Assignment 2

Li Lei, Ma Xianping
leili@link.cuhk.edu.cn, 221019087@link.cuhk.edu.cn

Please note that

- **Released date**: 12.00 a.m., 9.29
- **Due date**: 11.59 p.m., 10.14
- **To be responsible**: Li Lei(1-10), Ma Xianping(11-20)
- Late submission is **NOT** accepted.
- Please submit your answers as a PDF file with a name like `"118010XXX_HW2.pdf"` (Your student ID + HW No.). You may either typeset you answers directly using computers, or scan your handwritten answers. (We recommend you use the printers on campus to scan. If you use your smartphone to scan, please limit the file size $\leq$ 10MB).
- Please make sure that your submitted file is clear and readable. Submitted file that can not be opened or not readable will get 0 point.

Question 1. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$, calculate $(\mathbf{AB})^{10}$.

Question 2. Let $\mathbf{a}_1 = [1,1]^T, \mathbf{a}_2 = [1,-1]^T$, and matrix $\mathbf{A} == \begin{bmatrix} -2 & 6 \\ 0 & 4 \end{bmatrix}$. If $\mathbf{A} = k_1\mathbf{a}_1\mathbf{a}_1^T + k_2\mathbf{a}_1\mathbf{a}_2^T + k_3\mathbf{a}_2\mathbf{a}_1^T + k_4\mathbf{a}_2\mathbf{a}_2^T$, what are the values of $k_i, i = 1, \dots, 4$.

Question 3. Consider a 4×5 matrix $\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5]$. The reduced row-echelon form of $\mathbf{A}$ is

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 3 & 0 & -2 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{1}$$

Moreover, $\mathbf{x}_0 = (3, 2, 0, 2, 0)^T$ is a solution to $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{b} = (0, 5, 3, 4)^T$.

(a) Find the solution set of $\mathbf{Ax} = \mathbf{b}$.

(b) Recover $\mathbf{A}$ if we already know that $\mathbf{a}_1 = (2, 1, -3, -2)^T$ and $\mathbf{a}_2 = (-1, 2, 3, 1)^T$.

Question 4. Let $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 6 & 4 \end{bmatrix}$, express $\mathbf{A}$ as product of elementary matrices.

Question 5. Compute the matrix-matrix multiplication

(a) $diag(1, -2, 4) \begin{bmatrix} 3 & 4 & 1 \\ -2 & 0 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 4 & 1 \\ -2 & 0 & 1 \\ 1 & 2 & 2 \end{bmatrix} diag(1, -2, 4)$.

Question 6. Let $\mathbf{A} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. WHatcompute $\mathbf{A}^2, \mathbf{A}^3$. What will $\mathbf{A}^n$ be?

Question 7. Given $\mathbf{A} \in \mathbb{R}^{2\times 2}$. If $\mathbf{A}$ commutes with $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, show that if $\mathbf{A} = \begin{bmatrix} a & 0 \\ c & a \end{bmatrix}$ for some a and c.

Question 8. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$. Show that there is no elementary matrix $\mathbf{E}$ such that $\mathbf{C} = \mathbf{EA}$.

Question 9. Compute $\mathbf{AB}$ by using the following block partitioning.

$$\mathbf{A} = \left[\begin{array}{cc|cc} 2 & -1 & 3 & 1 \\ 1 & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \mathbf{B} = \left[\begin{array}{cc|c} 1 & 2 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 5 & 1 \\ 1 & -1 & 0 \end{array}\right]$$

Question 10. Compute the following by using block multiplication (all blocks are $k \times k$).

1. $\begin{bmatrix} \mathbf{I} & \mathbf{X} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{-X} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$
2. $[\mathbf{I} \quad \mathbf{X}^T][\mathbf{-X} \quad \mathbf{I}]^T$

Question 11. For any $\mathbf{A} \in \mathbb{R}^{n\times n}, \mathbf{B} \in \mathbb{R}^{n\times n}$, show that $\mathbf{AB} = \mathbf{BA}$ if and only if $(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}) = (\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{B})$.

Question 12. If $\mathbf{A}$ and $\mathbf{B}$ are symmetric, show that $\mathbf{AB}$ is symmetric if and only if $\mathbf{AB} = \mathbf{BA}$.

Question 13. If $\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$, shows that $\mathbf{A}^3 = \mathbf{I}$ and so find $\mathbf{A}^{-1}$.

Question 14. Find $\mathbf{A}$ if $(\mathbf{A}^T - 2\mathbf{I})^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$

Question 15. (a)Prove that: let $\mathbf{A}$ be a nonsingular matrix. Show that $\mathbf{A}^{-1}$ is also nonsingular and $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.

(b)Prove that: let $\mathbf{A}$ be a nonsingular matrix. Show that $\mathbf{A}^T$ is also nonsingular and $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$.

Question 16. Give $\mathbf{A}^{-1} = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 5 \\ -1 & 1 & 0 \end{bmatrix}$, Find a matrix $\mathbf{B}$ such that $\mathbf{AB} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

Question 17. Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}.$$

(a) Write the elementary matrices $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$ from the given $\mathbf{A}$ under the condition that $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$ contain $-a$, $-b$ and $-c$ respectively.

(b) Multiply $\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1$ to find the single matrix $\mathbf{E}$ that produces $\mathbf{EA} = \mathbf{I}$.

(c) Multiply $\mathbf{E}_1^{-1}\mathbf{E}_2^{-1}\mathbf{E}_3^{-1}$ to bring back $\mathbf{A}$. You will see that the multipliers a, b, c are mixed up in $\mathbf{E}$ but perfect in $\mathbf{A}$.

Question 18. Find an $\mathbf{LU}$-factorization for

(a) $\begin{bmatrix} 2 & 4 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{bmatrix}$.

(b) $\begin{bmatrix} -1 & -3 & 1 & 0 & -1 \\ 1 & 4 & 1 & 1 & 1 \\ 1 & 2 & -3 & -1 & 1 \\ 0 & -2 & -4 & -2 & 0 \end{bmatrix}$.

Question 19. Find a permutation matrix $\mathbf{P}$ and an $\mathbf{LU}$ decomposition of $\mathbf{PA}$ if $\mathbf{A}$ is $\begin{bmatrix} 0 & -1 & 2 \\ 0 & 0 & 4 \\ -1 & 2 & 1 \end{bmatrix}$.

Question 20. Solve the following system of equations using $\mathbf{LU}$ decomposition.

$$\begin{cases} 3x_1 + 3x_2 + x_3 - 4x_4 = 5 \\ 3x_1 + 5x_2 - 1x_3 - 3x_4 = 5 \\ -9x_1 - 3x_2 - 4x_3 + 16x_4 = -5 \\ 15x_1 + 13x_2 - 8x_3 - 21x_4 = -5 \end{cases}$$