Probability Theory
Homework Assignment 2

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1 Random Variables

Problem 4.25. Two coins are to be flipped. The first coin will land on heads with probability 0.6, the second with probability 0.7. Assume that the results of the flips are independent, and let X equal the total number of heads that result.

(a) Find $\mathbb{P}(X = 1)$.

(b) Determine $\mathbb{E}[X]$.

Solution. We denote the following:

- H − coin landed on heads;
- T − coin landed on tails.

(a) We have

$$\mathbb{P}(\text{coin landed on heads once}) = \mathbb{P}(\text{1st} - \text{H, 2nd} - \text{T} \cup \text{1st} - \text{T, 2nd} - \text{H})$$

These events are disjoint:

$$\mathbb{P}(\text{coin landed on heads once}) = \mathbb{P}(\text{1st} - \text{H, 2nd} - \text{T}) + \mathbb{P}(\text{1st} - \text{T, 2nd} - \text{H}).$$

Coin flips are independent:

$$\begin{aligned}\mathbb{P}(\text{1st} - \text{H, 2nd} - \text{T}) + \mathbb{P}(\text{1st} - \text{T, 2nd} - \text{H}) &= \\ &= \mathbb{P}(\text{1st} - \text{H})\mathbb{P}(\text{2nd} - \text{T}) + \mathbb{P}(\text{1st} - \text{T})\mathbb{P}(\text{2nd} - \text{H}).\end{aligned}$$

Probabilities are

$$\mathbb{P}(\text{2nd} - \text{T}) = 1 - \mathbb{P}(\text{2nd} - \text{H}) = 1 - 0.7 = 0.3.$$
$$\mathbb{P}(\text{1st} - \text{T}) = 1 - \mathbb{P}(\text{1st} - \text{H}) = 1 - 0.6 = 0.4.$$

Hence,

$$\mathbb{P}(X = 1) = \mathbb{P}(\text{coin landed on heads once}) = 0.6 \cdot 0.3 + 0.4 \cdot 0.7 = 0.46.$$

(b) Since X is the number of heads in result, we have to find

$$\mathbb{P}(X = 0), \quad \mathbb{P}(X = 1), \quad \mathbb{P}(X = 2)$$

in order to determine $\mathbb{E}[X]$.

(b.1) $\mathbb{P}(X = 0)$:

$$\begin{aligned}\mathbb{P}(X = 0) &= \mathbb{P}(\text{1st} - \text{T} \cap \text{2nd} - \text{T}) \\ &= \mathbb{P}(\text{1st} - \text{T})\mathbb{P}(\text{2nd} - \text{T}) \\ &= 0.4 \cdot 0.3 = 0.12.\end{aligned}$$

(b.2) $\mathbb{P}(X = 1)$ is computed in part (a).

(b.3) $\mathbb{P}(X = 2)$:

$$\begin{aligned}\mathbb{P}(X = 2) &= \mathbb{P}(\text{1st} - \text{H} \cap \text{2nd} - \text{H}) \\ &= \mathbb{P}(\text{1st} - \text{H})\mathbb{P}(\text{2nd} - \text{H}) \\ &= 0.6 \cdot 0.7 = 0.42.\end{aligned}$$

Hence,

$$\mathbb{E}[X] = \sum_{i=0}^{2} i \cdot p_i = \sum_{i=0}^{2} i \cdot \mathbb{P}(X = i) = 0 \cdot 12 + 1 \cdot 0.46 + 2 \cdot 0.42 = 1.3.$$

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Problem 4.27. An insurance company writes a policy to the effect that an amount of money A must be paid if some event E occurs within a year. If the company estimates that E will occur within a year with probability p, what should it charge the customer in order that its expected profit will be 10 percent of A?

Solution. We denote the following:

- C – number of money the company charges;
- X – profit.

According to the problem's conditions, there are 2 outcomes:

$$\begin{cases} C - A, & \text{if } E \text{ occurs,} \\ C, & \text{if } E \text{ doesn't occur.} \end{cases}$$

Since p is the probability of E occurring, then the opposite would be $1 - p$. Expected profit X can be found as expected value:

$$\mathbb{E}[X] = p(C - A) + (1 - p)C = -pA + C.$$

We can find C by solving the equation

$$\begin{aligned}\frac{1}{10} \cdot A &= C - pA \\ C &= A\left(p + \frac{1}{10}\right).\end{aligned}$$

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Problem 4.38. If $\mathbb{E}[X] = 1$ and $\mathrm{Var}(X) = 5$, find

(a) $\mathbb{E}[(2 + X)^2]$

(b) $\mathrm{Var}(4 + 3X)$

Solution.

(a) By the squared sum formula we have

$$\mathbb{E}\left[(2 + X)^2\right] = \mathbb{E}\left[4 + 4X + X^2\right].$$

Using the properties of expected value, we get

$$\mathbb{E}\left[4 + 4X + X^2\right] = \mathbb{E}[4] + \mathbb{E}[4X] + \mathbb{E}\left[X^2\right].$$

We know that

$$\mathrm{Var}(X) = \mathbb{E}\left[X^2\right] - \mathbb{E}\left[X\right]^2.$$

Hence,

$$\mathbb{E}[4] + 4 \cdot \mathbb{E}[X] + \mathrm{Var}(X) + \mathbb{E}[X]^2 = 4 + 4 \cdot 1 + 5 + 1 \cdot 1 = 14.$$

(b) By the properties of variance we have

$$\operatorname{Var}(4+3X) = \operatorname{Var}(3X) = 3^2 \cdot \operatorname{Var}(X) = 9 \cdot 5 = 45.$$

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Problem 4.50(a). Suppose that a biased coin that lands on heads with probability p is flipped 10 times. Given that a total of 6 heads results, find the conditional probability that the first 3 outcomes are

(a) h, t, t (meaning that the first flip results in heads, the second in tails, and the third in tails).

Solution. We donete the following:

- HTT – first 3 flips;
- $6H$ – 6 heads out of 10.

Taking that into account, we need to find

$$\mathbb{P}(HTT \mid 6H).$$

By Bayesian theorem we have

$$\mathbb{P}(HTT \mid 6H) = \frac{\mathbb{P}(6H \mid HTT)\mathbb{P}(HTT)}{\mathbb{P}(6H)}.$$

Since the outcome is 6 heads out of 10 flips, we use Binomial distribution:

$$\mathbb{P}(6H) = \binom{10}{6} p^6 (1-p)^4.$$

Let's take a closer look at the event HTT: the probability of heads in result is p, so the opposite is $1-p$. Hence, we have

$$\mathbb{P}(HTT) = p(1-p)^2.$$

As for $\mathbb{P}(6H \mid HTT)$, the first flip resulted in heads, meaning that another 5 flips resulted in heads are in the next 7 outcomes.

$$\mathbb{P}(6H \mid HTT) = \binom{7}{5} p^5 (1-p)^2.$$

Hence,

$$\begin{aligned}
\mathbb{P}(HTT \mid 6H) &= \frac{\mathbb{P}(6H \mid HTT)\mathbb{P}(HTT)}{\mathbb{P}(6H)} \\
&= \frac{\binom{7}{5} p^5 (1-p)^2 p (1-p)^2}{\binom{10}{6} p^6 (1-p)^4} \\
&= \binom{7}{5} \Big/ \binom{10}{6} = \frac{7!}{5!2!} \Big/ \frac{10!}{6!4!} = \frac{7!}{5!2!} \cdot \frac{6!4!}{10!} = \frac{6 \cdot 4 \cdot 3}{10 \cdot 9 \cdot 8} = \frac{1}{10} = 0.1.
\end{aligned}$$

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Problem 4.59. If you buy a lottery ticket in 50 lotteries, in each of which your chance of winning a prize is $\frac{1}{100}$, what is the (approximate) probability that you will win a prize

(a) at least once?

(b) exactly once?

(c) at least twice?

Solution. Let X be the number of prizes. We say that X has a binomial distribution with parameters n and p. By central limited theorem, X has Poison distribution as well:

$$X \sim \operatorname{Bin}(n, p) \sim \operatorname{Po}(\lambda),$$

where $\lambda = np$, then

$$\mathbb{P}(X = i) \approx e^{-\lambda} \frac{\lambda^i}{i!}.$$

(a) $\mathbb{P}(X \geqslant 1)$:

$$\mathbb{P}(X \geqslant 1) = 1 - \mathbb{P}(X = 0) = 1 - \frac{e^{-1/2}(1/2)^0}{0!} \approx 0.393.$$

(b) $\mathbb{P}(X = 1)$:

$$\mathbb{P}(X = 1) = \frac{e^{-1/2}(1/2)^1}{1!} \approx 0.303.$$

(c) $\mathbb{P}(X \geqslant 2)$:

$$\mathbb{P}(X \geqslant 2) = 1 - \mathbb{P}(X = 0) - \mathbb{P}(X = 1) \approx 0.393 - 0.303 \approx 0.09.$$

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2 Continuous Random Variables

Problem 5.7. The density function of X is given by

$$f(x) = \begin{cases} a + bx^2, & 0 \leqslant x \leqslant 1, \\ 0, & \text{otherwise.} \end{cases}$$

If $\mathbb{E}[X] = \frac{3}{5}$, find a and b.

Solution. The distribution density is non-negative $\forall x$ and is normalized to unity, i.e.

$$\int_{-\infty}^{\infty} f(x) \, \mathrm{d}\, x = 1. \tag{5.7.1}$$

Since we know the expression for the $f(x)$, we can calculate the probability that the value of x falls into the interval $[0, 1]$:

$$\mathbb{P}(0 \leqslant x \leqslant 1) = \int_0^1 \left(a + bx^2\right) \mathrm{d}\, x. \tag{5.7.2}$$

Combining (5.7.1) and (5.7.2), we have

$$\int_0^1 \left(a + bx^2\right) \mathrm{d}\, x = 1.$$

$$\begin{aligned} \int_0^1 \left(a + bx^2\right) \mathrm{d}\, x &= \int_0^1 a \, \mathrm{d}\, x + \int_0^1 bx^2 \, \mathrm{d}\, x = \\ &= a \int_0^1 1 \cdot \mathrm{d}\, x + b \int_0^1 x^2 \, \mathrm{d}\, x = ax|_0^1 + b \cdot \left.\frac{x^3}{3}\right|_0^1 = a + \frac{1}{3} b. \end{aligned} \tag{5.7.3}$$

Continuous expected value is defined as

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x) \, \mathrm{d}\, x.$$

Using the conditions of the problem, we obtain

$$\int_0^1 x(a + bx^2) \, \mathrm{d}\, x = \frac{3}{5}.$$

$$\begin{aligned} \int_0^1 x(a + bx^2) \, \mathrm{d}\, x &= \int_0^1 (ax + bx^3) \, \mathrm{d}\, x = \int_0^1 ax \, \mathrm{d}\, x + \int_0^1 bx^3 \, \mathrm{d}\, x = \\ &= a \int_0^1 x \, \mathrm{d}\, x + b \int_0^1 x^3 \, \mathrm{d}\, x = a \cdot \left.\frac{x^2}{2}\right|_0^1 + b \cdot \left.\frac{x^4}{4}\right|_0^1 = \frac{1}{2} a + \frac{1}{4} b. \end{aligned} \tag{5.7.4}$$

Combining the results from (5.7.3) and (5.7.4), we get the system of linear equations:

$$\begin{cases} a + \frac{1}{3} b &= 1 \\ \frac{1}{2} a + \frac{1}{4} b &= \frac{3}{5}. \end{cases}$$

By Gaussian elimination we have

$$\begin{pmatrix} 1 & 1/3 & 1 \\ 1/2 & 1/4 & 3/5 \end{pmatrix} \xrightarrow{(1)} \begin{pmatrix} 1 & 1/3 & 1 \\ 0 & 1/12 & 1/10 \end{pmatrix} \xrightarrow{(2)} \begin{pmatrix} 1 & 1/3 & 1 \\ 0 & 1 & 6/5 \end{pmatrix} \xrightarrow{(3)} \begin{pmatrix} 1 & 0 & 3/5 \\ 0 & 1 & 6/5 \end{pmatrix}.$$

(1) Subtract from the second row the first one multiplied by $\frac{1}{2}$;

(2) Multiply the second row by 12;

(3) Subtract from the first row the second one multiplied by $\frac{1}{3}$.

Hence,

$$a = \frac{3}{5}, \quad b = \frac{6}{5}.$$

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Problem 5.15(b, c, e). If X is a normal random variable with parameters $\mu = 10$ and $\sigma^2 = 36$, compute

(b) $\mathbb{P}(4 < X < 16)$;

(c) $\mathbb{P}(X < 8)$;

(e) $\mathbb{P}(X > 16)$.

Solution. Since X is a normal random variable, we can use transformation:

$$Z = \frac{X - \mu}{\sigma} \sim \mathrm{N}\left(\mu, \sigma^2\right).$$

(b) $\mathbb{P}(4 < X < 16)$:

$$\begin{aligned}\mathbb{P}(4 < X < 16) &= \mathbb{P}\left(\frac{4-10}{6} < \frac{X-\mu}{\sigma} < \frac{16-10}{6}\right)\\ &= \mathbb{P}(-1 < Z < 1)\\ &= \Phi(1) - \Phi(-1) = 0.8414 - 0.15866 = 0.68274.\end{aligned}$$

(c) $\mathbb{P}(X < 8)$:

$$\begin{aligned}\mathbb{P}(X < 8) &= P\left(\frac{X-\mu}{\sigma} < \frac{8-10}{6}\right)\\ &= \mathbb{P}\left(Z < -\frac{1}{3}\right)\\ &= \Phi\left(-\frac{1}{3}\right) = 0.3694.\end{aligned}$$

(e) $\mathbb{P}(X > 16)$:

$$\begin{aligned}\mathbb{P}(X > 16) &= \mathbb{P}\left(\frac{X-\mu}{\sigma} > \frac{16-10}{6}\right)\\ &= \mathbb{P}(Z > 1)\\ &= 1 - \Phi(1) = 0.1586.\end{aligned}$$

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Problem 5.19. Let X be a normal random variable with mean 12 and variance 4. Find the value of c such that $\mathbb{P}(X > c) = 0.1$.

Solution. Since X is a normal random variable, we can use transformation:

$$Z = \frac{X - \mu}{\sigma},$$

where μ is mean and $\sigma = \sqrt{\mathrm{Var}}$. Since $\mu = 12$ and $\sigma = \sqrt{\mathrm{Var}} = 2$, we have

$$\begin{aligned}\mathbb{P}(X > c) &= \mathbb{P}\left(\frac{X-12}{2} > \frac{c-12}{2}\right)\\ &= 1 - \mathbb{P}\left(\frac{X-12}{2} \leqslant \frac{c-12}{2}\right)\\ &= 1 - \Phi\left(\frac{c-12}{2}\right) = 0.1.\end{aligned}$$

From the standard normal table we have

$$\frac{c-12}{2} \approx 1.28.$$

Hence,

$$c = 14.56.$$

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Problem 5.25. Each item produced by a certain manufacturer is, independently, of acceptable quality with probability 0.95. Approximate the probability that at most 10 of the next 150 items produced are unacceptable.

Solution. Let X be a number of unacceptable items. We say that X has a binomial distribution with parameters n and p:

$$X \sim \mathrm{Bin}(n, p) \sim \mathrm{Bin}(150, 0.05)$$

Now we use the connection between distributions:

$$\mathrm{Bin}(n, p) \sim \mathrm{N}(\mu, \sigma^2),$$

where $\mu = np$ and $\sigma^2 = np(1-p)$.

$$X \sim \mathrm{N}(7.5, 7.125).$$

We denote U as an event of 10 of the next 150 items produced to be unacceptable.

$$\begin{aligned}
\mathbb{P}(U) &= \mathbb{P}(X \leqslant 10) \\
&= \mathbb{P}\left(\frac{X-\mu}{\sigma} \leqslant \frac{10-7.5}{2.67}\right) \\
&= \mathbb{P}(Z \leqslant 0.936) \\
&= \Phi(0.936) \\
&= 0.8264.
\end{aligned}$$

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Problem 5.33. The number of years a radio functions is exponentially distributed with parameter $\lambda = \frac{1}{8}$. If Jones buys a used radio, what is the probability that it will be working after an additional 8 years?

Solution. Let s be the number of years the radio functions. Then, what we need to find is

$$\mathbb{P}(X > 8 \mid X > s).$$

Since s is exponentially distributed, it has the memoryless property. That is, for some s and shift t we have

$$\mathbb{P}(X > s + t \mid X > s) = \mathbb{P}(X > t).$$

In the case of our problem, s is unknown and $t = 8$. The probability that the radio will last an additional 8 years is the same as if computing the probability that $X > 8$:

$$\begin{aligned}
\mathbb{P}(X > 8) &= \int_8^\infty \frac{1}{8} \exp\left(-\frac{1}{8}x\right) \mathrm{d}\,x = \\
&= \frac{1}{8}\int_8^\infty \exp\left(-\frac{1}{8}x\right) \mathrm{d}\,x = -\exp\left(-\frac{1}{8}x\right)\Big|_0^\infty .
\end{aligned}$$

We consider the upper limit as the limit of the function:

$$\lim_{x\to\infty}\left[-\exp\left(-\frac{1}{8}x\right)\right] = \lim_{x\to\infty}\left[1 \Big/ \exp\left(\frac{1}{8}x\right)\right] = 0.$$

Hence,

$$0 - \exp(0) = -\exp(1) = \exp(-1) \equiv e^{-1}.$$

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