Assignment 7

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Please note that

- **Released date**: 9th Dec, Mon.
- **Due date**: 16th Dec, Thu. by 11:59pm.
- Late submission is **NOT** accepted.
- Please submit your answers as a PDF file with a name like `"118010XXX_HW7.pdf"` (Your student ID + HW No.). You may either typeset you answers directly using computers, or scan your handwritten answers. (We recommend you use the printers on campus to scan. If you use your smartphone to scan, please limit the file size $\leq$ 10MB).
- Please make sure that your submitted file is clear and readable. Submitted file that can not be opened or not readable will get 0 point.

Question 1(Slide20). Find the distance from the point $\boldsymbol{b} = (1, 1, 1)$ to the plane $2x + 2y + 2z = 0$.

Question 2(Slide20). Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be linearly independent vectors in $\mathbb{R}^n$ and let $\mathcal{S} = \text{Span}\{\boldsymbol{x}, \boldsymbol{y}\}$. Construct a matrix by $\boldsymbol{A} = \boldsymbol{x}\boldsymbol{y}^T + \boldsymbol{y}\boldsymbol{x}^T$.

(a). Show that $\boldsymbol{A}$ is symmetric.

(b). Show that $N(\boldsymbol{A}) = \mathcal{S}^{\perp}$.

Question 3(Slide20). For the following linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, find the least square solutions.

(a).

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ -1 & -2 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

(b).

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 3 \\ -1 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} -2 \\ 0 \\ 8 \end{bmatrix}$$

Question 4(Slide20). Let $\boldsymbol{P} = \boldsymbol{A}(\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T$, where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and $\text{rank}(\boldsymbol{A}) = n$.

(a). Show that $\boldsymbol{P}^2 = \boldsymbol{P}$.

(b). Show that $\boldsymbol{P}^k = \boldsymbol{P}$, for $k = 1, 2, \cdots$.

(c). Show that $\boldsymbol{P}$ is symmetric.

Question 5(Slide20). Find the equation of the circle that gives the best least squares circle fit to the points $(-1, -2), (0, 2.4), (1.1, -4)$ and $(2.4, -1.6)$.

Question 6(Slide21). Let $\boldsymbol{x} \in \mathbb{R}^n$. $\|\boldsymbol{x}\|_\infty = \max\limits_{i=1,\cdots,n} |\boldsymbol{x}_i|$ and $\|\boldsymbol{x}\|_p = (\sum_{i=1}^n |\boldsymbol{x}_i|^p)^{1/p}$.

(a). Show that $\|\boldsymbol{x}\|_\infty \leq \|\boldsymbol{x}\|_2$.

(b). Show that $\|\boldsymbol{x}\|_2 \leq \|\boldsymbol{x}\|_1$.

Question 7(Slide21). Let $\boldsymbol{x} = (-1, -1, 1, 1)^T$ and $\boldsymbol{y} = (1, 1, 5, -3)^T$. Show that $\boldsymbol{x} \perp \boldsymbol{y}$. Calculate $\|\boldsymbol{x}\|_2$, $\|\boldsymbol{y}\|_2$, $\|\boldsymbol{x} + \boldsymbol{y}\|_2$ and verify that the Pythagorean law holds.

Question 8(Slide21). Show that for any $\boldsymbol{u}$ and $\boldsymbol{v}$ in a normed vector space

$$\|\boldsymbol{u} + \boldsymbol{v}\| \geq |\|\boldsymbol{u}\| - \|\boldsymbol{v}\||$$

Question 9(Slide21). Which of the following sets of vectors form an orthonormal basis for $\mathbb{R}^2$?

(a) $\{(1, 0)^T, (0, 1)^T\}$

(b) $\{(\frac{3}{5}, \frac{4}{5})^T, (\frac{5}{13}, \frac{12}{13})^T\}$

(c) $\{(1, -1)^T, (1, 1)^T\}$

(d) $\{(\frac{\sqrt{3}}{2}, \frac{1}{2})^T, (-\frac{1}{2}, \frac{\sqrt{3}}{2})^T\}$

Question 10(Slide21). Let $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ form an orthonormal basis for $\mathbb{R}^2$ and let $\boldsymbol{u}$ be a unit vector in $\mathbb{R}^2$. If $\boldsymbol{u}^T\boldsymbol{u}_1 = \frac{1}{2}$, determine the value of $|\boldsymbol{u}^T\boldsymbol{u}_2|$.

Question 11(Slide21). Let

$$\mathbf{z_1} = \begin{bmatrix} \frac{1+i}{2} \\ \frac{1-i}{2} \end{bmatrix} \quad \text{and} \quad \mathbf{z_2} = \begin{bmatrix} \frac{i}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

(a) Show that $\{\mathbf{z_1}, \mathbf{z_2}\}$ is an orthonormal set in $\mathbb{C}^2$.

(b) Write the vector $\mathbf{z} = \begin{bmatrix} 2+4i \\ -2i \end{bmatrix}$ as a linear combination of $\mathbf{z_1}$ and $\mathbf{z_2}$.

Question 12(Slide21). An $n \times n$ matrix $\mathbf{Q}$ is said to be an orthogonal matrix if the column vectors of $\mathbf{Q}$ form an orthonormal set in $\mathbb{R}^n$. Find a, b, c, d such that:

$$Q = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} & a \\ \frac{1}{2} & -\frac{1}{2} & 0 & b \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} & c \\ \frac{1}{2} & -\frac{1}{2} & 0 & d \end{bmatrix}$$

is an orthogonal matrix.

Question 13(Slide22). Find an orthonormal basis for each of the following matrices:

$$A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 0 & 1 \\ 2 & -4 & 2 \\ 4 & 0 & 0 \end{bmatrix}$$

Question 14(Slide22). Given $x_1 = \frac{1}{2}(1, 1, 1, -1)^T$ and $x_2 = \frac{1}{6}(1, 1, 3, 5)^T$, verify that these vectors form an orthonormal set in $\mathbb{R}^4$. Extend this set to an orthonormal basis for $\mathbb{R}^4$ by finding an orthonormal basis for the null space of

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 3 & 5 \end{bmatrix}$$

Question 15(Slide22). Use Gram-Schmidt algorithm to convert the given basis $\mathcal{B}$ of $\mathcal{V}$ into an orthogonal basis.

(a) $\mathcal{V} = \mathbb{R}^2$, $\mathcal{B} = \{(2, 1), (1, 2)\}$

(b) $\mathcal{V} = \mathbb{R}^3$, $\mathcal{B} = \{(0, 1, 1), (1, 1, 1), (1, -2, 2)\}$

Question 16(Slide23). Compute the eigenvalues and eigenvectors of A and A^2:

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix}$$

A^2 has the same ______________ as A. When A has eigenvalues λ_1 and λ_2, A^2 has eigenvalues__________. In this example, give a short explanation why is $\lambda_1^2+\lambda_2^2 = 13$

Question 17(Slide23). Find the eigenvalues and the corresponding eigenvectors for each of the following matrices:

$$\text{(a) } A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad \text{(b) } B = \begin{bmatrix} 4 & -5 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}, \quad \text{(c) } C = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

Question 18(Slide23). For an $n \times n$ matrix A, let λ be an eigenvalue of A. $f(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$ is a polynomial in x, proof $f(\lambda)$ is an eigenvalue of $f(A)$.

Question 19(Slide23). Let $\mathbf{B} = \mathbf{S^{-1}AS}$ and let $\mathbf{x}$ be an eigenvector of $\mathbf{B}$ belonging to an eigenvalue λ. Show that $\mathbf{Sx}$ as an eigenvector of $\mathbf{A}$ belonging to λ.

Question 20(Slide23). The 2×2 matrix $\mathbf{B}$ has eigenvalues 1,2 and the 2×2 matrix $\mathbf{C}$ has eigenvalues 3,4 and the 2×2 matrix $\mathbf{D}$ has eigenvalues 5,7. Find the eigenvalues of the following 4×4 matrix $\mathbf{A}$:

$$A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$$