第 2 周课后作业

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2024 年 3 月 11 日

1.

如图1所示，给出y方向上的受力示意图。

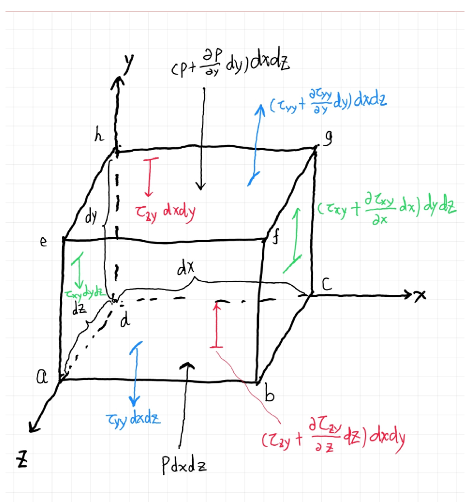

图 1: y 方向受力示意图

将作用在单位质量流体微团上的体积力记作 $\mathbf{f}$，其 y 方向分量为 $\mathbf{f}_y$。流体微团的体积为 $dxdydz$，所以

$$F_{Vy} = \rho f_y\,(dxdydz) \tag{1}$$

再观察y方向总的表面力

$$
\begin{aligned}
F_{Sy} = &\left[p - \left(p + \frac{\partial p}{\partial y}dy\right)\right]dxdz + \left[\left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x}dx\right) - \tau_{xy}\right]dydz \\
&+ \left[\left(\tau_{yy} + \frac{\partial \tau_{yy}}{\partial y}dy\right) - \tau_{yy}\right]dxdz + \left[\left(\tau_{zy} + \frac{\partial \tau_{zy}}{\partial z}dz\right) - \tau_{zy}\right]dxdy
\end{aligned} \tag{2}
$$

由此可以得到y方向上总的力

$$
F_y = \left(-\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right)dxdydz + \rho f_y dxdydz \tag{3}
$$

流体微团质量固定不变

$$
m = \rho dxdydz \tag{4}
$$

由于考虑的是运动的流体微团，故加速度由物质导数给出

$$
a_y = \frac{Dv}{Dt} \tag{5}
$$

再根据牛顿第二定律，我们可以得到

$$
\rho\frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho f_y \tag{6}
$$

观察公式6，对其左端进行改写。先根据物质导数的定义

$$
\rho\frac{Dv}{Dt} = \rho\frac{\partial v}{\partial t} + \rho\mathbf{V}\cdot\nabla v \tag{7}
$$

对式7的导数展开

$$
\rho\frac{\partial v}{\partial t} = \frac{\partial(\rho v)}{\partial t} - v\frac{\partial \rho}{\partial t} \tag{8}
$$

再根据散度的性质，对式7的另一项展开

$$
\rho\mathbf{V}\cdot\nabla v = \nabla\cdot(\rho v\mathbf{V}) - v\cdot\nabla(\rho\cdot\mathbf{V}) \tag{9}
$$

将式89代入式7，得到

$$
\begin{aligned}
\rho \frac{Dv}{Dt} &= \frac{\partial (\rho v)}{\partial t} - v \frac{\partial \rho}{\partial t} - v \nabla \cdot (\rho \mathbf{V}) + \nabla \cdot (\rho v \mathbf{V}) \\
&= \frac{\partial (\rho v)}{\partial t} - v \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) \right] + \nabla \cdot (\rho v \mathbf{V})
\end{aligned} \tag{10}
$$

可以发现，式10 方括号内即为连续性方程左端，为0。故

$$
\rho \frac{Dv}{Dt} = \frac{\partial (\rho v)}{\partial t} + \nabla \cdot (\rho v \mathbf{V}) \tag{11}
$$

最后将式11 代入式6，得到

$$
\frac{\partial (\rho v)}{\partial t} + \nabla \cdot (\rho v \mathbf{V}) = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho f_y \tag{12}
$$

至此得到y方向的守恒形式动量方程

2.

3.1

直接展开即可

$$
\begin{vmatrix} f1 & b1 & c1 & d1 \\ f2 & b2 & c2 & d2 \\ du & dy & 0 & 0 \\ dv & 0 & dx & dy \end{vmatrix} = dy \begin{vmatrix} f1 & b1 & c1 \\ f2 & b2 & c2 \\ du & dy & 0 \end{vmatrix} - dv \begin{vmatrix} b1 & c1 & d1 \\ b2 & c2 & d2 \\ dy & 0 & 0 \end{vmatrix} - dx \begin{vmatrix} f1 & b1 & d1 \\ f2 & b2 & d2 \\ du & dy & 0 \end{vmatrix}
$$

$$
\begin{aligned}
&= dydu\,(b_1c_2 - b_2c_1) - (dy)^2\,(f_1c_2 - f_2c_1) - dvdy\,(c_1d_2 - c_2d_1) \\
&- dxdu\,(b_1d_2 - b_2d_1) + dxdy\,(f_1d_2 - f_2d_1) \\
&= 0
\end{aligned}
$$

3.2

观察方程

$$
\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}
$$

移项，得到

$$\alpha\frac{\partial^2 T}{\partial x^2} - \frac{\partial T}{\partial t} = 0$$

可以使用一般二阶方程的判断方法

$$a = \alpha$$
$$b = 0$$
$$c = 0$$
$$D = b^2 - 4ac = 0$$

故这个方程是抛物型的

3.3

观察方程

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$

可以使用一般二阶方程的判断方法

$$a = 1$$
$$b = 0$$
$$c = 1$$
$$D = b^2 - 4ac = -4 < 0$$

故这个方程是椭圆型的

3.4

观察方程

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

移项，得到

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0$$

可以使用一般二阶方程的判断方法

$$
\begin{gathered}
a=1 \\
b=0 \\
c=-c^2 \\
D=b^2-4ac=4c^2>0
\end{gathered}
$$

故这个方程是双曲的

3.

我们先将这五个点都进行泰勒展开,展开到四次项

$$
\begin{aligned}
u_{i-2,j} &= u_i-2\Delta x\frac{\partial u}{\partial x}+2\Delta x^2\frac{\partial^2 u}{\partial x^2}-\frac{4\Delta x^3}{3}\frac{\partial^3 u}{\partial x^3}+\frac{2\Delta x^4}{3}\frac{\partial^4 u}{\partial x^4}+\mathcal{O}\left(\Delta x^5\right) \\
u_{i-1,j} &= u_i-\Delta x\frac{\partial u}{\partial x}+\frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}-\frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3}+\frac{\Delta x^4}{24}\frac{\partial^4 u}{\partial x^4}+\mathcal{O}\left(\Delta x^5\right) \\
u_{i,j} &= u_{i,j} \qquad (13)\\
u_{i+1,j} &= u_i+\Delta x\frac{\partial u}{\partial x}+\frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}+\frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3}+\frac{\Delta x^4}{24}\frac{\partial^4 u}{\partial x^4}+\mathcal{O}\left(\Delta x^5\right) \\
u_{i+2,j} &= u_i+2\Delta x\frac{\partial u}{\partial x}+2\Delta x^2\frac{\partial^2 u}{\partial x^2}+\frac{4\Delta x^3}{3}\frac{\partial^3 u}{\partial x^3}+\frac{2\Delta x^4}{3}\frac{\partial^4 u}{\partial x^4}+\mathcal{O}\left(\Delta x^5\right)
\end{aligned}
$$

现在只需要将Δx除了二次项其他次数项，通过加减全部消掉。然后便可以得到

$$
\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j}=\frac{-u_{i+2,j}+16u_{i+1,j}-30u_{i,j}+16u_{i-1,j}-u_{i-2,j}}{12\left(\Delta x\right)^2}+\mathcal{O}\left(\Delta x\right)^4 \qquad (14)
$$

故得证。