微积分H作业解答

第三周

题目1. (7.5.35) 将$f(x)=\frac{d}{dx}(\frac{e^x-1}{x})$展开成麦克劳林级数,并计算$\sum\limits_{n=1}^{+\infty}\frac{n}{(n+1)!}$.

解答: 令$g(x)=\frac{e^x-1}{x}$,因为$e^x=\sum\limits_{n=0}^{+\infty}\frac{x^n}{n!}$, 所以$g(x)=\frac{\sum\limits_{n=0}^{+\infty}\frac{x^n}{n!}-1}{x}=\sum\limits_{n=1}^{+\infty}\frac{x^{n-1}}{n!}$,
得$f(x)=\frac{d}{dx}g(x)=\sum\limits_{n=1}^{+\infty}\frac{d}{dx}\cdot\frac{x^{n-1}}{n!}=\sum\limits_{n=2}^{+\infty}\frac{(n-1)x^{n-2}}{n!}=\sum\limits_{n=1}^{+\infty}\frac{nx^{n-1}}{(n+1)!}$.
另一方面,$f(x)=\frac{d}{dx}g(x)=\frac{d}{dx}\cdot\frac{e^x-1}{x}=\frac{xe^x-e^x+1}{x^2}$, 得$\sum\limits_{n=1}^{+\infty}\frac{n}{(n+1)!}=f(1)=1$.

题目2. (7.5.36) 将下列函数展开成关于x的幂级数.并求其收敛区间:

(2) $(x+1)e^{2x}$;

(4) $\cos(x-\frac{\pi}{3})$;

(7) $\ln(x+\sqrt{1+x^2})$.

解答: (2) $(x+1)e^{2x}=(x+1)\sum\limits_{n=0}^{+\infty}\frac{(2x)^n}{n!}=\sum\limits_{n=0}^{+\infty}\frac{2^nx^{n+1}}{n!}+\sum\limits_{n=0}^{+\infty}\frac{2^nx^n}{n!}$
$=\sum\limits_{n=1}^{+\infty}\frac{2^{n-1}x^n}{(n-1)!}+(1+\sum\limits_{n=1}^{+\infty}\frac{2^nx^n}{n!})=1+\sum\limits_{n=1}^{+\infty}(\frac{2^{n-1}}{(n-1)!}+\frac{2^n}{n!})x^n=1+\sum\limits_{n=1}^{+\infty}\frac{(n+2)\cdot 2^{n-1}}{n!}x^n$.
而$\lim\limits_{n\to+\infty}|\frac{a_{n+1}}{a_n}|=\lim\limits_{n\to+\infty}\frac{(n+3)\cdot 2}{(n+2)(n+1)}=0$, 则收敛半径$r=+\infty$,收敛区间为$(-\infty,+\infty)$.

(4) 记$f(x)=\cos(x-\frac{\pi}{3})$,则$n$阶导数$f^{(n)}(x)=\cos(x+\frac{n\pi}{2}-\frac{\pi}{3})$.

得$f(x)=\sum\limits_{n=0}^{+\infty}\frac{f^{(n)}(0)}{n!}x^n=\sum\limits_{n=0}^{+\infty}\frac{\cos(\frac{n\pi}{2}-\frac{\pi}{3})}{n!}x^n$,

$(=\sum\limits_{n=0}^{+\infty}\frac{(-1)^n[(2n+1)x^{2n}+\sqrt{3}x^{2n+1}]}{2(2n+1)!}=\frac{1}{2}\sum\limits_{n=0}^{+\infty}(-1)^n\frac{x^{2n}}{(2n)!}+\frac{\sqrt{3}}{2}\sum\limits_{n=0}^{+\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!})$

而$\lim\limits_{n\to+\infty}\sqrt[n]{|a_n|}=\lim\limits_{n\to+\infty}\sqrt[n]{\frac{\cos(\frac{n\pi}{2}-\frac{\pi}{3})}{n!}}=0$,

则收敛半径$r=+\infty$,收敛区间为$(-\infty,+\infty)$.

(7) 记$f(x)=\ln(x+\sqrt{1+x^2})$,则$f'(x)=\frac{1}{\sqrt{1+x^2}}$,

在$(1+x)^\alpha$的展开式中取$\alpha=-\frac{1}{2}$可得,

$(1+x)^{-\frac{1}{2}}=1+\sum\limits_{n=1}^{+\infty}\frac{(-\frac{1}{2})(-\frac{1}{2}-1)...(-\frac{1}{2}-n+1)}{n!}x^n=1+\sum\limits_{n=1}^{+\infty}(-\frac{1}{2})^n\cdot\frac{1\cdot(1+2)...(1+2(n-1))}{n!}x^n$

$=1+\sum\limits_{n=1}^{+\infty}(-\frac{1}{2})^n\cdot\frac{(2n-1)!!}{n!}x^n=1+\sum\limits_{n=1}^{+\infty}(-1)^n\frac{(2n-1)!!}{2^n n!}x^n=1+\sum\limits_{n=1}^{+\infty}(-1)^n\frac{(2n-1)!!}{(2n)!!}x^n$,

且收敛区间为$(-1,1)$,令$x\to x^2$, 得$f'(x)=\frac{1}{\sqrt{1+x^2}}=1+\sum\limits_{n=1}^{+\infty}(-1)^n\frac{(2n-1)!!}{(2n)!!}x^{2n}$,

而$f(0)=0$,则$f(x)=\int f'(x)dx=x+\sum\limits_{n=1}^{+\infty}(-1)^n\frac{(2n-1)!!}{(2n+1)(2n)!!}x^{2n+1}$,

收敛区间仍为$(-1,1)$.

注记: 注意到此题的要求为展开成关于x的幂级数.

题目3. (7.5.37) 设$f(x)=\sin 3x\cos x$,计算$f^{(n)}(0)(n=1,2,...)$.

解答: $f(x)=\sin 3x\cos x=\frac{1}{2}(\sin 4x+\sin 2x)$

$=\frac{1}{2}[\sum\limits_{n=0}^{+\infty}(-1)^n\frac{(4x)^{2n+1}}{(2n+1)!}+\sum\limits_{n=0}^{+\infty}(-1)^n\frac{(2x)^{2n+1}}{(2n+1)!}]=\sum\limits_{n=0}^{+\infty}\frac{(-1)^n}{2(2n+1)!}(4^{2n+1}+2^{2n+1})x^{2n+1}$

由泰勒定理,$\frac{f^{(2n)}(0)}{(2n)!}=0$, $\frac{f^{(2n+1)}(0)}{(2n+1)!}=\frac{(-1)^n}{2(2n+1)!}(4^{2n+1}+2^{2n+1})$.

故$f^{(2n)}(0)=0$, $f^{(2n+1)}(0)=\frac{(-1)^n}{2}(4^{2n+1}+2^{2n+1})$.

题目4. (7.5.40) 利用函数的幂级数展开,计算下列极限:

(1) $\lim\limits_{x\to 0}\frac{2[\ln(1+x)-\sin x]+x^2}{x(\sqrt{1-2x}-1)\cdot\arcsin x}$;

(3) $\lim\limits_{x\to 0}\frac{x^2(1+\cos x)-2\sin^2 x}{x^4}$.

解答: (1) $\lim\limits_{x\to 0}\frac{2[\ln(1+x)-\sin x]+x^2}{x(\sqrt{1-2x}-1)\cdot\arcsin x}=\lim\limits_{x\to 0}\frac{2[(x-\frac{x^2}{2}+\frac{x^3}{3}+o(x^3))-(x-\frac{x^3}{6}+o(x^3))]+x^2}{x[(1-x+o(x))-1]\cdot(x+o(x))}$

$=\lim\limits_{x\to 0}\frac{2[-\frac{x^2}{2}+\frac{x^3}{2}+o(x^3)]+x^2}{-x(x+o(x))(x+o(x))}=\lim\limits_{x\to 0}\frac{x^3+o(x^3)}{-x(x^2+o(x^2))}=\lim\limits_{x\to 0}\frac{x^3+o(x^3)}{-x^3+o(x^3)}=-1.$

(3) $\lim\limits_{x\to 0}\frac{x^2(1+\cos x)-2\sin^2 x}{x^4}=\lim\limits_{x\to 0}\frac{x^2[1+(1-\frac{x^2}{2}+o(x^2))]-2(x-\frac{x^3}{6}+o(x^3))^2}{x^4}$

$=\lim\limits_{x\to 0}\frac{x^2(2-\frac{x^2}{2}+o(x^2))-2(x^2-\frac{x^4}{3}+o(x^6))}{x^4}=\lim\limits_{x\to 0}\frac{2x^2-\frac{x^4}{2}-2x^2+\frac{2x^4}{3}+o(x^4)}{x^4}$

$=\lim\limits_{x\to 0}\frac{\frac{x^4}{6}+o(x^4)}{x^4}=\frac{1}{6}.$

题目5. (7.5.42) 当$x\to 0$时,$\int_0^x e^t\cos t dt-x-\frac{x^2}{2}$与$Ax^n$为等价无穷小,

求常数A和n的值.

解答: 设$f(x)=\int_0^x e^t\cos t dt-x-\frac{x^2}{2}$,则$\lim\limits_{x\to 0}\frac{f(x)}{x^n}=A$.

注意到$f(0)=0$,则可以利用洛必达法则,有$\lim\limits_{x\to 0}\frac{f'(x)}{nx^{n-1}}=A$.

另一方面,$f'(x)=e^x\cos x-1-x=(1+x+\frac{x^2}{2}+\frac{x^3}{6}+o(x^3))(1-\frac{x^2}{2}+o(x^3))-1-x$

$=1+x-\frac{x^3}{3}+o(x^3)-1-x=-\frac{x^3}{3}+o(x^3).$

代入得$\lim\limits_{x\to 0}\frac{-\frac{x^3}{3}+o(x^3)}{nx^{n-1}}=A$, 只能是$\begin{cases}n-1=3,\\ \frac{-\frac{1}{3}}{n}=A,\end{cases}$ 得$A=-\frac{1}{12},n=4$.

题目6. (7.5.45) 将下列函数在指定点x_0处展开成泰勒级数:

(2) $\frac{2x+3}{x^2+3x}, x_0 = -2.$

解答: 令$t = x+2$,则$\frac{2x+3}{x^2+3x} = \frac{x+(x+3)}{x(x+3)} = \frac{1}{x} + \frac{1}{x+3} = \frac{1}{t-2} + \frac{1}{t+1}$

$= -\frac{1}{2}(1+(-\frac{t}{2}))^{-1} + (1+t)^{-1} = -\frac{1}{2}\sum\limits_{n=0}^{+\infty}(-1)^n(-\frac{t}{2})^n + \sum\limits_{n=0}^{+\infty}(-1)^n t^n$

$= \sum\limits_{n=0}^{+\infty}[(-\frac{1}{2})(-1)^n(-\frac{1}{2})^n + (-1)^n] = \sum\limits_{n=0}^{+\infty}[(-1)^n - \frac{1}{2^{n+1}}]t^n$

$= \sum\limits_{n=0}^{+\infty}[(-1)^n - \frac{1}{2^{n+1}}](x+2)^n$,收敛区间为$(-3,-1)$.

题目7. (7.7.53) 设$f(x) = x^2(0 \le x \le 1)$,而$S(x) = \sum\limits_{n=1}^{+\infty} b_n \sin n\pi x$

$(-\infty < x < +\infty)$, 其中$b_n = 2\int_0^1 f(x)\sin n\pi x dx (n = 1, 2, ...)$,求$S(-\frac{1}{2})$的值.

解答: 将$f(x)$延拓成$(-1,1]$上的奇函数$F(x) = \begin{cases} x^2, & 0 \le x \le 1; \\ -x^2, & -1 < x < 0. \end{cases}$,

再延拓成周期$T=2$的周期函数.

因为$F(x)$为奇函数,则$F(x)$的傅里叶级数展开式为$\sum\limits_{n=1}^{+\infty} B_n \sin n\pi x$,

其中傅里叶系数$B_n = \int_{-1}^1 F(x)\sin n\pi x dx = 2\int_0^1 f(x)\sin n\pi x dx = b_n$.

即$F(x) \sim S(x) = \sum\limits_{n=1}^{+\infty} b_n \sin n\pi x$,

故$S(-\frac{1}{2}) = \frac{F(-\frac{1}{2}+0)+F(-\frac{1}{2}-0)}{2} = F(-\frac{1}{2}) = -f(\frac{1}{2}) = -\frac{1}{4}$.

题目8. (7.7.54) 将下列周期为2π的函数展开成傅里叶级数:

(2) $f(x)=\begin{cases} x, & -\pi\le x<0,\\ 2x, & 0\le x<\pi; \end{cases}$

(5) $f(x)=\pi^2-x^2, -\pi\le x<\pi.$

解答: (2) $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_0^{\pi}2xdx+\frac{1}{\pi}\int_{-\pi}^{0}xdx=\frac{1}{\pi}\cdot\pi^2-\frac{1}{\pi}\cdot\frac{\pi^2}{2}=\frac{\pi}{2}$,

$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\frac{1}{\pi}\int_0^{\pi}2x\cos nxdx+\frac{1}{\pi}\int_{-\pi}^{0}x\cos nxdx$

$=(\frac{2}{\pi}(\frac{x\sin nx}{n})\big|_{x=0}^{\pi}-\frac{2}{n\pi}\int_0^{\pi}\sin nxdx)+(\frac{1}{\pi}(\frac{x\sin nx}{n})\big|_{x=-\pi}^{0}-\frac{1}{n\pi}\int_{-\pi}^{0}\sin nxdx)$

$=-\frac{2}{n\pi}\int_0^{\pi}\sin nxdx-\frac{1}{n\pi}\int_{-\pi}^{0}\sin nxdx=-\frac{2}{n\pi}(-\frac{\cos nx}{n})\big|_{x=0}^{\pi}-\frac{1}{n\pi}(-\frac{\cos nx}{n})\big|_{x=-\pi}^{0}$

$=\frac{2((-1)^n-1)}{n^2\pi}+\frac{1-(-1)^n}{n^2\pi}=\frac{(-1)^n-1}{n^2\pi}$,

$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx=\frac{1}{\pi}\int_0^{\pi}2x\sin nxdx+\frac{1}{\pi}\int_{-\pi}^{0}x\sin nxdx$

$=(\frac{2}{\pi}(-\frac{x\cos nx}{n})\big|_{x=0}^{\pi}-\frac{2}{n\pi}\int_0^{\pi}(-\cos nx)dx)+(\frac{1}{\pi}(\frac{-x\cos nx}{n})\big|_{x=-\pi}^{0}-\frac{1}{n\pi}\int_{-\pi}^{0}(-\cos nx)dx)$

$=\frac{2}{\pi}(-\frac{x\cos nx}{n})\big|_{x=0}^{\pi}+\frac{1}{\pi}(\frac{-x\cos nx}{n})\big|_{x=-\pi}^{0}=\frac{2}{\pi}(-\frac{\pi\cos n\pi}{n})+\frac{1}{\pi}(\frac{-\pi\cos(-n\pi)}{n})$

$=\frac{2}{\pi}\frac{\pi(-1)^{n+1}}{n}+\frac{1}{\pi}\frac{\pi(-1)^{n+1}}{n}=\frac{3\cdot(-1)^{n+1}}{n}$,

$f(x)\sim(\frac{a_0}{2}+\sum_{n=1}^{+\infty}a_n\cos nx+b_n\sin nx)=\frac{\pi}{4}+\sum_{n=1}^{+\infty}[\frac{(-1)^n-1}{n^2\pi}\cos nx+\frac{3\cdot(-1)^{n+1}}{n}\sin nx]$,

其中$x\ne(2k-1)\pi$.

(5) $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{\pi}(\pi^2-x^2)dx=\frac{1}{\pi}\cdot 2\pi^3-\frac{1}{\pi}\cdot(\frac{x^3}{3})\big|_{x=-\pi}^{\pi}$

$=2\pi^2-\frac{2\pi^2}{3}=\frac{4\pi^2}{3}$,

$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\frac{1}{\pi}\int_{-\pi}^{\pi}(\pi^2-x^2)\cos nxdx=\frac{2}{\pi}\int_0^{\pi}(\pi^2-x^2)\cos nxdx$

$=-\frac{2}{\pi}\int_0^{\pi}x^2\cos nxdx=(-\frac{2}{n\pi}x^2\sin nx)\big|_{x=0}^{\pi}+\frac{4}{n\pi}\int_0^{\pi}x\sin nxdx=\frac{4}{n\pi}\int_0^{\pi}x\sin nxdx$

$=(-\frac{4}{n^2\pi}x\cos nx)\big|_{x=0}^{\pi}+\frac{4}{n^2\pi}\int_0^{\pi}\cos nxdx=-\frac{4}{n^2\pi}\pi\cos n\pi=\frac{4\cdot(-1)^{n+1}}{n^2}$,

$b_n = 0,$

$f(x) \sim (\frac{a_0}{2} + \sum\limits_{n=1}^{+\infty} a_n \cos nx + b_n \sin nx) = \frac{2\pi^2}{3} + 4\sum\limits_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^2} \cos nx.$

题目9. (7.7.55) 设$f(x)$是周期为2π的函数,在指定区间内将$f(x)$展开成傅里叶级数:(2) $f(x) = x^2, (i) - \pi \le x < \pi; (ii) 0 \le x < 2\pi.$

解答: (i) 利用上题(5)的结论,$f(x) \sim \pi^2 - (\frac{2\pi^2}{3} + 4\sum\limits_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^2} \cos nx)$

$= \frac{\pi^2}{3} + 4\sum\limits_{n=1}^{+\infty} \frac{(-1)^n}{n^2} \cos nx.$

(ii) $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx == \frac{1}{\pi}\int_0^{2\pi} x^2 dx = \frac{1}{\pi}(\frac{x^3}{3})\Big|_{x=0}^{2\pi} = \frac{8\pi^2}{3},$

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx dx$

$= \frac{1}{\pi}(\frac{x^2 \sin nx}{n})\Big|_{x=0}^{2\pi} - \frac{2}{n\pi}\int_0^{2\pi} (x \sin nx)dx = -\frac{2}{n\pi}\int_0^{2\pi} (x \sin nx)dx$

$= -\frac{2}{n\pi}(\frac{-x\cos nx}{n})\Big|_{x=0}^{2\pi} - \frac{2}{n^2\pi}\int_0^{2\pi} (\cos nx)dx$

$= -\frac{2}{n\pi}(\frac{-2\pi\cos 2n\pi}{n}) = \frac{4}{n^2},$

$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx dx$

$= \frac{1}{\pi}(-\frac{x^2\cos nx}{n})\Big|_{x=0}^{2\pi} + \frac{2}{n\pi}\int_0^{2\pi} (x\cos nx)dx = -\frac{4\pi}{n} + \frac{2}{n\pi}\int_0^{2\pi} (x\cos nx)dx$

$= -\frac{4\pi}{n} + \frac{2}{n\pi}(\frac{x\sin nx}{n})\Big|_{x=0}^{2\pi} - \frac{2}{n^2\pi}\int_0^{2\pi} (\sin nx)dx = -\frac{4\pi}{n},$

$f(x) \sim \frac{4\pi^2}{3} + 4\sum\limits_{n=1}^{+\infty} (\frac{1}{n^2}\cos nx - \frac{\pi}{n}\sin nx).$

题目10. (7.7.56) 将函数$f(x) = \begin{cases} -\frac{\pi}{4}, & -\pi \le x < 0, \\ \frac{\pi}{4}, & 0 \le x < \pi; \end{cases}$

展开成傅里叶级数,并由此推出

(1) $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+...;$

(2) $\frac{\pi}{3}=1+\frac{1}{5}-\frac{1}{7}-\frac{1}{11}+\frac{1}{13}+\frac{1}{17}-\frac{1}{19}-\frac{1}{23}+...$

解答: $f(x)$奇函数,则$a_0=0, a_n=0,$

$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx=\frac{2}{\pi}\int_0^{\pi}\frac{\pi}{4}\cdot\sin nxdx=\frac{1}{2}(-\frac{\cos nx}{n})\Big|_{x=0}^{\pi}=\frac{1-(-1)^n}{2n},$

$f(x)\sim\sum\limits_{n=1}^{+\infty}\frac{1-(-1)^n}{2n}\sin nx.$

(1) $\frac{\pi}{4}=f(\frac{\pi}{2})=\frac{f(\frac{\pi}{2}+0)+f(\frac{\pi}{2}-0)}{2}=\sum\limits_{n=1}^{+\infty}\frac{1-(-1)^n}{2n}\sin\frac{n\pi}{2}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+....$

(2) $\frac{\pi}{3}=\frac{\pi}{4}+\frac{1}{3}\cdot\frac{\pi}{4}=(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+...)+\frac{1}{3}\cdot(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+...)$

$=1+\frac{1}{5}-\frac{1}{7}-\frac{1}{11}+\frac{1}{13}+\frac{1}{17}-\frac{1}{19}-\frac{1}{23}+....$