The Chinese University of Hong Kong, Shenzhen
School of Data Science

MAT3007 · Homework 2

Due: 11:59pm, September 28 (Thursday), 2023

Instructions:

- Homework problems must be carefully and clearly answered to receive full credit. Complete sentences that establish a clear logical progression are highly recommended.
- Please submit your assignment on Blackboard.
- The homework must be written in English.
- Late submission will not be graded.
- Each student must not copy homework solutions from another student or from any other source.
- For those questions that ask you to write MATLAB/Python codes to solve the problem. Please attach the code to the homework. You also need to clearly state (write or type) the optimal solution and the optimal value you obtained. However, you do not need to attach the outputs in the command window of MATLAB/Python.

Problem 1 (15pts). True or False

Consider an LP in its standard form and the corresponding constraint set $P = \{x | Ax = b, x \geq 0\}$. Suppose that the matrix A has dimensions $m \times n$ and that its rows are linearly independent. For each of the following statements, state whether it is true or false. Please explain your answers (if not true, please show a counterexample).

(a) The set of all optimal solutions (assuming existence) must be bounded;

(b) At every optimal solution, no more than m variables can be positive;

(c) If there is more than one optimal solution, then there are uncountably many optimal solutions.

Problem 2 (20pts). Graphical Method

Solve the following 2-dimensional linear optimization problem using the graphical method.

$$\begin{aligned} \text{maximize} \quad & x_1 + x_2 \\ \text{s.t.} \quad & -x_1 + x_2 \leq 2.5 \\ & x_1 + 2x_2 \leq 9 \\ & 0 \leq x_1 \leq 4 \\ & 0 \leq x_2 \leq 3 \end{aligned}$$

Which constraints are active at optimal solution? Also list all the vertices of the feasible region.

Problem 3 (20pts). Basic solutions and basic feasible solutions
Consider the following linear optimization problem:

$$\begin{aligned} \text{maximize} \quad & x_1 + 2x_2 + 4x_3 \\ \text{s.t.} \quad & x_1 + x_3 \le 8 \\ & x_2 + 2x_3 \le 15 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

(a) Transform it into standard form;

(b) Argue without solving this LP that there must exist an optimal solution with no more than 2 positive variables;

(c) List all the basic solutions and basic feasible solutions (of the standard form);

(d) Find the optimal solution by using the results in step (c).

Problem 4 (20pts). Vertex Covering Problem
Write a MATLAB/Python code to solve the vertex covering problem discussed in our Lecture (the graph is in Figure 1). When you solve it, use constraints $0 \le x_i \le 1$ rather than $x_i \in \{0, 1\}$. What are the optimal solution and optimal value returned by CVX (suppose we label the variables as $x_a, ..., x_j$)? What is the optimal value of the true problem? So whether one can remove the integer constraint when solving this problem?

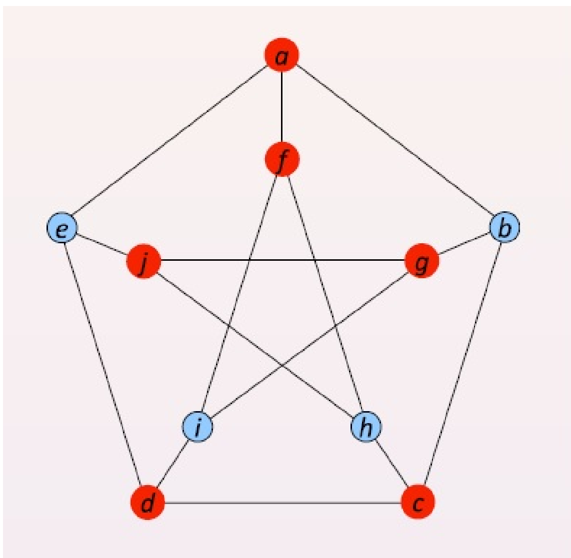

Figure 1: Graph for vertex covering

Problem 3 (25pts). A Robust LP Formulation
In this exercise, we consider the following optimization problem:

$$\min_{x\in\mathbb{R}^n} \; c^\top x \quad \text{subject to} \quad \|Ax-b\|_\infty \le \delta, \quad x \ge 0 \tag{1}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, and $\delta \ge 0$ are given and $\|y\|_\infty = \max_{1\le i\le p} |y_i|$ denotes the maximum norm of a vector $y \in \mathbb{R}^p$. In the case $\delta = 0$, problem (1) coincides with the standard form for linear programs. The choice $\delta > 0$ can be useful to model situations where A and/or b are not fully or exactly known, e.g., when A and/or b contains certain uncertainty (can be caused by noise). In this case, problem (1) belongs to the so-called robust optimization.

(a) Rewrite the optimization problem (1) as a linear problem.

(b) We now consider a specific application of problem (1).

The fruit store in Kuai Le Shi Jian is producing two different fruit salads A and B. The smaller fruit salad A consists of "1/4 mango, 1/8 pineapple, 3 strawberries"; the larger fruit salad B consists of "1/2 mango, 1/4 pineapple, 1 strawberry". The profits per fruit salad and the total number of fruits in stock are summarized in the following table:

	Mango	Pineapple	Strawberry	Net profit
Fruit salad A	1/4	1/8	3	10 RMB
Fruit salad B	1/2	1/4	1	20 RMB
Stock / Resources	25	10	120	

Suppose all fruits need to be processed and *completely used* to make the fruit salads A and B.

Given these constraints, formulate a linear program to maximize the total profits of the fruit store. Show that this program can be expressed in standard form

$$\min_{x\in\mathbb{R}^n} \; c^\top x \quad \text{subject to} \quad Ax = b, \quad x \ge 0,$$

with $n = 2$ and $m = 3$. In addition, is this linear programming solvable?

Note: Since we want to produce "complete fruit salads", the variables x_1 and x_2 should actually be modeled as integer variables: $x_1, x_2 \in \mathbb{Z}$. However, since we do not know how to deal with these integer constraints in general at this moment, you may just ignore them for now.

(c) One of the employee found some additional fruits in a storage crate and the manager of the fruit shop decides to determine the production plan by using the robust formulation (1). Consider the robust variant of the problem in part (b) with $\delta = 5$.

- Sketch the feasible set of this problem.
- Solve the problem graphically, i.e., Calculate the optimal value and the optimal solution set.
- Which constraints are active in the solution?
- Find one integer solution of this problem.