CSE 211: Discrete Mathematics **(Due: 17/01/21)**

Homework #4

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Course Policy: Read all the instructions below carefully before you start working on the assignment, and before you make a submission.

- It is not a group homework. Do not share your answers to anyone in any circumstance. Any cheating means at least -100 for both sides.
- Do not take any information from Internet.
- No late homework will be accepted.
- For any questions about the homework, send an email to gizemsungu@gtu.edu.tr
- The homeworks (both latex and pdf files in a zip file) will be submitted into the course page of Moodle.
- The latex, pdf and zip files of the homeworks should be saved as "Name_Surname_StudentId".{tex, pdf, zip}.
- If the answers of the homeworks have only calculations without any formula or any explanation -when needed- will get zero.
- Writing the homeworks on Latex is strongly suggested. However, hand-written paper is still accepted **IFF** hand writing of the student is clear and understandable to read, and the paper is well-organized. Otherwise, the assistant cannot grade the student's homework.

Problem 1 (15+15=30 points)

Consider the nonhomogeneous linear recurrence relation $a_n = 3a_{n-1} + 2^n$.

(a) Show that whether $a_n = -2^{n+1}$ is a solution of the given recurrence relation or not. Show your work step by step.
(Solution)

If $a_n = -2^{n+1}$, then $a_{n-1} = -2^{n+1-1} = -2^n$. We need to prove that $3a_{n-1} + 2^n = 3(-2^n) + 2^n$

$3a_{n-1} + 2^n = 3(-2^n) + 2^n$
$3a_{n-1} + 2^n$ = -3.2^n + 2^n
$3a_{n-1} + 2^n$ = 2^n(-3+1)
$3a_{n-1} + 2^n$ = 2^n.(-2)
$3a_{n-1} + 2^n = -2^{n+1}$
$3a_{n-1} + 2^n = a_n$

First, we accepted that $a_n = -2^{n+1}$, then showed that $3a_{n-1} + 2^n = -2^{n+1}$ which equals to a_n. So we proved that $a_n = -2^{n+1}$ is a solution of the given recurrence relation.

(b) Find the solution with $a_0 = 1$.
(Solution)

1. a_n - $3a_{n-1}$ = 2^n

2. P_r = r-3

3. Q_n = 2^n = 2^n . 1
$Q_r = (r-2)^{1+1} = (r-2)^2$

4. Q_r . P_r = (r-3)$(r-2)^2$
H_n = A.3^n + B.2^n + C.n.2^n

5. R_n = B.2^n + C.n.2^n

6. B.2^n + C.n.2^n - 3(B.2^{n-1} + C(n-1).2^{n-1}) = 2^n
B.2^n + C.n.2^n - 3(2^{n-1}(B + Cn - C) = 2^n (Multiply both sides by 1/2^{n-1})
2B + 2Cn - 3B - 3CN + 3C = 2
-B - Cn + 3C = 2 (C=0 because there is no element with 'n' in left side)
-B = 2, B = -2

7. H_n = A.3^n - 2.2^n

8. If a_0 = 1 = A - 2, then A = 3.
a_n = 3.3^n - 2.2^n
a_n = 3^{n+1} - 2^{n+1}

Problem 2 (35 points)

Solve the recurrence relation f(n) = 4f(n-1) - 4f(n-2) + n^2 for f(0) = 2 and f(1) = 5.
(Solution)

1. Associated linear homogeneous equation is f(n) = 4f(n-1) - 4f(n-2)

   P_r = r^2 - 4r + 4 = $(r-2)^2$

   r = 2(Double root)

3. f_n^h = ($\alpha_{1,0}$ + $\alpha_{1,1}$.n)r^n = ($\alpha_{1,0}$ + $\alpha_{1,1}$.n)2^n

4. f_n^p = An^2 + Bn + C = 4A$(n-1)^2$ + 4B(n-1) + 4C - 4A$(n-2)^2$ - 4B(n-2) - 4C + n^2

   An^2 + Bn + C = 4An^2 - 8An + 4A + 4Bn - 4B + 4C - 4An^2 + 16An - 16A -4Bn +8B -4C + n^2

   An^2 + Bn + C = n^2 + 8An - 12A + 4B

   An^2 = n^2 (A = 1) Bn = 8An (B = 8) C = -12A + 4B (C = 20)

   f_n^p = n^2 + 8n + 20

5. f_n = f_n^h + f_n^p

   f_n = ($\alpha_{1,0}$ + $\alpha_{1,1}$.n)2^n + n^2 + 8n + 20

6. f_0 = 2 = $\alpha_{1,0}$ + 20 ($\alpha_{1,0}$ = -18)

$f_1 = 5 =$ (-18 + $\alpha_{1,1}$).2 + 1 + 8 + 20 ($\alpha_{1,1} = 6$)

As a result, solution of the recurrence relation is:

f_n = -18.2^n + 6n.2^n + n^2 +8n +20

Problem 3 (20+15 = 35 points)

Consider the linear homogeneous recurrence relation $a_n = 2a_{n-1}$ - $2a_{n-2}$.
(a) Find the characteristic roots of the recurrence relation.
(Solution)

a_n - $2a_{n-1}$ + $2a_{n-2}$=0

r^2 - 2r + 2 = 0 (Characteristic equation)
Δ =$(-2)^2$ - 4.1.2
$\Delta = -4$

r$_1 = (2+2i)/2$, $r_2 = (2-2i)/2$
$r_1 = 1+i$, $r_2 = 1-i$
Characteristic roots are: 1+i and 1-i.

(b) Find the solution of the recurrence relation with $a_0 = 1$ and $a_1 = 2$.
(Solution)

$a_n = \alpha_1.r_1^n + \alpha_2.r_2^n$
$a_n = \alpha_1.(1+i)^n + \alpha_2.(1-i)^n$

$a_0 = 1 = \alpha_1.(1+i)^0 + \alpha_2.(1-i)^0$
$1 = \alpha_1 + \alpha_2$

$a_1 = 2 = \alpha_1.(1+i)^1 + \alpha_2.(1-i)^1$
$2 = \alpha_1 + \alpha_1$.i + α_2 - α_2.i
$2 = \alpha_1 + \alpha_2$ + i(α_1 - α_2) (Since $\alpha_1 + \alpha_2 = 1$, we can simplify the equation)
2 = 1 + i(α_1 - α_2)
1 = i(α_1 - α_2) (Multiply both sides by i)
i = -1(α_1 - α_2) = -α_1 + α_2

$1 = \alpha_1 + \alpha_2$
i = -α_1 + α_2

Summation of above equations gives us: 1 + i = $2\alpha_2$. So that, α_2 = (1+i)/2 and when we put this value to the first equation, we find: 1 = α_1 + (1+i)/2. So that, α_1 = (1-i)/2.

As a result, solution of the recurrence relation is:

a_n = ((1-i)/2) . $(1+i)^n$ + ((1+i)/2) . $(1-i)^n$