MAT3007 - Assignment 6

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November 17, 2023

Optimality Conditions for Unconstrained Problem I

$$
\begin{aligned}
f(x) &= x_1^3 - x_2^3 + 3x_1^2 + 3x_2^2 - 9x_1 \\
\Rightarrow \nabla f &= \begin{pmatrix} 3x_1^2 + 6x_1 - 9 \\ -3x_2^2 + 6x_2 \end{pmatrix} \\
\Rightarrow \nabla^2 f &= \begin{pmatrix} 6x_1 + 6 & 0 \\ 0 & -6x_2 + 6 \end{pmatrix}
\end{aligned}
$$

By First-Order Necessary Conditions (FONC), we must have $\nabla f(x^*) = 0 \Rightarrow 3x_1^2 + 6x_1 - 9 = 0, -3x_2^2 + 6x_2 = 0$. This implies that x^* is in $\{(x_1, x_2) | x_1 \in \{-3, 1\}, x_2 \in \{0, 2\}\}$. In other words $x^* \in \{(-3, 0), (-3, 2), (1, 0), (1, 2)\}$ by FONC.

By Second-Order Necessary Conditions (SONC), we must have the FONC conditions satisfied for x^*, and:

- For local minimizer, $\forall d \in R^n : d^T \nabla^2 f(x^*) d \geq 0$; equivalently, as $\nabla^2 f(x^*)$ is a symmetric matrix, all the eigenvalues in $\nabla^2 f(x^*)$ are non-negative. Eigenvalues for diagonal matrix is the diagonal entries, we have:

  - $6x_1 + 6 \geq 0 \Rightarrow x_1 \geq -1$
  - $-6x_2 + 6 \geq 0 \Rightarrow x_2 \leq 1$

  The only x^* that satisfies the above condition is $(1, 0)$.

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  - $-6x_2 + 6 \leq 0 \Rightarrow x_2 \geq 1$

  The only x^* that satisfies the above condition is $(-3, 2)$.

The above implies that $(-3, 0)$ and $(1, 2)$ are saddle points as they are neither local minimizer nor local maximizer.

By Second-Order Sucient Condition (SOSC), we must have the FONC conditions satisfied for x^*, and:

- For strict local minimizer, $\forall d \neq 0 : d^T \nabla^2 f(x^*) d > 0$ (positive definite); equivalently, as $\nabla^2 f(x^*)$ is a symmetric matrix, all the eigenvalues in $\nabla^2 f(x^*)$ are positive. Eigenvalues for diagonal matrix is the diagonal entries, we have:
  - $6x_1 + 6 > 0 \Rightarrow x_1 > -1$
  - $-6x_2 + 6 > 0 \Rightarrow x_2 < 1$
- For strict local maximizer, $\forall d \neq 0 : d^T \nabla^2 f(x^*) d < 0$ (negative definite); equivalently, as $\nabla^2 f(x^*)$ is a symmetric matrix, all the eigenvalues in $\nabla^2 f(x^*)$ are negative. Eigenvalues for diagonal matrix is the diagonal entries, we have:
  - $6x_1 + 6 < 0 \Rightarrow x_1 < -1$
  - $-6x_2 + 6 < 0 \Rightarrow x_2 > 1$

The above implies that $(1, 0)$ is the strict local minimizer and $(-3, 2)$ is the strict local maximizer.

Optimality Conditions for Unconstrained Problem II

(a)

$$\begin{aligned} f(x) &= x_1^3 - x_1(1 + x_2^2) + x_2^4 \\ \Rightarrow \nabla f &= \begin{pmatrix} 3x_1^2 - x_2^2 - 1 \\ 4x_2^3 - 2x_1 x_2 \end{pmatrix} \\ \Rightarrow \nabla^2 f &= \begin{pmatrix} 6x_1 & -2x_2 \\ -2x_2 & 12x_2^2 - 2x_1 \end{pmatrix} \end{aligned}$$

The $\nabla f(x) = 0$ implies the stationary points:

- $2x_2(2x_2^2 - x_1) = 0$, which implies $x_2 = 0$ or $x_1 = 2x_2^2$
- $3x_1^2 - x_2^2 - 1 = 0$
  - If $x_2 = 0$, then $x_1 = \pm\sqrt{\frac{1}{3}}$
  - If $x_1 = 2x_2^2$, then $x_2 = \pm\sqrt{\frac{1}{3}}$ and $x_1 = \frac{2}{3}$

The set of stationary points is $\{(-\frac{1}{\sqrt{3}}, 0), (\frac{1}{\sqrt{3}}, 0), (\frac{2}{3}, -\frac{1}{\sqrt{3}}), (\frac{2}{3}, \frac{1}{\sqrt{3}})\}$.

(b) We must have the FONC conditions satisfied for x^*, i.e., $x^* \in$ the set of stationary points. By Second-Order Necessary Conditions (SONC),

- $(-\frac{1}{\sqrt{3}}, 0) \Rightarrow \nabla^2 f = \begin{pmatrix} -2\sqrt{3} & 0 \\ 0 & \frac{2}{\sqrt{3}} \end{pmatrix}$. As both eigenvalues (derived by taking the diagonal entries as the matrix is diagonal) are neither ≥ 0 nor ≤ 0, we state $(-\frac{1}{\sqrt{3}}, 0)$ as a saddle point in $f(x)$.

- $(\frac{1}{\sqrt{3}}, 0) \Rightarrow \nabla^2 f = \begin{pmatrix} 2\sqrt{3} & 0 \\ 0 & -\frac{2}{\sqrt{3}} \end{pmatrix}$. As both eigenvalues (derived by taking the diagonal entries as the matrix is diagonal) are neither ≥ 0 nor ≤ 0, we state $(\frac{1}{\sqrt{3}}, 0)$ as a saddle point in $f(x)$.

- $(\frac{2}{3}, -\frac{1}{\sqrt{3}}) \Rightarrow \nabla^2 f = \begin{pmatrix} 4 & -\frac{2}{\sqrt{3}} \\ -\frac{2}{\sqrt{3}} & \frac{8}{3} \end{pmatrix}$. As $\lambda_1 + \lambda_2 = \frac{4}{3}$ and $\lambda_1 \lambda_2 = \frac{28}{3}$, $\nabla^2 f((\frac{2}{3}, -\frac{1}{\sqrt{3}}))$ is positive definite, $(\frac{2}{3}, -\frac{1}{\sqrt{3}})$ a strict local minimizer (categorized as local minimizer).

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KKT Conditions for Constrained Problem I

Lagrangian is derived as follows: $\mathcal{L}(x_1, x_2, \lambda) = (x_1 - 4)^2 + (x_2 - \frac{7}{2})^2 + \lambda_1(x_1^2 - x_2) + \lambda_2(x_1 + x_2 - 6)$

The KKT conditions:

1. Stationarity:
   - $\frac{\partial \mathcal{L}}{\partial x_1} = 2(x_1 - 4) + 2\lambda_1 x_1 + \lambda_2 = (2 + 2\lambda_1)x_1 + (-8 + \lambda_2) \geq 0$
   - $\frac{\partial \mathcal{L}}{\partial x_2} = 2(x_2 - \frac{7}{2}) - \lambda_1 + \lambda_2 = 2x_2 + (-\lambda_1 + \lambda_2 - 7) \geq 0$
2. Primal Feasibility:
   - $x_1^2 - x_2 \leq 0$
   - $x_1 + x_2 - 6 \leq 0$
   - $x_1 \geq 0$
   - $x_2 \geq 0$
3. Dual Feasibility:
   - $\lambda_1 \geq 0$
   - $\lambda_2 \geq 0$
4. Complementarity:
   - $x_1((2 + 2\lambda_1)x_1 + (-8 + \lambda_2)) = 0$
   - $x_2(2x_2 + (-\lambda_1 + \lambda_2 - 7)) = 0$

- $\lambda_1(x_1^2 - x_2) = 0$
- $\lambda_2(x_1 + x_2 - 6) = 0$

If $x_1 \neq 0$, then $(2+2\lambda_1)x_1 + (-8+\lambda_2) = 0$ by complementarity. Either $\lambda_1 = 0$ or $\lambda_1 \neq 0$.

- $\lambda_1 = 0$ implies $\lambda_2 = 8 - 2x_1$. If $\lambda_2 = 0$, then $x_1 = 4$ and $x_2 = \frac{7}{2}$ (a not feasible case). Otherwise, $x_1 + x_2 = 6$ and ($x_2 = 0$ or $x_2 = \frac{7-\lambda_2}{2}$). This implies $x_1 = 6$ (not a feasible case as $\lambda < 0$) or $x_1 = \frac{13}{4}$ (not a feasible case), respectively.
- $\lambda_1 \neq 0$ implies that $x_1^2 = x_2$. Then, $x_1^2 + x_1 - 6 = 0$ or $\lambda_2 = 0$.
  - $x_1^2 + x_1 - 6 = 0$, which implies ($x_1 = -3$ and $x_2 = 9$, a not primal feasible case) or ($x_1 = 2$ and $x_2 = 4$); further implying that $4\lambda_1 + \lambda_2 = 4$ and $-\lambda_1 + \lambda_2 = -1$ meaning that $\lambda_1 = 1$ and $\lambda_2 = 0$.
  - $\lambda_2 = 0$ (and $x_2 \neq 0$ due to $x_1 \neq 0$), which implies $2x_1^2 - \frac{4}{x_1} - 6 = 0$. This leads to $x_1 = -1$ (a not feasible case) or $x_1 = 2$ (mentioned previously).

If $x_1 = 0$, then $\lambda_1 = 0$ or $x_2 = 0$ by complementarity.

- If $\lambda_1 = 0$:
  - If $x_2 \neq 0$, then $x_2 = \frac{7-\lambda_2}{2}$ and ($\lambda_2 = 0$ or $x_2 = 6$). This implies that either ($x_2 = 6$ and $\lambda_2 = -5$, a not feasible case) or ($\lambda_2 = 0$ and $x_2 = \frac{7}{2}$)
  - If $x_2 = 0$ (below case with $\lambda_1 = 0$)
- If $x_2 = 0$, then $\lambda_2 = 0$ and λ_1 is not uniquely determined.

KKT pair $(x^*, \lambda^*) \in \{([2,4],[1,0]), ([0,\frac{7}{2}],[0,0])\} \cup \{([0,0],[\lambda_1,0]), \forall\lambda_1 \geq 0\}$

Failure of KKT Conditions for Constrained Problem II

The KKT conditions:

1. Stationarity:
   - $\frac{d(-x^2+x^3)}{dx} + \lambda\frac{dx^3(x+1)^3}{dx} = 0 \Rightarrow (-2x + 3x^2) + \lambda(3x^2(x+1)^2(2x+1)) = 0$
2. Primal Feasibility:
   - $x^3(x+1)^3 \leq 0$
3. Dual Feasibility:
   - $\lambda \geq 0$
4. Complementarity:
   - $\lambda x^3(x+1)^3 = 0$

By complementarity, we can deduce that either $x^3(x+1)^3 = 0$ or $\lambda = 0$.

- If $x^3(x+1)^3 = 0$, then either $x = 0$ or $x = -1$.
  - If $x = 0$, then λ is not uniquely determined by stationarity.
  - If $x = -1$, then the stationarity is not satisfied.
- If $\lambda = 0$, then either $x = 0$ (mentioned previously) or $x = \frac{2}{3}$.

By the above, the KKT points are $x = 0$ and $x = \frac{2}{3}$.
Consider $f(x^*)$ for the following $x^* \in \{-1, 0, \frac{2}{3}\}$:

- $x^* = -1 \Rightarrow f(x^*) = -2$
- $x^* = 0 \Rightarrow f(x^*) = 0$
- $x^* = \frac{2}{3} \Rightarrow f(x^*) = -\frac{4}{27}$

Indeed that $x = -1$ gives smaller objective value than $x = 0$ and $x = \frac{2}{3}$. The feasibility shows that x lies in $[-1, 0]$. We also notice that $\frac{df(x)}{dx} \geq 0$ in the domain; hence, $x = -1$ is a global minimizer. This demonstrates the failure of KKT conditions as $x = -1$, which leads into smaller objective value, is not considered.

KKT Conditions for Constrained Problem II

Lagrangian is derived as follows:
$\mathcal{L}(x_1, x_2, x_3, \lambda) = 2x_1 + x_2 + x_3 + \lambda(\frac{2}{x_1} + \frac{9}{x_2} + \frac{4}{x_3} - 1)$
The KKT conditions:

1. Stationarity:
   - $\frac{\partial \mathcal{L}}{\partial x_1} = 2 - \frac{2\lambda}{x_1^2} \geq 0$
   - $\frac{\partial \mathcal{L}}{\partial x_2} = 1 - \frac{9\lambda}{x_2^2} \geq 0$
   - $\frac{\partial \mathcal{L}}{\partial x_3} = 1 - \frac{4\lambda}{x_3^2} \geq 0$
2. Primal Feasibility:
   - $\frac{2}{x_1} + \frac{9}{x_2} + \frac{4}{x_3} \geq 1$
   - $x_1 \geq 0$
   - $x_2 \geq 0$
   - $x_3 \geq 0$
3. Dual Feasibility:
   - $\lambda \geq 0$

4. Complementarity:

   - $x_1(2 - \frac{2\lambda}{x_1^2}) = 0$
   - $x_2(1 - \frac{9\lambda}{x_2^2}) = 0$
   - $x_3(1 - \frac{4\lambda}{x_3^2}) = 0$
   - $\lambda(\frac{2}{x_1} + \frac{9}{x_2} + \frac{4}{x_3} - 1) = 0$

KKT points are found as followings.

Notice that $x_1, x_2, x_3 \neq 0 \Rightarrow x_1 = \sqrt{\lambda}, x_2 = 3\sqrt{\lambda}, x_3 = 2\sqrt{\lambda}$. By complementarity, $2\sqrt{\lambda} + 3\sqrt{\lambda} + 2\sqrt{\lambda} - \lambda = 0 \Rightarrow \lambda = 49$.

We confirm that there is only one KKT point: $(\sqrt{49}, 3\sqrt{49}, 2\sqrt{49}) = (7, 21, 14)$.