DDA6205 Spring 2024 - Assignment 1

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The tragedy of the commons (1968)

Suppose that n players each would like to have part of a shared resource. Each player i can send $x_i \in [0,1]$ units of flow along the channel. No player gets any benefit if the total bandwidth $\sum x_i$ exceeds the channel capacity 1. Otherwise the value for player i is $x_i(1-\sum_{j\neq i} x_j)$. What will happen?

Let's focus on one of the players, say player i. Assume that i knew the total units of flow sent by all other players, that is, total $= \sum_{j\neq i} x_j$; then, the benefit that player i gets if player i send x_i units of flow is given by:

$$\begin{aligned}\text{benefit}(x_i) &= x_i(1-\sum_{j\neq i} x_j)\\ &= x_i(1-\text{total})\end{aligned}$$

, for which it would be rational for player i to put x_i as high as possible while maintaining $x_i + \text{total} \leq 1$. This means the action that player i makes will result in $\sum_j x_j = 1$, achieving Nash Equilibrium, as no player can unilaterally increase their benefit by changing their strategy while the others keep theirs unchanged. If they were to unilaterally increase their flow, they would lose all their benefit. Moreover, if they were to unilaterally decrease their flow, they would reduce their benefit as well. Assuming the players are all rational in the act to maximize their individual utility while considering the actions of others, every player will contribute an equal portion of the total capacity, that is $x_i = \frac{1}{n}, \forall i$. With this, every player will receive $\frac{1}{n^2}$ benefit.

However, in terms of collective welfare, those n players are capable of obtaining:

$$\begin{aligned}\max_x \quad & \sum_i x_i(1-\sum_{j\neq i} x_j)\\ \text{s.t.} \quad & x_i \in [0,1], \forall i\\ & \sum_i x_i \leq 1\end{aligned}$$

. Take $x_i = \frac{1}{2(n-1)}, \forall i$, for example, then the collective welfare obtained is:

$$\sum_i x_i(1-\sum_{j\neq i} x_j) = \sum_i x_i(1-\frac{1}{2}) = \frac{1}{2}\sum_i x_i = \frac{n}{4(n-1)}$$

, for which when it is distributed over the n players, each player will receive $\frac{1}{4(n-1)}$ benefit, which is about $\frac{n-1}{4}$ times higher than what each player would obtain according to the individual rational behavior. The tragedy was demonstrated due to each player acting in their own self-interest, leading to over-utilizing the common resource.

Pollution game

n countries each face the choice of either passing legislation to control pollution or not. Pollution control has a cost of 3 for the country, but each country that pollutes adds 1 to the cost of all countries. What will happen?

Let's focus on one of the countries, say country i. Assume m out of the $n-1$ other countries choose to pollute. The cost to country i:

- If it chooses to control pollution is $m+3$ (as each polluting country adds 1 to the cost of all countries).

- If it chooses to pollute is $m+1$ (as it adds to its own cost as well as to the costs of others).

Definitely, $m+3 > m+1$. It would be rational for country i to choose to pollute no matter how many countries are there choosing to pollute as well. As a result, in a Nash Equilibrium, all countries would choose to pollute because no single country can reduce its cost by unilaterally deciding to control pollution while the others do not. If a country were to unilaterally switch from polluting to controlling pollution, its cost would increase from n to $(n-1)+3$. As a result, each country will incur a cost of n, assuming the countries are all rational in the act to minimize their expense.

However, in terms of collective welfare, those n countries are capable of reducing their total cost to $\sum_i 3 = 3n$ if they were to control the pollution altogether. Distributing the total cost over the n countries, each country only has a cost of 3, which is $\frac{n}{3}$ times smaller than what they would have to pay according to their individual rational behavior.

von Neumann-Morgenstern's Utility Theorem

A preference relation $\succeq$ has an expected utility representation iff it satisfies:

- Completeness: for any two lotteries x and y, at least one of $x \succeq y$ and $y \succeq x$ holds.
- Transitivity: if $x \succeq y$ and $y \succeq z$, then $x \succeq z$.
- Continuity: if $x \succeq y \succeq z$, then there exists $p \in [0,1]$ s.t. $px + (1-p)z \sim y$.
- Independence: For any lottery z and $p \in (0,1)$, $x \sim y$ iff $px + (1-p)z \sim py + (1-p)z$.

Prove the theorem.

Sufficiency

Assume $\succeq$ satisfies the axioms of completeness, transitivity, continuity, and independence. We want to show $\exists U : P \to \mathbb{R}$ such that for any lotteries $x, y \in P$, $x \succeq y$ only if $U(x) \geq U(y)$. Let C be the set of outcomes and P be the set of lotteries over C. Let $u : C \to \mathbb{R}$ be a function that maps the outcomes to a real number. Then,

1. For any outcomes $a, b \in C$ and any $p \in [0,1]$, let x be a lottery that results in a with probability p and b with probability $(1-p)$. By continuity, $\exists p$ such that $x \sim c, c \in C$. Then, define $u(c) = p \cdot u(a) + (1-p) \cdot u(b)$.

2. Define $U(x) = \sum_{c \in C} x(c) \cdot u(c)$ for a lottery x that is a probability distribution over outcomes in C, where $x(c)$ denotes the probability of c in x.

3. By completeness and transitivity,
   - If $x \succeq y$ and not $y \succeq x$, by definition, $U(x) > U(y)$.
   - If $x \sim y$, by definition, $U(x) = U(y)$.
   - If $x \succeq y$ and $y \succeq z$, $x \succeq z$, implying $U(x) \geq U(y)$ and $U(y) \geq U(z)$, hence $U(x) \geq U(z)$.

4. If $x \sim y$, then for any $z \in P$ and $p \in (0,1)$, $p \cdot x + (1-p) \cdot z \sim p \cdot y + (1-p) \cdot z$, by definition, $p \cdot U(x) + (1-p) \cdot U(z) \sim p \cdot U(y) + (1-p) \cdot U(z) \Rightarrow U(x) = U(y)$.

The above confirms that if a preference relation satisfies the specified axioms, it is possible to represent these preferences through a utility function, thus satisfying the conditions for an expected utility representation.

Necessity

Assume a utility function U exists, we want to show the preference relation $\succeq$ defined by U satisfies completeness, transitivity, continuity, and independence.

1. Both completeness and transitivity follow from the properties of real numbers and the construction of U.

2. Given $x \succeq y \succeq z$, we can find a p such that $U(px + (1-p)z) = U(y)$, hence $px + (1-p)z \sim y$.

3. If $x \sim y$, then for any z and p, $U(px + (1-p)z) = U(py + (1-p)z)$, so $px + (1-p)z \sim py + (1-p)z$.

By the aforementioned, the existence of a utility function U that orders preferences according to the expected utility principle inherently satisfies the von Neumann-Morgenstern axioms, thus establishing the necessity of these conditions for expected utility representation.

Existence of NE

Any game with a finite set of players and finite set of strategies has a NE of mixed strategies.

Suppose that A_i is a nonempty compact convex subset of $\mathbb{R}^k$ and u_i is continuous and quasi-concave in A_i; given any $a_{-i} \in A_{-i}$, $\forall i \in N$ for strategic game $(N, (A_i), (u_i))$. Then there exists a NE.

By applying Kakutani's fixed point theorem on BR() correspondence, prove the theorem.

Let A be a non-empty subset of a finite dimensional Euclidean space. Let $f : A \rightrightarrows 2^A$ be a correspondence, with $x \in A \mapsto f(x) \subseteq A$, satisfying the following conditions:

- A is a compact and convex set.
- $f(x)$ is non-empty for all $x \in A$.
- $f(x)$ is a convex-valued correspondence: for all $x \in A$, $f(x)$ is a convex set.
- $f(x)$ has a closed graph: that is, if $\{x^n, y^n\} \to \{x, y\}$ with $y^n \in f(x^n)$, then $y \in f(x)$.

Then, f has a fixed point, that is, there exists some $x \in A$, such that $x \in f(x)$.
— Kakutani's Fixed Point Theorem

Let A be $A_1 \times A_2 \times \ldots \times A_n$, then A is a nonempty, compact, convex subset of $\mathbb{R}^{nk}$ as A_i is a nonempty, compact, convex subset of $\mathbb{R}^k$ for all $i \in N$ (Cartesian product properties). We define the correspondence $f : A \to 2^A$ by $f(a) = BR_1(a_{-1}) \times BR_2(a_{-2}) \times \ldots \times BR_n(a_{-n})$, where $BR_i : A_{-i} \to 2^{A_i}$ is defined as the set of the best response correspondences to each strategy a_{-i} of the other players with the set of best responses in A_i. The necessary conditions are then implied as follows:

- A is a compact and convex set as implied in the beginning,
- Each $a_{-i}, BR_i(a_{-i})$ is non-empty due to the existence of at least one best response (as assumed about the players' strategy spaces and utility functions); subsequently, $f(x)$ is non-empty,
- u_i is quasi-concave $\forall i$ (ensuring convexity of $BR_i(a_{-i})$; subsequently, $f(x)$ is a convex-valued correspondence),

- u_i is continuous $\forall i$ (implying that the graph of BR_i (the set of points $(a_{-i}, a_i) \in A_{-i} \times A_i, a_i \in BR_i(a_{-i})$) is closed; subsequently, $f(x)$ has a closed graph).

With all the aforementioned, f has satisfied the conditions of Kakutani's Fixed Point Theorem, meaning that $\exists a \in A$ such that $a \in f(a)$. This further implies that there exists a Nash Equilibrium.