The Chinese University of Hong Kong, Shenzhen
School of Data Science

MAT3007 · Homework 6

Due: 11:59pm, Nov 17

Instructions:

- Homework problems must be carefully and clearly answered to receive full credit. Complete sentences that establish a clear logical progression are highly recommended.
- You must submit your assignment in Blackboard.
- The homework must be written in English.
- Late submission will not be graded.
- Each student **must not copy** homework solutions from another student or from any other source.

Problem 1 Optimality Conditions for Unconstrained Problem — I (20 pts).
Consider the function

$$f(x) = x_1^3 - x_2^3 + 3x_1^2 + 3x_2^2 - 9x_1$$

Use the first-order necessary condtion (FONC), second order necessary condition (SONC) and second order sufficient condition (SOSC) to find (i) saddle points, (ii) strict local minimizers and(iii) strict local maximizers.

Problem 2 Optimality Conditions for Unconstrained Problem — II (20 pts).
Consider the unconstrained optimization problem

$$\min_{x \in \mathbb{R}^2} \; f(x) = x_1^3 - x_1\left(1 + x_2^2\right) + x_2^4.$$

(a) Compute the gradient and Hessian of f and calculate all stationary points.

(b) For each stationary point, investigate whether it is a local maximizer, local minimizer, or saddle point and explain your answer.

Note: For a 2×2 Hessian, we can check the trace and determinant to verify their definiteness, as $\mathrm{tr}(Q) = \lambda_1 + \lambda_2$ and $\mathrm{let}(Q) = \lambda_1 \lambda_2$ for any matrix Q, where λ_1 and λ_2 are the two eigenvalues of Q.

Problem 3 KKT Conditions for Constrained Problem — I (20 pts).
Consider the following problem:

$$\begin{aligned} \underset{x_1, x_2 \in \mathbb{R}}{\text{minimize}} \quad & (x_1 - 4)^2 + \left(x_2 - \frac{7}{2}\right)^2, \\ \text{s.t.} \quad & x_2 - x_1^2 \geqslant 0, \\ & x_1 + x_2 \leqslant 6, \\ & x_1, x_2 \geqslant 0 \end{aligned}$$

(a) Write down the KKT optimality conditions.

(b) Find a KKT pair (x^*, λ^*) where $x^* = (x_1^*, x_2^*)$ and λ^* is the corresponding multiplier vector.

Problem 4 Failure of KKT Conditions for Constrained Problem — II (20 pts).
(**Note:** This example shows that a global minimizer may not satisfy the KKT conditions, though there are KKT points.)
Consider the optimization problem:

$$\begin{array}{ll} \text{minimize} & -x^2 + x^3 \\ \text{subject to} & x^3(x+1)^3 \leq 0. \end{array}$$

(a) Write down the KKT conditions for this problem.

(b) Find out the KKT pionts (the primal and dual variable pairs satisfying KKT conditions).

(c) Show that $x = -1$ is a global minimizer.

Problem 5 KKT Conditions for Constrained Problem — II (20 pts).
(**Note:** This problem is actually convex and any KKT points must be globally optimal, and we will study convex optimization soon.)
Consider the optimization problem:

$$\begin{array}{ll} \text{minimize} & 2x_1 + x_2 + x_3 \\ \text{subject to} & \frac{2}{x_1} + \frac{9}{x_2} + \frac{4}{x_3} \leq 1 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

(a) Write down the KKT conditions for this problem.

(b) Find the KKT points (the primal and dual variable pairs satisfying KKT conditions).