# Solving polynomial equations



• $\sqrt{3} = ?$


$x = \sqrt{3}$

This is not a genuine solution, because $\sqrt{3}$ is a suggestive symbol for the quantity whose square is 3, for which we still do not know what the quantity is.

In another viewpoint: $x^2 = 3$, $\underbrace{x^2 - 3}_{\text{a function of } x, \text{ denoted by } f(x).} = 0 \quad \cdots\cdots (*)$

We thus want to find some $x$ such that $f(x) = 0$.

| $x$ | $0$ | $1$ | $1\frac{1}{2}$ | $2$ |
|---|---|---|---|---|
| $f(x) = x^2 - 3$ | $-3$ | $-2$ | $-\frac{3}{4}$ | $1$ |

when $x$ varies continuously from $1\frac{1}{2}$ to 2, the function $x^2 - 3$ should vary continuously from $-\frac{3}{4}$ to 1, during which 0 must be passed through.

$\therefore x = 2$ is an approximate solution with error $< 0.5$.

That is to say, $x = 2$ makes $f(x)$ a small quantity (i.e. close to 0).

Since our target is to make $f(x)$ vanish, we could try to make $f(x)$ smaller by adjusting the value $x = 2$ a little:

$$f(2+\delta) = f(2) + \delta \cdot f'(2) + \underbrace{\cdots}_{\delta^2,\, \delta^3,\, \cdots \text{etc.}} \qquad (f'(x) = 2x)$$

$$= \underbrace{1 + \delta \cdot 4}_{} + \cdots$$

~ signifies "as small as"

if $\delta$ is such that "$\underbrace{\quad}$" $= 0$ (i.e. $\delta = -\frac{1}{4}$), then $\underbrace{f(2+\delta)}_{2-\frac{1}{4}} \sim \delta^2$.

Thus $f(2-\frac{1}{4})$ is as small as $(\frac{1}{4})^2$.

We could adjust the value of $x$ again to make $f(x)$ even [redacted] smaller.

$$f(2-\tfrac{1}{4}+\delta) = f(2-\tfrac{1}{4}) + \delta \cdot f'(2-\tfrac{1}{4}) + \cdots$$

$$= \underbrace{\frac{1}{16} + \delta \cdot \frac{7}{2}}_{} + \cdots$$

if $\delta$ is such that "$\underbrace{\quad}$" $= 0$ (i.e. $\delta = -\frac{1}{56}$), then $\underbrace{f(2-\frac{1}{4}+\delta)}_{2-\frac{1}{4}-\frac{1}{56}} \sim \delta^3$

$$f(2-\tfrac{1}{4}-\tfrac{1}{56}+\delta) = f(2-\tfrac{1}{4}-\tfrac{1}{56}) + \delta \cdot f'(2-\tfrac{1}{4}-\tfrac{1}{56}) + \cdots$$

$$\vdots$$

We could proceed as much as we wish.

Thus we have:

| | | | | | | |
|---|---|---|---|---|---|---|
| | 2 | 1.75 | 1.7321428... | ... | tends to | limit object |
| $x$ | 2 | $2-\frac{1}{4}$ | $2-\frac{1}{4}-\frac{1}{56}$ | ... | $\longrightarrow$ | $1.732\cdots$ |
| $f(x) = x^2-3$ | $f(2)$ | $f(2-\frac{1}{4})$ | $f(2-\frac{1}{4}-\frac{1}{56})$ | ... | $\longrightarrow$ | $f(1.732\cdots)$ <br> $\parallel$ <br> $0$ |
| | smaller and smaller, and | | | | $\longrightarrow$ | |

<u>Table.1</u>

That is to say, a solution of the equation $(*)$ is $x = 1.732\cdots$ (i.e. the limit object of the above number sequence) where the dots could be written down explicitly as many as we wish, by the above process.

The idea of the above method is:

find $x$ such that $f(x)=0$ ?

first, find $a$ such that $f(a)$ is small (by estimating)

adjust $a$ to $a+\delta_1$ such that $f(a+\delta_1)$ is smaller (find some $\delta_1$)

adjust $a+\delta_1$ to $a+\delta_1+\delta_2$ such that $f(a+\delta_1+\delta_2)$ is smaller and smaller

$\vdots$

Then the limit object $a+\delta_1+\delta_2+\cdots$ is the solution of $f(x)=0$ (See <u>Table.1</u>).

- More general cases.

The above method is universal: any equation could be written as $f(x)=0$ (where $f$ is a function of $x$), and then the above process could be applied.

The above method is even valid for the equations with un-determined coefficients. For example:

[redacted]

$$0 = f(x) = x^3 + ax^2 + a^2x - 1 \quad \text{where } a \text{ is an un-determined const.}$$

when $a=0$, we have $f(1)=0$. (when $a=0$, the equation $f(x)=0$ becomes an ordinary equation with definite constant coefficients, and has a solution $x=1$)

when $a$ is a tiny variable, we have $f(1)\sim a$ $(\dagger)$

By adjusting 1 to $1+\delta$, we hope to make $f(1+\delta)$ smaller:

$(\dagger)$: $f(x) =$ [monomials containing $a$] + [others]

when $a=0$: $x=1$ makes [others] vanish

So $f(1) =$ [monomials containing $a$]$\big|_{x=1}$

$$f(1+\delta) = f(1) + \delta\cdot f'(1) + \cdots \qquad (f'(x) = 3x^2 + a\cdot 2x + a^2)$$

$$= (a+\cdots) + \delta\cdot(3+\cdots) + \cdots$$

$$= \underbrace{a + \delta\cdot 3} + \cdots \to a^2,\ \delta\cdot a,\ \delta^2 \text{ and higher order terms}$$

if $\delta$ is such that "$\smile$" $=0$ (i.e. $\delta = -\frac{a}{3}$), then $\delta\sim a$

and $\underset{1-\frac{a}{3}}{f(1+\delta)} \sim a^2$

$$f(1-\tfrac{a}{3}+\delta) \doteq f(1-\tfrac{a}{3})+\delta\cdot f'(1-\tfrac{a}{3})+\cdots$$

$$\overset{(\ddagger)}{=} \underbrace{\tfrac{2}{3}a^2+\delta\cdot 3}+\cdots \rightarrow a^3,\ \delta\cdot a,\ \delta^2, \text{ and higher order terms}$$

if $\delta$ is such that "$\sim\sim$" $=0$ (i.e. $\delta=-\frac{2}{9}a^2$), then $\delta\sim a^2$ and $f(1-\frac{a}{3}+\delta)\sim a^3$

($\ddagger$): since $f(1-\frac{a}{3})\sim a^2$, the coeff. of the terms 1, $a$ in $f(1-\frac{a}{3})$ must be 0. Thus we only have to compute the $\sim a^2$ terms

$$1-\tfrac{a}{3}-\tfrac{2}{9}a^2$$

$$f(1-\tfrac{a}{3}-\tfrac{2}{9}a^2+\delta) = f(1-\tfrac{a}{3}-\tfrac{2}{9}a^2)+\delta\cdot f'(1-\tfrac{a}{3}-\tfrac{2}{9}a^2)+\cdots$$

$$\vdots$$

Thus the limit object $1-\frac{a}{3}-\frac{2}{9}a^2-\cdots$ is the solution for $f(x)=0$.

(Similar to Table.1)

Notice that the un-determined coeff. $a$ could vary freely and continuously, We thus actually achieved expressing $x$ explicitly as a function of $y$, under the polynomial restriction $x^3+yx^2+y^2x-1=0$. The solution is an infinite polynomial (i.e. a series) function $x=1-\frac{y}{3}-\frac{2}{9}y^2-\cdots$ which is not likely to be one of the known quantities (power, exponential, logarithm, triangular, etc.).

In fact, for each polynomial equation $f(x,y)=0$, the solution series is always a new function which is not known to us before, and thus deserves a new symbol (like sin, cos, log, $\xi$, $\cdots$) if it is frequently encountered.

## Appendix I. Convergence.

The sequence of quantities $1$, $1-\frac{a}{3}$, $1-\frac{a}{3}-\frac{2}{9}a^2$, $\cdots$ makes $x^3+ax^2+a^2x-1$ as small as $a$, $a^2$, $a^3$, $\cdots$ respectively.

Thus, under the following two restrictions of $a$, the statement that: "$x=1-\frac{a}{3}-\frac{2}{9}a^2-\cdots$ is the solution of the equation $x^3+ax^2+a^2x-1=0$" is both meaningful and true:

(1). $a$, $a^2$, $a^3$, $\cdots$ tends to vanish, i.e. $|a|<1$;

(2). $a$ is such that the above sequence of quantities tend to some definite quantity (i.e. the series $1-\frac{a}{3}-\frac{2}{9}a^2-\cdots$ converges).

This is why people are always interested in the convergence of a series —— it clarifies the range in which a solution formula (for some equation) is meaningful.