Assignment 6

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Please note that

- **Released date: 21 NOV, Sun.**
- **Due date: 3 DEC, Fri., by 11pm.**
- Late submission is **NOT** accepted.
- Please submit your answers as a PDF file with a name like "120010XXX ASS6.pdf" (Your student ID + ASS No.). You may either typeset you answers directly using computers, or scan your handwritten answers. (We recommend you use the printers on campus to scan. If you use your smartphone to scan, please limit the file size 10MB.)

Question 1(Slide 18). 1. Matrix A is called to be similar to matrix B if there is an invertible matrix S such that $A = S^{-1}BS$. Now prove the following properties of similarity:

(a) Any square matrix A is similar to itself.

(b) If B is similar to A, then A is similar to B

(c) If A is similar to B and B is similar to C, then A is similar to C.

Question 2(Slide 18). Let L be the linear transformation mapping $\mathbb{R}^3$ into $\mathbb{R}^2$ defined by

$$L(\boldsymbol{x}) = x_1 b_1 + (x_2 + x_3)\, b_2$$

for each $x \in \mathbb{R}^3$, where

$$b_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } b_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Find the matrix A representing L with respect to the ordered bases $\{e_1, e_2, e_3\}$ and $\{b_1, b_2\}$

Question 3(Slide 18). Find the matrix representation of the linear transformation $T(f(t)) = f(3t-2)$ from $\mathbb{P}_2$ to $\mathbb{P}_2$, where $\mathbb{P}_2 = \{a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$.

Question 4(Slide 18). Let L be the linear tranformation mapping $\mathbb{R}^3$ into $\mathbb{R}^3$ defined by $L(x) = Ax$, where

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}$$

and let

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

Find the transition matrix V corresponding to a change of basis from $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ to $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$, and use it to determine the matrix B representing L with respect to $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$

Question 5(Slide 18). Let L be the linear operator on $\mathbb{R}^3$ defined by

$$L(\mathbf{x}) = \begin{pmatrix} 2x_1 - x_2 - x_3 \\ 2x_2 - x_1 - x_3 \\ 2x_3 - x_1 - x_2 \end{pmatrix}$$

Determine the standard matrix representation A of L, and use A to find $L(\mathbf{x})$ for each of the following vectors $\mathbf{x}$: (a) $\mathbf{x} = (1,1,1)^T$ (b) $\mathbf{x} = (2,1,1)^T$ (c) $\mathbf{x} = (-5,3,2)^T$

Question 6(Slide 17). A linear transformation $L : V \to W$ is said to be one-to-one if $L(\mathbf{v}_1) = L(\mathbf{v}_2)$ implies that $\mathbf{v}_1 = \mathbf{v}_2$ (i.e., no two distinct vectors $\mathbf{v}_1, \mathbf{v}_2$ in V get mapped into the same vector $\mathbf{w} \in W$. Show that L is one-to-one if and only if $\ker(L) = \{0_V\}$.

Question 7(Slide 18). For the following linear transformations $T : \mathbb{R}^n \to \mathbb{R}^n$, find a matrix A such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^3$,

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - y \\ 3y \\ 4x + 5y \end{bmatrix}$$

(b) $T : \mathbb{R}^2 \to \mathbb{R}^2$, satisfying

$$T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, T\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

Question 8(Slide 18). Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$, and $\vec{v}_2 = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\vec{x}$ into $x_1\vec{v}_1 + x_2\vec{v}_2$. Find a matrix A such that $T(\vec{x})$ is $A\vec{x}$ for each $\vec{x}$

Question 9(Slide 17). Let $S : \mathbb{R}^p \to \mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations. Show that the mapping $\vec{x} \mapsto T(S(\vec{x}))$ is a linear transformation (from $\mathbb{R}^p$ to $\mathbb{R}^m$). [Hint: Compute $T(S(c\vec{u}+d\vec{v}))$ for $\vec{u}, \vec{v}$ in $\mathbb{R}^p$ and scalars c and d. Justify each step of the computation, and explain why this computation gives the desired conclusion.]

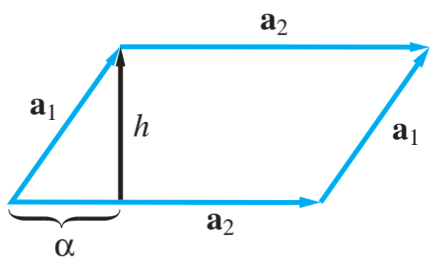
Question 10(Slide 17). Show that the transformation T defined by

$$T(x_1, x_2) = (x_1 - 2|x_2|, x_1 - 4x_2)$$

is not linear.

Question 11(Slide 19). Let $\mathbf{x} = \begin{bmatrix} \sqrt{2} \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 0 \\ \sqrt{5} \\ 0 \end{bmatrix}$ Find $\|\mathbf{x}\|$, $\|\mathbf{y}\|$ and $\|\mathbf{x}+\mathbf{y}\|$ and compare $\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$ and $\|\mathbf{x}+\mathbf{y}\|^2$

Question 12(Slide 19). Let A be a 2×2 matrix with linearly independent column vectors $\mathbf{a}_1$ and $\mathbf{a}_2$. If $\mathbf{a}_1$ and $\mathbf{a}_2$ are used to form a parallelogram P with altitude h (see the figure), show that $h^2 \|\mathbf{a}_2\|^2 = \|\mathbf{a}_1\|^2 \|\mathbf{a}_2\|^2 - \left(\mathbf{a}_1^T\mathbf{a}_2\right)^2$



Question 13(Slide 19). Let $\mathbf{a}$ and $\mathbf{b}$ be vectors in $\mathbb{R}^n$ such that their length are

$$\|\mathbf{a}\| = \|\mathbf{b}\| = 1$$

and the inner product

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^{\mathrm{T}}\mathbf{b} = -\frac{1}{2}$$

Then determine the length $\|\mathbf{a} - \mathbf{b}\|$. (Note that this length is the distance between $\mathbf{a}$ and $\mathbf{b}$.)

Question 14(Slide 19). a) Show that vectors $\mathbf{x} = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$ are orthogonal.

b) Find the constant a and b so that the vector $\mathbf{z} = \begin{bmatrix} a \\ b \\ 4 \end{bmatrix}$ is orthogonal to both vectors $\mathbf{x}$ and $\mathbf{y}$

Question 15(Slide 19). Find the constants a and b such that the vectors $\mathbf{u} = \begin{bmatrix} a \\ 4 \\ -b \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} a \\ 1 \\ b \end{bmatrix}$ are orthogonal and $a = b + 1$.

Question 16(Slide 19).

Find all values of θ such that the vectors $\mathbf{u} \begin{bmatrix} \sin(\theta) \\ \cos(\theta) \end{bmatrix}$ and $\mathbf{v} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are orthognal.

Question 17(Slide 19). Compute $W^{\perp}$, where

$$W = \text{Span} \left\{ \begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \right\}$$

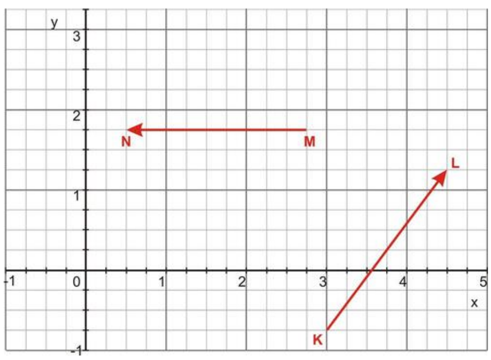
Question 18(Slide 19). State whether each of the following is true or false:

(a) If the subspaces V and W are orthogonal, then $V^{\perp}$ and $W^{\perp}$ are also orthogonal.

(b) If V is orthogonal to W and W orthogonal to Z then V is orthogonal to Z.

Question 19(Slide 19). Suppose that $S = \{0\}$ is the subspace of $\mathbb{R}^4$ containing only the origin. What is the orthogonal complement of S $\left(\text{i.e. } S^{\perp}\right)$? What is $S^{\perp}$ if S is the subspace of $\mathbb{R}^4$ spanned by the vector $(0, 0, 0, 1)$?

Question 20(Slide 19). Determine the vector projection of vector $\overrightarrow{MN}$ onto the vector $\overrightarrow{KL}$. Hint: use the given coordinates to represent vectors.


y
3
2
1
0
-1
-1
1
2
3
4
5
x
N
M
L
K