微积分H作业解答

第七周

题目1. (9.3.12) 求下列函数在指定点处的偏导数:

(1) 设$f(x,y)=x+(y-1)\arctan\frac{x}{y}$,求$f'_x(0,1)$和$f'_y(0,1)$.

解答: $f'_x(x,y)=1+(y-1)\frac{1}{1+\frac{x^2}{y^2}}\cdot\frac{1}{y}=1+\frac{y(y-1)}{x^2+y^2}$, 代入得$f'_x(0,1)=1$,

$f'_y(x,y)=\arctan\frac{x}{y}+(y-1)\frac{1}{1+\frac{x^2}{y^2}}\cdot(-\frac{x}{y^2})=\arctan\frac{x}{y}-\frac{x(y-1)}{x^2+y^2}$,代入得$f'_y(0,1)=0$

题目2. (9.3.13) 求下列函数对各个变量的一阶偏导数:

(2) $z=e^{\frac{y}{x}}(x+y)$;

(3) $z=\ln(2x+\sqrt{x^2+y^2})$.

解答: (2) $\frac{\partial z}{\partial x}=\frac{\partial e^{\frac{y}{x}}}{\partial x}\cdot(x+y)+e^{\frac{y}{x}}\cdot\frac{\partial(x+y)}{\partial x}=e^{\frac{y}{x}}\cdot(-\frac{y}{x^2})\cdot(x+y)+e^{\frac{y}{x}}=e^{\frac{y}{x}}\cdot\frac{x^2-xy-y^2}{x^2}$,

$\frac{\partial z}{\partial y}=\frac{\partial e^{\frac{y}{x}}}{\partial y}\cdot(x+y)+e^{\frac{y}{x}}\cdot\frac{\partial(x+y)}{\partial y}=e^{\frac{y}{x}}\cdot\frac{1}{x}\cdot(x+y)+e^{\frac{y}{x}}=e^{\frac{y}{x}}\cdot(\frac{y}{x}+2)$.

(3) $\frac{\partial z}{\partial x}=\frac{1}{2x+\sqrt{x^2+y^2}}\cdot\frac{\partial}{\partial x}(2x+\sqrt{x^2+y^2})=\frac{1}{2x+\sqrt{x^2+y^2}}\cdot(2+\frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}})$

$=\frac{1}{2x+\sqrt{x^2+y^2}}\cdot\frac{1}{\sqrt{x^2+y^2}}\cdot(2\sqrt{x^2+y^2}+x)=\frac{x+2\sqrt{x^2+y^2}}{2x\sqrt{x^2+y^2}+x^2+y^2}$,

$\frac{\partial z}{\partial y}=\frac{1}{2x+\sqrt{x^2+y^2}}\cdot\frac{\partial}{\partial y}(2x+\sqrt{x^2+y^2})=\frac{1}{2x+\sqrt{x^2+y^2}}\cdot\frac{1}{2}\frac{2y}{\sqrt{x^2+y^2}}=\frac{y}{2x\sqrt{x^2+y^2}+x^2+y^2}$.

题目3. (9.3.15) 设$f(x,y)=(x-1)(y-1)(x-2)(y-2)...(x-100)$

$(y-100)$,求$f_x'(1,0)$和$f_{xy}''(1,1)$.

解答: 记$g(x,y)=(x-2)(y-2)...(x-100)(y-100)$为$\mathbb{R}^2$上二元多项式,

则$f(x,y)=(x-1)(y-1)g(x,y)$.

$f_x'(x,y)=(y-1)g(x,y)+(x-1)\frac{\partial}{\partial x}[(y-1)g(x,y)]$

代入得$f_x'(1,0)=-g(1,0)=-99!\times 100!$,

$f_x'(x,y)=(y-1)g(x,y)+(x-1)\frac{\partial}{\partial x}[(x-1)g(x,y)]$

$f_{xy}''(x,y)=\frac{\partial}{\partial y}f_x'(x,y)=g(x,y)+(y-1)\frac{\partial g(x,y)}{\partial y}+(x-1)\frac{\partial^2}{\partial x\partial y}[(x-1)g(x,y)]$

代入得$f_{xy}''(1,1)=g(1,1)=(99!)^2$

注记: 因为g是多项式,则其导函数也均为多项式,在有界区域内的值有限,

这样才能保证形如$(x-1)\frac{\partial}{\partial x}[(y-1)g(x,y)]$的式子在代入$x=1$时为0.

题目4. (9.3.18) 设$f(x,y)=\begin{cases}\frac{x^3+y^2}{x^2+y^2}, & (x,y)\neq(0,0),\\ 0, & (x,y)=(0,0),\end{cases}$

讨论$f_x'(0,0)$与$f_y'(0,0)$是否存在;若存在,求出其值.

解答: 根据偏导数定义,$f_x'(0,0)=\lim\limits_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=\lim\limits_{\Delta x\to 0}\frac{\frac{(\Delta x)^3}{(\Delta x)^2}-0}{\Delta x}=1$,

同理,$f_y'(0,0)=\lim\limits_{\Delta y\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=\lim\limits_{\Delta y\to 0}\frac{\frac{(\Delta y)^2}{(\Delta y)^2}-0}{\Delta y}=\lim\limits_{\Delta y\to 0}\frac{1}{\Delta y}=\infty$,故不存在.

题目5. (9.3.20) 设$z = xy + xf(\frac{y}{x})$,且f可微,证明:$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = xy + z$.

解答: $\frac{\partial z}{\partial x} = y + f(\frac{y}{x}) + xf'(\frac{y}{x})(-\frac{y}{x^2}) = y + f(\frac{y}{x}) - \frac{y}{x} \cdot f'(\frac{y}{x})$,

$\frac{\partial z}{\partial y} = x + xf'(\frac{y}{x}) \cdot \frac{1}{x} = x + f'(\frac{y}{x})$,

代入得$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = xy + xf(\frac{y}{x}) - yf'(\frac{y}{x}) + yx + yf'(\frac{y}{x}) = 2xy + xf(\frac{y}{x}) = xy + z$.

题目6. (9.3.23) 设$z = e^{-(\frac{1}{x}+\frac{1}{y})}$,求证:$x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} = 2z$.

解答: $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}e^{-(\frac{1}{x}+\frac{1}{y})} = e^{-(\frac{1}{x}+\frac{1}{y})} \cdot \frac{\partial}{\partial x}(-(\frac{1}{x} + \frac{1}{y})) = \frac{1}{x^2}e^{-(\frac{1}{x}+\frac{1}{y})}$,

故$x^2\frac{\partial z}{\partial x} = e^{-(\frac{1}{x}+\frac{1}{y})} = z$, 而$z$中$x, y$对称,同理得$y^2\frac{\partial z}{\partial y} = e^{-(\frac{1}{x}+\frac{1}{y})} = z$,

综合得$x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} = 2z$.

题目7. (9.4.31) 求下列函数的偏导数:

(1) $z = \sqrt{u^2 + v^2}, u = x\sin y, v = e^{xy}$,求$\frac{\partial z}{\partial x}$ 和$\frac{\partial z}{\partial y}$;

(3) $z = \ln(x^2 + y^2), x = t\cos t, y = -\sin t$,求$\frac{dz}{dt}$.

解答: (1) 易得$\frac{\partial z}{\partial u} = \frac{u}{\sqrt{u^2+v^2}} = \frac{u}{z}, \frac{\partial z}{\partial v} = \frac{v}{\sqrt{u^2+v^2}} = \frac{v}{z}$,

故$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = \frac{1}{z}(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}) = \frac{1}{z}(x\sin^2 y + ye^{2xy})$,

$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = \frac{1}{z}(u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y}) = \frac{1}{z}(x^2\sin y\cos y + xe^{2xy})$.

(3) $\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} = \frac{2x}{x^2+y^2}\cdot(\cos t - t\sin t) - \frac{2y}{x^2+y^2}\cdot\cos t = \frac{2t\cos^2 t - 2(t^2-1)\sin t\cos t}{t^2\cos^2 t + \sin^2 t}$.

题目8. (9.4.33) 设$z = f(u, x+y), u = xe^y$,且f具有二阶连续偏导数, 求$\frac{\partial^2 z}{\partial x \partial y}$.

解答: $\frac{\partial z}{\partial x} = f_1' \cdot \frac{\partial u}{\partial x} + f_2' \cdot \frac{\partial (x+y)}{\partial x} = e^y f_1' + f_2'$,

$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y} \cdot \frac{\partial z}{\partial x} = \frac{\partial}{\partial y}(e^y f_1' + f_2') = e^y f_1' + e^y \frac{\partial f_1'}{\partial y} + \frac{\partial f_2'}{\partial y}$

$= e^y f_1' + e^y \cdot (f_{11}'' \cdot \frac{\partial u}{\partial y} + f_{12}'' \cdot \frac{\partial (x+y)}{\partial y}) + (f_{21}'' \cdot \frac{\partial u}{\partial y} + f_{22}'' \cdot \frac{\partial (x+y)}{\partial y})$

$= e^y f_1' + e^y \cdot (xe^y \cdot f_{11}'' + f_{12}'') + (xe^y \cdot f_{21}'' + f_{22}'')$

$= xe^{2y} f_{11}'' + (1+x)e^y f_{12}'' + f_{22}'' + e^y f_1'$.

最后的等号是由于f具有二阶连续偏导数,因此$f_{12}'' = f_{21}''$.

题目9. (9.4.34) 设$z = f(x^2 - y^2, x \sin y)$,且$f$具有二阶连续偏导数,

计算$\frac{\partial z}{\partial x}, \frac{\partial^2 z}{\partial x \partial y}$.

解答: $\frac{\partial z}{\partial x} = f_1' \cdot \frac{\partial (x^2-y^2)}{\partial x} + f_2' \cdot \frac{\partial (x \sin y)}{\partial x} = 2x f_1' + \sin y f_2'$,

$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y} \cdot \frac{\partial z}{\partial x} = \frac{\partial}{\partial y} \cdot (2x f_1' + \sin y f_2') = 2x \frac{\partial f_1'}{\partial y} + \cos y f_2' + \sin y \frac{\partial f_2'}{\partial y}$

$= 2x(f_{11}'' \cdot \frac{\partial (x^2-y^2)}{\partial y} + f_{12}'' \cdot \frac{\partial (x \sin y)}{\partial y}) + \cos y f_2' + \sin y \cdot (f_{21}'' \cdot \frac{\partial (x^2-y^2)}{\partial y} + f_{22}'' \cdot \frac{\partial (x \sin y)}{\partial y})$

$= 2x(-2y f_{11}'' + x \cos y f_{12}'') + \cos y f_2' + \sin y \cdot (-2y f_{21}'' + x \cos y f_{22}'')$

$= -4xy f_{11}'' + (2x^2 \cos y - 2y \sin y) f_{12}'' + x \sin y \cos y f_{22}'' + \cos y f_2'$.

最后的等号是由于f具有二阶连续偏导数,因此$f_{12}'' = f_{21}''$.

题目10. (9.4.38) 设$z=z(x,y)$有二阶连续偏导数,满足$6\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}-\frac{\partial^2 z}{\partial y^2}=0$.

如果引进变换$\begin{cases} u=x-2y, \\ v=x+3y, \end{cases}$ 试将上面方程变换为关于u,v的方程.

解答: 容易解得$\begin{cases} x=\frac{3u+2v}{5}, \\ y=\frac{v-u}{5}, \end{cases}$ 则有$z=z(x,y)=z(\frac{3u+2v}{5},\frac{v-u}{5})$

所以$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}=\frac{3}{5}\frac{\partial z}{\partial x}-\frac{1}{5}\frac{\partial z}{\partial y}$,

从而$\frac{\partial^2 z}{\partial u\partial v}=\frac{\partial}{\partial v}\cdot\frac{\partial z}{\partial u}=\frac{3}{5}\frac{\partial}{\partial v}\cdot\frac{\partial z}{\partial x}-\frac{1}{5}\frac{\partial}{\partial v}\cdot\frac{\partial z}{\partial y}$

$=\frac{3}{5}(\frac{2}{5}\frac{\partial^2 z}{\partial x^2}+\frac{1}{5}\frac{\partial^2 z}{\partial x\partial y})-\frac{1}{5}(\frac{2}{5}\frac{\partial^2 z}{\partial y\partial x}+\frac{1}{5}\frac{\partial^2 z}{\partial y^2})$

$=\frac{1}{25}(6\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}-\frac{\partial^2 z}{\partial y^2})$

最后的等号是由于z具有二阶连续偏导数,因此$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}$.

因此原方程可以变换为$\frac{\partial^2 z}{\partial u\partial v}=0$.

注记: 此题数据较为特殊，所以仅计算了$\frac{\partial^2 z}{\partial u\partial v}$即可完成方程变换. 对于一般情形,需要将$\frac{\partial^2 z}{\partial x\partial y},\frac{\partial^2 z}{\partial x^2},\frac{\partial^2 z}{\partial y^2}$ 均用$\frac{\partial^2 z}{\partial u\partial v},\frac{\partial^2 z}{\partial u^2},\frac{\partial^2 z}{\partial v^2}$表示, 再代回方程完成变换,可以参考书上例9.3.7.

题目11. (9.5.43) 设$\ln\sqrt{x^2+y^2}=\arctan\frac{y}{x}$,求$\frac{dy}{dx}$和$\frac{d^2y}{dx^2}$.

解答: 设$F(x,y)=\ln\sqrt{x^2+y^2}-\arctan\frac{y}{x}=\frac{1}{2}\ln(x^2+y^2)-\arctan\frac{y}{x}$,

则$F'_x(x,y)=\frac{1}{2}\cdot\frac{2x}{x^2+y^2}-\frac{1}{1+(\frac{y}{x})^2}\cdot(-\frac{y}{x^2})=\frac{x+y}{x^2+y^2}$,

$F'_y(x,y)=\frac{1}{2}\cdot\frac{2y}{x^2+y^2}-\frac{1}{1+(\frac{y}{x})^2}\cdot\frac{1}{x}=\frac{y-x}{x^2+y^2}$,

由隐函数定理,$\frac{dy}{dx}=-\frac{F'_x(x,y)}{F'_y(x,y)}=\frac{x+y}{x-y}$,

进一步有$\frac{d^2y}{dx^2}=\frac{d}{dx}(\frac{x+y}{x-y})=\frac{\partial}{\partial x}(\frac{x+y}{x-y})+\frac{\partial}{\partial y}(\frac{x+y}{x-y})\frac{dy}{dx}$

$=\frac{-2y}{(x-y)^2}+\frac{2x}{(x-y)^2}\cdot\frac{x+y}{x-y}=\frac{2(x^2+y^2)}{(x-y)^3}$.

题目12. (9.5.46) 设$u=f(x,y,z),g(x^2,e^y,z)=0,y=\sin x$,

其中f,g均有一阶连续偏导数, 且$\frac{\partial g}{\partial z}\neq 0$,求$\frac{du}{dx}$.

解答: $\frac{du}{dx}=f'_1+f'_2\cdot\frac{\partial y}{\partial x}+f'_3\cdot\frac{\partial z}{\partial x}$, 其中$\frac{\partial y}{\partial x}=\frac{\partial \sin x}{\partial x}=\cos x$.下面来计算$\frac{\partial z}{\partial x}:$

在等式$g(x^2,e^y,z)=0$两边对x求偏导数, 得$g'_1\frac{\partial(x^2)}{\partial x}+g'_2\frac{\partial(e^y)}{\partial x}+g'_3\frac{\partial z}{\partial x}=0$,

由于$g'_3\neq 0,y=\sin x$,解得$\frac{\partial z}{\partial x}=-\frac{1}{g'_3}(2xg'_1+e^{\sin x}\cos x\cdot g'_2)$,

代回得$\frac{du}{dx}=f'_1+\cos xf'_2-\frac{f'_3}{g'_3}(2xg'_1+e^{\sin x}\cos x\cdot g'_2)$.

题目13. (9.5.49) 设$\begin{cases}u^2-v+x=0,\\ u+v^2-y=0,\end{cases}$ 求du和dv.

解答: 对x求偏导数,得$\begin{cases}2u\frac{\partial u}{\partial x}-\frac{\partial v}{\partial x}+1=0,\\ \frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}=0,\end{cases}$ 解得$\begin{cases}\frac{\partial u}{\partial x}=\frac{-2v}{4uv+1},\\ \frac{\partial v}{\partial x}=\frac{1}{4uv+1},\end{cases}$

对y求偏导数,得$\begin{cases}2u\frac{\partial u}{\partial y}-\frac{\partial v}{\partial y}=0,\\ \frac{\partial u}{\partial y}+2v\frac{\partial v}{\partial y}-1=0,\end{cases}$ 解得$\begin{cases}\frac{\partial u}{\partial y}=\frac{1}{4uv+1},\\ \frac{\partial v}{\partial y}=\frac{2u}{4uv+1},\end{cases}$

故$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy=\frac{-2vdx+dy}{4uv+1},dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy=\frac{dx+2udy}{4uv+1}$.

注记: 也可以将d看作求导算子,作用在方程组上,得$\begin{cases} 2udu - dv + dx = 0, \\ du + 2vdv - dy = 0, \end{cases}$

解得$du = \frac{-2vdx+dy}{4uv+1}, dv = \frac{dx+2udy}{4uv+1}$.

题目14. (9.5.52) 设$z = z(x, y)$由方程$f(x + \frac{z}{y}, y + \frac{z}{x}) = 0$所确定,

且f具有二阶连续偏导数,证明:$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z - xy$.

解答: 在等式$f(x + \frac{z}{y}, y + \frac{z}{x}) = 0$两边对$x$求偏导数,

得$0 = f_1' \cdot \frac{\partial(x+\frac{z}{y})}{\partial x} + f_2' \cdot \frac{\partial(y+\frac{z}{x})}{\partial x} = f_1' \cdot (1 + \frac{1}{y}\frac{\partial z}{\partial x}) + f_2' \cdot (-\frac{z}{x^2} + \frac{1}{x}\frac{\partial z}{\partial x})$,

再对y求偏导数,$0 = f_1' \cdot \frac{\partial(x+\frac{z}{y})}{\partial y} + f_2' \cdot \frac{\partial(y+\frac{z}{x})}{\partial y} = f_1' \cdot (-\frac{z}{y^2} + \frac{1}{y}\frac{\partial z}{\partial y}) + f_2' \cdot (1 + \frac{1}{x}\frac{\partial z}{\partial y})$,

分别解得$\frac{\partial z}{\partial x} = \frac{-f_1' + \frac{z}{x^2}f_2'}{\frac{f_1'}{y} + \frac{f_2'}{x}}, \frac{\partial z}{\partial y} = \frac{\frac{z}{y^2}f_1' - f_2'}{\frac{f_1'}{y} + \frac{f_2'}{x}}$,

则$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{(-xf_1' + \frac{z}{x}f_2') + (\frac{z}{y}f_1' - yf_2')}{\frac{f_1'}{y} + \frac{f_2'}{x}} = \frac{\frac{z-xy}{y}f_1' + \frac{z-xy}{x}f_2'}{\frac{f_1'}{y} + \frac{f_2'}{x}} = z - xy$.