微积分H作业解答

第四周

题目1. (7.7.58) 将下列函数分别展开成正弦级数和余弦级数:

(4) $f(x) = \begin{cases} x-2, & 0 \le x < \frac{\pi}{2}, \\ 0, & \frac{\pi}{2} \le x \le \pi. \end{cases}$

解答: (4) 1. 若展开为正弦级数,则将$f(x)$进行奇函数延拓到$[-\pi,\pi]$,

有$a_0 = 0, a_n = 0$,

$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} (x-2)\sin nx dx$

$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\sin nx dx - \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \sin nx dx$

$= \frac{2}{\pi}(-\frac{x\cos nx}{n})\Big|_{x=0}^{\frac{\pi}{2}} + \frac{2}{n\pi}\int_0^{\frac{\pi}{2}} \cos nx dx - \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \sin nx dx$

$= \frac{2}{\pi}(-\frac{\frac{\pi}{2}\cos\frac{n\pi}{2}}{n}) + \frac{2}{n\pi}(\frac{\sin nx}{n})\Big|_{x=0}^{\frac{\pi}{2}} - \frac{4}{\pi}(-\frac{\cos nx}{n})\Big|_{x=0}^{\frac{\pi}{2}}$

$= -\frac{\cos\frac{n\pi}{2}}{n} + \frac{2\sin\frac{n\pi}{2}}{n^2\pi} + (\frac{4\cos\frac{n\pi}{2}}{n\pi} - \frac{4}{n\pi}) = \frac{(\pi-4)}{n\pi}\cos\frac{n\pi}{2} + \frac{2}{n^2\pi}\sin\frac{n\pi}{2} - \frac{4}{n\pi}$,

则$f(x) \sim \sum\limits_{n=1}^{+\infty} b_n \sin nx = \sum\limits_{n=1}^{+\infty} [\frac{(\pi-4)}{n\pi}\cos\frac{n\pi}{2} + \frac{2}{n^2\pi}\sin\frac{n\pi}{2} - \frac{4}{n\pi}]\sin nx.$

2. 若展开为余弦级数,则将$f(x)$进行偶函数延拓到$[-\pi,\pi]$,

有$b_n = 0, a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} (x-2)dx = \frac{2}{\pi}(\frac{\pi^2}{8} - \pi) = \frac{\pi}{4} - 2$,

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} (x-2)\cos nx dx$

$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\cos nx dx - \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \cos nx dx$

$= \frac{2}{\pi}(\frac{x\sin nx}{n})\Big|_{x=0}^{\frac{\pi}{2}} - \frac{2}{n\pi}\int_0^{\frac{\pi}{2}} \sin nx dx - \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \cos nx dx$

$= \frac{2}{\pi}(\frac{\frac{\pi}{2}\sin\frac{n\pi}{2}}{n}) - \frac{2}{n\pi}(-\frac{\cos nx}{n})\Big|_{x=0}^{\frac{\pi}{2}} - \frac{4}{\pi}(\frac{\sin nx}{n})\Big|_{x=0}^{\frac{\pi}{2}}$

$= \frac{\sin\frac{n\pi}{2}}{n} + (\frac{2\cos\frac{n\pi}{2}}{n^2\pi} - \frac{2}{n^2\pi}) - \frac{4\sin\frac{n\pi}{2}}{n\pi} = \frac{(\pi-4)}{n\pi}\sin\frac{n\pi}{2} + \frac{2}{n^2\pi}\cos\frac{n\pi}{2} - \frac{2}{n^2\pi},$

$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n\cos nx = (\frac{\pi}{8}-1) + \sum_{n=1}^{+\infty}[\frac{(\pi-4)}{n\pi}\sin\frac{n\pi}{2} + \frac{2}{n^2\pi}\cos\frac{n\pi}{2} - \frac{2}{n^2\pi}]\cos nx.$

题目2. (7.7.59) 将$f(x) = 2\pi^2 - x^2(-\pi \le x < \pi)$展开成傅里叶级数,

并计算级数$\sum_{n=1}^{+\infty}\frac{1}{n^2}$ 与级数$\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^2}$的值.

解答: 上次作业(7.7.54)(2)中已经得出:

$(\pi^2 - x^2)(-\pi \le x < \pi) \sim \frac{2\pi^2}{3} + 4\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^2}\cos nx,$

故$f(x) = \pi^2 + (\pi^2 - x^2) \sim \frac{5\pi^2}{3} + 4\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^2}\cos nx.$

令$x = \pi$,则$\frac{f(\pi+0)+f(\pi-0)}{2} = \frac{5\pi^2}{3} + 4\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^2}\cos n\pi = \frac{5\pi^2}{3} - 4\sum_{n=1}^{+\infty}\frac{1}{n^2},$

得$\sum_{n=1}^{+\infty}\frac{1}{n^2} = \frac{\frac{5\pi^2}{3} - f(\pi)}{4} = \frac{\frac{5\pi^2}{3} - \pi^2}{4} = \frac{\pi^2}{6}.$

令$x = 0$,则$\frac{f(0+0)+f(0-0)}{2} = \frac{5\pi^2}{3} + 4\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^2},$

得$\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^2} = \frac{f(0) - \frac{5\pi^2}{3}}{4} = \frac{2\pi^2 - \frac{5\pi^2}{3}}{4} = \frac{\pi^2}{12}.$

题目3. (8.1.2) 画图说明以下情形:

(1) 两个单位向量相加等于一个单位向量;

(2) 两个单位向量相减等于一个单位向量;

(3) 两个单位向量相加的模等于两个向量相减的模.

解答: 图略,分别取夹角为$\frac{2\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}$即可.

题目4. (8.1.4) 已知ΔABC上方的一点P,如图,G为ΔABC的重心,

试证:$\overrightarrow{PG} = \frac{1}{3}(\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC})$.

解答: 设BC的中点为D,则$\overrightarrow{AG} = 2\overrightarrow{GD}$,(书上已证)

则$\overrightarrow{AG} = \frac{2}{3}\overrightarrow{AD} = \frac{2}{3} \cdot \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC}) = \frac{1}{3}(\overrightarrow{AB} + \overrightarrow{AC})$,

从而$\overrightarrow{PG} = \overrightarrow{PA} + \overrightarrow{AG} = \overrightarrow{PA} + \frac{1}{3}(\overrightarrow{AB} + \overrightarrow{AC}) = \frac{1}{3}\overrightarrow{PA} + \frac{1}{3}(\overrightarrow{PA} + \overrightarrow{AB}) + \frac{1}{3}(\overrightarrow{PA} + \overrightarrow{AC})$

$= \frac{1}{3}(\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC})$.

注记: 该结论对任意点P均成立,当$P = G$时,可以得到$\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = 0$,是本题的推论,也是重心的性质之一,但不能直接利用此推论来证明本题.

题目5. (8.2.6) 用向量方法证明三角形的三条高交于一点.

解答: 设ΔABC的两条高BD, CE交于点H,

由高的定义可知$\overrightarrow{BH} \cdot \overrightarrow{AC} = 0, \overrightarrow{CH} \cdot \overrightarrow{AB} = 0$,

则$\overrightarrow{AH} \cdot \overrightarrow{BC} = (\overrightarrow{AB} + \overrightarrow{BH}) \cdot (\overrightarrow{BH} + \overrightarrow{HC}) = \overrightarrow{AB} \cdot \overrightarrow{BH} + \overrightarrow{BH} \cdot \overrightarrow{BH} + \overrightarrow{BH} \cdot \overrightarrow{HC}$

$= \overrightarrow{BH} \cdot (\overrightarrow{AB} + \overrightarrow{BH} + \overrightarrow{HC}) = \overrightarrow{BH} \cdot \overrightarrow{AC} = 0$,

故AH与BC垂直,即AH也为ΔABC的高,因此三条高交于一点H,得证!

题目6. (8.2.8) 已知$|\boldsymbol{a}|=2, |\boldsymbol{b}|=3, |\boldsymbol{a}+\boldsymbol{b}|=\sqrt{19}$,求$|\boldsymbol{a}-\boldsymbol{b}|$.

解答: $19=|\boldsymbol{a}+\boldsymbol{b}|^2=|\boldsymbol{a}|^2+|\boldsymbol{b}|^2+2\boldsymbol{a}\cdot\boldsymbol{b}=2^2+3^2+2\boldsymbol{a}\cdot\boldsymbol{b}$, 解得$\boldsymbol{a}\cdot\boldsymbol{b}=3$,

故$|\boldsymbol{a}-\boldsymbol{b}|=\sqrt{|\boldsymbol{a}|^2+|\boldsymbol{b}|^2-2\boldsymbol{a}\cdot\boldsymbol{b}}=\sqrt{2^3+3^2-2\cdot 3}=\sqrt{7}$.

题目7. (8.2.9) 设$(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}=3$, 求$[(\boldsymbol{a}+\boldsymbol{b})\times(\boldsymbol{b}+\boldsymbol{c})]\cdot(\boldsymbol{c}+\boldsymbol{a})$的值.

解答: $[(\boldsymbol{a}+\boldsymbol{b})\times(\boldsymbol{b}+\boldsymbol{c})]\cdot(\boldsymbol{c}+\boldsymbol{a})=[\boldsymbol{a}\times\boldsymbol{b}+\boldsymbol{a}\times\boldsymbol{c}+\boldsymbol{b}\times\boldsymbol{c}]\cdot(\boldsymbol{c}+\boldsymbol{a})$

$=(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}+(\boldsymbol{b}\times\boldsymbol{c})\cdot\boldsymbol{a}=2(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}=6$

注记: 注意到$\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}$中有两者相等时,必有$\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}$共面, 此时$(\boldsymbol{x}\times\boldsymbol{y})\cdot\boldsymbol{z}=0$.

题目8. (8.2.11) 设向量$\boldsymbol{a}+3\boldsymbol{b}$与$7\boldsymbol{a}-5\boldsymbol{b}$垂直, $\boldsymbol{a}-4\boldsymbol{b}$与$7\boldsymbol{a}-2\boldsymbol{b}$垂直,

求$\boldsymbol{a}$与$\boldsymbol{b}$的夹角θ.

解答: 由条件可知,$(\boldsymbol{a}+3\boldsymbol{b})\cdot(7\boldsymbol{a}-5\boldsymbol{b})=0, (\boldsymbol{a}-4\boldsymbol{b})\cdot(7\boldsymbol{a}-2\boldsymbol{b})=0$.

即$7|\boldsymbol{a}|^2-15|\boldsymbol{b}|^2+16\boldsymbol{a}\cdot\boldsymbol{b}=0, 7|\boldsymbol{a}|^2+8|\boldsymbol{b}|^2-30\boldsymbol{a}\cdot\boldsymbol{b}=0$,

将$\boldsymbol{a}\cdot\boldsymbol{b}$视为待定常数,解得$|\boldsymbol{a}|^2=|\boldsymbol{b}|^2=2\boldsymbol{a}\cdot\boldsymbol{b}$.

故$\cos\theta=\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|\cdot|\boldsymbol{b}|}=\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\sqrt{|\boldsymbol{a}|^2\cdot|\boldsymbol{b}|^2}}=\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\sqrt{(2\boldsymbol{a}\cdot\boldsymbol{b})^2}}=\frac{1}{2}$.

得$\theta=\frac{\pi}{3}$.

题目9. (8.3.12) 设有两点$M_1(x_1,y_1,z_1)$, $M_2(x_2,y_2,z_2)$,

M为M_1与M_2连线上一点, 且$\overrightarrow{M_1M}=\frac{1}{5}\overrightarrow{MM_2}$,求$M$的坐标.

解答: 设M的坐标为(x,y,z),

则$\overrightarrow{M_1M}=(x-x_1,y-y_1,z-z_1)$, $\overrightarrow{MM_2}=(x_2-x,y_2-y,z_2-z)$,

由$\overrightarrow{M_1M}=\frac{1}{5}\overrightarrow{MM_2}$,可得, $x-x_1=\frac{x_2-x}{5}$, $y-y_1=\frac{y_2-y}{5}$, $z-z_1=\frac{z_2-z}{5}$,

故$x=\frac{5x_1+x_2}{6}$, $y=\frac{5y_1+y_2}{6}$, $z=\frac{5z_1+z_2}{6}$, M的坐标为$(\frac{5x_1+x_2}{6},\frac{5y_1+y_2}{6},\frac{5z_1+z_2}{6})$.

题目10. (8.3.14) 已知$\boldsymbol{a}=(2,1,2)$, $\boldsymbol{b}=(1,3,5)$, 向量$\boldsymbol{c}$与$\boldsymbol{a},\boldsymbol{b}$共面, 且$\boldsymbol{a}\perp\boldsymbol{c}$,

求$\boldsymbol{c}$.

解答: 注意到$\boldsymbol{a},\boldsymbol{b}$不共线,故可设$\boldsymbol{c}=x\cdot\boldsymbol{a}+y\cdot\boldsymbol{b}$.其中$x,y\in\mathbb{R}$.

则$0=\boldsymbol{a}\cdot\boldsymbol{c}=x\cdot|\boldsymbol{a}|^2+y\cdot\boldsymbol{a}\cdot\boldsymbol{b}=9x+15y$, 得$(x,y)=(5t,-3t),\forall t\in\mathbb{R}$.

因此$\boldsymbol{c}=x\cdot\boldsymbol{a}+y\cdot\boldsymbol{b}=5t\cdot\boldsymbol{a}-3t\cdot\boldsymbol{b}=(7t,-4t,-5t),t\in\mathbb{R}$.

题目11. (8.3.16) 设有三点$A(1,1,1)$, $B(3,2,0)$, $C(2,4,1)$.

(1) 求ΔABC各边的长;

(2) 求ΔABC的面积;

(3) 求垂直于ΔABC所在平面的单位向量;

(4) 若$D(1,2,k)$与A,B,C共面,求k的值.

解答: $\overrightarrow{AB}=(2,1,-1)$, $\overrightarrow{AC}=(1,3,0)$, $\overrightarrow{BC}=(-1,2,1)$.

(1) $AB=|\overrightarrow{AB}|=\sqrt{2^2+1^2+(-1)^2}=\sqrt{6}$,

$AC=|\overrightarrow{AC}|=\sqrt{1^2+3^2+0^2}=\sqrt{10}$,

$BC=|\overrightarrow{BC}|=\sqrt{(-1)^2+2^2+1^2}=\sqrt{6}$.

(2) $\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 2 & 1 & -1\\ 1 & 3 & 0\end{vmatrix}=\begin{vmatrix}1 & -1\\ 3 & 0\end{vmatrix}\cdot\boldsymbol{i}-\begin{vmatrix}2 & -1\\ 1 & 0\end{vmatrix}\cdot\boldsymbol{j}+\begin{vmatrix}2 & 1\\ 1 & 3\end{vmatrix}\cdot\boldsymbol{k}$

$=3\boldsymbol{i}-\boldsymbol{j}+5\boldsymbol{k}=(3,-1,5)$.

则ΔABC的面积$S=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|=\frac{1}{2}\sqrt{3^2+(-1)^2+5^2}=\frac{\sqrt{35}}{2}$.

(3) 注意到$\overrightarrow{AB}\times\overrightarrow{AC}$为垂直于$\Delta ABC$所在平面的向量,

则垂直于ΔABC所在平面的单位向量$\boldsymbol{t}=\pm\frac{\overrightarrow{AB}\times\overrightarrow{AC}}{|\overrightarrow{AB}\times\overrightarrow{AC}|}=\pm\frac{1}{\sqrt{35}}(3,-1,5)$.

(4) 由共面可知$(\overrightarrow{AB}\times\overrightarrow{AC})\cdot\overrightarrow{AD}=0$, 即$(3,-1,5)\cdot(0,1,k-1)=0$,

得$-1+5(k-1)=0$,故$k=\frac{6}{5}$.