Yohandi - math homework week 12

Exercises 16.4

1. $M = -y = -a \sin t \quad N = x = a \cos t$

$\Rightarrow \frac{dy}{dt} = a \cos t \quad \Rightarrow \frac{dx}{dt} = -a \sin t$

$$\oint_C M dy - N dx = \int_0^{2\pi} (-a\sin t)(a\cos t) - (a\cos t)(-a\sin t) dt$$

$$= 0$$

$$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\, dy = \iint_R \left(\frac{\partial(-y)}{\partial x} + \frac{\partial x}{\partial y}\right) dx\, dy$$

$$= 0$$

$\therefore$ Equation (3) is verified

4. $M = -y = -a \sin t \quad N = x = a \cos t$

$\Rightarrow \frac{dy}{dt} = a \cos t \quad \Rightarrow \frac{dx}{dt} = -a \sin t$

$$\oint_C M dx + N dy = \int_0^{2\pi} (-a\sin t)(-a\sin t) + (a\cos t)(a\cos t)\, dt$$

$$= 2\pi a^2$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \iint_R \left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}\right) dx\, dy$$

$$= 2\pi a^2$$

$\therefore$ Equation (4) is verified

5. $M = x - y \quad N = y - x$

$\Rightarrow \text{flux}_{out} = \oint_C F \cdot n\, ds$

$$= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\, dy$$

$$= \iint_R \left(\frac{\partial(x-y)}{\partial x} + \frac{\partial(y-x)}{\partial y}\right) dx\, dy$$

$$= 2$$

$\Rightarrow \text{circulation}_{\text{counterclockwise}} = \iint_R \left(\frac{\partial(y-x)}{\partial x} - \frac{\partial(x-y)}{\partial y}\right) dx\, dy$

$$= 0$$

9. $M = xy + y^2 \quad N = x - y$

$\Rightarrow \text{flux}_{out} = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\, dy$

$$= \iint_R \left(\frac{\partial(xy+y^2)}{\partial x} + \frac{\partial(x-y)}{\partial y}\right) dx\, dy$$

$$= -\frac{11}{60}$$

$\Rightarrow \text{circulation}_{\text{counterclockwise}} = \iint_R \left(\frac{\partial(x-y)}{\partial x} - \frac{\partial(xy+y^2)}{\partial y}\right) dx\, dy$

$$= -\frac{7}{60}$$

19. $M = 2xy^3 \quad N = 4x^2y^2$

$\Rightarrow \text{circulation}_{\text{counterclockwise}}$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

$$= \iint_R \left(\frac{\partial(4x^2y^2)}{\partial x} - \frac{\partial(2xy^3)}{\partial y}\right) dx\, dy$$

$$= \frac{2}{33}$$

23. $M = 6y + x \quad N = y + 2x$

$$\iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \iint_R \left(\frac{\partial(y+2x)}{\partial x} - \frac{\partial(6y+x)}{\partial y}\right) dx\, dy$$

$$= -16\pi$$

25. $y = a \sin t \quad x = a \cos t$

$\frac{dy}{dt} = a \cos t \quad \frac{dx}{dt} = -a \sin t$

$$\text{Area} = \oint_C \frac{1}{2} x\, dy - \frac{1}{2} y\, dx$$

$$= \int_0^{2\pi} \frac{1}{2}(a\cos t)(a\cos t) - \frac{1}{2}(a\sin t)(-a\sin t)\, dt$$

$$= \pi a^2$$

27. $y = \sin^3 t \quad x = \cos^3 t$

$\frac{dy}{dt} = 3\sin^2 t \cos t \quad \frac{dx}{dt} = -3\cos^2 t \sin t$

$$\text{Area} = \oint_C \frac{1}{2} x\, dy - \frac{1}{2} y\, dx$$

$$= \int_0^{2\pi} \frac{1}{2}\cos^3 t(3\sin^2 t\cos t) - \frac{1}{2}\sin^3 t(-3\cos^2 t \sin t)\, dt$$

$$= \frac{3\pi}{8}$$

Exercises 16.5

1. $z = x^2+y^2 \le 4$

let $x = r\cos\theta$, $y = r\sin\theta$

$\cancel{|r|\le 2}$ with $\theta \in [0, 2\pi]$,

$\cancel{\Rightarrow \theta \in [0, 2\pi]}$ $z = x^2+y^2 = r^2 \le 4$

$\cancel{z = x^2+y^2 = r^2}$ $\Rightarrow |r| \le 2$

$r(r,\theta) = (r\cos\theta)i + (r\sin\theta)j + r^2k$

with $\theta \in [0, 2\pi]$ & $|r| \le 2$

5. $x^2+y^2+z^2 = 9$, $z = \sqrt{x^2+y^2}$

let $x = r\cos\theta\sin\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\phi$

for $z \ge \sqrt{x^2+y^2}$,

$\Rightarrow 3\cos\phi \ge \sqrt{9\cos^2\theta\sin^2\phi + 9\sin^2\theta\sin^2\phi}$

$\Rightarrow \tan\phi \le 1$, $\phi \in [0, \frac{\pi}{4}]$

$r(\theta) = (3\cos\theta\sin\phi)i + (3\sin\theta\sin\phi)j + (3\cos\phi)k$,

$\theta \in [0, 2\pi]$ & $\phi \in [0, \frac{\pi}{4}]$

7. $x^2+y^2+z^2 = 3$, $z \in [-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}]$

$\Rightarrow r = \sqrt{3}$

$\Rightarrow \sqrt{3}\cos\phi \in [-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}]$

$\Rightarrow \phi \in [\frac{\pi}{3}, \frac{2\pi}{3}]$

$r(\theta) = (\sqrt{3}\cos\theta\sin\phi)i + (\sqrt{3}\sin\theta\sin\phi)j + (\sqrt{3}\cos\phi)k$,

$\theta \in [0, 2\pi]$ & $\phi \in [\frac{\pi}{3}, \frac{2\pi}{3}]$

15. $(x-2)^2 + z^2 = 4$, $y \in [0, 3]$

let $x-2 = r\cos\theta$, $z = r\sin\theta$

with $\theta \in [0, 2\pi)$,

$\cancel{r^2 \le 4}$ $r^2 = 4$

$\cancel{\Rightarrow |r| \le 2}$ $\Rightarrow r = \pm 2$

$r(\theta) = (2+2\cos\theta, \lambda, 2\sin\theta)$

with $\lambda \in [0, 3]$ & $\theta \in [0, 2\pi)$

17. $y+2z = 2$, $x^2+y^2 = 1$

let $x = r\cos\theta$, $y = r\sin\theta$

with $\theta \in [0, 2\pi)$,

$r^2 = 1$

$\Rightarrow r = \pm 1$

since $y+2z = 2$ is inside $x^2+y^2 = 1$,

$|r_0| \le |r|$

$\Rightarrow r_0 \in [-1, 1]$ (we take radius ≥ 0)

$\Rightarrow z = 1 - \frac{1}{2}r_0\cos\theta$

$r(r,\theta) = (r\cos\theta)i + (r\sin\theta)j + (1 - \frac{1}{2}r\cos\theta)k$

with $r \in [0, 1]$, $\theta \in [0, 2\pi)$

$A = \iint_R |r_r \times r_\theta| \, dr\, d\theta$

$= \int_0^{2\pi}\int_0^1 \left| (\cos\theta i + \sin\theta j - \frac{\sin\theta}{2}k) \times (-r\sin\theta i + r\cos\theta j - \frac{r\cos\theta}{2}k) \right| dr\, d\theta$

$= \frac{\pi}{2}\sqrt{5}$

23. $z = 2 - x^2 - y^2$, $z = \sqrt{x^2+y^2}$

let $x = r\cos\theta$, $y = r\sin\theta$.

with $\theta \in [0, 2\pi)$,

$z = 2 - (x^2+y^2)$

$= 2 - r^2$

$\ge \sqrt{x^2+y^2}$

$\ge r$

since $2 - r^2 \ge r \Rightarrow r \in [-2, 1]$

(we take radius ≥ 0)

$A = \iint_R |r_r \times r_\theta| \, dr\, d\theta$

$= \int_0^{2\pi}\int_0^1 \left| (\cos\theta i + \sin\theta j - 2rk) \times (-r\sin\theta i + r\cos\theta j) \right| dr\, d\theta$

$= \frac{\pi}{6}(\sqrt{125} - 1)$

27. $r_r = \cos\theta\, i + \sin\theta\, j + k$

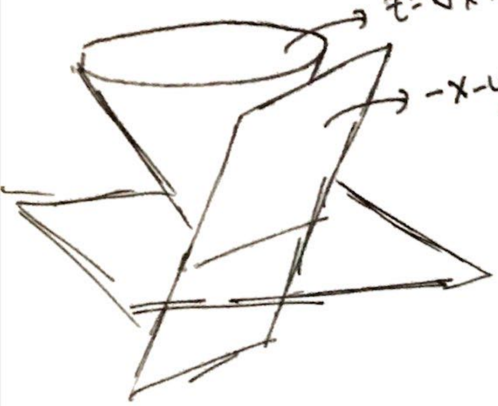
$r_\theta = -r\sin\theta\, i + r\cos\theta\, j$

$n(x,y,z)$

$n(2, \frac{\pi}{4}) = (\cos\frac{\pi}{4} i + \sin\frac{\pi}{4} j + 1) \times (-2\sin\frac{\pi}{4} i + 2\cos\frac{\pi}{4} j)$

$= -\sqrt{2}\, i - \sqrt{2}\, j + 2k$

$\Rightarrow -\sqrt{2}x - \sqrt{2}y + 2z = 0$



33 a. $x = a\cos\theta\cos\phi$, $y = b\sin\theta\cos\phi$, $z = c\sin\phi$

LHS:

$\Rightarrow \left(\frac{x^2}{a^2}\right) + \left(\frac{y^2}{b^2}\right) + \left(\frac{z^2}{c^2}\right)$

$= \cos^2\theta\cos^2\phi + \sin^2\theta\cos^2\phi + \sin^2\phi$

$= \cos^2\phi + \sin^2\phi$

$= 1$

b. $A = \iint_R |r_\theta \times r_\phi| \, d\theta\, d\phi$

$= \int_0^{\pi}\int_0^{2\pi} |(-a\sin\theta\cos\phi\, i + b\cos\theta\cos\phi\, j) \times (-a\cos\theta\sin\phi\, i - b\sin\theta\sin\phi\, j + c\cos\phi\, k)| \, d\theta\, d\phi$

$= \int_0^{\pi}\int_0^{2\pi} \sqrt{b^2c^2\cos^2\theta\cos^4\phi + a^2c^2\sin^2\theta\cos^4\phi + a^2b^2\sin^2\phi\cos^2\phi} \, d\theta\, d\phi$

37. $x^2+y^2-z=0$. $z=2$

let $x = r\cos\theta$, $y = r\sin\theta$.

$\Rightarrow r^2 \le 2$

$\Rightarrow r \in [-\sqrt{2}, \sqrt{2}]$ we take radius ≥ 0.

$r(r,\theta) = (r\cos\theta) i + (r\sin\theta) j + (2r) k$.

$A = \iint_R |r_r \times r_\theta| \, dr\, d\theta$

$= \int_0^{2\pi}\int_0^{\sqrt{2}} |(\cos\theta\, i + \sin\theta\, j + 2r\, k) \times (-r\sin\theta\, i + r\cos\theta\, j)| \, dr\, d\theta$

$= \frac{13\pi}{3}$

41. Surface area

$= \iint_R \frac{|\nabla f|}{|\nabla f \cdot k|} \, dA$

$= \int_0^2\int_0^{3x} \frac{|2x i - 2j - 2k|}{|(2x i - 2j - 2k) \cdot -2|} \, dy\, dx$

$= \sqrt{216} - \sqrt{8}$

Exercises 16.6

1. $r(x,z) = x i + x^2 j + z k$.

$|n| = |r_x \times r_z| = \sqrt{(2x)^2 + 1^2} = \sqrt{4x^2+1}$

$\iint_S G(x,y,z)\, d\sigma = \int_0^3\int_0^2 x\sqrt{4x^2+1} \, dx\, dz = \frac{17\sqrt{17}}{4} - \frac{1}{4}$

6. $\iint_R F(x,y,z)\, dS = \iint_R F(x,y,z) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + \left(\frac{\partial z}{\partial z}\right)^2} \, dy\, dx$

$= \iint_R \left(\sqrt{2(x^2+y^2)} - x\sqrt{2}\right) dy\, dx$

$= \int_0^{2\pi}\int_0 \left(\sqrt{2r^2} - r\cos\theta\sqrt{2}\right) r\, dr\, d\theta$

$= \frac{2\sqrt{2}}{3}\pi$

13. $\iint_R G(x,y,z)\, dS = \iint_R G(x,y,z) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + \left(\frac{\partial z}{\partial z}\right)^2} \, dy\, dx$

$= \int_0^1\int_0^{1-x} (2-x-y)\, 3\, dy\, dx$

$= 2$

19. $\iint_R F\cdot n\, d\sigma = \int_{-2}^{2}\int_{0}^{1} \langle y^4-8y^2+16, x, 3y^2-12\rangle$

$\cdot$

$\langle 0, 2y, 1\rangle\, dx\, dy$

$= -32$

23. $\iint_R F\cdot n\, d\sigma = \int_0^a\int_0^a \langle 2xy, 2yz, 2xz\rangle$

$\cdot$

$\langle 1, 1, 1\rangle\, dy\, dx$

$= \frac{13}{6}a^4$

25. $\iint_R F\cdot n\, d\sigma = \iint_R \langle xy, 0, -z\rangle$

$\cdot$

$\left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1\right\rangle dy\, dx$

$= \int_0^{2\pi}\int_0^1 \frac{r^2(1+r\cos^2\theta\sin\theta)}{r}\cdot r\, dr\, d\theta$

$= \frac{2\pi}{3}$

29. $\iint_R F\cdot n\, d\sigma = \iint_R \langle -1, 2, 3\rangle \cdot \langle 0, 0, 1\rangle \cdot 2\cdot 3\, dA$

$= 18$

37. $\iint_R F\cdot n\, d\sigma = \int_{-2}^{2}\int_{0}^{1} \langle y^4-8y^2+16, x, 3y^2-12\rangle$

$\cdot$

$\langle 0, 2y, 1\rangle\, dx\, dy$

$= -32$