Symmetries of a plane figure

I. How to describe symmetries

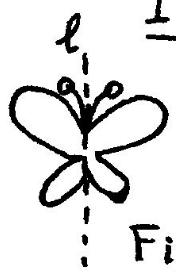


Fig.1

A butterfly is symmetric (Fig.1) w.r.t. the axis ℓ. A clearer description is that the two halves of the butterfly coincide with each other when [redacted] we fold this paper along the line ℓ, i.e. Fig.1 is un-changed under the operation of folding along ℓ.

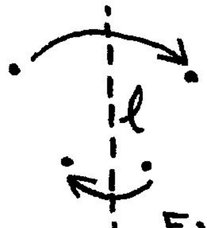


Fig.2

The operation of folding along ℓ sends every point on the plane to its destination point (Fig.2)

Hereafter, a dashed line represents the (operation of) folding along it (Fig.2).

Following this idea, a square is symmetric, because it is un-changed under each of the foldings shown in Fig.3.

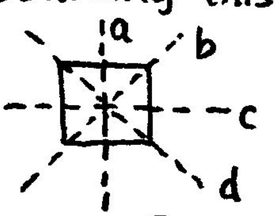


Fig.3

The square is un-changed under the individual foldings a and b, thus it is also un-changed under the whole operation of first performing a, then followed by performing b, which is called the composition of a and b, denoted by ba.
(2nd 1st)

In fact, the square is un-changed not only under the foldings a, b, c, d in Fig.3, but also under any composition of them (composed in any manner). All these operations are directly called the symmetries of the square. The collection of all the symmetries of the square is called the symmetry group of the square.

The symmetry group of a figure completely characterizes the symmetric properties (in the usual sense) of it: figures with the same symmetry group are of the same symmetry pattern, although they may look very different (Fig.4).

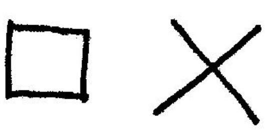


Fig.4 ⟶ figures with the same symmetry pattern.

II. Symmetries on a plane

Symmetry group of a square $= \{a, b, c, d$ and their compositions$\}$ (Fig.3)
So what operations are in the group exactly?

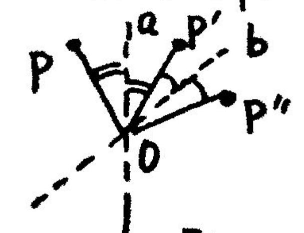


Fig.5

First, consider the composition of a and b (Fig.5).
By Fig.5, we see that the composition of a and b sends P to P'', where:

angle from OP to OP'' = (angle from a to b) $\times 2$.

Thus the composition ba with intersection O is a rotation of the whole plane around O, by twice the angle from a to b.

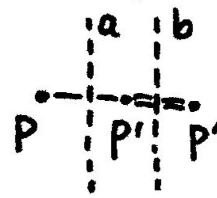


Fig.6

Similarly, the composition ba (with parallel axis) is a translation along a—b direction, by twice the displacement from a to b (Fig.6).

Notice that in Fig.7, the composition ba and $b'a'$ represent the same operation —— a rotation around O, by $2 \times \alpha$. That is to say, rotating a pair of intersecting dashed lines (by an arbitrary angle around their intersection point) will not change their composition.

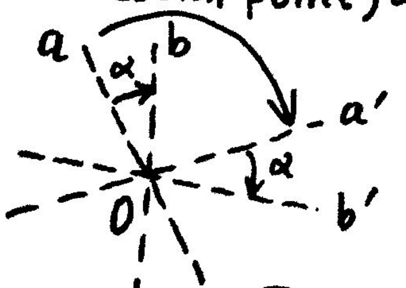


Fig.7

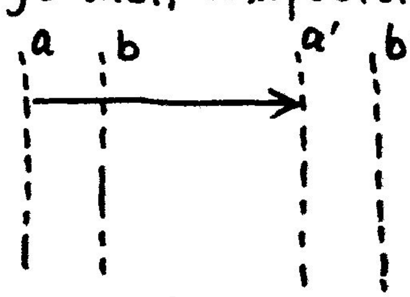


Fig.8

Similarly (Fig.8): translating a pair of parallel dashed lines (by an arbitrary distance along a—b direction) will not change their composition.

Keeping these two rules in mind, we could find out all the compositions of foldings on a plane —— they are of one of the following four types: folding, rotation, translation, or glide (Fig.9).

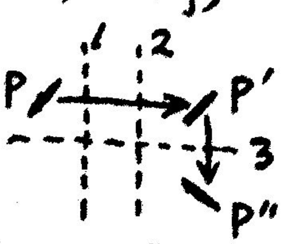
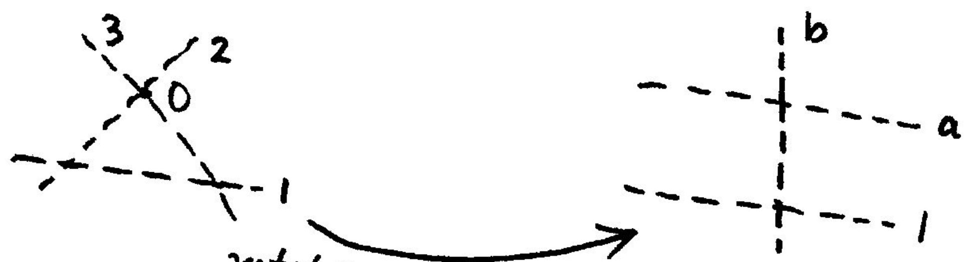


Fig.9

the composition 321 is a glide.

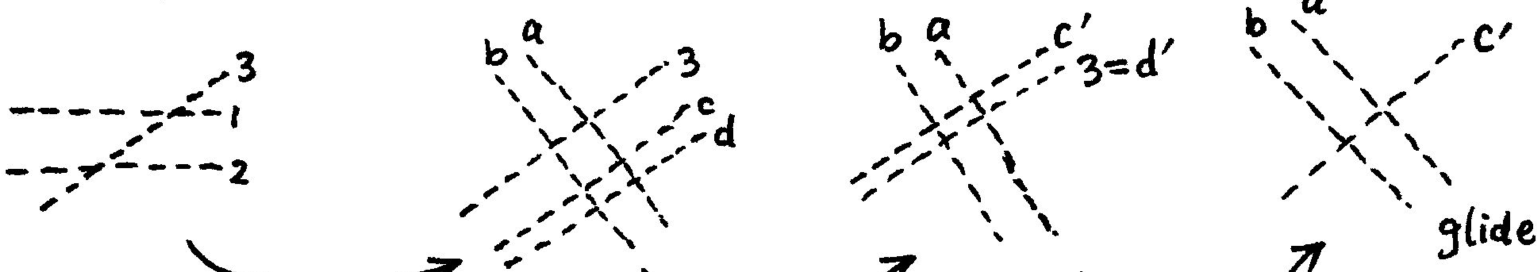
For example:

$321 \quad = \quad ba1$



rotate 32 to ba, such that a // 1

$321 \quad = \quad 3(dc)(ba)$ [redacted] $\quad = \quad 3(d'c')(ba) \quad = \quad c'ba$



decompose the translation 21 $\begin{cases} // 3 \\ \perp 3 \end{cases}$

translate dc to d'c' such that d' coincide with 3

3d' = twice a folding = identity operation