先推导其动力学方程：

$$F = E(x)A(x)\frac{\partial y}{\partial x}$$，根据 $F=ma$.

$$\rho(x)A(x)dx \cdot \frac{\partial^2 y}{\partial t^2} = (F + \frac{\partial F}{\partial x}dx) - F$$

$$\Rightarrow \rho(x)A(x)\cdot\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left[E(x)A(x)\frac{\partial y}{\partial x}\right]$$

分离变量法：$u(x,t) = X(x)\cdot T(t)$.

$$\rho(x)A(x)\cdot T''(t)\cdot X(x) = [E(x)A(x)X'(x)]'\cdot T(t)$$

$$\frac{T''(t)}{T(t)} = \frac{[E(x)A(x)X'(x)]'}{\rho(x)A(x)X(x)} = -\omega^2.$$ 讨论关于 x 的方程

$$[E(x)A(x)X'(x)]' + \omega^2\rho(x)A(x)X(x) = 0$$

设有两个不同固有频率 ω_i 和 ω_j 所对应的振型函数分别为 $X_i(x)$ X_j

$$\int_0^l X_j(x)[E(x)A(x)X'(x)]'dx = X_j(x)\cdot E(x)A(x)X_i'(x)\Big|_0^l - \int_0^l E(x)A(x)X_i'(x)X_j'(x)$$

① 两端弹性支撑，则有边界条件

$$\begin{cases} E(0)A(0)X'(0) = -k_1 \cancel{u(0,t)}X(0) \\ E(l)A(l)X'(l) = -k_2 \cancel{u(l,t)}X(l) \end{cases}$$

故 $\int \cancel{X_j(x)} - X_j(l)k_2u(l,t) + X_j(0)k_1u(0,t) - \int_0^l E(x)A(x)X_i'(x)X_j'(x)dx$

$$= \int_0^l -\omega_i^2\rho(x)A(x)X_i(x)X_j(x)$$

将i,j交换,有

$$-X_i(l)k_2u(l,t)+X_i(0)k_1u(0,t)-\int_0^l E(x)A(x)X_i'(x)X_j'(x)dx$$

$$=\int_0^l -\omega_j^2\cdot\rho(x)A(x)X_i(x)X_j(x)dx$$

两式相减：

$$\cancel{[X_i(l)-X_j(l)]\cdot k_2u(l,t)+[X_j(0)-X_i(0)]k_1u(0,t)}$$

$$\cancel{=\int(\omega_j^2-\omega_i^2)\int_0^l\rho(x)A(x)X_i(x)X_j(x)dx.}$$

得：$(\omega_j^2-\omega_i^2)\int_0^l\rho(x)A(x)X_i(x)X_j(x)dx=0$.

故正交性与一般情况相同

$$\int_0^l\rho(x)A(x)X_i(x)X_j(x)dx=\begin{cases}M_i & i=j\\ 0 & i\neq j\end{cases}$$ 其中 $M_i=\int_0^l\rho(x)A(x)X_i^2(x)dx$.

但关于刚度的正交性,则需要修正.

$$\int_0^l E(x)A(x)X_i'(x)X_j'(x)dx+k_2X_i(l)X_j(l)-k_1X_i(0)X_j(0)=\begin{cases}K_i & i=j\\ 0 & i\neq j\end{cases}$$

其中：$K_i=\int_0^l\omega_i^2\cdot\rho(x)A(x)X_i(x)X_j(x)dx$.

) 对于左端~~封~~固支,右端集中质量,采用类似的方法.

同样有 $[E(x)A(x)X'(x)]' + \omega^2\rho(x)A(x)X(x) = 0$.

$$\int_0^l X_j(x)[E(x)A(x)X_i'(x)]'dx = X_j(x)E(x)A(x)X_i'(x)\Big|_0^l - \int_0^l E(x)A(x)X_i'(x)X_j'(x)dx.$$

边界条件: $\begin{cases} \cancel{E(0)A(0)}X(0) = 0 \\ E(l)A(l)X'(l) = m_0\omega^2 X(l) \end{cases}$ → 代入后得:

$$\cancel{X_j(l)E(} -X_j(l)m_0\omega_i^2 X_i(l) + \int_0^l E(x)A(x)X_i'(x)X_j'(x)dx = \int_0^l \omega_i^2\rho(x)A(x)X_i(x)X_j(x)dx$$

i,j互换: $-\cancel{X_j(l)}m_0\omega_j^2 X_i(l)X_j(l) + \int_0^l E(x)A(x)X_i'(x)X_j'(x)dx = \omega_j^2\int_0^l \rho(x)A(x)X_i(x)X_j(x)dx$

两式相减,得到: $\int_0^l \rho(x)A(x)X_i(x)X_j(x)dx = -m_0 X_i(l)X_j(l) \quad (i \neq j)$

~~代回可以得到:~~ $\cancel{\int_0^l E(x)A(x)X_i'(x)X_j'(x)dx = 0}$.

故变成: $\int_0^l \rho(x)A(x)X_i(x)X_j(x)dx + m_0 X_i(l)X_j(l) = 0 \quad (i \neq j)$

代回有 $\int_0^l E(x)A(x)X_i'(x)X_j'(x)dx = 0 \quad (i \neq j)$

现在再考虑一下梁的情况。先推导梁的弯曲振动方程.

$M=-EI\frac{\partial^2 w}{\partial x^2}$　平衡方程：$\frac{\partial F_s}{\partial x}dx-\rho S(x)\frac{\partial^2 w}{\partial t^2}dx+f(x,t)dx=0$

又有. $\frac{\partial M}{\partial x}dx-F_s dx+f(x,t)\frac{(dx)^2}{2}=0$.

忽略 dx 平方项, $\Longrightarrow F_s=\frac{\partial M}{\partial x}$,

$$\Rightarrow \frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 w(x,t)}{\partial x^2}\right]+\rho S(x)\frac{\partial^2 w(x,t)}{\partial t^2}=f(x,t)$$

现取 $f(x,t)=0$, 得：$\frac{\partial^2}{\partial x^2}\left[E(x)I(x)\frac{\partial^2 w(x,t)}{\partial x^2}\right]+\rho(x)A(x)\frac{\partial^2 w(x,t)}{\partial t^2}=0$

分离变量 $w(x,t)=X(x)T(t)$

$$\Rightarrow \frac{\ddot{T}(t)}{T(t)}=-\frac{[E(x)I(x)X''(x)]''}{\rho(x)A(x)X(x)}=-\omega^2.$$

故对于X项, 有 $[E(x)I(x)X''(x)]''-\omega^2\rho(x)A(x)X(x)=0$

$$\int_0^l \phi_j(x)\left[EI(x)\cancel{\phi}''_i(x)\right]''dx$$

$$=\phi_j(x)\left[E(x)I(x)\cancel{\phi}''_i(x)\right]'\Big|_0^l-\left[\cancel{\phi}'_j(x)E(x)I(x)\cancel{\phi}''_i(x)\right]\Big|_0^l$$

$$+\int_0^l E(x)I(x)\phi''_j(x)\phi''_i(x)dx$$　　ϕ 改为 X

答① ~~一端~~ 一端固定 一端集中质量.

~~x=0~~ $X(0)=0 \quad X'(0)=0.$

集中质量: $\forall E(x)I(x)X''(l)=0, \quad E(x)I(x)X'''(l)=-m_0\omega^2X(l)$

故 $\cancel{X_j(l)}\ X_j(l)\cdot[-m_0\omega_i^2X_i(l)]-0+\int_0^l E(x)I(x)X_j''(x)X_i''(x)\,dx$

$=\int_0^l \omega_i^2\rho(x)A(x)X_i(x)X_j(x)\,dx$

i, j 互换. $X_i(l)\cdot[-m_0\omega_j^2X_j(l)]+\int_0^l E(x)I(x)X_i''(x)X_j''(x)\,dx$

$=\int_0^l \omega_j^2\rho(x)A(x)X_i(x)X_j(x)\,dx$

两式相减

$$(\omega_j^2-\omega_i^2)\cdot m_0X_i(l)X_j(l)=(\omega_i^2-\omega_j^2)\int_0^l\rho(x)A(x)X_i(x)X_j(x)\,dx$$

$$\Rightarrow \int_0^l\rho(x)A(x)X_i(x)X_j(x)\,dx+m_0X_i(l)X_j(l)=0 \quad (i\neq j)$$

代回原式, 有 $\int_0^l E(x)I(x)X_j''(x)X_i''(x)\,dx=0 \quad (i\neq j)$

② 两端弹簧支撑，变为

~~$E(x)I(x)\phi''$~~

~~$E(0)I(0)\phi''(0) = -k_1\phi'(0)$ $\quad EI\phi'''(0) = k_2\phi(0)$~~

$E(0)I(0)X''(0) = -k_1 X'(0) \qquad E(0)I(0)X'''(0) = k_2 X(0)$

$E(l)I(l)X''(l) = -k_3 X'(l) \qquad E(0)I(0)X'''(l) = k_4 X(l)$

$X_j(l)\cdot k_4 X_i(l) - X_j(0)\cdot k_2 X_i(0) + X_j'(l)\cdot k_3 X_i'(l)$

$- X_j'(0)\cdot k_1 X_i'(0) + \int_0^l E(x)I(x)X_j''(x)X_i''(x)\,dx$

$= \int_0^l \omega_i^2 \rho(x)A(x)X_i(x)X_j(x)\,dx$

同样 i,j 互换后两式相减，

得到 $\int_0^l \rho(x)A(x)X_i(x)X_j(x)\,dx = \begin{cases} M_i & i=j \\ 0 & i\neq j \end{cases}$ 与杆的结果相同.

代回，有 $\int_0^l E(x)I(x)X_j''(x)X_i''(x)\,dx + k_4 X_i(l)X_j(l) - k_2 X_j(0)X_i(0)$

$+ k_3 X_j'(l)\cdot X_i'(l) - k_1 X_j'(0)X_i'(0) = 0 \quad (i\neq j)$