Dilatation and vorticity of a fluid flow

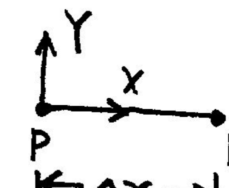
Abstract: we explore the concept of dilatation and vorticity in the fluid flow on a plane and in space.

- <u>Fluid flow on a plane</u>

Let fluid flow on a plane. Let a Cartesian coordinate system xy be set. For a fixed time t: let $\vec{V} = \binom{X}{Y}$ denotes the velocity vector at a general point $P\binom{x}{y}$, where X, Y are functions of $\binom{x}{y}$.

•• <u>The dilatation</u>

Consider a tiny segment PP' on x-direction (<u>Fig.1</u>)

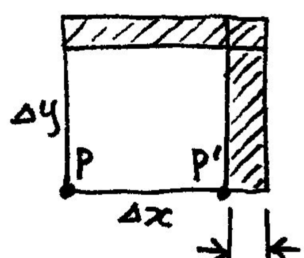


<u>Fig.1</u>

Since the difference of x-velocity: $X_{P'} - X_P = \partial_x X \cdot \Delta x$ (where $\partial_x X = \frac{\partial X}{\partial x}$), the stretched length of the segment PP' when time Δt elapsed is:

$$\partial_x X \cdot \Delta x \cdot \Delta t$$

The same holds for y-direction.
Thus $(\partial_x X \cdot \Delta x \cdot \Delta y + \partial_y Y \cdot \Delta y \cdot \Delta x) \cdot \Delta t$ is the increased area of the rectangular fluid element (<u>Fig.2</u>)



<u>Fig.2</u>

Its coefficient $\partial_x X + \partial_y Y = \begin{pmatrix}\partial_x \\ \partial_y\end{pmatrix} \cdot \begin{pmatrix}X \\ Y\end{pmatrix} = \vec{\nabla} \cdot \vec{V}$

$=$ the increase factor of the area of a rectangular fluid element at $P\binom{x}{y}$;

called the <u>dilatation</u> of the flow at the point $P\binom{x}{y}$.

The vorticity.

Consider the difference of velocities of P and P′ on the other direction —— the y-direction (Fig.3): $Y_{P'} - Y_P = \partial_x Y \cdot \Delta x$.

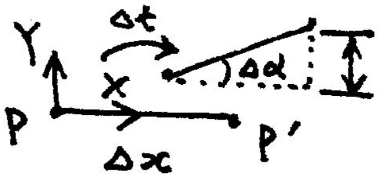


Fig 3

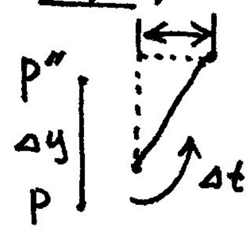


Fig.4

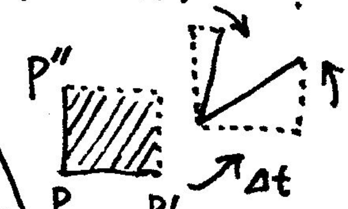


Fig.5

so, when time Δt elapsed, the segment PP′ inclines: the point P′ would be higher than P by $\partial_x Y \cdot \Delta x \Delta t$ (Fig.3)

Its coefficient
$$\begin{aligned}\partial_x Y &= \partial_x Y \cdot \Delta x \cdot \Delta t \div \Delta x \div \Delta t \\ &= \tan \Delta\alpha \div \Delta t \\ &= \Delta\alpha \div \Delta t \\ &= \text{the angular velocity of the rotation of PP}'\end{aligned}$$
(reference sense $y \uparrow\ \circlearrowleft^{+}\ \rightarrow x$)

For the y-direction (Fig.4): $X_{P''} - X_P = \partial_y X \cdot \Delta y$

The segment PP″ inclines: the point P″ would be ahead of P on x-direction by $\partial_y X \cdot \Delta y \cdot \Delta t$

(when X_P is a positive quantity, the velocity X_P on x-axis has a positive sense $\xrightarrow[X_P]{}\overset{+}{\rightarrow}$ x-axis. The same holds for $X_{P''}$, $\partial_y X \cdot \Delta y$.)

Its coefficient $\partial_y X$ = the angular velocity of the rotation of PP″
(reference sense $y \uparrow\ \circlearrowright^{+}\ \rightarrow x$)

Thus $\partial_x Y - \partial_y X = \begin{vmatrix} \partial_x & X \\ \partial_y & Y \end{vmatrix}$ $\boxed{\text{See Fig.5}}$

= the sum of angular velocities of the rotations of PP′ and PP″;
(reference sense $y \uparrow\ \circlearrowleft^{+}\ \rightarrow x$)

= twice the average angular velocity of the rotations of PP′ and PP″;

= twice the angular velocity of the rotation of the fluid element at $P\binom{x}{y}$;

called the vorticity of the flow at $P\binom{x}{y}$.

• Fluid flow in space

Let fluid flow in space. Let a Cartesian coordinate system xyz be set. For a fixed time t: let $\vec{V}=\begin{pmatrix}X\\Y\\Z\end{pmatrix}$ denotes the velocity vector of the flow at a general point $P\begin{pmatrix}x\\y\\z\end{pmatrix}$, where X, Y, Z are functions of $\begin{pmatrix}x\\y\\z\end{pmatrix}$.

·· The dilatation (Fig.6)

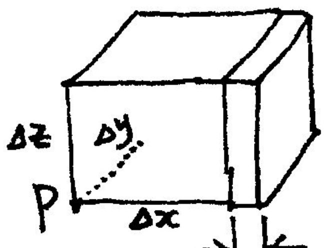


Fig.6

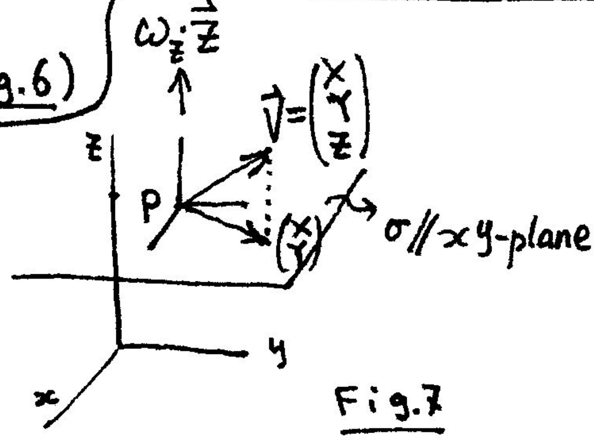


Fig.7

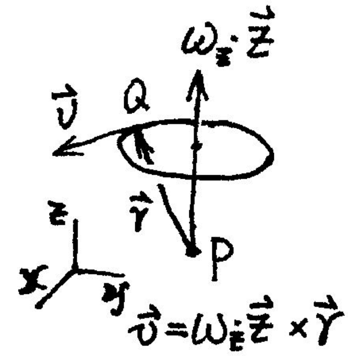


Fig.8

In Fig.6: $(\partial_x X\cdot\Delta x\,\Delta t)\Delta y\Delta z$ = increase volume in the direction of x-axis.

$$\partial_x X+\partial_y Y+\partial_z Z=\begin{pmatrix}\partial_x\\\partial_y\\\partial_z\end{pmatrix}\cdot\begin{pmatrix}X\\Y\\Z\end{pmatrix}=\vec{\nabla}\cdot\vec{V}$$

= the increase factor of the volume of a cuboid fluid element at $P\begin{pmatrix}x\\y\\z\end{pmatrix}$

= the dilatation of the flow at $P\begin{pmatrix}x\\y\\z\end{pmatrix}$

·· The vorticity

See Fig.7, let a tiny cuboid element fluid situates at $P\begin{pmatrix}x\\y\\z\end{pmatrix}$ (as its corner). The velocity field $\vec{V}$ has a shadow velocity field $\begin{pmatrix}X\\Y\end{pmatrix}$ on the plane σ, which induces a tendency of rotation at P on the plane, i.e. a tendency of rotation at P around $P\vec{z}$ (where $\vec{z}$ is the unit vector pointing to $+z$ axis), with angular velocity $\omega_z=\frac{1}{2}\begin{vmatrix}\partial_x & X\\\partial_y & Y\end{vmatrix}$.

The representation vector of this rotation is $\omega_z\vec{z}$ (See Appendix I).

Similar arguments apply to the direction of the other two coordinate planes.

Thus the point P is simultaneously affected by tendencies of three rotations, whose representation vectors are $\omega_x\vec{x}$, $\omega_y\vec{y}$, $\omega_z\vec{z}$ respectively. To determine the composed effect of these three rotations at P, we investigate the motion of a point Q in the neighbourhood of P, during a tiny interval of time Δt (Fig.8).

The total displacement of Q under the successive actions of these three rotations is (Fig.9):

$$(\omega_x \vec{x} \times \vec{r})\Delta t + (\omega_y \vec{y} \times \vec{r})\Delta t + (\omega_z \vec{z} \times \vec{r})\Delta t$$

$$= (\omega_x \vec{x} + \omega_y \vec{y} + \omega_z \vec{z}) \times \vec{r} \cdot \Delta t$$

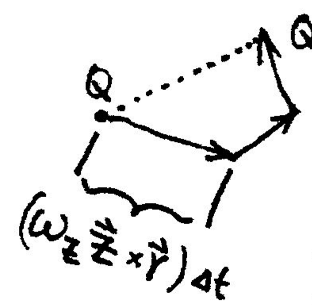


Fig.9

This shows that the composed instantaneous motion of the tendencies of the three rotations $\omega_x \vec{x}$, $\omega_y \vec{y}$, $\omega_z \vec{z}$ is also a rotation, whose representation vector is

$$\omega_x \vec{x} + \omega_y \vec{y} + \omega_z \vec{z} = \begin{pmatrix} \begin{vmatrix} \partial_y & Y \\ \partial_z & Z \end{vmatrix} \\ \begin{vmatrix} \partial_z & Z \\ \partial_x & X \end{vmatrix} \\ \begin{vmatrix} \partial_x & X \\ \partial_y & Y \end{vmatrix} \end{pmatrix} \cdot \frac{1}{2} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \cdot \frac{1}{2} = \vec{\nabla} \times \vec{V} \cdot \frac{1}{2}$$

Therefore, the vector

$$\vec{\nabla} \times \vec{V} = \begin{pmatrix} \begin{vmatrix} \partial_y & Y \\ \partial_z & Z \end{vmatrix} \\ \begin{vmatrix} \partial_z & Z \\ \partial_x & X \end{vmatrix} \\ \begin{vmatrix} \partial_x & X \\ \partial_y & Y \end{vmatrix} \end{pmatrix}$$

= the vector representation of this composed instantaneous rotation at P (with its length doubled), called the vorticity at P in the fluid flow $\vec{V}$, describes the instantaneous rotational motion at P completely: let a tiny cuboid be put at the point $P\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ (as its corner) (Fig.7), then under the fluid flow $\vec{V}$, it would rotate around the vector $\vec{\nabla} \times \vec{V}$ (right-hand rule), with twice the angular velocity being $|\vec{\nabla} \times \vec{V}|$ (i.e. the length of $\vec{\nabla} \times \vec{V}$).

(to be continued on the next page)

Appendix I. The representation of a rotation

Let an object rotates around an axis with angular velocity ω (Fig.10).

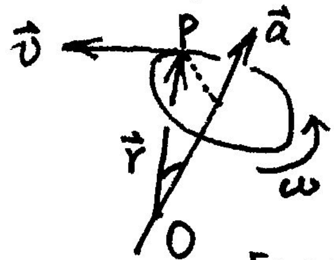


Fig.10

Let a unit vector $\vec{a}$ be on the axis and oriented through right-hand rule (Fig.11)

Let O be a choosen initial point, and $\vec{r} = \overrightarrow{OP}$ be the positioning vector of a point P.

Then the velocity vector of P under this rotation:

$$\vec{v} = \omega \vec{a} \times \vec{r}$$
$$= (\omega \vec{a}) \times \vec{r}.$$

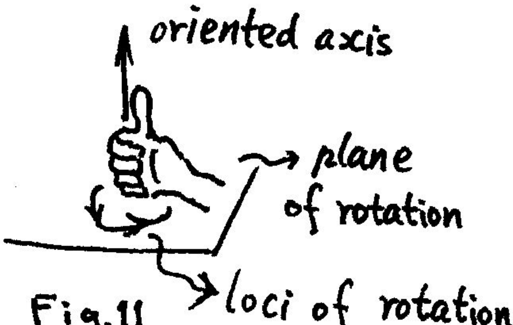


Fig.11

Thus the vector $\omega\vec{a}$ completely describes the rotational motion, and thus called the vector representation of this rotation:

the rotation is around the vector $\omega\vec{a}$ (through right-hand rule), with angular velocity being [redacted] its length ω; and to obtain the velocity vector $\vec{v}$ of any given point P, we only need to $\times \vec{r}$ (the positioning vector) to this vector.

Notice that we should distinguish this rotational [redacted] motion from the operation of a rotation, which is an operation of sending every point to its rotational destination (Fig.12).

Fig.12