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1. let u_1, u_2 be orthonormal basis for $\mathbb{C}^2$

if $z = (4+2i)u_1 + (6-5i)u_2$

then

(a) $u_1^H z = 4+2i$

$z^H u_1 = 4-2i$

$u_2^H z = 6-5i$

$z^H u_2 = 6+5i$

(b) $\|z\|^2 = z^H z$

$= 81$

$\|z\| = 9$

2. $B = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$

$p_B(\lambda) = \begin{vmatrix} 9-\lambda & 12 \\ 12 & 16-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda = 0$ or $\lambda = 25$

when $\lambda = 0$, $\text{Null}(B) = \text{Span}\left(\begin{bmatrix} -\frac{4}{3} \\ 1 \end{bmatrix}\right)$

when $\lambda = 25$, $\text{Null}(B - 25\lambda) = \text{Span}\left(\begin{bmatrix} 3/4 \\ 1 \end{bmatrix}\right)$

$\Rightarrow U = \begin{bmatrix} -4/5 & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}$ or $\begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$

$\Rightarrow U^{-1}BU = U^TBU = \begin{bmatrix} 0 & 0 \\ 0 & 25 \end{bmatrix}$ or $\begin{bmatrix} 25 & 0 \\ 0 & 0 \end{bmatrix}$

3. $A^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

(a) $A^T x = 0$

$\Rightarrow x = \alpha \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

$\Rightarrow N(A^T) = \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}\right)$

$\Rightarrow$ orthonormal basis for $N(A^T) = \left\{ \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\}$

(b) with $A = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$

$P = A(A^TA)^{-1}A^T = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

4. $f = a_1x^2 + a_2y^2 + a_3z^2 + a_4xy + a_5xz + a_6yz$.

then the matrix is $\begin{bmatrix} a_1 & a_4/2 & a_5/2 \\ a_4/2 & a_2 & a_6/2 \\ a_5/2 & a_6/2 & a_3 \end{bmatrix}$

(a) $\begin{bmatrix} 3 & -5/2 \\ -5/2 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 1/2 & -1 \\ 1/2 & 3 & 3/2 \\ -1 & 3/2 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & 1 & -2 \\ 1 & 2 & -7/2 \\ -2 & -7/2 & 5 \end{bmatrix}$

5. note that:

∘) A is positive definite $\Rightarrow$ $\det(A) > 0$ and all eigenvalues of A are positive

$\Rightarrow$ A is nonsingular since $\det(A) \neq 0$ & all eigenvalues of A are non zero

∘) A is symmetric $\Rightarrow$ A^{-1} is also symmetric

take $Ay = x$ where x is a non zero column vector.

$\Rightarrow$ $x^TA^{-1}x = y^TAy > 0$

$\therefore$, A^{-1} is also positive definite

6. given A & B are symmetric

then there exists P & Q s.t. $P^{-1}AP = D_1$ & $Q^{-1}BQ = D_2$

$\Rightarrow$ $A = PD_1P^{-1}$ & $B = QD_2Q^{-1}$

$= PD_1P^T$ & $B = QD_2Q^T$

A & B are similar $\Rightarrow$ $D_1 = D_2 = D$

$\Rightarrow$ $A = PDP^{-1}$ $\quad B = QDQ^{-1}$

$\Rightarrow$ $B = QP^{-1}(PDP^{-1})PQ^{-1} = MAM^{-1}$ $(M = QP^{-1})$ which is orthogonal

$M^T = (QP^{-1})^T = M^{-1}$

$\Rightarrow$ $MM^T = I$

$\Rightarrow$ M is orthogonal

7 (a). $H^T H = (I - 2uu^T)(I - 2uu^T)$

$= I - 4uu^T + 4u(u^T u)u^T$

$= I$

$\Rightarrow$ H is orthogonal

(b) $Hu = (I - 2uu^T)u$

$= u - 2uu^T u$

$= -u$

$\Rightarrow \lambda = -1$

8. $A^T A$ is positive definite $\Rightarrow$ all eigen values of A are positive

& $|A^T A| \neq 0$

$\Rightarrow \text{rank}(A^T A) = n$

~~$\text{rank}(A) = n \Rightarrow x^T(A^T A)x \neq 0$~~

$\text{rank}(A) = n$

$\Rightarrow \|Ax\|^2 \neq 0$

$\Rightarrow (A^T x^T)Ax \neq 0$

$\Rightarrow x^T(A^T A)x \neq 0$

$\Rightarrow$ A is positive definite

9. $D = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$

$p_D(\lambda) = \det(D - \lambda I) = (A - \lambda)(B - \lambda) \Rightarrow \lambda = \{\lambda_a \mid \lambda_a \in P_A(\lambda_a)\}$ or

$\lambda = \{\lambda_b \mid \lambda_b \in P_B(\lambda_b)\}$

$\Rightarrow \lambda = \{\lambda_{a1}, \lambda_{a2}, \ldots, \lambda_{b1}, \ldots, \lambda_{bn}\}$

since A & B are positive definite, $\lambda_c \in \lambda$ is > 0

$\Rightarrow$ D is also positive definite

10. $Q = \begin{bmatrix} 5 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 5 \end{bmatrix}$

$\det(Q) = 5\begin{vmatrix} 1 & -2 \\ -2 & 5 \end{vmatrix} - 2\begin{vmatrix} 2 & -2 \\ 4 & 5 \end{vmatrix} - 4\begin{vmatrix} 2 & 1 \\ -4 & -2 \end{vmatrix}$

$= 1$

$P_Q(\lambda) = \det(Q - \lambda I)$

$= \begin{vmatrix} 5-\lambda & 2 & -4 \\ 2 & 1-\lambda & -2 \\ -4 & -2 & 5-\lambda \end{vmatrix}$

$= 0$

$\Rightarrow \lambda = \{1, \pm 2\sqrt{6} + 5\}$

since $\det(Q) > 0$ & $\lambda_i \in \lambda > 0$

$\Rightarrow Q$ is positive definite