \frac{\pi}{2}\}$

b. $y = \tan(\arctan(x))$



domain: $\{x \in \mathbb{R}\}$

range: $\{y \in \mathbb{R}\}$

Exercises 8.2

2. $\int \theta \cos \pi\theta \, d\theta$

let $u = \theta$ $\quad \frac{dv}{d\theta} = \cos\pi\theta$

$\frac{du}{d\theta} = 1$ $\quad v = \frac{1}{\pi}\sin\pi\theta$

$= \frac{\theta}{\pi}\sin\pi\theta - \frac{1}{\pi}\int \sin\pi\theta \, d\theta$

$= \frac{\theta \sin\pi\theta}{\pi} + \frac{\cos\pi\theta}{\pi^2} + C$

6. $\int_1^e x^3 \ln(x) \, dx$

let $u = \ln(x)$ $\quad \frac{dv}{dx} = x^3$

$\frac{du}{dx} = \frac{1}{x}$ $\quad v = \frac{1}{4}x^4$

$= \frac{1}{4}x^4 \ln(x) \Big|_{x=1}^{e} - \int_1^e \frac{1}{4}x^3 \, dx$

$= \frac{1}{4}x^4 \ln(x) - \frac{1}{16}x^4 \Big|_{x=1}^{e}$

$= \frac{3}{16}e^4 + \frac{1}{16}$

11. $\int \tan^{-1}(y) \, dy$

let $u = \tan^{-1}(y)$ $\quad \frac{dv}{dy} = 1$

$\frac{du}{dy} = \frac{1}{y^2+1}$ $\quad v = y$

$= y\tan^{-1}(y) - \int \frac{y}{y^2+1} \, dy$

$= y\tan^{-1}(y) - \frac{1}{2}\ln|y^2+1| + C$

12. $\int \sin^{-1}(y) \, dy$

let $u = \sin^{-1}(y)$ $\quad \frac{dv}{dy} = 1$

$\frac{du}{dy} = \frac{1}{\sqrt{1-y^2}}$ $\quad v = y$

$= y\sin^{-1}(y) - \int \frac{y}{\sqrt{1-y^2}} \, dy$

$= y\sin^{-1}(y) + \sqrt{1-y^2} + C$

17. $\int (x^2 - 5x)e^x \, dx$

let $u = x^2 - 5x$ $\quad \frac{dv}{dx} = e^x$

$\frac{du}{dx} = 2x - 5$ $\quad v = e^x$

$= (x^2 - 5x)e^x - \int 2xe^x \, dx + \int 5e^x \, dx$

$= (x^2 - 5x - 2x + 2 + 5)e^x + C$

$= (x^2 - 7x + 7)e^x + C$

22. $\int e^{-y}\cos y \, dy$

let $u = e^{-y}$ $\quad \frac{dv}{dy} = \cos y$

$\frac{du}{dy} = -e^{-y}$ $\quad v = \sin y$

$= e^{-y}\sin y + \int e^{-y}\sin y \, dy$

let $u = e^{-y}$ $\quad \frac{dv}{dy} = \sin y$

$\frac{du}{dy} = -e^{-y}$ $\quad v = -\cos y$

$= e^{-y}\sin y + e^{-y}(-\cos y) - \int e^{-y}\cos y \, dy$

$= \frac{e^{-y}(\sin y - \cos y)}{2} + C$

23. $\int e^{2x}\cos 3x \, dx$

let $u = e^{2x}$ $\quad \frac{dv}{dx} = \cos 3x$

$\frac{du}{dx} = 2e^{2x}$ $\quad v = \frac{1}{3}\sin 3x$

$= \frac{1}{3}e^{2x}\sin 3x - \int \frac{2}{3}e^{2x}\sin 3x \, dx$

let $u = e^{2x}$ $\quad \frac{dv}{dx} = \sin 3x$

$\frac{du}{dx} = 2e^{2x}$ $\quad v = -\frac{1}{3}\cos 3x$

$= \frac{1}{3}e^{2x}\sin 3x + \frac{2}{9}e^{2x}\cos 3x - \int \frac{4}{9}e^{2x}\cos 3x \, dx$

$= \frac{3}{13}e^{2x}\sin 3x + \frac{2}{13}e^{2x}\cos 3x + C$

28. $\int \ln(x^2 + x) \, dx$

let $u = \ln(x^2 + x)$ $\quad \frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{2x+1}{x^2+x}$ $\quad v = x$

$= x\ln(x^2+x) - \int 2 - \frac{x+\frac{1}{2}}{x^2+x} + \frac{\frac{1}{2}}{(x+\frac{1}{2})^2 - \frac{1}{4}} \, dx$

$= x\ln(x^2+x) - 2x + \frac{\ln(x^2+x)}{2} + \frac{\ln(|x+1|) - \ln(|x|)}{2} + C$

$= x\ln(|x^2+x|) + \ln(|x+1|) - 2x + C$

29. $\int \sin(\ln(x)) \, dx$

let $u = \sin(\ln(x))$ $\quad \frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{\cos(\ln(x))}{x}$ $\quad v = x$

$= x\sin(\ln(x)) - \int \cos(\ln(x)) \, dx$

let $u = \cos(\ln(x))$ $\quad \frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{-\sin(\ln(x))}{x}$ $\quad v = x$

$= x\sin(\ln(x)) - x\cos(\ln(x)) - \int \sin(\ln(x)) dx$

$= \frac{x}{2}(\sin(\ln(x)) - \cos(\ln(x))) + C$

31. $\int \cancel{x}\sec(x^2) \frac{d(x^2)}{2\cancel{x}}$

$= \frac{1}{2}\ln|\sec(x^2) + \tan(x^2)| + C$

32. $\int \frac{\cos\sqrt{x}}{\cancel{\sqrt{x}}} \cdot \frac{d(\sqrt{x})}{\frac{1}{2\cancel{\sqrt{x}}}}$

$= 2\sin\sqrt{x} + C$

37. $\int \cancel{x^3}e^{x^4} \frac{d(x^4)}{4\cancel{x^3}}$

$= \frac{1}{4}e^{x^4} + C$

43. $\int \sqrt{x}\ln(x) \, dx$

let $u = \ln(x)$ $\quad \frac{dv}{dx} = \sqrt{x}$

$\frac{du}{dx} = \frac{1}{x}$ $\quad v = \frac{2}{3}x\sqrt{x}$

$= \frac{2}{3}x\sqrt{x}\ln(x) - \int \frac{2}{3}\sqrt{x} \, dx$

$= \frac{2}{3}x\sqrt{x}\ln(x) - \frac{4}{9}x\sqrt{x} + C$

45. $\int \cos\sqrt{x}\,dx$

let $u=\sqrt{x}$

$\frac{du}{dx}=\frac{1}{2\sqrt{x}}$

$=\int 2u\cos u\,du$

$=2u\sin u+2\cos u+C$

$=2\sqrt{x}\sin(\sqrt{x})+2\cos\sqrt{x}+C$

50. $\int_0^{1/\sqrt{2}} 2x\sin^{-1}(x^2)\frac{d(x^2)}{2x}$

$=x^2\sin^{-1}(x^2)+\sqrt{1-x^4}\Big|_{x=0}^{1/\sqrt{2}}$

$=\frac{\pi}{12}+\frac{\sqrt{3}}{2}-1$ (see number 12)

51. $\int x\tan^{-1}x\,dx$

$=$ let $u=\tan^{-1}x$ $\quad \frac{dv}{dx}=x$

$\frac{du}{dx}=\frac{1}{x^2+1}$ $\quad v=\frac{1}{2}x^2$

$=\frac{1}{2}x^2\tan^{-1}(x)-\int\left(\frac{1}{2}-\frac{1}{2(x^2+1)}\right)dx$

$=\frac{1}{2}x^2\tan^{-1}(x)-\frac{1}{2}x+\frac{1}{2}\tan^{-1}(x)+C$

53 $\int x\sin x\,dx$

$=-x\cos x+\sin x+C$

a. $\left|\int_0^{\pi}x\sin x\,dx\right|=|\pi|=\pi$

b. $\left|\int_{\pi}^{2\pi}x\sin x\,dx\right|=|-3\pi|=3\pi$

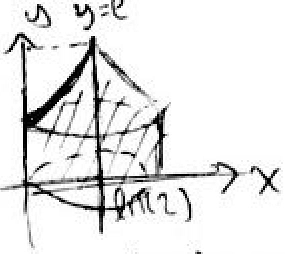
c. $\left|\int_{2\pi}^{3\pi}x\sin x\,dx\right|=|5\pi|=5\pi$

d. based on pattern

Area between curve from $n\pi\le x\le(n+1)\pi$

$=|(-1)^n(2n+1)\cdot\pi|=(2n+1)\pi$

(n is a non negative integer)

55. 

Volume $=2\pi\int_0^{\ln(2)}xe^{\ln(2)-x}\,dx$

$=4\pi\left[-(x+1)e^{-x}\right]_0^{\ln(2)}$

$=4\pi\left(\frac{1}{2}-\frac{\ln(2)}{2}\right)$

$=2\pi-2\pi\ln(2)$

61. $\int_0^{2\pi}2e^{-t}\cos t\,dt$

let $u=2e^{-t}$ $\quad \frac{dv}{dt}=\cos t$

$\frac{du}{dt}=-2e^{-t}$ $\quad v=\sin t$

$=2e^{-t}\sin t\Big|_{x=0}^{2\pi}+\int_0^{2\pi}2e^{-t}\sin t\,dt$

$\Rightarrow\int_0^{2\pi}2e^{-t}\sin t\,dt$

let $u=2e^{-t}$ $\quad \frac{dv}{dt}=\sin t$

$\frac{du}{dt}=-2e^{-t}$ $\quad v=-\cos t$

$=-2e^{-t}\cos t\Big|_{x=0}^{2\pi}-\int_0^{2\pi}2e^{-t}\cos t\,du$

$=\frac{2e^{-t}(\sin t-\cos t)}{2}\Big|_{x=0}^{2\pi}$

$=1-e^{-2\pi}$

Average $=\frac{1-e^{-2\pi}}{2\pi}$

63. $\int x^n\cos x\,dx$

let $u=x^n$ $\quad \frac{dv}{dx}=\cos x$

$\frac{du}{dx}=nx^{n-1}$ $\quad v=\sin x$

$=x^n\sin x-\int\sin x\cdot x^{n-1}\cdot n\,dx$

$=x^n\cdot\sin x-n\int x^{n-1}\cdot\sin x\,dx$

69. $\int_a^b\int_x^b f(t)\,dt\,dx$

let $u=\int_x^b f(t)\,dt$ $\quad \frac{dv}{dx}=1$

$\frac{du}{dx}=\frac{d\left(-\int_b^x f(t)dt\right)}{dx}$ $\quad v=x.$

$=-f(x)$

$=x\int_x^b f(t)\,dt\Big|_{x=a}^{b}+\int_a^b x f(x)\,dx$

$=\left(\underbrace{b\int_b^b f(t)dt}_{0}-a\int_a^b f(t)dt\right)+\int_a^b xf(x)\,dx$

$=\int_a^b(x-a)f(x)\,dx$

Exercises 8.3

3. $\int \cos^3 x \sin x \, \frac{d(\cos x)}{-\sin x}$

$= -\frac{1}{4}\cos^4 x + C$

14. $\int_0^{\pi/2} \sin^2 x \, dx$

$= \int_0^{\pi/2} \frac{1}{2} - \frac{\cos 2x}{2} \, dx$

$= \left[\frac{1}{2}x - \frac{\sin 2x}{4}\right]\Big|_0^{\pi/2}$

$= \frac{\pi}{4}$

15. $\int_0^{\pi/2} \sin^7 y \, \frac{d(\cos y)}{-\sin y}$

$= \int_0^{\pi/2} (1-\cos^2 y)^3 \, d(\cos y)$

$= \left[\cos y - \cos^3 y + \frac{3\cos^5 y}{5} - \frac{\cos^7 y}{7}\right]\Big|_0^{\pi/2}$

$= \frac{16}{35}$

19. $\int 16 \sin^2 x \cos^2 x \, dx$

$= \int 4 \sin^2 2x \, dx$

$= \int 2 - 2\cos 4x \, dx$

$= 2x - \frac{\sin 4x}{2} + C$

20. $\int_0^{\pi} 8 \sin^4 y \cos^2 y \, dy$

$= 8 \int_0^{\pi} \left(\frac{1-\cos 2y}{2}\right)^2 \left(\frac{1+\cos 2y}{2}\right)$

$= \int_0^{\pi} 1 - \cos 2y - \cos^2 2y + \cos^3 2y \, dy$

$= \int_0^{\pi} 1 - \cos 2y - \left(\frac{1+\cos 4y}{2}\right) dy + \int_0^{\pi} (1-\sin^2 2y) \frac{d(\sin 2y)}{2}$

$= \left[y - \frac{\sin 2y}{2} - \frac{y}{2} - \frac{\sin 4y}{8} + \frac{\sin 2y}{2} - \frac{\sin^3 2y}{6}\right]\Big|_0^{\pi}$

$= \frac{\pi}{2}$

23. $\int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}} \, dx$

$= \int_0^{2\pi} \sin \frac{1}{2}x \, dx$

$= -2\cos \frac{1}{2}x \Big|_{x=0}^{2\pi} = 4$

24. $\int_0^{\pi} \sqrt{2} \sin x \, dx$

$= -\sqrt{2}\cos x \Big|_{x=0}^{\pi} = 2\sqrt{2}$

27. $\int_{\pi/3}^{\pi/2} \frac{\sin x \sqrt{1+\cos x}\sqrt{1-\cos x}}{\sqrt{1-\cos x}} \, dx$

$= \int_{\pi/3}^{\pi/2} \sin x \sqrt{1+\cos x} \, \frac{d(1+\cos x)}{-\sin x}$

$= -\frac{2}{3}\left[(1+\cos x)^{3/2}\right]_{\pi/3}^{\pi/2}$

$= -\frac{2}{3} + \frac{1}{2}\sqrt{6}$

30. $\int_{\pi/2}^{3\pi/4} \sqrt{\sin^2 x + \cos^2 x - 2\sin x \cos x} \, dx$

$= \int_{\pi/2}^{3\pi/4} \sin x - \cos x \, dx$

$= -\cos x - \sin x \Big|_{x=\pi/2}^{3\pi/4}$

$= 1$

33. $\int \sec^2 x \tan x \, \frac{d(\tan x)}{\sec^2 x}$

$= \frac{1}{2}\tan^2 x + C$

34. $\int \sec x \tan^2 x \, dx$

Let $u = \tan x$, $\frac{dv}{dx} = \sec x \tan x$

$\frac{du}{dx} = \sec^2 x$, $v = \sec x$

$= \sec x \tan x - \int \sec^3 x \, dx$

$= \sec x \tan x - \int \sec x \tan^2 x \, dx - \int \sec x \, dx$

$= \frac{\sec x \tan x - \ln|\sec x + \tan x|}{2} + C$

35. $\int \sec^3 x \tan x \, \frac{d(\sec x)}{\sec x \tan x}$

$= \frac{1}{3}\sec^3 x + C$

44. $\int \sec^6 x \, dx$

$= \int (\tan^2 x + 1)^2 \, d(\tan x)$

$= \frac{1}{5}\tan^5 x + \frac{2}{3}\tan^3 x + \tan x + C$

53. $\int_{-\pi}^{\pi} \sin^2 3x$

$= \int_{-\pi}^{\pi} \frac{1}{2}\left[\cos(0) - \cos(6x)\right] dx$

$= \left[\frac{1}{2}x - \frac{1}{12}\sin 6x\right]_{-\pi}^{\pi}$

$= \pi$

54. $\int_0^{\pi/2} \sin x \cos x \, dx$

$= \frac{1}{2}\int_0^{\pi/2} \sin 2x \, dx$

$= -\frac{1}{4}\cos 2x \Big|_{x=0}^{\pi/2}$

$= \frac{1}{2}$

56. $\int_{-\pi/2}^{\pi/2} \cos x \cos 7x \, dx$

$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\pi/2} \cos(6x) + \cos(8x)\, dx$

$= \left[\frac{1}{12}\sin(6x) + \frac{1}{16}\sin(8x)\right]_{-\pi/2}^{\pi/2}$

$= 0$

58. $\int \cos^2 2\theta \cdot \sin\theta \cdot \frac{d(\cos\theta)}{-\sin\theta}$

$= -\int (2\cos^2\theta - 1)^2 \, d(\cos\theta)$

$= -\frac{4}{5}\cos^5\theta + \frac{4}{3}\cos^3\theta - \cos\theta + C$

59. $\int \cos^3\theta \cdot 2\sin\theta\cos\theta \; \frac{d(\cos\theta)}{-\sin\theta}$

$= -\frac{2}{5}\cos^5\theta + C$

61. $\int \sin\theta \cdot \cos\theta \cos 3\theta \, d\theta$

$= \frac{1}{2}\int \sin 2\theta \cdot \cos 3\theta \, d\theta$

$= \frac{1}{4}\int \sin(-\theta) + \sin(5\theta)\, d\theta.$

$= \frac{\cos\theta}{4} - \frac{\cos 5\theta}{20} + C$

64. $\int \tan^3 x \sec x \; \frac{d(\sec x)}{\sec x \tan x}$

$= \int \sec^2 x - 1 \quad d(\sec x)$

$= \frac{1}{3}\sec^3 x - \sec x + C$

67. $\int x \sin^2 x \, dx$

$= \int x\left(\frac{1-\cos 2x}{2}\right) dx$

$= \frac{1}{4}x^2 - \frac{1}{2}\int x\cos 2x \, dx$

let $u = x$ $\quad \frac{dv}{dx} = \cos 2x$

$\frac{du}{dx} = 1$ $\quad v = \frac{1}{2}\sin 2x$

$= \frac{1}{4}x^2 - \frac{1}{4}x\sin 2x + \frac{1}{4}\int \sin 2x \, dx$

$= \frac{1}{4}x^2 - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + C$

Exercises 8.4

2. $\int \frac{3}{\sqrt{1+9x^2}}\,dx$

let $3x = \tan\theta$

$3 = \sec^2\theta \cdot \frac{d\theta}{dx}$

$= \int \sec\theta \, d\theta$

$= \ln|\sec\theta + \tan\theta| + C$

$= \ln|\sqrt{9x^2+1} + 3x| + C$

3. $\int_{-2}^{2} \frac{dx}{4+x^2}$

let $x = 2\tan\theta$

$1 = 2\sec^2\theta \frac{d\theta}{dx}$

$= \frac{1}{2}\theta \Big|_{x=-2}^{2}$

$= \frac{1}{2}\theta \Big|_{\theta=\arctan(-1)}^{\arctan(1)}$

$= \pi/4$

5. $\int_0^{3/2} \frac{dx}{\sqrt{9-x^2}}$

let $x = 3\sin\theta$

$1 = 3\cos\theta \frac{d\theta}{dx}$

$= \theta \Big|_{x=0}^{3/2}$

$= \theta \Big|_{\theta=0}^{\arcsin(\frac{1}{2})}$

$= \pi/6$

8. $\int \sqrt{1-9t^2}\,dt$

let $3t = \sin\theta$

$3 = \cos\theta \frac{d\theta}{dt}$

$= \frac{1}{3}\int \cos^2\theta \, d\theta$

$= \frac{1}{3}\int \left(\frac{1+\cos 2\theta}{2}\right) d\theta$

$= \frac{1}{6}\theta + \frac{1}{12}\sin 2\theta + C$

$= \frac{\arcsin(3t)}{6} + \frac{t\sqrt{1-9t^2}}{2} + C$

9. $\int \frac{dx}{\sqrt{4x^2-49}}$

let $2x = 7\sec\theta$

$2 = 7\sec\theta \frac{d\theta}{dx} \cdot \tan\theta$

$\int \frac{1}{2}\sec\theta \, d\theta$

$= \frac{1}{2}\ln|\sec\theta + \tan\theta|$

$= \frac{1}{2}\ln\left|\frac{2x}{7} + \frac{\sqrt{4x^2-49}}{7}\right| + C$

$= \frac{1}{2}\ln|2x + \sqrt{4x^2-49}| + C$

10. $\int \frac{5\,dx}{\sqrt{25x^2-9}}$

let $5x = 3\sec\theta$

$5 = 3\sec\theta\tan\theta \cdot \frac{d\theta}{dx}$

$= \int \sec\theta \, d\theta$

$= \ln|\sec\theta + \tan\theta| + C$

$= \ln\left|\frac{5x}{3} + \frac{\sqrt{25x^2-9}}{3}\right| + C$

$= \ln|5x + \sqrt{25x^2-9}| + C$

12. $\int \frac{\sqrt{y^2-25}}{y^3}\,dy$

let $y = 5\sec\theta$

$1 = 5\sec\theta\tan\theta \frac{d\theta}{dy}$

$= \int \frac{\tan^2\theta}{5\sec^2\theta}\,d\theta$

$= \frac{1}{5}\int \left(\frac{1-\cos 2\theta}{2}\right) d\theta$

$= \frac{1}{10}\theta - \frac{\sin 2\theta}{20} + C$

$= \frac{1}{10}\operatorname{arcsec}\left(\frac{y}{5}\right) - \frac{\sqrt{y^2-25}}{2y^2} + C$

15. $\int \frac{x}{\sqrt{9-x^2}}\,dx$

let $x = 3\sin\theta$

$1 = 3\cos\theta \cdot \frac{d\theta}{dx}$

$= -3\cos\theta + C$

$= -\sqrt{9-x^2} + C$

17. $\int \frac{x^3\,dx}{\sqrt{x^2+4}}$

let $x = 2\tan\theta$

$1 = 2\sec^2\theta \cdot \frac{d\theta}{dx}$

$= \int 8\tan^3\theta\sec\theta \, \frac{d(\sec\theta)}{\sec\theta\tan\theta}$

$= \int 8\sec^2\theta - 8 \, d(\sec\theta)$

$= \frac{8}{3}\sec^3\theta - 8\sec\theta + C$

$= \frac{(x^2+4)\sqrt{x^2+4}}{3} - 4\sqrt{x^2+4} + C$

21. $\int \sqrt{\frac{x+1}{1-x}}\,dx$

$= \int \frac{x+1}{\sqrt{1-x^2}}\,dx$

let $x = \sin\theta$

$1 = \cos\theta \cdot \frac{d\theta}{dx}$

$= \int \sin\theta + 1 \, d\theta$

$= -\cos\theta + \theta + C$

$= -\sqrt{1-x^2} + \arcsin(x) + C$

26. $\int \frac{x^2}{(x^2-1)^2\sqrt{x^2-1}}\,dx$

let $x = \sec\theta$

$1 = \sec\theta\tan\theta \frac{d\theta}{dx}$

$= \int \frac{\sec^3\theta}{\tan^4\theta}\,d\theta$

$= \int \frac{\cos\theta}{\sin^4\theta} \cdot \frac{d(\sin\theta)}{\cos\theta}$

$= -\frac{1}{3\sin^3\theta} + C$

$= -\frac{x^3}{3(\sqrt{x^2-1})^3} + C$

36. $\displaystyle\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^t\,dt}{(1+e^{2t})^{3/2}}$

let $e^t = \tan\theta$.

$e^t = \sec^2\theta \cdot \frac{d\theta}{dt}$

$= \displaystyle\int_{\arctan(3/4)}^{\arctan(4/3)} \cos\theta\, d\theta$

$= \sin\theta \Big|_{\theta=\arctan(3/4)}^{\arctan(4/3)}$

$= \frac{1}{5}$

37. $\displaystyle\int_{1/12}^{1/4} \frac{2}{\sqrt{t}\,(1+4t)}\, dt$

let $2\sqrt{t} = \tan\theta$.

$\frac{1}{\sqrt{t}} = \sec^2\theta \cdot \frac{d\theta}{dt}$

$= \displaystyle\int_{\arctan\left(\frac{2}{\sqrt{12}}\right)}^{\arctan\left(\frac{2}{2}=1\right)} 2\, d\theta$

$= \frac{\pi}{6}$

44. $\displaystyle\int \frac{\sqrt{1-(\ln x)^2}}{x\ln x}\, dx$

let $\ln x = \sin\theta$.

$\frac{1}{x} = \cos\theta \cdot \frac{d\theta}{dx}$

$= \displaystyle\int \frac{\cos^2\theta}{\sin\theta}\, d\theta$

$= \displaystyle\int \frac{1-\sin^2\theta}{\sin\theta}\, d\theta$

$= \cos\theta - \ln|\csc\theta + \cot\theta| + C$

$= \sqrt{1-(\ln x)^2} - \ln\left|\frac{1+\sqrt{1-(\ln x)^2}}{\ln x}\right| + C$

46. $\displaystyle\int \sqrt{\frac{x}{1-x^3}}\, dx$.

let $x^{3/2} = \sin\theta$

$\frac{3}{2}x^{1/2} = \cos\theta \cdot \frac{d\theta}{dx}$

$= \frac{2}{3}\theta + C$

$= \frac{2}{3}\arcsin(x^{3/2}) + C$

47. $\displaystyle\int \sqrt{x}\sqrt{1-x}\, dx$

let $\sqrt{x} = \sin\theta$.

$\frac{1}{2\sqrt{x}} = \cos\theta \cdot \frac{d\theta}{dx}$

$= \displaystyle\int 2\sin^2\theta\cos^2\theta\, d\theta$

$= \frac{1}{2}\displaystyle\int \left(\frac{1-\cos 4\theta}{2}\right) d\theta$

$= \frac{1}{4}\theta - \frac{\sin 4\theta}{16} + C$

$= \frac{1}{4}\arcsin\sqrt{x} - \frac{1}{4}(\sqrt{x}\sqrt{1-x}(1-2x)) + C$

57. $\displaystyle\int x^3\sqrt{1-x^2}\, dx$

a. let $u = \sqrt{1-x^2}$ $\quad \frac{dv}{dx} = x^3$

$\frac{du}{dx} = \frac{-x}{\sqrt{1-x^2}}$ $\quad v = \frac{1}{4}x^4$

$= \frac{1}{4}x^4\sqrt{1-x^2} + \displaystyle\int \frac{x^5}{4\sqrt{1-x^2}}\, dx$

let $u = 1-x^2$

$\frac{du}{dx} = -2x$

$= \frac{1}{4}x^4\sqrt{1-x^2} - \frac{1}{8}\displaystyle\int \frac{(1-u)^2}{\sqrt{u}}\, du$

$= \frac{1}{4}x^4\sqrt{1-x^2} - \frac{1}{20}(1-x^2)^{5/2} + \frac{1}{6}(1-x^2)^{3/2} - \frac{1}{4}\sqrt{1-x^2}$

b. let $u = 1-x^2$, $\frac{du}{dx} = -2x$

$= \displaystyle\int -\frac{1}{2}(1-u)\sqrt{u}\, du$

$= -\frac{1}{3}(1-x^2)^{3/2} + \frac{1}{5}(1-x^2)^{5/2} + C$

c. let $x = \sin\theta$., $\frac{dx}{d\theta} = \cos\theta$.

$= \displaystyle\int \sin^3\theta\cos^2\theta \quad \frac{d(\cos\theta)}{-\sin\theta}$

$= -\displaystyle\int (1-\cos^2\theta)\cos^2\theta\, d(\cos\theta)$

$= -\frac{1}{3}\cos^3\theta + \frac{1}{5}\cos^5\theta + C = -\frac{1}{3}(1-x^2)^{3/2} + \frac{1}{5}(1-x^2)^{5/2} + C$

58. a. $f'(x) = \frac{y_{boat} - f(x)}{0-x}$ ...(1)

$(y_{boat} - f(x))^2 + (0-x)^2 = 10^2$ ...(2)

$\Rightarrow f'(x) = \frac{\pm\sqrt{100-x^2}}{-x} = \mp\frac{\sqrt{100-x^2}}{x}$

we take $f'(x) < 0$ (based on graph)

$\therefore f'(x) = -\frac{\sqrt{100-x^2}}{x}$

b. $f(x) = \displaystyle\int f'(x)\, dx$

let $x = 10\sin\theta$.

$1 = 10\cos\theta \cdot \frac{d\theta}{dx}$

$= -10\displaystyle\int \frac{\cos^2\theta}{\sin\theta}\, d\theta$

$= -10\displaystyle\int \frac{1-\sin^2\theta}{\sin\theta}\, d\theta$

$= 10\ln|\csc\theta + \cot\theta| - 10\cos\theta + C$

$= 10\ln\left|\frac{10}{x} + \frac{\sqrt{100-x^2}}{x}\right| - \sqrt{100-x^2} + C$

when $x = 10$, $f(x) = 0$

$0 = 10 \cdot \ln|1+0| - 0 + C$

$C = 0$

$\therefore f(x) = 10\ln\left|\frac{10}{x} + \frac{\sqrt{100-x^2}}{x}\right| - \sqrt{100-x^2}$