yohandi - homework for week 4

Exercises 4.1

1. absolute minimum at $x = c_2$
   absolute maximum at $x = b$
   it is because for every point $x \in [a,b]$, $f(c_2) \le f(x) \le f(b)$.

2. absolute minimum at $x = b$
   absolute maximum at $x = c$
   it is because for every point $x \in [a,b]$, $f(b) \le f(x) \le f(c)$.

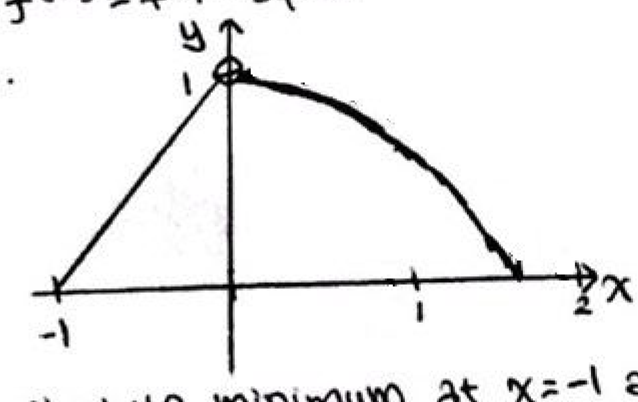
3. absolute minimum, none
   absolute maximum at $x = c$
   it is because for every point $x \in (a,b)$, $f(x) \le f(c)$.
   there doesn't exist any point $x_0 \in (a,b)$ such that $f(x_0) \le f(x)$ for every $x \in (a,b)$.

4. absolute minimum, none
   absolute maximum, none
   there doesn't exist any point $x_0 \in (a,b)$ such that $f(x_0) \le f(x)$ or $f(x) \le f(x_0)$ for every $x \in (a,b)$.

5. absolute minimum at $x = a$
   absolute maximum at $x = c$
   it is because for every point $x \in [a,b]$, $f(a) \le f(x) \le f(c)$.

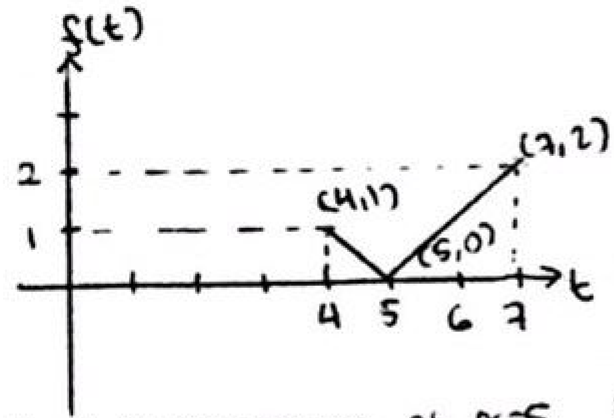
6. absolute minimum at $x = c$
   absolute maximum at $x = a$
   it is because for every point $x \in [a,b]$ $f(c) \le f(x) \le f(a)$

20.



absolute minimum at $x = -1$ and $x = \frac{\pi}{2}$
absolute maximum, none
It is because for every $x \in [-1, \frac{\pi}{2}], x \neq 0$, $f(-1) = f(\frac{\pi}{2}) \le f(x)$.
there doesn't exist any point $x_0 \in [-1, \frac{\pi}{2}]$, $x_0 \neq 0$ such that $f(x) \le f(x_0)$ for every $x \in [-1, \frac{\pi}{2}], x \neq 0$.

36.



absolute minimum at $x = 5$ $y = 0$
absolute maximum at $x = 7$ $y = 2$
It is because for every $x \in [4,7]$, $f(5) \le f(x) \le f(7)$

42. $f(x) = 6x^2 - x^3$

$\frac{df(x)}{dx} = 12x - 3x^2$

critical points when $\frac{df(x)}{dx} = 0$

$\Rightarrow 12x - 3x^2 = 0$

$3x(4 - x) = 0$

$x = 0$ and $x = 4$

45. $y = x^2 + \frac{2}{x}$

$\frac{dy}{dx} = 2x - \frac{2}{x^2}$

critical points when $\frac{dy}{dx} = 0$

$\Rightarrow 2x - \frac{2}{x^2} = 0$

$x^3 = 1$

$x = 1$

$\frac{dy}{dx}$ doesn't exist $\Rightarrow x = 0$

Since $f(0)$ doesn't exist,
$x = 0$ is not a critical point

50. $y = x^3 - 2x + 4$

$\frac{dy}{dx} = 3x^2 - 2$

critical points when $\frac{dy}{dx} = 0$

$\Rightarrow 3x^2 - 2 = 0$

$x = \pm\frac{1}{3}\sqrt{6}$

$$\begin{array}{ccccc} + & & - & & + \\ \hline & -\frac{1}{3}\sqrt{6} & & \frac{1}{3}\sqrt{6} & \end{array}$$

local maxima when $x = -\frac{1}{3}\sqrt{6}$, $y = \sqrt{6}(\frac{2}{3} - \frac{2}{9}) + 4$
$= \frac{4}{9}\sqrt{6} + 4$

$\Rightarrow$ at $(-\frac{1}{3}\sqrt{6}, \frac{4}{9}\sqrt{6} + 4)$

local minima when $x = \frac{1}{3}\sqrt{6}$, $y = \sqrt{6}(\frac{2}{9} - \frac{2}{3}) + 4$
$= -\frac{4}{9}\sqrt{6} + 4$

$\Rightarrow$ at $(\frac{1}{3}\sqrt{6}, -\frac{4}{9}\sqrt{6} + 4)$

51. $y = x^3 + x^2 - 8x + 5$

$\frac{dy}{dx} = 3x^2 + 2x - 8$

critical points when $\frac{dy}{dx}$

$\Rightarrow 3x^2 + 2x - 8 = 0$

$(3x-4)(x+2) = 0$

$x = \frac{4}{3} \quad x = -2$

$+ \quad - \quad +$

$-2 \quad \frac{4}{3}$

local maxima when $x = -2$, $y = 17$

$\Rightarrow$ at $(-2, 17)$

local minima when $x = 4/3$, $y = -44/27$

$\Rightarrow$ at $(4/3, -44/27)$

67. $f(x) = (x-2)^{2/3}$

$f'(x) = \frac{2}{3}(x-2)^{-1/3} \cdot 1$

$= \frac{2}{3\sqrt[3]{x-2}}$

a. $f'(2)$ is undefined, $f'(2)$ doesn't exist

b. critical points when $f'(x) = 0$, it is when $x \to 2$

therefore, a local extreme value occurs at $x = 2$.

with $f(2) = 0$ and $f(x) \geq f(2)$ for all $x \in (-\infty, \infty)$. (extreme minima at $x = 2$)

c. no, it is because $(-\infty, \infty)$ is an open interval.

d. for $f(x) = (x-a)^{2/3}$:

$\Rightarrow f'(a)$ doesn't exist

$\Rightarrow$ a local extreme value occurs at $x = a$ with $f(a) = 0$ and $f(x) \geq f(a)$ for all $x \in (-\infty, \infty)$. (extreme minima at $x = a$)

68. $f(x) = |x^3 - 9x|$

$$= \begin{cases} x^3 - 9x, & -3 \leq x \leq 0 \text{ or } x \geq 3 \\ -x^3 + 9x, & \text{otherwise} \end{cases}$$

$$f'(x) = \begin{cases} 3x^2 - 9, & -3 \leq x \leq 0 \text{ or } x \geq 3 \\ -3x^2 + 9, & \text{otherwise} \end{cases}$$

a. $f'(0^-) = -9 \neq 9 = f'(0^+)$. therefore $f'(0)$ doesn't exist

b. $f'(3^-) = -18 \neq 18 = f'(3^+)$. therefore $f'(3)$ doesn't exist

c. $f'(-3^-) = -18 \neq 18 = f'(-3^+)$. therefore $f'(-3)$ doesn't exist

d. all critical points when $f'(x) = 0$ or $f'(x)$ is undefined

$\Rightarrow f'(x) = 0$

$3x^2 - 9 = 0$

$x = \pm\sqrt{3}$

$\Rightarrow f'(x)$ is undefined

$x = \{-3, 0, 3\}$

local minima when $x = -3$, $y = 0$

$x = 0$, $y = 0$

$x = 3$, $y = 0$

$\Rightarrow$ at $(-3, 0)$, $(0, 0)$ and $(3, 0)$

local maxima when $x = -\sqrt{3}$, $y = 6\sqrt{3}$

$x = \sqrt{3}$, $y = 6\sqrt{3}$

$\Rightarrow$ at $(-\sqrt{3}, 6\sqrt{3})$ and $(\sqrt{3}, 6\sqrt{3})$

Exercises 4.2

13. It is because the function is discontinuous at $f(1) = 0 \neq \lim_{x \to 1^-} f(x) = 1$

In this case, Rolle's theorem fails since the domain must continuous for $x \in [0,1]$

14a(iii). $y = x^3 - 3x^2 + 4$

$$\frac{dy}{dx} = 3x^2 - 6x = 3x(x-2)$$

when $\frac{dy}{dx} = 0$, $x = 0$ or $x = 2$

b. let $f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$

$$f'(x) = nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \ldots + a_1$$

Rolle's Theorem state that if there exist x_1 and x_2 $(x_1 < x_2)$ such that $f(x_1) = f(x_2)$, there exist $f'(c) = 0$ $(x_1 \leq c \leq x_2)$.

now, since $nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \ldots + a_1$ represent the derivative of $f(x)$ and it also satisfy the requirements:
$\Rightarrow f(x)$ is continuous at $(-\infty, \infty)$
$\Rightarrow f'(x)$ is differentiable at $(-\infty, \infty)$

it is proved.

18. let $f(x) = x^3 + ax^2 + bx + c$

$$f'(x) = 3x^2 + 2ax + b$$

critical points when $f'(x) = 0$

$$\Rightarrow x_{1,2} = \frac{-2a \pm \sqrt{4a^2 - 12b}}{6}$$

since $f(x)$ has only 2 critical points there will be at most 2 turning points resulting $f(x) = 0$ has at most 3 solutions

20. $f(x) = x^3 + \frac{4}{x^2} + 7$, $(-\infty, 0)$

$$f'(x) = 3x^2 - \frac{8}{x^3}$$

critical points when $f'(x) = 0$ or $f'(x)$ is undefined

$$f'(x) = 0 = 3x^2 - \frac{8}{x^3}$$

$$\frac{8}{3} = x^5 \Rightarrow x = \sqrt[5]{\frac{8}{3}}$$ (out of intervals)

$f(x)$ is undefined $\Rightarrow x = 0$ (out of intervals)

now since there's no turning point, $f(x)$ has at most 1 solution for $f(x) = 0$.

$\lim_{x \to 0^-} f(x) = \infty > 0$ and

$\lim_{x \to -\infty} f(x) = -\infty < 0$.

Since $f(x)$ is continuous, by IVT theorem there exist $x_0 \in (-\infty, 0)$ such that $f(x_0) = 0$

23. $r(\theta) = \theta + \sin^2\left(\frac{\theta}{3}\right) - 8$, $(-\infty, \infty)$

$$r'(\theta) = 1 + \frac{1}{3}\sin\left(\frac{2\theta}{3}\right)$$

critical points when $r'(\theta) = 0$

$$\Rightarrow \sin\left(\frac{2\theta}{3}\right) = -3$$ (no solution for θ)

now since there's no turning point, $r(\theta)$ has at most 1 solution for $r(\theta) = 0$.

$\lim_{\theta \to -\infty} r(\theta) = -\infty < 0$ and $\lim_{\theta \to \infty} r(\theta) = \infty > 0$.

since $r(\theta)$ is continuous, by IVT theorem there exist $\theta \in (-\infty, \infty)$ such that $f(\theta) = 0$

27. Since $f'(x) = 0$, $f(x)$ must be a constant function, resulting $f(a) = f(b)$ for every $a < b$.
therefore for all $x \in (-\infty, \infty)$,

$$f(x) = f(-1) = 3$$

33a. let $g(x) = \frac{1}{x} \Rightarrow g'(x) = -\frac{1}{x^2} = y'$

since $g'(x) = y'$, by Corollary 2,

$$y = g(x) + C$$
$$= \frac{1}{x} + C$$

b. let $g(x) = x + \frac{1}{x} \Rightarrow g'(x) = 1 - \frac{1}{x^2} = y'$

since $g'(x) = y'$, by Corollary 2,

$$y = g(x) + C$$
$$= x + \frac{1}{x} + C$$

c. let $g(x) = 5x - \frac{1}{x} \Rightarrow g'(x) = 5 + \frac{1}{x^2} = y'$

since $g'(x) = y'$, by Corollary 2,

$$y = g(x) + C$$
$$= 5x - \frac{1}{x} + C$$

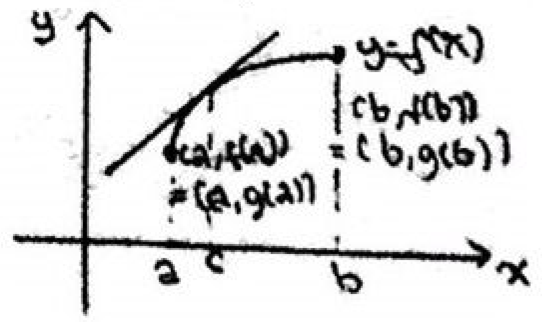
60. since $f(x)$ and $g(x)$ are differentiable on $[a,b]$.

by Cauchy Mean Value Theorem, there exist $c \in (a,b)$ such that:

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{g(b)-g(a)}{g(b)-g(a)} = 1$$

$$f'(c) = g'(c)$$

when $x = c$, tangent line to graph of f and tangent line to graph g are parallel.



61. $f'(x) \leq 1$ for $1 \leq x \leq 4$.

by MVT, there exist x such that

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = f'(x)$$

for $x \in (x_0, x_1)$

$$f(x_1) - f(x_0) = f'(x)(x_1 - x_0)$$
$$f(4) - f(1) = f'(x) \cdot 3 \leq 3$$
$$f(4) - f(1) \leq 3$$

63. let $f(t) = \cos t$ on $[0, x]$.

It is known that $f(t)$ is continuous at every point and also differentiable on $(0, x)$, by MVT,

there exist x_0 such that

$$f'(x_0) = \frac{\cos x - \cos 0}{x - 0} = \frac{\cos(x) - 1}{x}$$

for $x_0 \in (0, x)$

$$f'(t) = -\sin t \Rightarrow |f'(t)| \leq 1$$

$$|f'(x_0)| \leq 1$$
$$\left|\frac{\cos(x) - 1}{x}\right| \leq 1$$
$$|\cos(x) - 1| \leq |x| \text{ for } x \in [0, \infty)$$

since both $|\cos(x) - 1|$ and $|x|$ are symmetric to the y-axis,

$|\cos(x) - 1| \leq |x|$ is valid for $x \in (-\infty, \infty)$

65. let's suppose that both $f(x)$ and $g(x)$ exist in $[a,b]$,

$\rightarrow$ we know that for every point the functions have the same rate of change

$$\Rightarrow f'(x) = g'(x)$$

by Corollary 2,

$$f(x) = g(x) + C$$
$$f(a) = g(a) + C$$
$$g(a) = g(a) + C$$ (since they start at the same point)

$$\Rightarrow C = 0$$
$$\Rightarrow f(x) = g(x)$$ (identical)

Exercises 4.3

15. Increasing on: $(-2,0)$, $(2,4)$
decreasing on: $(-4,-2)$, $(0,2)$

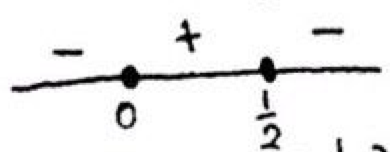
absolute maximum at $(-4,2)$
absolute minimum at $(2,-3)$
local maximum at $(0,1)$, $(4,-1)$
local minimum at $(-2,0)$

16. Increasing on: $(-4,-3.1)$, $(-1.5,1)$, $(2,4)$
decreasing on: $(-3.1,-1.5)$, $(1,2)$
absolute maximum at $(4,2)$
absolute minimum at $(-1.5,-1)$
local maximum at $(-3.1,1)$ and $(1,1)$
local minimum at $(-4,0)$ and $(2,0)$

17. increasing on: $(-4,-1)$, $(0.5,2)$, $(2,4)$
decreasing on: $(-1,0.5)$
absolute maximum at $(4,3)$
no absolute minimum
local maximum at $(-1,2)$ and $(2,1)$
local minimum at $(-4,-1)$ and $(0.5,-1)$

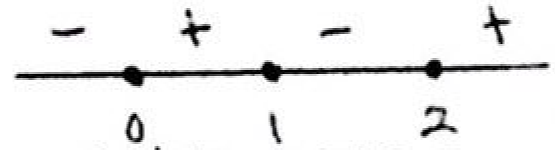
18. increasing on: $(-4,-2.5)$, $(-1,1)$, $(3,4)$
decreasing on: $(-2.5,-1)$, $(1,3)$
no absolute maximum
no absolute minimum
local maximum at $(-2.5,1)$, $(1,2)$ and $(4,2)$
local minimum at $(-1,0)$ and $(3,1)$

23. $f(\theta)=3\theta^2-4\theta^3$
$f'(\theta)=6\theta-12\theta^2$
critical points when $f'(\theta)=0$
$6\theta-12\theta^2=0$
$6\theta(1-2\theta)=0$
$\theta=0 \quad \theta=\frac{1}{2}$



increasing on: $(0,\frac{1}{2})$
decreasing on: $(-\infty,0)$, $(\frac{1}{2},\infty)$
no absolute maximum
no absolute minimum
local maximum at $(\frac{1}{2},\frac{1}{4})$
local minimum at $(0,0)$

28. $g(x)=x^4-4x^3+4x^2$
$g'(x)=4x^3-12x^2+8x$
critical points when $g'(x)=0$
$4x(x^2-3x+2)=0$
$x=0 \quad x=1 \quad x=2$

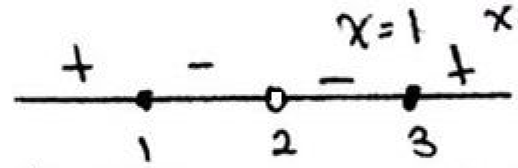


no absolute maximum
increasing on: $(0,1)$, $(2,\infty)$
decreasing on: $(-\infty,0)$, $(1,2)$

absolute minimum at $(0,0)$ and $(2,0)$
local maximum at $(1,1)$
local minimum at $(0,0)$ and $(2,0)$

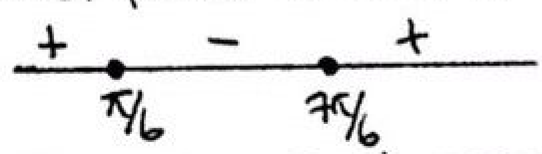
35. $f(x)=\dfrac{x^2-3}{x-2}=x+2+\dfrac{1}{x-2}$

$f'(x)=1-\dfrac{1}{(x-2)^2}$
critical points when $f'(x)=0$ and $f'(x)$ is undefined
$(x-2)^2=1$
$x=1 \quad x=3$ $\qquad x=2$

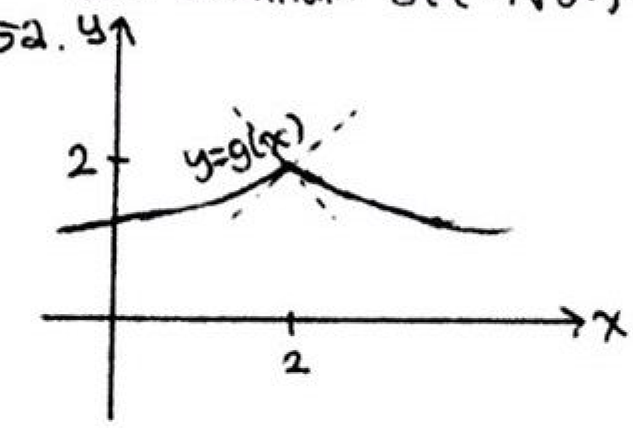


increasing on: $(-\infty,1)$, $(3,\infty)$
decreasing on: $(1,2)$, $(2,3)$
no absolute maximum
no absolute minimum
local maximum at $(1,2)$
local minimum at $(3,6)$

55a. $f(x)=\sqrt{3}\cos x+\sin x=2(\sin\frac{\pi}{3}\cos x+\cos\frac{\pi}{3}\sin x)$
$=2\sin(x+\frac{\pi}{3})$
$f'(x)=2\cos(x+\frac{\pi}{3})$
critical points when $f'(x)=0$, $x=\frac{\pi}{6} \quad x=\frac{7\pi}{6}$



local maximum at $(\frac{\pi}{6},2)$, $(2\pi,\sqrt{3})$
local minimum at $(0,\sqrt{3})$, $(\frac{7\pi}{6},-2)$

65a.


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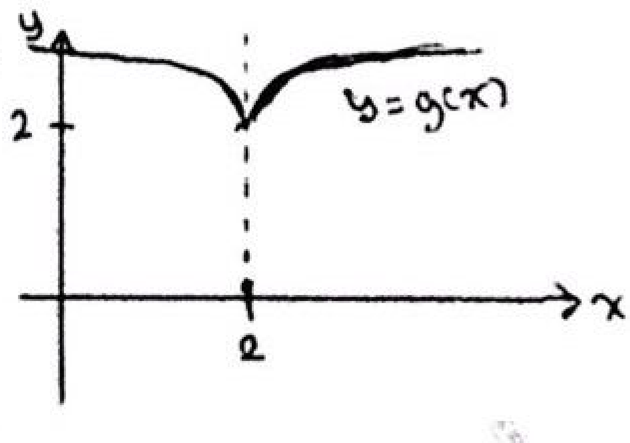


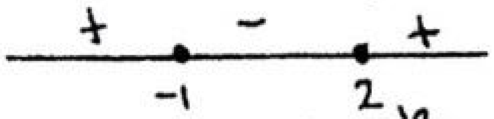
Exercises 4.4

1. $y = \frac{x^3}{3} - \frac{x^2}{2} - 2x + \frac{1}{3}$

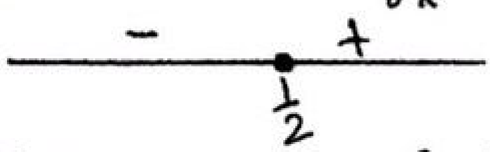
$\frac{dy}{dx} = x^2 - x - 2$

$\frac{d^2y}{dx^2} = 2x - 1$

critical points when $\frac{dy}{dx} = 0 \Rightarrow x = 2, x = -1$



inflection points when $\frac{d^2y}{dx^2} = 0 \Rightarrow x = \frac{1}{2}$



increasing on: $(-\infty, -1), (2, \infty)$

decreasing on: $(-1, 2)$

local maxima at $(-1, 3/2)$

local minima at $(2, -3)$

inflection points at $(\frac{1}{2}, -3/4)$

concave up on $(\frac{1}{2}, \infty)$

concave down on $(-\infty, \frac{1}{2})$

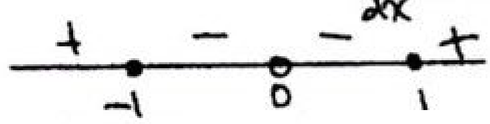
4. $y = \frac{9}{14}(x^{7/3} - 7x^{1/3})$

$\frac{dy}{dx} = \frac{3}{2}(x^{4/3} - x^{-2/3})$

$\frac{d^2y}{dx^2} = 2x^{1/3} + x^{-5/3}$

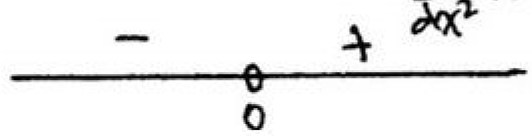
critical points when $\frac{dy}{dx} = 0 \Rightarrow x = -1, x = 1$

$\frac{dy}{dx}$ is undefined $\Rightarrow x = 0$



inflection points when $\frac{d^2y}{dx^2} = 0 \Rightarrow x \in \emptyset$

$\frac{d^2y}{dx^2}$ is undefined $\Rightarrow x = 0$



increasing on: $(-\infty, -1], (1, \infty)$

decreasing on: $(-1, 1]$

local maxima at $(-1, \frac{27}{7})$

local minima at $(1, -\frac{27}{7})$

inflection points at $(0, 0)$

concave up on $(0, \infty)$

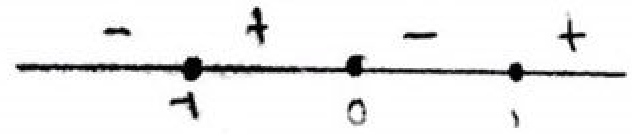
concave down on $(-\infty, 0)$

17. $y = x^4 - 2x^2 = x^2(x^2 - 2)$

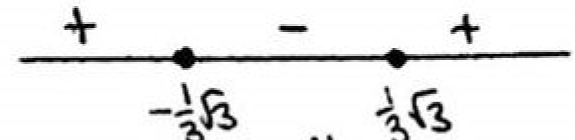
$\frac{dy}{dx} = 4x^3 - 4x$

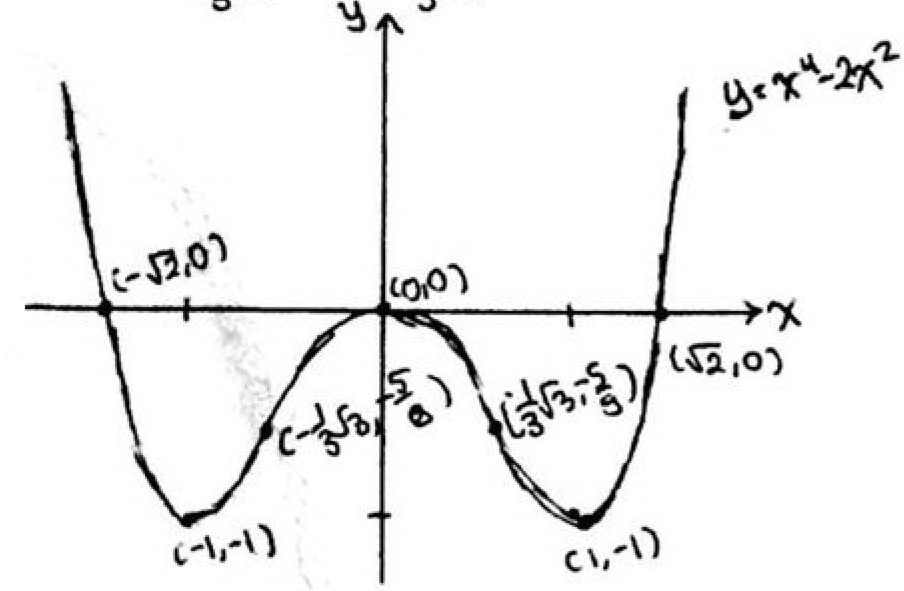
$\frac{d^2y}{dx^2} = 12x^2 - 4$

critical points when $\frac{dy}{dx} = 0 \Rightarrow x = -1, x = 0, x = 1$



inflection points when $\frac{d^2y}{dx^2} = 0 \Rightarrow x = \pm\frac{1}{3}\sqrt{3}$





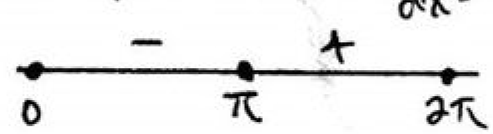
23. $y = x + \sin x \qquad 0 \le x \le 2\pi$

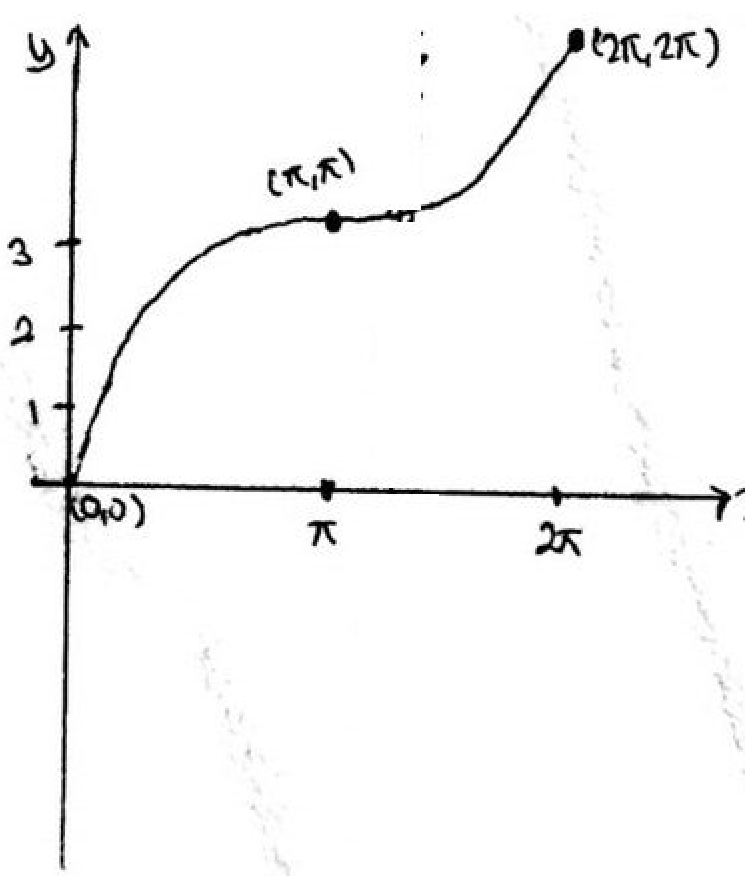
$\frac{dy}{dx} = 1 + \cos x$

$\frac{d^2y}{dx^2} = -\sin x$

critical points when $\frac{dy}{dx} = 0 \Rightarrow x = \pi$

inflection points when $\frac{d^2y}{dx^2} = 0 \Rightarrow x = 0, x = \pi, x = 2\pi$





32. $y = \frac{\sqrt{1-x^2}}{2x+1} = f(x)$ $\quad 1-x^2 \ge 0 \Rightarrow -1 \le x \le 1$

$\frac{dy}{dx} = \frac{-(x+2)}{(2x+1)^2\sqrt{1-x^2}}$

$\frac{d^2y}{dx^2} = \frac{-4x^3-12x^2+7}{(2x+1)^3(1-x^2)^{3/2}}$

critical points when $\frac{dy}{dx} = 0 \Rightarrow x = -2$

$\frac{dy}{dx}$ is undefined $\Rightarrow x = -1, x = -\frac{1}{2}, x = 1$

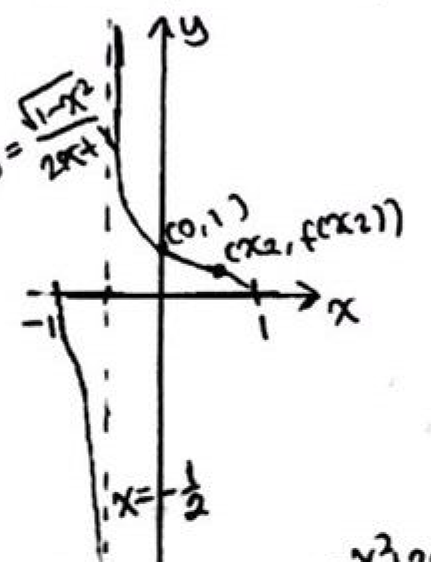
inflection points when $\frac{d^2y}{dx^2} = 0 \Rightarrow$ $x_1 \approx -0.9165$, $x_2 \approx 0.6888$, $x_3 \approx -2.7723$

$\frac{d^2y}{dx^2}$ is undefined $\Rightarrow x = -1, x = -\frac{1}{2}, x = 1$

$\frac{dy}{dx}$: $- \; - \quad - \quad -$

$-1 \quad x_1 \quad -\frac{1}{2} \quad x_2 \quad 1$

$\frac{d^2y}{dx^2}$: $+ \; - \quad + \quad -$



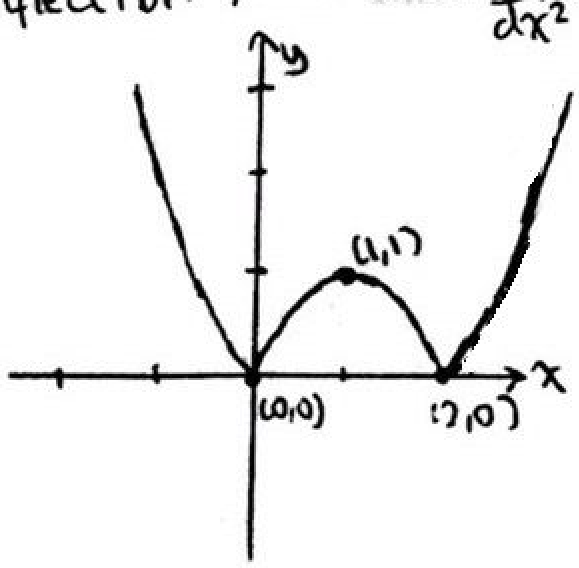
46. $y = |x^2-2x| \begin{cases} -x^2+2x, & 0 \le x \le 2 \\ x^2-2x, & \text{otherwise} \end{cases}$

$\frac{dy}{dx} = \begin{cases} -2x+2, & 0 \le x \le 2 \\ 2x-2, & \text{otherwise} \end{cases}$

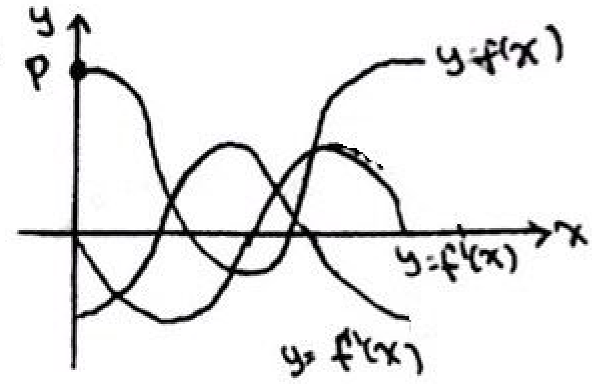
$\frac{d^2y}{dx^2} \begin{cases} -2, & 0 \le x \le 2 \\ 2, & \text{otherwise} \end{cases}$

critical points when $\frac{dy}{dx} = 0 \Rightarrow x = 1$

inflection points when $\frac{d^2y}{dx^2} = 0 \Rightarrow x = \emptyset$



73.



76. $y = \frac{x^2-49}{x^2+5x-14} = \frac{x-7}{x-2}$; $x \ne 2, x \ne -7$

$= 1 + \frac{-5}{x-2}$

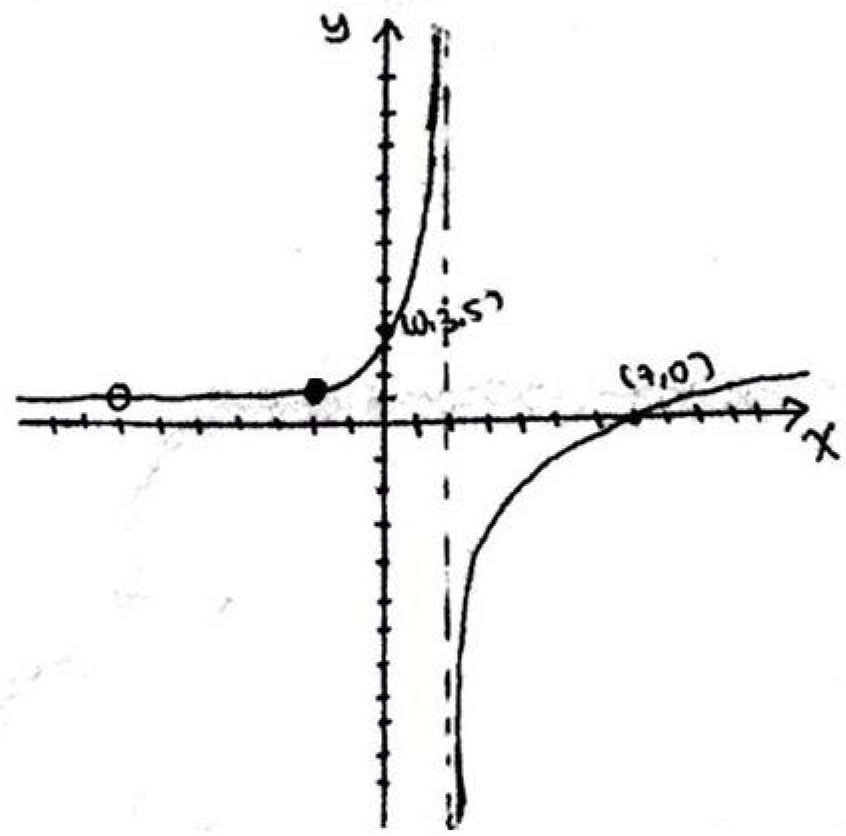
$y' = 5(x-2)^{-2}$ $\qquad$ y' doesn't exist $\Rightarrow x = 2$

$y'' = -10(x-2)^{-3}$

$\frac{dy}{dx}$: $+ \quad +$

2

$\frac{d^2y}{dx^2}$: $+ \quad -$



91. $y = -\frac{x^2-2}{x^2-1} = -1 + \frac{1}{x^2-1}$ $\quad (x \ne 1, x \ne -1)$

$y' = -2x(x^2-1)^{-2}$

$y'' = -2[(x^2-1)^{-2} - 4x^2(x^2-1)^{-3}]$

$= \frac{2(3x^2+1)}{(x^2-1)^3}$

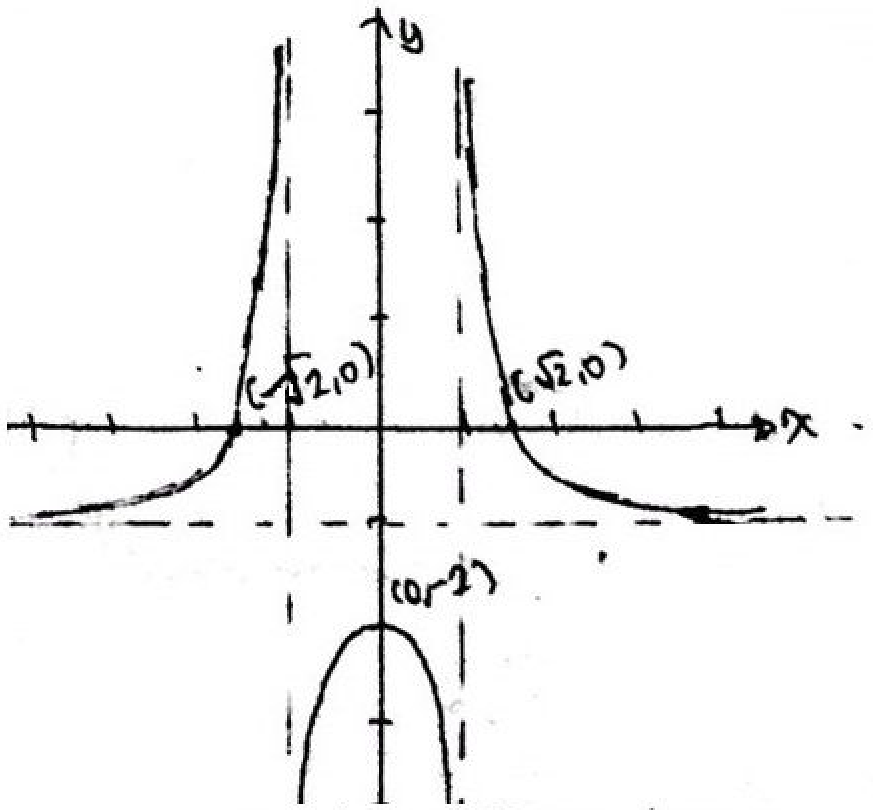
$y' = 0 \Rightarrow x = 0$

y' doesn't exist $\Rightarrow x = \pm 1$

$\lim_{x \to \pm\infty} y = -1$ (horizontal asymptote)

$\frac{dy}{dx}$: $+ \quad + \quad - \quad -$

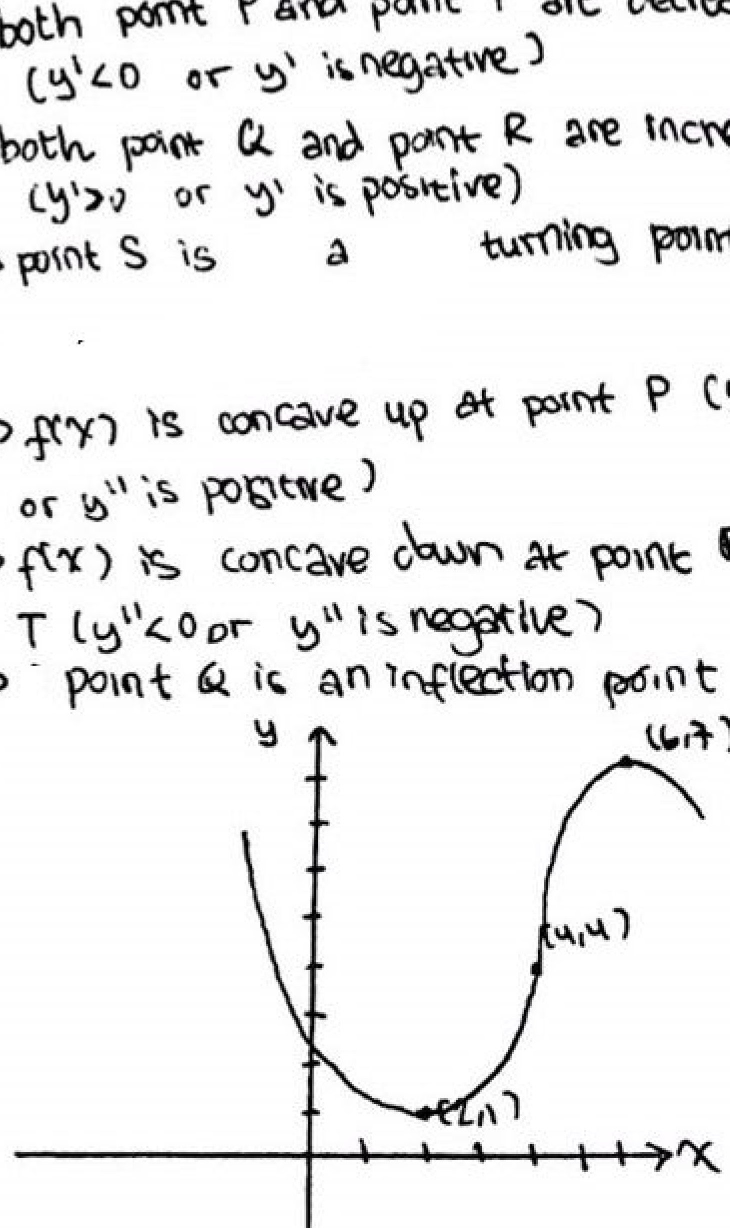
$\frac{d^2y}{dx^2}$: $+ \quad -1 \quad - \quad 0 \quad - \quad 1 \quad +$



93. => both point P and point T are decreasing. ($y'<0$ or y' is negative)
=> both point Q and point R are increasing. ($y'>0$ or y' is positive)
=> point S is a turning point ($y'=0$)

=> $f(x)$ is concave up at point P ($y''>0$ or y'' is positive)
=> $f(x)$ is concave down at point R, S and T ($y''<0$ or y'' is negative)
=> point Q is an inflection point ($y''=0$)

95.



98a. toward the origin:
=> $0 \le t < 1.5$
=> $4 < t < 10$
=> $12 < t < 13.5$
away from origin:
=> $1.5 < t < 4$
=> $10 < t < 12$
=> $13.5 < t \le 16$
b. $t=4$, $t=12$
c. $t=1.5$, $t=6$, $t=8$, $t=11$, $t=13.5$
d. positive:
=> $0 \le t < 1.5$
=> $6 < t < 8$
=> $11 < t < 13.5$
negative:
=> $1.5 < t < 6$
=> $8 < t < 11$
=> $13.5 < t \le 16$

99. inflection point located on $(x_0, f(x_0))$ where $x_0 \approx 60000$ units
=> there is a marginal cost change (from decreasing to increasing)

104. Since the second derivative form is continuous at every x in domain and never equals to 0,
↳ the function doesn't have any inflection point
=> the function is concave or convex for every $x : D$

108. inflection points when $f''(x)$ is 0

$$y = f(x) = ax^3 + bx^2 + cx + d \quad (a \ne 0)$$

$$y' = f'(x) = 3ax^2 + 2bx + c$$

$$y'' = f''(x) = 6ax + 2b = 0$$

$$x = -\frac{b}{3a}$$

∴ for every cubic function $\left(x = -\frac{b}{3a}, y = f\left(-\frac{b}{3a}\right)\right)$ exist as the inflection point of the function