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MAT2040 - Homework 4

1 (a) $\begin{bmatrix} \boxed{1} & 1 & 1 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & \boxed{1} \end{bmatrix}$ since 1^{st}, 2^{nd}, & 3^{rd} columns are the pivots, the bases are $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

since there are 3 bases that form matrix A, $\dim(A) = 3 \Rightarrow A \subseteq \mathbb{R}^3$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}$

, since 1^{st} & 3^{rd} columns are the pivots, the bases are $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

since there are less than 3 bases that form matrix A, $\dim(A) \neq 3 \Rightarrow A \not\subseteq \mathbb{R}3$

(c) $\begin{bmatrix} 1 & 3 & -3 \\ 0 & 2 & -5 \\ -2 & -4 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 3 & -3 \\ 0 & \boxed{2} & -5 \\ 0 & 0 & 0 \end{bmatrix}$

since 1^{st} & 2^{nd} columns are the pivots, the bases are $\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} \right\}$

since there are less than 3 bases that form matrix A, $\dim(A) \neq 3 \Rightarrow A \not\subseteq \mathbb{R}3$

2. $\begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 0 & 6 & 5 \\ 0 & \boxed{2} & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

here, x_3 & x_4 are the variables that free

$$x = x_3 \begin{bmatrix} -6 \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix}$$

3. convert polynomials to systems of matrix

(a) $\begin{bmatrix} 0 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}$

hence, the dimension is 3

(b) $\begin{bmatrix} 0 & -1 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 0 & 0 & -2 \\ 0 & \boxed{1} & 0 & 2 \\ 0 & 0 & \boxed{1} & 1 \end{bmatrix}$

hence, the dimension is 3

(c) $\begin{bmatrix} 0 & -1 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & -1 \\ 0 & 0 & 0 \end{bmatrix}$

hence, the dimension is 2

(d) $\begin{bmatrix} 0 & -2 \\ 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

hence, the dimension is 2

4. Transition Matrix $= V^{-1}U$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 1 \\ 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$$

5(a) Transition Matrix $= U^{-1}V$

$$= \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 1 \\ 7 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & -2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

(b) $[x]_u = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ -2 \end{bmatrix}$

6. Transition Matrix B to C $= C^{-1}B$

$$= \begin{bmatrix} 1 & -2 \\ -5 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 7 & -3 \\ 5 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}$$

Transition Matrix C to B $= B^{-1}C$

$$= \begin{bmatrix} 7 & -3 \\ 5 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -2 \\ -5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix}$$

7(a) $\begin{bmatrix} \boxed{1} & -3 & 3 \\ 0 & \boxed{4} & -6 \\ 0 & 0 & \boxed{3} \end{bmatrix}$

since the 1^{st}, 2^{nd}, & 3^{rd} columns are the pivots, the bases are $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix} \right\}$.

this implies that B spans $\mathbb{R}^3$

(b) $[x]_B = B^{-1}Ex = \begin{bmatrix} 1 & -3 & 3 \\ 0 & 4 & -6 \\ 0 & 0 & 3 \end{bmatrix}^{-1} \begin{bmatrix} -8 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix}$

(c) $[y]_E = E^{-1}By = \begin{bmatrix} 1 & -3 & 3 \\ 0 & 4 & -6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -10 \\ 9 \end{bmatrix}$

8. Null(A) is the solution set of $Ax=0$

$$\begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 0 & 6 & 5 \\ 0 & \boxed{2} & .5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x = x_3 \begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \text{basis} = \left\{ \begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

9(a) $[A|b] = \left[\begin{array}{cc|c} 3 & 6 & 1 \\ 1 & 2 & 1 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & -2 \end{array}\right]$

since the system is inconsistent, b is not in the column space of A

(b) $[A|b] = \left[\begin{array}{cc|c} 0 & 1 & 2 \\ 1 & 0 & 5 \\ 0 & 1 & 2 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$

since the system is consistent, b is in the column space of A

(c) $[A|b] = \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 2 & 4 & 10 \\ 1 & 2 & 5 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$

since the system is consistent, b is in the column space of A

10 (a) $\begin{bmatrix} 1 & 2 & 3 & 5 & 0 & 2 & 4 \\ 2 & 1 & 3 & 4 & 7 & 8 & 9 \\ 1 & 1 & 2 & 3 & 2 & 2 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & -14 & 14 \\ 0 & 1 & 1 & 2 & 0 & 8 & -5 \\ 0 & 0 & 0 & 0 & 1 & 4 & -2 \end{bmatrix}$

since 1^{st}, 2^{nd}, and 5^{th} columns are the pivots, then:

$$\text{Col}(A) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix} \right\}$$

(b) $$x = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 14 \\ -8 \\ 0 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_7 \begin{bmatrix} -14 \\ 5 \\ 0 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Null}(A) = \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 14 \\ -8 \\ 0 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -14 \\ 5 \\ 0 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$\dim(\text{Null}(A)) = 4$

11(a) $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 4 & 7 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\text{Row}(A) = \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \right\}$

$\text{Col}(A) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix} \right\}$

$\text{Null}(A) = \left\{ \begin{matrix} -2 \\ 0 \\ 1 \end{matrix} \right\}$

(b) $\begin{bmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -\frac{10}{7} \\ 0 & 1 & 0 & -\frac{2}{7} \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$\text{Row}(A) = \left\{ \begin{bmatrix} -3 \\ 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 8 \\ 4 \\ 2 \end{bmatrix} \right\}$

$\text{Col}(A) = \left\{ \begin{bmatrix} -3 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} \right\}$

$\text{Null}(A) = \left\{ \begin{bmatrix} \frac{10}{7} \\ \frac{2}{7} \\ 0 \\ 1 \end{bmatrix} \right\}$

(c) $\begin{bmatrix} 1 & 3 & -2 & 1 \\ 2 & 1 & 3 & 2 \\ 3 & 4 & 5 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -\frac{13}{20} \\ 0 & 1 & 0 & \frac{21}{20} \\ 0 & 0 & 1 & \frac{3}{4} \end{bmatrix}$

$\text{Row}(A) = \left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} \right\}$

$\text{Col}(A) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix} \right\}$

$\text{Null}(A) = \left\{ \begin{bmatrix} \frac{13}{20} \\ -\frac{21}{20} \\ -\frac{3}{4} \\ 1 \end{bmatrix} \right\}$

12. $\begin{bmatrix} 1 & 2 & 2 & 3 & 1 & 4 \\ 2 & 4 & 5 & 5 & 4 & 9 \\ 3 & 6 & 7 & 8 & 5 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 5 & -3 & 0 \\ 0 & 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

since 1st, 3rd, and 6th columns are the pivots, $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 9 \\ 9 \end{bmatrix}$ form ~~column~~ column space of A. conversely, $\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$ form the dependent variables.

$\rightarrow \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}$

13(a) According to the theorem,

$\dim(A) + \dim(N(A)) = n$

where A is a $m \times n$ matrix.

since $N(A) = \{0\} \Rightarrow \dim(N(A)) = 0$

$\Rightarrow \dim(A) = n$

This shows that all columns in A are linearly independent.

This also shows that the column space of A doesn't span $\mathbb{R}^m$

(b) case b is not in the column space of A:

$\Rightarrow$ there doesn't exist any combinations of $c_1a_1 + c_2a_2 + \dots + c_na_n = b$

$\Rightarrow$ no solution for $Ax=b$

case b is in the column space of A:

$\Rightarrow$ the solution x for $Ax=b$ is unique.

proof.

assume x_1 & x_2 are the solutions of $Ax=b$ such that $x_1 = x_2$.

consider:

~~$\Rightarrow A(x-y)$~~

$\Rightarrow A(x_1-x_2) = Ax_1 - Ax = b-b = 0$

$\Rightarrow x_1 = x_2$

which contradicts with our assumptions

14. $a_3 = 2a_1 + a_2 = 2\begin{bmatrix}-3\\5\\2\\1\end{bmatrix} + \begin{bmatrix}4\\-3\\3\\-1\end{bmatrix} = \begin{bmatrix}-2\\7\\11\\1\end{bmatrix}$

$a_4 = a_1 + 4a_2 = \begin{bmatrix}-3\\5\\2\\1\end{bmatrix} + 4\begin{bmatrix}4\\-3\\3\\-1\end{bmatrix} = \begin{bmatrix}13\\-7\\30\\-3\end{bmatrix}$

15. $U = \begin{bmatrix}1&0&1&0&2\\0&1&2&0&-1\\0&0&0&1&3\\0&0&0&0&0\end{bmatrix}$

16(a) According to the theorem,

$\dim(A) + \dim(N(A)) = n$

where A is a $m \times n$ matrix.

$\Rightarrow \dim(N(A)) = n - \dim(A) = n - \text{rank}(A)$ ~~A~~

$\Rightarrow \dim(N(A)) = 3-3 = 0$

$\Rightarrow N(A) = \{0\}$

(b) $\Rightarrow \alpha y_1 + \beta y_2 + \gamma y_3 = 0$, $\alpha, \beta, \gamma \in \mathbb{R}$

$\Rightarrow \alpha(Ax_1) + \beta(Ax_2) + \gamma(Ax_3) = 0$

$\Rightarrow A(\alpha x_1) + A(\beta x_2) + A(\gamma x_3) = 0$

$\Rightarrow A(\alpha x_1 + \beta x_2 + \gamma x_3) = 0$

$\Rightarrow \alpha x_1 + \beta x_2 + \gamma x_3 = 0$

$\Rightarrow (\alpha, \beta, \gamma) = (0,0,0)$ since $\{x_1, x_2, x_3\}$ are linearly independent

$\Rightarrow \{y_1, y_2, y_3\}$ are linearly independent

(c) $\dim(\mathbb{R}^5) = 5$

$\Rightarrow$ every basis $\mathbb{R}^5$ has 5 elements

$\Rightarrow \{y_1, y_2, y_3\}$ doesn't span $\mathbb{R}^5$

$\Rightarrow \{y_1, y_2, y_3\}$ doesn't form a basis for $\mathbb{R}^5$

17. $A_{m\times n}$ has rank equal to n

let $A = [a_1\ a_2\ \dots\ a_n]$ and $x = (x_1, \dots, x_n)^T \neq 0$

$\Rightarrow$ assume $y = 0$, $\{a_1, a_2, \dots a_n\}$ is linearly independent,

$Ax = y \Rightarrow Ax = 0$

$\Rightarrow [a_1\ \dots\ a_n]\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix} = 0$

$\Rightarrow x_1a_1 + \dots + x_na_n = 0$

$\Rightarrow \{a_1, \dots, a_n\}$ is linearly ~~independent~~ dependent

which contradicts

$\Rightarrow y = 0$

18(a) let $x \in N(BA)$.

$\Rightarrow$ B is non singular

$\Rightarrow Bx = 0$ has only trivial solution

$\Rightarrow x \in N(BA)$

$\Rightarrow BAx = 0$

$\Rightarrow B(Ax) = 0$

$\Rightarrow Ax = 0$

$\Rightarrow x \in N(A)$

$\Rightarrow N(BA) \subseteq N(A)$

let $x \in N(A)$

$\Rightarrow Ax=0$

$\Rightarrow B(Ax)=0$

$\Rightarrow (BA)x=0$

$\Rightarrow x \in N(BA)$

$\Rightarrow N(A) \subseteq N(BA)$

all those imply $N(A)=N(BA)$

$\Rightarrow \dim(N(A)) = \dim(N(BA))$

$\Rightarrow n-\text{rank}(A) = n-\text{rank}(BA)$

$\Rightarrow \text{rank}(A) = \text{rank}(BA)$

(b) C is nonsingular $\Leftrightarrow C^T$ is non singular

$\Rightarrow \text{rank}(A) = \text{rank}(A^T)$
$= \text{rank}(C^T A^T)$
$= \text{rank}(AC)$

19. $N(A-B) \in \mathbb{R}^n$

$\Rightarrow (A-B)x=0 \quad , x \in \mathbb{R}^n$

$\Rightarrow (A-B)=0$

$\Rightarrow A=B$

20.(a) let $x=[x_1 \ldots x_m]^T \neq 0$ &
$y^T=[y_1 \ldots y_n] \neq 0^T$

$\Rightarrow A = xy^T$
$= \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} y^T$
$= \begin{bmatrix} x_1 y^T \\ \vdots \\ x_m y^T \end{bmatrix}$

$\Rightarrow \text{Row}(A) = \text{span}\{x_1 y^T, \ldots, x_m y^T\}$

$\Rightarrow \text{Row}(A) = \text{span}\{y^T\}$

$\Rightarrow y^T$ is a basis for row space A

$\Rightarrow A = xy^T$
$= x[y_1 \ldots y_n]$
$= [y_1 x \ldots y_n x]$

$\Rightarrow \text{Col}(A) = \text{span}\{y_1 x, \ldots, y_n x\}$

$\Rightarrow \text{Col}(A) = \text{span}\{x\}$

$\Rightarrow x$ is a basis for column space A

(b) $x \in \mathbb{R}^m, y \in \mathbb{R}^n \Rightarrow xy^T$ is a $m \times n$ matrix
in (a) we proved y^T is basis for row space $\Rightarrow \text{rank}(A)=1$

~~$\Rightarrow N(A) = n - \text{rank}(A)$~~

$\Rightarrow \dim(N(A)) = n - \text{rank}(A)$
$= n-1$