微积分H作业解答

第九周

题目1. (9.6.60) 已知三角形的周长为$2s$,求面积的最大值.

解答: 设三角形的三条边分别为x, y, z,则题目转化为:

已知$x+y+z=2s$,求$\sqrt{s(s-x)(s-y)(s-z)}$的最大值.

为便于计算,我们求$(s-x)(s-y)(s-z)$的最大值,

令$L(x,y,z,\lambda)=(s-x)(s-y)(s-z)+\lambda(2s-x-y-z)$,

则$\frac{\partial L}{\partial x}=-(s-y)(s-z)-\lambda$,由$\frac{\partial L}{\partial x}=0$可得$\lambda=-(s-y)(s-z)$,

对y, z有类似的结果,则$\lambda=-(s-y)(s-z)=-(s-x)(s-z)=-(s-x)(s-y)$,

得到驻点$x=y=z=\frac{2s}{3}$,经检验为极大值点,并取得最大值,此时面积为$\frac{\sqrt{3}}{9}s^2$.

题目2. (9.6.62) 设$z=z(x,y)$是由方程$x^2-6xy+10y^2-2yz-z^2+18=0$

所确定的函数,求函数$z=z(x,y)$的极值点和极值.

解答: 将z看作函数,将方程分别对$x.y$求偏导,

得$2x-6y-2y\frac{\partial z}{\partial x}-2z\frac{\partial z}{\partial x}=0$ 和$-6x+20y-2z-2y\frac{\partial z}{\partial y}-2z\frac{\partial z}{\partial y}=0$,

令$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial y}=0$,则上述两方程化为$2x-6y=0;-6x+20y-2z=0$,

因此在驻点处有$z=y=\frac{x}{3}$,代回原方程得$z^2=9,z=\pm 3$,

注意到原方程可以写为$(x-3y)^2+(y-z)^2=2(z^2-9)$,则$|z|\geq 3$,

故$(9,3)$为极小值点,极小值为3, $(-9,-3)$为极大值点,极大值为-3.

题目3. (9.6.65) 求原点到曲面$S:z^2=xy+x-y+6$上点的最短距离.

解答: 即求$\sqrt{x^2+y^2+z^2}$的最小值,为计算方便,我们求$x^2+y^2+z^2$的最小值,

取$L(x,y,z,\lambda)=x^2+y^2+z^2+\lambda(z^2-xy-x+y-6)$,

令$\frac{\partial L}{\partial x}=2x-\lambda(y+1)=0;\frac{\partial L}{\partial y}=2y+\lambda(1-x)=0;\frac{\partial L}{\partial z}=2z+2z\lambda=0$,

解得驻点为$\lambda=-1,x=-1,y=1,z=\pm\sqrt{3}$,

或$\lambda=-\frac{14+2\sqrt{7}}{7},x=1\pm\sqrt{7},y=-1\mp\sqrt{7},z=0$,

代入并逐一检验得最小值为$\sqrt{5}$.

题目4. (9.7.73) 求下列向量函数的导数:

(3) $\boldsymbol{r}(t)=(t\cos t,t\sin t,t(\cos t-\sin t))$.

解答: $\boldsymbol{r}'(t)=(\cos t-t\sin t,\sin t+t\cos t,(\cos t-\sin t)-t(\sin t+\cos t))$.

题目5. (9.7.74) 设$\boldsymbol{r}(t)=(\frac{2t}{1+t^2},\frac{1-t^2}{1+t^2},1)$,证明: $\boldsymbol{r}(t)$与$\boldsymbol{r}'(t)$之间的夹角为定值.

解答: $\boldsymbol{r}'(t)=(\frac{2(1-t^2)}{(1+t^2)^2},-\frac{4t}{(1+t^2)^2},0)$. 有$\boldsymbol{r}(t)\cdot\boldsymbol{r}'(t)=0$,故夹角为定值$\frac{\pi}{2}$.

题目6. (9.7.76) 设$f(x,y,z)=\frac{1}{\sqrt{x^2+y^2+z^2}}$,$\boldsymbol{l}=(-1,2,-2)$, 求$\frac{\partial f}{\partial \boldsymbol{l}}|_{(1,2,2)}$和$\mathrm{grad}f|_{(1,2,2)}$.

解答: 记$r=\sqrt{x^2+y^2+z^2}$,则$\frac{\partial f}{\partial x}=-\frac{1}{r^2}\cdot\frac{x}{r}=-\frac{x}{r^3}$, 得$\frac{\partial f}{\partial x}|_{(1,2,2)}=-\frac{1}{27}$,

同理可得$\frac{\partial f}{\partial y}|_{(1,2,2)}=\frac{\partial f}{\partial z}|_{(1,2,2)}=-\frac{2}{27}$,与方向$\boldsymbol{l}$同向的单位向量$\boldsymbol{l}^0=(-\frac{1}{3},\frac{2}{3},-\frac{2}{3})$,

因此$\frac{\partial f}{\partial \boldsymbol{l}}|_{(1,2,2)}=\frac{\partial f}{\partial x}|_{(1,2,2)}\cos\alpha+\frac{\partial f}{\partial y}|_{(1,2,2)}\cos\beta+\frac{\partial f}{\partial z}|_{(1,2,2)}\cos\gamma$

$=-\frac{1}{27}\cdot(-\frac{1}{3})-\frac{2}{27}\cdot\frac{2}{3}-\frac{2}{27}\cdot(-\frac{2}{3})=\frac{1}{81}$.

且$\mathrm{grad}f|_{(1,2,2)}=(\frac{\partial f}{\partial x}|_{(1,2,2)},\frac{\partial f}{\partial y}|_{(1,2,2)},\frac{\partial f}{\partial z}|_{(1,2,2)})=-\frac{1}{27}(1,2,2)$.

题目7. (9.7.79) 求下列函数在指定点处的梯度:

(2) $f(x,y,z)=\frac{x}{x^2+y^2+z^2}$,在点$M(-3,0,1)$处.

解答: $\frac{\partial f}{\partial x}=\frac{-x^2+y^2+z^2}{(x^2+y^2+z^2)^2}$, $\frac{\partial f}{\partial y}=\frac{-2xy}{(x^2+y^2+z^2)^2}$, $\frac{\partial f}{\partial z}=\frac{-2xz}{(x^2+y^2+z^2)^2}$,

则$\mathrm{grad}f|_M=(\frac{\partial f}{\partial x}|_M,\frac{\partial f}{\partial y}|_M,\frac{\partial f}{\partial z}|_M)=(-\frac{2}{25},0,\frac{3}{50})$.

题目8. (9.7.80) 设$f(x,y,z)=axy^2+byz+cz^2x^3$在点$M(1,2,-1)$处沿$z$轴正向的方向导数为点$M$处所有方向导数的最大值,且其最大值为64,求常数$a,b,c$的值.

解答: $\frac{\partial f}{\partial x}=ay^2+3cz^2x^2$, $\frac{\partial f}{\partial y}=2axy+bz$, $\frac{\partial f}{\partial z}=by+2czx^3$,

由题意得$\mathrm{grad}f|_M=(\frac{\partial f}{\partial x}|_M,\frac{\partial f}{\partial y}|_M,\frac{\partial f}{\partial z}|_M)=(4a+3c,4a-b,2b-2c)=(0,0,64)$,

解得$a=6,b=24,c=-8$.

题目9. (9.7.82) 设$f(x,y)=\sqrt[3]{x^3+y^3}$,证明: $f(x,y)$在点$(0,0)$处沿任何方向的方向导数均存在,但$f(x,y)$在点$(0,0)$处不可微.

解答: 设方向$\boldsymbol{l}=(\cos\alpha,\sin\alpha)(0\le\alpha<2\pi)$,

则$\frac{\partial f}{\partial \boldsymbol{l}}|_{(0,0,0)}=\lim\limits_{\rho\to 0}\frac{f(\rho\cos\alpha,\rho\sin\alpha)-f(0,0)}{\rho}=\sqrt[3]{\cos^3\alpha+\sin^3\alpha}$,

因此函数$f(x,y)$在点$(0,0)$处沿任何方向的方向导数均存在.

特别地有$\frac{\partial f}{\partial x}|_{(0,0)}=\frac{\partial f}{\partial y}|_{(0,0)}=1$,

则$f(x,y)$在点$(0,0)$处可微等价于$\lim\limits_{(x,y)\to(0,0)}\frac{\sqrt[3]{x^3+y^3}-x-y}{\sqrt{x^2+y^2}}=0$,

但当$x=y$时,重极限为$\frac{\sqrt[3]{2}-2}{\sqrt{2}}\neq 0$,故$f(x,y)$在点$(0,0)$处不可微.

题目10. (9.8.83) 求下列曲线在指定点处的切线方程:

$$(1)\ \begin{cases} x=2\cos t,\\ y=2\sin t, \qquad t=\frac{\pi}{3};\\ z=6t,\end{cases}$$

$$(4)\ \begin{cases} x^2+y^2+z^2=50,\\ x^2+y^2=z^2,\end{cases} \qquad 点(3,4,5).$$

解答: (1) 曲线$\boldsymbol{r}(t)=(2\cos t,2\sin t,6t)$,向量函数导数为$\boldsymbol{r}'(t)=(-2\sin t,2\cos t,6)$,

当$t=\frac{\pi}{3}$时,切点为$\boldsymbol{r}(\frac{\pi}{3})=(1,\sqrt{3},2\pi)$,切向量$\boldsymbol{r}'(\frac{\pi}{3})=(-\sqrt{3},1,6)$,

则切线方程为$\frac{x-1}{-\sqrt{3}}=\frac{y-\sqrt{3}}{1}=\frac{z-2\pi}{6}$.

(4) 令$F(x,y,z)=x^2+y^2+z^2-50, G(x,y,z)=x^2+y^2-z^2$,

则切线方程为$\begin{cases} F'_x(3,4,5)\cdot(x-3)+F'_y(3,4,5)\cdot(y-4)+F'_z(3,4,5)\cdot(z-5)=0, \\ G'_x(3,4,5)\cdot(x-3)+G'_y(3,4,5)\cdot(y-4)+G'_z(3,4,5)\cdot(z-5)=0, \end{cases}$

即$\begin{cases} 6(x-3)+8(y-4)+10(z-5)=0, \\ 6(x-3)+8(y-4)-10(z-5)=0, \end{cases}$,化简得$\begin{cases} 3x+4y+5z=50, \\ 3x+4y-5z=0, \end{cases}$

题目11. (9.8.84) 在曲线$C:\boldsymbol{r}(t)=(t,\frac{1}{2}t^2,\frac{1}{3}t^3)$上求一点,使得该点处切线与平面$x-2y+z=4$平行,并求该点处的切线方程.

解答: $\boldsymbol{r}'(t)=(1,t,t^2)$,平面$x-2y+z=4$的法向量为$\overrightarrow{n}=(1,-2,1)$,

由$\boldsymbol{r}'(t)\cdot\overrightarrow{n}=(1,t,t^2)\cdot(1,-2,1)=(t-1)^2=0$解得$t=1$,

则切点为$\boldsymbol{r}(1)=(1,\frac{1}{2},\frac{1}{3})$,切向量为$\boldsymbol{r}'(1)=(1,1,1)$,

切线方程为$x-1=y-\frac{1}{2}=z-\frac{1}{3}$.

题目12. (9.8.85) 证明:螺旋线$\boldsymbol{r}(t)=(a\cos t,a\sin t,bt)$上任意一点处的切线与$z$轴成定角.

解答: $\boldsymbol{r}'(t)=(-a\sin t,a\cos t,b)$, 设$\overrightarrow{z}=(0,0,1)$, $\boldsymbol{r}'(t)$与$\overrightarrow{z}$的夹角为θ,

则有$\cos\theta=\frac{\boldsymbol{r}'(t)\cdot\overrightarrow{z}}{|\boldsymbol{r}'(t)|\cdot|\overrightarrow{z}|}=\frac{b}{\sqrt{a^2+b^2}}$,与$t$无关,得夹角$\theta$为定值.

题目13. (9.8.88)求曲面$S: x^2+2y^2-3z^2=3$在点$(2,-1,1)$处的切平面方程.

解答: 设$F(x,y,z)=x^2+2y^2-3z^2-3$,则$F'_x=2x, F'_y=4y, F'_z=-6z$,

在点$(2,-1,1)$处切平面的法向量为$(F'_x, F'_y, F'_z)|_{(2,-1,1)}=(4,-4,-6)$,

得切平面方程为$4(x-2)-4(y+1)-6(z-1)=0$,即$2x-2y-3z-3=0$.

题目14. (9.8.95) 设$z=f(x,y)$在$\mathbb{R}^2$上连续,且满足

$$\lim_{(x,y)\to(1,2)} \frac{f(x,y)+x-2y+6}{(x-1)^2+(y-2)^2}=2$$

(1) 求曲面$z=f(x,y)$在点$(1,2)$处的切平面方程;

(2) 点$(1,2)$是否为函数$z=f(x,y)$的极值点,为什么?

解答: (1) 显然$\lim\limits_{(x,y)\to(1,2)}[(x-1)^2+(y-2)^2]=0$,

则$\lim\limits_{(x,y)\to(1,2)}[f(x,y)+x-2y+6]=0$,得$f(1,2)=-3$.

且$f(x,y)=-x+2y-6+2(x-1)^2+2(y-2)^2+o(\rho^2)$,其中$\rho=\sqrt{(x-1)^2+(y-2)^2}$,

即$f(x,y)=f(1,2)-(x-1)+2(y-2)+o(\rho)$,

因此函数$z=f(x,y)$在点$(1,2)$处可微,且$dz|_{(1,2)}=-dx+2dy$,

故曲面$z=f(x,y)$在点$(1,2)$处的切平面方程为$z+3=-(x-1)+2(y-2)$;

(2) 不一定.

由(1)知,$\frac{\partial z}{\partial x}|_{(1,2)}=-1\neq 0, \frac{\partial z}{\partial y}|_{(1,2)}=2\neq 0$则$(1,2)$不是$z$的极值点.