MAT 3007 – Optimization

Assignment 7

Due: 11:59pm, Nov. 24 (Friday), 2023

Instructions:

- Homework problems must be carefully and clearly answered to receive full credit. Complete sentences that establish a clear logical progression are highly recommended.
- Please submit your assignment on Blackboard.
- The homework must be written in English.
- Late submission will not be graded.
- Each student must not copy homework solutions from another student or from any other source.

Problem 1 (Convex Sets): (approx. *25* points)
In this exercise, we study convexity of various sets.

a) Verify whether the following sets are convex or not and explain your answer!

$$\Omega_1 = \{x \in \mathbb{R}^n : \alpha \le (a^\top x)^2 \le \beta\}, \quad \alpha, \beta \in \mathbb{R},\ 0 < \alpha \le \beta,\ a \in \mathbb{R}^n,$$
$$\Omega_2 = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : x^\top x \le t^2\}.$$

b) Decide whether the following statements are true or false. Explain your answer and either present a proof / verification or a counter-example.

- The intersection of two convex sets $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ is always a convex set.
- Let $\Omega \subset \mathbb{R}^n$ be a convex set and suppose that the set $S := \{(x,t) \in \Omega \times \mathbb{R} : f(x) \le t\} \subset \mathbb{R}^n \times \mathbb{R}$ is convex. Then, $f : \Omega \to \mathbb{R}$ is a convex function.

Problem 2 (Convex Compositions): (approx. *20* points)
Either prove or find a counterexample for each of the following statements (you can assume that all functions are twice continuously differentiable if needed):

a) If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are concave, then the composition $f \circ g : \mathbb{R}^n \to \mathbb{R}$, $(f \circ g)(x) = f(g(x))$ is concave.

b) Let $\Omega \subset \mathbb{R}^n$ be a convex set and suppose that $g : \Omega \to \mathbb{R}$ is concave and $f : I \to \mathbb{R}$ is concave and nondecreasing where $I \supseteq g(\Omega)$ is an interval containing $g(\Omega)$. Then, $f \circ g$ is convex.

c) If $f : \mathbb{R} \to \mathbb{R}$ is convex, then $x \mapsto |f(x)|$ is a convex function on $\mathbb{R}$.

Problem 3 (Convex Functions): (approx. *30* points)

In this exercise, convexity properties of different functions are investigated.

a) Let $r : \mathbb{R}^n \to \mathbb{R}$ be defined as $r(x) = \max_i |x_i|$. Show that r is a convex function.

b) Verify that the following functions are convex over the specified domain:

- $f : \mathbb{R} \times \mathbb{R}_{++} \to \mathbb{R}$, $f(x) := x_1^2/x_2$, where $\mathbb{R}_{++} := \{x \in \mathbb{R} : x > 0\}$.
- $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) := \frac{1}{2}\|Ax - b\|^2 + \mu\|Lx\|_\infty$, where $A \in \mathbb{R}^{m\times n}$, $L \in \mathbb{R}^{p\times n}$, $b \in \mathbb{R}^m$, and $\mu > 0$ are given and $\|y\|_\infty := \max_{i=1,\dots,p} |y_i|$, $y \in \mathbb{R}^p$.
- $f : \mathbb{R}^{n+1} \to \mathbb{R}$, $f(x, y) := \frac{\lambda}{2}\|x\|^2 + \sum_{i=1}^m \max\{0, 1 - b_i(a_i^\top x + y))\}$, where $a_i \in \mathbb{R}^n$ and $b_i \in \{-1, 1\}$ are given data points for all $i = 1, ..., m$ and $\lambda > 0$ is a parameter.

c) Let us set $f(x) = \|x\|_1 := \sum_{i=1}^n |x_i|$ and define $g : \mathbb{R}^n \to \mathbb{R}$, $g(x) := \max_{y\in\mathbb{R}^n} y^\top x - f(y)$.

Calculate $g(x)$ explicitly and verify that the function g is convex.

Problem 4: (approx. *25* points)

a) Let $A \in \mathbb{R}^{4\times 4}$ be a symmetric matrix with nonnegative components and $A1 = 1$, i.e., each row of the matrix A has sum 1. Prove that $I - A$ is positive semidefinite.

b) Let $a \in \mathbb{R}^n$. We define $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) := \log(1 + \exp(a^\top x))$. Show that f is a convex function.

c) Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, and $d \in \mathbb{R}$. Prove that the nonconvex optimization problem

$$\begin{array}{ll} \min\limits_{x\in\mathbb{R}^n} & \dfrac{\|Ax - b\|}{c^\top x + d} \\ \text{subject to} & \|x\| \le 1, c^T x + d > 0 \end{array} \tag{1}$$

is equivalent to the convex optimization problem

$$\begin{array}{ll} \min\limits_{y\in\mathbb{R}^n, t} & \|Ay - bt\| \\ \text{subject to} & \|y\| \le t \\ & c^\top y + dt = 1 \end{array} \tag{2}$$

d) Use `CVX` (in `MATLAB` or `Python`) to solve problem (2) with

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad c = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \quad d = 1$$