STA2001 Assignment 2:

Question 3 ,7 and computer exercise are optional questions. Please submit your solution in PDF form on BB before due time, otherwise you will get 0 for that assignment

1. Prove the properties of the conditional probability
   (a) $P(A|B) \geq 0$ and $P(A|B) \leq 1$
   (b) $P(B|B) = 1$
   (c) for mutually exclusive events $A_1, A_2, A_3, \cdots, A_k$, then
   $P(A_1 \cup A_2 \cup \cdots \cup A_k|B) = P(A_1|B) + P(A_2|B) + \cdots + P(A_k|B)$
   (d) $P(A'|B) = 1 - P(A|B)$
   (e) $P(A \cup C|B) = P(A|B) + P(C|B) - P(A \cap C|B)$

2. (1.3-8) An urn contains 17 balls marked LOSE and 3 balls marked WIN. You and an opponent take turns selecting a single ball at random from the urn without replacement.The person who selects the third WIN ball wins the game. It does not matter who selected the first two WIN balls.
   (a) If you draw first, find the probability that you win the game on your second draw.
   (b) If you draw first, find the probability that your opponent wins the game on his second draw.
   (c) If you draw first, what is the probability that you win? Hint: You could win on your second, third, fourth, $\cdots$, or tenth draw, but not on your first.
   (d) Would you prefer to draw first or second? Why?

3. (1.3-9) An urn contains four balls numbered 1 through 4. The balls are selected one at a time without replacement. A match occurs if the ball numbered m is the mth ball selected. Let the event Ai denote a match on the ith draw, i = 1, 2, 3, 4.
   (a) Show that $P(A_i) = \frac{3!}{4!}$ for each i.
   (b) Show that $P(A_i \cap A_j) = \frac{2!}{4!}, i \neq j$.
   (c) Show that $P(A_i \cap A_j \cap A_k) = \frac{1!}{4!}, i \neq j, i \neq k, j \neq k$.
   (d) Show that the probability of at least one match is $P(A_1 \cup A_2 \cup A_3 \cup A_4) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!}$.

(e) Extend this exercise so that there are n balls in the urn. Show that the probability of at least one match is
$P(A_1 \cup A_2 \cup \cdots \cup A_n)$
$= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots + \frac{(-1)^{n+1}}{n!}$
$= 1 - (1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!})$.

(f) What is the limit of this probability as n increases without bound?

4. (1.3-13) In the gambling game craps, a pair of dice is rolled and the outcome of the experiment is the sum of the points on the up sides of the six-sided dice. The bettor wins on the first roll if the sum is 7 or 11. The bettor loses on the first roll if the sum is 2, 3, or 12. If the sum is 4, 5, 6, 8, 9, or l0, that number is called the bettors point. Once the point is established, the rule is as follows: If the bettor rolls a 7 before the point, the bettor loses; but if the point is rolled before a 7, the bettor wins.

   (a) List the 36 outcomes in the sample space for the roll of a pair of dice. Assume that each of them has a probability of 1/36.

   (b) Find the probability that the bettor wins on the first roll. That is, find the probability of rolling a 7 or 11, P(7 or 11).

   (c) Given that 8 is the outcome on the first roll, find the probability that the bettor now rolls the point 8 before rolling a 7 and thus wins. Note that at this stage in the game the only outcomes of interest are 7 and 8. Thus find $P(8|7or8)$.

   (d) The probability that a bettor rolls an 8 on the first roll and then wins is given by $P(8)P(8|7or8)$. Show that this probability is (5/36)(5/11).

   (e) Show that the total probability that a bettor wins in the game of craps is 0.49293. Hint: Note that the bettor can win in one of several mutually exclusive ways: by rolling a 7 or an 11 on the first roll or by establishing one of the points 4, 5, 6, 8, 9, or 10 on the first roll and then obtaining that point on successive rolls before a 7 comes up.

5. (1.4-6) Show that if A, B, and C are mutually independent, then the following pairs of events are independent: A and $(B \cap C)$, A and $(B \cup C)$, A' and $(B \cap C')$. Show also that $A', B', and C'$ are mutually independent.

6. (1.4-11) Let A and B be two events.

   (a) If the events A and B are mutually exclusive, are A and B always independent? If the answer is no, can they ever be independent? Explain.

   (b) If $A \subset B$, can A and B ever be independent events? Explain.

7. (1.4-12) Flip an unbiased coin eight independent times. Compute the probability of

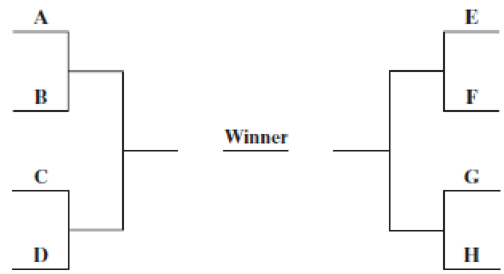
   (a) HHHTHTTH

(b) TTHHHHTT

(c) HTHTHTHT

(d) Four heads occurring in the eight trials.

8. (1.4-18) An eight-team single-elimination tournament is set up as follows: For example,



eight students (called $A - H$) set up a tournament among themselves. The top-listed student in each bracket calls heads or tails when his or her opponent flips a coin. If the call is correct, the student moves on to the next bracket.

(a) How many coin flips are required to determine the tournament winner?

(b) What is the probability that you can predict all of the winners?

(c) In NCAA Division I basketball, after the "play-in" games, 64 teams participate in a single-elimination tournament to determine the national champion. Considering only the remaining 64 teams, how many games are required to determine the national champion?

(d) Assume that for any given game, either team has an equal chance of winning. (That is probably not true.) On page 43 of the March 22, 1999, issue, Time claimed that the "mathematical odds of predicting all 63 NCAA games correctly is 1 in 75 million." Do you agree with this statement? If not, why not?

9. (1.5-6) A life insurance company issues standard, preferred, and ultrapreferred policies. Of the companys policyholders of a certain age, 60% have standard policies and a probability of 0.01 of dying in the next year, 30% have preferred policies and a probability of 0.008 of dying in the next year, and 10% have ultrapreferred policies and a probability of 0.007 of dying in the next year. A policyholder of that age dies in the next year.What are the conditional probabilities of the deceased having had a standard, a preferred, and an ultrapreferred policy?

10. (1.5-10) Suppose we want to investigate the percentage of abused children in a certain population. To do this, doctors examine some of these children taken at random from that population. However, doctors are not perfect: They sometimes classify an abused child (A^+) as one not abused (D^-) or they classify a nonabused child (A^-) as one that is

abused (D^+). Suppose these error rates are $P(D^-|A^+) = 0.08$ and $P(D^+|A^-) = 0.05$, respectively; thus, $P(D^+|A^+) = 0.92$ and $P(D^-|A^-) = 0.95$ are the probabilities of the correct decisions. Let us pretend that only 2% of all children are abused; that is, $P(A^+) = 0.02$ and $P(A^-) = 0.98$.

(a) Select a child at random. What is the probability that the doctor classifies this child as abused? That is, compute
$P(D^+) = P(A^+)P(D^+|A^+) + P(A^-)P(D^+|A^-)$.

(b) Compute $P(A^-|D^+) and P(A^+|D^+)$.

(c) Compute $P(A^-|D^-) and P(A^+|D^-)$.

(d) Are the probabilities in (b) and (c) alarming? This happens because the error rates of 0.08 and 0.05 are high relative to the fraction 0.02 of abused children in the population.