yohandi - assignment 5

1a. $X \sim U(0,10)$

$$f(x) = \begin{cases} \frac{1}{10-0} = \frac{1}{10} & , x \in [0,10] \\ 0 & , \text{otherwise} \end{cases}$$

b. $P(X \geq 8) = 1 - P(X \leq 8) = 1 - F(8) = 1 - \frac{8-0}{10-0} = \frac{1}{5}$

c. $P(2 \leq X < 8) = P(X \leq 8) - P(X \leq 2) = \frac{8-0}{10-0} - \frac{2-0}{10-0} = \frac{3}{5}$

d. $E(X) = \frac{0+10}{2} = 5$

e. $Var(X) = \frac{(10-0)^2}{12} = \frac{25}{3}$

2. $M(t) = \begin{cases} \frac{e^{5x} - e^{4x}}{t(5-4)} & , t \neq 0 \\ 1 & , t = 0 \end{cases}$ is a moment generating function of $X \sim U(4,5)$

a. $E(x) = \frac{4+5}{2} = \frac{9}{2}$

b. $Var(x) = \frac{(5-4)^2}{12} = \frac{1}{12}$

c. $P(4.2 < X \leq 4.7) = P(X \leq 4.7) - P(X \leq 4.2) = \frac{4.7-4}{5-4} - \frac{4.2-4}{5-4} = \frac{1}{2}$

3a. $P(a + (b-a)Y \leq w) = P(Y \leq \frac{w-a}{b-a})$

as $Y \sim U(0,1)$,

$$P(Y \leq \frac{w-a}{b-a}) = \frac{(\frac{w-a}{b-a}) - 0}{1-0} = \frac{w-a}{b-a}$$

cdf:

$$F_Y(\frac{w-a}{b-a}) = \begin{cases} 0 & , w < 0 \\ \frac{w-a}{b-a} & , 0 \leq w \leq 1 \\ 1 & , w > 1 \end{cases}$$

b. from the above cdf function, it can be concluded that:

$W \sim U(a,b)$

4. $E(n,x) = \int_0^{200} \frac{1}{200} \text{profit}(x) \, dx$

$$= \int_0^n \frac{1}{200}(x - \frac{1}{2}(n-x)) dx + \int_n^{200} \frac{1}{200}(n - 5(x-n)) dx$$

$$= \frac{1}{200} \left[(\frac{x^2}{2} - \frac{1}{2}xn + \frac{x^2}{4}) \Big|_{x=0}^{n} + (nx - \frac{5}{2}x^2 + 5nx) \Big|_{x=n}^{200} \right]$$

$$= \frac{1}{200} \left[\frac{n^2}{4} + 1200n - 100000 - 6n^2 + \frac{5}{2}n^2 \right]$$

$$= 6n - 500 - \frac{13}{800} n^2$$

$$\frac{d}{dn} E_n(x) = 6 - \frac{13}{400} n = 0$$

$$\Rightarrow n = 184.6$$

as n is integer and the expected value is a quadratic function,

$\Rightarrow$ it is symmetric to its peak value

therefore $\lceil 184.6 \rceil = 185$ watermelons are needed to maximize the profit

5(a) $f(x) = \frac{x^3}{4}$, $0 < x < c$

(i) $\int_0^c \frac{x^3}{4} dx = 1$

$$\frac{x^4}{16} \Big|_{x=0}^{c} = 1$$

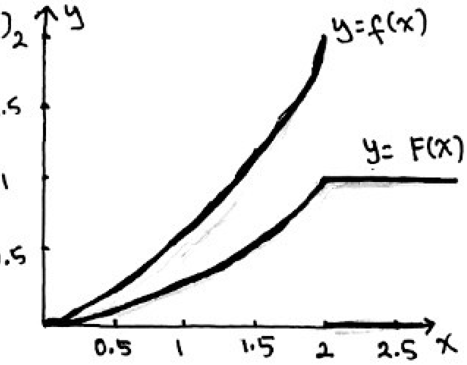
$$c^4 = 16$$

$c = -2$ (rejected) $\quad c = 2$

(ii) $P(X \leq x) = \int_0^x \frac{t^3}{4} dt = \frac{x^4}{16}$

$$F(x) = \begin{cases} 0 & , x < 0 \\ \frac{x^4}{16} & , 0 \leq x \leq 2 \\ 1 & , x > 2 \end{cases}$$

(iii)



(iv) $\mu = \int_{-\infty}^{\infty} x f(x)\, dx$

$= \int_0^2 \frac{x^4}{4}\, dx = \frac{8}{5}$

$\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2$

$= \int_0^2 \frac{x^5}{4}\, dx - \left(\frac{8}{5}\right)^2 = \frac{8}{75}$

(b) $f(x) = \frac{3}{16} x^2, \quad -c<x<c$

(i) $\int_{-c}^{c} \frac{3}{16} x^2\, dx = 1$

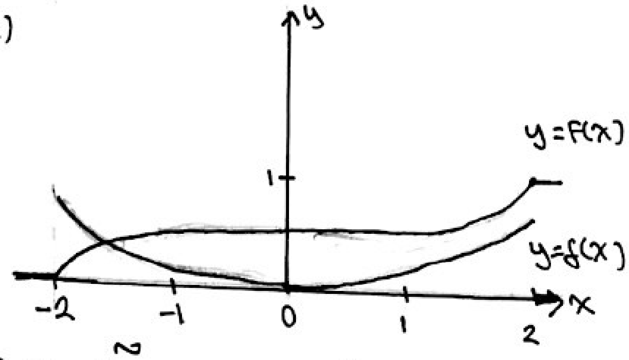
$\frac{x^3}{16}\Big|_{x=-c}^{c} = 1$

$c=2$

(ii) $P(X \le x) = \int_{-c}^{x} \frac{3}{16} t^2\, dt = \frac{x^3+8}{16}$

$F(x) = \begin{cases} 0, & x<-2 \\ \frac{x^3}{16} + \frac{1}{2}, & -2 \le x \le 2 \\ 1, & x>2 \end{cases}$

(iii)



(iv) $\mu = \int_{-\infty}^{\infty} x f(x)\, dx = \int_{-2}^{2} \frac{3x^3}{16}\, dx = 0$

$\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2 = \int_{-2}^{2} \frac{3x^4}{16}\, dx = \frac{12}{5}$

(c) $f(x) = \frac{c}{\sqrt{x}}, \quad 0<x<1$

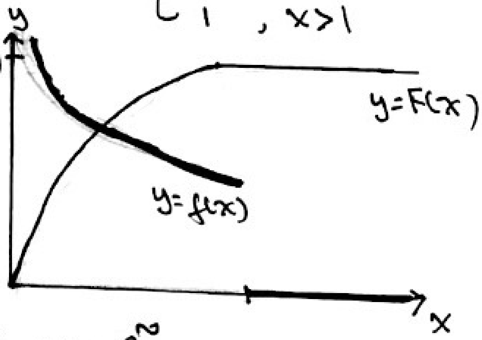
(i) $\int_0^1 \frac{c}{\sqrt{x}}\, dx = 2c\Big|_{x=0}^{1} = 1$

$\Rightarrow c = \frac{1}{2}$

(ii) $P(X \le x) = \int_0^x \frac{1}{2\sqrt{t}}\, dt = \sqrt{x}$

$F(x) = \begin{cases} 0, & x<0 \\ \sqrt{x}, & 0 \le x \le 1 \\ 1, & x>1 \end{cases}$

(iii)



(iv) $\mu = \int_{-\infty}^{\infty} x f(x)\, dx$

$= \int_0^1 \frac{\sqrt{x}}{2}\, dx = \frac{1}{3}$

$\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2$

$= \int_0^1 \frac{x\sqrt{x}}{2}\, dx - \left(\frac{1}{3}\right)^2 = \frac{4}{45}$

6. $P(X \le x) = \int_{-1}^{x} \frac{(t+1)}{2}\, dt = \frac{x^2}{4} + \frac{x}{2} + \frac{1}{4}$

a) $\pi_{0.64} \Rightarrow P(X \le x) = 0.64$

$\Rightarrow x = \frac{3}{5}$

b) $\pi_{0.25} \Rightarrow P(X \le x) = 0.25$

$\Rightarrow x_1 = 0 \qquad x = -2$ (rejected)

c) $\pi_{0.81} \Rightarrow P(X \le x) = 0.81$

$\Rightarrow x = \frac{4}{5}$

7. for every $i \in [1,k]$,

$$\int_{-\infty}^{\infty} f_i(x)\,dx = 1$$

a) assume that $\sum_{i=1}^{k} c_i f_i(x)$ is the pdf of a continuous-type RV on S,

$$\int_{-\infty}^{\infty} \sum_{i=1}^{k} c_i f_i(x)\,dx$$

$$= \sum_{i=1}^{k} c_i \underbrace{\int_{-\infty}^{\infty} f_i(x)\,dx}_{1}$$

$$= \sum_{i=1}^{k} c_i$$

$$= 1$$

since $\int_{-\infty}^{\infty} f_s(x)\,dx = 1$,

$\Rightarrow$ $f_s(x) = \sum_{i=1}^{k} c_i f_i(x)$ is true

b) $$E(X) = \int_{-\infty}^{\infty} x_i \sum_{i=1}^{k} c_i f_i(x_i)\,dx$$

$$= \sum_{i=1}^{k} c_i \underbrace{\int_{-\infty}^{\infty} x_i f_i(x_i)\,dx}_{\mu_i}$$

$$= \sum_{i=1}^{k} c_i \cdot \mu_i$$

c) for every $i \in [1,k]$,

$$Var(x_i) = \int_{-\infty}^{\infty} x_i^2 f_i(x_i)\,dx - E(x_i)^2$$

$$\Rightarrow \int_{-\infty}^{\infty} x_i^2 f_i(x_i)\,dx = Var(x_i) + E(x_i)^2 = \sigma_i^2 + \mu_i^2$$

$$Var(x) = \int_{-\infty}^{\infty} x_i^2 \sum_{i=1}^{k} c_i f_i(x_i)\,dx - E(x)^2$$

$$= \sum_{i=1}^{k} c_i \underbrace{\int_{-\infty}^{\infty} x_i^2 f_i(x_i)\,dx}_{\sigma_i^2+\mu_i^2} - E(x)^2$$

$$= \sum_{i=1}^{k} c_i(\sigma_i^2+\mu_i^2) - \left(\sum_{i=1}^{k} c_i \cdot \mu_i\right)^2$$

8. note that for every exponential dist.,

$$M(t) = \frac{1}{1-\theta t}, \quad t < \frac{1}{\theta}$$

a. $M(t) = \frac{1}{1-3t}$, $t < \frac{1}{3}$ $\Rightarrow \theta = 3$

$$f(x) = \frac{1}{\theta} \cdot e^{-x/\theta} = \frac{1}{3} \cdot e^{-x/3}$$

$\mu = \theta = 3$

$\sigma^2 = \theta^2 = 9$

b. $M(t) = \frac{3}{3-t} = \frac{1}{1-t/3}$, $t < 3 \Rightarrow \theta = \frac{1}{3}$

$$f(x) = \frac{1}{\theta} \cdot e^{-x/\theta} = 3e^{-3x}$$

$\mu = \theta = \frac{1}{3}$

$\sigma^2 = \theta^2 = \frac{1}{9}$

9. $X \sim e(\frac{1}{\theta})$,

$$\Rightarrow P(X > x+y \mid X > x) = \frac{P(X > x+y)}{P(X > x)}$$

$$= \frac{\int_{x+y}^{\infty} \frac{1}{\theta} \cdot e^{-\frac{t}{\theta}}\,dt}{\int_{x}^{\infty} \frac{1}{\theta} \cdot e^{-\frac{t}{\theta}}\,dt}$$

$$= \frac{-e^{-\frac{t}{\theta}}\Big|_{t=x+y}^{\infty}}{-e^{-\frac{t}{\theta}}\Big|_{t=x}^{\infty}}$$

$$= e^{-y}$$

$$= \int_{y}^{\infty} \frac{1}{\theta} \cdot e^{-\frac{t}{\theta}}\,dt$$

$$= P(X > y)$$

10. $$P(X > x+y \mid X > x) = \frac{P(X > x+y)}{P(X > x)} = \frac{1-F(x+y)}{1-F(x)}$$

$P(X > y) = 1 - F(y)$

$\Rightarrow 1 - F(x+y) = (1-F(x))(1-F(y))$

let $g(x) = 1 - F(x)$,

$g(x+y) = g(x)\,g(y)$

$\Rightarrow g(x) = a^{cx}$

$\Rightarrow F(x) = 1 - a^{cx}$

$$\lim_{x\to\infty} F(x) = 1 - \lim_{x\to\infty} a^{cx} = 1$$

$\Rightarrow c = -\lambda$, $\lambda > 0$

for $a = e$,

$F(x) = 1 - e^{-\lambda x}$, $\lambda > 0$ & $x > 0$

yohandi - assignment 5 (computer-based)

1. In theory, for every $i \in [1,n]$ such that $T_i \sim Exp(\lambda_i)$,

$$T = T_1 + T_2 + \ldots + T_n$$

$$f_T(x) = \left(\prod_{i=1}^{n} \lambda_i\right)\left(\sum_{i=1}^{n} \frac{e^{-\lambda_i x}}{\prod_{\substack{j=1,\\ \lambda_j \neq \lambda_i}}^{n} (\lambda_j - \lambda_i)}\right)$$

(proof will be provided on the last page)

in this case $n = 3$ and $\lambda_i = \lambda_j$, $i \neq j$

$$\Rightarrow f_T(x) = \frac{\lambda^3 . x^2 e^{-\lambda x}}{2!}$$

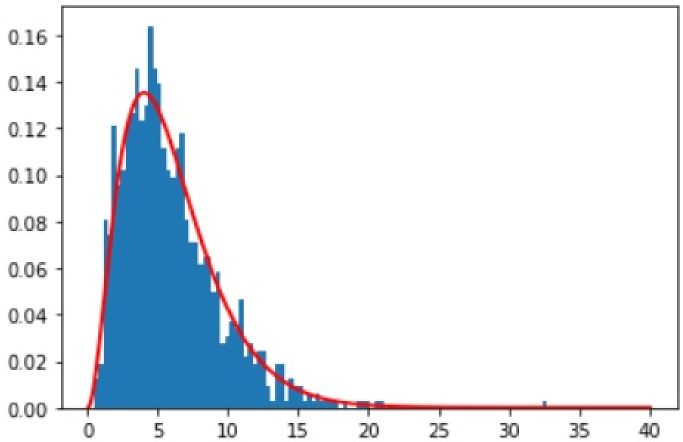
for $X \sim \Gamma(\kappa, \theta) \equiv Gamma(\kappa, \theta)$,

$$f(x; \kappa, \theta) = \frac{x^{\kappa-1} e^{-\frac{x}{\theta}}}{\theta^\kappa \Gamma(\kappa)}, x > 0 \text{ and } \kappa, \theta > 0$$

$$f(x; 3, \tfrac{1}{\lambda}) = \frac{\lambda^3 . x^2 . e^{-\lambda x}}{2}$$

therefore, $T \sim Gamma(3, 2)$.

2 & 3 & 4.



here, red line represents the theoretical pdf of T

5. The Erlang distribution has two parameters "κ" (denotes "shape") and "λ" (denotes "rate").

-> $\kappa = 1$ simplifies to the exponential distribution

-> sum of κ independent exponential with mean $1/\lambda$ each simplifies to the gamma distribution

```python
import numpy as np
import matplotlib.pyplot as plt
import scipy.special as sps

T1 = random.exponential(scale = 2, size = 1000)
T2 = random.exponential(scale = 2, size = 1000)
T3 = random.exponential(scale = 2, size = 1000)
T = T1 + T2 + T3
plt.hist(T, density = True, bins = 100)

count, bins = np.histogram(random.gamma(shape = 3, scale = 2, size = 100000), 100000, density = True)
y = bins ** (3 - 1) * (np.exp(-bins / 2) / (sps.gamma(3) * 2 ** 3))
plt.plot(bins, y, linewidth = 2, color = 'r')

plt.show()
```

proof :

-> for $n=2$,

$$f_{x_1+x_2}(x) = f_{x_1}(x) * f_{x_2}(x) = \int_0^x \lambda_1 \cdot e^{-\lambda_1(x-t)} \cdot \lambda_2 \cdot e^{-\lambda_2 \cdot t} dt = \lambda_1 \cdot \lambda_2 \cdot \frac{e^{-\lambda_2 x} - e^{-\lambda_1 x}}{\lambda_1 - \lambda_2}$$

(it is true)

-> assume that

$$f_{x_1+x_2+\dots x_n}(x) = f_{x_1+x_2+\dots+x_{n-1}}(x) * f_{x_n}(x) = \left(\prod_{i=1}^{n-1} \lambda_i\right)\left(\sum_{j=1}^{n-1} \frac{e^{-\lambda_j x}}{\prod_{k \neq j, k=1}^{n-1} (\lambda_k - \lambda_j)}\right) * f_{x_n}(x)$$

is true for $n \geq 3$ & "n" $= n-1$.

Since the coefficient of $e^{-\lambda_n x}$ fits the coefficients in our lemma,

$$-\sum_{j=1}^{n-1} \frac{1}{\prod_{k \neq j, k=1}^{n} (\lambda_k - \lambda_j)} = \frac{1}{\prod_{k=1}^{n-1} (\lambda_k - \lambda_n)},$$

equivalently,

$$\sum_{j=1}^{n} \frac{1}{\prod_{k \neq j, k=1}^{n} (\lambda_k - \lambda_j)} = 0$$

$$\Rightarrow \sum_{j=1}^{n} \prod_{\substack{k \neq l \neq j \\ k,l=1}}^{n} (\lambda_k - \lambda_l) = \sum_{j=1}^{n} \prod_{\substack{j \neq k \neq l \neq j \\ k,l=1}}^{n} (\lambda_k - \lambda_l) \prod_{\substack{k=j \neq l \\ k,l=1}}^{n} (\lambda_k - \lambda_l)$$

which equals to 0 if and only if :

$$\sum_{j=1}^{n} \prod_{\substack{j \neq k > l \neq j \\ k,l=1}}^{n} (\lambda_k - \lambda_l)(-1)^j = 0$$

therefore,

$$\begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^{n-2} \\ 1 & \lambda_2 & \lambda_2^2 & \dots & \lambda_2^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \dots & \lambda_n^{n-2} \end{vmatrix} = 0$$

It is true of $n \in \mathbb{N}$