An outline of Maxwell equations

Abstract:

Maxwell equations (Integral form) $\xrightarrow[\text{Stokes' Thm}]{\text{Gauss' Thm}}$ Maxwell equations (differential form)

$\downarrow$ when $\vec{E}$ and $\vec{B}$ vary

electro-magnetic wave equation

- The Maxwell equations

The corner stones of electro-magnetic theory are Maxwell equations. Let $\vec{E}=\begin{pmatrix}X\\Y\\Z\end{pmatrix}$ be an electric field in space, where each of X, Y, Z are functions of position $\begin{pmatrix}x\\y\\z\end{pmatrix}$; $\vec{B}$ be a magnetic field in space; C be the boundary curve of the surface S; $\oint$ indicate the base curve or surface or volume is a closed one.

For a fixed time:

(E). $\oint_S \vec{E}\cdot\vec{N}\Delta s = q\cdot\frac{1}{\varepsilon}$

Gauss' law: the total amount of $\vec{E}$ passing through a closed surface S (called the out-flow of $\vec{E}$) is proportional to the electric charge enclosed by S.

(See Appendix I)

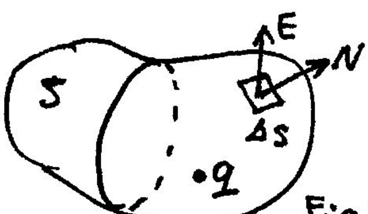


Fig.1.

(B). $\oint_S \vec{B}\cdot\vec{N}\Delta s = 0$

Gauss' law: for any closed surface, the enclosed magnetic charge is zero, i.e. there is no magnetic charged particle.

As time varies:

(ΔE). $\Delta_t \int_S \vec{E}\cdot\vec{N}\Delta s = \oint_C \vec{B}\cdot\vec{T}\Delta c\cdot\frac{1}{\varepsilon\mu}$

$\left(\Delta_t = \frac{\Delta\cdot}{\Delta t}\right)$

Ampere-Maxwell law: as the out-flow of $\vec{E}$ on an open surface S changes with time, a magnetic field $\vec{B}$ is generated; and the rate of that change is proportional to the potential energy of $\vec{B}$ around the boundary C of S.

(i.e. the work done by $\vec{B}$ when an imaginary unit magnetic charge goes around C —— called the circulation of $\vec{B}$ on C)

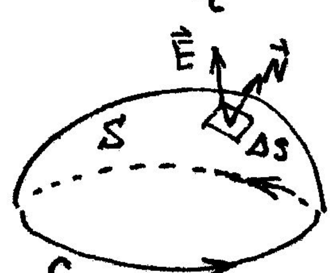


Fig.2

(ΔB). $\Delta_t \int_S \vec{B}\cdot\vec{N}\Delta s = \oint_C \vec{E}\cdot\vec{T}\Delta c\cdot(-1)$ Faraday's law:

as the out-flow of $\vec{B}$ on an open surface S changes with time, an electric field $\vec{E}$ is generated, and the rate of that change is proportional to the potential energe of $\vec{E}$ around the boundary C of S.

While (E) and (B) are mathematical descriptions of intensity of a source that 'emits' $\vec{E}$ and $\vec{B}$, respectively (See Appendix I), the formulae (ΔE) and (ΔB) came from experimental facts.

There are two types of integrals involved in Maxwell equations: one is for the calculation of out-flow, and the other for that of the circulation. Both of them could be calculated by dividing the base geometric objects into small pieces.

- For the out-flow of $\vec{E}$ on a closed surface:

$$\oint_S \vec{E}\cdot\vec{N}\Delta s \underset{(*)}{=\!=\!=} \oint_{I} + \oint_{II} = \oint_{\Delta V} + \oint_{\Delta V'} + \cdots$$

Fig.3

(*) At any point on the common boundary of I and II:

$\vec{N}$ and $\vec{N}'$ are of opposite direction

$\vec{E}\cdot\vec{N}\Delta s$ and $\vec{E}\cdot\vec{N}'\Delta s$ cancel each other.

In Fig.3, the out-flow of the tiny cuboid in the left-right side is

$$(\partial_x X\cdot\Delta x)\Delta y\Delta z \qquad \left(\text{remember } \vec{E}=\begin{pmatrix}X\\Y\\Z\end{pmatrix}\right)$$

where $\partial_x X\cdot\Delta x$ is the difference of x-component of $\vec{E}$ from left side to right side. Similar holds in the front-back sides (i.e. y-direction) and up-bottom sides (i.e. z-direction). Thus the total out-flow of the cuboid is $(\partial_x X + \partial_y Y + \partial_z Z)\Delta x\Delta y\Delta z$.

i.e. for a tiny cuboid: out-flow $=$ dilatation ~~~~ $\vec{\nabla}\cdot\vec{E}$, except that the out-flow contains volume as a factor.

We thus have reduced the out-flow of $\vec{E}$ on S to that on the tiny cuboid, which is (the dilatation) $\times \Delta V$:

$$\oint_S \vec{E}\cdot\vec{N}\Delta S = \int_{(S)} \vec{\nabla}\cdot\vec{E}\,\Delta V \quad \text{(called \underline{Gauss' Thm})}$$

$$= q\cdot\frac{1}{\varepsilon} \qquad \text{(Maxwell equ. (E))}$$

where (S) is the solid enclosed by S.

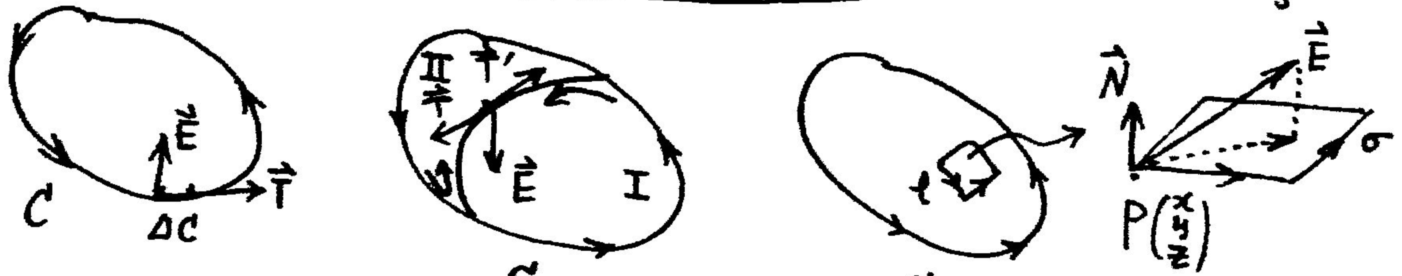
Let S be shrinked to the tiny cuboid at P (Fig.3), then we have:

$$\vec{\nabla}\cdot\vec{E}\,\Delta V = q\cdot\frac{1}{\varepsilon}$$

$$\vec{\nabla}\cdot\vec{E} = \rho\cdot\frac{1}{\varepsilon}$$ where ρ is the point density of electric charge at P.

Similarly: $\vec{\nabla}\cdot\vec{B} = 0$ (which follows from the calculation of $\oint_S \vec{B}\cdot\vec{N}\Delta S$)

- For the circulation of $\vec{E}$ on a closed curve C:



$$\oint_C \vec{E}\cdot\vec{T}\Delta C \overset{}{\underset{(**)}{=\!=}} \oint_{\mathrm{I}} + \oint_{\mathrm{II}} = \oint_{\ell} + \oint_{\ell'} + \cdots$$

Fig. 4

(**): At any point on the common boundary of I and II:

$\vec{T}$ and $\vec{T}'$ are of opposite direction,

$\vec{E}\cdot\vec{T}\Delta C$ and $\vec{E}\cdot\vec{T}'\Delta C$ cancel each other.

In Fig.4, the circulation of $\vec{E}$ along the tiny rectangle

$=$ the circulation of shadow $\vec{E}'$ of $\vec{E}$ (on the plane σ), along the tiny rectangle

$\overset{\text{Appendix II}}{=\!=}$ (the vorticity of $\vec{E}'$ at P, on the plane σ) $\cdot\,\Delta S$

$\overset{\text{Appendix III}}{=\!=}$ (the vorticity of $\vec{E}$ at P, in space) $\cdot\,\vec{N}\cdot\Delta S$

$$= (\vec{\nabla}\times\vec{E})\cdot\vec{N}\,\Delta S$$

We thus have reduced the circulation of $\vec{E}$ along c' to that along the tiny rectangle, which is (the planar vorticity)$\cdot \Delta S$:

$$\oint_c \vec{E}\cdot\vec{T}\Delta c = \int_{(c)} (\vec{\nabla}\times\vec{E})\cdot\vec{N}\Delta S \quad \text{(called \underline{Stokes' Thm})}$$
$$= \Delta_t \int_{(c)} \vec{B}\cdot\vec{N}\Delta S\cdot(-1) \quad \text{(Maxwell equ. } (\Delta B))$$

where (c') is any surface enclosed by c' (i.e. whose boundary is c'). Let c' be shrinked to the tiny rectangle at P (Fig.4), we then have:

$$\partial_t \vec{B}\cdot\vec{N}\Delta S = (\vec{\nabla}\times\vec{E})\cdot\vec{N}\Delta S\cdot(-1) \quad \cdots\cdots \quad (+)$$
$$\partial_t \vec{B} = \vec{\nabla}\times\vec{E}\cdot(-1)$$

(Notice that the direction of the tiny rectangle is arbitrary, i.e. (+) holds for arbitrary $\vec{N}$)

Similarly, $\partial_t \vec{E} = \vec{\nabla}\times\vec{B}\cdot\frac{1}{\varepsilon\mu}$ (which follows from the calculation of $\oint_c \vec{B}\cdot\vec{T}\Delta c$)

According to the above calculation, there are two forms of Maxwell equations —— the integral form and the point-differential form.

Integral form

$$\left.\begin{aligned} &(E)\quad \oint_s \vec{E}\cdot\vec{N}\Delta S = q\cdot\frac{1}{\varepsilon} \\ &(B)\quad \oint_s \vec{B}\cdot\vec{N}\Delta S = 0 \end{aligned}\right\} \xrightarrow[\text{Gauss' Thm}]{\text{as } S \text{ becomes a tiny cuboid}} \begin{aligned} &\vec{\nabla}\cdot\vec{E} = \rho\cdot\frac{1}{\varepsilon} \quad (E') \\ &\vec{\nabla}\cdot\vec{B} = 0 \quad (B') \end{aligned}$$

$$\left.\begin{aligned} &(\Delta E)\quad \Delta_t \int_s \vec{E}\cdot\vec{N}\Delta S = \oint_c \vec{B}\cdot\vec{T}\Delta c\cdot\frac{1}{\varepsilon\mu} \\ &(\Delta B)\quad \Delta_t \int_s \vec{B}\cdot\vec{N}\Delta S = \oint_c \vec{E}\cdot\vec{T}\Delta c\cdot(-1) \end{aligned}\right\} \xrightarrow[\text{Stokes' Thm}]{\text{as } S \text{ becomes a tiny rectangle}} \begin{aligned} &\partial_t\vec{E} = \vec{\nabla}\times\vec{B}\cdot\frac{1}{\varepsilon\mu} \quad (\Delta E') \\ &\partial_t\vec{B} = \vec{\nabla}\times\vec{E}\cdot(-1) \quad (\Delta B') \end{aligned}$$

The last two equations $(\Delta E')$ and $(\Delta B')$ implies that the variation of $\vec{E}$ (or $\vec{B}$) in vaccum normally leads to the wave behaviour of both $\vec{E}$ and $\vec{B}$ (called electro-magnetic wave).

Let $\vec{E}$ vary, then $\vec{B}$ is generated, which (normally) would also vary.
And concerning the variation of $\vec{E}$ and $\vec{B}$:

both $(\Delta E')$ $\quad \partial_t \vec{E} = \vec{\nabla} \times \vec{B} \cdot \frac{1}{\varepsilon\mu}$

and $(\Delta B')$ $\quad \partial_t \vec{B} = \vec{\nabla} \times \vec{E} \cdot (-1)$ $\quad$ hold.

Plug the first into the second:

$$\partial_t \left(\vec{\nabla} \times \vec{B} \cdot \frac{1}{\varepsilon\mu}\right) = \partial_t \partial_t \vec{E}$$

$$= \vec{\nabla} \times \partial_t \vec{B} \cdot \frac{1}{\varepsilon\mu}$$

$$\overset{(\Delta B')}{=\!=} \vec{\nabla} \times \left(\vec{\nabla} \times \vec{E} \cdot (-1)\right) \frac{1}{\varepsilon\mu}$$

regard components $\partial_x, \partial_y, \partial_z$ of $\vec{\nabla}$ as numbers

$$= \left[(\vec{\nabla} \cdot \vec{E})\vec{\nabla} - (\vec{\nabla} \cdot \vec{\nabla})\vec{E}\right] \cdot (-1) \cdot \frac{1}{\varepsilon\mu}$$

Each component of the above equality still holds when we interpret $\partial_x, \partial_y, \partial_z$ as operators, as long as we put $\partial_x, \partial_y, \partial_z$ in the front in each product. $\partial_x X \partial_x \longrightarrow \partial_x \partial_x X$, etc.

$$= \left[\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - (\vec{\nabla} \cdot \vec{\nabla})\vec{E}\right] \cdot (-1) \cdot \frac{1}{\varepsilon\mu}$$

$\vec{\nabla}$ acts on the [redacted] function $\vec{\nabla} \cdot \vec{E}$

$$\underset{(E')\ \vec{\nabla} \cdot \vec{E} = 0}{\overset{\text{in vaccum}}{=\!=\!=}} \left[0 - (\vec{\nabla} \cdot \vec{\nabla})\vec{E}\right] \cdot (-1) \cdot \frac{1}{\varepsilon\mu}$$

$$= (\vec{\nabla} \cdot \vec{\nabla})\vec{E} \cdot \frac{1}{\varepsilon\mu}.$$

i.e. we have $\partial_t \partial_t \vec{E} = (\vec{\nabla} \cdot \vec{\nabla})\vec{E} \cdot \frac{1}{\varepsilon\mu}$.

the x-component X of $\vec{E}$ satisfies $\partial_t \partial_t X = (\vec{\nabla} \cdot \vec{\nabla}) X \cdot \frac{1}{\varepsilon\mu}$

which is the wave equation in space.

i.e. $X = X\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ behaves like a scalar wave in space.

The same is true for the other components of $\vec{E}$ and $\vec{B}$.

And the speed of this <u>electro-magnetic wave</u> is $\sqrt{\frac{1}{\varepsilon\mu}} = 3 \times 10^8$ (m/s) which equals to the speed of light measured by astronomical methods!

This makes us suspect that the everyday-seen light might be a kind of electro-magnetic wave!

The verification and ramifications of this conjecture open a gate to a new realm of our understanding of physical world.

Appendix I. Out-flow and the intensity of a source.

A star emits light at night. The intensity of its light decays in the same way as $\frac{1}{r^2}$ (Fig. 5). (At an instant, all the light diffuses on a sphere, whose surface area varies the same way as r^2).

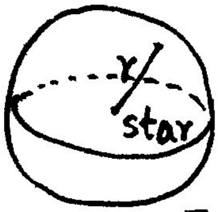


Fig.5

The sum of amount of light over the whole sphere indicates the intensity of the light source.

This sum should keep un-changed when we consider a larger sphere, or even an arbitrary closed surface enclosing the light source (Fig.6).

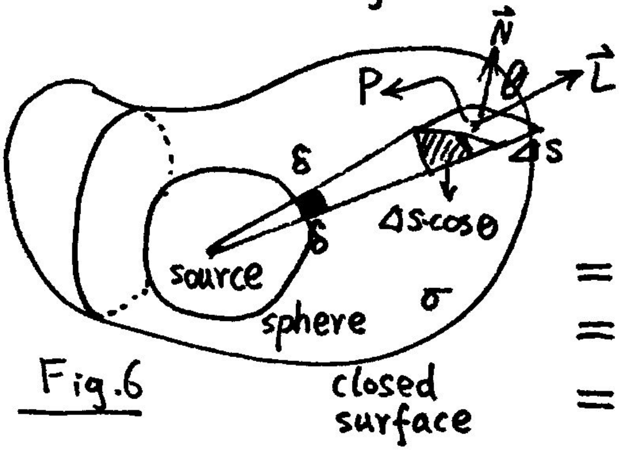


Fig.6

$\vec{L}$ = intensity of light at P.

$\vec{N} \perp \Delta S$ (unit vector $\vec{N}$)

amount of light through δ on the sphere

$= $ amount of light through ΔS on σ

$= $ amount of light through $\Delta S \cdot \cos\theta$

$= |\vec{L}| (\Delta S \cdot \cos\theta)$

$= \vec{L} \cdot \vec{N} \Delta S$

Total amount of light passing through σ

$= \oint_\sigma \vec{L} \cdot \vec{N} \Delta S$ = the out-flow of light field $\vec{L}$

(regarding $\vec{L}$ as a velocity field of a fluid flow)

The above arguement is also valid for gravity and electric force: both of them obey the inverse-square law;

intensity of a gravity source $=$ mass enclosed by σ

$= \oint_\sigma \vec{G} \cdot \vec{N} \Delta S$;

intensity of an electric source $=$ amount of charge enclosed by σ

$= \oint_\sigma \vec{E} \cdot \vec{N} \Delta S$.

Appendix II. Circulation of a tiny rectangle on a plane.

Let an oriented tiny rectangle on a plane be choosen (Fig.7).

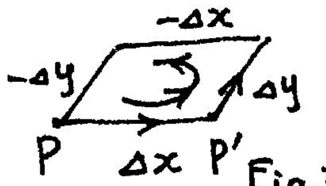


Fig.7

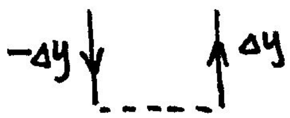
For convenience, let a pair of successive sides (along the orientation) be the positive direction of x and y axes, respectively (Δx, Δy in Fig.7).

Let $\vec{E} = \binom{X}{Y}$ be an electric field on the plane. We want to know the circulation of $\vec{E}$ along the oriented rectangle.

Notice that X is the abscissa of the component of $\vec{E}$ on x-axis,

Δx is the abscissa of the vector $\overrightarrow{PP'}$ on x-axis,

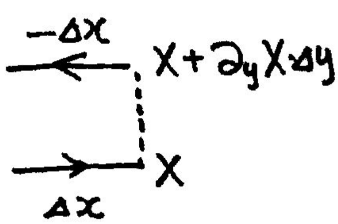
so $X \cdot \Delta x \begin{cases} + & \text{if } X \text{ and } \Delta x \text{ are of the same direction.} \\ - & \text{if } X \text{ and } \Delta x \text{ are of opposite direction.} \end{cases}$



y-components of $\vec{E}$: $Y \qquad Y + \partial_x Y \cdot \Delta x$

total work done (by $\vec{E}$ on unit charge) along the two sides

$= \partial_x Y \cdot \Delta x \cdot \Delta y$



total work done (by $\vec{E}$ on unit charge) along the two sides

$= -\partial_y X \cdot \Delta y \cdot \Delta x$

$\therefore$ circulation of $\vec{E}$ along the tiny rectangle

$$= (\partial_x Y - \partial_y X)\,\Delta x\,\Delta y = \begin{vmatrix} \partial_x & X \\ \partial_y & Y \end{vmatrix} \Delta x\,\Delta y$$

$$= (\text{vorticity of } \vec{E} \text{ at } P) \times \text{area}$$

i.e. for a tiny rectangle:

circulation = vorticity, except that the circulation contains area as a factor.

Appendix III. Projecting the vorticity.

The vector $\vec{\nabla}\times\vec{E}$ is the vorticity vector at the point $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ under the flow $\vec{E}$, which describes the instantaneous tendency of rotation at this point, and thus should be an intrinsic property of the flow, and should be independent of the choice of coordinate system.

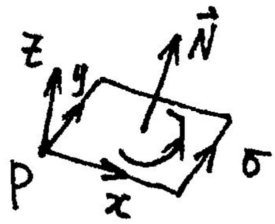


Fig.8

For a chosen tiny rectangle (Fig.8), one could establish a special 3-dimensional Cartesian coordinate system based on two adjacent sides of this tiny rectangle (Fig.8), analyze the tendency of rotation at P caused by the shadow flow on each of the coordinate planes xy, yz, zx, compose them, and finally obtain $\vec{\nabla}\times\vec{E}$.

Then, the vorticity (i.e. tendency of instantaneous rotation) at P on the plane $xy = \sigma$ caused by the shadow flow of $\vec{E}$ on σ

$=$ the z-component of the total vorticity $\vec{\nabla}\times\vec{E}$

$=$ the projection of $\vec{\nabla}\times\vec{E}$ on unit vector $\vec{z}$ (i.e. on unit vector $\vec{N}$)

$= (\vec{\nabla}\times\vec{E})\cdot\vec{N}$

which is again independent of the choice of the coordinate system.