STA2001 Assignment 9: Functions of Single RV, Multi RVs and the MGF technique

1. 5.1-4. The pdf of X is $f(x) = 2x$, $0 < x < 1$.
   (a) Find the cdf of X.
   (b) Describe how an observation of X can be simulated.
   (c) Simulate 10 observations of X.

2. 5.1-9. Statisticians frequently use the **extreme value distribution** given by the cdf

$$F(x) = 1 - \exp[-e^{(x-\theta_1)/\theta_2}], \quad -\infty < x < \infty.$$

A simple case is when $\theta_1 = 0$ and $\theta_2 = 1$, giving

$$F(x) = 1 - \exp[-e^{x}], \quad -\infty < x < \infty.$$

Let $Y = e^X$ or $X = \ln Y$; then the support of Y is $0 < y < \infty$.

(a) Show that the distribution of Y is exponential when $\theta_1 = 0$ and $\theta_2 = 1$.

(b) Find the cdf and the pdf of Y when $\theta_1 \neq 0$ and $\theta_2 > 0$.

(c) Let $\theta_1 = \ln\beta$ and $\theta_2 = 1/\alpha$ in the cdf and pdf of Y. What is this distribution?

(d) As suggested by its name, the extreme value distribution can be used to model the longest home run, the deepest mine, the greatest flood, and so on. Suppose the length X (in feet) of the maximum of someones home runs was modeled by an extreme value distribution with $\theta_1 = 550$ and $\theta_2 = 25$.What is the probability that X exceeds 500 feet?

3. 5.1-13. If the distribution of X is $N(\mu, \sigma^2)$, then $M(t) = E(e^{tX}) = \exp(\mu t + \sigma^2 t^2/2)$. We then say that $Y = e^X$ has a lognormal distribution because X = lnY.

   (a) Show that the pdf of Y is

$$g(y) = \frac{1}{y\sqrt{2\pi\sigma^2}} \exp[-(\ln y - \mu)^2/2\sigma^2], \quad 0 < y < \infty.$$

   (b) Using M(t), find (i) $E(Y) = E(e^X) = M(1)$, (ii) $E(Y^2) = E(e^{2X}) = M(2)$, and (iii) $\mathrm{Var}(Y)$.

4. 5.3-11. Let X_1, X_2, X_3 be three independent random variables with binomial distributions $b(4, 1/2)$, $b(6, 1/3)$, and $b(12, 1/6)$, respectively. Find

   (a) $P(X_1 = 2, X_2 = 2, X_3 = 5)$.

   (b) $E(X_1X_2X_3)$.

   (c) The mean and the variance of $Y = X_1 + X_2 + X_3$.

5. 5.3-17. In considering medical insurance for a certain operation, let X equal the amount (in dollars) paid for the doctor and let Y equal the amount paid to the hospital. In the past, the variances have been $\mathrm{Var}(X) = 8100$, $\mathrm{Var}(Y) = 10,000$, and $\mathrm{Var}(X + Y) = 20,000$. Due to increased expenses, it was decided to increase the doctors fee by \$500 and increase the hospital charge Y by 8%. Calculate the variance of $X + 500 + (1.08)Y$, the new total claim.

6. 5.3-19. Two components operate in parallel in a device, so the device fails when and only when both components fail. The lifetimes, X_1 and X_2, of the respective components are independent and identically distributed with an exponential distribution with $\theta = 2$. The cost of operating the device is $Z = 2Y_1 + Y_2$, where $Y_1 = \min(X_1, X_2)$ and $Y_2 = \max(X_1, X_2)$. Compute E(Z).

7. 5.3-21. Flip $n = 8$ fair coins and remove all that came up heads. Flip the remaining coins (that came up tails) and remove the heads again. Continue flipping the remaining coins until each has come up heads. We shall find the pmf of Y, the number of trials needed. Let X_i equal the number of flips required to observe heads on coin i, $i = 1, 2, ..., 8$. Then $Y = \max(X_1, X_2, \ldots, X_8)$.

   (a) Show that $P(Y \leq y) = [1 - (1/2)^y]^8$.

   (b) Show that the pmf of Y is defined by $P(Y = y) = [1 - (1/2)^y]^8 - [1 - (1/2)^{y-1}]^8$, $y = 1, 2, \ldots$

   (c) ~~Use a computer algebra system such as Maple or Mathematica to show that the mean of Y is~~ $\cancel{E(Y) = 13,315,424/3,011,805 = 4.421}$. **No need to do this one**

   (d) What happens to the expected value of Y as the number of coins is doubled?

8. 5.4-3. Let X_1, X_2, X_3 be mutually independent random variables with Poisson distributions having means 2, 1, and 4, respectively.

   (a) Find the mgf of the sum $Y = X_1 + X_2 + X_3$.

   (b) How is Y distributed ?

   (c) Compute $P(3 \leq Y \leq 9)$.

9. 5.4-8. Let $W = X_1 + X_2 + \cdots + X_h$, a sum of h mutually independent and identically distributed exponential random variables with mean θ. Show that W has a gamma distribution with parameters $\alpha = h$ and θ, respectively.

10. 5.4-9. Let X and Y, with respective pmfs $f(x)$ and $g(y)$, be independent discrete random variables, each of whose support is a subset of the nonnegative integers $0, 1, 2, \ldots$ Show that the pmf of $W = X + Y$ is given by the **convolution formula**

$$h(w) = \sum_{x=0}^{w} f(x)g(w-x), \quad w = 0, 1, 2, \ldots$$

Hint: Argue that $h(w) = P(W = w)$ is the probability of the $w + 1$ mutually exclusive events $(x, y = w - x)$, $x = 0, 1, \ldots, w$.