Multifractal Scaling in the Bak-Tang-Wiesenfeld Sandpile and Edge Events

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A widely applicable analysis of numerical data shows that, while the distribution of avalanche areas obeys finite size scaling, that of toppling numbers is universally characterized by a full, nonlinear multifractal spectrum. Boundary effects determine an unusual dependence on system size of the moment scaling exponents of the conditional toppling distribution at a given area. This distribution is also multifractal in the bulk regime. The resulting picture brings to light unsuspected physics of this long-studied prototype model.

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Finite size scaling (FSS) [1] is a widely adopted framework for the description of finite, large systems near criticality. In the past decade, after the work of Bak *et al.* [2], much attention has been devoted to a class of models in which criticality is spontaneously generated by the dynamics itself. This self-organized criticality (SOC) has been advocated as a paradigm for a wide range of phenomena, from earthquakes to interface depinning, economics, and biological evolution [3]. The prototype model of SOC is the two dimensional (2D) Bak, Tang, and Wiesenfeld (BTW) sandpile [2,4], which represents a system driven by a slow external influx, dissipated at the borders through a local, nonlinear mechanism.

In spite of its relative analytical tractability [4–7], the 2D BTW resisted, so far, all attempts based on the FSS ansatz, to fully and exactly characterize its scaling [6]. Numerical approaches, also assuming FSS [8], led to rather scattered and sometimes contradictory results [9–11], which hardly reconcile with theoretical conjectures. Thus, with its intriguing intractability, BTW scaling remains a formidable challenge for nonequilibrium theory [12], and it is very important to check if FSS works in this context.

In this Letter, by a new strategy of data collection and interpretation, we determine to what extent FSS can be applied, or rather has to be modified, for a correct and consistent description of the 2D BTW. Our results are striking and largely unexpected: while compelling evidence is obtained that the probability distribution functions (PDFs) of some quantities obey FSS, for other magnitudes, whose fractal dimensions can widely fluctuate within the nonlinear dynamics, this is definitely excluded. We demonstrate that relations between different key quantities do not reduce to standard power laws, as in FSS, and are substantially influenced by the infrared cutoff, i.e., the system size.

To each site i of the BTW on an $L \times L$ square lattice box Λ, we associate an integer height $z_i > 0$, the number of grains. When $z_i > z_c = 4$, site i topples: $z_i \to z_i - 4$, while for the nearest neighbors j of i, $z_j \to z_j + 1$. At the boundary less than four neighbors are upgraded with consequent grain dissipation. Further instabilities can be created by the first toppling. An avalanche is the set of the s topplings necessary to reach a stable system configuration after addition of one grain ($z_k \to z_k + 1$ at some randomly chosen $k \in \Lambda$); a is the number of lattice sites toppling at least once during the avalanche. A sequence of avalanches is created by successive additions. After many grains, thanks to border dissipation, the BTW reaches a steady state. We analyzed up to 10^8 avalanches in this state for $L = 128$, 256, 512, and 1024.

The PDFs for a and s do not reveal characteristic scales intermediate between lattice spacing and linear pile size L. FSS postulates for them the forms

$$\begin{aligned} P_s(s, L) &= s^{-\tau_s} F_s(s/L^{D_s}), \\ P_a(a, L) &= s^{-\tau_a} F_a(a/L^{D_a}), \end{aligned} \tag{1}$$

and usually assumes that s and a are simply related by a power law ($s \sim a^{\gamma}$). This and $P_a da = P_s ds$ further imply

$$\gamma = D_s/D_a, \qquad \gamma = (\tau_a - 1)/(\tau_s - 1), \tag{2}$$

for the single exponents τ and fractal dimensions D of each PDF. Without assuming the FSS form (1), scaling of P_s is most generally described by the multifractal spectrum [13] associated with the integrated log probability:

$$f(\alpha) = \frac{\log[\int_s^\infty P_s(x, L)\, dx]}{\log(L)}, \tag{3}$$

where $\alpha = \log(s)/\log(L)$ is fixed and $L \to \infty$ is implied. f is the Legendre transform of the moment scaling function $\sigma(q)$ defined by

$$\begin{aligned} \langle s^q \rangle_L &= \int P_s(s, L) s^q\, ds \\ &= \int e^{[f(\alpha) + q\alpha] \log L}\, d\alpha \sim L^{\sigma(q)}, \end{aligned} \tag{4}$$

i.e., $\sigma(q) = \sup_\alpha [f(\alpha) + q\alpha]$. Analogous definitions apply to the spectrum $g(\beta)$ ($\beta = \log a / \log L$) and the

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moment exponent $\rho(q)$ of P_a. If Eqs. (1) hold, $f(\alpha) = -(\tau_s - 1)\alpha$ for $0 < \alpha < D_s$, and $f = -\infty$ for $\alpha > D_s$. Consistently, $\sigma(q) = D_s(q - \tau_s + 1)$ for $q > \tau_s - 1$, and $\sigma(q) = 0$ for $q < \tau_s - 1$. So, within FSS, both f and σ are piecewise linear functions of their arguments [14]. Analogous linear forms apply to the corresponding a quantities. Being extrapolated for $L \to \infty$ on the basis of Eq. (3), σ and ρ are very asymptotic characterizations of the PDFs. Thus, a reliable way to establish if Eqs. (1) hold is by checking the above linearity of σ and ρ in significant ranges of q. For P_a, a constant gap $\Delta\rho(q) = \rho(q+1) - \rho(q) \sim 2.02 \pm 0.03 \sim D_a$ establishes already for $q = 1$ (Table I). To the contrary, for P_s, $\sigma'(1) \sim 2.5$, while $\Delta\sigma$ steadily increases from $\Delta\sigma(1) \sim 2.70$ [15] to $\Delta\sigma(8) \sim 2.92$. Thus, unlike for P_a, and in violation of FSS, for P_s there is clearly no constant gap in the range $q \geq 1$. Since σ' increases from $\sigma'(1) \sim 2.5$ to $\sigma'(\infty) \sim 3.0$ [16], we expect f to be $> -\infty$ for $\alpha \lesssim 3.0$, and to be nonlinear for $2.5 \lesssim \alpha \lesssim 3.0$. Avalanches in this range are rare, dissipative edge events [17], which determine essentially all the multiscaling structure of P_s with their nontrivial, self-similar intermittency pattern.

Figure 1 reports g and f as obtained by the data collapse technique in Ref. [8]. The linear form $g(\beta) = -(\tau_a - 1)\beta$ is well verified for $\beta \lesssim 1.5$ with an estimated slope -0.19 ± 0.01. For $\beta > 1.5$, the collapse gets worse. One expects $D_a = 2$ and, in fact, the poorer g collapse for $\beta \lesssim 2$ is consistent with the infinite discontinuity of a FSS spectrum with $D_a = 2$: curves for various L smooth out to different degrees such discontinuity, and underestimate g for $\beta \lesssim 2$. Assuming a linear g in the whole domain $0 \leq \beta \leq 2$, and $\tau_a = 6/5$ exactly, as suggested by the estimated initial slope, one gets $\langle a \rangle_L \sim L^{\rho(1)}$ with $\rho(1) = \sup_\beta[g(\beta) + \beta] = -2/5 + 2 = 1.6$, in nice agreement with our estimate $\rho(1) = 1.59 \pm 0.02$. All this strongly supports FSS for P_a. Additional evidence comes from the excellent standard FSS collapse of this PDF (Fig. 1, inset).

For $\alpha \lesssim 2$ the collapsed f is very close to linear and overlaps with the expected g [$f(2) \sim -0.39$]. Thus, an acceptable FSS form of P_s should assume $\tau_s = \tau_a$, in order to be consistent with the well collapsed, initial part of the plots. Within such assumption, $D_s = 2.5$ would be imposed by the exact result $\sigma(1) = 2$ [4] [we find $\sigma(1) = 1.99 \pm 0.02$]. Indeed, $\sup_\alpha[f(\alpha) + \alpha] = \sup_\alpha[4\alpha/5] = 2$ in such a case, the sup being attained right at $\alpha = D_s = 2.5$. Thus, the hypothetical linear f should have support in $0 \leq \alpha \leq 2.5$, with $f(5/2) = -1/2$. According to Eqs. (1) and (4), for such P_s of FSS form, one would also have $D_s(2 - \tau_s) = 2 = \sigma(1)$. However, a manifest inconsistency would arise. Indeed, Eqs. (2) would imply both $\gamma = 1$, and $\gamma = 5/4$. On the other hand, a linear f with support in $\alpha \leq 2.5$ would not agree with the plots which show that $f > -\infty$ at least up to $\alpha \sim 3.0$. Indeed, curves for various L collapse rather well for $\alpha \lesssim 3.0$, and clearly suggest for f a support bounded by $\alpha \sim 3.0$, fully consistent with the trend of $\Delta\sigma$ (Table I). This bound follows also from the leftward trend of the curves for increasing L in the region $\alpha \gtrsim 3$, where collapse gets worse. We conclude that a FSS τ_s exponent, so extensively discussed in the past decade, does not exist, because it cannot be simultaneously consistent with the initial slope $\sim -1/5$ of f, $\Delta\sigma(\infty) \sim 3$, and $\sigma(1) = 2$. Full consistency is recovered by assuming that f is indeed linear, with the same slope $-1/5$ as g, only up to $\alpha = 2.5$ [$f(2.5) = -1/2$], but has a nonlinear, continuous drop in the range $2.5 < \alpha < 3.0$. This drop should be such to still allow $\sigma(1)$ to be determined by a sup at $\alpha = 2.5$. The slight underestimation by the plot for $\alpha \sim 2.5$ [$f(5/2) \sim -0.57$] should again be imputed to round off and slower L convergence in correspondence to the major bending of f [18]. A discontinuity in the slope of f at $\alpha = 2.5$ could be expected in view of the fact that dissipating avalanches alone contribute to the spectrum for $\alpha > 2.5$. Thus, P_s is genuinely multifractal, and its universal features are described by a full, nonlinear f spectrum, rather than by just two parameters, such as τ_s and D_s in Eq. (1).

The striking difference between P_s and P_a suggests unusual structure for the conditional PDF, C, such that $P_s(s, L) = \int da\, C(s|a, L) P_a(a, L)$. FSS would imply

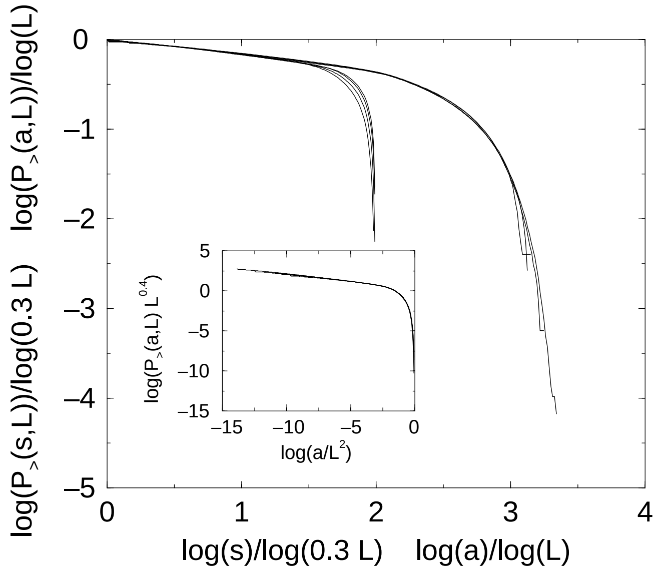

FIG. 1. Multifractal collapses for g and f; a standard FSS collapse of $P_>(a, L)$ is in the inset. The subscript $>$ indicates integrated PDF.

TABLE I. Gaps at different q for P_a and P_s. The estimated uncertainty is ± 0.03.

q	1	2	3	4	5	6	7
$\Delta\rho$	2.02	2.02	2.02	2.03	2.02	2.02	2.02
$\Delta\sigma$	2.70	2.83	2.87	2.91	2.91	2.92	2.92

$C \sim \delta(s - a^{\gamma})$ for $a < L^2$ and $L \to \infty$, L entering simply as an upper cutoff. In fact C is a complex, broad PDF. Figure 2 reports values of α vs β for $L = 1024$ avalanches. Ratios $\gamma = \alpha/\beta$ range between 1 ($s \geq a$) and $\sim$1.25, and their spread is not modified appreciably by sampling data for progressively larger L's. For $\beta \lesssim 2$ the range of γ's also shifts upwards and broadens. If a relation $s \sim a^{\gamma}$ would hold, data should coalesce into a straight line with slope γ. To the contrary, points are quite spread and form an open angle, rather than a narrow strip. A striplike shape is instead shown by data for the radius of gyration, r, of the surface covered by the avalanche (Fig. 2). In fact, the r PDF satisfies FSS like P_a, and the conditional PDF of r at given a is $\sim\delta(r - a^{1/2})$.

At fixed β, C moments can be assumed to scale as $\langle s^q \rangle_{a,L} = \int ds\, C(s|a,L) s^q \sim L^{\beta\kappa(q,\beta)}$. One would hope exponents such as κ to be well defined, i.e., independent of β [$\kappa(q,\beta) = \tilde{\kappa}(q)$], for $L \to \infty$. This would allow moments to be expressed as well-defined power laws in a ($\langle s^q \rangle_{a,L} \sim a^{\tilde{\kappa}(q)}$, $a < L^2$), independent of the L to which they refer. Furthermore, within FSS, one should find $\beta\kappa(q,\beta)/q = \beta\tilde{\kappa}(q)/q = \beta\gamma$, independent of q. In curves a of Fig. 3, we plot $\log(\langle s^q \rangle_{a,L}^{1/q})/\log L$ versus β for various q and $L = 1024$. The curves, which correspond to $\beta\kappa(q,\beta)/q$ asymptotically, do not overlap. Moreover, for each q the plots have pronounced curvature, especially for $\beta \gtrsim 1.7$. This curvature does not decrease appreciably, or increase signaling crossovers, if one considers progressively larger L's. The β dependence of the spread in q of the various curves a of Fig. 3 implies that for C it makes sense to talk about multiscaling only with reference to a given β. If $\beta\kappa/q$ would be linear in β, one could discuss multiscaling globally, for all β's. The β dependence of κ can be viewed as a boundary effect and suggests that only for rather low β's (in principle, $\beta \to 0$) one can consider the scaling as strictly representative of a bulk behavior. We verified the existence of bulk multiscaling for γ. A quantitative measure of this multiscaling can be obtained through the spectral density $h(\gamma,\beta) = \lim_{L\to\infty} \log C_>(\beta,\gamma,L)/\beta \log L$, where $C_> = \int_{L^{\beta\gamma}}^{\infty} dx\, C(x|a = L^{\beta}, L)$ is the probability of avalanches with $\log a/\log L = \beta$ and $\log s/\log a \geq \gamma$. h was computed by Legendre transforming the $L \to \infty$ extrapolated $\kappa(q,\beta)$ with respect to q [17]. Figure 4 shows that indeed, in the bulk limit, h has support in $1 \lesssim \gamma \lesssim 5/4$, confirming the multiscaling. Similar plots for increasing β show intervals of γ values remaining wide and shifting upwards, as already suggested by the plots in curves a of Fig. 3. $\gamma = 5/4$ was conjectured as a unique exponent for all avalanches in Ref. [6]. The unjustified assumption there was a sharp PDF for the maximum number of topplings in an avalanche at fixed a. This PDF is broad instead, as shown above for C. Different, but again unique, γ values [19], sometimes definitely larger than 1.25 [15], were conjectured in the recent literature on the basis of FSS and numerical results. That such values could be singled out by analyses based on FSS should not surprise, in view of the complex structure of C elucidated above. We showed here that there exists instead an entire continuous range of possible γ values, also in the bulk scaling regime.

The β nonlinearity of the curves a of Fig. 3 is crucial for the overall consistency of scaling properties. Since $\langle s^q \rangle_L = \int da\, P_a(a,L) \langle s^q \rangle_{a,L}$, we must have $\sigma(q) = \sup_{\beta}[g(\beta) + \beta\kappa(q,\beta)]$. With $g = -\beta/5$, as argued above, the sup for $q = 1$ should fall at $\beta = 2$ (curves a of Fig. 3), and be equal to $-2/5 + 2\kappa(1,2)$. Thus, one must have $\kappa(1,2) = 1.2$: if $\kappa(1,\beta)$ would keep constantly its low-β value ($\simeq$1.04, see also [19]) up to $\beta = 2$, we could not find $\sigma(1) = 2$. To the contrary, our determinations consistently extrapolate to $\kappa(1,2) \simeq 1.20$ (curves a of Fig. 3). For sure dissipating edge avalanches

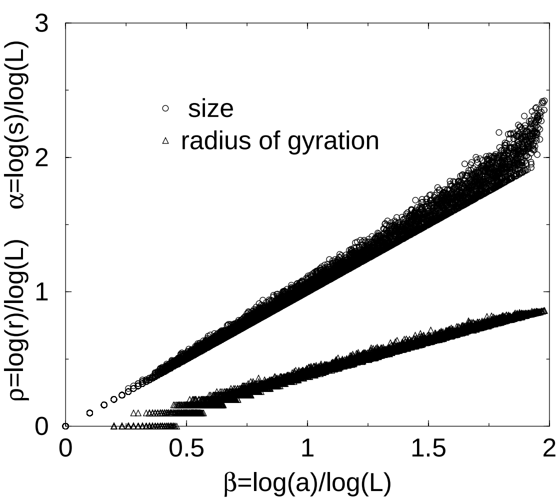

FIG. 2. (β,α) points for avalanches with $L = 1024$ (upper); similar plot for $(\beta, \log r/\log L)$ (lower). The slope in the latter case is $D_a^{-1} = 0.50 \pm 0.01$.

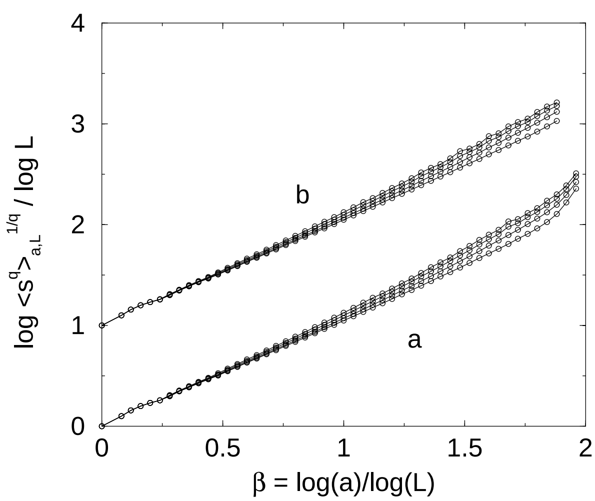

FIG. 3. The lower set of curves (a) refers to all avalanches, while the upper one (b), shifted by 1 along y axes, pertains to the nondissipative ones. $q = 1, 4, 10, 16$ moving upwards.

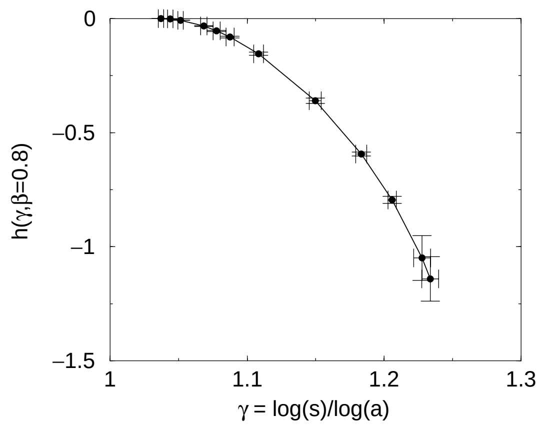

FIG. 4. $\beta = 0.8$ is the smallest β for which meaningful extrapolations could be obtained from our box sizes.

are responsible for most of the bending of the curves a of Fig. 3 near to $\beta = 2$. However, even if one eliminates from the sample these avalanches, the resulting curves b (Fig. 3) are still only approximately straight. For example, after such elimination one has $\kappa(1,1) \simeq 1.05$, and $\kappa(1,1.8) \simeq 1.08$. Since in this case also $\sigma(1) \simeq 1.8$, dissipative avalanches appear to be essential for the fulfillment of the Laplacian conservation constraint $\sigma(1) = 2$.

In summary, the long-standing puzzle of 2D BTW scaling finds a solution in a strong violation of FSS by both P_s and C. While P_a obeys FSS, with $D_a = 2$ and $\tau_a = 6/5$ as most plausible exponents, multiscaling holds for P_s and C. In spite of the several exactly known properties, the belief that the 2D BTW should be "easily" solvable reveals unjustified. The unusual scaling pattern discovered here is not found in more simple systems, such as directed sandpiles [12], and enhances the paradigmatic role of the 2D BTW. Indeed, this model belongs to the class of complex statistical problems, such as turbulence [20], which can correctly be described only within a multiscaling framework. Most recently, results of the present analysis have been used to elucidate quantitative connections between BTW scaling and turbulence in 3D [21].

The establishment of a possible multifractal character for models like the BTW is a crucial step towards the solutions of universality issues, which are prerequisites for setting up a theory of SOC. In the context of sandpiles, a main, open controversy [15,22] concerns whether or not one can put in the same universality class the BTW and the Manna model. In the latter each toppling gives rise to grain emission in randomly, rather than deterministically, chosen directions. By applying our analysis to the two-state Manna model [23], we obtained compelling evidence that it obeys FSS: $\Delta\sigma$ and $\Delta\rho$ are constant for high q, g and f spectra are linear, and the curves a corresponding to those in Fig. 3 are straight and overlap nicely. This means that the 2D Manna model and the BTW belong to even qualitatively different universality classes. Thus, varying degrees of determinism in avalanche dynamics can produce radically different scaling behaviors in SOC models.

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