Probability Theory
Homework Assignment 3

Bulat Khamitov, MSSA191

October 21, 2019

1 Jointly Distributed Random Variables

Problem 6.11. A television store owner figures that 45 percent of the customers entering his store will purchase an ordinary television set, 15 percent will purchase a plasma television set, and 40 percent will just be browsing. If 5 customers enter his store on a given day, what is the probability that he will sell exactly 2 ordinary sets and 1 plasma set on that day?

Solution. We denote the following:

- X_1 – ordinary set;
- X_2 – plasma set;
- X_3 – just browsing.

Every person *buys* or *does not buy* a TV set independently from each other. Hence, we can say that random vector (X_1, X_2, X_3) has multidimensional distribution

$$\mathbb{P}(X_1 = x_1, X_2 = x_2, \ldots, X_r = x_r) = \frac{n!}{n_1! n_2! \ldots n_r!} p_1^{n_1} p_2^{n_2} \ldots p_r^{n - \sum n_i}. \tag{6.11.1}$$

Taking into account the conditions of the problem, we obtain

$$X_1 = 2, \quad X_2 = 1, \quad X_3 = 5 - X_1 - X_2 = 2.$$

Hence, using (6.11.1), we have

$$\mathbb{P}(X_1 = 2, X_2 = 1, X_3 = 2) = \frac{5!}{2!1!2!}(0.45)^2 \cdot (0.15)^2 \cdot (0.40)^2 = 0.1457.$$

∎

Problem 6.20. The joint density of X and Y is given by

$$f(x, y) = \begin{cases} x e^{-(x+y)}, & x > 0, \quad y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Are X and Y independent? If, instead, $f(x, y)$ were given by

$$f(x, y) = \begin{cases} 2, & 0 < x < y, \quad 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

would X and Y be independent?

Solution. We need to check if $f(x, y) = f_X(x) f_Y(y)$ holds.

1. For $x > 0$ we have

$$f_X(x) = \int_0^\infty xe^{-(x+y)}\,\mathrm{d}\,y = xe^{-x} \cdot \left(-e^{-y}\right)\Big|_0^\infty = xe^{-x}.$$

For $y > 0$ we have

$$f_Y(y) = \int_0^\infty xe^{-(x+y)}\,\mathrm{d}\,x = e^{-y}\int_0^\infty xe^{-x}\,\mathrm{d}\,x.$$

We use partial integration:

$$\begin{aligned} u &= x & \mathrm{d}\,v &= e^{-x}, \\ \mathrm{d}\,u &= \mathrm{d}\,x & v &= -e^{-x}. \end{aligned}$$

Then, we have

$$e^{-y}\left(uv\Big|_0^\infty - \int_0^\infty v\,\mathrm{d}\,u\right) = e^{-y}\left(-xe^{-x}\Big|_0^\infty + \int_0^\infty e^{-x}\,\mathrm{d}\,x\right) = e^{-y}.$$

We obtained that $f(x, y) = f_X(x)f_Y(y)$. Hence, X and Y are independent.

2. We do similar for the second case. For $0 < y < 1$ we have

$$f_X(x) = \int_x^1 2\,\mathrm{d}\,y = 2\int_x^1 \mathrm{d}\,y = 2\cdot y\Big|_x^1 = 2 - 2x.$$

And for $0 < x < y$ we have

$$f_Y(y) = \int_0^y 2\,\mathrm{d}\,x = 2\int_0^y \mathrm{d}\,x = 2\cdot x\Big|_y^0 = 2y.$$

We obtained that $f(x, y) = (2 - 2x)2y \neq f_X(x)f_Y(y)$. Hence, X and Y are *not* independent.

■

Problem 6.24(a-c). Consider independent trials, each of which results in outcome i, $i = 0, 1, \ldots, k$, with probability p_i, $\sum_{i=0}^k p_i = 1$. Let N denote the number of trials needed to obtain an outcome that is not equal to 0, and let X be that outcome.

(a) Find $\mathbb{P}(N = n)$, $n \geqslant 1$;

(b) Find $\mathbb{P}(X = j)$, $j = \overline{1, k}$;

(c) Show that $\mathbb{P}(N = n, X = j) = \mathbb{P}(N = n)\mathbb{P}(X = j)$.

Solution.

(a) The condition implies that in first $n - 1$ trials we obtain outcomes equal to 0 and in n-th trial we obtain any other outcome rather than 0. N is a geometric random variable with parameter of success $1 - p_0$ (not getting 0 as the outcome):

$$N \sim \mathrm{Geom}(p_0) \sim N = \begin{cases} 0, & p_0 \\ 1, & 1 - p_0 \end{cases}$$

with probability mass function

$$\mathbb{P}(N = n) = p_0^{n-1}(1 - p_0).$$

(b) The condition on X is that is does not equal to 0. If we define new random variable Y to be the outcome of the trial, then it will have distribution such that $\mathbb{P}(Y = y) = p_y$:

$$\mathbb{P}(X = j) = \mathbb{P}(Y = j \mid Y \neq 0) = \frac{\mathbb{P}(Y = j)}{\mathbb{P}(Y \neq 0)} = \frac{p_j}{1 - p_0}.$$

(c) The condition implies that $n-1$ trials have 0 as the outcome, the n-th trial has j as the outcome. Since the trials are independent, the probability of this is

$$\mathbb{P}(N=n, X=j) = p_0^{n-1} p_i.$$

We show that

$$\mathbb{P}(N=n)\mathbb{P}(X=j) = \left(p_0^{n-1}(1-p_0)\right)\left(\frac{p_j}{1-p_0}\right) = p_0^{n-1}p_i = \mathbb{P}(N=n, X=j).$$

■

Problem 6.31(a). According to the U.S. National Center for Health Statistics, 25.2 percent of males and 23.6 percent of females never eat breakfast. Suppose that random samples of 200 men and 200 women are chosen. Approximate the probability that

(a) at least 110 of these 400 people never eat breakfast.

Solution. Let M and W be the number of men and women that never eat breakfast respectively. From exact definition we have

$$\begin{aligned} M &\sim \mathrm{Bin}(n, p_M) \sim \mathrm{Bin}(200, 0.252), \\ W &\sim \mathrm{Bin}(n, p_W) \sim \mathrm{Bin}(200, 0.236). \end{aligned}$$

We will use normal approximation due to the complexity of doing approximations with binomial one:

$$\begin{aligned} M &\sim \mathrm{N}(\mu_M, \sigma_M^2), \\ W &\sim \mathrm{N}(\mu_W, \sigma_W^2), \end{aligned}$$

where

$$\begin{aligned} \mu_M &= \mathbb{E}[M] = np_M = 200 \cdot 0.252 = 50.4,\ \sigma_M^2 = \mathrm{Var}[M] = np_M(1-p_M) = 200 \cdot 0.252(1-0.252) = 37.7, \\ \mu_W &= \mathbb{E}[W] = np_W = 200 \cdot 0.236 = 47.2,\ \sigma_W^2 = \mathrm{Var}[W] = np_W(1-p_W) = 200 \cdot 0.236(1-0.236) = 36.1. \end{aligned}$$

We want to find $\mathbb{P}(M+W \geqslant 110)$. We approximate $M+W$ by a normal random variable distribution with $\mu = 50.4 + 47.2 = 97.6$ and $\sigma^2 = 37.7 + 36.1 = 73.8$ using the property of the sum of 2 independent variables:

$$\begin{aligned} \mathbb{P}(M+W \geqslant 110) = \mathbb{P}(110 \leqslant M+W) &= \mathbb{P}\left(\frac{100-97.6}{\sqrt{73.8}} \leqslant \frac{M+W-97.6}{\sqrt{73.8}}\right) \\ &= 1 - \Phi\left(\frac{100-97.6}{\sqrt{73.8}}\right) \\ &= 1 - \Phi(1.44) = 1 - 0.9251 = 0.0749. \end{aligned}$$

■

Problem 6.40. The joint probability mass function of X and Y is given by

$$\begin{array}{ll} p(1,1) = 1/8 & p(1,2) = 1/4 \\ p(2,1) = 1/8 & p(2,2) = 1/2 \end{array}$$

(a) Compute the conditional mass function of X given $Y = i$, $i = 1, 2$;

(b) Are X and Y independent?

(c) Compute $\mathbb{P}(XY \leqslant 3)$, $\mathbb{P}(X+Y > 2)$, $\mathbb{P}(X/Y > 1)$.

Solution.

(a) Suppose we are given that $Y = 1$. Probability for that is

$$\mathbb{P}(Y = 1) = p(1,1) + p(2,1) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}.$$

Hence,

$$\begin{aligned}
\mathbb{P}(X = 1 \mid Y = 1) &= \frac{\mathbb{P}(X = 1, Y = 1)}{\mathbb{P}(Y = 1)} = \frac{p(1,1)}{\mathbb{P}(Y = 1)} = \frac{1/8}{1/4} = \frac{1}{2}, \\
\mathbb{P}(X = 2 \mid Y = 1) &= \frac{\mathbb{P}(X = 2, Y = 1)}{\mathbb{P}(Y = 1)} = \frac{p(2,1)}{\mathbb{P}(Y = 1)} = \frac{1/8}{1/4} = \frac{1}{2}.
\end{aligned}$$

Suppose we are given that $Y = 2$. Probability for that is

$$\mathbb{P}(Y = 2) = p(1,2) + p(2,2) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}.$$

Hence,

$$\begin{aligned}
\mathbb{P}(X = 1 \mid Y = 2) &= \frac{\mathbb{P}(X = 1, Y = 2)}{\mathbb{P}(Y = 2)} = \frac{p(1,2)}{\mathbb{P}(Y = 2)} = \frac{1/4}{3/4} = \frac{1}{3}, \\
\mathbb{P}(X = 2 \mid Y = 2) &= \frac{\mathbb{P}(X = 2, Y = 2)}{\mathbb{P}(Y = 2)} = \frac{p(2,2)}{\mathbb{P}(Y = 2)} = \frac{1/2}{3/4} = \frac{2}{3}.
\end{aligned}$$

(b) We see that

$$\mathbb{P}(X = 2) = p(2,1) + p(2,2) = \frac{1}{8} + \frac{1}{2} = \frac{5}{8},$$

which *is not* equal to $\mathbb{P}(X = 2 \mid Y = 2)$. Hence, X and Y are *not* independent.

(c)

$$\begin{aligned}
\mathbb{P}(XY \leqslant 3) &= 1 - \mathbb{P}(XY = 4) = 1 - P(X = 2 \mid Y = 2) \\
&= 1 - \mathbb{P}(X = 2 \mid Y = 2) \\
&= 1 - \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{2}.
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}(X + Y > 2) &= 1 - \mathbb{P}(X + Y = 2) = 1 - \mathbb{P}(X = 1, Y = 1) \\
&= 1 - \mathbb{P}(X = 1, Y = 1)\mathbb{P}(Y = 1) \\
&= 1 - \frac{1}{2} \cdot \frac{1}{4} = \frac{7}{8}.
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}(X/Y > 1) &= \mathbb{P}(X > Y) = \mathbb{P}(X = 2, Y = 1) \\
&= \mathbb{P}(X = 2, Y = 1)\mathbb{P}(Y = 1) \\
&= \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}.
\end{aligned}$$

■

2 Properties of Expectation

Problem 7.22. *How many times would you expect to roll a fair die before all* 6 *sides appeared at least once?*

Solution. We define random variable X_i, $i = \overline{1,6}$ as the number of additional throws that are needed to jump from $i-1$ collected distinct numbers to i distinct numbers. Because of the nature of the problem, X has geometric distribution with parameter of success $\frac{6-i+1}{6}$:

$$X \sim \text{Geom}\left(\frac{6-i+1}{6}\right).$$

The total number of needed throws is $N = \sum_i X_i$. Hence,

$$\mathbb{E}[N] = \sum_i \mathbb{E}[X_i] = \sum_i 1 \Big/ \frac{6-i+1}{6} = \sum_i \frac{6}{6-i+1} = 14.7$$

■

Problem 7.30. *If X and Y are independent and identically distributed with mean μ and variance σ^2, find $\mathbb{E}\left[(X-Y)^2\right]$.*

Solution.

$$\begin{aligned}
\mathbb{E}\left[(X-Y)^2\right] &= \mathbb{E}[X^2 - 2XY + Y^2] = \mathbb{E}[X^2] - 2\,\mathbb{E}[X]\,\mathbb{E}[Y] + \mathbb{E}[Y^2] \\
&= \operatorname{Var} X + \mathbb{E}[X]^2 - 2\,\mathbb{E}[X]\,\mathbb{E}[Y] + \operatorname{Var} Y + \mathbb{E}[Y]^2 \\
&= \sigma^2 + \mu^2 - 2\mu^2 + \sigma^2 + \mu^2 \\
&= 2\sigma^2.
\end{aligned}$$

■

Problem 7.39. *Let $X_1 \ldots$ be independent with common mean μ and common variance σ^2, and set $Y_n = X_n + X_{n+1} + X_{n+2}$. For $j \geqslant 0$, find $\operatorname{Cov}(Y_n, Y_{n+j})$.*

Solution.

- $j = 0$:
$$\operatorname{Cov}(Y_n, Y_n) = \operatorname{Var}(X_n + X_{n+1} + X_{n+2}) = 3\sigma^2. \tag{1}$$

- $j = 1$:
$$\begin{aligned}
\operatorname{Cov}(Y_n, Y_{n+1}) &= \operatorname{Cov}(X_n + X_{n+1} + X_{n+2}, X_{n+1} + X_{n+2} + X_{n+3}) \\
&= \operatorname{Var}(X_{n+1}) + \operatorname{Var}(X_{n+2}) \\
&= 2\sigma^2.
\end{aligned}$$

- $j \geqslant 3$:
$$\operatorname{Cov}(Y_n, Y_{n+j}) = 0.$$

■

Problem 7.49. *There are two misshapen coins in a box; their probabilities for landing on heads when they are flipped are, respectively, .4 and .7. One of the coins is to be randomly chosen and flipped* 10 *times. Given that two of the first three flips landed on heads, what is the conditional expected number of heads in the* 10 *flips?*

Solution. We denote the following:

- X – number of Heads within first 10 tosses;
- Y – number of Heads in range within toss number from 4 to 10;
- A – first coin has been chosen;
- B – second coin has been chosen;
- C – 2 Heads within first 2 tosses.

We need to find

$$\mathbb{E}_C[X] = 2 + \mathbb{E}_C[X].$$

By the law of total probability we have

$$\mathbb{E}_C[X] = \mathbb{E}_C[Y \mid A]\mathbb{P}_C(A) + \mathbb{E}_C[Y \mid B]\mathbb{P}_C(B).$$

By Bayes rule we have

$$\mathbb{P}_C(A) = \mathbb{P}(A \mid C)\frac{\mathbb{P}(C \mid A)\mathbb{P}(A)}{\mathbb{P}(C)} = \frac{\mathbb{P}(C \mid A)}{\mathbb{P}(C \mid B) + \mathbb{P}(C \mid B)}$$

Let us take a closer look:

$$\mathbb{P}(C \mid A) = \mathrm{Bin}(n = 3, p = 0.7) = \binom{3}{2} \cdot (0.4)^2 \cdot (0.6),$$
$$\mathbb{P}(C \mid B) = \mathrm{Bin}(n = 3, p = 0.4) = \binom{3}{2} \cdot (0.7)^2 \cdot (0.3).$$

Hence,

$$\mathbb{P}(A \mid C) = \frac{32}{81}.$$

That implies that

$$\mathbb{P}_C(B) = \mathbb{P}(B \mid C) = 1 - \mathbb{P}(A \mid C) = \frac{49}{81}.$$

Finally,

$$\mathbb{E}_C[T \mid A] = \mathbb{E}[Y \mid A] = 7 \cdot 0.4 = 2.8,$$
$$\mathbb{E}_C[T \mid B] = \mathbb{E}[Y \mid B] = 7 \cdot 0.7 = 4.9.$$

Hence,

$$\mathbb{E}_C[X] = \mathbb{E}[X \mid C] \approx 6.0704.$$

■

Problem 7.61(a-c). *Let $X_1 \ldots$ be independent random variables with the common distribution function F, and suppose they are independent of N, a geometric random variable with parameter p. Let $M = \max(X_1, \ldots, X_N)$.*

(a) Find $\mathbb{P}(M \leqslant x)$ by conditioning on N;

(b) Find $\mathbb{P}(M \leqslant x \mid N = 1)$;

(c) Find $\mathbb{P}(M \leqslant x \mid N > 1)$.

Solution.

(a)

$$\mathbb{P}(M \leqslant x) = \sum_{n=1}^{\infty} \mathbb{P}(M \leqslant x \mid N = n)\mathbb{P}(N = n) = \sum_{n=1}^{\infty} F^n(x)p(1-p)^{n-1} = \frac{pF(x)}{1 - (1-p)F(x)}.$$

(b)

$$\mathbb{P}(M \leqslant x \mid N = 1) = F(x).$$

(c)

$$\mathbb{P}(M \leqslant x \mid N > 1) = F(x)\mathbb{P}(M \leqslant x).$$

■