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~~EIE2050 Assignment 8~~

MAT2040 Assignment 7

1. $D = \dfrac{|2\cdot 1+2\cdot 1+2\cdot 1|}{\sqrt{2^2+2^2+2^2}} = \sqrt{3}$

2 a) $A^T = (xy^T+yx^T)^T$
$= (xy^T)^T + (yx^T)^T$
$= yx^T + xy^T$
$= A$

b) $z \in N(A) \iff Az = 0$
$= (xy^T+yx^T)z = 0$
$= x(y^Tz)+y(x^Tz) = 0$
$\iff y^Tz = x^Tz = 0$
$\iff z \in S^\perp$

$\Rightarrow N(A) = S^\perp$

3a) $A^TAx = A^Tb$
$$\begin{bmatrix}1&2&-1\\2&4&-2\end{bmatrix}\begin{bmatrix}1&2\\2&4\\-1&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$
$$= \begin{bmatrix}1&2&-1\\2&4&-2\end{bmatrix}\begin{bmatrix}3\\2\\1\end{bmatrix}$$
$$\Rightarrow \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} + \alpha\begin{bmatrix}2\\1\end{bmatrix}$$

b) $A^TAx = A^Tb$
$$\begin{bmatrix}1&-1&1\\1&3&2\\3&1&4\end{bmatrix}\begin{bmatrix}1&1&3\\-1&3&1\\1&2&4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$
$$= \begin{bmatrix}1&-1&1\\1&3&2\\3&1&4\end{bmatrix}\begin{bmatrix}-2\\0\\8\end{bmatrix}$$
$$\Rightarrow \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}2\\1\\0\end{bmatrix} + \alpha\begin{bmatrix}-2\\-1\\1\end{bmatrix}$$

4a) $P^2 = A\underbrace{(A^TA)^{-1}A^T\cdot A}_{I}(A^TA)^{-1}A^T$
$= A(A^TA)^{-1}A^T$
$= P$ (shown)

b) When $k=1$,
$P^k = P^1 = P$ (true)
assume $P^k = P$ is true for $k=n$
when $k=n+1$,
$P^{n+1} = P^n\cdot P = P\cdot P = P^2 = P$ (by part (a))
$\therefore$ shown

c) $P^T = (A(A^TA)^{-1}A^T)^T$
$= A((A^TA)^{-1})^TA^T$
$= A((A^TA)^T)^{-1}A^T$
$= A(A^TA)^{-1}A^T$
$= P$
$\therefore P$ is symmetric

5. $F: (x-c_1)^2+(y-c_2)^2 = r^2$
$x^2+y^2-2xc_1-2yc_2+\underbrace{c_1^2+c_2^2-r^2}_{c_3} = 0$

$(-1,-2) \mapsto F \Rightarrow 5+2c_1+4c_2+c_3=0$
$(0,2.4) \mapsto F \Rightarrow (2.4)^2-4.8c_2+c_3=0$
$(1.1,-4) \mapsto F \Rightarrow (1.1)^2+(-4)^2-2.2c_1+8c_2+c_3=0$
$(2.4,-1.6) \mapsto F \Rightarrow (2.4)^2+(-1.6)^2-4.8c_1+3.2c_2+c_3=0$

$$\Rightarrow \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0.575\\-0.643\\-6.602\end{bmatrix} \Rightarrow r = 2.725$$

$\Rightarrow (x-0.575)^2+(y+0.643)^2 = 2.725^2$

6. a) $|x_i|^2 \le \sum_{j=1}^n |x_j|^2$ for every $i=1,\dots,n$

$\Rightarrow |x_i| = \sqrt{|x_i|^2} \le \sqrt{\sum_{j=1}^n |x_j|^2}$ for every $i=1,\dots,n$

$\Rightarrow \max_{1\le i\le n} |x_i| \le \|x\|_2$

$\Rightarrow \|x\|_\infty \le \|x\|_2$ (shown)

b) $(|x_1|+\dots+|x_n|)^2 \ge x_1^2+\dots+x_n^2$
$\Rightarrow \left(\sum_{i=1}^n |x_i|\right)^2 \ge \sum_{i=1}^n x_i^2$
$\Rightarrow \sum_{i=1}^n |x_i| \ge \left(\sum_{i=1}^n x_i^2\right)^{1/2}$
$\Rightarrow \|x\|_1 \ge \|x\|_2$ (shown)

7. $x \cdot y = (-1)\cdot 1 + (-1)\cdot 1 + 1\cdot 5 + 1\cdot(-3) = 0$

$\Rightarrow x \perp y$

$\|x\|_2 = \sqrt{(-1)^2+(-1)^2+1^2+1^2} = 2$

$\|y\|_2 = \sqrt{1^2+1^2+5^2+(-3)^2} = 6$

$\|x+y\|_2 = \sqrt{0^2+0^2+6^2+(-2)^2} = 2\sqrt{10}$

Since $2^2+6^2 = (2\sqrt{10})^2$,

$\Rightarrow \|x\|_2^2 + \|y\|_2^2 = \|x+y\|_2^2$

$\Rightarrow$ Pythagorean law holds

8. $\|u\| = \|u+v-v\| \le \|u+v\| + \|-v\|$ --- (1)

$\|v\| = \|v+u-u\| \le \|u+v\| + \|-u\|$ --- (2)

$\|-a\| = \|a\|$ for every vector a --- (3)

by (1), (2), & (3),

$\|u+v\| \ge \|u\| - \|v\|$

&

$\|u+v\| \ge \|v\| - \|u\|$

$\Rightarrow \|u+v\| \ge \big| \|u\| - \|v\| \big|$

9a) $(1,0)^T \cdot (0,1)^T = 0$ & $\|(1,0)\| = \|(0,1)\| = 1$

$\Rightarrow \{(1,0)^T, (0,1)^T\}$ forms an orthonormal basis

b) $(\frac{3}{5}, \frac{4}{5})^T \cdot (\frac{5}{13}, \frac{12}{13})^T \neq 0$

$\Rightarrow \{(\frac{3}{5}, \frac{4}{5})^T, (\frac{5}{13}, \frac{12}{13})^T\}$ doesn't form an orthonormal basis

c) $(1,-1)^T \cdot (1,1)^T = 0$ & $\|(1,-1)\| = \|(1,1)\| \neq 1$

$\Rightarrow \{(1,-1)^T, (1,1)^T\}$ doesn't form an orthonormal basis

d) $(\frac{\sqrt{3}}{2}, \frac{1}{2})^T \cdot (-\frac{1}{2}, \frac{\sqrt{3}}{2})^T = 0$ & $\|(\frac{\sqrt{3}}{2}, \frac{1}{2})\| = \|(-\frac{1}{2}, \frac{\sqrt{3}}{2})\| = 1$

$\Rightarrow \{(\frac{\sqrt{3}}{2}, \frac{1}{2})^T, (-\frac{1}{2}, \frac{\sqrt{3}}{2})^T\}$ forms an orthonormal basis

10. $u = c_1 \cdot u_1 + c_2 \cdot u_2 = (u^T u_1)\cdot u_1 + (u^T u_2)\cdot u_2$

$\|u\| = (u^T u_1)^2 + (u^T u_2)^2$

$\Rightarrow 1 = \frac{1}{4} + (u^T u_2)^2$

$\Rightarrow |u^T u_2| = \frac{\sqrt{3}}{2}$

11. a) $(\frac{1+i}{2}, \frac{1-i}{2})^T \cdot (\frac{i}{\sqrt{2}}, -\frac{1}{\sqrt{2}})^T = 0$

&

$\|(\frac{1+i}{2}, \frac{1-i}{2})^T\| = \|(\frac{i}{\sqrt{2}}, \frac{-1}{\sqrt{2}})^T\| = 1$

$\Rightarrow \{z_1, z_2\}$ is an orthonormal set in $\mathbb{C}^2$

b) $z = \begin{bmatrix} 2+4i \\ -2i \end{bmatrix} = c_1 \begin{bmatrix} \frac{1+i}{2} \\ \frac{1-i}{2} \end{bmatrix} + c_2 \begin{bmatrix} \frac{i}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$

$\Rightarrow c_1 = 4$ & $c_2 = 2\sqrt{2}$

$\Rightarrow z = 4 \begin{bmatrix} \frac{1+i}{2} \\ \frac{1-i}{2} \end{bmatrix} + 2\sqrt{2} \begin{bmatrix} \frac{i}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$

12. $Q \cdot Q^T = Q^T \cdot Q = I$

$\Rightarrow \begin{bmatrix} 1+a^2 & ab & ac & ad \\ ab & \frac{1}{2}+b^2 & bc & \frac{1}{2}+bd \\ ac & bc & 1+c^2 & cd \\ ad & \frac{1}{2}+bd & cd & \frac{1}{2}+d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow a = 0$ & $b = \pm\frac{1}{\sqrt{2}}$ & $c = 0$ & $d = \pm\frac{1}{\sqrt{2}}$

13. $A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix} \Rightarrow q_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ $q_2 = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$ $q_3 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$

$Q = [q_1, q_2, q_3]$

$B = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 0 & 1 \\ 2 & -4 & 2 \\ 4 & 0 & 0 \end{bmatrix} \Rightarrow q_1 = \begin{bmatrix} \frac{1}{5} \\ \frac{2}{5} \\ \frac{2}{5} \\ \frac{4}{5} \end{bmatrix}$ $q_2 = \begin{bmatrix} -2/5 \\ 1/5 \\ -4/5 \\ 2/5 \end{bmatrix}$ $q_3 = \begin{bmatrix} -4/5 \\ 2/5 \\ 2/5 \\ -1/5 \end{bmatrix}$

$Q = [q_1, q_2, q_3]$

14. $x_1 \cdot x_2 = \frac{1}{2}(1,1,1,-1)^T \cdot \frac{1}{6}(1,1,3,5)^T = 0$

$\|x_1\| = \sqrt{(\frac{1}{2})^2 + (\frac{1}{2})^2 + (\frac{1}{2})^2 + (-\frac{1}{2})^2} = 1$

$\|x_2\| = \sqrt{(\frac{1}{6})^2 + (\frac{1}{6})^2 + (\frac{3}{6})^2 + (\frac{5}{6})^2} = 1$

$\Rightarrow \{x_1, x_2\}$ forms an orthonormal basis

$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 3 & 5 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

$\Rightarrow x = \alpha \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 4 \\ 0 \\ -3 \\ 1 \end{bmatrix}$

$\Rightarrow \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$ forms a basis for N(A)

$\Rightarrow q_1 = \frac{a_1}{\|a_1\|} = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0)^T$

$\Rightarrow q_2 = \frac{a_2 - \langle a_2, q_1\rangle q_1}{\|a_2 - \langle a_2, q_1\rangle q_1\|} = (\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}, -\frac{1}{\sqrt{2}}, \frac{1}{3\sqrt{2}})^T$

$\Rightarrow \{q_1, q_2\}$ forms an orthonormal basis for $N(A)$

assume $\{x_1, x_2, q_1, q_2\}$ forms the basis:

$c_1x_1 + c_2x_2 + c_3q_1 + c_4q_2 = 0$

$\Rightarrow c \doteq (0,0,0,0)^T$

$\Rightarrow$ the assumption is true

$\therefore \{x_1, x_2, q_1, q_2\}$ forms an orthonormal basis for $\mathbb{R}^4$

15a) $v_1 = u_1$ $\quad v_2 = u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1} u_1$

$= (2,1)$ $\quad = (-\frac{3}{5}, \frac{6}{5})$

$\Rightarrow \{\frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}\} = \{(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})^T, (-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})^T\}$

forms an orthogonal basis

b) $v_1 = u_1 = (0,1,1)$

$v_2 = u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1}(u_1) = (1,0,0)$

$v_3 = u_3 - \frac{u_3 \cdot v_1}{v_1 \cdot v_1}(v_1) - \frac{u_3 \cdot v_2}{v_2 \cdot v_2}(v_2) = (0,-2,2)$

$\Rightarrow \{\frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|}\} = \{(0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})^T,$

$(1,0,0)^T, (0, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})^T\}$ forms an orthogonal basis

16. $P_A(\lambda) = \begin{vmatrix} -1-\lambda & 3 \\ 2 & -\lambda \end{vmatrix} = \lambda^2 + \lambda - 6 = 0$

$\Rightarrow \lambda = -3 \Rightarrow \mathrm{Null}(A+3I) = \mathrm{Span}(\begin{bmatrix} -3 \\ 2 \end{bmatrix})$

$\Rightarrow \lambda = 2 \Rightarrow \mathrm{Null}(A-2I) = \mathrm{Span}(\begin{bmatrix} 1 \\ 1 \end{bmatrix})$

$P_{A^2}(\lambda) = \begin{vmatrix} 7-\lambda & -3 \\ -2 & 6-\lambda \end{vmatrix} = \lambda^2 - 13\lambda + 36$

$\Rightarrow \lambda = 4 \Rightarrow \mathrm{Null}(A^2 - 4I) = \mathrm{Span}(\begin{bmatrix} 1 \\ 1 \end{bmatrix})$

$\Rightarrow \lambda = 9 \Rightarrow \mathrm{Null}(A^2 - 9I) = \mathrm{Span}(\begin{bmatrix} -3 \\ 2 \end{bmatrix})$

$\Rightarrow A$ & A^2 share the same eigen vectors

$\Rightarrow A$ has λ_1 & λ_2 as eigenvalues, A^2 has λ_1^2 & λ_2^2

17a) $P_A(\lambda) = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 1 & 1-\lambda & 0 \\ 1 & 1 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda)^2 = 0$

$\Rightarrow \lambda = 1 \Rightarrow \mathrm{Null}(A-I) = \mathrm{Span}(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix})$

$\Rightarrow \lambda = 2 \Rightarrow \mathrm{Null}(A-2I) = \mathrm{Span}(\begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix})$

$\therefore \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is the eigenvector when $\lambda = 1$ &

$\begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}$ —— 11 —— $\lambda = 2$

b) $P_B(\lambda) = \begin{vmatrix} 4-\lambda & -5 & 1 \\ 1 & -\lambda & -1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = -\lambda(\lambda^2 - 3\lambda + 2) = 0$

$\Rightarrow \lambda = 0 \Rightarrow \mathrm{Null}(B) = \mathrm{Span}(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix})$

$\Rightarrow \lambda = 1 \Rightarrow \mathrm{Null}(B-I) = \mathrm{Span}(\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix})$

$\Rightarrow \lambda = 2 \Rightarrow \mathrm{Null}(B-2I) = \mathrm{Span}(\begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix})$

$\therefore \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is the eigenvector when $\lambda = 0$ &

$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ —— 11 —— $\lambda = 1$ &

$\begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$ —— 11 —— $\lambda = 2$

c) $P_C(\lambda) = \begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 4$

$\Rightarrow \lambda = 1-2i \Rightarrow \mathrm{Null}(C - (1-2i)I) = \mathrm{Span}(\begin{bmatrix} i \\ 1 \end{bmatrix})$

$\Rightarrow \lambda = 1+2i \Rightarrow \mathrm{Null}(C - (1+2i)I) = \mathrm{Span}(\begin{bmatrix} -i \\ 1 \end{bmatrix})$

$\therefore \begin{bmatrix} i \\ 1 \end{bmatrix}$ is the eigenvector when $\lambda = 1-2i$ &

$\begin{bmatrix} -i \\ 1 \end{bmatrix}$ —— 11 —— $\lambda = 1+2i$

18. let v be the eigenvector of A with λ

$\Rightarrow f(A)v = \sum_{n=0}^{m} a_n A^n v = \sum_{n=0}^{m} a_n \lambda^n v = (\sum_{n=0}^{m} a_n \lambda^n) v$

$= f(\lambda) v$

$\Rightarrow f(A)v = f(\lambda)v$

$\therefore f(\lambda)$ is an eigenvalue of the matrix $f(A)$

19. $Bx = \lambda x$

$\Rightarrow S^{-1}ASx = \lambda x$

$\Rightarrow SS^{-1}ASx = S\lambda x$

$\Rightarrow A(Sx) = \lambda(Sx)$

$\therefore Sx$ is an eigenvector of A belonging to λ

20. $P_A(\lambda) = \begin{vmatrix} B-\lambda I & C \\ 0 & D-\lambda I \end{vmatrix}$

$= (B-\lambda I)(D-\lambda I)$

$= 0$

$\Rightarrow \lambda \in \{1, 2, 5, 7\}$