5-8: $T=\frac{1}{2}ml^2(\dot\theta_1^2+\dot\theta_2^2+\dot\theta_3^2)$

$$V=mgl(3-\cos\theta_1-\cos\theta_2-\cos\theta_3)+\frac{1}{2}kh^2\left[(\sin\theta_2-\sin\theta_1)^2+(\sin\theta_3-\sin\theta_2)^2\right]$$

V~~按~~小量变换一下

$$M=\begin{pmatrix} ml^2 & 0 & 0 \\ 0 & ml^2 & 0 \\ 0 & 0 & ml^2 \end{pmatrix},\quad K=$$

$$V=\frac{1}{2}mgl(\theta_1^2+\theta_2^2+\theta_3^2)+\frac{1}{2}kh^2(\theta_1^2+2\theta_2^2+\theta_3^3-2\theta_1\theta_2-2\theta_2\theta_3)$$

$$K=\begin{pmatrix} mgl+kh^2 & -kh^2 & 0 \\ -kh^2 & mgl+2kl^2 & -kh^2 \\ 0 & -kh^2 & mgl+kh^2 \end{pmatrix}$$

特征方程: $|K-\omega^2 M|=0$

$$\begin{vmatrix} mgl+kh^2-m\omega^2l^2 & -kh^2 & 0 \\ -kh^2 & mgl+2kl^2-m\omega^2l^2 & -kh^2 \\ 0 & -kh^2 & mgl+kh^2-m\omega^2l^2 \end{vmatrix}=0$$

$$\Rightarrow\begin{cases} mgl+kh^2-m\omega^2l^2=0 \\ (mgl+2kh^2-m\omega^2l^2)(mgl+kh^2-m\omega^2l^2)=2kh^2 \end{cases}$$

解得:
$$\begin{cases} \omega_1=\sqrt{\dfrac{mgl+kh^2}{ml^2}} \\ \omega_2=\sqrt{\dfrac{g}{l}} \\ \omega_3=\cancel{\sqrt{\dfrac{3kh^2+glm}{lm}}}\ \dfrac{1}{l}\sqrt{\dfrac{mgl+3h^2k}{m}} \end{cases}\quad \xrightarrow{(K-\omega_i^2M)A=0}\quad \phi^{(1)}=\begin{pmatrix}-1\\0\\1\end{pmatrix}\quad \phi^{(2)}=\begin{pmatrix}1\\1\\1\end{pmatrix}\quad \phi^{(3)}=\begin{pmatrix}1\\-2\\1\end{pmatrix}$$

5-11 $\quad Kx_j = \omega_j^2 Mx_j \xrightarrow{\text{左乘} KM^{-1}}$

$$KM^{-1}Kx_j = \omega_j^2 KM^{-1}Mx_j = \omega_j^2 Kx_j \xrightarrow{\text{左乘} x_i^T}$$

$$x_i^T KM^{-1}Kx_j = \omega_j^2 x_i^T \cdot Kx_j = 0 \xrightarrow{\text{左乘} KM^{-1}}$$ （先左乘再加上 x_i^T）

$$x_i^T (KM^{-1})^2 \cdot Kx_j = \omega_j^2 x_i^T [KM^{-1}K] x_j = 0$$

重复n次使有 $x_i^T (KM^{-1})^n Kx_j = 0$

同理，若改为左乘 MK^{-1}，

即可得. $x_i^T (MK^{-1})^n Mx_j = 0$

6.2.17 由 $D\ddot{x}+x=0 \Rightarrow x=-D\ddot{x}$

$V=\frac{1}{2}x^TKx \Rightarrow V=\frac{1}{2}\ddot{x}^TD^TKD\ddot{x}=\frac{1}{2}\ddot{x}^TMD\ddot{x}$

又 $x=A\sin(\omega t+\varphi) \Rightarrow V_{max}=\frac{1}{2}\omega^4A^TMDA$

又 $T_{max}=\frac{1}{2}\omega^2A^TMA \Rightarrow T_{max}=V_{max}$

故 $\omega^2=R_D(A)=\dfrac{A^TMA}{A^TMDA}$

考虑 $A=\phi^{(i)}$. $D\phi^{(i)}=FM\phi^{(i)}=K^{-1}M\phi^{(i)}$

又 $K\phi^{(i)}=\omega_i^2M\phi^{(i)} \Rightarrow D\phi^{(i)}=\frac{1}{\omega_i^2}\phi^{(i)}$

故 $R_D(\phi^{(i)})=\dfrac{\phi^{(i)T}M\phi^{(i)}}{\frac{1}{\omega_i^2}\phi^{(i)T}M\phi^{(i)}}=\omega_i^2$

取假设振型 $\psi=\sum_{j=1}^{n}a_j\phi_N^{(j)}=\Phi_N\cdot a$

又 $D\Phi_N=\frac{1}{\omega_i^2}\cdot\Phi_N=\Phi_N\cdot\Lambda^{-1}$

故 $R_D(\psi)=\dfrac{a^T\Phi_N^TM\Phi_N\cdot a}{a^T\Phi_N^TMD\Phi_Na}=\dfrac{a^T\Phi_N^TM\Phi_Na}{a^T\Phi_N^TM\Phi_N\Lambda^{-1}\cdot a}$

$=\dfrac{a^TE\cdot a}{a^TE\cdot\Lambda^{-1}\cdot a}=\dfrac{\sum_{j=1}^{n}a_j^2}{\sum_{j=1}^{n}\frac{a_j^2}{\omega_j^2}}$, 得证.

6.2.22

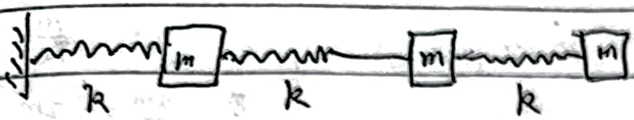
$$R(\psi)-R_D(\psi)=\frac{\sum_{j=1}^{n}a_j^2w_j^2}{\sum_{j=1}^{n}a_j^2}-\frac{\sum_{j=1}^{n}a_j^2}{\sum_{j=1}^{n}\frac{a_j^2}{w_j^2}}$$

$$=\frac{\sum_{j=1}^{n}a_j^2w_j^2\times\sum_{j=1}^{n}\frac{a_j^2}{w_j^2}-\left(\sum_{j=1}^{n}a_j^2\right)^2}{\sum_{j=1}^{n}a_j^2\times\sum_{j=1}^{n}\frac{a_j^2}{w_j^2}}$$

又∵对任意实数 u_j 和 v_j，有

$$\sum_{j=1}^{n}u_j^2\times\sum_{j=1}^{n}v_j^2\geqslant\left(\sum_{j=1}^{n}u_jv_j\right)^2.$$

故 $R(\psi)-R_D(\psi)\geqslant 0$.



振型：$x^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $x^{(2)} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $x^{(3)} = \begin{pmatrix} 0 \\ a \\ 1 \end{pmatrix}$

1、求精确 ω_1：

$$M = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \qquad K = \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \qquad (K - \omega^2 m) \cdot A = 0$$

特征方程：$|K - \omega^2 M| = 0$

$$\begin{vmatrix} 2k-\omega^2 m & -k & 0 \\ -k & 2k-\omega^2 m & -k \\ 0 & -k & k-\omega^2 m \end{vmatrix} = 0 \Rightarrow (2k-\omega^2 m)^2 (k-\omega^2 m) - k^2(3k - 2\omega^2 m) = 0$$

$$\Rightarrow \omega^6 - 5\frac{k}{m}\cdot\omega^4 + 6\left(\frac{k}{m}\right)^2 \cdot \omega^2 - \left(\frac{k}{m}\right)^3 = 0$$

取 $\lambda = \left(\frac{m}{k}\right) \cdot \omega^2 \Rightarrow \lambda^3 - 5\lambda^2 + 6\lambda - 1 = 0$

得到 $\lambda_1 = 0.198 \Rightarrow \left(\frac{m}{k}\right) \cdot \omega_1^2 = 0.198 \Rightarrow \boxed{\omega_1 = 0.445\sqrt{\frac{k}{m}}}$

$$\begin{bmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \xrightarrow{\text{取} x_3 = 1} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0.445 \\ 0.802 \\ 1 \end{pmatrix}$$

邓克利法

柔度矩阵：$F=K^{-1}=\begin{bmatrix} \frac{1}{k} & \frac{1}{k} & \frac{1}{k} \\ \frac{1}{k} & \frac{2}{k} & \frac{2}{k} \\ \frac{1}{k} & \frac{2}{k} & \frac{3}{k} \end{bmatrix}$ $\quad$ $D=FM=\begin{bmatrix} \frac{m}{k} & \frac{m}{k} & \frac{m}{k} \\ \frac{m}{k} & \frac{2m}{k} & \frac{2m}{k} \\ \frac{m}{k} & \frac{2m}{k} & \frac{3m}{k} \end{bmatrix}$

$\tilde{\omega}_1^2=\frac{k}{m}\quad \tilde{\omega}_2^2=\frac{k}{2m}\quad \tilde{\omega}_3^2=\frac{k}{3m}$

$\frac{1}{\omega_1^2}=\sum_{i=1}^{3}\frac{1}{\tilde{\omega}_i^2}=\frac{6m}{k}\Rightarrow \omega_1=\frac{1}{\sqrt{6}}\sqrt{\frac{k}{m}}=0.408\sqrt{\frac{k}{m}}$

误差：$\frac{|0.408-0.445|}{0.445}=8.3\%$

※稍微分析一下三个振型，$x^{(2)}=\begin{bmatrix}1\\2\\3\end{bmatrix}$ 比较接近精确值

故瑞利法中选用此振型，取 $\psi=\begin{bmatrix}1\\2\\3\end{bmatrix}$

$R_I(\psi)=\frac{\psi^TK\psi}{\psi^TM\psi}=\frac{3k}{14m}$，故 $\omega_1=\sqrt{\frac{3}{14}}\cdot\sqrt{\frac{k}{m}}=0.462\sqrt{\frac{k}{m}}$

相对误差：$\frac{0.462-0.445}{0.445}=5.2\%$

$R_{II}(\psi)=\frac{\psi^TM\psi}{\psi^TMD\psi}=\frac{14m}{\frac{70}{k}m^2}=\frac{1}{5}\frac{k}{m}\Rightarrow \omega_1=\frac{1}{\sqrt{5}}\sqrt{\frac{k}{m}}=0.447\sqrt{\frac{k}{m}}$

相对误差：$\frac{0.447-0.445}{0.445}=0.45\%$

包括上周作业，以及上上上周老师出差未交的那次作业。