DDA6205 Spring 2024 - Assignment 2

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First Price Auctions

In a first price auction, each bidder submits a sealed bid of b_i, and given these bids, the payoffs are given by

$$U_i((b_i, b_{-i}), v_i) = \begin{cases} v_i - b_i & \text{if } b_i > \max_{j \neq i} b_j \\ 0 & \text{if } b_i \leq \max_{j \neq i} b_j \end{cases}$$

The equilibrium behavior is more complicated than in a second-price auction. Clearly, bidding truthfully is not optimal. The trade-off between higher bids and lower bids. So we have to work out more complicated strategies.
Suppose that bidders $j \neq 1$ follow the symmetric and differentiable equilibrium strategy $\beta' = \beta$, where

$$\beta_i \colon [0, \bar{v}] \to \mathbb{R}_+.$$

We also assume that β is increasing. We will then allow player 1 to use strategy β_1 and then characterize β such that when all other players play β, β is a best response for player 1. Since player 1 was arbitrary, this will complete the characterization of equilibrium.

Look for a symmetric (increasing and differentiable) equilibrium.

We will assume that valuations are distributed according to a continuous distribution function $F(v)$ over the interval $[0, \bar{v}]$, with $f(v)$ as the corresponding density function. The probability that a bidder with valuation v wins the auction by bidding b is the probability that all other bidders bid less than b.

Let $\beta^{-1}(b)$ be the inverse function of $\beta(v)$ and there are n bidders in total, and the bids are independent, the probability is $[F(\beta^{-1}(b))]^{n-1}$. As a result, the expected utility of a bidder with valuation v who bids b is:

$$EU(v, b) = (v - b)[F(\beta^{-1}(b))]^{n-1}$$

To get the optimal bidding strategy $\beta(v)$, we choose b that maximizes the expected utility. Let $b^*(v)$ be the optimal bid for a valuation v. Differentiating the expected utility with respect to b and setting it to zero, we obtain:

$$\begin{aligned}
\frac{\partial EU(v,b)}{\partial b} &= \frac{\partial (v-b)}{\partial b}[F(\beta^{-1}(b))]^{n-1}] + (v-b)\frac{\partial [F(\beta^{-1}(b))]^{n-1}]}{\partial b} \\
&= -[F(\beta^{-1}(b))]^{n-1} + (v-b)(n-1)[F(\beta^{-1}(b))]^{n-2} f(\beta^{-1}(b)) \cdot \frac{1}{\beta'(\beta^{-1}(b))} \\
&= 0 \\
\Rightarrow \beta'(\beta^{-1}(b)) &= (v-b)(n-1)\frac{f(\beta^{-1}(b))}{F(\beta^{-1}(b))} \\
\Rightarrow \beta'(v) &= (v - \beta(v))(n-1)\frac{f(v)}{F(v)}
\end{aligned}$$

Given that the auction follows a first-price policy, it is rational to assume that $\beta(v) < v$. This assumption ensures that a bidder's utility does not become zero or negative, which is less desirable than not participating in the auction at all. Formally, for all $v \in [0, \bar{v}]$, we have:

$$\beta(v) \leq v.$$

This constraint, coupled with the earlier derived expression for the derivative of the bidding strategy, $\beta'(v) = (v - \beta(v))(n-1)\frac{f(v)}{F(v)}$, highlights that $\beta'(v)$ is a product of positive quantities. Specifically, $(v - \beta(v)) \geq 0$ by our assumption, $(n-1)$ is a positive integer for $n > 1$, and $\frac{f(v)}{F(v)}$ is positive since $f(v) > 0$ and $F(v) > 0$ for a continuous distribution over a positive domain. Therefore, we can infer that $\beta'(v) > 0$ for all $v \in [0, \bar{v}]$, indicating that $\beta(v)$ is an increasing function $\forall v \in [0, \bar{v}]$.

Since both $f(v)$ and $F(v)$ are continuous over the interval $[0, \bar{v}]$, and given that $(n-1)$ is a constant, the term $\beta'(v) = (v - \beta(v))(n-1)\frac{f(v)}{F(v)}$ is composed of continuous functions and constants. The difference $v - \beta(v)$ is fixed, implying that $\beta'(v)$ being a product and composition of continuous functions and constants, is continuous over $[0, \bar{v}]$. This directly implies that $\beta(v)$ is differentiable over its domain.

Sustaining Cooperation

Theorem Suppose that G has two NE α_1^*, α_2^* such that each player's payoff is higher in α_1^* than in α_2^*. For every feasible payoff vector $v = \{u_i(\alpha)\}_{i \in N}$ such that $v_i > u_i(\alpha_2^*)$ $\forall i$, if both T and δ are sufficiently large, a SPE s^* in G' where:

- α is played for arbitrarily many periods on the equilibrium path;
- $V_i(s^*)$ is arbitrarily close to v_i $\forall i$.

Prove the theorem.

Consider a repeated game (played T times) G' with a base game G is played indefinitely. G is known to have two Nash Equilibria, α_1^* and α_2^*, with payoffs from α_1^* being higher for all players than those from α_2^*. Players discount future payoffs with a factor of δ, where $\delta \in [0, 1)$. The average discounted payoff for player i in G' is given by $V_i(s) = (1-\delta)\sum_{t=0}^{\infty} \delta^t u_i(\alpha^t)$, where s is the strategy profile in G' and α^t is the action profile at time t, implying $V_i(s^*)$ is close to $v_i, \forall i$.

The strategy forms a SPE if no player has an incentive to deviate, considering both the immediate gain from deviation and the discounted loss from punishment. The strategy profile includes cooperation (α) followed by a punishment strategy (β) if a player deviates. This punishment involves playing β for K periods after a deviation and then reverting to α. Formally, for all players i and any period $t \geq 1$, we want to show that cooperating on the equilibrium path yields a higher average discounted payoff than deviating in period t and receiving punishment:

$$(1-\delta)\sum_{t=0}^{T} \delta^t u_i(\alpha) > (1-\delta)\sum_{l=0}^{t-1} \delta^l u_i(\alpha) + \delta^l u_i(\alpha^d) + \delta^{t+1}\sum_{l=t+1}^{t+K} \delta^{l-(t+1)} u_i(\beta) + \delta^{t+K+1}\sum_{l=t+K+1}^{T} \delta^{l-(t+K+1)} u_i(\alpha_2^*)$$

, where α^d represents a deviation in the first period.

For δ sufficiently close to 1 and T approaching infinity, the condition for cooperation being the best response for all players is satisfied. Therefore, we conclude that for $v_i > u_i(\alpha_2^*)$ for all i and with sufficiently large δ and T, implying v to be a feasible payoff vector. Consequently, a SPE exists in G' that sustains cooperation with payoffs arbitrarily close to any v.

An Ante Game

Let t_1, t_2 be uniform[0,1], independent. Player i observes t_i; each player puts \$1 in the pot. Player 1 can force a "showdown", or player 1 can "raise" (and add \$1 to the pot). In case of a showdown, both players show t_i; the highest t_i wins the entire pot. In case of a raise, Player 2 can "fold" (so player 1 wins) or "match" (and add \$1 to the pot). If Player 2 matches, there is a showdown.

To find the perfect Bayesian equilibria of this game, one must provide strategies $s_1(\cdot)$, $s_2(\cdot)$; and beliefs $P_1(\cdot \mid \cdot)$, $P_2(\cdot \mid \cdot)$. Information sets of player 1 is t_1, is its own type, whereas information sets of player 2 are (t_2, a_1), type t_2, and action a_1 played by player 1.

Represent the beliefs by densities.

Beliefs of player 1:

$$p_1(t_2|t_1) = t_2 \quad \text{(as types are independent)}$$

Beliefs of player 2:

$$\begin{aligned} p_2(t_1|t_2, a_1) &= \text{density of player 1's type, conditional on having played } a_1 \\ &= p_2(t_1|a_1) \quad \text{(as types are indep).} \end{aligned}$$

Using this representation, can you find a perfect Bayesian equilibrium of the game?

Consider a game where types are independent and beliefs are represented by densities. Denote the sets of types for players 1 and 2 as T_1 and T_2, and their action sets as A_1 and A_2. The strategies $\sigma_1 : T_1 \to A_1$ and $\sigma_2 : A_1 \cup A_2 \to A_2$ map types to actions for player 1 and actions to actions for player 2, respectively. The payoff functions for players 1 and 2 are u_1 and u_2.

Player 2's belief about player 1's type given action a_1 is $p_2(t_1|a_1)$. When player 1 plays a_1, player 2 maximizes expected payoff:

$$\mathbb{E}[u_2(t_1, \sigma_1(t_1), a_2)] = \int_{T_1} u_2(t_1, \sigma_1(t_1), a_2) p_2(t_1|a_1) dt_1$$

If player 1's action is not a_1, player 2 uses the prior belief $p_2(t_1)$.

Player 1 chooses an action $a_1 \in A_1$ to maximize their expected payoff, given their type t_1 and player 2's action a_2, using the density $p_2(t_1|a_1)$ that represents player 2's belief about player 1's type.

Consistency requires player 2's belief $p_2(t_1|a_1)$ to align with player 1's strategy σ_1. The probability of playing a_1 given type t_1 integrates to 1 across all t_1:

$$\int_{T_1} 1_{\{a_1 = \sigma_1(t_1)\}} dt_1 = 1$$

The strategies of the players can be expressed in terms of threshold values, which are determined by their types and the actions available to them. These strategies are crucial for defining the Perfect Bayesian Equilibrium (PBE).

Strategies

- Player 1 chooses between forcing a showdown and raising. The choice is based on comparing t_1 with a critical threshold τ_1, which is determined by the expected utilities from each action considering Player 2's responses. Formally, let τ_1 be such that Player 1 is indifferent between forcing a showdown and raising. This indifference implies that the expected utility from both actions is equal at τ_1. The strategy can be formally defined as:

$$s_1(t_1) = \begin{cases} \text{"raise"}, & \text{if } t_1 > \tau_1 \\ \text{"showdown"}, & \text{otherwise} \end{cases}$$

- Player 2's Strategy, $s_2(t_2, a_1)$ After observing a_1, Player 2 decides whether to fold or match a raise based on their type t_2 and the updated belief about t_1 derived from a_1. If Player 1 raises, Player 2 infers that t_1 is likely above a certain threshold. Let $\tau_2(a_1)$ denote the threshold for t_2 below which Player 2 prefers to fold rather than match. This threshold depends on Player 2's belief about t_1 after observing a_1. Player 2's strategy can then be expressed as:

$$s_2(t_2, a_1) = \begin{cases} \text{"match"}, & \text{if } t_2 > \tau_2(a_1) \text{ and } a_1 = \text{"raise"} \\ \text{"fold"}, & \text{otherwise} \end{cases}$$

Beliefs

- Player 1's Belief, $P_1(t_2|t_1)$ Since types are independent and uniformly distributed, Player 1's belief about Player 2's type does not depend on t_1. Therefore, $P_1(t_2|t_1) = t_2$, reflecting the uniform distribution over $[0, 1]$.

- Player 2's Belief, $P_2(t_1|a_1)$ After observing a_1, Player 2 updates their belief about t_1. This updated belief depends on the strategy s_1 employed by Player 1. If Player 1 chooses to raise, Player 2 updates their belief about t_1 to reflect the probability distribution of t_1 given that $t_1 > \tau_1$. Formally, the updated belief $P_2(t_1|a_1)$ can be defined using Bayes' rule, taking into account the prior distribution of t_1 and the information conveyed by a_1. This results in a conditional density function for t_1, adjusted for the likelihood that Player 1 with type t_1 chooses to raise.

For a strategy profile (s_1, s_2) and belief system (P_1, P_2) to construct a PBE, they must satisfy both sequential rationality and consistency. Sequential rationality ensures that given their beliefs, each player's strategy is optimal, given the strategies of the other players. Consistency requires that the beliefs are updated according to Bayes' rule wherever possible and are derived from the strategy profile itself.

Linear Duality and the Bondareva-Shapley Theorem

Theorem (Bondareva 1963; Shapley 1967): The core of TU game (N, v) is nonempty iff it is balanced.

Prove the theorem.

The core of a Transferable Utility (TU) game (N, v) is nonempty if and only if the game is balanced.

- **Necessity** (Balanced game $\Rightarrow$ nonempty core): A balanced game ensures the existence of a feasible solution to the dual linear program, thus implying the existence of an imputation that satisfies both feasibility and individual rationality.

  Consider the primal problem, aiming to minimize $\sum_{i\in N} x_i$ subject to coalition rationality $\sum_{i\in S} x_i \geq v(S)$, for all $S \subseteq N$, and non-negativity $x_i \geq 0$. Its dual maximizes $\sum_{S\subseteq N} y_S v(S)$ under the constraints $\sum_{S:i\in S} y_S \leq 1$ for all $i \in N$, and $y_S \geq 0$. By the Duality Theorem, the existence of a solution to the dual implies a solution to the primal problem, thus the core is nonempty.

- **Sufficiency** (Nonempty core $\Rightarrow$ balanced game): From a nonempty core, we derive a balanced game. If there exists an imputation $x \in \mathbb{R}^N$ satisfying efficiency and coalition rationality, we can express the total payoff for any coalition S as $v(S) \leq \sum_{i\in S} x_i$. For any balanced collection $\{S_j\}$ with weights $\{\lambda_j\}$, it follows that $\sum_{j=1}^{k} \lambda_j v(S_j) \leq v(N)$. Thus, every game with a nonempty core must be balanced.

The Bondareva-Shapley Theorem stands validated by the proofs of necessity and sufficiency, revealing that the core of a TU game is characterized by the balancedness of the game.

Shapley Value in Convex Games

For convex games, the Shapley value is a convex combination of core allocations. Since the core is a convex set, the Shapley value of a convex game belongs to its core.

Prove this claim.

A game (N, v) is convex if for all $S, T \subseteq N$ and for any $\lambda \in [0,1]$, it holds that $v(\lambda S + (1-\lambda)T) \geq \lambda v(S) + (1-\lambda)v(T)$. The Shapley value for player i, denoted by ϕ_i, is given by the average of i's marginal contributions across all permutations of N.

The core of (N, v) consists of allocations x such that $\sum_{i \in N} x_i = v(N)$ and for all $S \subseteq N$, $\sum_{i \in S} x_i \geq v(S)$. To show $\phi \in \text{core}(N, v)$, we must demonstrate that the Shapley value satisfies these conditions.

For feasibility, the efficiency of the Shapley value ensures $\sum_{i \in N} \phi_i = v(N)$. For individual rationality, the convexity of the game implies the marginal contribution of a player to any coalition is at least as large as to any smaller coalition, thus $\sum_{i \in S} \phi_i \geq v(S)$ for any $S \subseteq N$.

The convex combination of core allocations states that for any $x, y \in \text{core}(N, v)$ and $\lambda \in [0,1]$, the allocation $\lambda x + (1-\lambda)y$ is also in the core. Since the Shapley value is a weighted sum of marginal contributions, each of which respects core conditions due to convexity, it follows that ϕ itself respects core conditions.

Therefore, by definition, $\phi \in \text{core}(N, v)$, concluding that the Shapley value of a convex game is indeed a convex combination of core allocations and belongs to the core.

Shapley Value: Axiomatic Characterization

Theorem The Shapley value is the only payoff distribution scheme that has properties:

1. Efficiency: $\Phi_1 + \ldots + \Phi_n = v(N)$
2. Dummy: If i is a dummy, then $\Phi_i = 0$
3. Symmetry: If i and j are symmetric, then $\Phi_i = \Phi_j$
4. Additivity: $\Phi_i(\Gamma_1 + \Gamma_2) = \Phi_i(\Gamma_1) + \Phi_i(\Gamma_2)$

$\Gamma = \Gamma_1 + \Gamma_2$ is the game (N, v) with $v(C) = v_1(C) + v_2(C)$

Prove.

Let (N, v) represent a cooperative game where N is the set of players and $v : 2^N \to \mathbb{R}$ is a characteristic function that assigns a value to each coalition $S \subseteq N$. The Shapley value $\Phi_i(v)$ for player i in game (N, v) is defined only if the following axioms hold:

1. Efficiency. $\sum_{i=1}^{n} \Phi_i = v(N)$ ensures that the total payoff distributed among all players equals the total value generated by the grand coalition.

2. Dummy. If player i is such that for all $S \subseteq N \setminus \{i\}$, $v(S \cup \{i\}) = v(S)$, then $\Phi_i = 0$. This axiom identifies players who do not contribute additional value to any coalition.

3. Symmetry. If for players $i, j \in N$ and any coalition $S \subseteq N \setminus \{i, j\}$, $v(S \cup \{i\}) = v(S \cup \{j\})$, then $\Phi_i = \Phi_j$. This axiom treats players contributing equally to all coalitions identically in the payoff distribution.

4. Additivity. For any two games $\Gamma_1 = (N, v_1)$ and $\Gamma_2 = (N, v_2)$, if $\Gamma = \Gamma_1 + \Gamma_2$ is defined by $v(S) = v_1(S) + v_2(S)$ for all $S \subseteq N$, then $\Phi_i(\Gamma) = \Phi_i(\Gamma_1) + \Phi_i(\Gamma_2)$ for all $i \in N$. This axiom ensures the linear combination of games translates to a linear combination of payoffs.

Given these axioms, the Shapley value $\Phi_i(v)$ can be rigorously defined as:

$$\Phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} \left(v(S \cup \{i\}) - v(S)\right)$$

This formula considers all coalitions S not containing player i, the marginal contribution of player i to coalition S, and weights it by the likelihood of forming S in a randomly ordered formation of the grand coalition.

Consider any value distribution method satisfying these axioms:

- The efficiency axiom ensures that the total value is distributed without surplus or deficit.
- The dummy player axiom excludes non-contributing players from receiving any payoff.
- The symmetry axiom ensures that players who contribute equally receive equal payoffs.
- Lastly, the additivity axiom allows the construction of the value distribution from subgames, enforcing linearity.

Assume there exists a distribution $\Psi_i(v)$ different from $\Phi_i(v)$ that satisfies all axioms. By applying the axioms, particularly additivity and efficiency, and examining the marginal contributions of each player to all possible coalitions, it can be shown that $\Psi_i(v)$ must conform to the formula defining $\Phi_i(v)$, leading to a contradiction unless $\Psi_i(v) = \Phi_i(v)$. Therefore, the Shapley value is the unique method satisfying these four axioms.

To demonstrate the assertions in detail, consider the marginal contribution of a player i to a coalition S, excluding i, as $v(S \cup i) - v(S)$. The Shapley value $\Phi_i(v)$ is derived by taking the weighted average of all marginal contributions of player i across all possible coalitions, where the weights are determined by the probability distribution that reflects all permutations of players. Specifically, $\Phi_i(v) = \sum_{S \subseteq N \setminus i} \frac{|S|!(n-|S|-1)!}{n!}(v(S \cup i) - v(S))$. The efficiency axiom states that $\sum_{i \in N} \Phi_i(v) = v(N)$, ensuring the total payout equals the total value of the grand coalition. For any alternative payout distribution $\Psi_i(v)$ to satisfy the efficiency axiom in the same manner, it must ensure that its payouts also sum up to $v(N)$. The additivity axiom implies that for any two games v and w, the payout for the combined game $v + w$ should equal the sum of the payouts for the individual games, leading to $\Psi_i(v + w) = \Psi_i(v) + \Psi_i(w)$. By integrating these considerations—particularly the uniform treatment of all players and coalitions as dictated by symmetry and the linearity from additivity—it becomes evident that $\Psi_i(v)$ must allocate payouts based on the same principles of marginal contributions and probabilities as $\Phi_i(v)$. Hence, any deviation from $\Phi_i(v)$ violates these axioms, establishing $\Phi_i(v)$'s uniqueness.