微积分H作业解答

第八周

题目1. (9.3.24) 求下列函数的全微分:

(2) $u=(x+\sin y)^z(x>1)$;

(3) $u=\sqrt[z]{\frac{x}{y}}$,求$du|_{(1,1,1)}$.

解答: (2) $du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz$

$=z(x+\sin y)^{z-1}dx+z\cos y(x+\sin y)^{z-1}dy+\ln(x+\sin y)(x+\sin y)^z dz.$

(3) $du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz$

$=\frac{1}{yz}(\frac{x}{y})^{\frac{1}{z}-1}dx-\frac{x}{y^2z}(\frac{x}{y})^{\frac{1}{z}-1}dy-\frac{\ln(\frac{x}{y})}{z^2}(\frac{x}{y})^{\frac{1}{z}}dz,$

代入$(x,y,z)=(1,1,1)$得,$du|_{(1,1,1)}=dx-dy.$

题目2. (9.3.27) 设$f(x,y)=\begin{cases}\frac{xy^3}{x^2+y^4}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0,\end{cases}$
讨论$f(x,y)$在$(0,0)$处的连续性、可偏导性和可微性.

解答: (1) 由于$|\frac{xy^3}{x^2+y^4}|=|y|\frac{|xy^2|}{x^2+y^4}\le|y|\frac{\frac{x^2+y^4}{2}}{x^2+y^4}=\frac{|y|}{2}$, 而$\lim\limits_{(x,y)\to(0,0)}\frac{|y|}{2}=0$,

则$\lim\limits_{(x,y)\to(0,0)}|\frac{xy^3}{x^2+y^4}|=0$, 即$f(x,y)$在$(0,0)$处连续.

(2) 由偏导数的定义, $f'_x(0,0)=\lim\limits_{\Delta x\to0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=\lim\limits_{\Delta x\to0}\frac{0-0}{\Delta x}=0$,

$f'_y(0,0)=\lim\limits_{\Delta y\to0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=\lim\limits_{\Delta y\to0}\frac{0-0}{\Delta y}=0$,

故$f(x,y)$在$(0,0)$处可偏导,偏导数为$f'_x(0,0)=f'_y(0,0)=0$.

(3) f在$(0,0)$处可微当且仅当$\lim\limits_{(\Delta x,\Delta y)\to(0,0)}\frac{\Delta z-[f'_x(0,0)\Delta x+f'_y(0,0)\Delta y]}{\sqrt{(\Delta x)^2+(\Delta y)^2}}=0$,

但$\Delta z=f(\Delta x,\Delta y)-f(0,0)=\frac{\Delta x(\Delta y)^3}{(\Delta x)^2+(\Delta y)^4}$, $f'_x(0,0)=f'_y(0,0)=0$,

则$\lim\limits_{(\Delta x,\Delta y)\to(0,0)}\frac{\Delta z-[f'_x(0,0)\Delta x+f'_y(0,0)\Delta y]}{\sqrt{(\Delta x)^2+(\Delta y)^2}}=\lim\limits_{(\Delta x,\Delta y)\to(0,0)}\frac{\Delta x(\Delta y)^3}{((\Delta x)^2+(\Delta y)^4)\sqrt{(\Delta x)^2+(\Delta y)^2}}$

当$\Delta x=k(\Delta y)^2$时, 极限为$\lim\limits_{\Delta y\to0}\frac{k(\Delta y)^5}{(k^2+1)(\Delta y)^4\sqrt{k^2(\Delta y)^4+(\Delta y)^2}}$

$=\lim\limits_{\Delta y\to0}\frac{k}{(k^2+1)\sqrt{k^2(\Delta y)^2+1}}=\frac{k}{k^2+1}$,

对不同的k极限不同，故$(\Delta x,\Delta y)\to(0,0)$时极限不存在,

因此f在$(0,0)$处不可微.

题目3. (9.3.28) 设$z=f(x,y)$可微,且$dz=\frac{3(xdy-ydx)}{(x-y)^2}$, $f(1,0)=1$.

求$f(x,y)$的表达式.

解答: $\frac{\partial f}{\partial x}=-\frac{3y}{(x-y)^2}$, 则$f(x,y)=-3y\int\frac{dx}{(x-y)^2}+\phi(y)=\frac{3y}{x-y}+\phi(y)$,

代入$f(1,0)=1$得,$\phi(0)=1$,

另一方面$\frac{\partial f}{\partial y}=\frac{3x}{(x-y)^2}$,即$\frac{3x}{(x-y)^2}+\phi'(y)=\frac{3x}{(x-y)^2}$,得$\phi'(y)=0$

故$\phi(y)=\phi(0)=1$, $f(x,y)=\frac{3y}{x-y}+\phi(y)=\frac{3y}{x-y}+1=\frac{x+2y}{x-y}$.

题目4. (9.4.40) 设函数$z=f(x,y)$在点$(1,1)$处可微,且$f(1,1)=1$,

$\frac{\partial f}{\partial x}|_{(1,1)}=2, \frac{\partial f}{\partial y}|_{(1,1)}=3$. 若$\varphi(x)=f(x,f(x,x))$,求$\frac{d}{dx}[\varphi(x)]^3|_{x=1}$.

解答: $\varphi'(x)=\frac{\partial f}{\partial x}(x,f(x,x))+\frac{\partial f}{\partial y}(x,f(x,x))\cdot\frac{df}{dx}(x,x)$

$=\frac{\partial f}{\partial x}(x,f(x,x))+\frac{\partial f}{\partial y}(x,f(x,x))\cdot(\frac{\partial f}{\partial x}(x,x)+\frac{\partial f}{\partial y}(x,x))$, 由于$f(1,1)=1$,

则$\varphi'(1)=\frac{\partial f}{\partial x}|_{(1,1)}+\frac{\partial f}{\partial y}|_{(1,1)}\cdot(\frac{\partial f}{\partial x}|_{(1,1)}+\frac{\partial f}{\partial y}|_{(1,1)})=2+3\times(2+3)=17$,

故$\frac{d}{dx}[\varphi(x)]^3|_{x=1}=3\varphi^2(x)\varphi'(x)|_{x=1}=3\varphi^2(1)\varphi'(1)=3\times 17=51$.

题目5. (9.6.69) 求下列函数在指定点处的泰勒展开式:

(1) $f(x,y)=xy^2$在点$P(2,1)$处(二阶);

(3) $f(x,y)=\sin(x^2+y^2)$在点$P(0,0)$处(二阶).

解答: (1) 注意到f有任意阶连续偏导数,则求偏导数无需考虑次序.

$f'_x=y^2, f'_x(2,1)=1; f'_y=2xy, f'_y(2,1)=4;$

$f''_{xx}=0; f''_{xy}=2y, f''_{xy}(2,1)=2; f''_{yy}=2x, f''_{yy}(2,1)=4;$

$f'''_{xxx}=f'''_{xxy}=f'''_{yyy}=0; f'''_{xyy}=2;$

得f在点$P(2,1)$处的二阶泰勒展开为:

$$\begin{aligned}
f(x,y)&=f(2,1)+f'_x(2,1)\cdot(x-2)+f'_y(2,1)\cdot(y-1)\\
&\quad+\tfrac{1}{2!}[f''_{xx}(2,1)\cdot(x-2)^2+2f''_{xy}(2,1)\cdot(x-2)(y-1)+f''_{yy}(2,1)\cdot(y-1)^2]\\
&\quad+\tfrac{1}{3!}[f'''_{xxx}(2+\theta(x-2),1+\theta(y-1))\cdot(x-2)^3\\
&\qquad+3f'''_{xxy}(2+\theta(x-2),1+\theta(y-1))\cdot(x-2)^2(y-1)
\end{aligned}$$

$$+3f'''_{xyy}(2+\theta(x-2),1+\theta(y-1))\cdot(x-2)(y-1)^2$$

$$+f'''_{yyy}(2+\theta(x-2),1+\theta(y-1))\cdot(y-1)^3]$$

$$=2+(x-2)+4(y-1)+\tfrac{1}{2}[0+4(x-2)(y-1)+4(y-1)^2]$$

$$+\tfrac{1}{6}[0+0+6(x-2)(y-1)^2+0]$$

$$=2+(x-2)+4(y-1)+2(x-2)(y-1)+2(y-1)^2+(x-2)(y-1)^2.$$

(3) 注意到f有任意阶连续偏导数,则求偏导数无需考虑次序.

$f'_x=2x\cos(x^2+y^2), f'_x(0,0)=0; f'_y=2y\cos(x^2+y^2), f'_y(0,0)=0;$

$f''_{xx}=2\cos(x^2+y^2)-4x^2sin(x^2+y^2), f''_{xx}(0,0)=2;$

$f''_{xy}=-4xy\sin(x^2+y^2), f''_{xy}(0,0)=0;$

$f''_{yy}=2\cos(x^2+y^2)-4y^2sin(x^2+y^2), f''_{yy}(0,0)=2;$

$f'''_{xxx}=-12x\sin(x^2+y^2)-8x^3\cos(x^2+y^2);$

$f'''_{xxy}=-4y\sin(x^2+y^2)-8x^2y\cos(x^2+y^2);$

$f'''_{xyy}=-4x\sin(x^2+y^2)-8xy^2\cos(x^2+y^2);$

$f'''_{yyy}=-12y\sin(x^2+y^2)-8y^3\cos(x^2+y^2);$

得f在点$P(0,0)$处的二阶泰勒展开为:

$$f(x,y)=f(0,0)+f'_x(0,0)\cdot x+f'_y(0,0)\cdot y$$

$$+\tfrac{1}{2!}[f''_{xx}(0,0)\cdot x^2+2f''_{xy}(0,0)\cdot xy+f''_{yy}(0,0)\cdot y^2]$$

$$+\tfrac{1}{3!}[f'''_{xxx}(\theta x,\theta y)\cdot x^3+3f'''_{xxy}(\theta x,\theta y)\cdot x^2y$$

$$+3f'''_{xyy}(\theta x,\theta y)\cdot xy^2+f'''_{yyy}(\theta x,\theta y)\cdot y^3]$$

$$=x^2+y^2-2(x^2+y^2)^2\sin[(\theta x)^2+(\theta y)^2]-\tfrac{4}{3}(x^2+y^2)^3\cos[(\theta x)^2+(\theta y)^2].$$

其中$0<\theta<1$.

题目6. (9.6.56) 判断下列函数是否有极值,若有,请判断是极大值还是极小值;并求极值: (1) $f(x,y)=x^2-xy+y^2-2x+y$;

(5) $f(x,y)=3axy-x^3y^3(a>0)$.

解答: (1) 令$f'_x=2x-y-2=0$; $f'_y=2y-x+1=0$;解得驻点$P(1,0)$,

而$f_{xx}=2, f_{xy}=-1, f_{yy}=2$,

则$AC-B^2=3>0$且$A>0$,在点$(1,0)$取得极小值$f(1,0)=-1$.

(5) 令$f'_x=3ay-3x^2y^3=0, f'_y=3ax-3x^3y^2=0$;

解得$(x,y)=(0,0)$或$xy=\pm\sqrt{a}$,

而$f_{xx}=-6xy^3, f_{xy}=3a-9x^2y^2, f_{yy}=-6x^3y$,

当$(x,y)=(0,0)$时,$B^2-AC=9a^2>0$,则点$(0,0)$不是极值点.

当$xy=\pm\sqrt{a}$时,$B^2-AC=36a^2-36a^2=0$,还需另作判断,

令$t=xy$,看作一个整体,则$f(t)=3at-t^3, \frac{df}{dt}=3a-3t^2, \frac{d^2f}{dt^2}=-6t$,

可得$f(t)$在$t=\sqrt{a}$时取得极大值,在$t=-\sqrt{a}$时取得极小值.

故当$xy=\sqrt{a}$时f取得极大值$2a\sqrt{a}$,当$xy=-\sqrt{a}$时f取得极小值$-2a\sqrt{a}$.

题目7. (9.6.57) 求下列函数在指定区域内的最值:

(2) $z=x^2y(4-x-y), D=\{(x,y)|0\le y\le 6-x, 0\le x\le 6\}$.

解答: 记$z=f(x,y)$,令$f'_x=8xy-3x^2y-2xy^2=0$; $f'_y=4x^2-x^3-2x^2y=0$;

在区域D内部有驻点$P(2,1)$, $f(2,1)=4$.

$f''_{xx} = 8y - 6xy - 2y^2, f''_{xy} = 8x - 3x^2 - 4xy, f''_{yy} = -2x^2,$

在点$(2,1)$处有$B^2 - AC = 16 - 48 < 0, A = -6 < 0$,则$f(2,1)$为极大值.

再考虑D的边界,当$x = 0$或$y = 0$时,$f(x,y) \equiv 0$;(也包含了$x = 6$情形)

最后考虑$y = 6 - x, 0 < x < 6$的情形,

此时$z = f(x, 6-x) = -2x^2(6-x), \frac{dz}{dx} = 6x(x-4),$

则$f(x, 6-x)$在$(0,4)$上单调递减,在$(4,6)$上单调递增,

在这部分边界上的最小值为$f(4,2) = -64$,最大值为$f(0,6) = f(6,0) = 0$.

由于D有界,则最值必在边界或驻点处取得,

综上对比可得z在D上的最大值为$f(2,1) = 4$,最小值为$f(4,2) = -64$.