Homework 9

Huang Yuchong
3210105423

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Problem

Weighted residual methods are frequently used to obtain approximate solutions for differential and partial differential equations. This homework is to find approximate solutions for the following differential equation

$$\frac{d^2u}{dx^2} - \frac{1}{x} = 0, \quad 1 \le x \le 2$$

with boundary conditions $u(1) = 0$ and $u(2) = 0$. The exact solution to this problem is

$$u = x \ln x + (2 \ln 2)(1 - x)$$

Answer the following questions:

1) Write down a polynomial approximate function that satisfies the boundary conditions. This approximate function should have three unknowns. Write down the residual function for the 1st and 2nd order approximate.
2) Using the approximate function with only 1 unknown (first order approximation), determine the unknown and hence the approximate function by using the point collocation method at x=3/2.
3) Using the approximate function with 2 unknowns (second order approximation), determine the unknowns and hence the approximate function by using the point collocation method at x=4/3 and 5/3.
4) For first order approximation, write down the weight functions, determine the unknowns and hence the approximate function by using the Galerkin method.
5) For second order approximation, write down the weight functions, determine the unknowns and hence the approximate function by using the Galerkin method.
6) Compare your approximate solutions in 3) and 5) with the exact solution at points x=1.33, 1.5, 1.67.

Solution

1)

Assuming an approximate solution that already satisfies BCs with three unknowns.

$$\overline{u} = (x-1)(x-2)\left(a_1 + a_2 x + a_3 x^2\right)$$

First order approximation:

$$\overline{u} = a_1 (x-1)(x-2)$$
$$R_1 = 2a_1 - \frac{1}{x}$$

Second order approximation:

$$\overline{u} = (a_1 + a_2 x)(x-1)(x-2)$$
$$R_2 = 2a_1 + 6a_2 x - 6a_2 - \frac{1}{x}$$

2)

$$R_1(\frac{3}{2}) = 2a_1 - \frac{2}{3} = 0$$

So, we have $a_1 = \frac{1}{3}$. Therefore, $\overline{u} = \frac{1}{3}(x-1)(x-2)$

3)

$$R_2(\frac{4}{3}) = 2a_1 + 8a_2 - 6a_2 - \frac{3}{4} = 0$$
$$R_2(\frac{5}{3}) = 2a_1 + 10a_2 - 6a_2 - \frac{3}{5} = 0$$

So, we have $a_1 = \frac{9}{20}, a_2 = -\frac{3}{40}$ Therefore, $\overline{u} = (\frac{9}{20} - \frac{3}{40}x)(x-1)(x-2)$

4)

First order approximation:

$$\overline{u} = a_1 (x-1)(x-2)$$
$$W_1 = N_1 = (x-1)(x-2)$$
$$R_1 = 2a_1 - \frac{1}{x}$$
$$\int_1^2 W_1 R_1 dx = -\frac{1}{3}a_1 + \frac{3}{2} - 2\ln 2 = 0$$

So, we get $a_1 = \frac{9}{2} - 6\ln 2$. Therefore, $\overline{u} = \left(\frac{9}{2} - 6\ln 2\right)(x-1)(x-2)$

5)

Second order approximation:

$$
\begin{aligned}
\overline{u} &= (a_1 + a_2 x)(x-1)(x-2) \\
W_1 &= N_1 = (x-1)(x-2) \\
W_2 &= N_2 = x(x-1)(x-2) \\
R_2 &= 2a_1 + 6a_2 x - 6a_2 - \frac{1}{x} \\
\int_1^2 W_1 R_2 dx &= -\frac{1}{3}a_1 - \frac{1}{2}a_2 + \frac{3}{2} - 2\ln 2 = 0 \\
\int_1^2 W_2 R_2 dx &= -\frac{1}{2}a_1 - \frac{4}{5}a_2 + \frac{1}{6} = 0
\end{aligned}
$$

So, we have

$$a_1 = 67 - 96\ln 2, a_2 = 60\ln 2 - \frac{125}{3}$$

Therefore,

$$\overline{u} = \left[67 - 96\ln 2 + \left(60\ln 2 - \frac{125}{3}\right)x\right](x-1)(x-2)$$

6)

Call the solution in 3) as F, and G in 5). And the results are as below.

	1.33	1.5	1.67
F	-0.07744	-0.08438	-0.07180
G	-0.07835	-0.08528	-0.07250
Exact solution	-0.07819	-0.08495	-0.07240
Error of F(%)	0.96	0.67	0.83
Error of G(%)	0.20	0.39	0.14

As we can see, Both solutions exhibit a high degree of precision, with the solution in 5) being more precise.