MAT3007 - Assignment 4

Yohandi [SID: 120040025]

Problem 1.

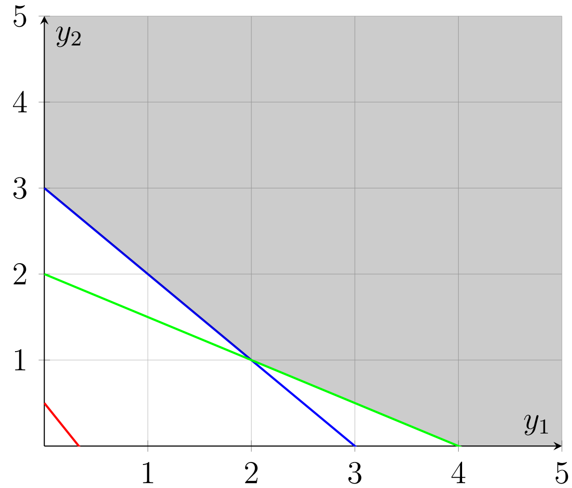
The linear program is equivalent to:

$$\begin{aligned}
\texttt{min} \quad & -3x_1 - x_2 - 4x_3 \\
\texttt{s.t.} \quad & -x_1 - 3x_2 - x_3 \geq -5 \\
& -x_1 - 2x_2 - 2x_3 \geq -7 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}$$

(a) Then, the corresponding dual problem is as follows:

$$\begin{aligned}
\texttt{max} \quad & -5y_1 - 7y_2 \\
\texttt{s.t.} \quad & -y_1 - y_2 \leq -3 \\
& -3y_1 - 2y_2 \leq -1 \\
& -y_1 - 2y_2 \leq -4 \\
& y_1, y_2 \geq 0
\end{aligned}$$

(b) The graph for the dual problem is as follows:



The points in the feasible regions are:

- $(0, 3)$, for which the objective values is -21.
- $(2, 1)$, for which the objective value is -17.
- $(4, 0)$, for which the objective value is -20.

From the objective values, the maximum value is obtained to be -17.

(c) By Complementary Conditions Theorem, for feasible points x and y to the primal and dual problems. Then x and y are optimal iff:

- $y_i \cdot (a_i^T x - b_i) = 0, \forall i$, and
- $x_j \cdot (A_j^T y - c_j) = 0, \forall j$.

This implies that:

$$(y_1 + y_2 - 3)x_1 = 0 \quad (1)$$
$$(3y_1 + 2y_2 - 1)x_2 = 0 \quad (2)$$
$$(y_1 + 2y_2 - 4)x_3 = 0 \quad (3)$$
$$(x_1 + 3x_2 + x_3 - 5)y_1 = 0 \quad (4)$$
$$(x_1 + 2x_2 + 2x_3 - 7)y_2 = 0 \quad (5)$$

Notice that, graphically, $(3y_1 + 2y_2 - 1) \neq 0$; consequently, $x_2 = 0$ by (2).
Moreover, as $y_1, y_2 \neq 0$, we have:

- $x_1 + x_3 - 5 = 0$
- $x_1 + 2x_3 - 7 = 0$

$\Rightarrow x_1 = 3, x_3 = 2.$

Problem 2.

Denote:

- x_1 as the number of breads that should be consumed
- x_2 as the number of milk that should be consumed
- x_3 as the number of fishes that should be consumed
- x_4 as the number of potatoes that should be consumed

(a) Then, the linear program is derived as follows:

$$\begin{array}{rl} \texttt{min} & 3x_1 + 4x_2 + 9x_3 + x_4 \\ \texttt{s.t.} & 4x_1 + 6x_2 + 20x_3 + x_4 \geq 30 \\ & 7x_1 + 10x_2 + 30x_4 \geq 40 \\ & 130x_1 + 120x_2 + 150x_3 + 70x_4 \geq 400 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

(b) Python code is as follows:

```python
import cvxpy as cp

if __name__ == "__main__":
    n = 4
    X = cp.Variable(n, nonneg=True)

    A = [
        [4, 6, 20, 1],
        [7, 10, 0, 30],
        [130, 120, 150, 70]
    ]
```

```python
    b = [30, 40, 400]
    c = [3, 4, 9, 1]

    consts = [A[i] @ X >= b[i] for i in range(len(A))]
    obj = cp.Minimize(c @ X)

    problem = cp.Problem(obj, consts)
    problem.solve()

    for i in range(n):
        print(f"x{i+1} = {X[i].value}")
    print(f"Objective value = {problem.value}")
```

Console output:

```
x1 = 8.285481553091176e-11
x2 = 3.4691239602491257e-09
x3 = 1.3599999990523963
x4 = 2.7999999999180485
Objective value = 15.040000005514676
```

The optimal solution is obtained by having $x_1 = 0, x_2 = 0, x_3 = 1.36, x_4 = 2.8$, which leads to an optimal value 15.04.

(c) The corresponding dual problem is as follows:

$$
\begin{aligned}
\texttt{max} \quad & 30y_1 + 40y_2 + 400y_3 \\
\texttt{s.t.} \quad & 4y_1 + 7y_2 + 130y_3 \leq 3 \\
& 6y_1 + 10y_2 + 120y_3 \leq 4 \\
& 20y_1 + 150y_3 \leq 9 \\
& y_1 + 30y_2 + 70y_3 \leq 1 \\
& y_1, y_2, y_3 \geq 0
\end{aligned}
$$

We may interpret y_1, y_2, y_3 as protein, carbohydrate, and calory nutrient pills, respectively. Pharmaceutical companies will produce each pill and sell them each for a certain price.

(d) Python code is as follows:

```python
import cvxpy as cp

if __name__ == "__main__":
    m = 3
    Y = cp.Variable(m, nonneg=True)
```

```python

    A_dual = [
        [4, 7, 130],
        [6, 10, 120],
        [20, 0, 150],
        [1, 30, 70]
    ]
    b_dual = [3, 4, 9, 1]
    c_dual = [30, 40, 400]

    consts_dual = [A_dual[i] @ Y <= b_dual[i] for i in range(len(A_dual))]
    obj_dual = cp.Maximize(c_dual @ Y)

    problem_dual = cp.Problem(obj_dual, consts_dual)
    problem_dual.solve()

    for i in range(m):
        print(f"y{i+1} = {Y[i].value}")
    print(f"Objective value (Dual) = {problem_dual.value}")
```

Console output:

```
y1 = 0.38400000000195056
y2 = 4.114046520811263e-12
y3 = 0.008800000009860889
Objective value (Dual) = 15.040000004167434
```

The optimal solution is obtained by having $y_1 = 0.384, y_2 = 0, y_3 = 0.0088$, which leads to an optimal value 15.04.

Problem 3.

Suppose the first statement $\exists x \in R^N$ such that $Ax = b$ and $x \geq 0$ holds true. Then,

$$\begin{aligned} A^T y \geq 0 \Rightarrow x^T A^T y &\geq 0 \\ = (Ax)^T y &\geq 0 \\ = b^T y \geq 0 \\ = y^T b \geq 0 \end{aligned}$$

, which contradicts the second statement that says $\exists y$ such that $y^T b < 0$.

Suppose the first statement $\exists x \in R^N$ such that $Ax = b$ and $x \geq 0$ does not hold true. Consider an auxiliary linear program below:

$$\begin{aligned} \texttt{min} \quad & z \\ \texttt{s.t.} \quad & Ax + z[1, \ldots, 1]^T = b \\ & x \geq 0, z \in R \end{aligned}$$

If the optimal solution to this primal auxiliary linear program is greater than 0, then $Ax \neq b, \forall x \geq 0$, contradicting the first statement; otherwise, the optimal value to the following dual problem is negative.

Then, the corresponding dual problem is as follows:

$$\begin{aligned} \texttt{max} \quad & y^T b \\ \texttt{s.t.} \quad & A^T y \leq 0 \\ & [1, \ldots, 1]^T y \leq 1, y \in R^m \end{aligned}$$

$A^T y \leq 0 \Rightarrow -A^T y \geq 0$. Due to the negativity of the dual problem, we may conclude that $y^T b < 0$. Hence, the second statement is shown true in this case.

Problem 4.

(a) The corresponding dual problem to the primal problem is as follows:

$$\begin{aligned} \texttt{max} \quad & -c^T y \\ \texttt{s.t.} \quad & M^T y \leq c \\ & y \geq 0 \end{aligned}$$

As $\max -c^T y \equiv \min c^T y$ and $M^T = -M$, we see that the dual problem is exactly the same as the primal problem.

(b)

($\Rightarrow$) The optimal solution indicates that there exists at least one feasible solution.

($\Leftarrow$) If the primal problem has a feasible solution x, so does y in the dual problem (due to the equivalence of both problems). By the weak duality theorem, both problems have optimal solution(s).

Hence, proved two directions.