CSC4120 Spring 2024 - Written Homework 1

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Asymptotic notation

Problem 1.

Consider the following functions:

- $f_1(n) = (\log_2(n))^2$
- $f_2(n) = \log_e(2^{\log_2(n)})$
- $f_3(n) = \log_2(n!)$
- $f_4(n) = 5^{(n+\log_2(n))}$

Questions:

1. Compute the tight asymptotic bounds $\Theta(\cdot)$ for each of the above functions.
2. Rank the above functions by increasing the order of growth. Please explicitly include the comparison in your answer.

1. The tight asymptotic bounds for all of the above functions are computed as follows:
   - As $\exists c_0 = c_1 = \log_2^2(b), n_0 = 2$ such that $c_0 \log_b^2(n) \leq \log_2^2(n) \leq c_1 \log_b^2(n), \forall n \geq n_0$. Then, $f_1(n) = \Theta(\log_b^2(n))$ by definition provided that $b > 1$. The asymptotic behavior is the same regardless of the logarithm's base due to the proportional relationship between logarithms of different bases, implying $f_1(n) = \Theta(\log^2 n)$.
   - $f_2(n)$ can be simplified to be $\ln(2^{\log_2(n)}) = \ln(n)$. As $\exists c_0 = c_1 = \log_b(e), n_0 = 2$ such that $c_0 \ln(n) \leq \log_b(n) \leq c_1 \ln(n), \forall n \geq n_0$. Then, $f_2(n) = \Theta(\log_b(n))$ by definition provided that $b > 1$. The asymptotic behavior is the same regardless of the logarithm's base due to the proportional relationship between logarithms of different bases, implying $f_2(n) = \Theta(\log n)$.
   - By Stirling's approximation, $f_3(n) = \log_2(n!) \approx \log_2(\sqrt{2\pi n}(\frac{n}{e})^n) = \frac{1}{2} + \frac{\log_2(\pi)}{2} + \frac{\log_2(n)}{2} + n\log_2(n) - n\log_2(e)$.
     Let $g(n) = \frac{1}{2} + \frac{\log_2(\pi)}{2} + \frac{\log_2(n)}{2} - n\log_2(e)$, then:
     $$\frac{dg(n)}{dn} = \frac{1-2n}{2\ln(2)n}$$
     Notice that $g(-1) < 0$ and $\frac{dg(n)}{dn} < 0$ when $n \geq 1$ due to division of negative number with positive number; therefore, $g(n) < 0, \forall n \geq 1$. This implies that $\exists c_0 = 1, n_0 = 1$ such that $f_3(n) = n\log_2(n) + g(n) \leq c_0 n \log_2(n), \forall n \geq n_0$. By definition, $f_3(n) = \mathcal{O}(n \log_2 n)$.
     Let $h(n) = g(n) + \frac{1}{2} n \log_2(n)$, then:
     $$\frac{dh(n)}{dn} = \frac{1-2n}{2\ln(2)n} + \frac{\ln(n)+1}{2\ln(2)}$$

Notice that $h(3) > 0$ and $\frac{dh(n)}{dn} > 0$ when $n \geq 2$ due to outgrow by $n\ln(n)$ to $2n$; therefore, $h(n) > 0, \forall n \geq 3$. This implies that $\exists c_0 = \frac{1}{2}, n_0 = 3$ such that $f_3(n) = h(n) + \frac{1}{2}n\log_2(n) \geq c_0 n\log_2(n), \forall n \geq 3$. By definition, $f_3(n) = \Omega(n\log_2 n)$.
As $f_3(n) = \mathcal{O}(n\log_2 n)$ and $f_3(n) = \Omega(n\log_2 n)$, $f_3(n) = \Theta(n\log_2 n) = \Theta(n\log n)$.

- $f_4(n)$ can be simplified to be $5^n \cdot (5^{\log_5(2)})^{\log_2(n^{\log_2(5)})} = 5^n \cdot n^{\log_2(5)}$. $\exists c_0 = c_1 = 1, n_0 = 2$ such that $c_0 5^n \cdot n^{\log_2(5)} \leq f(n) \leq c_1 5^n \cdot n^{\log_2(5)}, \forall n \geq n_0$. By definition, $f_4(n) = \Theta(5^n \cdot n^{\log_2(5)})$.

2. The increasing order of growth for the above functions is $f_2(n), f_1(n), f_3(n), f_4(n)$. The comparisons are included as follows:

   - $\lim_{n\to\infty} \frac{f_2(n)}{f_1(n)} = \lim_{n\to\infty} \frac{\ln n}{\log_2^2 n} = \lim_{n\to\infty} \frac{\ln(2)}{\log_2(n)} = 0$, which indicates that $f_2(n)$'s growth is slower than $f_1(n)$'s.
   - $\lim_{n\to\infty} \frac{f_1(n)}{f_3(n)} \leq \lim_{n\to\infty} \frac{f_1(n)}{\frac{1}{2}n\log_2(n)} = \lim_{n\to\infty} \frac{2\log_2(n)}{n} = 0$ as n outgrows $\log_2(n)$. As $f_1(n) > 0$ and $f_3(n) > 0$ when $n > 1$, $\lim_{n\to\infty} \frac{f_1(n)}{f_3(n)} = 0$, which indicates that $f_1(n)$'s growth is slower than $f_3(n)$'s.
   - $\lim_{n\to\infty} \frac{f_3(n)}{f_4(n)} \leq \lim_{n\to\infty} \frac{n\log_2(n)}{5^n n^{\log_2(5)}} = \lim_{n\to\infty} \frac{1}{5^n} \lim_{n\to\infty} \frac{1}{n^{\log_2(5)-2}} \lim_{n\to\infty} \frac{\log_2(n)}{n} = 0 \cdot 0 \cdot 0 = 0$ due to n outgrowing $\log_2(n)$. As $f_3(n) > 0$ and $f_4(n) > 0$ when $n > 1$, $\lim_{n\to\infty} \frac{f_3(n)}{f_4(n)} = 0$, which indicates that $f_3(n)$'s growth is slower than $f_4(n)$'s.

Problem 2.

Consider the following functions:

- $f_1(n) = (n^{\frac{1}{\log_2(n)}})$
- $f_2(n) = (\log_2(n))!$
- $f_3(n) = \log_e(\log_e(n))$
- $f_4(n) = 2^{\sqrt{2\log_2(n)}}$

Are the statements (a) - (c) true? Indicate the reason.

(a) The asymptotic complexity of f_1 is $\Theta(n^{\frac{1}{\log_2(n)}})$

(b) The asymptotic complexity of f_1 is $\mathcal{O}(1)$

(c) The asymptotic complexity of f_2 is $\Theta([(\log_2(n)]^{(\log_2(n)+\frac{1}{2})} \cdot e^{-\log_2(n)})$

(d) Rank the above functions in terms of increasing order of growth. Please include the comparison in your answer explicitly

(a) As $\exists c_0 = 1, c_1 = 1, n_0 = 2$ such that $c_0 n^{\frac{1}{\log_2(n)}} \leq f_1(n) \leq c_1 n^{\frac{1}{\log_2(n)}}, \forall n \geq n_0$, the statement "The asymptotic complexity of f_1 is $\Theta(n^{\frac{1}{\log_2(n)}})$" is true by definition.

(b) As $\exists c_0 = 2, n_0 = 1$ such that $f_1(n) = n^{\frac{1}{\log_2(n)}} = n^{\log_n(2)} = 2 \leq c_0, \forall n \geq n_0$, the statement "The asymptotic complexity of f_1 is $\mathcal{O}(1)$" is true by definition.

(c) By Stirling's approximation, $f_2(n) = (\log_2(n))! \approx \sqrt{2\pi \log_2(n)}(\frac{\log_2(n)}{e})^{\log_2(n)}$. Let $g(n) = \log_2(n)^{\log_2(n)+\frac{1}{2}} \cdot e^{-\log_2(n)}$. As $\exists c_0 = 2, c_1 = 3, n_0 = 2$ such that $c_0 g(n) \leq f(n) \leq c_1 g(n), \forall n \geq n_0$, the statement "The asymptotic complexity of f_2 is $\Theta([(\log_2(n)]^{(\log_2(n)+\frac{1}{2})} \cdot e^{-\log_2(n)})$" is true.

(d) The increasing order of growth for the above functions is $f_1(n), f_3(n), f_4(n), f_2(n)$. The comparisons are included as follows:

- $\lim_{n\to\infty} \frac{f_1(n)}{f_3(n)} = \lim_{n\to\infty} \frac{n^{\frac{1}{\log_2(n)}}}{\ln\ln(n)} = \lim_{n\to\infty} \frac{2}{\ln\ln(n)} = 0$, which indicates that $f_1(n)$'s growth is slower than $f_3(n)$'s.
- $\lim_{n\to\infty} \frac{f_3(n)}{f_4(n)} = \lim_{n\to\infty} \frac{\ln\ln(n)}{2^{\sqrt{2\log_2(n)}}} \underset{\text{by L'Hôpital's rule}}{=} \lim_{n\to\infty} \frac{1}{\sqrt{\ln(n)\ln(2)} \cdot 2^{\sqrt{2\log_2(n)} - \frac{1}{2}}} = 0$, which indicates that $f_3(n)$'s growth is slower than $f_4(n)$'s.
- $\lim_{n\to\infty} \frac{f_4(n)}{f_2(n)} \leq \lim_{n\to\infty} \frac{n^2}{f_2(n)}$ as $n \geq \sqrt{2} \Rightarrow \sqrt{\log_2(n^2)} \leq \log_2(n^2) \Rightarrow f_4(n) \leq n^2$. Claim that $\forall m \geq 13$, $(m-1)! = \Gamma(m) > (2^m)^2 = 4^m$. Prove by induction:
  - Base case:
    As $\Gamma(x)$ is increasing when $x > 1$, the claim holds true for $\forall m \in [13, 14)$, $(m-1)! \geq 12! = 479001600 > 4^{14} = 268435456$.
  - Inductive step:
    Assume that $\forall m \in [k, k+1)$, $(k-1)! > 4^{k+1}$ is true. Then $\forall m \in [k+1, k+2)$, $k! = k \cdot (k-1)! > k \cdot 4^{k+1} > 4^{k+2}$ is also true as $k \geq 13 > 4$. Hence, proven.

  Let $m = \log_2(n)$, then $(\log_2(n) - 1)! > 4^{\log_2(n)} = n^2$. Then, $\lim_{n\to\infty} \frac{n^2}{\log_2(n)!} = \lim_{n\to\infty} \frac{1}{\log_2(n)} \frac{n^2}{(\log_2(n)-1)!} = 0 \cdot c = 0$ for some finite constant c. This is supported by the fact that $\lim_{n\to\infty} \log_2(n) = +\infty \geq 13$. The previous implies that $\lim_{n\to\infty} \frac{f_4(n)}{f_2(n)} \leq 0$. As $f_4(n) > 0$ and $f_2(n) > 0$ when $n > 2$, $\lim_{n\to\infty} \frac{f_4(n)}{f_2(n)} = 0$, which indicates that $f_4(n)$'s growth is slower than $f_2(n)$'s.

Problem 3.

Let $f(n)$ and $g(n)$ be asymptotically non-negative functions. Prove that if $h(n) = \max\{f(n), g(n)\}$, then $h(n) = \Theta(f(n) + g(n))$.

Suppose $h(n) \neq \Theta(f(n) + g(n))$, which implies that there is no $c_0, c_1 > 0, n_0$ such that $c_0(f(n) + g(n)) \leq h(n) \leq c_1(f(n) + g(n)), \forall n \geq n_0$. However, if $h(n)$ were to be $\max\{f(n), g(n)\}$:

- If $f(n) \leq g(n)$, then $\exists c_0 = \frac{1}{2}, c_1 = 1, n_0$ that satisfies:
  - $\frac{1}{2}(f(n) + g(n)) \leq \frac{1}{2}2g(n) = g(n) = \max\{f(n), g(n)\} = h(n), \forall n \geq n_0$
  - $h(n) = \max\{f(n), g(n)\} = g(n) \leq f(n) + g(n), \forall n \geq n_0$

- If $f(n) > g(n)$, then $\exists c_0 = \frac{1}{2}, c_1 = 1, n_0$ that satisfies:
    - $\frac{1}{2}(f(n) + g(n)) \leq \frac{1}{2}2f(n) = f(n) = \max\{f(n), g(n)\} = h(n), \forall n \geq n_0$
    - $h(n) = \max\{f(n), g(n)\} = f(n) \leq f(n) + g(n), \forall n \geq n_0$

Those contradict our supposition about $h(n) \neq \Theta(f(n) + g(n))$; consequently, $h(n)$ can't be $\max\{f(n), g(n)\}$. As it has been shown that $h(n) \neq \Theta(f(n)+g(n)) \Rightarrow h(n) \neq \max\{f(n), g(n)\}$, the equivalent claim follows $h(n) = \max\{f(n), g(n)\} \Rightarrow h(n) = \Theta(f(n) + g(n))$; hence, proven.

Problem 4.

Suppose that our list of size n is being sorted with insertion sort, and each element is at most k away from its final position (k-sorted list). What is the asymptotic complexity if

a) $k = \mathcal{O}(1)$,

b) $k = \frac{n}{2}$, and why?

Insertion sort works as follows:

1. Iterates from 1 to $n - 1$ and compare each element to its predecessor.
2. If the comparison shows the correct sort, then the algorithm ends.
3. Otherwise, shift the position of the found element until it finds the smaller element as its predecessor.
4. Repeat.

(a) As each element is at most k away from its final position and $\exists c_0, n_0$ such that $k \leq c_0, \forall n \geq n_0$, we can conclude that the asymptotic complexity is $\mathcal{O}(n)$. This is true because there will be at most k comparisons for each of the $n - 1$ elements. Let $g(n)$ be such a function, then choose c_0, n_0 that establishes the definition of k being $\mathcal{O}(1)$, implying $g(n) \leq k(n - 1) \leq kn \leq c_0 n, \forall n \geq n_0$. By definition, $g(n) = \mathcal{O}(n)$.

(b) In a similar way, as each element is at most $k = \frac{n}{2}$ away from its final position, we can conclude that the asymptotic complexity is $\mathcal{O}(n^2)$. As $\exists c_1 = \frac{1}{2}, n_1$ such that $k \leq c_1 n, \forall n \geq n_1$, the previous conclusion is true because there will be at most k comparisons for each of the $n - 1$ elements. Let $h(n)$ be such a function, then choose c_1, n_1 that establishes the definition of k being $\mathcal{O}(n)$, implying $h(n) \leq k(n-1) \leq kn \leq c_1 n^2, \forall n \geq n_1$. By definition, $h(n) = \mathcal{O}(n^2)$. Notice that the asymptotic complexity uses $\mathcal{O}$ instead of Θ as the problem uses the term "at most".

Recurrences

Problem 5.

Solve $T(n) = T(\sqrt{n}) + 1$ and give a Θ bound.

Let $n = 2^m$ and $S(m) = T(2^m)$, then:

$$T(2^m) = T(2^{\frac{m}{2}}) + 1 \Rightarrow S(m) = S\left(\frac{m}{2}\right) + 1 \underset{\text{by Master's Theorem}}{\Rightarrow} S(m) = \Theta(\log m)$$

Therefore, $T(n) = T(2^m) = S(m) = \Theta(\log m) = \Theta(\log \log n)$.

Problem 6.

Are the following statements true? Indicate the reason.

(a) The solution of $T(n) = T(n-1) + n$ is $\mathcal{O}(n^2)$.

(b) The solution of $T(n) = 2T(n-1) + 1$ is $\mathcal{O}(n \log n)$ or $\mathcal{O}(2^n)$.

(c) The recurrence $T(n) = T(\frac{n}{3}) + T(\frac{2n}{3}) + cn$ $(c > 0)$ is

I. $\Omega(n \log n)$
II. $\mathcal{O}(n \log n)$

(d) The recurrence $T(n) = 4T(\frac{n}{2}) + cn^2\sqrt{n}$ $(c > 0)$ is

I. $\Theta(n^2)$
II. $\Theta(n^2\sqrt{n})$

(e) The recurrence $T(n) = T(\frac{n}{2}) + T(\frac{n}{4}) + T(\frac{n}{8}) + n$ is

I. $\Theta(n \log n)$
II. $\Theta(n)$

(a) The expansion of $T(n)$ is as follows:

$$\begin{aligned}
T(n) &= T(n-1) + n \\
&= T(n-2) + (n-1) + n \\
&\vdots \\
&= \Theta(1) + 2 + \ldots + (n-1) + n \\
&= \mathcal{O}(1) + 2 + \ldots + (n-1) + n
\end{aligned}$$

This implies $\exists c_0, n_0$ such that:

$$\begin{aligned} T(n) &\leq c_0 + 2 + \ldots + (n-1) + n, \forall n \geq n_0 \\ &= c_0 - 1 + \frac{n(n+1)}{2}, \forall n \geq n_0 \\ &\leq c_0 + n^2, \forall n \geq \max(1, n_0) \end{aligned}$$

We can choose $c_1 = \max(1, c_0) + \lim_{\epsilon \to 0^+} \epsilon$ and $n_1 = \max(1, n_0, \sqrt{\frac{c_1}{c_1 - 1}})$ such that $T(n) \leq c_1 n^2, \forall n \geq n_1 \underset{\text{by definition}}{\Rightarrow} T(n) = \mathcal{O}(n^2)$. This further implies that the statement "The solution of $T(n) = T(n-1) + n$ is $\mathcal{O}(n^2)$" is true.

(b) The expansion of $T(n)$ is as follows:

$$\begin{aligned} T(n) &= 2T(n-1) + 1 \\ &= 4T(n-2) + 2 + 1 \\ &\vdots \\ &= 2^{n-1}\Theta(1) + 2^{n-2} + \ldots + 2 + 1 \end{aligned}$$

- $T(n) = 2^{n-1}\Theta(1) + 2^{n-2} + \ldots + 2 + 1$ implies $\exists c_0, n_0$ such that:

$$\begin{aligned} T(n) &\geq c_0 2^{n-1} + 2^{n-2} + \ldots + 2 + 1, \forall n \geq n_0 \\ &\geq 2^{n-2}, \forall n \geq n_0 \end{aligned}$$

  For a function $f(n) = 2^{n-2} - cn\log n$, it can be claimed that $\lim_{n\to\infty} f(n) = +\infty$ as $\lim_{n\to\infty} \frac{2^{n-2}}{cn\log n} = +\infty$ provided that $c > 0$. When $c \leq 0$, $f(n)$ is simply a positive function due to the addition of two positive values. Both cases lead to a conclusion where $2^{n-2} > cn\log n$ when $n \to \infty$; therefore, one can't find such c_1, n_1 such that $T(n) \leq c_1 n \log n, \forall n \geq n_1$. The statement "The solution of $T(n) = 2T(n-1) + 1$ is $\mathcal{O}(n\log n)$" is false.

- $T(n) = 2^{n-1}\Theta(1) + 2^{n-2} + \ldots + 2 + 1$ implies $\exists c_0, n_0$ such that:

$$\begin{aligned} T(n) &\leq c_0 2^{n-1} + 2^{n-2} + \ldots + 2 + 1, \forall n \geq n_0 \\ &\leq \max(1, c_0)(2^{n-1} + 2^{n-2} + \ldots + 2 + 1), \forall n \geq n_0 \\ &\leq \max(1, c_0)\ 2^n, \forall n \geq n_0 \end{aligned}$$

  We can choose $c_1 = \max(1, c_0)$ and $n_1 = \max(1, n_0)$ such that $T(n) \leq c_1 2^n, \forall n \geq n_1 \underset{\text{by definition}}{\Rightarrow} T(n) = \mathcal{O}(2^n)$. This further implies that the statement "The solution of $T(n) = 2T(n-1) + 1$ is $\mathcal{O}(2^n)$" is true.

(c) Both statements will be proved using the substitution method.

I. Claim that $T(n) \geq c_0 n \log n, \forall n \geq n_0$. Then,

$$
\begin{aligned}
T(n) &= T(\frac{n}{3}) + T(\frac{2n}{3}) + cn \\
&\geq c_0 \frac{n}{3} \log(\frac{n}{3}) + c_o \frac{2n}{3} \log(\frac{2n}{3}) + cn, \forall n \geq n_0 \\
&\geq c_0 n \log(\frac{n}{3}) + cn, \forall n \geq n_0 \\
&\geq c_0 n \log n - c_0 n \log 3 + cn, \forall n \geq n_0
\end{aligned}
$$

The claim is correct when $cn \geq c_0 n \log 3 \Rightarrow c_0 \leq \frac{c}{\log 3}$. The statement "The recurrence $T(n) = T(\frac{n}{3}) + T(\frac{2n}{3}) + cn$ $(c > 0)$ is $\Omega(n \log n)$" is true as $\exists c_0$ (such as $\frac{c}{\log 3}$) and $n_0 > 1$ that holds the claim.

II. Claim that $T(n) \leq c_0 n \log n, \forall n \geq n_0$. Then,

$$
\begin{aligned}
T(n) &= T(\frac{n}{3}) + T(\frac{2n}{3}) + cn \\
&\leq c_0 \frac{n}{3} \log(\frac{n}{3}) + c_0 \frac{2n}{3} \log(\frac{2n}{3}) + cn, \forall n \geq n_0 \\
&\leq c_0 n \log(\frac{2n}{3}) + cn, \forall n \geq n_0 \\
&\leq c_0 n \log n + c_0 n \log 2 - c_0 n \log 3 + cn, \forall n \geq n_0
\end{aligned}
$$

The claim is correct when $c_0 n \log 2 + cn \leq c_0 n \log 3 \Rightarrow c_0 \geq \frac{c}{\log 3 - \log 2}$. The statement "The recurrence $T(n) = T(\frac{n}{3}) + T(\frac{2n}{3}) + cn$ $(c > 0)$ is $\mathcal{O}(n \log n)$" is true as $\exists c_0$ (such as $\frac{c}{\log 3 - \log 2}$) and $n_0 > 1$ that holds the claim.

(d) By Master's Theorem, $T(n) = \Theta(n^2\sqrt{n})$. Therefore, the statement "The recurrence $T(n) = 4T(\frac{n}{2}) + cn^2\sqrt{n}$ $(c > 0)$ is $\Theta(n^2\sqrt{n})$" is true.

The previous implies that $\exists c_0, n_0$ such that $T(n) \geq c_0 n^2 \sqrt{n}, \forall n \geq n_0$. Assume that $\exists c_1, n_1$ such that $T(n) \leq c_1 n^2, \forall n \geq n_1$. For $n > \max(1, n_0, n_1, (\frac{c_1}{c_0})^2)$, the condition $c_0 n^2\sqrt{n} > c_1 n^2$ invalidates the possibility of $T(n)$ satisfying both $\Omega(n^2\sqrt{n})$ and $\mathcal{O}(n^2)$ simultaneously. Consequently, the statement "The recurrence $T(n) = 4T(\frac{n}{2}) + cn^2\sqrt{n}$ $(c > 0)$ is $\Theta(n^2)$" must be false.

(e) Claim that $T(n) \leq c_0 n, \forall n \geq n_0$. By substitution method,

$$
\begin{aligned}
T(n) &= T(\frac{n}{2}) + T(\frac{n}{4}) + T(\frac{n}{8}) + n \\
&\leq c_0 \frac{n}{2} + c_0 \frac{n}{4} + c_0 \frac{n}{8} + n \\
&\leq \frac{7c_0}{8} n + n
\end{aligned}
$$

The claim is correct when $\frac{7c_0}{8} n + n \leq c_0 n \Rightarrow c_0 \geq 8$. Therefore, $T(n) = \mathcal{O}(n)$ as $\exists c_0$ (such as $c_0 = 8$) and $n_0 > 1$ that holds the claim.

Assume $\exists c_1, n_1$ such that $T(n) \geq c_1 n \log n, \forall n \geq n_1$. For $n > \max(1, n_0, n_1, \frac{c_0}{c_1})$, the condition $c_1 n \log n > c_0 n$ invalidates the possibility of $T(n)$ satisfying both $\mathcal{O}(n)$ and $\Omega(n \log n)$ simultaneously. Consequently, the statement "The recurrence $T(n) = T(\frac{n}{2}) + T(\frac{n}{4}) + T(\frac{n}{8}) + n$ is $\Theta(n \log n)$" must be false.

Similarly, claim that $T(n) \geq c_1 n, \forall n \geq n_1$. By substitution method,

$$\begin{aligned} T(n) &= T(\frac{n}{2}) + T(\frac{n}{4}) + T(\frac{n}{8}) + n \\ &\geq c_1 \frac{n}{2} + c_1 \frac{n}{4} + c_1 \frac{n}{8} + n \\ &\geq \frac{7c_1}{8} n + n \end{aligned}$$

The claim is correct when $\frac{7c_1}{8} n + n \geq c_1 n \Rightarrow c_1 \leq 8$. Therefore, $T(n) = \Omega(n)$ as $\exists c_1$ (such as $c_1 = 8$) and $n_1 > 1$ that holds the claim. $T(n) = \mathcal{O}(n)$ and $T(n) = \Omega(n)$ is equivalent to $T(n) = \Theta(n)$. With that, the statement "The recurrence $T(n) = T(\frac{n}{2}) + T(\frac{n}{4}) + T(\frac{n}{8}) + n$ is $\Theta(n)$" is true.