# Networks II

# Project I Report

Bulat Khamitov

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## 1 Task Description

Solve the problem with your number with the Monte Carlo method. If you are able to solve it also theoretically, compare both results.

### Problem 9

A convex shape $L$ is given. What is the probability that the randomly selected four points in $L$ will determine a convex quadrilateral? Solve for $L$ being an equilateral triangle. [Hint 1].

## 2 Analytical Solution

### 2.1 Sylvester's Four Point Problem

The problem proposed for a student to solve, in its general form represents nothing but *Sylvester's four point problem* [1]. The solution below is based on paragraph 2.2.6 from [2].

We consider a convex domain $L$. We assume that four points are selected independently and randomly inside $L$. The probability is proportional to the area of $L$. What is the probability that these four points form a convex quadrilateral?

Without any loss of generality, we suppose that 3 points do not fall on one the same line. These 3 points form a triangle. If the 4-th point falls inside this triangle, then the 4 points form not a convex quadrilateral, but a re-entrant one (non-convex). The three points forming the triangle or the fourth point can be selected in

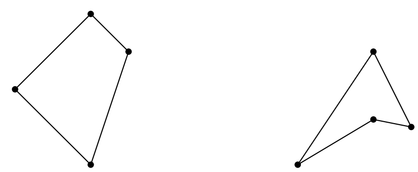


**Figure 1:** Convex and re-entrant quadrilaterals.

$$\binom{4}{3} = \binom{4}{1} = 4$$

ways. Let $X$, $Y$ and $Z$ be the three points. Then the probability $p$ of the four points forming a re-entrant quadrilateral is given by

$$\begin{aligned} p &= 4\left[\frac{\text{expexted area of the triangle } XYZ}{\text{area of the convex figure } L}\right] \\ &= \frac{4}{S}\,\mathbb{E}[\text{area of } XYZ], \end{aligned}$$

where $\mathbb{E}$ denotes the expected value and $S$ is the area of $L$. Then

$$p^* = 1 - p$$

is the probability of the four points forming a convex quadrilateral.

Let $p_1$ be the same probability as $p$ when one of the three points is on the boundary of $L$. From Crofton's theorem on measures [2] we have

$$\mathrm{d}\, f = \frac{4}{S}[p_1 - p]\,\mathrm{d}\, S.$$

Let $Z$ be the point on the boundary. This means that $X$ and $Y$ are independently chosen inside $L$ and $Z$ is uniformly distributed over the perimeter of $L$.

Let the triangle $T$ be $ABC$. Let the point $Z$ be on the side $BC$. Let $T_1$ and $T_2$ denote the triangles $ABZ$ and $AZC$ respectively. Let $X$ and $Y$ be the random points within $T$. There are 4 possible cases:

- $X, Y \in T_1$;
- $X, Y \in T_2$;
- $X \in T_1$, $Y \in T_2$;

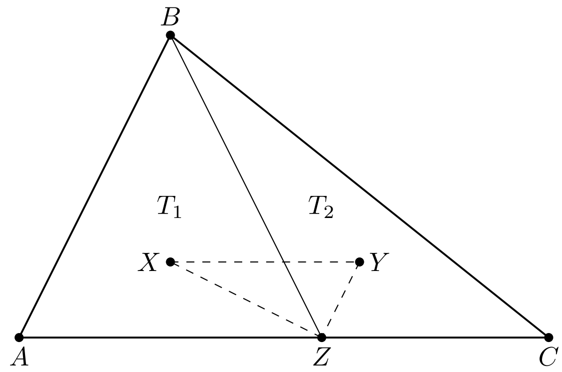



**Figure 2:** 2 random points inside a triangle.

- $X \in T_2$, $Y \in T_1$.

Since $Z$ is fixed on $BC$, by $\mathbb{E}_z$(area of $XYZ$) we denote the expected value of the area of $XYZ$. Then we have

$$\begin{aligned}
\mathbb{E}_z(\text{area of } XYZ) = {} & \mathbb{E}_z(\text{area of } XYZ \mid X, Y \in T_1)\mathbb{P}(X \in T_1)\mathbb{P}(Y \in T_1) \\
& + \mathbb{E}_z(\text{area of } XYZ \mid X, Y \in T_2)\mathbb{P}(X \in T_2)\mathbb{P}(Y \in T_2) \\
& + \mathbb{E}_z(\text{area of } XYZ \mid X \in T_1, Y \in T_2)\mathbb{P}(X \in T_1)\mathbb{P}(Y \in T_2) \\
& + \mathbb{E}_z(\text{area of } XYZ \mid X \in T_2, Y \in T_1)\mathbb{P}(X \in T_2)\mathbb{P}(Y \in T_1),
\end{aligned}$$

where $\mathbb{P}(\cdot)$ denotes the probability of the event $(\cdot)$. Then we have

$$\begin{aligned}
\mathbb{E}_z(\text{area of } XYZ) = {} & \mathbb{E}_z(\text{area of } XYZ \mid X, Y \in T_1)\frac{S_1^2}{S^2} \\
& + \mathbb{E}_z(\text{area of } XYZ \mid X, Y \in T_2)\frac{S_2^2}{S^2} \\
& + 2\,\mathbb{E}_z(\text{area of } XYZ \mid X \in T_1, Y \in T_2)\frac{S_1 S_2}{S^2},
\end{aligned}$$

where $S_1$, $S_2$ and $S$ are the areas of the triangles $T_1$, $T_2$ and $T$. After several steps, we get the following:

$$\begin{aligned}
\mathbb{E}_z(\text{area of } XYZ \mid X \in T_1, Y \in T_2) &= \frac{1}{9}S, \\
\mathbb{E}_z(\text{area of } XYZ \mid X, Y \in T_1) &= \frac{4}{27}S_1, \\
\mathbb{E}_z(\text{area of } XYZ \mid X, Y \in T_2) &= \frac{4}{27}S_2.
\end{aligned}$$

We get for $X$, $Y$ belonging to the triangle $T = ABC$ and $Z$ a fixed point on the boundary

$$\mathbb{E}(\text{area of } XYZ \mid X, Y, Z \in T) = \frac{1}{S^2}\left\{\frac{4}{27}S_1^3 + \frac{4}{27}S_2^3 + \frac{2}{9}SS_1S_2\right\}.$$

Then from the area of the triangle $T$ is

$$S = \frac{1}{2}ah_1 = \frac{1}{2}bh_2 = \frac{1}{2}ch_3.$$

Since $S_2 = S - S_1$, we write

$$\frac{1}{S^2}\left\{\frac{4}{27}S_1^3 + \frac{4}{27}S_2^3 + \frac{2}{9}SS_1S_2\right\} = \frac{4}{27}S - \frac{2}{9}S_1 + \frac{2}{9}\frac{S_1^2}{S}.$$

Let $a + b + c = \delta$ and let $Z$ be uniformly distributed over $[0,\, \delta]$.

$$
\begin{aligned}
S_1 &= \frac{xh_1}{2}, \quad 0 \leqslant x \leqslant a \\
&= (x-a)\frac{h_2}{2}, \quad a \leqslant x \leqslant a+b \\
&= (x-a-b)\frac{h_3}{2}, \quad a+b \leqslant x \leqslant a+b+c = \delta.
\end{aligned}
$$

Then,

$$
\begin{aligned}
-\frac{2}{9}\int S_1 \frac{\mathrm{d}\,x}{\delta} &= -\frac{2}{9}\left\{\frac{h_1}{2}\int_0^a x\,\mathrm{d}\,x + \frac{h}{2}\int_a^{a+b}(x-a)\,\mathrm{d}\,x + \frac{h_3}{2}\int_{a+b}^{a+b+c}(x-a-b)\,\mathrm{d}\,x\right\} \\
&= -\frac{2}{9\delta}\left\{\frac{h_1^2}{4}a^2 + \frac{h_2^2}{4}b^2 + \frac{h_3^2}{4}c^2\right\}.
\end{aligned}
$$

$$
\frac{2}{9}\int S_1^2 \frac{\mathrm{d}\,x}{\delta} = \frac{2}{9S\delta}\left\{\frac{h_1^2}{12}a^3 + \frac{h_2^2}{12}b^3 + \frac{h_3^2}{12}c^3\right\} = \frac{2}{9\delta}\left\{\frac{h_1a^2}{6} + \frac{h_2b^2}{6} + \frac{h_3c^2}{6}\right\}.
$$

Hence,

$$
-\frac{2}{9}\int S_1 \frac{\mathrm{d}\,x}{\delta} + \frac{2}{9S}\int S_1^2 \frac{\mathrm{d}\,x}{\delta} = -\frac{1}{54}(h_1a^2 + h_2b^2 + h_3c^2) = -\frac{S}{27\delta}(a+b+c) = -\frac{S}{27}.
$$

Then from the unconditional expectation of the area $XYZ$ when $Z$ is on the boundary of the triangle is

$$
\frac{4}{27}S - \frac{S}{27} = \frac{1}{9}S.
$$

There are 3 possibilities of $Z$ or $X$ or $Y$ being on the boundary. Hence, the unconditional expectation of the area $XYZ$ when any one of the 3 points is on the boundary of the triangle is

$$
\frac{3}{9}S = \frac{1}{3}S \to p_1 = \frac{1}{3}.
$$

$$
\mathrm{d}\,p = \frac{4}{S}(p_1 - p)\,\mathrm{d}\,S \to \frac{\mathrm{d}}{\mathrm{d}\,S}(S^4p) = 4S^3p_1 = \frac{4}{3}S^3.
$$

That is, $S^4p = \frac{S^4}{3} + c$, where $c = 0$, giving

$$
p = \frac{1}{3}.
$$

Hence,

$$
p^* = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}.
$$

## 2.2 Combinatorial Approach

Another generalization asks the probability that $n$ randomly selected points in a fixed convex domain $L \in \mathbb{R}^2$ are the vertices of a convex $n$-gon. The solution is

$$
P_n = \frac{2^n(3n-3)!}{[(n-1)!]^3(2n)!}
$$

for a triangular domain [3]. Having $n = 4$ points would give us

$$
P_4 = \frac{2^4(12-3)!}{[3!]^3 8!} = \frac{144}{216} = \frac{2}{3}.
$$

# 3 Monte-Carlo Simulation

## 3.1 Triangle Point Picking

For a triangle with vertices $(A, B, C)$, we construct a point on its surface by generating two random numbers, $r_1$ and $r_2$, between 0 and 1, and evaluating the following equation [4]:

$$P = (1 - \sqrt{r_1})A + \sqrt{r_1}(1 - r_2)B + \sqrt{r_1}r_2C.$$

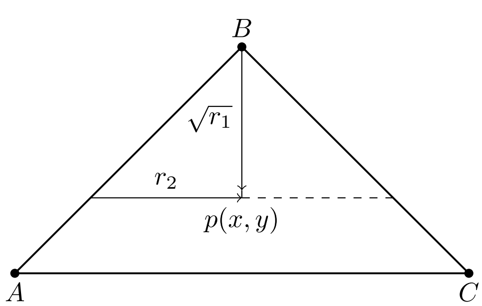
Intuitively, $\sqrt{r_1}$ sets the percentage from vertex $A$ to the opposing edge, while $r_2$ represents the percentage along that edge. Taking the $\sqrt{r_1}$ gives a uniform random point with respect to surface area.

```python
import random

def tri_sample(A, B, C):
    r1 = random.random()
    r2 = random.random()

    s1 = math.sqrt(r1)

    x = A[0] * (1.0 - s1) + B[0] * (1.0 - r2) * s1 + C[0] * r2 * s1
    y = A[1] * (1.0 - s1) + B[1] * (1.0 - r2) * s1 + C[1] * r2 * s1

    return (x, y)
```

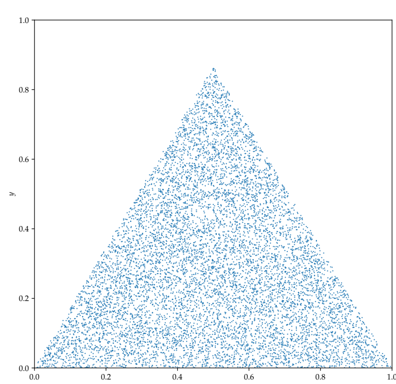
**Listing 1:** Sampling a point inside a triangle.

For simplicity, we consider a unit triangle. Hence, the vertices $A$, $B$ and $C$ have the following coordinates:

$$A = (0.0,\ 0.0), \quad B = \left(0.5,\ \frac{\sqrt{3}}{2}\right), \quad C = (1.0,\ 0.0).$$




**(a)** Sampling a random point in a triangle.




**(b)** Equilateral triangle with $N = 10000$ points.

## 3.2 Quadrilateral Check

For any triple points of $A$, $B$ and $C$ in the plane, we can determine whether the angle $A{-}B{-}C$ makes a counterclockwise or a clockwise turn. We call 4 points and compute these signs for each of the four triples. If all signs are equal or there are 2 positive and 2 negative signs, the convex hull is a quadrilateral. If there are 3 positive and 1 negative sign, the convex hull is a triangle.

We assume that no 3 points are collinear.

```
def sign(x):
""" Return the sign of a finite number x. """
    if x > 0:
        return 1
    elif x < 0:
        return -1
    else:
        return 0
```

**Listing 2:** Sign definition function.

```
def ccw(A, B, C):
""" Return 1 if A-B-C is a counterclockwise turn,
          -1 for clockwise,
           0 if the points are collinear (or not all distinct). """
    disc = (A[0] - C[0]) * (B[1] - C[1]) - (A[1] - C[1]) * (B[0] - C[0])
    return sign(disc)
```

**Listing 3:** Turn definition function.

Then goes the classification of a quadruple of points:

```
def classify_points(A, B, C, D):
""" Return 1 if a convex hull of A, B, C and D is a quadrilateral,
          -1 if a triangle,
           0 if any three of A, B, C and D are collinear (or if not all points are distinct).
"""
    return ccw(A, B, C) * ccw(A, B, D) * ccw(A, C, D) * ccw(B, C, D)
```

**Listing 4:** Point classification function.

### 3.3 Monte-Carlo Method

Classical definition of probability:

$$\mathbb{P} = \frac{\text{number of favorable outcomes}}{\text{total number of possible outcomes}}.$$

In our case, favorable outcome would be each time a convex quadrilateral is formed. The procedure is quite simple:

1. On a given triangle domain pick 4 random points;
2. Check if they form a convex quadrilateral;
3. If they do, increase a special counter per 1;
4. Repeat previous steps 10.000 times;
5. Using the above classical definition of probability, compute the chances of forming a convex quadrilateral.

## 4 Comparison of Results

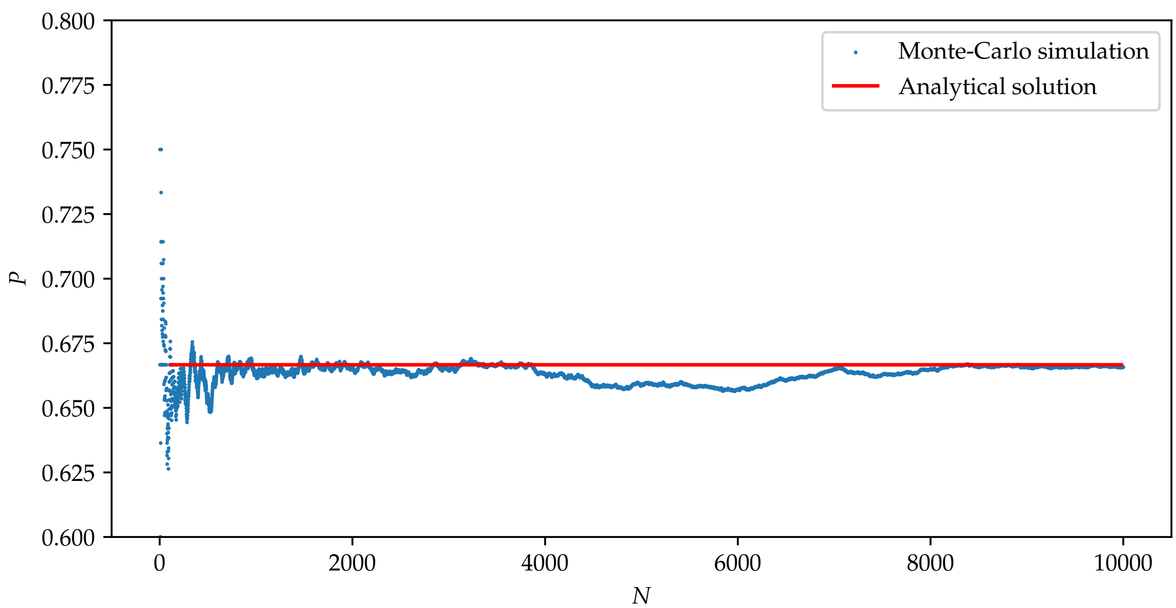



**Figure 4:** Monte-Carlo simulation compared to the analytical solution.

As we can see, the simulation results tend to match the theoretical one, when the number of choices approaches 10.000.

## References

[1] Sylvester, J. J., Problem 1491, *The Educational Times*, London, (April, 1864).

[2] Mathai, A. M., An Introduction to Geometrical Probability: Distributional Aspects with Applications (Statistical Distributions & Models with Applications), *CRC Press*, Australia, (December, 1999).

[3] Valtr, P., The Probability that $n$ Random Points in a Triangle are in Convex Position, *Combinatorica 16*, (567-573, 1996).

[4] Osada, R., Funkhouser, T., Chazelle, B., and Dobkin, D. Shape Distributions, *Association for Computing Machinery*, New York, USA, (October 2002).