Assignment 3

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Slide 8: Q1-Q4, Slide 9: Q5-Q6, Slides 6 & 8: Q7-Q10

Slide 10: Q11-Q15, Slide 11: Q16-Q18, Slide 12: Q19-Q20

Please note that

- **Released date: 15th Oct., Fri.**
- **Due date: 28th Oct., Thur., by 11:59pm.**
- Late submission is **NOT** accepted.
- Please submit your answers as a PDF file with a name like "120010XXX ASS1.pdf" (Your student ID + ASS No.). You may either typeset you answers directly using computers, or scan your handwritten answers. (We recommend you use the printers on campus to scan. If you use your smartphone to scan, please limit the file size 10MB.)

Question 1. Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$, if it exists

Question 2. Find the inverse of the matrix $A = \begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}$ and solve the system

$$\begin{cases} 8x_1 + 6x_2 = 2 \\ 5x_1 + 4x_2 = -1 \end{cases}$$

Question 3. Find A if $(A^T - I)^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$

Question 4. Is matrix $A = \begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}$ invertible? If yes, find the inverse of A. If not, show the reason. (Hint:Perform Gauss-Jordan elimination)

Question 5. As for the linear system

$$\begin{cases} x_1 + 3x_2 - 5x_3 - 2x_4 = 0 \\ -3x_1 - 2x_2 + x_3 + x_4 = 0 \\ -11x_1 - 5x_2 - x_3 + 2x_4 = 0 \\ 5x_1 + x_2 + 3x_3 = 0 \end{cases},$$

find the homogeneous solutions.

Question 6. As for the linear system

$$\begin{cases} x_1 + 2x_2 - 3x_3 - 4x_4 = -5 \\ 3x_1 - x_2 + 5x_3 + 6x_4 = -1 \\ -5x_1 - 3x_2 + x_3 + 2x_4 = 11 \\ -9x_1 - 4x_2 - x_3 = 17 \end{cases},$$

find the particular solution and homogeneous solution.

Question 7. Let A^N=0 for a certain integer $N > 0$, prove that $(I - A)^{-1} = I + A + A^2 + \cdots + A^{N-1}$.

Question 8. Show that if A is symmetric nonsingular matrix, then A^{-1} is also symmetric.

Question 9. Let $A = \begin{bmatrix} 2 & 1 \\ 6 & 4 \end{bmatrix}$.

(1)Express A as product of elementary matrices.

(2)Express A^{-1} as product of elementary matrices.

Question 10. Show that an $n \times n$ system $Ax = b$ has a unique solution if and only if A is nonsingular.

Question 11. Determine whether the following vectors are linearly independent:

(a) $\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

(b) $\begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

(c) $\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

(d) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

(e) $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$

Question 12. Find a maximal linearly independent subset of the following set:

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -5 \\ 9 \\ -4 \\ -7 \end{bmatrix} \right\}.$$

Question 13. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ be linearly independent vectors in $\mathbb{R}^n$ and let

$$\mathbf{y}_1 = \mathbf{x}_2 - \mathbf{x}_1, \mathbf{y}_2 = \mathbf{x}_3 - \mathbf{x}_2, \mathbf{y}_3 = \mathbf{x}_3 - \mathbf{x}_1$$

Are $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$ linearly independent? Prove your answer.

Question 14. Let $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k$ be linearly independent vectors in a vector space V.

(a) If we add a vector $\mathbf{x}_{k+1}$ to the collection, will we still have a linearly independent collection of vectors? Explain.

(b) If we delete a vector, say, $\mathbf{x}_k$, from the collection, will we still have a linearly independent collection of vectors? Explain.

Question 15. Given

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 2 \\ 6 \\ 6 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} -9 \\ -2 \\ 5 \end{bmatrix}.$$

(a) Is $\mathbf{x} \in \text{Span}(\mathbf{x}_1, \mathbf{x}_2)$? why?

(b) Is $\mathbf{y} \in \text{Span}(\mathbf{x}_1, \mathbf{x}_2)$? why?

Question 16. Suppose $(x_1, x_2) + (y_1, y_2)$ is defined to be $(x_1 + y_2, x_2 + y_1)$. With the usual multiplication $c\boldsymbol{x} = (cx_1, cx_2)$, which of the eight axioms of a vector space (see Definition 11.1 in Lecture 11) are **not** satisfied?

Question 17. Determine if the set $S = \{(x, y, z)^T | xy = z^2\}$ is a subspace of $\mathbb{R}^3$.

Question 18. Assume U, V are subspaces of a vector space W. Define $U + V = \{\mathbf{z} = \mathbf{u} + \mathbf{v} | \mathbf{u} \in U, \mathbf{v} \in V\}$, i.e. each vector in $U + V$ is the sum of one vector in U and one vector in V. Prove that $U + V$ is a subspace of W.

Question 19. Find the basis for

$$V = \operatorname{Span}\left(\begin{bmatrix} 1 \\ 0 \\ -3 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 6 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \\ 1 \\ 2 \end{bmatrix} \right).$$

Question 20. Let

$$V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 \middle| x_1 + x_2 + x_3 + x_4 = 0 \right\}.$$

By the definition of basis, show that

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

is a basis for V. Also find out the dimension of V.