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Critical behavior of sandpile models with sticky grains

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Abstract

We revisit the question whether the critical behavior of sandpile models with sticky grains is in the directed percolation universality class. Our earlier theoretical arguments in favor, supported by evidence from numerical simulations [P.K. Mohanty, D. Dhar, Phys. Rev. Lett. 89 (2002) 104303], have been disputed by Bonachela et al. [Phys. Rev. E 74 (2006) 050102] for sandpiles with no preferred direction. We discuss possible reasons for the discrepancy. Our new results of longer simulations of the one-dimensional undirected model fully support our earlier conclusions.

Keywords: Self-organized criticality; Sandpile models; Directed percolation

After the pioneering work of Bak et al. [1] in 1987, sandpile models have been studied extensively in the past two decades, both as paradigms of self-organized critical systems in general [2], and also as models of real granular matter [3]. Many different types of sandpile models with different toppling rules have been studied [4]: deterministic and stochastic, with or without preferred direction, different instability criteria [5], or particle distribution rules [6], with fixed energy [7] etc. Most of these models could only be studied numerically, and for a while it seemed that each new variation studied belonged to a new universality class of critical behavior. Though not complete, a broad picture of the different universality classes of self-organized critical behavior has emerged in recent years [8,9].

In an earlier paper [10], we have argued that the generic behavior of sandpile models is in the universality class of directed percolation (DP), and models with deterministic toppling rules like the original BTW model, and models with stochastic toppling rules like the Manna models, are unstable to a perturbation of introduction of "stickiness" in the toppling rules, and under renormalization, the flows are directed towards the DP-fixed point. These arguments are reasonable, but not rigorous, and we presented a detailed study of a specific model, where some of the steps in the arguments could be shown to be valid, and we used detailed Monte Carlo simulations to check our conclusions.

Some of the conclusions of this paper have recently been disputed by Bonachela et al. [11]. These authors contend that while the directed sandpiles with sticky grains show DP behavior in the SOC limit, our arguments do not apply to *undirected* sandpiles, where, even with stickiness, the critical behavior continues to be the same

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as that of the Manna model, i.e., in the universality class of the directed percolation with a conservation law (hereafter referred to as the Manna/C-DP universality class).

In this paper, we will try discussing these conflicting claims, and also present some data from more recent extensive simulations, which supports our original conclusions. We start by defining the model precisely, and then summarize the arguments of Ref. [10]. We then discuss the simulations of Ref. [11], and finally present the results of our new more extensive simulations.

First the precise definition of the model. We consider the directed model on a $(1+1)$-dimensional square lattice, for definiteness. Generalizations to higher dimensions are straightforward. The sites on an $L \times M$ torus are labeled by euclidean coordinates (i,j) with $(i+j)$ even and j increasing downward. At each site (i,j), there is a non-negative integer $h_{i,j}$ to be called the height of the pile at that site. Initially all $h_{i,j}$ are zero. The system is driven by choosing a site at random and increasing the height at that site by one.

The 'stickiness' of the grains is characterized by a parameter p, and its role in the dynamics of sandpiles is defined as follows: A site is said to become unstable at time t, if at least one particle is added to it at time t, and its height becomes greater than 1. A site (i,j) made unstable at time t relaxes at the time $(t+1)$ stochastically: With probability $(1-p)$, it becomes stable *without losing any grains*, and the added particle(s) sticks to the existing grains. Otherwise (with probability p), the site topples, and the height at the site decreases by two, and the site *becomes stable*. We introduce bulk dissipation: at each toppling, with probability δ both grains from the toppling are lost, otherwise (with probability $1-\delta$), the two grains are transferred to the two downward neighbors $(i \pm 1, j+1)$.

We relax all the unstable sites by parallel dynamics. An unstable site is relaxed in one step, independent of whether it received one or more grains at the previous time step. Once a site has relaxed, it remains stable until perturbed again by new grains coming to the site. This relaxation process is repeated until all sites become stable, and then a new grain is added.

The model is specified by two parameters p and δ. For this model, the following results can be proved [10]:

(i) Depending on the values of p and δ, two different behaviors are possible. There is a threshold $p^*(\delta)$, such that for $p > p^*(\delta)$, there is a steady state in the system, but for $p < p^*(\delta)$, no steady state is possible, and the mean height of the pile grows linearly with time [Fig. 1].

(ii) The boundary line $p = p^*(\delta)$ is exactly given in terms of the function $\bar{n}_{DP}(\lambda)$, which gives the mean number of infected and boundary sites in a cluster in a directed site percolation process on the same lattice with infection probability λ. In particular, this boundary line meets the $\delta = 0$ line at $p = p_c$, where p_c is the exact directed site percolation threshold for the square lattice.

(iii) At the boundary line $p = p^*(\delta)$, the distribution of sizes of avalanches is exactly the same as in the DP process, with a infection probability $\lambda = p(1-\delta)$.

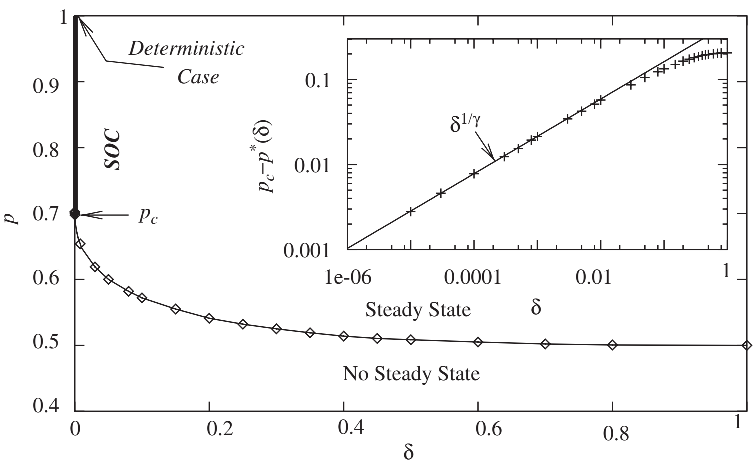

Fig. 1. The phase diagram in the p–δ plane. The inset shows the variation of $[p_c - p^*(\delta)]$ versus δ, with the straight line showing the theoretical fit (from [10]).

(iv) In the regime where the steady state exists, mean cluster size is finite for non-zero δ, and we get SOC behavior only for $p_c \leqslant p \leqslant 1$, and $\delta \to 0$.
(v) The model can be solved exactly along three of the boundary lines, $p = 1$, $\delta = 1$ and $p = 0$.

We find that correlations between heights at different sites in the steady state are quite weak, and the steady state is almost a product measure state. We argued that if these correlations are *irrelevant*, one can study the avalanche process in a background where these correlations are *absent*. Then the avalanche process becomes a Domanay–Kinzel process [12] with two parameters (p_1, p_2), where $p_2 = p$ and p_1 is determined in terms of the concentration of sites with height zero in the steady state. If the correlations in height are indeed irrelevant, the avalanche distribution function for all p along the SOC line in Fig. 1 would show DP exponents, except the end point $p = 1, \delta = 0$, where the model has deterministic toppling rules.

We then extended the arguments given to undirected sandpiles, by considering the time-evolution of a d-dimensional pile as a infection process on a $(d+1)$-dimensional lattice. It can be proved that point (i)-(iv) listed above continue to hold for the undirected case, and $p^*(\delta = 0)$ is exactly given by the critical threshold of the corresponding DP process.

Again, if we can neglect correlations between heights of pile at different sites in the steady state, the avalanche exponents in the model can be seen to be same as DP exponents. Correlations between heights in this case are stronger than in the directed case, as height at a site remains unchanged if there is no activity, and is obviously correlated in time. Thus, it is not clear that these are also irrelevant in the renormalization group sense in our problem. In fact, Bonachela et al. argue that these are not. However, we note that because our toppling rules do not depend on the height, so long as it is greater than 1, most of these correlations do not come into picture, as probabilities of different topplings depend only whether the height at a site was zero or non-zero before the addition of the new particle that made it unstable. The agreement of the results of our numerical simulations with the theoretical predictions shows that indeed, for determining the critical exponents, neglecting the effect of these correlations is justified.

Bonachela et al. have argued that while the arguments in Ref. [10] are correct for the directed model, the neglect of correlations is not valid in the case of undirected models. They presented Monte Carlo evidence based on simulations of the fixed-energy sandpile (FES) model, and also some non-rigorous arguments in support of their proposition.

Let us discuss the simulations of the FES first. It should be noted that one cannot get any avalanche exponents directly from such simulations, as by definition, the FES shows a single avalanche that never stops, and there is no distribution of avalanche sizes to quantify. The avalanche exponents can only be inferred indirectly, by determining some other exponents, and then using scaling theory to relate them to the conventional avalanche exponents. For example, exponent governing the decay of activity-activity correlation function with time in the steady state of the fixed-energy sandpile is used to determine the exponent $\beta/\nu_{\parallel}$.

The FES sandpile model can be obtained from our model by considering a uniform addition of particles at rate ε per site per unit time, with $\varepsilon \ll \delta$, and take the double limit $\varepsilon \to 0$, and $\delta \to 0$. In the usual slowly driven sandpiles, one takes the limit $\varepsilon \to 0$ first, and then $\delta \to 0$ limit. In the FES case, we have to take the limit $\varepsilon \to 0$ with mean activity $\bar{a} = \varepsilon/(2\delta)$ held fixed, and then take the limit $\bar{a} \to 0$. The different order of limits can lead to different scaling behaviors.

In Fig. 2, we have shown a schematic space–time plot of the evolution of the activity in a part of the sandpile in different cases (the y-axis is time, and it increases upwards). In the case of slowly driven sandpile case , with $\varepsilon \ll \delta$, we see that different activity clusters are disconnected from each other, and show a wide variation of sizes [panel (a)]. If ε is comparable to δ, there are overlapping activity clusters [panel (b)]. However, there still are disconnected clusters of activity, and one can find events where activity starts from inert background (due to addition of a particle), and the avalanche activity so generated diffuses around, and may later die, or merge with the infinite avalanche. In (c), we show the FES activity cluster. There are no disconnected activity clusters, and the large-scale structure of the cluster is different from (b), in that no activity can start from inert state unless a neighbor was active. Thus, the large-scale structure of clusters is clearly different in the three cases.

In particular, we note that the avalanche distributions in the conventional slowly driven sandpile are determined by the properties of disconnected finite clusters (when $\varepsilon \ll \delta$, the probability that the added new

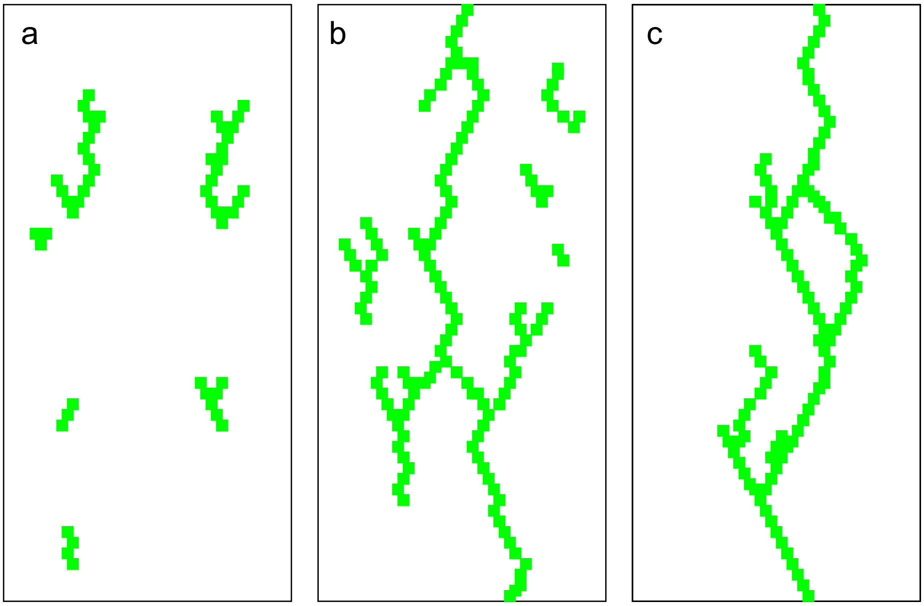

Fig. 2. Schematic representation of time-evolution in the model in different cases: (a) non-overlapping avalanche clusters for very small ε (b) an infinite avalanche cluster and disconnected finite clusters for intermediate ε, (c) avalanche cluster in the fixed energy sandpile.

particle will be added to an active site is small). In contrast, the FES properties are determined by the properties of the single infinite cluster.

We note, as argued in Ref. [10], that on the line $p = p^*(\delta)$, the probabilities of all clusters in the sticky sandpile model are exactly those of DP clusters, with the infection probability λ given as an explicit function of p and δ. This is so, as on this line, the fraction of sites with height 0 is zero in the steady state, and every infected site has the same probability $\lambda = p(1-\delta)$ of infecting a downward neighbor. If $p > p^*(\delta)$ by a small amount Δp, the mean height would be large, and concentration of sites with height 0 in the steady state is small of order Δp. The size of a cluster s is the number of topplings during a single avalanche. Clearly the difference in cluster size distribution $|\mathrm{Prob}(s) - \mathrm{Prob}_{DP}(s)|$ tend to zero as Δp tends to zero, and the probabilities for small avalanches tend to the DP value near the boundary $p = p^*(\delta)$.

We also note the activity cluster in the FES for large average energy E is the same as the infinite DP-cluster with site percolation parameter p, as each site having at least one particle coming to it would topple with a probability p. Thus, in the limit of large mean energy, all n-point correlation functions of the activity in the steady state are exactly the same as in the DP process with infection parameter $\lambda = p$.

The only possibility for seeing the Manna exponents in the avalanche statistics then is to look for it in some intermediate range of values of s, $s_{min} \gg s \gg s_{max}$, where s_{min} is the crossover scale above which DP behavior is presumably lost, and s_{max} is the upper cutoff imposed by finite value of δ. Here s_{min} is expected to diverge as $(\Delta p)^{-a}$ near the critical line. As the difference between the DP and C-DP exponents is small, s_{min} would be expected to be large, and the difference (even if present) is difficult to establish convincingly by simulations.

Bonachela et al. have also studied the problem using a numerical study of the coupled Langevin equations for the density fields for activity and particles. However, this study is also a numerical integration of the stochastic evolution equations, and is qualitatively not different from a Monte Carlo simulation working with coarse-grained variables instead of the original variables defined on the lattice sites.

Finally, we discuss our more recent Monte Carlo data for the test case of one-dimensional undirected sandpile model with sticky grains. We monitored the average transverse cluster size as a function of the number of topplings in the cluster. This is expected to be less sensitive to the upper cutoff on the s. We did our simulations for $p = 0.85, \delta = 10^{-5}$ on a line of length $L = 4096$ with periodic boundary conditions. This value of δ is a factor 10 smaller than the values used in Ref. [10], which allows us to generate much larger clusters. The chosen value of p (and a small δ) is half-way between p_c and 1, and one would expect the effects of

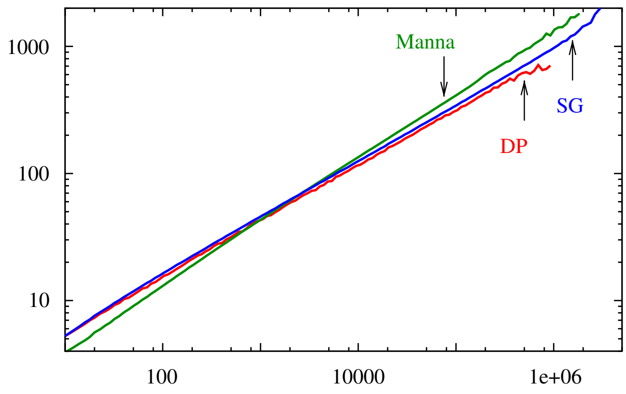

Fig. 3. Log–log plot of mean transverse size of cluster as a function of number of topplings in the cluster for the sandpile model with sticky grains (SG) in one dimension for $p = 0.85$ (blue line). For a comparison, we also plot simulation results for DP clusters (red line), and Manna model clusters (green line).

crossovers from DP and deterministic limits to be small. The mean height in the steady state at this value of p is 1.977(5), and approximately 8.3% of the sites are of zero height. Thus, any deviations from the DP behavior coming from the presence of such sites should be measurable. The results of data taken for 2×10^6 avalanches are shown in Fig. 3. For comparison, we have also plotted the results for DP clusters with $p = 0.7$, and for the Manna model. It is clear that the observed slope is much closer to DP than to that for Manna clusters.

The theoretical arguments for neglecting, or not neglecting the correlations are not fully convincing, and appeal to numerical simulations for their justification. Extracting critical exponents from simulation data could be complicated by crossover effects, as the difference in the avalanche exponents for the Manna and DP cases is not large. We hope that further work on this problem will clarify the situation, and lead to better insight into the problem.

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