yohandi - homework for Ch 10

Exercises 10.1

30. $\lim\limits_{n\to\infty} \frac{2n+1}{-3\sqrt{n}+1} = \lim\limits_{n\to\infty} \frac{2\sqrt{n}+\frac{1}{\sqrt{n}}}{-3+\frac{1}{\sqrt{n}}}$

$= \infty$ (diverge)

31. $\lim\limits_{n\to\infty} \frac{1-5n^4}{n^4+8n^3} = \lim\limits_{n\to\infty} \frac{-5+\frac{1}{n^4}}{1+\frac{8}{n}}$

$= -5$ (converge)

44. $\lim\limits_{n\to\infty} \frac{n\pi}{\sec(n\pi)} = \lim\limits_{n\to\infty} n\pi\,(-1)^{(n \bmod 2)}$

$= \pm\infty$ (diverge)

49. $\lim\limits_{n\to\infty} \frac{\ln(n+1)}{\sqrt{n}}$ "$\frac{\infty}{\infty}$" apply L'Hôpital

$= \lim\limits_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{2\sqrt{n}}} = \lim\limits_{n\to\infty} \frac{2}{\sqrt{n}+\frac{1}{\sqrt{n}}} = 0$ (converge)

58. $\lim\limits_{n\to\infty} e^{\ln((n+4)^{\frac{1}{n+4}})}$

$= e^{\lim\limits_{n\to\infty} \frac{\ln(n+4)}{n+4}}$ "$\frac{\infty}{\infty}$" apply L'Hôpital

$= e^{\lim\limits_{n\to\infty} \frac{1}{n+4}} = 1$ (converge)

66. $\lim\limits_{n\to\infty} \frac{n!}{2^n \cdot 3^n} = \lim\limits_{n\to\infty} \prod_{k=1}^{n} \frac{k}{6}$

$= \lim\limits_{n\to\infty} \left(\prod_{k=7}^{n} \frac{k}{6} \cdot \prod_{k=1}^{6} \frac{k}{6} \right)$

$\geq \lim\limits_{n\to\infty} \left(\frac{7}{6}\right)^n \cdot \prod_{k=1}^{6} \frac{k}{6} = \infty$

(diverge)

73. $\lim\limits_{n\to\infty} \frac{3^n \cdot 6^n}{2^{-n} \cdot n!} = \lim\limits_{n\to\infty} \frac{36^n}{n!}$

$= \lim\limits_{n\to\infty} \prod_{k=1}^{n} \frac{36}{k} = \lim\limits_{n\to\infty} \left(\prod_{k=1}^{36} \frac{36}{k} \cdot \prod_{k=37}^{n} \frac{36}{k} \right)$

$\leq \lim\limits_{n\to\infty} \left(\frac{36}{37}\right)^n \prod_{k=1}^{36} \frac{36}{k} = 0$

since $\lim\limits_{n\to\infty} a_n$ can't be negative,

the limit converges to 0

80. $\lim\limits_{n\to\infty} e^{\ln((3^n+5^n)^{1/n})}$

$= \lim\limits_{n\to\infty} e^{\frac{\ln(3^n+5^n)}{n}}$ "$\frac{\infty}{\infty}$" apply L'Hôpital

$= \lim\limits_{n\to\infty} e^{\frac{\ln(3)\cdot 3^n + \ln(5)\, 5^n}{3^n+5^n}}$

$= \lim\limits_{n\to\infty} e^{\frac{\ln(5)\cdot(\frac{5}{3})^n}{(\frac{5}{3})^n}} = e^{\ln(5)} = 5$ (converge)

87. $\lim\limits_{n\to\infty} n-\sqrt{n^2-n}$

$= \lim\limits_{n\to\infty} \frac{n^2-(n^2-n)}{n+\sqrt{n^2-n}}$

$= \lim\limits_{n\to\infty} \frac{1}{1+\sqrt{1-\frac{1}{n}}} = \frac{1}{2}$ (converge)

90. $\lim\limits_{n\to\infty} \int_1^n \frac{1}{x^p}\,dx$, $p>1$

$= \lim\limits_{n\to\infty} \frac{1}{1-p} x^{1-p} \Big|_{x=1}^{n} = \frac{1}{p-1}$ (converge)

93. $\lim\limits_{n\to\infty} a_{n+1} = \sqrt{8+2a_n}$

$= \lim\limits_{n\to\infty} a_n^2 - 2a_n - 8 = 0$

$= \lim\limits_{n\to\infty} (a_n-4)(a_n+2) = 0$

$a_n = 4$ $\quad a_n = -2$
(this is not possible since $a_n \geq 0$)

98. let $a_n = \underbrace{\sqrt{1+\sqrt{1+\dots}}}_{n}$

$\lim\limits_{n\to\infty} a_{n+1} = \sqrt{1+a_n}$

$\lim\limits_{n\to\infty} a_n^2 - a_n - 1 = 0$

$\lim\limits_{n\to\infty} a_n = \frac{1\pm\sqrt{5}}{2} = \frac{1+\sqrt{5}}{2}$ (since $a_n \geq 1$)

101 a. $x_0 = 1$

$$x_1 = 1 - \frac{1^2-2}{2.1} = \frac{3}{2}$$

$$x_2 = \frac{3}{2} - \frac{(\frac{3}{2})^2-2}{2.\frac{3}{2}} = \frac{17}{12}$$

$$x_3 = \frac{17}{12} - \frac{(\frac{17}{12})^2-2}{2.\frac{17}{12}} = \frac{577}{408}$$

$\vdots$

$$\lim_{n\to\infty} x_n = 1{,}41421\ldots = \sqrt{2}$$
(converge)

b. $x_0 = 1$

$$x_1 = 1 - \frac{\tan(1)-1}{\sec^2(1)} = 0{,}83727787\ldots$$

$$x_2 = x_1 - \frac{\tan(x_1)-1}{\sec^2(x_1)} = 0{,}788180293\ldots$$

$$x_3 = x_2 - \frac{\tan(x_2)-1}{\sec^2(x_2)} = 0{,}78540592\ldots$$

$\vdots$

$$\lim_{n\to\infty} x_n = 0{,}78539816\ldots = \frac{\pi}{4}$$
(converge)

c. $x_0 = 1$

$x_1 = 0$

$x_2 = -1$

$\vdots$

$$\lim_{n\to\infty} x_n = 1-n = -\infty$$ (diverge)

113.

n	a_n
1	6
2	18
3	36
4	54
5	64,8
6	64,8
7	55,542.

the sequence is not monotonic since both a_5 and a_6 are maximum points.

119. $$\lim_{n\to\infty} ((-1)^n+1)\left(\frac{n+1}{n}\right)$$

$$= \lim_{n\to\infty} ((-1)^n+1) \cdot \lim_{n\to\infty} \left(\frac{n+1}{n}\right)$$

$$= \begin{cases} 0 & , n \bmod 2 = 1 \\ 2 & , n \bmod 2 = 0 \end{cases}$$

(diverge)

123. $$\lim_{n\to\infty} \frac{4^{n+1}+3^n}{4^n}$$

$$= \lim_{n\to\infty} 4 + \left(\frac{3}{4}\right)^n$$

$= 4$ (converge)

Exercises 10.2

1. $S_\infty = \dfrac{2}{1-\frac{1}{3}} = 3$

3. $S_\infty = \dfrac{1}{1+\frac{1}{2}} = \dfrac{2}{3}$

4. The series is diverge as $|r| \geq 1$

5. $S_n = \dfrac{1}{2\cdot 3} + \dfrac{1}{3\cdot 4} + \cdots + \dfrac{1}{(n+1)(n+2)}$

$= \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \cdots + \left(\frac{1}{n+1}-\frac{1}{n+2}\right)$

$= \frac{1}{2} - \frac{1}{n+2}$

$S_\infty = \frac{1}{2}$

13. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n} + \sum_{n=0}^{\infty}\left(-\frac{1}{5}\right)^n$

$= \dfrac{1}{1-\frac{1}{2}} + \dfrac{1}{1+\frac{1}{5}}$

$= \dfrac{17}{6}$

19. $S_n = \dfrac{23}{100} + \dfrac{23}{100^2} + \cdots + \dfrac{23}{100^n}$

$S_\infty = \dfrac{\frac{23}{100}}{1-\frac{1}{100}} = \dfrac{23}{99}$

37. $\displaystyle\sum_{n=1}^{\infty}\left(\ln\sqrt{n+1} - \ln\sqrt{n}\right)$

$= \displaystyle\lim_{n\to\infty} \ln\sqrt{n} - \ln\sqrt{1}$

$= \infty$ (diverge)

41. $\displaystyle\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)}$

$= \displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{4n-3} - \frac{1}{4n+1}\right)$

$= \dfrac{1}{4-3} - \displaystyle\lim_{n\to\infty}\frac{1}{4n+1}$

$= 1$

44. $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$

$= \displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right)$

$= \dfrac{1}{1^2} - \displaystyle\lim_{n\to\infty}\frac{1}{(n+1)^2}$

$= 1$

52. The series is diverge as:

$|r_{min}| = \left|\displaystyle\lim_{n\to\infty}\frac{(n+1)}{n}\right| = 1 \geq 1$

53. $\displaystyle\sum_{n=0}^{\infty}\cos\left(\frac{n\pi}{2}\right)$

$= \displaystyle\lim_{n\to\infty} f(n) = \begin{cases} 1, & n \bmod 4 = 0 \\ 1, & n \bmod 4 = 1 \\ 0, & n \bmod 4 = 2 \\ 0, & n \bmod 4 = 3 \end{cases}$

(diverge)

54. $\displaystyle\sum_{n=0}^{\infty}\left(-\frac{1}{5}\right)^n$

$= \dfrac{1}{1+\frac{1}{5}} = \dfrac{5}{6}$

55. $\displaystyle\sum_{n=0}^{\infty} e^{-2n}$

$= \dfrac{1}{1-\frac{1}{e^2}} = \dfrac{e^2}{e^2-1}$

56. $\displaystyle\sum_{n=1}^{\infty}\ln\left(\frac{1}{3^n}\right)$

$= \displaystyle\lim_{n\to\infty}\ln\left(\frac{1}{3}\cdot\frac{1}{3^2}\cdots\frac{1}{3^n}\right)$

$= -\infty$ (diverge)

61. $\displaystyle\sum_{n=0}^{1000}\frac{n!}{1000^n} + \sum_{n=1001}^{\infty}\frac{n!}{1000^n}$,

The series is diverge as for $n \geq 1001$,

$|r| = \left|\dfrac{n!}{1000^n}\right| \geq 1$

62. $\lim_{n\to\infty} \frac{n^n}{n!} = \lim_{n\to\infty} \frac{n.n.\dots.n}{n(n-1)\dots 1}$

$= \infty$ (diverge)

64. $\lim_{n\to\infty} \frac{2^n+4^n}{3^n+4^n} = \lim_{n\to\infty} \frac{\left(\frac{2}{4}\right)^n+1}{\left(\frac{3}{4}\right)^n+1}$

$= 1$ diverge as the limit doesn't converge to 0

66. $\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+1}\right)$

$= \lim_{n\to\infty} \ln\left(\frac{n!}{3.5.\dots(2n+1)}\right)$

$= -\infty$ (diverge)

71. $\lim_{n\to\infty} 3\left(\frac{x-1}{2}\right)^n$,

the limit goes to 0 as $\left|\frac{x-1}{2}\right| < 1$

(i.e. $-1 < x < 3$). otherwise, it diverges

$\sum_{n=0}^{\infty} 3\left(\frac{x-1}{2}\right)^n = \frac{3}{1-\left(\frac{x-1}{2}\right)} = \frac{6}{3-x}$

72. $\lim_{n\to\infty} \frac{(-1)^n}{2}\left(\frac{1}{3+\sin x}\right)^n$

$= \lim_{n\to\infty} \frac{(-1)^n}{2} \cdot \lim_{n\to\infty}\left(\frac{1}{3+\sin x}\right)^n$

$= \lim_{n\to\infty} \frac{(-1)^n}{2} . 0 = 0$ (converge)

$S_\infty = \frac{\frac{1}{2}}{1+\frac{1}{3+\sin x}} = \frac{3+\sin x}{8+2\sin x}$

73. $\lim_{n\to\infty} 2^n x^n = \lim_{n\to\infty} (2x)^n$

$|r| = |2x| < 1$

$|x| < \frac{1}{2} \Rightarrow -\frac{1}{2} < x < \frac{1}{2}$

$S_\infty = \frac{1}{1-2x}$

80 a. $\sum_{n=-1}^{\infty} \frac{5}{(n+2)(n+3)}$

b. $\sum_{n=3}^{\infty} \frac{5}{(n-2)(n-1)}$

c. $\sum_{n=20}^{\infty} \frac{5}{(n-19)(n-18)}$

85. let $a_n = \left(\frac{1}{3}\right)^n$ and $b_n = \left(\frac{1}{2}\right)^n$

$A = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$ $\quad B = \frac{1}{1-\frac{1}{2}} = 2$

as $S_\infty = \frac{1}{1-\frac{2}{3}} = 3 \neq \frac{A}{B} = \frac{3}{4}$

87. Note that sum of divergent series is -infinite or infinite, substracting or adding some finite number won't change the trend of the series.

88. $\sum(a_n+b_n) = \sum a_n + \sum b_n$

assume that $\sum(a_n+b_n)$ is a convergent series:

$\sum(a_n+b_n) - \sum(a_n) = \sum b_n$

this show that $\sum b_n$ is a convergent series which contradicts with the main statement.

$\therefore \sum(a_n+b_n)$ diverges

95 a. $[0,1]$

$[0,\frac{1}{3}][\frac{2}{3},1]$

$[0,\frac{1}{9}][\frac{2}{9},\frac{1}{3}][\frac{2}{3},\frac{7}{9}][\frac{8}{9},1]$

$[0,\frac{1}{27}][\frac{2}{27},\frac{1}{9}][\frac{2}{9},\frac{7}{27}][\frac{8}{27},\frac{1}{3}]$ ......

Cantor $= \{0,\frac{1}{27},\frac{2}{27},\frac{1}{9},\frac{2}{9},\frac{7}{27},\frac{8}{27},\frac{1}{3},\frac{2}{3},\frac{7}{9},\frac{8}{9},1\}$

b. $S_n = \frac{1}{3}+\frac{2}{9}+\frac{4}{27}+\dots+\frac{2^{n-1}}{3^n}$

$S_\infty = \frac{\frac{1}{3}}{1-\frac{2}{3}} = 1$ (it is true)

Exercises 10.3

4. $\frac{d}{dx}\left(\frac{1}{x+4}\right) = -\frac{1}{(x+4)^2}$

from the slope of function, we can conclude that $\frac{1}{x+4}$ is decreasing as $\frac{d}{dx} < 0$

$$\int_1^\infty \frac{1}{x+4}\,dx = \ln|x+4|\Big|_{x=1}^\infty = \infty \text{ (diverge)}$$

6. $\frac{d}{dx}\left(\frac{1}{x\ln^2 x}\right) = -\frac{\ln(x)+2}{x^2\ln^3(x)}$

as $f'(2)<0$ and $e^{-2}<2$, the function is decreasing for $x \geq 2$

$$\int_2^\infty \frac{dx}{x(\ln x)^2} = \int_2^\infty \frac{d(\ln(x))}{\ln^2 x} = -\frac{1}{\ln(x)}\Big|_{x=2}^\infty$$

$$= \frac{1}{\ln(2)} \text{ (converge)}$$

13. $\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$ (diverge)

16. $\frac{d}{dx}\left(\frac{2}{x\sqrt{x}}\right) = \frac{-3}{x^{5/2}}$

the function decreases as $x^{5/2} > 0$

for $x > 1$.

$$\int_1^\infty \frac{-2\,dx}{x\sqrt{x}} = -\int_1^\infty \frac{2}{x\sqrt{x}}\,dx = \frac{4}{\sqrt{x}}\Big|_{x=1}^\infty$$

$$= -4 \text{ (converge)}$$

19. $\sum_{n=1}^\infty \frac{\ln(n)}{n} = \sum_{n=1}^{2} \frac{\ln(n)}{n} + \sum_{n=3}^\infty \frac{\ln(n)}{n}$

$\frac{d}{dx}\left(\frac{\ln(n)}{n}\right) = \frac{1-\ln(x)}{x^2}$

from the slope, the function decreases for $x > e$.

$$\int_3^\infty \frac{\ln(x)}{x}\,dx = \frac{\ln^2(x)}{2}\Big|_{x=3}^\infty = \infty \text{ (diverge)}$$

21. $\sum_{n=1}^\infty \left(\frac{2}{3}\right)^n = \frac{\frac{2}{3}}{1-\frac{2}{3}} = 2$ (converge)

24. $\frac{d}{dx}\left(\frac{1}{2x-1}\right) = -\frac{2}{(2x-1)^2}$

since $f'(x)<0$ for every $x \in \mathbb{R}$, f is a decreasing function

$$\int_1^\infty \frac{1}{2x-1}\,dx = \frac{1}{2}\ln(2x-1)\Big|_{x=1}^\infty = \infty \text{ (diverge)}$$

28. $\frac{d}{dx}\left(\frac{1}{x+\sqrt{x}}\right) = -\frac{\frac{1}{2\sqrt{x}}+1}{(x+\sqrt{x})^2}$

since $f'(x)<0$ for every $x \in \mathbb{R}$, f is a decreasing function

$$\int_1^\infty \frac{dx}{x+\sqrt{x}} = \int_1^\infty 2\frac{d(\sqrt{x}+1)}{(\sqrt{x}+1)} = 2\ln|\sqrt{x}+1|\Big|_{x=1}^\infty$$

$$= \infty \text{ (diverge)}$$

32. $\frac{d}{dx}\left(\frac{1}{x(1+\ln^2 x)}\right) = -\frac{(\ln(x)+1)^2}{x^2(\ln^2(x)+1)^2}$

since $f'(x)<0$ for every $x \in \mathbb{R}$, f is a decreasing function

$$\int_1^\infty \frac{dx}{x(1+\ln^2 x)} = \int_1^\infty \frac{d(\ln(x))}{1+\ln^2(x)}$$

$$= \arctan(\ln(x))\Big|_{x=1}^\infty$$

$$= \frac{\pi}{2} \text{ (converge)}$$

37. $\frac{d}{dx}\left(\frac{8\arctan(x)}{1+x^2}\right) = -\frac{8(2x\arctan(x)-1)}{(x^2+1)^2}$

let's assume that c is the solution for $2x\arctan(x)-1=0$

$$\sum_{n=1}^\infty \frac{8\arctan(n)}{1+n^2} = \underbrace{\sum_{n=1}^{\lceil c\rceil} f(n)}_{\text{converge}} + \sum_{n=\lceil c\rceil+1}^\infty f(n)$$

Now that x and $\arctan(x)$ increases as x increases, $(2x\arctan(x)-1)>0$ for $x>c$. This shows that $f'(x)<0$ for $x>c$.

$$\int_{\lceil c\rceil+1}^\infty \frac{8\arctan(x)}{x^2+1}\,dx = 4\arctan^2(x)\Big|_{x=\lceil c\rceil+1}^\infty$$

$$= \pi^2 - 4\arctan^2(\lceil c\rceil+1)$$

(converge)

47a. Since $f(x)=\frac{1}{\sqrt{x+1}}$ is a decreasing function,

$f(a)<f(b)$ when $a>b$.

we know that $L \le \int_a^b f(x)dx \le U$ with

$$\left.\begin{array}{l} U=\sum_{n=a}^{b-1} f(n) \\ L=\sum_{n=a+1}^{b} f(n) \end{array}\right\} \text{where } f \text{ as a decreasing function}$$

and therefore $\int_0^{50} f(x)\,dx \ge \sum_{n=1}^{50} f(n)$ and.

$\int_1^{51} f(x)\,dx \le \sum_{n=1}^{50} f(n)$.

$$\therefore \int_1^{51} f(x)\,dx \le \sum_{n=1}^{50} f(n) = S_{50} \le \int_0^{50} f(x)\,dx.$$

b. to make $S_n > 1000$, $\int_1^{n+1} f(x)\,dx$ must

also greater than 1000.

$$\int_1^{n+1} f(x)\,dx = 2\sqrt{x+1}\Big|_{x=1}^{n+1} = 2\sqrt{n+2}-2\sqrt{2} > 1000$$

$$\Rightarrow \sqrt{n+2} > 500+\sqrt{2}$$

$$\Rightarrow n > 251414$$

48a. error $= \sum_{n=31}^{\infty}\frac{1}{n^4} < \int_{30}^{\infty}\frac{1}{x^4}dx \approx 1.23 \cdot 10^{-5}$

b. error $= \sum_{n=k+1}^{\infty}\frac{1}{n^4} < \int_{k}^{\infty}\frac{1}{x^4}dx \approx 1 \cdot 10^{-6}$

$$\frac{1}{3k^3} \approx 1 \cdot 10^{-6} \Rightarrow k=70$$

49, error $= \sum_{n=k+1}^{\infty}\frac{1}{n^3} < \int_{k}^{\infty}\frac{1}{x^3}dx = 0.01$

$$\frac{1}{2k^2} < 0.01$$

$$\Rightarrow k=8.$$

$$S_8 = \sum_{n=1}^{8}\frac{1}{n^3} \approx 1{,}195$$

53. denote $b_n = \sum_{k=2^n}^{2^{n+1}-1} a_n$

Since a_n is a non-increasing sequence,

$a_{2^n} \le a_{2^n-1} \le \dots \le a_{2^{n-1}+1}$.

therefore,

$L \cdot a_{2^n} \le b_n \le L \cdot a_{2^{n-1}}$ $\left(L = 2^n - 2^{n-1} = 2^{n-1}\right)$

$$\Rightarrow 2^n \cdot a_{2^n} \le \frac{1}{2} b_n \quad \forall n \ge 1$$

$$\Rightarrow 2^{n-1} \cdot a_{2^{n-1}} \ge b_n \quad \forall n \ge 2$$

$\therefore \sum_{n=2}^{\infty} b_n$ converges if and only

if $\sum_{n=2}^{\infty} 2^{n-1} \cdot a_{2^{n-1}}$ converges

(note that if $\lim_{n\to\infty} a_{2^n} = 0$,

$$\sum_{n=2}^{\infty} 2^{n-1} \cdot a_{2^{n-1}} = \sum_{n=1}^{\infty} 2^n \cdot a_{2^n})$$

55a. $\int_2^{\infty} \frac{d(\ln x)}{(\ln x)^p} = \frac{1}{1-p}(\ln x)^{1-p}\Big|_{x=2}^{\infty}$

$$= \begin{cases} \frac{1}{p-1}\ln(2)^{1-p}, & p>1 \\ \infty, & p \le 1 \end{cases}$$

b. since $\frac{1}{x(\ln x)^p}$ is a decreasing

function for $p>1$, $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^p}$

and $\int_2^{\infty}\frac{d(\ln x)}{\ln^p x}$ are both convergent

57. we know that $f(x) = \frac{1}{x}$ is a decreasing function with property:

$\Rightarrow \int_a^b f(x)\,dx \leq \sum_{n=a}^{b-1} f(n)$

$\Rightarrow \int_a^b f(x)\,dx \geq \sum_{n=a+1}^{b} f(n)$

a). $1 + \int_1^n \frac{1}{x}\,dx \geq 1 + \sum_{k=2}^{n} \frac{1}{k}$

$1 + \ln(n) \geq 1 + \frac{1}{2} + \ldots + \frac{1}{n}$ ...(1)

$\int_1^{n+1} \frac{1}{x}\,dx \leq \sum_{k=1}^{n} \frac{1}{k}$

$\ln(n+1) \leq 1 + \frac{1}{2} + \ldots + \frac{1}{n}$ ...(2)

$\ln(n+1) - \ln(n) = \ln\left(\frac{n+1}{n}\right) > \ln(1) = 0$ ..(3)

(1) & (2) & (3):

$0 < \ln(n+1) - \ln(n) \leq 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln(n) \leq 1$

b) graphically,

$\frac{1}{n+1} < \int_n^{n+1} \frac{1}{x}\,dx = \ln|x| \Big|_{x=n}^{n+1}$

$= \ln(n+1) - \ln(n)$

$\Rightarrow \frac{1}{n+1} + \ln(n) - \ln(n+1) < 0$

$\Rightarrow \left(1 + \frac{1}{2} + \ldots + \frac{1}{n+1} - \ln(n+1)\right) - \left(1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln(n)\right) < 0$

let $a_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln(n)$

$\therefore a_{n+1} < a_n$ (a_n decreases as n increases)

Exercises 10.4

1. $\sum_{n=1}^{\infty} \frac{1}{n^2+30} < \sum_{n=1}^{\infty} \frac{1}{n^2}$ (converge)

5. $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{3/2}} < \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ (converge)

9. let $a_n = \frac{n-2}{n^3-n^2+3}$, $b_n = \frac{1}{n^2}$

$\lim_{n\to\infty} \left[\frac{a_n}{b_n} = \frac{n^3-2n^2}{n^3-2n^2+3}\right] = 1$

since $\sum_{n=1}^{\infty} b_n$ converges,

$\sum_{n=1}^{\infty} a_n$ also converges

16. let $a_n = \ln(1+\frac{1}{n^2})$, $b_n = \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} n^2 \ln(1+\frac{1}{n^2})$

$= \ln(\lim_{n\to\infty} (1+\frac{1}{n^2})^{n^2})$

$= \ln(e)$

$= 1$

Since $\sum_{n=1}^{\infty} b_n$ converges, $\sum_{n=1} a_n$ also converges

19. $\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n} < \sum_{n=1}^{\infty} \frac{1}{2^n}$ (converge)

20. $\sum_{n=1}^{\infty} \frac{1+\cos n}{n^2} < \sum_{n=1}^{\infty} \frac{2}{n^2}$ (converge)

24. let $a_n = \frac{5n^3-3n}{n^2(n-2)(n^2+5)}$, $b_n = \frac{1}{n^2}$

$\lim_{n\to\infty} \left[\frac{a_n}{b_n} = \frac{5n^5-3n^3}{n^2(n-2)(n^2+5)}\right] = 5$

Since $\sum_{n=1}^{\infty} b_n$ converges, $\sum_{n=1}^{\infty} a_n$ also converges

25. $\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n < \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ (converge)

33. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}} < \sum_{n=2}^{\infty} \frac{1}{n^{3/2}}$ (converge)

36. $\sum_{n=1}^{\infty} \frac{n+2^n}{n^2 2^n} = \sum_{n=1}^{\infty} \frac{1}{n.2^n} + \sum_{n=1}^{\infty} \frac{1}{n^2}$

$< \sum_{n=1}^{\infty} \frac{1}{2^n} + \sum_{n=1}^{\infty} \frac{1}{n^2}$ (converge)

42. let $a_n = \frac{\ln n}{\sqrt{n}.e^n}$, $b_n = \frac{1}{e^n}$

$\lim_{n\to\infty} \left[\frac{a_n}{b_n} : \frac{\ln n}{\sqrt{n}}\right] = \lim_{n\to\infty} \frac{\ln(n)}{\sqrt{n}}$ "$\frac{\infty}{\infty}$" apply L'Hôpital

$= \lim_{n\to\infty} \frac{2\sqrt{n}}{n} = 0$

Since $\sum_{n=1}^{\infty} b_n$ converges, $\sum_{n=1}^{\infty} a_n$ also converges

43. note that:

$\frac{1}{n!} = \frac{1}{n(n-1)(n-2)!} \le \frac{1}{n(n-1)}$, $n \ge 2$

Since $(n-2)! \ge 1$ for $n \ge 2$

$\sum_{n=2}^{\infty} \frac{1}{n!} \le \sum_{n=2}^{\infty} \frac{1}{n(n-1)}$

let $a_n = \frac{1}{n(n-1)}$, $b_n = \frac{1}{n^2}$

$\lim_{n\to\infty} \left[\frac{a_n}{b_n} = \frac{n^2}{n^2-n}\right] = 1$

Since $\sum_{n=1}^{\infty} b_n$ converges, $\sum_{n=1}^{\infty} a_n$ also converges

46. note that:

$\tan(x) \ge x$ for $x \in [0,1]$

since $\frac{1}{n} \in [0,1]$ for $n \ge 1$,

$\sum_{n=1}^{\infty} \tan(\frac{1}{n}) \ge \sum_{n=1}^{\infty} \frac{1}{n}$ (diverge)

47. $\sum_{n=1}^{\infty} \frac{\arctan(n)}{n^{1.1}} < \sum_{n=1}^{\infty} \frac{\frac{\pi}{2}}{n^{1.1}}$ (converge)

51. let $a_n = \frac{1}{n\sqrt[n]{n}}$, $b_n = \frac{1}{n}$.

$\lim_{n\to\infty} \left[\frac{a_n}{b_n} = \frac{1}{\sqrt[n]{n}}\right] = \lim_{n\to\infty} e^{-\frac{1}{n}\ln(n)}$

$= e^{\lim_{n\to\infty} -\frac{\ln(n)}{n}}$

$= 1$

Since $\sum_{n=1}^{\infty} b_n$ diverges, $\sum_{n=1}^{\infty} a_n$ also diverges

57. → by Limit Comparison Test,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty \quad \text{show } \sum_{n=1}^{\infty} a_n \text{ diverges}$$

$$\text{if } \sum_{n=1}^{\infty} b_n \text{ diverges}$$

since $\sum_{n=1}^{\infty} a_n$ doesn't diverge, $\sum_{n=1} b_n$ also doesn't diverge (i.e. converges)

→ furthermore,

as $a_n > 0$ and $b_n > 0$ for $n \geq N$, let's consider the sigma from N to infinite.

$$\text{as } \frac{a_n}{b_n} > c > 0,$$

$$a_n > c \cdot b_n$$

$\sum_{n=N}^{\infty} a_n$ converges then $\sum_{n=N}^{\infty} b_n$ converges

58. let $c_n = a_n^2$, $d_n = a_n$

$$\lim_{n\to\infty} \left[\frac{c_n}{d_n} = a_n\right] = 0 \quad \left(\text{as } \sum_{n=1}^{\infty} a_n \text{ converges}\right)$$

since $\sum_{n=1}^{\infty} d_n$ converges, $\sum_{n=1}^{\infty} c_n$ also converges

63. $$\frac{d_1}{10} + \frac{d_2}{10^2} + \dots = \sum_{n=1}^{\infty} \frac{d_n}{10^n}$$

since $d_n \in [0,9]$

$$\sum_{n=1}^{\infty} \frac{d_n}{10^n} \leq \sum_{n=1}^{\infty} \frac{9}{10^n} = \frac{\frac{9}{10}}{1-\frac{1}{10}} = 1$$

$$\sum_{n=1}^{\infty} \frac{d_n}{10^n} \geq \sum_{n=1}^{\infty} 0 = 0$$

(converges)

64. note that:

$$\sin(x) \leq x \quad \text{for } x \geq 0$$

$$\sum_{n=1}^{\infty} \sin(a_n) \leq \sum_{n=1}^{\infty} a_n \quad (\text{since } a_n > 0)$$

(converges)

70. let $a_n = \frac{1}{\sqrt{n} \cdot \ln n}$, $b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \left[\frac{a_n}{b_n} = \frac{\sqrt{n}}{\sqrt{\ln n}}\right] = \lim_{n\to\infty} \sqrt{\frac{n}{\ln n}}$$

$$= \sqrt{\lim_{n\to\infty} \frac{n}{\ln n}}$$

$$\text{“} \frac{\infty}{\infty} \text{”}$$

apply L'Hôpital

$$= \sqrt{\lim_{n\to\infty} n}$$

$$= \infty$$

since $\sum_{n=2}^{\infty} b_n$ diverges, $\sum_{n=2}^{\infty} a_n$ also diverges