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Self-organized criticality: Sandpiles, singularities, and scaling

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ABSTRACT We present an overview of the statistical mechanics of self-organized criticality. We focus on the successes and failures of hydrodynamic description of transport, which consists of singular diffusion equations. When this description applies, it can predict the scaling features associated with these systems. We also identify a hard driving regime where singular diffusion hydrodynamics fails due to fluctuations and give an explicit criterion for when this failure occurs.

Self-organized criticality (SOC), introduced by Bak et al. (1) in the context of simple sandpile cellular automata, is an intriguing hypothesis that may yet explain the dynamic origin of nontrivial scaling in a large class of nonequilibrium systems. Nontrivial scaling refers to self-similarity that extends over a broad range and can be manifest in spatial features (fractals), time series ($1/f$ noise), and event-size distributions for systems in which individual events are well defined. The “criticality” in SOC refers to the case in which there are no characteristic scales, as in equilibrium systems at critical points, so that scaling extends from some short scale cutoff (e.g., the lattice spacing) to a cutoff set by the system size. While effort has been invested toward applying these ideas to real systems, to date the most unambiguous realizations of SOC have been obtained in simple models, and progress has been made toward developing theoretical methods to analyze these systems. The primary motivation for the extensive efforts directed toward self-organizing automata is the hope that an understanding of the scaling properties in models will shed light upon similar features in real systems.

Analytic treatments of self-organizing systems have proceeded predominantly along two distinct lines. The first involves construction of exact solutions to specific automata that can be characterized as Abelian (2, 3). Based on this attribute the stationary distribution of the configurations can be generated, and event-size distributions (4) and correlation functions (5) can be calculated. The second approach and the one that we will focus on here involves analysis of transport properties and hydrodynamics of the systems, which we have found to exhibit rather novel features that are described below. In comparison to the exact solutions, this approach has the advantage of providing a direct link to an underlying critical phenomenon (6). Additionally, this approach is applicable to a broader class of systems, which includes those discussed in refs. 2–5. It is, however, predicated on the existence of a conserved quantity, the transport of which is an essential feature of the system, as is the case for a preponderance of self-organizing systems that have been studied to date. It is worth noting that there has been considerable debate about the necessity of a conservation law for SOC (7–9). The possibility

of a hydrodynamic description of nonconserving SOC systems remains an open issue.

The sandpile models first introduced in ref. 1 have become the paradigm for SOC. Each site on a finite lattice is occupied by a certain number of sand grains, and sand is added randomly to the system. When a specified stability condition is violated, avalanches occur, which instantaneously relax the system to a stable state. When an avalanche intersects the boundaries, sand falls off the edge. The scaling properties of the distribution of avalanche sizes first attracted interest in these models. However, the most primitive description of SOC models is that a conserved quantity, such as sand, is added to an open system, and discrete toppling events transport the sand to the dissipative boundaries.

This coarse description does not, in any meaningful way, distinguish SOC from other systems in which a conserved quantity is being transported, such as classical heat flow. What does distinguish self-organizing models is the statistical mechanics of this transport. In particular, it can be proven in some cases and verified in simulations for others that transport in many self-organizing systems is described by diffusion equations with diffusion coefficients that possess singularities at critical densities of the conserved quantity (10). The origin of this singularity lies in a phase transition in a perfectly conventional equilibrium setting (which we will refer to as a *closed system*) analogous to statistical mechanical systems such as percolation.

The scaling behavior exhibited by self-organizing systems is apparent when the singular diffusion equation is applied to the open driven system. With appropriate boundary conditions the solution yields a prediction for what the steady-state density of the conserved quantity should be as a function of system size, and we find that in the thermodynamic limit it is approaching the singularity. This is the special feature of SOC systems—the open system approaches the critical point of the associated closed system in the thermodynamic limit. The scaling of avalanche distributions is a direct result of the scaling properties associated with this critical point. When hydrodynamics holds in SOC, the largest events diverge with system size, but the scaling is sublinear, so that relative to the system size the largest events are becoming local as the system size diverges. An equally intriguing fact is that the hydrodynamic description can break down when the density approaches the singularity too quickly. In contrast to the hydrodynamic case, in this case the event-size distribution cuts off at a size that scales linearly with the system size. It is also in this regime where our theoretical understanding of SOC remains undeveloped.

The paper is organized as follows. In *Sandpiles and Scaling* we briefly discuss scaling in the context of equilibrium statistical mechanical models and compare this with the behavior of a few specific sandpile models. In *Equilibrium Statistical Mechanics of Sandpiles* we describe equilibrium and near-

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Abbreviations: SOC, self-organized criticality; LL model, limited local model; GSI, generic scale invariance; BTW model, Bak, Tang, Wiesenfeld model.

equilibrium properties of self-organizing systems—this is where singular-diffusion hydrodynamics is introduced. In *Open Driven Systems* we describe the predictions that singular diffusions yield for the scaling properties associated with SOC and contrast these predictions with those that would be associated with more conventional transport equations. Finally, in *Fluctuations and the Failure of Hydrodynamics* we describe and characterize the hard-driving regime in which the singular-diffusion description fails.

Sandpiles and Scaling

Our theoretical understanding of the origins of scaling is currently most well established in the context of equilibrium critical phenomena (see, for example, ref. 11). In this section we contrast this behavior with that observed for sandpile models.

The typical properties of equilibrium systems can be illustrated by considering site percolation, which is perhaps the simplest statistical mechanical model with a critical point (see, for example, ref. 12). We consider a two-dimensional integer lattice in which each site is either occupied or vacant. Clusters are defined to be sets of sites that are connected by a path of nearest-neighbor occupied sites. A random configuration is generated by independently designating each site to be occupied with probability p or vacant with probability $1 - p$. We focus on the size of the cluster containing the origin and ask how the probability that the origin is in an infinite cluster depends upon p . It can be proven that for any p in excess of a critical value p_c there is positive probability of the origin being in the infinite cluster; for any p below p_c this probability is zero.

Now, suppose that lightning strikes the origin and electrocutes all sites in the cluster connected to the origin. What is the distribution of the number of sites electrocuted? For $p < p_c$, the probability that more than s sites are electrocuted scales like $1/s^\kappa e^{-\zeta(p)s}$, where $\zeta(p) > 0$. Similarly, when $p > p_c$, then the probability that more than s sites but fewer than an infinite number of sites are electrocuted scales like $1/s^\kappa e^{-\gamma(p)s^b}$ for some positive number b that depends upon dimension, and where $\gamma(p) > 0$. What is most important to this discussion, however, is that both $\zeta(p)$ and $\gamma(p)$ decrease to zero as $p \rightarrow p_c$. At p_c the probability of more than s sites being electrocuted scales as a power law $1/s^\kappa$ for some positive κ . The key point is that cluster sizes (spatial correlations) decay exponentially fast, except at the critical point, where power laws are expected.

Next, we describe the behavior of sandpile models in terms of the original model introduced by Bak, Tang, and Wiesenfeld (1), which we refer to as the BTW model, and a model introduced by Kadanoff *et al.* (13), which is referred to as the limited local (LL) model. While many variations of these models have been studied by these groups and others, the basic phenomena observed are conveniently illustrated by these two systems.

The BTW model is defined on a two-dimensional $N \times N$ integer lattice. Note that the system size enters explicitly in the definition of all SOC models, and we will see that it plays a crucial role in the scaling behavior that is observed. To each site i on the lattice, an integer variable $h(i)$ represents the height of the column of sand on the site. There is a fixed (spatially uniform) constant, h_c , which represents the threshold at which individual sites become unstable. The system evolves according to the following rules: A grain of sand is dropped onto a randomly selected site i , so that $h(i) \rightarrow h(i) + 1$. If $h(i) \geq h_c$, then site i loses four grains [$h(i) \rightarrow h(i) - 4$], and each of the nearest neighbors j gains one grain [$h(j) \rightarrow h(j) + 1$]. The edges of the lattice are open, so that if an instability occurs on the boundary, then one grain (or two grains if the site is a corner) is removed from the system during the toppling event. The toppling rule is iterated for each unstable site until all sites

are below threshold. It is a key point, and another general feature of SOC systems, that the system fully stabilizes before another grain is added, so that there is a complete separation of time scales between addition and relaxation events. This is designed to mimic situations where the driving rate is much slower than the relaxation rate in real systems; a generalization of these results to the case in which the separation of time scale is not infinite is considered in ref. 14.

The aggregate of the toppling events associated with one added grain defines a single avalanche, and many properties of the avalanche can be measured. For example, the avalanche size is defined by the number of sites that topple, the flip number corresponds to the number of individual toppling events, and the drop number corresponds to the number of grains that fall off the edge. Scaling behavior of these quantities can easily be measured numerically. Fig. 1*a* illustrates a scaling collapse of the distribution of avalanche sizes $P(s)$ as a function of size s for various system sizes N^2 . The distribution is well described by a power law with an exponential cutoff, similar to the percolation cluster-size distribution described above. However, for the sandpile models it is the system size rather than a parameter like p that sets the cutoff in the range of scaling that is observed. For the BTW model, this cutoff scales linearly with the system size, so that the distribution in avalanche sizes is essentially a pure power law ranging from the smallest (a single lattice site) to the largest scale. It was this power law that nucleated the interest in sandpile automata.

Our second example is the LL model, and in this case we will consider a one-dimensional system of N sites. As in the BTW model, the state of each site can be described by a height variable $h(i)$, and sand is added one grain at a time to randomly chosen sites: $h(i) \rightarrow h(i) + 1$. However, in the LL model the stability is prescribed in terms of slope variables $z(i) \equiv h(i) - h(i + 1)$. When $z(i)$ is greater than or equal to a uniform threshold z_c , a fixed number of grains n fall from site i to site $i + 1$, so that $h(i) \rightarrow h(i) - n$ and $h(i + 1) \rightarrow h(i + 1) + n$. This toppling process is iterated until all of the sites have reached stability before another grain of sand is added to the system. Note that the dynamics described here in terms of height variables can be defined equivalently in terms of slope. The addition rule becomes $z(i) \rightarrow z(i) + 1$ and $z(i - 1) \rightarrow z(i - 1) - 1$, unless $i = 1$ (the first site), where the transition is: $z(i) \rightarrow z(i + 1)$, with no change in the slope of any other site. When a toppling occurs at site i , $2n$ units of slope are removed from site i , and each neighbor gains n units: $z(i) \rightarrow z(i) - 2n$, $z(i + 1) \rightarrow z(i + 1) + n$, and $z(i - 1) \rightarrow z(i - 1) + n$, unless $i = 1$ [in which case $z(i) \rightarrow z(i) - 2n$, $z(i + 1) \rightarrow z(i + 1) + n$] or $i = N$ [where $z(i) \rightarrow z(i) - n$, and $z(i - 1) \rightarrow z(i - 1) + n$]. The stability criterion for the LL model is in terms of slope, and for this model defining the dynamics in terms of slope is most natural.[‡] Note that the only events that do not conserve the total slope on the system are those that involve addition or toppling at the left edge $i = 1$. In addition, in the LL model the interior toppling events ($i \neq 1, N$) transport slope symmetrically in each direction.

Like the BTW model, the LL model exhibits a distribution of avalanche sizes in which the large-scale cutoff increases with system size N . The interesting new feature of this model is that

[‡]In terms of slope, the following algorithm yields the postavalanche configuration (6). Any site with slope less than $z_c - n$ is called a trough and has the property that it can be hit by n grains of sand and remain subcritical. If sand lands on site i such that $z(i) \geq z_c$, then the new configuration is obtained by locating the nearest troughs to the left (site i_L) and right (site i_R) and the site i_c that is the reflection of site i in the interval $[i_L, i_R]$. The final configuration is constructed by removing n units of slope from sites i and i_c and adding n units to the troughs at sites i_L and i_R . Of course, avalanches intersecting the boundary require special provisions. This algorithm accelerates simulations considerably and illuminates the symmetry of the slope transport.

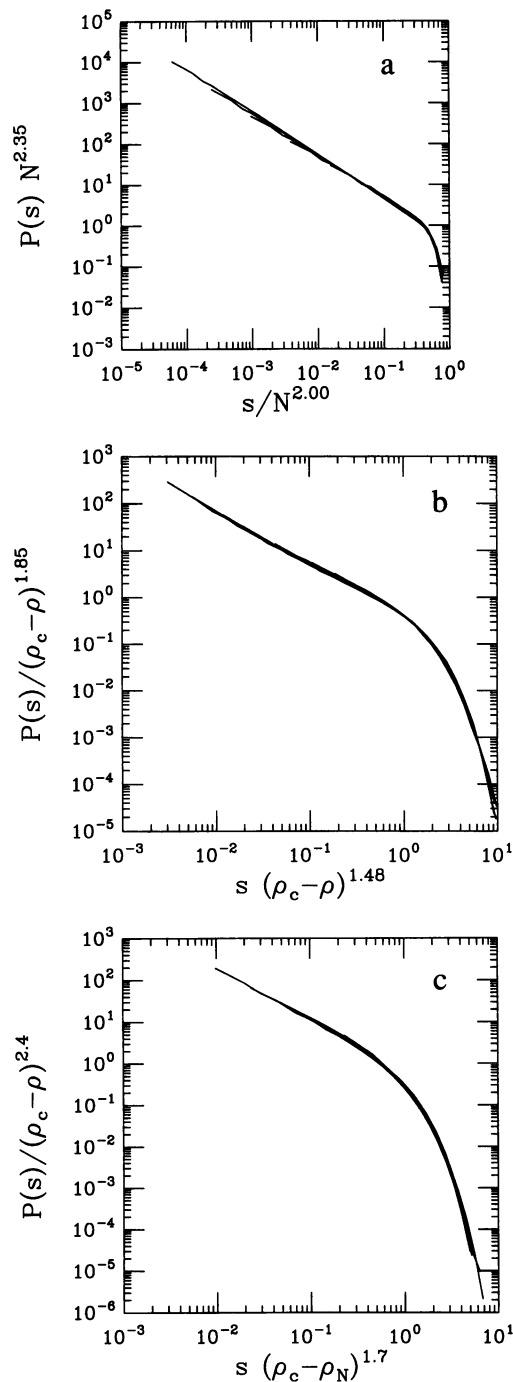


FIG. 1. Scaling collapse of the distribution $P(s)$ of events involving s sites. (a) Results for the original BTW sandpile model, as formulated in ref. 1 with $h_c = 4$ (we set $h_c = 4$ for all results reported). The distribution consists of a power law that cuts off sharply at a size that scales linearly with the size N^2 of the system. This case corresponds to $d_A = 2$ as discussed in the *Open Driven Systems* section. (b) Our results for the closed system, where data is rescaled by the distance from the singularity. A data collapse is obtained when the x axis is rescaled by $\xi^d = (\rho_c - \rho)^{-d\nu}$, and from our fit we extract the exponent $\nu = 0.74$ in Eq. 1, in agreement with results obtained in ref. 15. (c) Our results for the open system with $d_B = 1$. In this case we estimate that $\nu = 0.85$, in reasonable agreement with the closed-system result. The agreement is better for larger system sizes, where the density profile is more uniform over a greater fraction of the system (i.e., in large systems edge effects are much less important in the overall distribution of events), and we attribute the discrepancy between $\nu = 0.74$ and $\nu = 0.85$ to these kinds of finite size effects. Note also that the middle quarter of the system, where the density is most uniform, sets the scaling value of $\rho_c - \rho_N$, which yields the best collapse.

this cutoff scales like $N^{1/3}$ —i.e., with a nontrivial exponent, so that while the largest event is increasing with N , on the scale of the system size the largest events look local in the limit of $N \rightarrow \infty$.

In equilibrium systems, the range of scaling is determined by some measure of the distance from the critical point. In contrast, for sandpile models the range of scaling is set by the system size. Thus we are led to ask whether the power laws and nontrivial scaling observed for SOC systems are associated with identifiable phase transitions, and, if so, why are these systems, which have no *a priori* tuning parameters analogous to the percolation parameter p or other obvious constraints on the value of the conserved quantity, driven to the critical point?

Equilibrium Statistical Mechanics of Sandpiles

Sandpile models are much easier to define than they are to analyze. (i) In their original form described above they are open systems in which the relevant conserved quantity is being driven in some way. Therefore, they are by definition out of equilibrium—perhaps far out of equilibrium. (ii) While the toppling rules are local, avalanches consist of iterations of these rules, and the result is a system in which the range of interaction is bounded *a priori* only by the system size (i.e., for any range up to the system size one can construct configurations so that an avalanche will span this range). Below we define equilibrium analogs by considering variants of the models on closed systems. In these systems the integral of the conserved local density (ρ) remains fixed, and there is no net transport. These modified systems have the advantage that the range of interaction decays exponentially beyond a length scale that depends on the density ρ . Furthermore, as we will see, they undergo phase transitions as a function of ρ . This feature will ultimately lead to a connection between the nonequilibrium systems and an equilibrium phase transition.

We first define the equilibrium version of BTW on an $N \times N$ lattice with periodic boundary conditions. The system is initialized at a given density of height ρ by randomly distributing ρN^2 grains of sand on the sites and relaxing the system via the toppling rule defined in the previous section until all sites are stable. In the nonequilibrium version of the BTW model all avalanches are triggered by the addition of a grain of sand. Because sand is the conserved quantity, in our closed version we must prescribe a rule for initiating avalanches without violating the conservation law. We do this by randomly moving one grain of sand from a selected site to a nearest neighbor (16, 17). The toppling rule is then iterated on the periodic lattice until the system is fully relaxed. In this way, there is no net gain or loss of the total amount of sand in the system.

Similarly, for the LL model we define an equilibrium analog on a periodic lattice of N sites with a fixed slope density ρ by initially depositing ρN “grains of slope” on the system. Here it is essential that we define the model in terms of slope, which is the relevant conserved quantity. Note that with periodic boundary conditions, both the addition and toppling rules conserve slope. Furthermore, slope transport during avalanches is symmetric. However, in the original addition rule, if site i were hit by a grain of sand, it gained a unit of slope, while the site to the left, $i - 1$, lost a unit of slope. Implementing this rule directly would result in a net drift in the closed system. To avoid this peculiarity associated with the original driving mechanism, we initiate avalanches by transferring one unit of slope randomly between nearest-neighbor pairs of sites with no preference for transfer to the left or right. When this exchange results in a site that is at or above threshold, the toppling rule is iterated until the system has fully stabilized.

It is clear that systems with a larger height (or slope) density will, on average, sustain larger avalanches, simply due to the

fact that instabilities require an excess of the conserved quantity. Simulations of the closed-system BTW model yield the results depicted in Fig. 1*b*, in which the probability of an avalanche involving s sites, $P(s)$, is plotted as a function of s . For large-enough systems, the location of the sharp cutoff is independent of system size but increases with the value of ρ . This cutoff defines the event-size correlation length $\xi(\rho)$. Similar results are obtained for the closed-system LL model as a function of the slope density. Note that $\xi(\rho)$ is not a measure of the height–height (or slope–slope) correlations of the system, which could also be defined. Instead, ξ measures the characteristic connectivity of sites in avalanches, just as the correlation length in percolation is a measure of the characteristic connectivity of sites in clusters. [For percolation the second kind of correlation function is trivial. Because sites are independently chosen to be occupied or vacant, there is *a priori* no correlation between the occupation numbers (0 or 1) from site to site. This kind of trivial behavior is observed for some, but not all, SOC models.]

A striking feature of the closed systems, which can be extracted from the avalanche size distributions, is the existence of a singularity in $\xi(\rho)$ at a critical density ρ_c :

$$\xi(\rho) \sim \frac{1}{(\rho_c - \rho)^\nu}, \quad [1]$$

where $\nu > 0$. This divergence of the event-size correlation length distinguishes self-organizing models from other systems transporting a conserved quantity. The critical point ρ_c is the criticality that underlies the scaling behavior in SOC models, and the value of ρ_c typically does not coincide with the value of the stability threshold (e.g., h_c or z_c). For densities above ρ_c the behavior of the system is ill-defined in terms of avalanches because there is a finite probability of an infinite avalanche—one in which the iteration of the toppling rule never stops. Thus, when $\rho \geq \rho_c$ in our closed-system simulations, after an initial transient the infinite event is eventually triggered.[§] This crossover from avalanche dynamics below ρ_c to a continuous sliding state above ρ_c has an analog in the depinning transitions that are observed in systems such as charge density waves.[¶]

The hydrodynamic limits of these systems describe transport on time scales that are much longer than the duration of individual avalanches and length scales that are much greater than $\xi(\rho)$. We must consider situations in which $\rho < \rho_c$, so that the individual avalanches contributing to the transport are effectively bounded by $\xi(\rho) < \infty$. To describe transport in the closed systems, we focus on the relaxation of nonequilibrium density profiles to the equilibrium state characterized by constant density. In refs. 10 and 25 it was shown that certain SOC models relax according to a *singular* diffusion equation of the form:

$$\frac{\partial \rho}{\partial t} = \nabla \cdot [D(\rho) \nabla \rho], \quad [2]$$

where

$$D(\rho) \sim \frac{1}{(\rho_c - \rho)^\phi}. \quad [3]$$

[§]It is possible to define a closed *finite* system in which an infinite event is never realized as a consequence of a transient sequence of finite events, although in the thermodynamic limit the infinite event is realized with probability 1 when the density is above the singularity; this occurs when the addition and toppling rules satisfy a no-passing constraint, as discussed in ref. 18. Our triggering rule violates no passing, so that the infinite event eventually occurs.

[¶]In fact, there is an intriguing relationship between the closed version of the BTW model and the Fukuyama, Lee, Rice (FLR) model of charge density waves (refs. 15 and 19–24).

In ref. 26 this description was proven rigorously for a simple class of two-state models, and in ref. 25 this description was shown to apply to a larger set of systems. For the models that we have defined above, the symmetry of the toppling rules suggests a diffusion limit by analogy with symmetric random walks. The diverging correlation $\xi(\rho)$ is an indication that for larger values of ρ the transition lengths are increasing, which ultimately leads to the diffusion singularity. For the LL model the order of the singularity is $\phi = 4$, which was established first numerically (10) and later derived by using scaling arguments (27). For the BTW model, the order of the singularity is $\phi = 2.3$, which was determined numerically in ref. 25.

To obtain these results numerically we monitor the relaxation of a small-amplitude spatial perturbation $a(x, t)$ to the equilibrium (flat) profile of the system $\rho = \bar{\rho}$, a constant. Linearizing Eq. 2 in $a(x, t)$, the relaxation of any individual mode is given by $\exp(-\lambda D(\bar{\rho})t)$, where λ is the eigenvalue of the mode. By averaging over realizations for a given value of $\bar{\rho}$ and finally varying $\bar{\rho}$, we reconstruct the diffusion coefficient $D(\rho)$. These results can be confirmed in an equilibrium setting by monitoring the displacement of a tagged particle, as discussed in ref. 28. The width σ of the Gaussian distribution of displacements of a tagged particle after time t also depends on the value of ρ and diverges at ρ_c with a singularity of order $\phi - 1$ in all cases that we have studied—either rigorously or numerically.

Finally, it is worth noting that the foundations of this analysis have been developed as part of an ongoing field of research in mathematics that is involved in the study of hydrodynamic (i.e., continuum) limits of interacting particle systems (see, for example, refs. 29–31; for a review, see ref. 32). The singular diffusion description of these automata is a space–time law of large numbers obtained by scaling space by N^{-1} , so that an N^d site system is scaled into a d -dimensional unit hypercube, scaling mass (i.e., the conserved quantity) by N^{-d} , so that the density remains finite as $N \rightarrow \infty$, and scaling time by N^2 , so that N^2 transitions correspond to a unit interval of time in the diffusion limit. Note that the microscopic time scale corresponds to the rate at which avalanches are initiated and not to the rate of individual toppling events, which are by definition instantaneous.

Denoting the height at site i and at time t by $h(i, t)$, then under the rescaling above, a local height density can be defined by averaging h over a box of size ε :

$$\hat{h}(x, t) = \frac{1}{(\varepsilon N)^d} \sum_{i \sim x} h\left(\frac{i}{N}, tN^2\right), \quad [4]$$

where d is the dimension, and $i \sim x$ refers to all sites i such that i/N is inside a box of size ε centered at x . The singular-diffusion description is essentially the statement that $\hat{h}(x, t)$ converges weakly to $\rho(x, t)$, which satisfies Eq. 2 when the system size diverges ($N \rightarrow \infty$) and then the box size shrinks ($\varepsilon \rightarrow 0$). As for any law of large numbers, randomness vanishes only in the limit, and the density of the finite system fluctuates about the limiting partial differential equation. Description of these fluctuations is essentially a central limit theorem around the hydrodynamic limit—an issue that will be discussed in more detail in the last section.

Open Driven Systems

The singular-diffusion equation (Eq. 2) is derived for closed systems. Determination of transport equations for open systems from first principles is complicated by the fact that the open systems support a (possibly large) flux, and there are no *a priori* bounds on the transition lengths. Nonetheless, if the open system is not too far from equilibrium, it is plausible that

the singular-diffusion equation may still apply and yield an accurate description of the system.

In this section we study properties of the stationary solution of the singular-diffusion equation with boundary conditions associated with the driving mechanisms. This approach will yield a prediction for the average steady-state density $\langle \rho N \rangle$ as a function of system size N . The key result is that $\langle \rho N \rangle \rightarrow \rho_c$ as $N \rightarrow \infty$ at a rate that depends on the dimension, the driving mechanism, and the order of the singularity ϕ . In other words, SOC systems approach the phase transition of the closed system in the thermodynamic limit. The singular-diffusion equation with boundary conditions explains this in many cases. The regimes in which fluctuations result in failure of the singular-diffusion description are discussed in the last section.

Before proceeding to more general considerations, we first consider a concrete example in $d = 1$ for a specific set of boundary conditions. Recall that to obtain the diffusion limit we have rescaled space by $1/N$, so that $x = 1$ corresponds to site N on the discrete lattice, and consider the cases: (i) zero density at the right of the system: $\rho(1) = 0$, and (ii) fixed input flux αN at the left of the system: $D(\rho) \partial \rho / \partial x|_{x=0} = -\alpha N$. The boundary conditions can be interpreted directly in terms of the underlying automaton. Condition *i* corresponds to an open right edge—i.e., the conserved quantity leaves the system when toppling events extend to that edge. Condition *ii*, which fixes a flux at the left edge, corresponds to the addition of the conserved quantity at a fixed rate α in the automaton. Because flux scales like $[\text{mass}] \times [\text{time}]^{-1}$, if the automaton is driven at constant rate α independent of system size, then when the flux is rescaled in the diffusion limit, the fact that mass is scaled by $1/N^d$ while time is scaled by N^2 implies that in the hydrodynamic limit in $d = 1$ the rescaled flux picks up a factor of N relative to the unscaled flux α .

Subject to these boundary conditions, the steady-state solution to the singular-diffusion equation is easily obtained from Eq. 2. For example, when $\phi > 1$ we obtain

$$\rho_N(x) = \rho_c \left(1 - \frac{1}{[1 + (\phi - 1)\rho_c^{\phi-1}\alpha N(1-x)]^{1/(\phi-1)}} \right), \quad [5]$$

which is illustrated in Fig. 2a for $\phi = 3$, $\rho_c = 1$, $\alpha = 1$, and $N = 512$. Note that in the steady-state solution $\partial \rho / \partial t = 0$, and the flux is translation invariant, so that condition *ii* is satisfied across the entire system. It is useful to compare Eq. 5 to the corresponding steady-state solution of the more familiar linear equation with constant diffusion coefficient (here we are discussing linear diffusion without reference to the behavior of an underlying particle system):

$$\frac{\partial \rho}{\partial t} = D \nabla^2 \rho, \quad [6]$$

which is illustrated in Fig. 2b for $D = 1$, $\alpha = 1$, and $N = 512$. The behavior of Eq. 6 is well known; the density profile is linear:

$$\rho_N(x) = \frac{\alpha N}{D}(1-x), \quad [7]$$

and the gradient is proportional to the flux. When the diffusion coefficient is constant, the increasing flux with increasing N must be compensated by an increasing gradient. The solution described by Eq. 5 for the singular-diffusion equation is quite different: across most of the system the steady-state density scales like

$$\rho_c - \rho_N(x) \sim \frac{1}{N^{1/(\phi-1)}}, \quad [8]$$

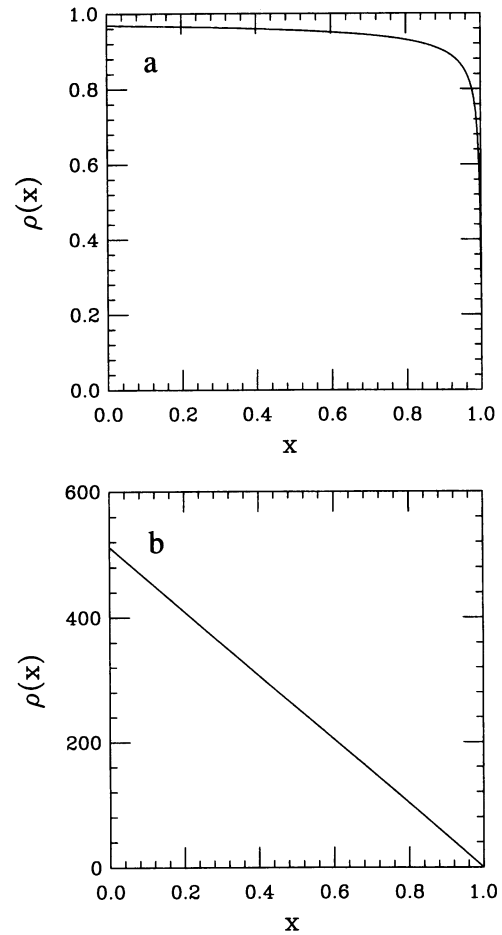


FIG. 2. Steady-state solutions to the singular diffusion equation (a) and the conventional diffusion equation (b) are compared for boundary conditions that scale in a manner characteristic of SOC systems. In this case, density is fixed at $x = 1$, and flux scales with system size N . As N increases in a, the profile becomes increasingly flat with the density approaching ρ_c , whereas in b the gradient is proportional to N .

and an increase in the diffusion coefficient accommodates the increasing flux. In fact, in Eq. 5 the gradient decreases as N increases. As the density is pushed closer to the singularity, the gradient is limited by the fact that the density cannot exceed ρ_c . The only significant gradient occurs in a boundary layer at the right edge in which the density falls sharply to zero to satisfy the boundary condition *i*.

For most SOC systems it is difficult to extract an analytical expression for the boundary conditions associated with the external driving mechanism. For example, in the BTW model the flux is input across the entire system each time a grain is added. In the LL model the conserved quantity enters and leaves the system at the same edge, while each interior addition of sand causes a net drift of the slope to the left. However, even in these more complicated cases, the steady-state profiles bear a striking resemblance to Fig. 2a. Furthermore, even without detailed boundary conditions it is possible to predict the rate at which the steady-state density converges to the singularity. For this analysis we assume that some of the boundaries are open, so that transport to a $d - 1$ dimensional edge results in dissipation. Our results are then obtained by relating the time scales for addition and transport of the conserved quantity.

It is useful to have a mechanism by which one can incrementally increase the rate of addition (flux) of the conserved quantity from the closed-system value of zero to something more interesting. This can be done for the general automaton on a d -dimensional integer lattice of linear dimension N as follows. In each time step there are two possibilities: (i)

addition events—a unit of the conserved quantity (such as height in BTW) is added to the system, possibly inducing an avalanche, and (ii) stirring events—a unit of the conserved quantity is exchanged between randomly selected nearest-neighbor pairs of sites, possibly inducing an avalanche. During a stirring event, the conserved quantity remains unchanged as long as the avalanche does not intersect the open boundaries. To prescribe the probability of addition events relative to stirring events we introduce an exponent d_A (for “dimension of addition”) so that (i) with probability N^{d_A}/N^d an addition occurs; and (ii) with probability $1 - N^{d_A}/N^d$ a stirring event occurs. In the original BTW model all events were induced by addition rules so $d_A = d = 2$. However, the BTW rule could be modified by adding sand only a fraction, say $1/N$, of the times at which avalanches are initiated and stirring otherwise. In that case we would have $d_A = 1$. In the LL model only additions of sand at the left-most site $i = 1$ changed the amount of conserved quantity, so that $d_A = 0$, although we can modify the rules in this case as well, so that the value of d_A can be varied.

Now assuming that the singular-diffusion description is valid, we again let $\rho_N(x)$ denote the stationary density on a system of size N (rescaled into a unit hypercube), but now the goal is simply to calculate the convergence rate b of the spatially averaged steady-state density:

$$\rho_c - \langle \rho_N \rangle \sim \frac{1}{N^b}, \quad [9]$$

which is the generalization of Eq. 8. We determine the exponent b by balancing the time scale associated with the driving (addition) mechanism with the time scale associated with diffusive transport.

First, the time scale of the diffusion mechanism is:

$$\tau_D \sim N^{2-b\phi} \quad [10]$$

which is obtained from the singular diffusion by substituting Eq. 9 and $x \rightarrow Nx$ into Eqs. 2 and 3. To ascertain the time scale associated with addition events τ_A we begin by recalling that on the closed system consisting of N^d sites, each unit of conserved quantity contributes mass N^{-d} in the hydrodynamic limit. Recalling Eq. 9, we focus on changes in the density of order N^{-b} , so that the contribution of a unit of slope is $N^b/N^d = N^{b-d}$. The rescaled addition rate is, therefore, N^{b-d+d_A} , and the reciprocal yields the scaling for the addition time scale:

$$\tau_A \sim N^{d-b-d_A}. \quad [11]$$

If τ_D and τ_A scaled differently as the system size diverges, one of the mechanisms would overwhelm the other, and we would have a contradiction. Therefore, equating the exponents of the two times scales yields a value for b , the rate at which $\langle \rho_N \rangle \rightarrow \rho_c$:

$$b = \frac{2 - d + d_A}{\phi - 1}. \quad [12]$$

This central result establishes the rate of convergence of the density to the critical value ρ_c as a function of system size in terms of the order of the diffusion singularity, the dimension of the system, and the driving rate. Note that b increases with driving rate d_A and decreases with the order of the singularity ϕ . The latter fact is simply a statement that if $D(\rho)$ increases more rapidly as $\rho \rightarrow \rho_c$, then the density need not increase as quickly to accommodate the flux. Furthermore, this general result is consistent with that obtained above for the specific boundary conditions leading to Eq. 5 (where $d_A = 0$ and $d = 1$) and is consistent with simulations when the diffusion limit is valid (25).

Finally, in situations where one is fortunate enough to know details of the equilibria at all densities, the scaling of avalanche distributions can be calculated. When the boundary conditions are known precisely, it may be possible to calculate the complete distribution: the probability of an avalanche of size k in a system of size N is computed by estimating the transition probabilities in terms of the local densities $\rho_N(x)$ of each of the sites. In those cases where the analysis has been done, the distribution satisfies finite size scaling (see, for example, ref. 10). When the detailed form of $\rho_N(x)$ is not known, the validity of the diffusion limit implies that the event-size correlation length scales like $\xi(\langle \rho_N \rangle)$, consistent with the value of ξ on the closed system at a density given by the average steady-state density of the open system. Deviations from this scaling are a key indication of the breakdown of the diffusion limit.

Fluctuations and the Failure of Hydrodynamics

When we assume the validity of the singular-diffusion equation for the open system, there are two points at which this assumption can break down. First, the validity of the equation rests on local equilibrium. That is, the open driven system must locally resemble the closed system at the same density in the limit of large system size, so that at each location x the transition probabilities and correlation functions on the open system must be the same as on the closed system at a density given by the local density $\rho(x)$. Furthermore, because the range of interaction (avalanche size) is increasing with system size, the relevant condition for local equilibrium is not restricted to finite range-correlation functions as it would be for nearest-neighbor exchange-type interactions; instead it is necessary to verify that the open- and closed-system correlations agree over a range that grows with system size. We have numerically observed the apparent failure of local equilibrium in the LL model (25). The second source of failure of singular-diffusion hydrodynamics is associated with fluctuations. In particular, because the density is approaching the critical value as $N \rightarrow \infty$, fluctuations may cause the local density to exceed ρ_c at times, at which point transport becomes nonlocal. In this regime a diffusion operator, even a singular one, cannot possibly describe the transport. This type of breakdown is observed in the original BTW model, and we will focus our attention on this case in more detail below.

Before embarking on a discussion of fluctuations in self-organizing systems, we begin with a brief initial remark on another scenario in which fluctuations in systems with a conservation law have been studied—namely, generic scale invariance (GSI) (33–35). The theory of GSI identifies an origin for power law decay of spatial-correlation functions for systems with conservation laws. A starting point of GSI, based upon certain symmetries and conditioning on local transport in a system, is that fluctuations can be described by driven nonlinear diffusion equations; for example,

$$\frac{\partial h}{\partial t} = c_1 \nabla^2 \parallel h + c_2 \nabla^2 \perp h - \lambda (\nabla \parallel h)^2 + \eta(x, t), \quad [13]$$

where $\eta(x, t)$ is a noise term that satisfies the conservation law, and \parallel and \perp denote orthogonal directions. In this equation the nonlinearity is the first in a series of terms arising in a gradient expansion. Dynamic renormalization group techniques lead to exponents describing the scaling of various quantities.

The scaling laws associated with GSI originate from the fact that systems with conservation laws are, in a certain sense, always critical—that is, the presence of a conservation law induces slow relaxation of perturbations. When such a system is subjected to a driving mechanism so that the system must transport the conserved quantity from one boundary to another, the resulting anisotropy coupled with the slow relaxation results in power law decay of spatial correlations. The primary

achievement of GSI is a clear description of how the presence of a conservation law, anisotropies, and nonlinearities can result in nontrivial scaling of spatial correlations.

Initially, GSI was proposed as a possible characterization of the fluctuations in self-organizing systems. However, in GSI there is no reference to distributions of avalanche sizes or of the scaling of the density of the conserved quantity with system size. The focus is entirely upon the height–height correlation functions, and even systems with purely nearest-neighbor interactions (events) can exhibit this phenomenon. Ultimately it was found that this method does not adequately describe the SOC models (36).

The role of fluctuations in SOC is very different from that in Eq. 13 due to the singularities and system-size dependence of the density. The singular diffusion equation is derived in the $N \rightarrow \infty$ limit of the closed system, where the density is fixed. However, for any finite system there are fluctuations about the hydrodynamic limit that arise simply because the local density (Eq. 4) is a finite sum of random variables (e.g., heights at each site). If these fluctuations h_N are small, then in $d = 1$ they should satisfy the linearized equation

$$\frac{\partial h_N}{\partial t} = D(\rho_N) \frac{\partial^2 h_N}{\partial x^2} + \frac{\partial}{\partial x} [(2\chi(\rho_N)D(\rho_N))^{1/2} \eta(x,t)], \quad [14]$$

which is obtained by letting $\rho(x,t) = \rho_N(x) + h_N(x,t)$ in Eq. 2, where $\rho_N(x)$ is the steady-state solution analogous to Eq. 5, and keeping terms to leading order in $h_N(x,t)$. Here $\eta(x,t)$ is a space–time white noise, and the density-dependent coefficient of $\eta(x,t)$ involves $\chi(\rho_N)$, the susceptibility, which follows from the fluctuation-dissipation theorem.

In contrast to equations like Eq. 13, which have been studied in the context of GSI, here the diffusion coefficient depends explicitly on the steady-state density. On the closed system this dependence does not influence any scaling properties because the density is *a priori* bounded away from the singularity, and fluctuations are always higher order and vanish in the limit. In contrast, on the open system the density is converging to ρ_c as $N \rightarrow \infty$, so that the divergence of $D(\rho_N)$ in the limit will clearly affect the scaling of correlations—a possibility that is absent in GSI. Most importantly, fluctuations are potentially dangerous to the validity of the diffusion limit or any other local description, so that it may be impossible to truncate the gradient expansion to obtain an equation analogous to Eq. 14. In particular, if the convergence rate to ρ_c is too large (i.e., if the driving rate d_A is too large), then fluctuations can push the density over the singularity (where the characteristic avalanche size diverges), and local operators such as those in the diffusion description could not describe the system. Thus analysis of the magnitude of the fluctuations gives us a criterion for hydrodynamic failure.

As described in the previous section, the density of the system converges to the critical value as $\rho_c - \langle \rho_N \rangle \sim N^{-b}$, where b is given by Eq. 12, provided that the singular-diffusion equation is valid. In the simplest cases the fluctuations in the empirical density are central limit, scaling as $N^{-d/2}$. Noncentral limit fluctuations result in a modification of the exponent $d/2$ and occur when the sandpile models sustain an additional phase transition at ρ_c associated with the divergence of a height–height or slope–slope correlation length (25). Note that divergence of the avalanche-size correlation length $\xi(\rho)$ does not imply nontrivial behavior of correlation functions of the underlying height or slope variables. The BTW model is an example of a system in which $\xi(\rho)$ diverges, whereas the height–height correlation length remains finite, and the fluctuations are thus central limit. In contrast, the LL model exhibits both diverging avalanche sizes and diverging slope–slope correlations when $\rho = \rho_c$.

To determine the condition on the driving rate d_A , which sets the crossover between the success and failure of the

diffusion limit, we compare the rate at which diffusion predicts that the density converges to the singularity with the rate at which fluctuations decay. For simplicity, we focus on the central limit case, in which fluctuations preclude the singular-diffusion description if $b \geq d/2$; recalling Eq. 12, this becomes

$$d_A \geq d \left[\frac{\phi - 1}{2} + 1 \right] - 2. \quad [15]$$

This condition can also be established directly from the derivation of Eq. 14. Equality in Eq. 15 coincides with the point at which *all higher order terms in the gradient expansion become relevant* and of equal order. In other words, when d_A is small, singular diffusions describe the system, and fluctuations are irrelevant. When d_A is larger than the bound in Eq. 15, hydrodynamics fails, and fluctuations are dominant. Of course, because transport (avalanches) is nonlocal in this regime, the system is not characterized by anything like the local gradient expansion given in Eq. 13, which is an intrinsic aspect of GSI.

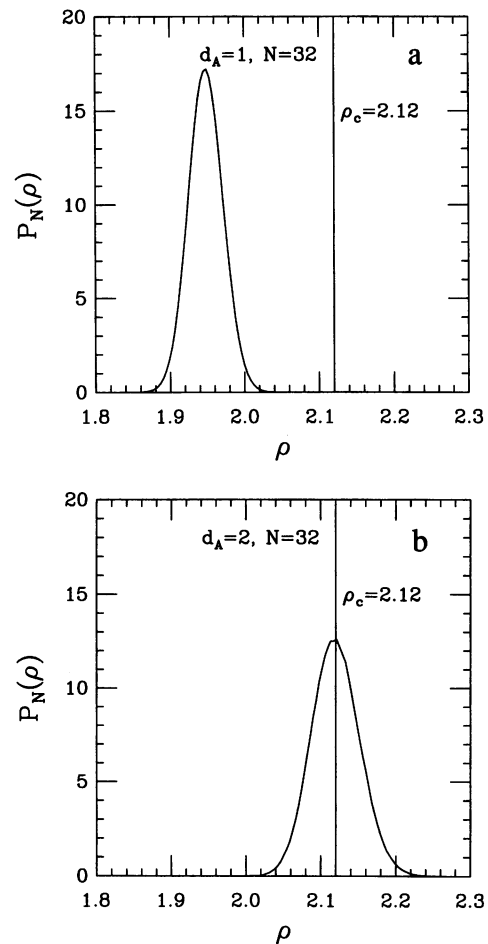


FIG. 3. Distribution of densities taken from the middle quarter of the system (where the density is most spatially uniform) in the BTW model when hydrodynamics holds ($d_A = 1$) (a) and when hydrodynamics fails ($d_A = 2$; the usual case) (b). The vertical line at $\rho = 2.12$ represents the diffusion singularity. In each case, the density distribution is Gaussian; however, in a the mean density converges to the singularity more slowly than the width converges to zero, whereas b corresponds to the opposite case. In both cases, as $N \rightarrow \infty$, the density converges to a point mass at ρ_c : $P(\rho) \rightarrow \delta(\rho_c - \rho_N)$. However, the relative rates of convergence of the mean vs. the width determine whether or not the diffusion limit holds. To simulate the case $d_A = 1$, after each avalanche is complete, we initiate the next event by randomly choosing a site, adding a grain with probability $1/N$, and stirring (i.e., moving a grain to a randomly selected nearest-neighbor site) with probability $1 - 1/N$.

Applying the failure condition to the BTW model (central limit fluctuations and diffusion singularity $\phi = 2.3$), we see that Eq. 15 implies that fluctuations can be ignored, and diffusion holds when $d_A < 1.3$, and fluctuations dominate when $d_A \geq 1.3$, which is consistent with numerical simulations in ref. 25. The two cases are illustrated in Fig. 3. Each plot illustrates the distribution of the density in a finite box, in which the mean density is converging to the singularity at ρ_c as $N \rightarrow \infty$. However, Fig. 3a shows a case ($d_A = 1$) in which the convergence to the singularity is slower than the decay in the size of the fluctuations, whereas Fig. 3b shows a situation ($d_A = 2$, which corresponds to the original model) in which fluctuations decay more slowly than the gap to the singularity.

Finally, there is an interesting contrast between the avalanche size distributions in the regime where the singular-diffusion description applies and where it fails. The two cases are illustrated for the BTW model in Fig. 1. When hydrodynamics holds as in Fig. 1c, we can collapse the distributions for various system sizes N with respect to $\rho_c - \langle \rho_N \rangle$ and obtain an estimate of the event-size correlation-length exponent ν in $\xi(\langle \rho_N \rangle) \sim (\rho_c - \langle \rho_N \rangle)^{-\nu}$, which is consistent with the value of ν obtained for the closed system (Fig. 1b). Note that this implies that as a function of system size $\xi(N) \sim 1/N^{b\nu}$, where b is given in Eq. 12, which diverges sublinearly with N . In other words, relative to the size of the system the largest events are becoming local in the limit. In contrast, when hydrodynamics fails as in Fig. 1a, we can no longer collapse the plots using the distance from the critical point as a scaling variable. In fact, for some of the system sizes the mean density is above the singularity.** However, in this case we can still collapse the data by scaling the axes by the system size, and we find that the characteristic length scale in the event-size distribution scales linearly with the system size N . Because diffusion operators are intrinsically local, it is clear that in this regime the diffusion limit must fail. It remains an open and intriguing question to determine what replaces the diffusion description in this case.

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**If the mean density is computed in the middle portion of the BTW model, then with the conventional driving the density converges to the singularity from above. Other models, like the unlimited models discussed in ref. 13, approach the singularity from above because the rules yield a constraint that once the density fluctuates above the singularity it never again will fall below.

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