yohandi - assignment 9

1a. $F(x) = \int_{-\infty}^{x} f(t)\,dt$

$= \int_{0}^{x} 2t\,dt$

$= x^2$

b. considering $Y = X^2$, we obtain:

$\cdot\!> u(x) = x^2$

$\cdot\!> v(y) = \cancel{\sqrt{x}} = \sqrt{y}$

$\Rightarrow g(y) = f(v(y))\left|\frac{dv(y)}{dy}\right|$

$= 2\cdot\sqrt{y}\cdot\frac{1}{2\sqrt{y}}$

$= 1$

this implies that $Y \sim U(0, 1)$

where $Y = X^2$

c.

0.45340	0.96606
0.38684	0.69644
0.45830	0.65078
0.85490	0.67374
0.44689	0.25594

2a. considering $Y = e^X$, we obtain:

$\cdot\!> u(x) = e^x$

$\cdot\!> v(y) = \ln(y)$

$\Rightarrow g(y) = f(v(y))\left|\frac{dv(y)}{dy}\right|$

$= e^{\ln(y) - y}\cdot\frac{1}{y}$

$= e^{-y}$

with mean $= 1$, $Y \sim Exp(1)$

b. $\Rightarrow g(y)$

$= f(v(y))\left|\frac{dv(y)}{dy}\right|$

$= \frac{1}{\theta_2} e^{\frac{\ln(y)-\theta_1}{\theta_2} - e^{\frac{\ln(y)-\theta_1}{\theta_2}}}\cdot\frac{1}{y}$

$= \frac{1}{\theta_2 \cdot y}\exp\left(\frac{\ln(y)-\theta_1}{\theta_2} - \exp\left(\frac{\ln(y)-\theta_1}{\theta_2}\right)\right)$

$G(y) = 1 - F(v(y)) = 1 - e^{-e^{(y-\theta_1)/\theta_2}}$

c. for $\theta_1 = \ln(\beta)$ and $\theta_2 = \frac{1}{\alpha}$,

$g(y) = \frac{1}{\alpha y}\exp\left(\frac{\ln(y)-\ln(\beta)}{\frac{1}{\alpha}} - \exp\left(\frac{\ln(y)-\ln(\beta)}{\frac{1}{\alpha}}\right)\right)$

$= \frac{1}{\alpha y}\exp\left(\alpha\ln\left(\frac{y}{\beta}\right) - \exp\left(\alpha\ln\left(\frac{y}{\beta}\right)\right)\right)$

$G(y) = 1 - \exp(-\exp(\alpha(y - \ln(\beta))))$

Y follows the Weibull distribution

d. $P(X > 500) = 1 - \left(1 - \exp\left(-\exp\left(\frac{500-550}{25}\right)\right)\right) \approx 0.8734$

3a. considering $Y = e^X$, we obtain:

$\cdot\!> u(x) = e^x$

$\cdot\!> v(y) = \ln(y)$

$\Rightarrow g(y) = f(v(y))\left|\frac{dv(y)}{dy}\right|$

$= \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{1}{2}\cdot\left(\frac{\ln(y)-\mu}{\sigma}\right)^2\right)\cdot\frac{1}{y}$

$= \frac{1}{y\sqrt{2\pi\sigma^2}}\exp\left(\frac{-(\ln(y)-\mu)^2}{2\sigma^2}\right)$

b.(i) $E(Y) = E(e^X) = M(1) = \exp\left(\mu + \frac{\sigma^2}{2}\right)$

(ii) $E(Y^2) = E(e^{2X}) = M(2) = \exp(2\mu + 2\sigma)$

(iii) $Var(Y) = E(Y^2) - E(Y)^2 = \exp(2\mu + 2\sigma) - \exp(2\mu + \sigma^2)$

4a. $P(X_1 = 2, X_2 = 2, X_3 = 5)$

$= P(X_1 = 2)\cdot P(X_2 = 2)\cdot P(X_3 = 5)$

$= \binom{4}{2}\left(\frac{1}{2}\right)^4\cdot\binom{6}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^4\cdot\binom{12}{5}\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right)^7$

$\approx 3.5\cdot 10^{-3}$

b. $E\left(\prod_{i=1}^{3} X_i\right) = \prod_{i=1}^{3} E(X_i) = 4\left(\frac{1}{2}\right)\cdot 6\left(\frac{1}{3}\right)\cdot 12\left(\frac{1}{6}\right) = 8$

c. $E\left(\sum_{i=1}^{3} X_i\right) = \sum_{i=1}^{3} E(X_i) = 4\left(\frac{1}{2}\right) + 6\left(\frac{1}{3}\right) + 12\left(\frac{1}{6}\right) = 6$

d. $Var\left(\sum_{i=1}^{3} X_i\right) = \sum_{i=1}^{3} Var(X_i) = 4\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + 6\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) + 12\left(\frac{1}{6}\right)\left(\frac{5}{6}\right) = 4$

5. $Cov(x, y) = \dfrac{Var(x+Y) - (Var(x) + Var(Y))}{2}$

$= 950$

$Var(X + 500 + 1.08\,Y)$

$= Var(x) + 1.08^2 Var(Y) + 2(1.08)\,Cov(X, Y)$

$= 21816$

6. $E(Z) = E(2Y_1 + Y_2)$

$= 2E(Y_1) + E(Y_2)$

$G_1(y) = P(\min(X_1, X_2) \leq y)$

$= 1 - P(\min(X_1, X_2) > y)$

$= 1 - (1 - (1 - e^{-\frac{y}{2}}))^2$

$= 1 - e^{-y}$

$g_1(y) = e^{-y}$

$E(Y_1) = \int_0^\infty y \cdot g_1(y)\, dy = 1$

$G_2(y) = P(\max(X_1, X_2) \leq y)$

$= (1 - e^{-\frac{y}{2}})^2$

$g_2(y) = e^{-\frac{y}{2}} - e^{y}$

$E(Y_2) = \int_0^\infty y \cdot g_2(y)\, dy = 3$

$\Rightarrow E(Z) = 2 \cdot 1 + 3 = 5$

7a. $P(Y \leq y) = P(\max_{i=1}^{8} x_i \leq y)$

$= P(X \leq y)^8$

$= (1 - (\frac{1}{2})^y)^8$

b. $P(Y = y) = P(Y \leq y) - P(Y \leq y-1)$

$= (1 - (\frac{1}{2})^y)^8 - (1 - (\frac{1}{2})^{y-1})^8$

this works as y is discrete

d. $E(Y) = \sum_{y=1}^{\infty} y((1-(\frac{1}{2})^y)^{16} - (1-(\frac{1}{2})^{y-1})^{16})$

≈ 5.3774

8a. $M(t) = E(e^{tY})$

$= E(e^{tX_1} \cdot e^{tX_2} \cdot e^{tX_3})$

$= M_1(t) \cdot M_2(t) \cdot M_3(t)$

$= e^{2(e^t - 1)} \cdot e^{1\cdot(e^t-1)} \cdot e^{4(e^t-1)}$

$= e^{7(e^t - 1)}$

b. $Y \sim Poi(7)$

c. $P(3 \leq Y \leq 9) = \sum_{i=3}^{9} p(y) = \sum_{i=3}^{9} \frac{7^i \cdot e^{-7}}{i!} \approx 0.801$

9. $M(t) = E(e^{tW})$

$= E(e^{tX_1} \cdot e^{tX_2} \cdots e^{tX_n})$

$= M_X(t)^n$

$= \dfrac{1}{(1 - t\theta)^n}$

$W \sim Gamma(n, \theta)$

10. $M(t) = E(e^{tW})$

$= E(e^{tX}) \cdot E(e^{tY})$

$= \sum_{x=0}^{\infty} f(x) \cdot e^{xt} \cdot \sum_{y=0}^{\infty} g(y) \cdot e^{yt}$

$= \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} e^{(x+y)t} f(x) \cdot g(y)$

$= \sum_{w=0}^{\infty} \underbrace{\sum_{k=0}^{w} f(k) \cdot g(w-k)}_{h(w)} \cdot e^{wt}$

$h(w)$ is the pmf of W,

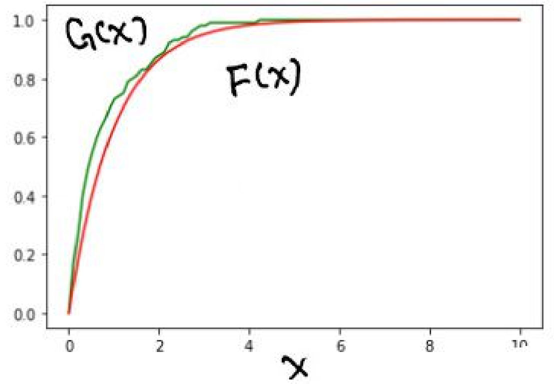
$h(w) = \sum_{k=0}^{w} f(k) \cdot g(w-k)$, $w = 0, 1, 2, \ldots$

yohandi - assignment 9 (computer-based)

Theorem [Random Number Generator],

Let $Y \sim U(0,1)$ and $F(x)$ have the properties of a cdf of a continuous RV with $F(a)=0$, $F(b)=1$. Moreover, $F(x)$ is strictly increasing such that $F(x): (a,b) \to [0,1]$, where a could be $-\infty$, b could be ∞. Then $X = F^{-1}(Y)$ is continuous RV with cdf $F(x)$

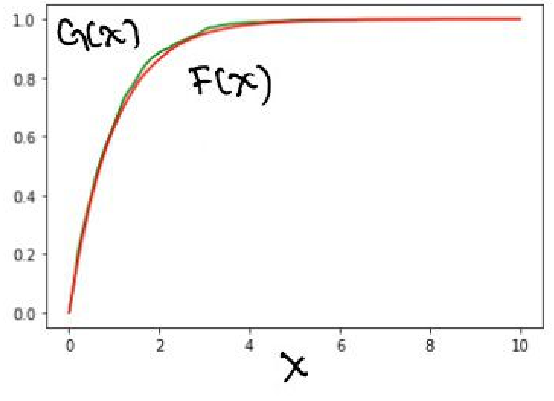
- case $N=100$:



here,
- green color represents $G(x) = \frac{N(x)}{100}$
- red color represents $F(x) = 1 - e^{-x}$

from those two functions it can be seen how $F(x)$ is able to "represent" the scattered points $(x_i, G(x_i))$ with $i \in [1, 100]$.

- case $N=1000$:



here,
- green color represents $G(x) = \frac{N(x)}{1000}$
- red color represents $F(x) = 1 - e^{-x}$

similar with the case where $N=100$. However, the value error for every $i \in [1, 1000]$ such that $|F_{1000}(x_i) - G_{1000}(x_i)| < \varepsilon_{1000} < |F_{100}(x_i) - G_{100}(x_i)|$.

```python
import math
import matplotlib.pyplot as plt
import numpy as np

def simulateExperiment(N):
    def G(x0):
        return len([i for i in range(N) if x_i[i] < x0]) / N

    y_i = np.random.uniform(0, 1, N)
    x_i = [-math.log(1 - y_i[i]) for i in range(N)]

    x = np.linspace(0, 10, 100)
    G_x = [G(i) for i in x]
    F_x = [1 - math.exp(-i) for i in x]

    plt.plot(x, G_x, color = 'green')
    plt.plot(x, F_x, color = 'red')
    plt.show()

simulateExperiment(100)
simulateExperiment(1000)
```