

Networks II

Project I Report

Bulat Khamitov

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1 Task Description

Solve the [problem](#) with your [number](#) with the Monte Carlo method. If you are able to solve it also theoretically, compare both results.

Problem 9

A convex shape L is given. What is the probability that the randomly selected four points in L will determine a convex quadrilateral? Solve for L being an equilateral triangle. [\[Hint 1\]](#).

2 Analytical Solution

2.1 Sylvester's Four Point Problem

The problem proposed for a student to solve, in its general form represents nothing but *Sylvester's four point problem* [1]. The solution below is based on paragraph 2.2.6 from [2].

We consider a convex domain L . We assume that four points are selected independently and randomly inside L . The probability is proportional to the area of L . What is the probability that these four points form a convex quadrilateral?

Without any loss of generality, we suppose that 3 points do not fall on one the same line. These 3 points form a triangle. If the 4-th point falls inside this triangle, then the 4 points form not a convex quadrilateral, but a re-entrant one (non-convex). The three points forming the triangle or the fourth point can be selected in



Figure 1: Convex and re-entrant quadrilaterals.

$$\binom{4}{3} = \binom{4}{1} = 4 \quad (1)$$

ways. Let X , Y and Z be the three points. Then the probability p of the four points forming a re-entrant quadrilateral is given by

$$\begin{aligned} p &= 4 \left[\frac{\text{expexted area of the triangle } XYZ}{\text{area of the convex figure } L} \right] \\ &= \frac{4}{S} \mathbb{E}[\text{area of } XYZ], \end{aligned} \quad (2)$$

where \mathbb{E} denotes the expected value and S is the area of L . Then

$$p^* = 1 - p \quad (3)$$

is the probability of the four points forming a convex quadrilateral.

Let p_1 be the same probability as p when one of the three points is on the boundary of L . From Crofton's theorem on measures [2] we have

$$df = \frac{4}{S} [p_1 - p] dS. \quad (4)$$

Let Z be the point on the boundary. This means that X and Y are independently chosen inside L and Z is uniformly distributed over the perimeter of L .

Let the triangle T be ABC . Let the point Z be on the side BC . Let T_1 and T_2 denote the triangles ABZ and

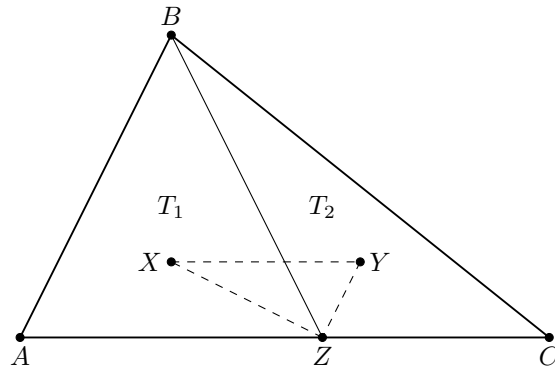


Figure 2: 2 random points inside a triangle.

AZC respectively. Let X and Y be the random points within T . There are 4 possible cases:

- $X, Y \in T_1$;
- $X, Y \in T_2$;
- $X \in T_1, Y \in T_2$;
- $X \in T_2, Y \in T_1$.

Since Z is fixed on BC , by $\mathbb{E}_z(\text{area of } XYZ)$ we denote the expected value of the area of XYZ . Then we have

$$\begin{aligned}\mathbb{E}_z(\text{area of } XYZ) &= \mathbb{E}_z(\text{area of } XYZ \mid X, Y \in T_1)\mathbb{P}(X \in T_1)\mathbb{P}(Y \in T_1) \\ &\quad + \mathbb{E}_z(\text{area of } XYZ \mid X, Y \in T_2)\mathbb{P}(X \in T_2)\mathbb{P}(Y \in T_2) \\ &\quad + \mathbb{E}_z(\text{area of } XYZ \mid X \in T_1, Y \in T_2)\mathbb{P}(X \in T_1)\mathbb{P}(Y \in T_2) \\ &\quad + \mathbb{E}_z(\text{area of } XYZ \mid X \in T_2, Y \in T_1)\mathbb{P}(X \in T_2)\mathbb{P}(Y \in T_1),\end{aligned}\tag{5}$$

where $\mathbb{P}(\cdot)$ denotes the probability of the event (\cdot) . Then we have

$$\begin{aligned}\mathbb{E}_z(\text{area of } XYZ) &= \mathbb{E}_z(\text{area of } XYZ \mid X, Y \in T_1)\frac{S_1^2}{S^2} \\ &\quad + \mathbb{E}_z(\text{area of } XYZ \mid X, Y \in T_2)\frac{S_2^2}{S^2} \\ &\quad + 2\mathbb{E}_z(\text{area of } XYZ \mid X \in T_1, Y \in T_2)\frac{S_1 S_2}{S^2},\end{aligned}\tag{6}$$

where S_1 , S_2 and S are the areas of the triangles T_1 , T_2 and T . After several steps, we get the following:

$$\begin{aligned}\mathbb{E}_z(\text{area of } XYZ \mid X \in T_1, Y \in T_2) &= \frac{1}{9}S, \\ \mathbb{E}_z(\text{area of } XYZ \mid X, Y \in T_1) &= \frac{4}{27}S_1, \\ \mathbb{E}_z(\text{area of } XYZ \mid X, Y \in T_2) &= \frac{4}{27}S_2.\end{aligned}\tag{7}$$

We get for X, Y belonging to the triangle $T = ABC$ and Z a fixed point on the boundary

$$\mathbb{E}(\text{area of } XYZ \mid X, Y, Z \in T) = \frac{1}{S^2} \left\{ \frac{4}{27}S_1^3 + \frac{4}{27}S_2^3 + \frac{2}{9}SS_1S_2 \right\}.\tag{8}$$

Then from the area of the triangle T is

$$S = \frac{1}{2}ah_1 = \frac{1}{2}bh_2 = \frac{1}{2}ch_3.\tag{9}$$

Since $S_2 = S - S_1$, we write

$$\frac{1}{S^2} \left\{ \frac{4}{27}S_1^3 + \frac{4}{27}S_2^3 + \frac{2}{9}SS_1S_2 \right\} = \frac{4}{27}S - \frac{2}{9}S_1 + \frac{2}{9}\frac{S_1^2}{S}.\tag{10}$$

Let $a + b + c = \delta$ and let Z be uniformly distributed over $[0, \delta]$.

$$\begin{aligned}S_1 &= \frac{xh_1}{2}, \quad 0 \leq x \leq a \\ &= (x - a)\frac{h_2}{2}, \quad a \leq x \leq a + b \\ &= (x - a - b)\frac{h_3}{2}, \quad a + b \leq x \leq a + b + c = \delta.\end{aligned}\tag{11}$$

Then,

$$\begin{aligned}-\frac{2}{9} \int S_1 \frac{dx}{\delta} &= -\frac{2}{9} \left\{ \frac{h_1}{2} \int_0^a x dx + \frac{h}{2} \int_a^{a+b} (x - a) dx + \frac{h_3}{2} \int_{a+b}^{a+b+c} (x - a - b) dx \right\} \\ &= -\frac{2}{9\delta} \left\{ \frac{h_1^2}{4}a^2 + \frac{h_2^2}{4}b^2 + \frac{h_3^2}{4}c^2 \right\}.\end{aligned}\tag{12}$$

$$\frac{2}{9} \int S_1^2 \frac{dx}{\delta} = \frac{2}{9S\delta} \left\{ \frac{h_1^2}{12} a^3 + \frac{h_2^2}{12} b^3 + \frac{h_3^2}{12} c^3 \right\} = \frac{2}{9\delta} \left\{ \frac{h_1 a^2}{6} + \frac{h_2 b^2}{6} + \frac{h_3 c^2}{6} \right\}. \quad (13)$$

Hence,

$$-\frac{2}{9} \int S_1 \frac{dx}{\delta} + \frac{2}{9S} \int S_1^2 \frac{dx}{\delta} = -\frac{1}{54} (h_1 a^2 + h_2 b^2 + h_3 c^2) = -\frac{S}{27\delta} (a + b + c) = -\frac{S}{27}. \quad (14)$$

Then from the unconditional expectation of the area XYZ when Z is on the boundary of the triangle is

$$\frac{4}{27} S - \frac{S}{27} = \frac{1}{9} S. \quad (15)$$

There are 3 possibilities of Z or X or Y being on the boundary. Hence, the unconditional expectation of the area XYZ when any one of the 3 points is on the boundary of the triangle is

$$\frac{3}{9} S = \frac{1}{3} S \rightarrow p_1 = \frac{1}{3}. \quad (16)$$

$$dp = \frac{4}{S} (p_1 - p) dS \rightarrow \frac{d}{dS} (S^4 p) = 4S^3 p_1 = \frac{4}{3} S^3. \quad (17)$$

That is, $S^4 p = \frac{S^4}{3} + c$, where $c = 0$, giving

$$p = \frac{1}{3}. \quad (18)$$

Hence,

$$p^* = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}. \quad (19)$$

2.2 Combinatorial Approach

Another generalization asks the probability that n randomly selected points in a fixed convex domain $L \in \mathbb{R}^2$ are the vertices of a convex n -gon. The solution is

$$P_n = \frac{2^n (3n - 3)!}{[(n - 1)!]^3 (2n)!} \quad (20)$$

for a triangular domain [3]. Having $n = 4$ points would give us

$$P_4 = \frac{2^4 (12 - 3)!}{[3!]^3 8!} = \frac{144}{216} = \frac{2}{3}. \quad (21)$$

3 Monte-Carlo Simulation

3.1 Triangle Point Picking

For a triangle with vertices (A, B, C) , we construct a point on its surface by generating two random numbers, r_1 and r_2 , between 0 and 1, and evaluating the following equation [4]:

$$P = (1 - \sqrt{r_1})A + \sqrt{r_1}(1 - r_2)B + \sqrt{r_1}r_2C. \quad (22)$$

Intuitively, $\sqrt{r_1}$ sets the percentage from vertex A to the opposing edge, while r_2 represents the percentage along that edge. Taking the $\sqrt{r_1}$ gives a uniform random point with respect to surface area.

Listing 1: Sampling a point inside a triangle.

```
def tri_sample(A, B, C):
    """ Return the point sampled uniformly in the triangle ABC. """
    r1 = random.random()
    r2 = random.random()

    s1 = math.sqrt(r1)

    x = A[0] * (1.0 - s1) + B[0] * (1.0 - r2) * s1 + C[0] * r2 * s1
    y = A[1] * (1.0 - s1) + B[1] * (1.0 - r2) * s1 + C[1] * r2 * s1

    return (x, y)
```

For simplicity, we consider a unit triangle. Hence, the vertices A , B and C have the following coordinates:

$$A = (0.0, 0.0), \quad B = \left(0.5, \frac{\sqrt{3}}{2}\right), \quad C = (1.0, 0.0). \quad (23)$$

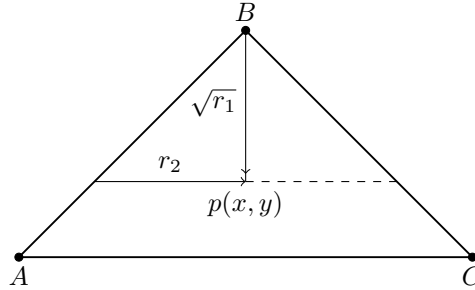


Figure 3: Sampling a random point in a triangle.

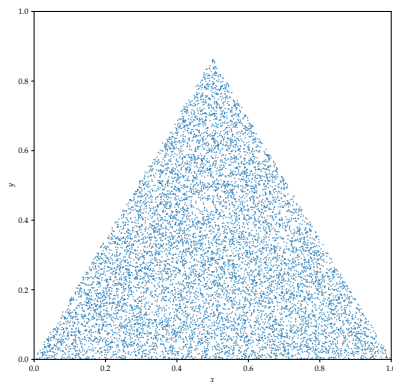


Figure 4: Equilateral triangle generated with $N = 10000$ points.

3.2 Quadrilateral Check

For any triple points of A , B and C in the plane, we can determine whether the angle $A-B-C$ makes a counterclockwise or a clockwise turn. We call 4 points and compute these signs for each of the four triples. If all signs are equal or there are 2 positive and 2 negative signs, the convex hull is a quadrilateral. If there are 3 positive and 1 negative sign, the convex hull is a triangle.

We assume that no 3 points are collinear.

Listing 2: Sign helper function.

```
def sign(x):  
    """ Return the sign of a finite number x. """  
    if x > 0:  
        return 1  
    elif x < 0:  
        return -1  
    else:  
        return 0
```

Listing 3: Turn definition.

```
def ccw(A, B, C):  
    """ Return 1 if A-B-C is a counterclockwise turn,  
        -1 for clockwise,  
        0 if the points are collinear (or not all distinct). """  
    disc = (A[0] - C[0]) * (B[1] - C[1]) - (A[1] - C[1]) * (B[0] - C[0])  
    return sign(disc)
```

Then goes the classification of a quadruple of points:

Listing 4: Point classification.

```
def classify_points(A, B, C, D):  
    """ Return 1 if a convex hull of A, B, C and D is a quadrilateral,  
        -1 if a triangle,  
        0 if any three of A, B, C and D are collinear (or if not all points are distinct). """  
    return ccw(A, B, C) * ccw(A, B, D) * ccw(A, C, D) * ccw(B, C, D)
```

3.3 Monte-Carlo

Classical definition of probability:

$$\mathbb{P} = \frac{\text{number of favorable outcomes}}{\text{total number of possible outcomes}}. \quad (24)$$

In our case, favorable outcome would be each time a convex quadrilateral is formed. The procedure is quite simple:

1. On a given triangle domain pick 4 random points;
2. Check if they form a convex quadrilateral;
3. If they do, increase a special counter per 1;
4. Repeat previous steps 10.000 times;
5. Using the above classical definition of probability, compute the chances of forming a convex quadrilateral.

4 Comparison of Results

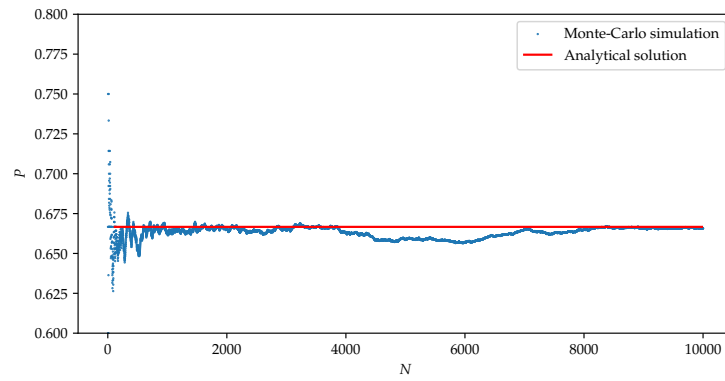


Figure 5: Monte-Carlo simulation compared to the analytical solution.

As we can see, the simulation results tend to match the theoretical one, when the number of choices approaches 10.000.

References

- [1] Sylvester, J. J., Problem 1491, *The Educational Times*, London, (April, 1864).
- [2] Mathai, A. M., An Introduction to Geometrical Probability: Distributional Aspects with Applications (Statistical Distributions & Models with Applications), *CRC Press*, Australia, (December, 1999).
- [3] Valtr, P., The Probability that n Random Points in a Triangle are in Convex Position, *Combinatorica* 16, (567-573, 1996).
- [4] Osada, R., Funkhouser, T., Chazelle, B., and Dobkin, D. Shape Distributions, *Association for Computing Machinery*, New York, USA, (October 2002).