Home Assignment 5

1. 3.1-3. Customers arrive randomly at a bank teller's window. Given that one customer arrived during a particular 10-minute period, let X equal the time within the 10 minutes that the customer arrived. If X is $U(0, 10)$, find
   (a) The pdf of X.
   (b) $P(X \geq 8)$.
   (c) $P(2 \leq X < 8)$.
   (d) $E(X)$.
   (e) $Var(X)$.

2. 3.1-4. If the mgf of X is
$$M(t) = \frac{e^{5t} - e^{4t}}{t}, t \neq 0 \quad \text{and} \quad M(0) = 1$$
   find (a) $E(X)$, (b) $Var(X)$, and (c) $P(4.2 < X \leq 4.7)$.

3. 3.1-5. Let Y have a uniform distribution $U(0, 1)$, and let
$$W = a + (b - a)Y, \quad a < b.$$
   (a) Find the cdf of W.
   Hint: Find $P[a + (b - a)Y \leq w]$.
   (b) How is W distributed?

4. 3.1-6. A grocery store has n watermelons to sell and makes \$1.00 on each sale. Say the number of consumers of these watermelons is a random variable with a distribution that can be approximated by
$$f(x) = \frac{1}{200}, \quad 0 < x < 200,$$
   a pdf of the continuous type. If the grocer does not have enough watermelons to sell to all consumers, she figures that she loses \$5.00 in goodwill from each unhappy customer. But if she has surplus watermelons, she loses 50 cents on each extra watermelon. What should n be to maximize profit? Hint: If $X \leq n$, then her profit is $(1.00)X + (-0.50)(n - X)$; but if $X > n$, her profit is $(1.00)n + (-5.00)(X - n)$. Find the expected value of profit as a function of n, and then select n to maximize that function.

5. 3.1-8. For each of the following functions, (i) find the constant c so that $f(x)$ is a pdf of a random variable X, (ii) find the cdf, $F(x) = P(X \leq x)$, (iii) sketch graphs of the pdf $f(x)$ and the distribution function $F(x)$, and (iv) find μ and σ^2:
   (a) $f(x) = x^3/4, 0 < x < c$.
   (b) $f(x) = (3/16)x^2, -c < x < c$.
   (c) $f(x) = c/\sqrt{x}, 0 < x < 1$. Is this pdf bounded?

6. 3.1-16. Let $f(x) = (x+1)/2, -1 < x < 1$. Find (a) $\pi_{0.64}$, (b) $q_1 = \pi_{0.25}$, and (c) $\pi_{0.81}$.

7. 3.1-21. Let $X_1, X_2, \cdots, X_k$ be random variables of the continuous type, and let $f_1(x), f_2(x), \cdots, f_k(x)$ be their corresponding pdfs, each with sample space $S = (-\infty, \infty)$. Also, let $c_1, c_2, \cdots, c_k$ be nonnegative constants such that $\sum_{i=1}^{k} c_i = 1$.
   (a) Show that $\sum_{i=1}^{k} c_i f_i(x)$ is a pdf of a continuous-type random variable on S.
   (b) If X is a continuous-type random variable with pdf $\sum_{i=1}^{k} c_i f_i(x)$ on S, $E(X_i) = \mu_i$, and $Var(X_i) = \sigma_i^2$ for $i = 1, \cdots, k$, find the mean and the variance of X.

8. 3.2-1. What are the pdf, the mean, and the variance of X if the moment-generating function of X is given by the following?
   (a) $M(t) = \frac{1}{1-3t}, t < 1/3$.
   (b) $M(t) = \frac{3}{3-t}, t < 3$.

9. 3.2-3. Let X have an exponential distribution with mean $\theta > 0$. Show that

$$P(X > x + y | X > x) = P(X > y).$$

10. 3.2-4. Let $F(x)$ be the cdf of the continuous-type random variable X, and assume that $F(x) = 0$ for $x \leq 0$ and $0 < F(x) < 1$ for $0 < x$. Prove that if

$$P(X > x + y | X > x) = P(X > y), y \geq 0$$

then

$$F(x) = 1 - e^{-\lambda x}, 0 < x.$$

Hint: Show that $g(x) = 1 - F(x)$ satisfies the functional equation

$$g(x + y) = g(x)g(y).$$

which implies that $g(x) = a^{cx}$.