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MAT2040 Homework 3

1. $$\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 1 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 & 0 \\ 0 & -7 & -12 & -4 & -4 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right]$$

(first matrix: left block A, right block I)

$$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 1 & 0 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{9}{2} & 7 & -\frac{3}{2} \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{array}\right]$$

(last matrix: left block I, right block A^{-1})

$A^{-1} = \begin{bmatrix} -\frac{9}{2} & 7 & -\frac{3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{bmatrix}$ as obtained by performing Gauss-Jordan elimination

2. $$\left[\begin{array}{cc|cc} 8 & 6 & 1 & 0 \\ 5 & 4 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 8 & 6 & 1 & 0 \\ 0 & \frac{1}{4} & -\frac{5}{8} & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 8 & 0 & 16 & -24 \\ 0 & \frac{1}{4} & -\frac{5}{8} & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & -\frac{5}{2} & 4 \end{array}\right]$$

(first matrix: left block A, right block I; last matrix: left block I, right block A^{-1})

$A^{-1} = \begin{bmatrix} 2 & -3 \\ -\frac{5}{2} & 4 \end{bmatrix}$ as obtained by performing Gauss-Jordan elimination

• $AX = B \Rightarrow X = A^{-1}B = \begin{bmatrix} 2 & -3 \\ -\frac{5}{2} & 4 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ -9 \end{bmatrix}$

3. $$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right]$$

(first matrix: left block $(A^T-I)^{-1}$, right block I)

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{array}\right]$$

(left block I, right block A^T-I)

$A^T - I = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$ as obtained by performing Gauss-Jordan elimination

• $A^T = I + \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 9 & 3 & 1 \\ 10 & 5 & 1 \\ \frac{7}{2} & \frac{3}{2} & \frac{3}{2} \end{bmatrix}$

$\Rightarrow A = \begin{bmatrix} 9 & 10 & \frac{7}{2} \\ 3 & 5 & \frac{3}{2} \\ 1 & 1 & \frac{3}{2} \end{bmatrix}$

4. $$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ -2 & 6 & -4 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 2 & -2 & 2 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 10 & -2 & 1 \end{array}\right]$$

(first matrix: left block A, right block I)

we can see that $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ is not a row-equivalent with I as it only has 2 column pivots, therefore, A is not invertible.

5. $$\left[\begin{array}{cccc|c} 1 & 3 & -5 & -2 & 0 \\ -3 & -2 & 1 & 1 & 0 \\ -11 & -5 & -1 & 2 & 0 \\ 5 & 1 & 3 & 0 & 0 \end{array}\right] \xrightarrow[R_4 = R_4 - 5R_1]{R_2 = R_2 + 3R_1,\ R_3 = R_3 + 11R_1} \left[\begin{array}{cccc|c} 1 & 3 & -5 & -2 & 0 \\ 0 & 7 & -14 & -5 & 0 \\ 0 & 28 & -56 & -20 & 0 \\ 0 & -14 & 28 & 10 & 0 \end{array}\right] \xrightarrow{R_3 = R_3 - 4R_2,\ R_4 = R_4 + 2R_2} \left[\begin{array}{cccc|c} 1 & 3 & -5 & -2 & 0 \\ 0 & 7 & -14 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

we see that x_3 & x_4 are free $\Rightarrow$ $x_2 = 2x_3 + \frac{5}{7}x_4$, $x_1 = -3(2x_3 + \frac{5}{7}x_4) + 5x_3 + 2x_4$ $\Rightarrow X = \left[-p - \frac{q}{7},\ 2p + \frac{5}{7}q,\ p,\ q\right]^T$, $p, q \in \mathbb{R}$

6. $\left[\begin{array}{cccc|c} 1 & 2 & -3 & -4 & -5 \\ 3 & -1 & 5 & 6 & -1 \\ -5 & -3 & 1 & 2 & 11 \\ -9 & -4 & -1 & 0 & 17 \end{array}\right] \xrightarrow[R_4=R_4+9R_1]{R_2=R_2-3R_1,\ R_3=R_3+5R_1} \left[\begin{array}{cccc|c} 1 & 2 & -3 & -4 & -5 \\ 0 & -7 & 14 & 18 & 14 \\ 0 & 7 & -14 & -18 & -14 \\ 0 & 14 & -28 & -36 & -28 \end{array}\right] \xrightarrow[R_4=R_4+2R_2]{R_3=R_3+R_2} \left[\begin{array}{cccc|c} 1 & 2 & -3 & -4 & -5 \\ 0 & -7 & 14 & 18 & 14 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

x_3 & x_4 are free $\Rightarrow$ $x_2 = -2+2x_3+\frac{18}{7}x_4$

$x_1 = -5-2(-2+2x_3+\frac{18}{7}x_4)+3x_3+4x_4 \Rightarrow x = \underbrace{\begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \end{bmatrix}}_{\text{particular}} + \underbrace{p\begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + q\begin{bmatrix} -\frac{8}{7} \\ \frac{18}{7} \\ 0 \\ 1 \end{bmatrix}}_{\text{homogenous}}$

7. $(I-A)(I+A+A^2+\dots+A^{N-1}) = (I+A+A^2+\dots+A^{N-1}) - (A+A^2+A^3+\dots+A^N)$

$= I - \underbrace{A^N}_{0}$

$= I$

by definition $(I-A)^{-1} = (I+A+A^2+\dots+A^{N-1})$

8. Let A is the symmetric non-singular matrix and B is its inverse

$I_n = AB \Rightarrow In = In^T = A^TB^T = AB^T \Rightarrow AB = AB^T \Rightarrow B = B^T \Rightarrow A^{-1} = (A^{-1})^T$ (true)

$= BA$

9. (1) $A = \begin{bmatrix} 2 & 1 \\ 6 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

we have $A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}^{-1}$

$= \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

(2) $A^{-1} = (A)^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

10. A is invertible implies there exists A^{-1}

$Ax=b \Rightarrow x = A^{-1}b$ (unique)

$\therefore$ A is nonsingular $\Rightarrow$ unique solution

$Ax=b$ has unique solution implies

$x = A^{-1}b$ is the only solution

(A is invertible)

$\therefore$ unique solution $\Rightarrow$ A is nonsingular

11. (a) $\left[\begin{array}{cc|c} 2 & 3 & 0 \\ 1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$ linearly independent

(b) $\left[\begin{array}{ccc|c} -2 & 1 & 2 & 0 \\ 1 & 3 & 4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -2 & 1 & 2 & 0 \\ 0 & \frac{7}{2} & 5 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -2 & 0 & \frac{4}{7} & 0 \\ 0 & \frac{7}{2} & 5 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -\frac{2}{7} & 0 \\ 0 & \frac{7}{2} & 5 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -\frac{2}{7} & 0 \\ 0 & 1 & \frac{10}{7} & 0 \end{array}\right]$ linearly dependent

(c) $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ linearly independent

(d) $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$ linearly independent

(e) $\left[\begin{array}{cccc|c} 2 & 3 & 2 & 4 & 0 \\ 1 & 2 & 2 & 5 & 0 \\ -2 & -2 & 0 & 6 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 2 & 3 & 2 & 4 & 0 \\ 0 & \frac{1}{2} & 1 & 3 & 0 \\ 0 & 1 & 2 & 10 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 2 & 3 & 2 & 4 & 0 \\ 0 & \frac{1}{2} & 1 & 3 & 0 \\ 0 & 0 & 0 & 4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & \frac{3}{2} & 1 & 2 & 0 \\ 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \rightarrow$

$\left[\begin{array}{cccc|c} 1 & \frac{3}{2} & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$ linearly dependent

12. $\left[\begin{array}{ccccc|c} 1 & -2 & 3 & -4 & -5 & 0 \\ 0 & 3 & -6 & 3 & 9 & 0 \\ -2 & -3 & 1 & 1 & -4 & 0 \\ -1 & 4 & -3 & 2 & -7 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccccc|c} 1 & -2 & 3 & -4 & -5 & 0 \\ 0 & 1 & -2 & 1 & 3 & 0 \\ 0 & -7 & 7 & -7 & -14 & 0 \\ 0 & 6 & -6 & 6 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccccc|c} 1 & -2 & 3 & -4 & -5 & 0 \\ 0 & 1 & -2 & 1 & 3 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & -10 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccccc|c} 1 & -2 & 3 & -4 & -5 & 0 \\ 0 & 1 & -2 & 1 & 3 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -7 & 0 \end{array}\right]$

since the 1st column, 2nd column, 3rd column, and 5th column are the pivots we have $\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -5 \\ 9 \\ -4 \\ -7 \end{bmatrix} \right\}$ as the maximum linearly independent subset

13. $\underbrace{\left[\begin{array}{ccc|c} -1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]}_{\text{system to matrix}} \rightarrow \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ linearly dependent as final row are all zeroes

14 (a) Depends on the value of the vector. We firstly have to check the linear independece of x_{k+1} toward x_i for every $i = 1, 2, \dots, k$. If x_k is linearly independent toward all x_i, then we will still have a linearly independent collection of vectors in the vector space V. Else, we will not.

(b) Yes, we will still have a linearly independent collection of vectors in the vector space V. This is because vector x_i is still linearly independent to x_j for every $1 \leq i < j < k$.

15(a) $\underbrace{\left[\begin{array}{cc|c} -1 & 3 & 2 \\ 2 & 4 & 6 \\ 3 & 2 & 6 \end{array}\right]}_{\text{system to matrix}} \rightarrow \left[\begin{array}{cc|c} -1 & 3 & 2 \\ 0 & 10 & 10 \\ 0 & 11 & 12 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -3 & -2 \\ 0 & 1 & 1 \\ 0 & 11 & 12 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -3 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right]$ inconsistent

since the system is inconsistent, there doesn't exist α and β such that $\begin{bmatrix} 2 \\ 6 \\ 6 \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + \beta \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$. this implies that $x \notin \text{Span}(x_1, x_2)$

(b) $\underbrace{\left[\begin{array}{cc|c} -1 & 3 & -9 \\ 2 & 4 & -2 \\ 3 & 2 & 5 \end{array}\right]}_{\text{system to matrix}} \rightarrow \left[\begin{array}{cc|c} -1 & 3 & -9 \\ 0 & 10 & -20 \\ 0 & 11 & -22 \end{array}\right] \rightarrow \begin{bmatrix} 1 & -3 & 9 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} 1 & -3 & 9 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$ consistent

since the system is consistent, there exists α and β such that $\begin{bmatrix} -9 \\ -2 \\ 5 \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + \beta \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$ that is $\alpha = 3$ and $\beta = -2$, this implies that $y \in \text{Span}(x_1, x_2)$

16. set $x=(x_1,x_2)$ and $y=(y_1,y_2)$

(1) $x+y=(x_1+y_2,\ x_2+y_1)$

$y+x=(y_1+x_2,\ y_2+x_1)$

since $x+y \neq y+x$, the addition is not commutative

(2) $x+(y+z)=x+(y_1+z_2,\ y_2+z_1)$

$=(x_1+y_2+z_1,\ x_2+y_1+z_2)$

$(x+y)+z=(x_1+y_2,\ x_2+y_1)+z$

$=(x_1+y_2+z_2,\ x_2+y_1+z_1)$

since $x+(y+z) \neq (x+y)+z$, the addition is not associative

(3) $(c_1+c_2)x=((c_1+c_2)x_1,\ (c_1+c_2)x_2)$

$c_1x+c_2x=(c_1x_1+c_2x_2,\ c_1x_2+c_2x_1)$

since $(c_1+c_2)x \neq c_1x+c_2x$, the addition is not constantly distributive

17. suppose we have vector u and v.

$u,v \in S$, $u=\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $v=\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$.

consider $u+v=\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{bmatrix}$, notice that

$(x_1+x_2)(y_1+y_2)=x_1y_1+x_1y_2+x_2y_1+x_2y_2$

$=z_1^2+x_1y_2+x_2y_1+z_2^2$

$\neq (z_1+z_2)^2$

this implies that S is not a subspace of $\mathbb{R}^3$

18. suppose we have vector x and y that $x \in U$ and $y \in V$.

consider $a+x$ and $b+y$ be such two vectors,

- $a+x+b+y=(a+b)+(x+y)$

since $(a+b) \in U$ and $(x+y) \in V$, $U+V$ is closed under addition

- $k(x+y)=kx+ky$

since $kx \in U$ and $ky \in V$, $U+V$ is closed under scalar multiplication

- $0+0=0$

since $0 \in U+V$,

0 is the identity element of W

this implies that $U+V$ is a subspace of W

19. system to matrix,

$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ -3 & 0 & 6 & -1 & 0 \\ 3 & 4 & -2 & 1 & 0 \\ 2 & 0 & -4 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 9 & 9 & 8 & 0 \\ 0 & -5 & -5 & -8 & 0 \\ 0 & -6 & -6 & -4 & 0 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & -4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

since the 1st, 2nd, & 4th columns are pivot we have $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \\ 1 \\ 2 \end{bmatrix} \right\}$ as the basis

20. suppose we choose $x_1, x_2, \& x_3$ as the free variables,

$x_1+x_2+x_3+x_4=0$

$x_4=-x_1-x_2-x_3$

$$V=x_1\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}+x_2\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}+x_3\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

$$=\alpha\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}+\beta\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}+\gamma\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix},$$

$\alpha, \beta, \gamma \in \mathbb{R}$

therefore, we have $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$

as the basis.

$\dim V=3$ (as we have 3 bases)