Assignment 4

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Please note that

- **Released date: 22th Oct, Fri.**
- **Due date: 7th Nov., Sun., by 11:59pm.**
- Late submission is **NOT** accepted.
- Please submit your answers as a PDF file with a name like "120010XXX ASS4.pdf" (Your student ID + ASS No.). You may either typeset you answers directly using computers, or scan your handwritten answers. (We recommend you use the printers on campus to scan. If you use your smartphone to scan, please limit the file size 10MB.)

Question 1. (Slide 12) Determine whether the following sets are bases for $\mathbb{R}^3$.

(a)

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$$

(b)

$$\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$$

(c)

$$\begin{bmatrix}1\\0\\-2\end{bmatrix}, \begin{bmatrix}3\\2\\-4\end{bmatrix}, \begin{bmatrix}-3\\-5\\1\end{bmatrix}$$

Question 2. (Slide 12) Find a basis for the following solution set:

$$\left\{ \mathbf{x} \in \mathbb{R}^4 \middle| \begin{bmatrix}-2 & 4 & -2 & -4\\ 2 & -6 & -3 & 1\\ -3 & 8 & 2 & -3\end{bmatrix}\mathbf{x} = 0 \right\}$$

Question 3. (Slide 12) In each of the following, find the dimension of the subspace of P_3 spanned by the given vectors:

(a) $x, x-1, x^2+1$

(b) $x, x-1, x^2+1, x^2-1$

(c) $x^2, x^2-x-1, x+1$

(d) $2x, x-2$

Question 4. (Slide 12) Let $\{x, 1\}$ and $\{2x-1, 2x+1\}$ be ordered bases for P_2, find the transition matrix representing the change in coordinates from $\{2x-1, 2x+1\}$ to $\{x, 1\}$.

Question 5. (Slide 12) Let

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, u_3 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, v_1 = \begin{bmatrix} 4 \\ 6 \\ 7 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix},$$

(a) Find the transition matrix from $\{v_1, v_2, v_3\}$ to $\{u_1, u_2, u_3\}$

(b) If $\mathbf{x} = 2v_1 + 3v_2 - 4v_3$, determine the coordinate of $\mathbf{x}$ with respect to $\{u_1, u_2, u_3\}$.

Question 6. (Slide 12) Let $B = \left\{ \begin{bmatrix} 7 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \end{bmatrix} \right\}$, $C = \left\{ \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right\}$ be bases for $\mathbb{R}^2$. Find the transition matrix from B to C and the transition matrix from C to B.

Question 7. (Slide 12) Let $b_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, b_2 = \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix}, b_3 = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$.

(a) Show that the set $\mathbf{B} = \{b_1, b_2, b_3\}$ is a basis of $\mathbb{R}^3$.

(b) Given a vector $\mathbf{x} = \begin{bmatrix} -8 \\ 2 \\ 3 \end{bmatrix}$ in $\mathbb{R}^3$. Find the coordinate vector of $\mathbf{x}$ with respect to the basis $\mathbf{B}$.

(c) Given the coordinate of $\mathbf{y}$ with respect to the basis $\mathbf{B}$ is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. What is the vector $\mathbf{y}$ in $\mathbb{R}^3$.

Question 8. (Slide 13) Find a basis for the following subspace:

$$\text{Null}\left(\begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}\right)$$

Question 9. (Slide 13) For each of the following choices of A and b, determine whether b is in the column space of A.

(a) $A = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & 2 \end{bmatrix}, b = \begin{bmatrix} 5 \\ 10 \\ 5 \end{bmatrix}$

Question 10. (Slide 13) If $A = \begin{bmatrix} 1 & 2 & 3 & 5 & 0 & 2 & 4 \\ 2 & 1 & 3 & 4 & 7 & 8 & 9 \\ 1 & 1 & 2 & 3 & 2 & 2 & 5 \end{bmatrix}$

(a) Find a basis for $Col(A)$

(b) Find a basis for $Null(A)$ and find the dimension of $Null(A)$.

Question 11. (Slide 13, 14) For each of the following matrices, find a basis for the row space, a basis for the column space, and a basis for the null space.

(a)

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 4 & 7 & 8 \end{bmatrix}$$

(b)

$$\begin{bmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1 & 3 & -2 & 1 \\ 2 & 1 & 3 & 2 \\ 3 & 4 & 5 & 6 \end{bmatrix}$$

Question 12. (Slide 13) Let

$$A = \begin{bmatrix} 1 & 2 & 2 & 3 & 1 & 4 \\ 2 & 4 & 5 & 5 & 4 & 9 \\ 3 & 6 & 7 & 8 & 5 & 9 \end{bmatrix}$$

Compute the reduced row echelon form U of A. Which column vectors of A correspond to the dependent variables of U? Which column vectors form a basis for the column space of A? Write each of the remaining column vectors of A as a linear combination of these basis vectors.

Question 13. (Slide 13) Let A be an $m \times n$ matrix with $m > n$. Let $\mathbf{b} \in \mathbb{R}^m$ and suppose that $N(A) = \{\mathbf{0}\}$.

(a) What can you conclude about the column vectors of A? Are they linearly independent? Do they span $\mathbb{R}^m$? Explain.

(b) How many solutions will the system $A\mathbf{x} = \mathbf{b}$ have if $\mathbf{b}$ is not in the column space of A? How many solutions will there be if $\mathbf{b}$ is in the column space of A? Explain.

Question 14. (Slide 13) Let A be a 4×4 matrix with reduced row echelon form given by

$$U = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

If

$$\mathbf{a}_1 = \begin{bmatrix} -3 \\ 5 \\ 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{a}_2 = \begin{bmatrix} 4 \\ -3 \\ 7 \\ -1 \end{bmatrix}$$

find $\mathbf{a}_3$ and $\mathbf{a}_4$.

Question 15. (Slide 13) Let A be a 4×5 matrix. If $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_4$ are linearly independent and

$$\mathbf{a}_3 = \mathbf{a}_1 + 2\mathbf{a}_2, \qquad \mathbf{a}_5 = 2\mathbf{a}_1 - \mathbf{a}_2 + 3\mathbf{a}_4$$

determine the reduced row echelon form of A.

Question 16. (Slide 14) Let A be a 5×3 matrix of rank 3 and let $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ be a basis for $\mathbb{R}^3$.

(a) Show that $N(A) = \{\mathbf{0}\}$.

(b) Show that if $\mathbf{y}_1 = A\mathbf{x}_1$, $\mathbf{y}_2 = A\mathbf{x}_2$, and $\mathbf{y}_3 = A\mathbf{x}_3$ then $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ are linearly independent.

(c) Do the vectors $\mathbf{y}_1$, $\mathbf{y}_2$, $\mathbf{y}_3$ from part (b) form a basis for $\mathbb{R}^5$? Explain.

Question 17. (Slide 14) Let A be an $m \times n$ matrix with rank equal to n. Show that if $\mathbf{x} \neq 0$ and $\mathbf{y} = A\mathbf{x}$, then $\mathbf{y} \neq 0$.

Question 18. (Slide 14) Let A be an $m \times n$ matrix.

(a) Show that if B is a nonsingular $m \times m$ matrix, then BA and A have the same null space and hence the same rank.

(b) Show that if C is a nonsingular $n \times n$ matrix, then AC and A have the same rank.

Question 19. (Slide 14) Show that if A and B are $n \times n$ matrices and $N(A - B) = \mathbb{R}^n$, then $A = B$.

Question 20. (Slide 13, 14) Let $\mathbf{x}$ and $\mathbf{y}$ be nonzero vectors in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, and let $A = \mathbf{x}\mathbf{y}^T$.

(a) Show that $\{\mathbf{x}\}$ is a basis for the column space of A and that $\{\mathbf{y}\}$ is a basis for the row space of A.

(b) What is the dimension of $N(A)$?