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MAT2040 Assignment 5

Q1. (a)(i) $\det(A) = (1)\cdot(-1) - (2)(3) = -7$

(ii) $\operatorname{adj}(A) = \begin{bmatrix} -1 & -3 \\ -2 & 1 \end{bmatrix}^T = \begin{bmatrix} -1 & -2 \\ -3 & 1 \end{bmatrix}$

(iii) $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \begin{bmatrix} \frac{1}{7} & \frac{2}{7} \\ \frac{3}{7} & -\frac{1}{7} \end{bmatrix}$

(b)(i) $\det(A) = (3)(4) - (1)(2) = 10$

(ii) $\operatorname{adj}(A) = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}^T = \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix}$

(iii) $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \begin{bmatrix} \frac{2}{5} & -\frac{1}{10} \\ -\frac{1}{5} & \frac{3}{10} \end{bmatrix}$

(c)(i) $\det(A) = 1\cdot((1)(-1) - (1)(2)) - 3((2)(-1) - (1)(-2)) + 1\cdot((2)(2) - (1)(-2))$

$= 3$

(ii) $\operatorname{adj}(A) = \begin{bmatrix} -3 & 0 & 6 \\ 5 & 1 & -8 \\ 2 & 1 & -5 \end{bmatrix}^T = \begin{bmatrix} -3 & 5 & 2 \\ 0 & 1 & 1 \\ 6 & -8 & -5 \end{bmatrix}$

(iii) $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \begin{bmatrix} -1 & \frac{5}{3} & \frac{2}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} \\ 2 & -\frac{8}{3} & -\frac{5}{3} \end{bmatrix}$

(d)(i) $\det(A) = 1\cdot 1\cdot 1 = 1$

(ii) $\operatorname{adj}(A) = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

(iii) $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

Q2 (a) $\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{b}$

$\det(A) = (1)(-1) - (2)(3) = -7$

$x_1 = \frac{\det\left(\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}\right)}{\det(A)} = \frac{(-1)(3) - (2)(1)}{-7} = \frac{5}{7}$

$x_2 = \frac{\det\left(\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}\right)}{\det(A)} = \frac{(1)(1) - (3)(3)}{-7} = \frac{8}{7}$

(b) $\underbrace{\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 2 \\ 5 \end{bmatrix}}_{b}$

$\det(A) = (2)(2) - (3)(3) = -5$

$x_1 = \frac{\det\left(\begin{bmatrix} 2 & 3 \\ 5 & 2 \end{bmatrix}\right)}{\det(A)} = \frac{(2)(2) - (3)(5)}{-5} = \frac{11}{5}$

$x_2 = \frac{\det\left(\begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix}\right)}{\det(A)} = \frac{(2)(5) - (2)(3)}{-5} = -\frac{4}{5}$

(c) $\underbrace{\begin{bmatrix} 2 & 1 & -3 \\ 4 & 5 & 1 \\ -2 & -1 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 0 \\ 8 \\ 2 \end{bmatrix}}_{B}$

$\det(A) = 2((5)(4) - (1)(-1)) - 1((4)(4) - (1)(-2)) - 3((4)(-1) - (5)(-2))$

$= 6$

$x_1 = \frac{\det\left(\begin{bmatrix} 0 & 1 & -3 \\ 8 & 5 & 1 \\ 2 & -1 & 4 \end{bmatrix}\right)}{\det(A)} = 4$

$x_3 = \frac{\det\left(\begin{bmatrix} 2 & 1 & 0 \\ 4 & 5 & 8 \\ -2 & -1 & 2 \end{bmatrix}\right)}{\det(A)} = 2$

$x_2 = \frac{\det\left(\begin{bmatrix} 2 & 0 & -3 \\ 4 & 8 & 1 \\ -2 & 2 & 4 \end{bmatrix}\right)}{\det(A)} = -2$

(d) $\underbrace{\begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ -2 & 2 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 5 \\ -8 \end{bmatrix}}_{b}$

$\det(A) = 1((1)(-1) - (1)(2)) + 3((2)(-1) - (1)(-2)) + 1((2)(2) - (1)(-2))$

$= 3$

$x_1 = \frac{\det\left(\begin{bmatrix} 1 & 3 & 1 \\ 5 & 1 & 1 \\ -8 & 2 & -1 \end{bmatrix}\right)}{\det(A)} = 2$

$x_3 = \frac{\det\left(\begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 5 \\ -2 & 2 & -8 \end{bmatrix}\right)}{\det(A)} = 2$

$x_2 = \frac{\det\left(\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 1 \\ -2 & -8 & -1 \end{bmatrix}\right)}{\det(A)} = -1$

Q3. $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \\ 3 & 4 & 3 \end{bmatrix}$ $\det(A) = 1((2)(3) - (1)(4)) - 2((2)(3) - (1)(3)) + 3((2)(4) - (2)(3))$

$= 2$

$\operatorname{adj}(A) = \begin{bmatrix} 2 & -3 & 2 \\ 6 & -6 & 2 \\ -4 & 5 & -2 \end{bmatrix}^T = \begin{bmatrix} 2 & 6 & -4 \\ -3 & -6 & 5 \\ 2 & 2 & -2 \end{bmatrix}$

$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \begin{bmatrix} 1 & 3 & -2 \\ -3/2 & -3 & 5/2 \\ 1 & 1 & -1 \end{bmatrix}$

$B = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ $\det(B) = (2)(3) - (1)(5) = 1$

$\operatorname{adj}(B) = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}^T = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

$B^{-1} = \frac{1}{\det(B)} \operatorname{adj}(B) = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

$\therefore AXB = C \Rightarrow XB = A^{-1}C \Rightarrow X = A^{-1}CB^{-1}$

$X = \begin{bmatrix} 1 & 3 & -2 \\ -3/2 & -3 & 5/2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 10 & -4 \\ -10 & 4 \end{bmatrix}$

Q4. $P = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}$ $\det(P) = \det\left(\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}\right) = 1\cdot 2 = 2$

$\operatorname{adj}(P) = \begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix}^T = \begin{bmatrix} 4 & -2 \\ -1 & 1 \end{bmatrix}$

$P^{-1} = \frac{1}{\det(P)} \operatorname{adj}(P) = \begin{bmatrix} 2 & -1 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$\therefore AP = P\Lambda \Rightarrow A = P\Lambda P^{-1}$

$A = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$

Lemma: $A^n = \begin{bmatrix} 2-2^n & 2^n-1 \\ 2-2^{n+1} & 2^{n+1}-1 \end{bmatrix}$

proof by induction:

for $n=1$,

$$A^1 = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 2-2^1 & 2^1-1 \\ 2-2^2 & 2^2-1 \end{bmatrix} \text{ (true)}$$

assume

$$A^k = \begin{bmatrix} 2-2^k & 2^k-1 \\ 2-2^{k+1} & 2^{k+1}-1 \end{bmatrix}$$

is true for $n=k$, $k \geq 1$

for $n=k+1$,

$$A^{k+1} = A^k \cdot A$$
$$= \begin{bmatrix} 2-2^k & 2^k-1 \\ 2-2^{k+1} & 2^{k+1}-1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 2-2\cdot 2^k & (2-2^k)+3(2^k-1) \\ 2-2\cdot 2^{k+1} & (2-2^k)+3(2^{k+1}-1) \end{bmatrix}$$
$$= \begin{bmatrix} 2-2^{k+1} & 2^{k+1}-1 \\ 2-2^{k+2} & 2^{k+2}-1 \end{bmatrix} \text{ (true)}$$

hence, it is proved.

Q5. we know that:

$\Rightarrow A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$

$\Rightarrow \operatorname{adj}(A) = \det(A) A^{-1}$

$\Rightarrow \operatorname{adj}^{-1}(A) = \frac{1}{\det(A)} A$

$\Rightarrow \operatorname{adj}^{-1}(A) = \det(A^{-1}) A$ (proved) __(1)

$\Rightarrow (A^{-1})^{-1} = \frac{1}{\det(A^{-1})} \operatorname{adj}(A^{-1})$

$\Rightarrow A = \frac{1}{\det(A^{-1})} \operatorname{adj}(A^{-1})$

$\Rightarrow \operatorname{adj}(A^{-1}) = \det(A^{-1}) A$ (proved) --(2)

$\underline{(1) \& (2)}$

$\operatorname{adj}^{-1}(A) = \operatorname{adj}(A^{-1})$

note that since $\operatorname{adj}^{-1}(A) = \operatorname{adj}(A^{-1})$, A^{-1} must be first exist (i.e. A must be invertible). since A is nonsingular, this condition is satisfied.

note also that $\operatorname{adj}^{-1}(A) = \frac{1}{\det(A)} A$ exists if $\det(A) \neq 0$. in other words A is nonsingular.

hence, it is shown.

Q6 (a) $\det(AB) = \det(A)\det(B) = -3 \cdot -4 = 12$

(b) $\det(5A) = 5^3 \det(A) = 5^3(-3) = -375$

(c) $\det(B^T) = \det(B) = -4$

(d) $\det(A^{-1}) = \frac{1}{\det(A)} = -\frac{1}{3}$

(e) $\det(A^3) = \det(A)^3 = -3^3 = -27$

Q7. $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

LHS:

$$\det(A+B) = \det\left(\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}\right)$$
$$= (a_{11}+b_{11})(a_{22}+b_{22}) - (a_{12}+b_{12})(a_{21}+b_{21})$$

RHS:

$$\det(A)+\det(B)+\det(C)+\det(D)$$
$$= (a_{11}a_{22}) - (a_{12}a_{21}) + (b_{11}b_{22}) - (b_{12}b_{21}) + (a_{11}b_{22}) - (a_{12}b_{21}) + (b_{11}a_{22}) - (b_{12}a_{21})$$
$$= a_{11}(a_{22}+b_{22}) - a_{12}(a_{21}+b_{21}) + b_{11}(a_{22}+b_{22}) - b_{12}(a_{21}+b_{21})$$
$$= (a_{11}+b_{11})(a_{22}+b_{22}) - (a_{12}+b_{12})(a_{21}+b_{21})$$

since LHS $\equiv$ RHS, it is shown.

Q8 (a) $\det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 1$

(b) $\det\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\right) = -\det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = -1$

(c) $\det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = k \cdot \det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = k$

Q9. $V = [V_1; V_2, V_3] = \begin{bmatrix} 5 & -3 & 2 \\ -7 & 3 & -7 \\ 9 & -5 & 5 \end{bmatrix}$

$$\det(V) = 5((3)(5)-(-7)(-5)) + 3((-7)(5)-(-7)(9)) + 2((-7)(-5)-(3)(9)) = 0$$

since $\det(V) = 0$, V is singular. this implies that V is not able to be converted into an identity matrix by using only elementary row operations. hence, V_1, V_2, and V_3 are linearly dependent

Q10. $\det(U^TU) = \det(I)$

$\Rightarrow \det(U^TU) = 1$

$\Rightarrow \det(U^T)\det(U) = 1$

$\Rightarrow \det(U)^2 = 1$

$\Rightarrow \det(U) \in \{-1, 1\}$

Q11(a) True, if the columns of A are linearly dependent, A is a singular matrix, $\det(A)=0$

(b) False, consider $A=\begin{bmatrix}1&0\\0&1\end{bmatrix}$.

$\det(A^{-1})=1\neq(-1)\det(A)=-1$

(c) False, only for triangular matrix

(d) False, consider $A=\begin{bmatrix}2&3\\4&6\end{bmatrix}$

(e) False, consider $A=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, $B=\begin{bmatrix}-1&0\\0&-1\end{bmatrix}$

$\det(A)=1$ $\qquad$ $\det(B)=1$

$\det(A+B)=\det\left(\begin{bmatrix}0&0\\0&0\end{bmatrix}\right)=0$

here, $\det(A+B)\neq\det(A)+\det(B)$

Q12.(a) $\det(M_{21})=\det\left(\begin{bmatrix}5&2\\-2&1\end{bmatrix}\right)=(5)(1)-(2)(-2)=9$

$\det(M_{22})=\det\left(\begin{bmatrix}1&2\\1&1\end{bmatrix}\right)=(1)(1)-(2)(1)=-1$

$\det(M_{23})=\det\left(\begin{bmatrix}1&5\\1&-2\end{bmatrix}\right)=(1)(-2)-(5)(1)=-7$

(b) $A_{21}=(-1)^{2+1}\det(M_{21})=-9$

$A_{22}=(-1)^{2+2}\det(M_{22})=-1$

$A_{23}=(-1)^{2+3}\det(M_{23})=7$

(c) $\det(A)=a_{21}A_{21}+a_{22}A_{22}+a_{23}A_{23}$

$=2(-9)+4(-1)+(-1)7$

$=-29$

Q13(a) consider 2nd row of A,

$\det(A)=(-1)(-1)\left((1)(6)-(-2)(0)\right)=6$

(b) $\det(A^4)=\det(A)^4=1296$

(c) $x_1=\dfrac{\det\left(\begin{bmatrix}16&1&-2\\-3&0&0\\-8&0&6\end{bmatrix}\right)}{\det(A)}=2$

$x_2=\dfrac{\det\left(\begin{bmatrix}3&16&-2\\-1&-2&0\\5&-8&6\end{bmatrix}\right)}{\det(A)}=4$

$x_3=\dfrac{\det\left(\begin{bmatrix}3&1&16\\-1&0&-2\\5&0&-8\end{bmatrix}\right)}{\det(A)}=-3$

Q14(a) $\det(A)=-3\det\left(\begin{bmatrix}4&-7&3&-5\\0&2&0&0\\5&5&2&-3\\0&9&-1&2\end{bmatrix}\right)$

$=(-3)2\det\left(\begin{bmatrix}4&3&-5\\5&2&-3\\0&-1&2\end{bmatrix}\right)$

$=-6\left(((4)(-3)-(-5)(5))\right.$
$\left.2((4)(2)-(3)(5))\right)$

$=6$

(b) $\det(B)=-2\det\left(\begin{bmatrix}3&2&4&0\\0&-4&1&0\\-5&6&7&1\\2&3&2&0\end{bmatrix}\right)$

$=(-2)(-1)\det\left(\begin{bmatrix}3&2&4\\0&-4&1\\2&3&2\end{bmatrix}\right)$

$=2\left(-4((3)(2)-(4)(2))-((3)(3)-(2)(2))\right)$

$=6$

Q15. consider block matrix:

$\Rightarrow E=\begin{bmatrix}I_k&0\\0&B\end{bmatrix}$

$\Rightarrow F=\begin{bmatrix}A&0\\0&I_{n-k}\end{bmatrix}$

$\Rightarrow EF=\begin{bmatrix}I_k&0\\0&B\end{bmatrix}\begin{bmatrix}A&0\\0&I_{n-k}\end{bmatrix}=\begin{bmatrix}A&0\\0&B\end{bmatrix}=C$

$\Rightarrow \det(C)=\det(EF)=\det(E)\det(F)$

$\Rightarrow \det(E)=\underbrace{(-1)^{1+1}}_{1}\det\left(\begin{bmatrix}I_{k-1}&0\\0&B\end{bmatrix}\right)$

$=\underbrace{(-1)^{1+1}}_{1}\det\left(\begin{bmatrix}I_{k-2}&0\\0&B\end{bmatrix}\right)$

$\vdots$

$=(-1)^{1+1}\det(B)$

$=\det(B)$

$\Rightarrow \det(F)=\underbrace{(-1)^{n+n}}_{1}\det\left(\begin{bmatrix}A&0\\0&I_{n-k-1}\end{bmatrix}\right)$

$=\underbrace{(-1)^{(n-1)+(n-1)}}_{1}\det\left(\begin{bmatrix}A&0\\0&I_{n-k-2}\end{bmatrix}\right)$

$\vdots$

$=(-1)^{(k+1)+(k+1)}\det(A)$

$=\det(A)$

$\Rightarrow \det(C)=\det(A)\det(B)$

Q16. $L_1: U\to V$ &

$L_2: V\to W$ be linear transformatrons.

$L=L_2\circ L_1$ defined as $L(u)=L_2(L_1(u))$ for every $u\in U$

$\Rightarrow$ for every $u\in U$, $\underbrace{L_1(u)}_{v}\in V$

$\Rightarrow$ for every $v\in V$, $L_2(L_1(u))\in W\Rightarrow L(u)\in W$

therefore, L is a map from U to W.

proof:

(i) let u_1 & u_2 as vectors s.t. $u_1, u_2 \in U$

$$L(u_1+u_2) = L_2(L_1(u_1+u_2))$$
$$= L_2(L_1(u_1)+L_1(u_2))$$
$$= L_2(L_1(u_1)) + L_2(L_1(u_2))$$
$$= L(u_1) + L(u_2)$$

(ii) let α be a scalar s.t. $\alpha \neq 0$

$$L(\alpha u) = L_2(L_1(\alpha u))$$
$$= L_2(\alpha L_1(u))$$
$$= \alpha L_2(L_1(u))$$
$$= \alpha L(u)$$

both (i) & (ii) proved the linear transformation L is valid.

$\therefore L$ is a linear transformation from U to W

Q17. (a) $\ker(L) = \{[0,0,0]^T\}$

(b) $\ker(L) = \{[0,0,c]^T \mid c \in \mathbb{R}\}$

(c) $\ker(L) = \{[0,c_1,c_2]^T \mid c_1, c_2 \in \mathbb{R}\}$

Q18. $V = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix}$

(a) $$\det(V) = \det\left(\begin{bmatrix} 1 & x_1 & x_1^2 \\ 0 & x_2-x_1 & x_2^2-x_1^2 \\ 0 & x_3-x_1 & x_3^2-x_1^2 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} 1 & x_1 & x_1^2 \\ 0 & x_2-x_1 & x_2^2-x_1^2 \\ 0 & 0 & (x_3^2-x_1^2)-(x_3-x_1)(x_2+x_1) \end{bmatrix}\right)$$

$$= 1 \cdot (x_2-x_1)(x_3-x_1)(x_3+x_1-x_2-x_1)$$

$$= (x_2-x_1)(x_3-x_1)(x_3-x_2)$$

(b) V is non-singular if $\det(V) \neq 0$.
this implies that $x_1 \neq x_2$, $x_1 \neq x_3$, and $x_2 \neq x_3$.

Q19. it is given that:

$$A = [[L(e_1)]_B \ [L(e_2)]_B]$$

where:

$$L(e_1) = L(1,0)^T = b_1 + b_3$$

and

$$L(e_2) = L(0,1)^T = b_2 + b_3$$

$\Rightarrow [L(e_1)]_B = (1,0,1)^T$

$\Rightarrow [L(e_2)]_B = (0,1,1)^T$

$\Rightarrow A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$

Q20. $A^{-1} = \frac{1}{\det(A)} \cdot adj(A)$

$\Rightarrow adj(A) = \det(A) \cdot A^{-1}$

$\Rightarrow \det(adj(A)) = \det(A^{-1} \cdot \det(A))$

$= \det^n(A) \cdot \det^{-1}(A)$

$= \det^{n-1}(A)$