yohandi - homework week 10

Exercises 7.3

12. $e^{-0.3t} = 27$

$-0.3t = \ln(27)$

$t = -10\ln(3)$

b. $e^{kt} = \frac{1}{2}$

$kt = \ln(\frac{1}{2})$

$t = \frac{1}{k}\ln(\frac{1}{2})$

$= -\frac{1}{k}\ln(2)$

c. $e^{(\ln 0.2)t} = 0.4$

$0.2^t = 0.4$

$t = \frac{\ln 0.4}{\ln 0.2}$

8. $y = e^{(4\sqrt{x}+x^2)}$

$\ln(y) = 4\sqrt{x} + x^2$ $\quad \frac{d(\ldots)}{dx}$

$\frac{\frac{dy}{dx}}{y} = \frac{2}{\sqrt{x}} + 2x$

$\frac{dy}{dx} = e^{(4\sqrt{x}+x^2)}(\frac{2}{\sqrt{x}} + 2x)$

14. $y = \ln(3\theta . e^{-\theta}) = \ln(3) + \ln(\theta) + (-\theta)$

$\frac{dy}{d\theta} = \frac{1}{\theta} - 1$

16. $y = \theta^3 e^{-2\theta}\cos 5\theta$

$\ln(y) = 3\ln(\theta) + (-2\theta) + \ln(\cos 5\theta)$

$\frac{\frac{dy}{d\theta}}{y} = \frac{3}{\theta} - 2 - \frac{5\sin 5\theta}{\cos 5\theta}$

$\frac{dy}{d\theta} = (\theta^3 . e^{-2\theta} . \cos 5\theta)\left[\frac{3}{\theta} - 2 - 5\tan 5\theta\right]$

24. $y = \int_{e^{4\sqrt{x}}}^{e^{2x}} \ln t \, dt$

$\frac{dy}{dx} = \ln(e^{2x}) . 2e^{2x} - \ln(e^{4\sqrt{x}}) . \frac{2}{\sqrt{x}} . e^{4\sqrt{x}}$

$= 4xe^{2x} - 8e^{4\sqrt{x}}$

26. $\ln(xy) = e^{x+y}$

$\frac{(xy)'}{xy} = (1+y')e^{x+y}$

$\frac{y + x . \frac{dy}{dx}}{xy} = e^{x+y} + \frac{dy}{dx} . e^{x+y}$

$\frac{dy}{dx} = \frac{xy . e^{x+y} - y}{x - xy . e^{x+y}}$

27. $e^{2x} = \sin(x+3y)$

$2e^{2x} = \cos(x+3y)\left[1 + 3\frac{dy}{dx}\right]$

$\frac{dy}{dx} = \frac{1}{3}\left[\frac{2e^{2x} - \cos(x+3y)}{\cos(x+3y)}\right]$

$= \frac{2e^{2x}}{3\cos(x+3y)} - \frac{1}{3}$

29. $\int(e^{3x} + 5e^{-x})\,dx$

$= \int e^{3x}\frac{d(3x)}{3} + \int 5e^{-x}\frac{d(-x)}{-1}$

$= \frac{1}{3}e^{3x} - 5e^{-x} + C$

38. $\int \frac{e^{-\sqrt{r}}}{\sqrt{r}} \frac{d(\sqrt{r})}{-\frac{1}{2\sqrt{r}}}$

$= -2e^{-\sqrt{r}} + C$

42. $\int \frac{e^{-1/x^2}}{x^3} \frac{d(-\frac{1}{x^2})}{\frac{2}{x^3}}$

$= \frac{1}{2}e^{-1/x^2} + C$

48. $\int_0^{\sqrt{\ln\pi}} 2xe^{x^2}\cos(e^{x^2}) \frac{d(e^{x^2})}{2xe^{x^2}}$

$= \left[\sin(e^{x^2})\right]_0^{\sqrt{\ln\pi}}$

$= -\sin(1)$

49. $\int \frac{e^r}{1+e^r} \frac{d(1+e^r)}{e^r}$

$= \ln(1+e^r) + C$

50. $\int \frac{dx}{1+e^x}$

$= \int \frac{1+e^x}{1+e^x}dx - \int \frac{e^x}{1+e^x} \frac{d(1+e^x)}{e^x}$

$= x - \ln(1+e^x) + C$

56. $y = 3^{-x}$

$\ln(y) = -x\ln(3)$

$\frac{\frac{dy}{dx}}{y} = -\ln(3)$

$\frac{dy}{dx} = -\ln(3) . 3^{-x}$

59. $y = x^{\pi}$

$\ln(y) = \pi \ln(x)$

$\dfrac{\frac{dy}{dx}}{y} = \dfrac{\pi}{x}$

$\dfrac{dy}{dx} = \pi x^{\pi - 1}$

62. $y = (\ln \theta)^{\pi}$

$\ln(y) = \pi \ln(\ln \theta)$

$\dfrac{\frac{dy}{d\theta}}{y} = \pi \cdot \dfrac{\frac{1}{\theta}}{\ln\theta}$

$\dfrac{dy}{d\theta} = \dfrac{\pi}{\theta} (\ln\theta)^{\pi - 1}$

65. $y = 2^{\sin 3t}$

$\ln(y) = \sin(3t) \cdot \ln(2)$

$\dfrac{\frac{dy}{dt}}{y} = 3\cos(3t) \cdot \ln(2)$

$\dfrac{dy}{dt} = 3\ln(2) \cdot \cos(3t) \cdot 2^{\sin(3t)}$

70. $y = \dfrac{\ln e^{x}}{\ln 25} - \dfrac{\ln \sqrt{x}}{\ln 5}$

$\dfrac{dy}{dx} = \dfrac{1}{\ln 25} - \dfrac{\frac{1}{2x}}{\ln 5}$

$= \dfrac{1}{\ln(25)} - \dfrac{1}{2\ln(5) x}$

74. $y = \dfrac{\ln(5)}{2} \cdot \dfrac{\ln\left(\frac{7x}{3x+2}\right)}{\ln(5)}$

$\dfrac{dy}{dx} = \dfrac{1}{2} \cdot \dfrac{3x+2}{7x} \cdot \dfrac{7(3x+2) - 7x(3)}{(3x+2)^2}$

$= \dfrac{1}{x(3x+2)}$

75. $y = \theta \sin\left(\frac{\ln(\theta)}{\ln(7)}\right)$

$\dfrac{dy}{d\theta} = \sin\left(\dfrac{\ln(\theta)}{\ln(7)}\right) + \theta \cdot \cos\left(\dfrac{\ln(\theta)}{\ln(7)}\right) \cdot \dfrac{1}{\theta \ln(7)}$

$= \sin(\log_7 \theta) + \dfrac{\cos(\log_7 \theta)}{\ln 7}$

82. $y = t \cdot \dfrac{\ln(e^{(\sin t)(\ln 3)})}{\ln(3)}$

$\dfrac{dy}{dt} = \dfrac{1}{\ln(3)} \left[\ln(3) \cdot \sin(t) + \ln(3) \cdot t \cos(t)\right]$

$= \sin(t) + t \cdot \cos(t)$

85. $\displaystyle\int_0^1 2^{-\theta} \cdot \frac{d(2^{-\theta})}{-\ln(2) \cdot 2^{-\theta}}$

$= \left[\dfrac{2^{-\theta}}{-\ln(2)}\right]_0^1$

$= \dfrac{1}{2\ln(2)}$

88. $\displaystyle\int_1^4 \frac{2^{\sqrt{x}}}{\sqrt{x}} \cdot \frac{d(2^{\sqrt{x}})}{2^{\sqrt{x}} \frac{1}{2\sqrt{x}} \cdot \ln(2)}$

$= \left[\dfrac{2 \cdot 2^{\sqrt{x}}}{\ln(2)}\right]_1^4$

$= \dfrac{4}{\ln(2)}$

91. $\displaystyle\int_2^4 x^{2x}(1 + \ln x) \frac{d(x^{2x})}{x^{2x} \cdot 2(\ln(x) + 1)}$

$= \left[\frac{1}{2} x^{2x}\right]_2^4$

$= 2^{15} - 2^{3} = 32760$

96. $\displaystyle\int_1^e x^{(\ln 2) - 1} dx$

$= \dfrac{1}{\ln(2)} \left[x^{\ln(2)}\right]_1^e$

$= \dfrac{1}{\ln(2)}$

99. $\displaystyle\int_1^4 \frac{\ln(2) \cdot \ln(x)}{x \cdot \ln(2)} \cdot \frac{d(\ln(x))}{\frac{1}{x}}$

$= \left[\frac{1}{2} \ln^2(x)\right]_1^4$

$= 2\ln^2(2)$

111. $y=(x+1)^x$

$\ln(y)= x.\ln(x+1)$

$$\frac{\frac{dy}{dx}}{y}=\ln(x+1)+\frac{x}{x+1}$$

$$\frac{dy}{dx}=(x+1)^x.\ln(x+1)+x.(x+1)^{x-1}$$

114. $y=t^{\sqrt{t}}$

$\ln(y)=\sqrt{t}\,\ln(t)$

$$\frac{\frac{dy}{dt}}{y}=\frac{\ln(t)}{2\sqrt{t}}+\frac{\sqrt{t}}{t}$$

$$\frac{dy}{dt}=t^{\sqrt{t}}\frac{(\ln(t)+2)}{2\sqrt{t}}$$

116. $y=x^{\sin x}$

$\ln(y)=\sin(x).\ln(x)$

$$\frac{\frac{dy}{dx}}{y}=\cos(x).\ln(x)+\frac{\sin(x)}{x}$$

$$\frac{dy}{dx}=x^{\sin x}\left[\cos(x).\ln(x)+\frac{\sin(x)}{x}\right]$$

118. $y=(\ln(x))^{\ln(x)}$

$\ln(y)=\ln(x).\ln(\ln(x))$

$$\frac{\frac{dy}{dx}}{y}=\frac{\ln(\ln(x))}{x}+\ln(x)\cdot\frac{\frac{1}{x}}{\ln(x)}$$

$$\frac{dy}{dx}=\frac{\ln(x)^{\ln(x)}}{x}\left[\ln(\ln(x))+1\right]$$

121. $f(x)=xe^{-x}$

a. $f'(x)=e^{-x}-x\cdot e^{-x}$

$f'(x)=0$

$e^{-x}-x\,e^{-x}=0$

$(1-x).e^{-x}=0$

$\Rightarrow x=1$

extreme value with $x=1$ is $f(1)=\frac{1}{e}$

b. $f''(x)=-e^{-x}-e^{-x}+xe^{-x}$

$f''(x)=0$

$(-2+x)e^{-x}=0$

$\Rightarrow x=2$

inflection point at $(2,f(2))=(2,\frac{2}{e^2})$

123. $f(x)=x^2\ln(\frac{1}{x})$

$f'(x)=2x\ln(\frac{1}{x})+x^2(-\frac{1}{x})$

$\Rightarrow f'(x)=0$

$2x\ln(\frac{1}{x})-x=0$

$x(2\ln(\frac{1}{x})-1)=0$

$x:\{0,e^{-1/2}\}$

not in domain

$f''(x)=2\ln(\frac{1}{x})-1-1$

Since $f''(e^{-1/2})<0$ and $f'(e^{-1/2})=0$,

$(e^{-1/2},\frac{1}{2e})$ is an absolute maximum point

131. $\int_0^{\pi/4}\sqrt{1+(\frac{dy}{dx})^2}\,dx$

note that: $y=\ln(\cos x)$

$\frac{dy}{dx}=\frac{-\sin x}{\cos x}$

$=-\tan x$

$=\int_0^{\pi/4}\sqrt{1+\tan^2 x}\,dx$

$=\left[\ln|\sec x+\tan x|\right]_0^{\pi/4}$ #

$=\ln(\sqrt{2}+1)$

135a. at x near 0,

$a=0$

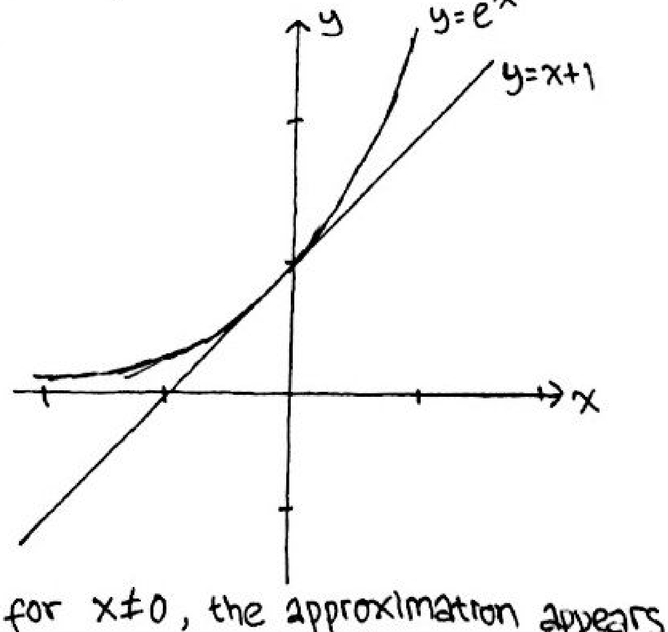
$f(x)=f(a)+f'(a)(x-a)$

$\Rightarrow e^x\approx e^a+e^a(x-a)$

$\Rightarrow e^x\approx 1+x$

b. error $=|e^{0,2}-(1,2)|=0,02140$

c.



for $x\neq 0$, the approximation appears to underestimate e^x.
(since e^x is a concave up function)

1362. $y = e^x$

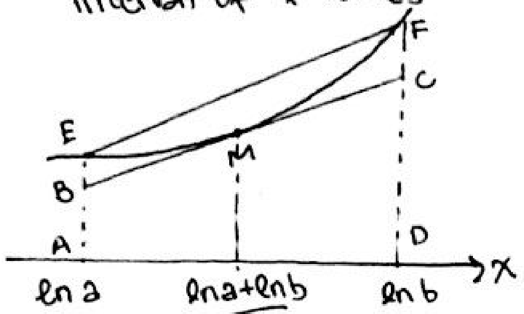
$$\frac{dy}{dx} = e^x$$

$$\frac{d^2y}{dx^2} = e^x$$

for every value of x, $\frac{d^2y}{dx^2} > 0$

$\Rightarrow e^x$ is concave up over every interval of x-values

b.



Area ABCD $= |AD| \cdot e^{\frac{\ln a + \ln b}{2}}$

$= e^{\frac{\ln a + \ln b}{2}} \cdot (\ln(b) - \ln(a))$

Area AEMFD $= \int_{\ln a}^{\ln b} e^x \, dx$

Area AEFD $= |AD| \cdot \frac{1}{2}(e^{\ln a} + e^{\ln b})$

$= \frac{e^{\ln a} + e^{\ln b}}{2} \cdot (\ln(b) - \ln(a))$

Since Area ABCD < Area AEMFD < AEFD

$$\Rightarrow e^{\frac{\ln a + \ln b}{2}}(\ln(b) - \ln(a)) < \int_{\ln(a)}^{\ln(b)} e^x \, dx < \frac{e^{\ln a} + e^{\ln b}}{2}(\ln(b) - \ln(a))$$

c. $$e^{\frac{\ln a + \ln b}{2}}(\ln(b) - \ln(a)) < \int_{\ln(a)}^{\ln(b)} e^x \, dx < \frac{e^{\ln(a)} + e^{\ln(b)}}{2}(\ln(b) - \ln(a))$$

since $\ln(b) - \ln(a) > 0$

$$e^{\frac{\ln a + \ln b}{2}} < \frac{\left[e^x\right]_{\ln(a)}^{\ln(b)}}{\ln(b) - \ln(a)} < \frac{e^{\ln(a)} + e^{\ln(b)}}{2}$$

$$\sqrt{e^{\ln a} \cdot e^{\ln b}} < \frac{b-a}{\ln(b) - \ln(a)} < \frac{a+b}{2}$$

$$\therefore \sqrt{ab} < \frac{b-a}{\ln(b) - \ln(a)} < \frac{a+b}{2}$$

Exercises 7.5

5. $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{2\sin^2 \frac{1}{2}x}{x^2} = \frac{1}{2}$

16. $\lim_{x\to 0} \frac{\sin x - x}{x^3}$ "$\frac{0}{0}$" indeterminate form

$= \lim_{x\to 0} \frac{\cos x - 1}{3x^2}$ "$\frac{0}{0}$" indeterminate form

$= \lim_{x\to 0} \frac{-\sin x}{6x} = -\frac{1}{6}$

20. $\lim_{x\to 1} \frac{x-1}{\ln(x) - \sin(\pi x)}$ "$\frac{0}{0}$" indeterminate form

$= \lim_{x\to 1} \frac{1}{\frac{1}{x} - \pi\cos(\pi x)} = \frac{1}{\pi+1}$

21. $\lim_{x\to 0} \frac{x^2}{\ln(\sec x)}$ "$\frac{0}{0}$" indeterminate form

$= \lim_{x\to 0} \frac{2x}{\frac{\sec x \tan x}{\sec x}} = 2$

26. $\lim_{x\to (\frac{\pi}{2})^-} \left[\left(\frac{\pi}{2} - x\right)\tan x = \frac{(\frac{\pi}{2} - x)}{\cot x}\right]$

"$\frac{0}{0}$" indeterminate form

$= \lim_{x\to (\frac{\pi}{2})^-} \frac{-1}{-\csc^2 x} = 1$

28. $\lim_{\theta\to 0} \frac{(1/2)^\theta - 1}{\theta}$ "$\frac{0}{0}$" indeterminate form

$= \lim_{\theta\to 0} \frac{\ln(\frac{1}{2})\cdot(\frac{1}{2})^\theta}{1} = -\ln(2)$

30. $\lim_{x\to 0} \frac{3^x - 1}{2^x - 1}$ "$\frac{0}{0}$" indeterminate form

$= \lim_{x\to 0} \frac{\ln(3)\cdot 3^x}{\ln(2)\cdot 2^x} = \frac{\ln(3)}{\ln(2)} = \log_2(3)$

31. $\lim_{x\to \infty} \frac{\ln(x+1)}{\log_2 x}$ "$\frac{\infty}{\infty}$" indeterminate form

$= \lim_{x\to \infty} \frac{\frac{1}{x+1}}{\frac{1}{\ln(2)}\cdot\frac{1}{x}} = \ln(2)$

38. $\lim_{x\to 0^+} \ln(x) - \ln(\sin(x))$

$= \lim_{x\to 0^+} \ln\left[\frac{x}{\sin(x)}\right)$

since ln is a continuous function

$= \ln\left(\lim_{x\to 0^+} \frac{x}{\sin(x)}\right)$

$= \ln(1) = 0$

40. $\lim_{x\to 0^+} \left(\frac{3x+1}{x} - \frac{1}{\sin x}\right)$

$= \lim_{x\to 0^+} 3 + \frac{\sin x - x}{x\sin x}$

$= 3 + \lim_{x\to 0^+} \frac{\sin x - x}{x\sin x}$ "$\frac{0}{0}$" indeterminate form

$= 3 + \lim_{x\to 0^+} \frac{\cos x - 1}{\sin x + x\cos x}$ "$\frac{0}{0}$" indeterminate form

$= 3 + \lim_{x\to 0^+} \frac{-\sin x}{2\cos x - x\sin x} = 3$

41. $\lim_{x\to 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln(x)}\right)$

$= \lim_{x\to 1^+} \frac{\ln(x) - x + 1}{x\ln(x) - \ln(x)}$ "$\frac{0}{0}$" indeterminate form

$= \lim_{x\to 1^+} \frac{\frac{1}{x} - 1}{\ln(x) + 1 - \frac{1}{x}}$ "$\frac{0}{0}$" indeterminate form

$= \lim_{x\to 1^+} \frac{-\frac{1}{x^2}}{\frac{1}{x} + \frac{1}{x^2}} = -\frac{1}{2}$

45. $\lim_{t\to \infty} \frac{e^t + t^2}{e^t - t}$ "$\frac{\infty}{\infty}$" indeterminate form

$= \lim_{t\to \infty} \frac{e^t + 2t}{e^t - 1}$ "$\frac{\infty}{\infty}$" indeterminate form

$= \lim_{t\to \infty} \frac{e^t + 2}{e^t} = \lim_{t\to \infty} 1 + \frac{2}{e^t} = 1$

49. $\lim_{\theta\to 0} \frac{\theta - \sin\theta\cos\theta}{\tan\theta - \theta}$ "$\frac{\infty}{\infty}$" indeterminate form

$= \lim_{\theta\to 0} \frac{1 - \cos 2\theta}{\sec^2\theta - 1} = \lim_{\theta\to 0} \frac{2\sin^2\theta}{\left(\frac{1-\cos^2\theta}{\cos^2\theta}\right)} = 2$

51. $\lim_{x\to 1^+} x^{\frac{1}{(1-x)}}$

$= e^{\ln(\lim_{x\to 1^+} x^{\frac{1}{(1-x)}})}$

$= e^{\lim_{x\to 1^+} \frac{1}{1-x}\cdot\ln(x)}$ "$\frac{0}{0}$" indeterminate form

$= e^{\lim_{x\to 1^+} \frac{\frac{1}{x}}{-1}}$

$= e^{-1} = \frac{1}{e}$

53. $\lim_{x\to\infty} (\ln(x))^{1/x}$

$= e^{\ln\left(\lim_{x\to\infty} \ln(x)^{1/x}\right)}$

$= e^{\lim_{x\to\infty} \frac{1}{x}\cdot \ln(\ln(x))}$ "$\frac{\infty}{\infty}$" indeterminate form

$= e^{\lim_{x\to\infty} \frac{\frac{1}{x}}{\ln(x)}}$

$= e^{0} = 1$

59. $\lim_{x\to 0^+} x^x$

$= e^{\ln\left(\lim_{x\to 0^+} x^x\right)}$

$= e^{\lim_{x\to 0^+} x.\ln(x)}$

$= e^{0} = 1$

60. $\lim_{x\to 0^+} (1+\frac{1}{x})^x$

$= e^{\ln\left(\lim_{x\to 0^+} (1+\frac{1}{x})^x\right)}$

$= e^{\lim_{x\to 0^+} \cdot x \ln(1+\frac{1}{x})}$

$= e^{\lim_{x\to 0^+} \frac{\ln(1+\frac{1}{x})}{\frac{1}{x}}}$ "$\frac{\infty}{\infty}$" indeterminate form

$= e^{\lim_{x\to 0^+} \frac{1}{1+\frac{1}{x}}} = e^{0} = 1$

61. $\lim_{x\to\infty} \left(\frac{x+2}{x-1}\right)^x$

$= e^{\ln\left(\lim_{x\to\infty} \left(\frac{x+2}{x-1}\right)^x\right)}$

$= e^{\lim_{x\to\infty} x.\ln\left(\frac{x+2}{x-1}\right)}$

$= e^{\lim_{x\to\infty} \frac{\ln\left(\frac{x+2}{x-1}\right)}{\frac{1}{x}}}$ "$\frac{0}{0}$" indeterminate form

$= e^{\lim_{x\to\infty} \frac{\frac{1}{x+2}-\frac{1}{x-1}}{-\frac{1}{x^2}}}$

$= e^{\lim_{x\to\infty} \frac{3x^2}{x^2+x-2}}$

$= e^{3}$

66. $\lim_{x\to 0^+} \sin(x).\ln(x)$

$\lim_{x\to 0^+} \cdot \frac{\ln(x)}{\csc(x)}.$ "$\frac{-\infty}{\infty}$" indeterminate form

$= \lim_{x\to 0^+} \cdot \frac{\frac{1}{x}}{-\csc(x).\cot(x)}$

$= \lim_{x\to 0^+} \cdot - \frac{\sin(x).\tan(x)}{x}$

$= 0$

67. $\lim_{x\to\infty} \frac{\sqrt{9x+1}}{\sqrt{x+1}}$

$= \lim_{x\to\infty} \sqrt{\frac{9+\frac{1}{x}}{1+\frac{1}{x}}}$

$= \sqrt{9} = 3$

68. $\lim_{x\to 0^+} \sqrt{\frac{x}{\sin x}}$

$= \sqrt{\lim_{x\to 0^+} \cdot \frac{x}{\sin x}}$

$= \sqrt{1} = 1$

75. the form of $\lim_{x\to 3} \frac{x-3}{x^2-3}$ is not indeterminate

- L'Hôpital doesn't work on the lim

=> a is wrong, b is correct

80. $\lim_{x\to 0} \left(\frac{\tan 2x}{x^3} + \frac{a}{x^2} + \frac{\sin bx}{x}\right)$

$= \lim_{x\to 0} \left(\frac{\tan 2x + x.a + x^2 \sin(bx)}{x^3}\right)$ "$\frac{0}{0}$" indeterminate form

$= \lim_{x\to 0} \frac{2\sec^2 2x + a + 2x\sin(bx) + bx^2\cos(bx)}{3x^2}$

this form must be an indeterminate form (i.e. $2\sec^2 2x + a + 2x\sin(bx) + bx^2\cos(bx) = 0$) else the result would lead to $-\infty$ or ∞.

$\Rightarrow 2\sec^2 2(0) + a = 0$

$\Rightarrow a = -2$

$= \lim_{x\to 0} \frac{2\sec^2 2x - 2 + 2x\sin(bx) + bx^2\cos(bx)}{3x^2}$

"$\frac{0}{0}$" indeterminate form

$$= \lim_{x\to 0} \frac{8\sec^2 2x \tan 2x + 2\sin(bx) + 4bx\cos(bx) - x^2b^2\sin bx}{6x}$$

$$\Rightarrow 8\cdot\frac{2}{6} + \frac{b}{3} + \frac{4b}{6} - 0 = 0$$

$$\Rightarrow b = -\frac{8}{3}$$

84a. $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$

$$= e^{\ln\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right)}$$

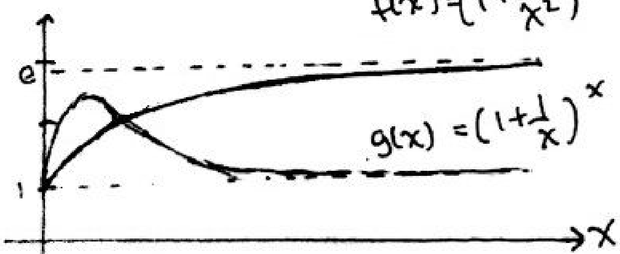
$$= e^{\lim_{x\to\infty} x\cdot\ln\left(1+\frac{1}{x}\right)}$$

$$= e^{\lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}} \quad \text{"}\frac{\infty}{\infty}\text{" indeterminate form}$$

$$= e^{\lim_{x\to\infty} \frac{1}{1+\frac{1}{x}}}$$

$$= e^1 = e$$

b.



f tends to 1 as x tends to ∞

while g tends to e as x tends to ∞

c. $\lim_{x\to\infty} \left(1+\frac{1}{x^2}\right)^{x^2\cdot\frac{1}{x}}$

$$= e^{\lim_{x\to\infty}\frac{1}{x}}$$

$$= e^0 = 1$$

85. $\lim_{k\to\infty} \left(1+\frac{r}{k}\right)^k$

$$= e^{\ln\left(\lim_{k\to\infty}\left(1+\frac{r}{k}\right)^k\right)}$$

$$= e^{\lim_{k\to\infty} k\cdot\ln\left(1+\frac{r}{k}\right)}$$

$$= e^{\lim_{k\to\infty} \frac{\ln\left(1+\frac{r}{k}\right)}{\frac{1}{k}}} \quad \text{"}\frac{0}{0}\text{" indeterminate form}$$

$$= e^{\lim_{k\to\infty} \frac{r}{1+\frac{r}{k}}}$$

$$= e^r$$

88. $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = e^{\lim_{x\to 0} -\frac{1}{x^2}}$$

$$= e^{-\infty}$$

$$= 0 = f(0)$$

$\Rightarrow f(x)$ is continuous at $x=0$

$$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \frac{e^{-\frac{1}{(0+h)^2}}}{h} = \frac{\frac{1}{h}}{e^{\frac{1}{(0+h)^2}}}$$

"$\frac{\infty}{\infty}$" indeterminate form

$$f'(0) = \lim_{h\to 0}\frac{-\frac{1}{h^2}}{-\frac{2}{h^3}e^{\frac{1}{h^2}}}$$

$$= \lim_{h\to 0}\frac{h}{2}\cdot e^{-\frac{1}{h^2}}$$

$$= 0$$

Exercises 7.8

1a. $\lim_{x\to\infty} \frac{e^x}{x-3} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{x\to\infty} \frac{e^x}{1} = \infty$ (slower)

b. $\lim_{x\to\infty} \frac{e^x}{x^3+\sin^2 x} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=}{}^{\times 3} \lim_{x\to\infty} \frac{e^x}{6+[0,1]} = \infty$ (slower)

c. $\lim_{x\to\infty} \frac{e^x}{\sqrt{x}} = \lim_{y\to\infty} \frac{e^{y^2}}{y} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{y\to\infty} 2ye^{y^2} = \infty$ (slower)

d. $\lim_{x\to\infty} \frac{e^x}{4^x}$

$= \lim_{x\to\infty} e^{x\cdot\ln(\underbrace{\frac{e}{4}}_{<1})} = 0$ (faster)

e. $\lim_{x\to\infty} \frac{e^x}{(\frac{3}{2})^x}$

$= \lim_{x\to\infty} e^{x\ln(\underbrace{\frac{2e}{3}}_{>1})} = \infty$ (slower)

f. $\lim_{x\to\infty} \frac{e^x}{e^{x/2}} = \lim_{x\to\infty} e^{x/2} = \infty$ (slower)

g. $\lim_{x\to\infty} 2\cdot\frac{e^x}{e^x} = 2$ (same rate)

h. $\lim_{x\to\infty} \frac{e^x}{\ln(x)}\cdot\ln(10) = \lim_{x\to\infty} \ln(10)\, x\cdot e^x = \infty$ (slower)

20. $\lim_{x\to\infty} \frac{e^x}{\sum_{i=0}^{n} a_i x^i}$ "$\frac{\infty}{\infty}$" indeterminate form

$= \lim_{x\to\infty} \frac{e^x}{\sum_{i=0}^{n-1} (i+1) a_{i+1}\, x^i}$ "$\frac{\infty}{\infty}$" indeterminate form

$\vdots$ n-1 more times

$= \lim_{x\to\infty} \frac{e^x}{\sum_{i=0}^{0} n!\cdot a_{i+n}\cdot x^i} = \lim_{x\to\infty} \frac{e^x}{n!\cdot a_n} = \infty$

$\Rightarrow$ the function e^x outgrows any polynomial

21a. $\lim_{x\to\infty} \frac{\ln(x)}{x^{1/n}}$, "$\frac{\infty}{\infty}$" indeterminate form

$= \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{n}\cdot x^{(1/n-1)}}$

$= \lim_{x\to\infty} \frac{n}{x^{1/n}}$

$= 0$

$\Rightarrow$ $\ln x$ grows slower as $x\to\infty$ than $x^{1/n}$ for any $n>0$

22. $\lim_{x\to\infty} \frac{\ln(x)}{\sum_{i=0}^{n} a_i x^i}$ "$\frac{\infty}{\infty}$" indeterminate form

$= \lim_{x\to\infty} \frac{\frac{1}{x}}{\sum_{i=0}^{n-1} (i+1)\, a_{i+1}\cdot x^i}$

$= \lim_{x\to\infty} \frac{1}{\sum_{i=1}^{n} i\cdot a_i\, x^i}$

for $n>0$

$\lim_{x\to\infty} \frac{1}{\sum_{i=1}^{n} i\cdot a_i\cdot x^i}$

$= \frac{1}{\infty} = 0$

for $n=0$

$\lim_{x\to\infty} \frac{1}{\sum_{i=1}^{0} i\cdot a_i\cdot x^i}$

$= \frac{1}{0} = \infty$

this shows that $\ln(x)$ grows slower than any polynomial except for $n=0$ (constant polynomial).