yohandi - Homework for week 2

Exercises 10.5

1. $\rho = \lim_{n\to\infty} \left|\frac{2}{n+1}\right| = 0 < 1$ (converge)

6. $\rho = \lim_{n\to\infty} \left| \frac{3\ln(n)}{\ln(n+1)} \right| = 3 > 1$ (diverge)

9. $\rho = \lim_{n\to\infty} \left|\frac{7^{1/n}}{2n+5}\right| = 0 < 1$ (converge)

13. $\rho = \lim_{n\to\infty} \left| \frac{-8^{1/n}}{(3+(1/n))^2} \right| = \frac{1}{9} < 1$ (converge)

18. $\rho = \lim_{n\to\infty} \left| (-1)\, n^{2/n} . e^{-1} \right| = \frac{1}{e} < 1$ (converge)

19. $\rho = \lim_{n\to\infty} \left| \frac{n+1}{e} \right| = \infty > 1$ (diverge)

22. $\lim_{n\to\infty} \left(\frac{n-2}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{1}{\left(\frac{n}{-2}\right)}\right)^{-\frac{n}{2}(-\frac{2}{n})(n)}$

$= e^{-2} > 0$ (diverge)

23. $\sum_{n=1}^{\infty} \frac{\frac{2}{1.25^n} + \frac{(-1)^n}{1.25^n}}{1} = \sum_{n=1}^{\infty} \frac{2}{1.25^n} + \sum_{n=1}^{\infty} \left(-\frac{1}{1.25}\right)^n$

$= \frac{\frac{8}{5}}{1-\frac{4}{5}} + \frac{-\frac{4}{5}}{1+\frac{4}{5}}$

$= \frac{68}{9}$ (converge)

33. $\rho = \lim_{n\to\infty} \left| \frac{(n+2)(n+3)}{(n+1)} \cdot \frac{1}{(n+1)(n+2)} \right| = 0 < 1$ (converge)

34. $\rho = \lim_{n\to\infty} \left| e^{-1}\, n^{3/n} \right| = \frac{1}{e} < 1$ (converge)

40. $\rho = \lim_{n\to\infty} \left| \frac{n^{1/n}}{\ln^2 n} \right| = \frac{1}{\infty} = 0 < 1$ (converge)

47. $\rho = \lim_{n\to\infty} \left| \frac{3n-1}{2n+5} \right| = \frac{3}{2} > 1$ (diverge)

53. $\lim_{n\to\infty} \left(\frac{1}{3}\right)^{n!} = 1 > 0$ (diverge)

58. $\lim_{n\to\infty} \left| (-1) \frac{n!}{n^n} \right| = 0 < 1$ (converge)

62. $\sum_{n=1}^{\infty} \frac{1.3.\ldots.(2n-1)}{2.4.\ldots.(2n)(3^n+1)} < \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{2}$ (converge)

63. by ratio test,

$\rho = \lim_{n\to\infty} \left| \left(\frac{n}{n+1}\right)^p \right| = 1$ (inconclusive)

by root test,

$\rho = \lim_{n\to\infty} \left| \left(\frac{1}{n}\right)^{p/n} \right| = 1^p = 1$ (inconclusive)

therefore, both ratio test and root test fail to provide information about convergence

65. $a_n = \begin{cases} \frac{n}{2^n} & \text{if } n \text{ is a prime number} \\ \frac{1}{2^n} & \text{otherwise} \end{cases}$

$\rho = \lim_{n\to\infty} |a_n^{1/n}| = \begin{cases} \frac{n^{1/n}}{2} = \frac{1}{2} \\ \frac{1}{2}. \end{cases}$

$= \frac{1}{2} < 1$ (converge)

Exercises 10.6

3. $u_n = \frac{1}{n3^n}$

1) since $n>0$ and $3^n>0$, u_n is positive for $n \in \mathbb{N}$

2) $\frac{u_{n+1}}{u_n} = \frac{n \cdot 3^n}{(n+1)3^{n+1}} = \frac{n}{3(n+1)} \le 1$

$u_{n+1} \le u_n$

3) $\lim_{n\to\infty} \frac{1}{n \cdot 3^n} = 0$

therefore, $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ converges

6. $u_n = \frac{n^2+5}{n^2+4}$

3) $\lim_{n\to\infty} \frac{n^2+5}{n^2+4} = 1 \neq 0$

since the third requirement is not fulfilled, $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ diverges

14. $u_n = \frac{3\sqrt{n+1}}{\sqrt{n}+1}$

3) $\lim_{n\to\infty} \frac{3\sqrt{n+1}}{\sqrt{n}+1} = \lim_{n\to\infty} \frac{3\sqrt{1+\frac{1}{n}}}{1+\frac{1}{\sqrt{n}}} = 3 \neq 0$

since the third requirement is not fulfilled, $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ diverges

17. $\sum_{n=1}^{\infty} \left| (-1)^n \frac{1}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} > \int_2^{\infty} \frac{1}{\sqrt{x}}\,dx = \infty$

take u_n as $\frac{1}{\sqrt{n}}$,

1). since $n>0$ and $\sqrt{n}>0$. u_n is positive for $n \in \mathbb{N}$

2). $u_n' = -\frac{1}{2n\sqrt{n}}$, $u_n' < 0$ for all n which implies that u_n is a non-increasing function

3). $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

therefore, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$ converges conditionally

19. $\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{n}{n^3+1} \right| = \sum_{n=1}^{\infty} \frac{n}{n^3+1} < \sum_{n=1}^{\infty} \frac{1}{n^2}$

$< \int_1^{\infty} \frac{1}{x^2}\,dx = 1$

(converge) absolutely

20. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n!}{2^n}$

as $\lim_{n\to\infty} \frac{n!}{2^n} = \infty$ (diverge)

22. $\sum_{n=1}^{\infty} \left| (-1)^n \frac{\sin n}{n^2} \right| = \sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right| \le \sum_{n=1}^{\infty} \frac{1}{n^2}$

$< \int_1^{\infty} \frac{1}{x^2}\,dx = 1$

(converge) absolutely

23. $\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{3+n}{5+n} \right| = \sum_{n=1}^{\infty} \frac{3+n}{5+n}$.

as $\lim_{n\to\infty} \frac{3+n}{5+n} = 1 \neq 0$, the sum diverges

26. $\sum_{n=1}^{\infty} \left| (-1)^{n+1} (\sqrt[n]{10}) \right| = \sum_{n=1}^{\infty} 10^{1/n}$

since $\lim_{n\to\infty} 10^{1/n} = 1 \neq 0$, the sum diverges

28. $\sum_{n=2}^{\infty} \left| (-1)^{n+1} \frac{1}{n \ln n} \right| = \sum_{n=2}^{\infty} \frac{1}{n \ln n}$

$> \int_3^{\infty} \frac{1}{x \ln x}\,dx = \infty$

take $u_n = \frac{1}{n \ln n}$

1.) since $n > 0 \Rightarrow \frac{1}{n \ln n} > 0$.

u_n is positive for $n \in \mathbb{N}$

2.) $u_n' = -\frac{\ln(n)+1}{n^2 \ln^2(n)}$

since $n^2 \ln^2(n) > 0$ and $\ln(n) > 0$,

$u_n' < 0 \Rightarrow u_n$ is a decreasing function

3.) $\lim_{n\to\infty} \frac{1}{n \ln n} = 0$

therefore $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n \ln n}$ converges conditionally

29. $\sum_{n=1}^{\infty} \left| (-1)^n \frac{\tan^{-1}(n)}{n^2+1} \right| = \sum_{n=1}^{\infty} \frac{\tan^{-1}(n)}{n^2+1} < \sum_{n=1}^{\infty} \frac{\pi}{2n^2}$

(converge absolutely)

33. $\sum_{n=1}^{\infty} \left| \frac{(-100)^n}{n!} \right| = \sum_{n=1}^{\infty} \frac{100^n}{n!}$

$\rho = \lim_{n\to\infty} \frac{100}{n+1} = 0 < 1$ (converge absolutely)

36. $\sum_{n=1}^{\infty} \left| \frac{\cos(n\pi)}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$

$-\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$, take $u_n = \frac{1}{n}$

1) as $n > 0$, $\frac{1}{n} > 0$. u_n is positive

2) $\frac{u_{n+1}}{u_n} = \frac{n}{n+1} < 1$

$u_{n+1} < u_n$ (decreasing)

3) $\lim_{n\to\infty} \frac{1}{n} = 0$

therefore, $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n}$ converge conditionally

43. $\lim_{n\to\infty} (-1)^n (\sqrt{n+\sqrt{n}} - \sqrt{n})$

$= \lim_{n\to\infty} (-1)^n \frac{\sqrt{n}}{\sqrt{n+\sqrt{n}}+\sqrt{n}}$

$= \lim_{n\to\infty} (-1)^n \frac{1}{2}$ (diverge)

49. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ converges (proved at number 36)

$E = \left| \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} - \sum_{n=1}^{4} \frac{(-1)^{n+1}}{n} \right|$

$= \left| \sum_{n=5}^{\infty} \frac{(-1)^{n+1}}{n} \right|$

$\leq \left| \frac{(-1)^6}{5} \right| = \frac{1}{5}$

50. $E = \left| \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{10^n} - \sum_{n=1}^{4} (-1)^{n+1} \frac{1}{10^n} \right|$

$= \left| \frac{\frac{1}{10}}{1+\frac{1}{10}} - 0{,}0909 \right|$

$= 9{,}\overline{09} \cdot 10^{-6}$

53. $E = \left| \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2+3} - \sum_{n=1}^{k} (-1)^n \frac{1}{n^2+3} \right|$

$\leq \left| \frac{(-1)^{k+1}}{(k+1)^2+3} \right| \approx 0{,}001$

$k > 30$

61. Since the series satisfies the conditions of Theorem 15, $u_{n+1} \leq u_n$

$\sum_{n=k+1}^{\infty} (-1)^{n+1} u_n = (-1)^{n+1} \left[(u_{k+1} - u_{k+2}) + (u_{k+3} - u_{k+4}) + \ldots \right]$

since $u_{k+2} \leq u_{k+1}$ the sign is determined by $(-1)^{n+1}$ (proved).

66. let $a_n = (-1)^{n+1} \frac{1}{n}$

$b_n = (-1)^{n+1} \frac{1}{\ln(n)}$

1) for $n > 1$, $\frac{1}{n} > 0$ and $\ln(n) > 0$

$\Rightarrow$ both a_n and b_n are positive

2) $\frac{a_{n+1}}{a_n} = \frac{n}{n+1} < 1$ $\quad$ $\frac{b_{n+1}}{b_n} = \frac{\ln(n)}{\ln(n+1)} < 1$

$a_{n+1} < a_n$ $\quad$ $b_{n+1} < b_n$

$\Rightarrow$ both a_n and b_n are decreasing functions

3). $\lim_{n\to\infty} \frac{1}{n} = 0$ $\quad$ $\lim_{n\to\infty} \frac{1}{\ln(n)} = 0$

therefore, both $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ and $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ converge

However $\sum_{n=1}^{\infty} \frac{1}{n \ln n} > \int_2^{\infty} \frac{1}{x \ln x} dx$

$= \infty$

(diverge)

Exercises 10.7

2. $\sum_{n=0}^{\infty}(x+5)^n$

note that this is a power series

(a)&(b).

converges when $|(x+5)|<1$

$\Rightarrow -6<x<-4$

(c).

Since the interval for converge and absolutely converge are the same, the series will never converge conditionally

4. $\sum_{n=1}^{\infty}\frac{(3x-2)^n}{n}$

$\rho=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=|3x-2|$

for $|3x-2|=1$,

$\sum_{n=1}^{\infty}\frac{1}{n}$ which diverges and

$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ which converges

for $|3x-2|<1$,

$\sum_{n=1}^{\infty}\frac{(3x-2)^n}{n}$ converges as $x\in(\frac{1}{3},1)$

(a). $x\in[\frac{1}{3},1)$

(b) $x\in(\frac{1}{3},1)$

(c). $x=\frac{1}{3}$

16. $\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{\sqrt{n}+3}$

$\rho=\lim_{n\to\infty}\left|\frac{(-1)\,x^{\frac{n+1}{n}}}{1}\right|=\lim_{n\to\infty}|x|=|x|$

for $|x|=1$,

$\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n}+3}$ which converges and

$\sum_{n=0}^{\infty}\frac{(-1)^{2n+1}}{\sqrt{n}+3}$ which diverges

for $|x|<1$,

$\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{\sqrt{n}+3}$ converges as $x\in(-1,1)$

(a). $x\in(-1,1]$

(b). $x\in(-1,1)$

(c). $x=1$

22. $\sum_{n=1}^{\infty}\frac{(-1)^n 3^{2n}(x-2)^n}{3n}$

$\rho=\lim_{n\to\infty}\left|\frac{(-1).3^2(x-2)}{1}\right|=\lim_{n\to\infty}|9(x-2)|$

for $|9(x-2)|=1$,

$\sum_{n=1}^{\infty}\frac{(-1)^n}{3n}$ which converges and

$\sum_{n=1}^{\infty}\frac{1}{3n}$ which diverges

for $|9(x-2)|<1$,

$\sum_{n=1}^{\infty}\frac{(-1)^n 3^{2n}(x-2)^n}{3n}$ converges as $x\in(\frac{17}{9},\frac{19}{9})$

(a) $x\in(\frac{17}{9},\frac{19}{9}]$

(b) $x\in(\frac{17}{9},\frac{19}{9})$

(c) $x=\frac{19}{9}$

26. $\sum_{n=0}^{\infty}n!\,(x-4)^n$

$\rho=\lim_{n\to\infty}|(n+1)(x-4)|$

for $\lim_{n\to\infty}|(n+1)(x-4)|=1$,

$\sum_{n=0}^{\infty}n!\lim_{x\to0}x^n$ which converges

for $\lim_{n\to\infty}|(n+1)(x-4)|<1$,

$\sum_{n=0}^{\infty}n!(x-4)^n$ converges as $x=4$

(a). $x=4$

(b). $x=4$

(c.) Since the interval for converge and absolutely converge are the same, the series will never converge conditionally

29. $\sum_{n=2}^{\infty}\frac{x^n}{n(\ln n)^2}$

$\rho=\lim_{n\to\infty}\left|\frac{x}{n^{1/n}\ln^{2/n}n}\right|=\lim_{n\to\infty}|x|=|x|$

for $|x|=1$,

$\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ which converges and

$\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$ which converges

for $|x|<1$,

$\sum_{n=2}^{\infty} \frac{x^n}{n\ln^2 n}$ converges as $x\in(-1,1)$

(a). $x\in[-1,1]$

(b) $x\in[-1,1]$

(c) the series never converge conditionally

35. $\sum_{n=1}^{\infty} \frac{1+2+\dots+n}{1^2+2^2+\dots+n^2} x^n = \sum_{n=1}^{\infty} \frac{\frac{n(n+1)}{2}}{\frac{n(n+1)(2n+1)}{6}} x^n$

$= \sum_{n=1}^{\infty} \frac{3x^n}{2n+1}$

$\rho = \lim_{n\to\infty} |x| = |x|$

for $|x|=1$,

$\sum_{n=1}^{\infty} \frac{3}{2n+1}$ which diverges and

$\sum_{n=1}^{\infty} \frac{3(-1)^n}{2n+1}$ which converges

for $|x|<1$,

$\sum_{n=1}^{\infty} \frac{3x^n}{2n+1}$ converges as $x\in(-1,1)$

(a) $x\in[-1,1)$

(b) $x\in(-1,1)$

(c) $x=-1$

39. $\sum_{n=1}^{\infty} \left(\frac{x}{2}\right)^n \frac{(n!)^2}{(2n)!}$

$\rho = \lim_{n\to\infty} \left|\left(\frac{x}{2}\right) \frac{(n+1)^2}{(2n+2)(2n+1)}\right| = \lim_{n\to\infty} \left|\frac{x}{8}\right| = \left|\frac{x}{8}\right|$

$\left|\frac{x}{8}\right|<1 \Rightarrow -8<x<8$

radius of convergence is 8

46. as the sum is a power series

$r = |\ln(x)|<1 \Rightarrow \frac{1}{e}<x<e$

$S_\infty = \frac{1}{1-\ln x}$

47. as the sum is a power series

$r = \left|\frac{x^2+1}{3}\right| < 1 \Rightarrow -\sqrt{2}<x<\sqrt{2}$

$S_\infty = \frac{1}{1-\left(\frac{x^2+1}{3}\right)} = \frac{3}{2-x^2}$

50a. $f(x) = \frac{5}{3-x}$

$= \frac{\left(\frac{5}{3}\right)}{1-\left(\frac{x}{3}\right)}$

$= \sum_{n=0}^{\infty} \frac{5}{3}\left(\frac{x}{3}\right)^n \quad (-3<x<3)$

b. $g(x) = \frac{3}{x-2}$

$= \frac{-3/2}{1-\frac{x}{2}}$

$= \sum_{n=0}^{\infty} \left(-\frac{3}{2}\right)\left(\frac{x}{2}\right)^n \quad (-2<x<2)$

51. $g(x) = \frac{3}{x-2} = \frac{3}{3-(-(x-5))} = \frac{1}{1-\left(-\frac{(x-5)}{3}\right)}$

$= \sum_{n=0}^{\infty} \left(-\frac{(x-5)}{3}\right)^n \quad (2<x<8)$

53. $f(x) = 1-\frac{1}{2}(x-3)+\frac{1}{4}(x-3)^2+\dots+\left(-\frac{1}{2}\right)^n(x-3)^n+\dots$

$= \sum_{n=0}^{\infty} \left(-\frac{(x-3)}{2}\right)^n$

$= \frac{1}{1+\frac{(x-3)}{2}} = \frac{2}{x-1}$,

converge when $\left|-\frac{(x-3)}{2}\right|<1 \quad 1<x<5$

$f'(x) = -\frac{1}{2}+\frac{1}{2}(x-3)+\dots+\left(-\frac{1}{2}\right)^n n(x-3)^{n-1}+\dots$

$= \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n n(x-3)^{n-1}$

$= \frac{-\frac{1}{2}}{1-\left(1-\frac{(x-1)^2}{4}\right)} = -\frac{2}{(x-1)^2}$,

converge when $1<x<5$

Exercises 10.8

3. $f(x) = \ln(x) \quad P_0(x) = 0$

$f'(x) = \frac{1}{x} \quad P_1(x) = (x-1)$

$f''(x) = -\frac{1}{x^2} \quad P_2(x) = (x-1) - \frac{1}{2}(x-1)^2$

$f'''(x) = \frac{2}{x^3} \quad P_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$

7. $f(x) = \sin(x) \quad P_0(x) = \frac{1}{2}\sqrt{2}$

$f'(x) = \cos(x) \quad P_1(x) = \frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}(x-\frac{\pi}{4})$

$f''(x) = -\sin(x) \quad P_2(x) = \frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}(x-\frac{\pi}{4}) - \frac{1}{4}\sqrt{2}(x-\frac{\pi}{4})^2$

$f'''(x) = -\cos(x) \quad P_3(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x-\frac{\pi}{4}) - \frac{\sqrt{2}}{4}(x-\frac{\pi}{4})^2 - \frac{\sqrt{2}}{12}(x-\frac{\pi}{4})^3$

23. $f(x) = x^3 - 2x + 4$

$f'(x) = 3x^2 - 2$

$f''(x) = 6x$

$f'''(x) = 6$

$f''''(x) = 0$

$\vdots$

$x^3 - 2x + 4 = 8 + 10(x-2) + 6(x-2)^2 + (x-2)^3$

13. $f(x) = \frac{1}{1+x}$

$f'(x) = -\frac{1}{(1+x)^2}$

$f''(x) = \frac{2}{(1+x)^3}$

$f'''(x) = -\frac{6}{(1+x)^4}$

$\vdots$

$f^{(k)}(x) = (-1)^k \frac{(k!)}{(1+x)^{k+1}}$

$\frac{1}{1+x} = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(k)}(0)}{k!}x^k + \dots$

$= 1 + (-x) + x^2 + \dots + (-1)^k x^k + \dots$

$= \sum_{n=0}^{\infty} (-x)^{n-1}$

20. $f(x) = \frac{e^x - e^{-x}}{2}$

$f'(x) = \frac{e^x + e^{-x}}{2}$

$f''(x) = \frac{e^x - e^{-x}}{2}$

$\vdots$

$f^{(k)}(x) = \frac{e^x + (-1)^{k+1} e^{-x}}{2}$

$\sinh(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(k)}(0)}{k!}x^k + \dots$

$= 0 + x + 0 + \dots + \left[\frac{f^{(k)}(0)}{k!}x^k = \begin{cases} \frac{x^k}{k!}, & k \text{ is odd} \\ 0, & \text{otherwise} \end{cases}\right] + \dots$

$= \sum_{n=1}^{\infty} \frac{x^{(2n-1)}}{(2n-1)!}$

31. $f(x) = \cos(2x + (\frac{\pi}{2}))$

$f'(x) = -2\sin(2x + (\frac{\pi}{2}))$

$f''(x) = -4\cos(2x + (\frac{\pi}{2}))$

$f'''(x) = 8\sin(2x + (\frac{\pi}{2}))$

$\vdots$

$f^{(k)}(x) = \begin{cases} (-4)^{k/2}(-1), & k \text{ is even} \\ 0, & \text{otherwise} \end{cases}$

$\cos(2x + (\frac{\pi}{2})) = f(\frac{\pi}{4}) + f'(\frac{\pi}{4})(x-\frac{\pi}{4}) + \frac{f''(\frac{\pi}{4})(x-\frac{\pi}{4})^2}{2!} + \dots + \frac{f^{(k)}(\frac{\pi}{4})(x-\frac{\pi}{4})^k}{k!} + \dots$

$= \sum_{n=1}^{\infty} \frac{(x-\frac{\pi}{4})^{(2n-2)}(-1)(-4)^{\frac{(2n-2)}{2}}}{(2n-2)!}$

33. $f(x) = \cos(x) + \frac{2}{x-1}$

$f'(x) = -\sin(x) - \frac{2}{(x-1)^2}$

$f''(x) = -\cos(x) + \frac{4}{(x-1)^3}$

$\vdots$

let $a(k,x) = \begin{cases} \cos(x), & k \text{ is even} \\ \sin(x), & k \text{ is odd} \end{cases}$

$b(k,x) = (-1)^{\lfloor \frac{k+1}{2} \rfloor}$

$c(k,x) = \frac{2k!}{(x-1)^{k+1}} \cdot (-1)^k$

$f^{(k)}(x) = a(k,x)\, b(k,x) + c(k,x)$

$\cos(x) + \frac{2}{x-1} = f(0) + f'(0)\, x + \frac{f''(0)}{2!} x^2 + \dots$

$= -1 - 2x - \frac{5}{2}x^2 + \dots$

$= (1 - \frac{1}{2}x^2 + \dots) - 2(1 + x + x^2 + \dots)$

converge when $r = |x| < 1 \Rightarrow x \in (-1,1)$

40. i) $E(a) = 0$

$f(a) - g(a) = 0$

$f(a) = g(a)$

$= b_0$

ii) $\lim_{x \to a} \frac{E(x)}{(x-a)^n}$ "$\frac{0}{0}$"

apply L'Hôpital

$\lim_{x \to a} \frac{f'(x) - g'(x)}{n(x-a)^{n-1}}$ "$\frac{0}{0}$"

apply L'Hôpital

$\vdots$

$\lim_{x \to a} \frac{f^{(n)}(x) - g^{(n)}(x)}{n!} = 0.$

this shows that $f^{(k)}(a) = g^{(k)}(a)$ for $k = 0, 1, 2, \dots, n$

note that $g^{(k)}(a) = b_k$

$\Rightarrow f^{(k)}(a) = b_k.$

$g(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)\,(x-a)^n}{n!}$

$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$

Exercises 10.9

7. $\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$

$\ln(1+x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2(n+1)}}{n+1}$

$= x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \dots$

10. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$\frac{1}{2-x} = \frac{(\frac{1}{2})}{1-(\frac{x}{2})} = \frac{1}{2}\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$

$= \frac{1}{2} + \frac{x}{4} + \frac{x^2}{8} + \dots$

14. $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

$\sin x - x + \frac{x^3}{3!} = (-x + \frac{x^3}{3!}) + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

$= \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots$

16. $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

$x^2\cos(x^2) = x^2 \cdot \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!}$

$= x^2 - \frac{x^6}{2!} + \frac{x^{10}}{4!} - \dots$

18. $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

$\sin^2(x) = \frac{1}{2} - \frac{\cos(2x)}{2}$

$= \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!}$

$= \frac{(2x)^2}{2\cdot 2!} - \frac{(2x)^4}{2\cdot 4!} + \frac{(2x)^6}{2\cdot 6!} - \dots$

22. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$\frac{2}{(1-x)^3} = \sum_{n=0}^{\infty} \frac{d^2 x^n}{dx^2} = \sum_{n=0}^{\infty} n(n-1)x^{n-2}$

$= 2 + 6x + 12x^2 + \dots$

25. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$e^x + \frac{1}{1+x} = e^x + \frac{1}{1-(-x)}$

$= \sum_{n=0}^{\infty} \left[\frac{x^n}{n!} + (-x)^n\right]$

$= 2 + \frac{3}{2}x^2 - \frac{5}{6}x^3 + \dots$

33. $f(x) = e^{\sin x}$

$f'(x) = \cos x \, e^{\sin x}$

$f''(x) = -\sin x e^{\sin x} + \cos^2 x e^{\sin x}$

$f'''(x) = e^{\sin x}\cos x(\cos^2 x - 3\sin x - 1)$

$e^{\sin x} = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$

$= 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \dots$

35. $f(x) = \sin x$

$\vdots$

$f^{(4)}(x) = \sin x$

$|R_3(0.1)| \le \frac{0.1^4}{4!} = 4{,}1\bar{6} \cdot 10^{-6}$

36. $f(x) = e^x$

$\vdots$

$f^{(5)}(x) = e^x$

$|R_4(0{,}5)| \le e^{\frac{1}{2}} \cdot \frac{0.5^5}{5!} \approx 4{,}29 \cdot 10^{-4}$

39. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

$$= x - \sum_{n=1}^{\infty} \frac{x^{4n-1}}{(4n-1)!} - \frac{x^{4n+1}}{(4n+1)!}$$

since x is an increasing function,

$$\frac{x^{4n-1}}{(4n-1)!} > \frac{x^{4n+1}}{(4n+1)!}$$

therefore, $\sin x < x$ for $x \in (0, 10^{-3})$

max error occurs when $x \to 10^{-3}$,

$$E_{max} = |x - \sin x| = \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \dots$$

$$< 1,\bar{6} \cdot 10^{-10}$$

45. note that:

for every $n \in \mathbb{N}$,

$$f(x) = f(a) + f'(a)(x-a) + \dots + f^{(n)}(c)(x-a)$$

where $c \in (\min(x,a), \max(x,a))$

for $n=1$,

$$f(x) = f(a) + f'(c)(x-a)$$

therefore, MVT is special case of Taylor's Theorem

47. $f(x) = f(a) + f'(a)(x-a) + \frac{f''(c_2)}{2}(x-a)^2$

as $f'(a) = 0$,

$$f(x) = \underbrace{f(a)}_{\text{constant}} + \frac{f''(c_2)}{2}(x-a)^2$$

a) suppose $c_1 \in (a-\delta, a+\delta)$,

$$f(c_1) = f(a) + \frac{f''(c_2)}{2}(c_1 - a)^2$$

assume that $f(c_1) \leq f(a)$:

$$f(c_1) - f(a) \leq 0$$

$$\frac{f''(c_2)}{2}(c_1-a)^2 \leq 0$$

since $(c_1-a)^2 \geq 0$,

$f''(c_2) \leq 0$ (which is true)

b) similar with proof in part (a)

in which this case assume $f(c_1) \geq f(a)$,

$$f(c_1) \geq f(a)$$

$$\vdots$$

$f''(c_2) \geq 0$ (which is true)

52 note that for even function f,

$$f(x) = f(-x)$$

$$f'(x) = -f'(-x)$$

$\Rightarrow f'$ is an odd function

note that for odd function f,

$$f(x) = -f(-x)$$

$$f'(x) = f'(-x)$$

$\Rightarrow f'$ is an even function

Suppose we have a function g which is also an odd function,

as $g(x) = -g(-x)$

$$g(0) = -g(0)$$

$$2g(0) = 0$$

$$g(0) = 0$$

as $f(x) = \sum_{n\geq 0}^{\infty} a_n x^n$,

$$f(0) = a_0$$

$$f'(0) = a_1$$

$$\vdots$$

$$f^{(k)}(0) = a_k$$

(a) even function f,

$$f'(0) = a_1 = 0$$

$$f'''(0) = a_3 = 0$$

$$\vdots$$

$f^{(k)}(0) = a_k = 0$ (k is odd)

(b) odd function f,

$$f(0) = a_0 = 0$$

$$f''(0) = a_2 = 0$$

$$\vdots$$

$f^{(k)}(0) = a_k = 0$ (k is even)