Geometry of conic sections

Abstract: the [redacted] sundial inspires people to investigate the conic sections. We explore the geometric properties of conic sections in a coherent way.

- The sun-dial.

A sun-dial (Fig.1) tells the time and the date by the shadow of a pointer (called the gnomon), which points to the Polaris, on a horizontal plane (Fig.1), (called the [redacted] dial plate).

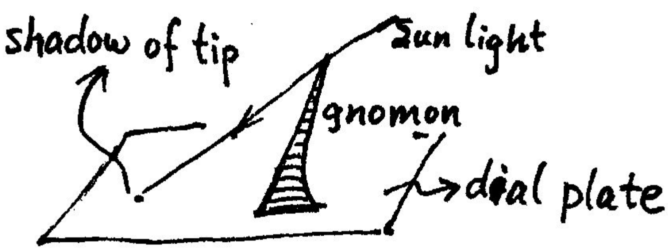


Fig.1

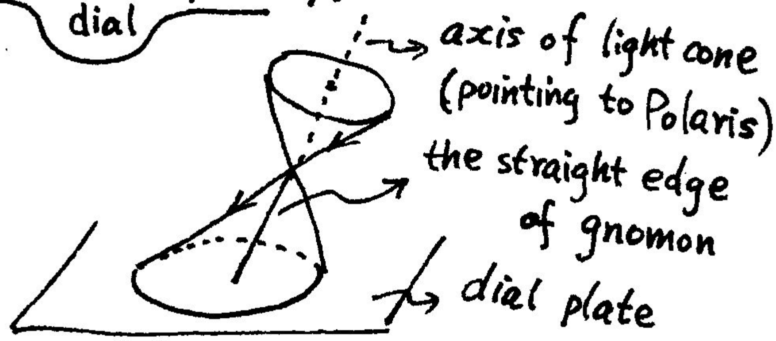


Fig.2

The loci of the shadow of the gnomon tip is a curve — the section of the upright circular cone of sun light, cut by the dial plate. (See Fig.2). See Appendix I for more details on the sun-dial.

This inspires us to investigate the conic sections (i.e. the sections obtained by cutting an upright circular cone with a plane).

- Conic sections and Dandelin spheres

Let a light source illuminate a sphere S, the light thus form an upright circular cone (Fig.3). Let a plane touch the sphere. Then the shadow of the sphere S on the plane is a conic section.

As we rotates the plane (while keeping touching the sphere S), three shapes of conic sections are found (Fig.3).

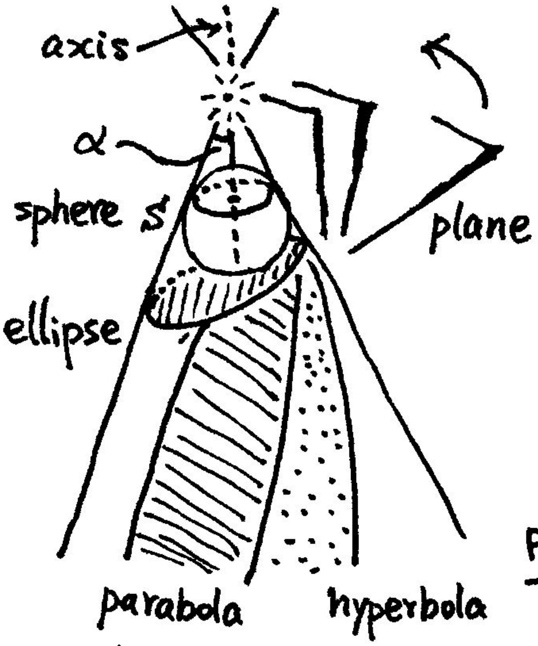


Fig.3

The properties of the three types of conic sections could be discovered by investigating the <u>Dandelin spheres</u>, i.e. the spheres which have the same characteristic as the given one in <u>Fig.3</u> —— simultaneously touching the light cone and the conic section.

Ellipse:

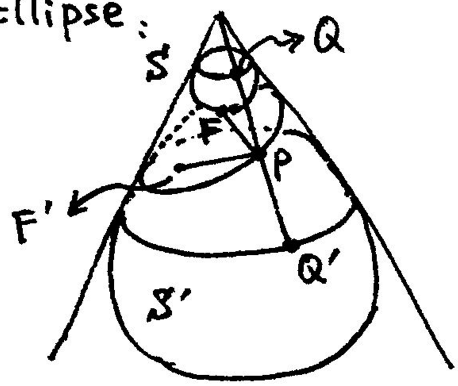


Fig.4

$\left.\begin{matrix} PF \\ PQ \end{matrix}\right\}$ both touching S

$$\left.\begin{matrix} PF & PF' & PF+PF' \\ =PQ & =PQ' & =PQ+PQ'=QQ' \text{ (a const.)} \end{matrix}\right\}(*)$$

This (*) holds for every point P on the section curve.

This gives us a description of the ellipse in terms of the line segments within the plane of the conic section.

This also leads to an apparatus for drawing ellipse (<u>Fig.6</u>).

Hyperbola:

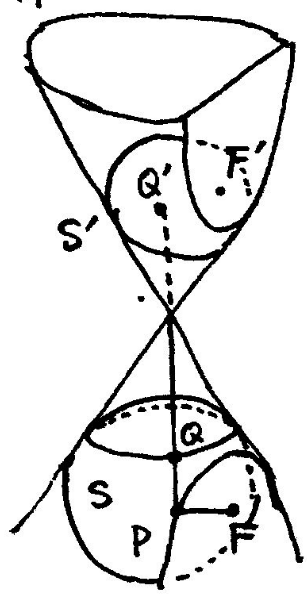


Fig.5

Similar to the right side situation of ellipse.

$$\begin{matrix} PF & PF' & PF'-PF \\ =PQ & =PQ' & =PQ'-PQ=QQ' \text{ (a const.)} \end{matrix}$$

apparatus for drawing hyperbola (See. <u>Fig.7</u>)

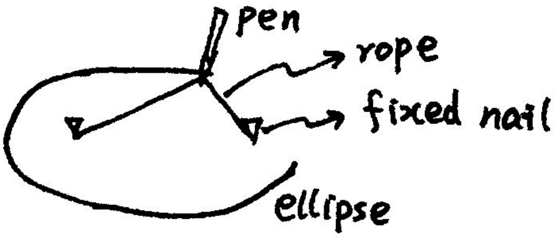


Fig.6

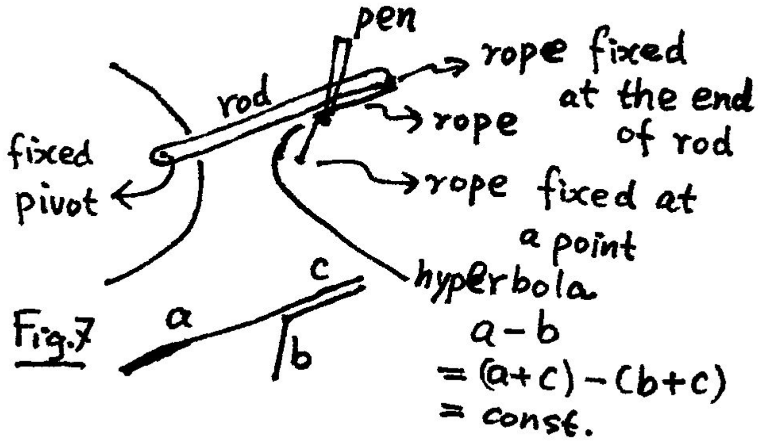


Fig.7

• The eccentricity.

In Fig.3, as the plane rotates upwards from the horizontal position, (while keeps touching the sphere S), the shape of the shadow conic section deforms from the circle and is determined by the relation between α and β, where

α = the angle of the edge of the cone to the axis;
β = the angle of the plane to the axis.

β	$90°$	α^+	α	α^-
shape	circle	ellipse	parabola	hyperbola

A more suitable and quantified description of this phenomenon is given by the eccentricity $\frac{\cos\beta}{\cos\alpha}$ (denoted by e):

$e=\frac{\cos\beta}{\cos\alpha}$	0	1^-	1	1^+
shape	circle	ellipse	parabola	hyperbola

The eccentricity describes quantitatively the deviation of a conic section from a circle.

We pick the ratio of cosines because this ratio could be written in terms of the line segments within the plane of the conic section (Fig.8)

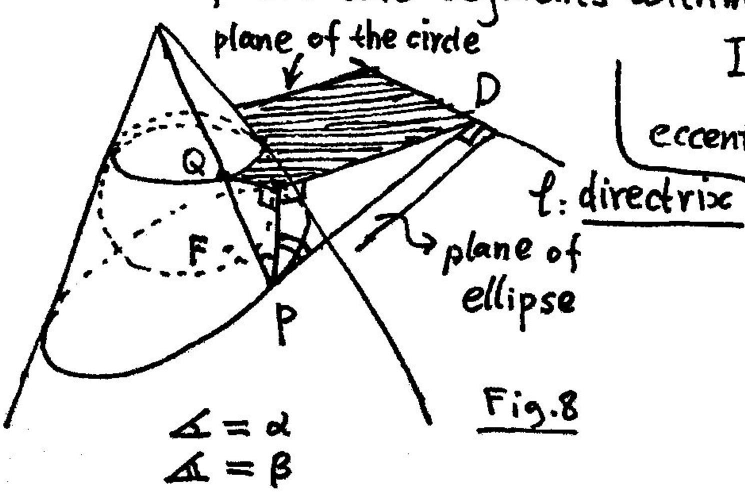


Fig.8

In Fig.8: $\frac{\cos\beta}{\cos\alpha}=\frac{PQ}{PD}=\frac{PF}{PD}$, i.e.

$$\text{eccentricity}=\frac{\text{distance: P to focus}}{\text{distance: P to directrix}}=\text{const.}$$

Noticing that the above arguements applies to a general conic section regardless its shape, we thus obtain another and unified way of describing all types of the conic sections in terms of the line segments which totally situate within the plane of the conic section,

a conic section is the loci of a point, for which the ratio:

$$\frac{\text{distance to } F}{\text{distance to } \ell} = \text{const.} \begin{cases} 0 & \text{circle} \\ 1^{-} & \text{ellipse} \\ 1 & \text{parabola} \\ 1^{+} & \text{hyperbola} \end{cases}$$

where F is a pre-given point (serves as the focus);

ℓ is a pre-given straight line (serves as the directrix).

This suggests an apparatus for drawing a parabola (Fig. 9).

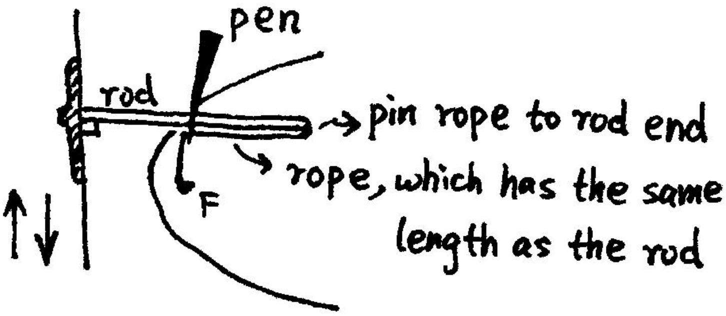


Fig. 9

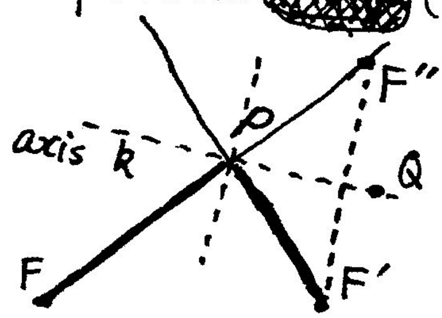


Fig. 10

- The focus property of conic sections

For any point P on an ellipse, we could connect P to the two focus, extend them to straight lines, and make the figure more complete by drawing the two axes of symmetry (Fig. 10)

Thus PF' could flip to PF'' which completes a straight line with PF. Thus on the axis k of symmetry:

P is on the ellipse, i.e. $PF + PF' = c$ (a const. when P moves along the ellipse);

and any point Q other than P must be outside the ellipse, since

$$QF + QF' > c. \quad (\text{See Appendix II}).$$

So the axis k is the touching line of the ellipse at P, and thus, for an elliptic mirror, if $\overrightarrow{FP}$ is an incident light beam, then $\overrightarrow{PF'}$ should be the light beam reflected by the elliptic mirror. (Fig. 11)

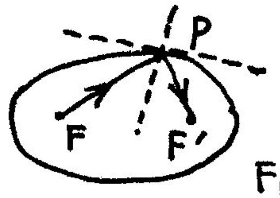
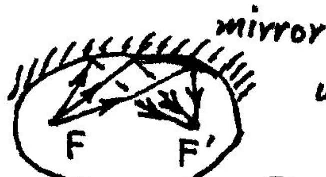


Fig.11



where the arrowed segments represent light beams

Fig.12

Noticing that the above arguement applies to any point P on the ellipse, we thus have discovered the following optical property of the ellipse, which justifies the name focus for F and F′ (Fig.12).

Hyperbola and parabola have similar optical properties for similar reasons.

Hyperbola:

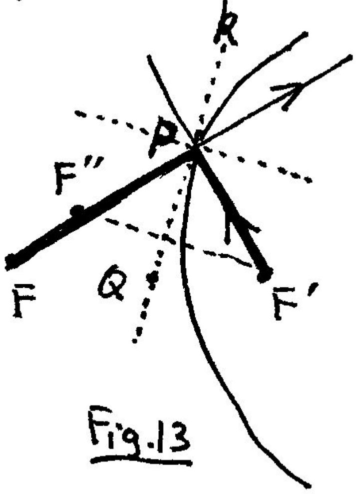
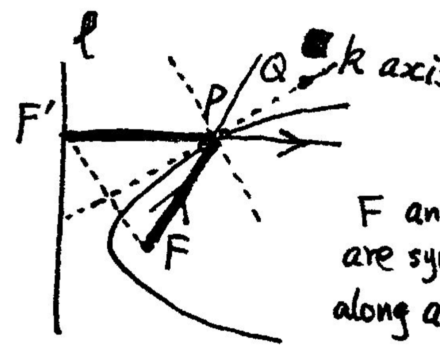


Fig.13

Parabola:



F and F′ are symm. along axis k.

Fig.14

	Hyperbola	Parabola
condition for P being on the curve:	$PF - PF'$ $= PF - PF'' = \text{const.}$	$PF =$ the height of P on ℓ
for any other point Q on the axis ....., thus Q is outside the curve.	$QF - QF'$ $< PF - PF'$ (the const.)	$QF >$ the height of Q on ℓ ($QF = QF'$)

The axis k is thus the touching line of the curve at the point P.
The route of a light beam reflected by a mirror with the shape of the curve is shown by the arrowed lines in the corresponding figures.

Some applications of the conic section mirrors in the real world are shown in Appendix III.

Appendix I. The motion of the sun and the sundial.

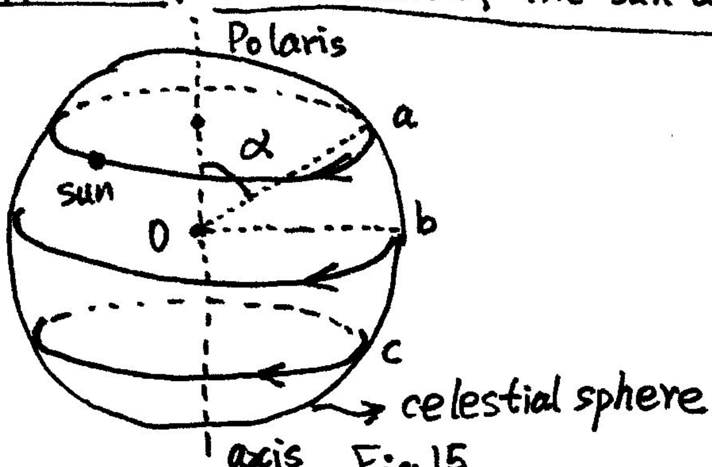


Fig.15

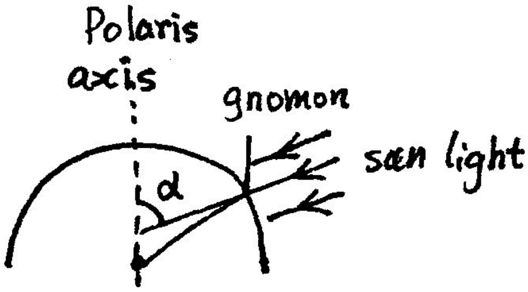


Fig.16

The loci of the sun on days a, b, c on the celestial sphere are drawn in Fig.15, where the days a, b, c are called summer solstice, equinox, winter solstice respectively. Since the sun is far far away from earth, (i.e. the radius of the celestial sphere is very long), the earth could be regarded as a tiny sphere centered at O, and thus the sun-light on earth is considered to be parallel (Fig.16). When the sun completes a circle on day a on celestial sphere, the sun light completes a circle around the Polaris axis, and also around the gnomon axis, with a constant angle α (Fig.16). Thus, a beam of sun light through the tip of the gnomon would sweep out an upright cone around the gnomon axis with the vertex angle α.

Appendix II. A point outside the ellipse.

Ellipse is the loci of P, for which $PF + PF' = C$ (a given const.)

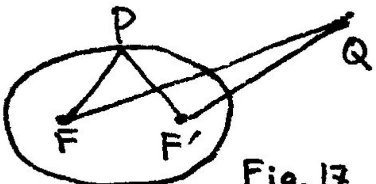


Fig. 17

For a point Q outside and far away from the ellipse, we obviously have:

$$QF + QF' > C \quad (\text{Fig.17})$$

From this, we infer that $XF + XF' > C$ should be the criterion for a point X being outside the ellipse. In fact, when a point X moves continuously from Q to another point outside the ellipse, the value of $XF + XF'$ should stay above C (otherwise, it must go through C, which indicate that X must go through the ellipse.).

Appendix III. Some applications of conic section mirrors.

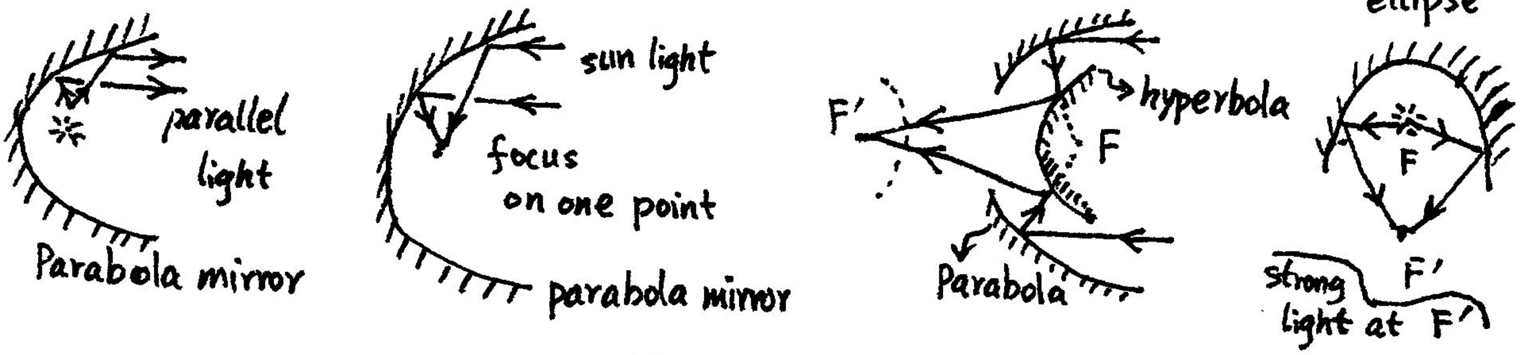


Fig. 18