On transcendental quantities

- **Area of a circle and trigonometric functions**

In [redacted] $\frac{1}{4}$ circle:

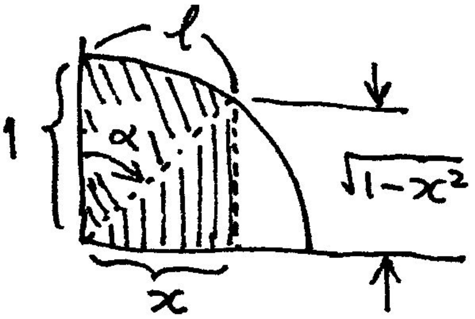

Fig.1

shaded area (for a fixed x)

$$= x\cdot\sqrt{1-x^2}\cdot\frac{1}{2} + \ell\cdot 1\cdot\frac{1}{2}$$

$$= \int_{0\sim x} \sqrt{1-t^2}\,\Delta t \cdots\cdots (*)$$

By Binomial Theorem, $\sqrt{1-x^2}$ is an infinite polynomial, and thus $(*)$ is essentially a polynomial equality:

$$\sqrt{1-x^2} = [1+(-x^2)]^{\frac{1}{2}} = 1 - x^2\cdot\frac{1}{2} - x^4\cdot\frac{1}{8} - x^6\cdot\frac{1}{16} - \cdots$$

$$\int\sqrt{1-t^2}\,\Delta t = \quad t - t^3\cdot\frac{1}{6} - t^5\cdot\frac{1}{40} - t^7\cdot\frac{1}{112} - \cdots$$

$(*)$ becomes $x\cdot\left(1 - x^2\cdot\frac{1}{2} - x^4\cdot\frac{1}{8} - x^6\cdot\frac{1}{16} - \cdots\right)\cdot\frac{1}{2} + \ell\cdot 1\cdot\frac{1}{2}$

$$= x - x^3\cdot\frac{1}{6} - x^5\cdot\frac{1}{40} - x^7\cdot\frac{1}{112} - \cdots$$

thus $\ell = x + x^3\cdot\frac{1}{6} + x^5\cdot\frac{3}{40} + x^7\cdot\frac{5}{112} + \cdots \quad (\dagger)$

$=$ the length of <u>arc</u> whose (subtending angle's) <u>sin</u> is <u>x</u>.

$\triangleq$ <u>arc sin$x$</u>.

i.e. arcsin$x$ could be obtained by simply performing the basic algebraic operations $+, -, \times$ on x.

In the polynomial equation $(\dagger)$, regarding x as the un-known variable, we could solve x in terms of ℓ (<u>Appendix I</u>), obtaining,

$$x = \ell - \ell^3\cdot\frac{1}{6} + \ell^5\cdot\frac{1}{120} - \cdots$$

$=$ the sin of (the angle subtending) ℓ

$\triangleq \sin\ell$.

[redacted] Therefore $\cos\ell = \sqrt{1-\sin^2\ell} = (1+(-\sin^2\ell))^{\frac{1}{2}}$

$$= 1 + (-\sin^2\ell)\cdot\frac{1}{2} - (-\sin^2\ell)^2\cdot\frac{1}{8} + \cdots$$

$$= 1 - \left(\ell - \ell^3\cdot\frac{1}{6} + \ell^5\cdot\frac{1}{120} - \cdots\right)^2\cdot\frac{1}{2} - \left(\ell - \ell^3\cdot\frac{1}{6} + \ell^5\cdot\frac{1}{120} - \cdots\right)^4\cdot\frac{1}{8} + \cdots$$

$$= 1 - \ell^2\cdot\frac{1}{2} + \ell^4\cdot\frac{1}{24} - \ell^6\cdot\frac{1}{720} + \cdots$$

In this polynomial equation of two variables ℓ and $\cos\ell \triangleq y$, regarding ℓ as the un-known variable, we could solve ℓ in terms of y (see Appendix II), obtaining $\ell = \frac{\pi}{2} - y - y^3 \cdot \frac{1}{6} - y^5 \cdot \frac{3}{40} - \cdots$

$= $ the length of arc whose (subtending angle's) cos is y.

$\triangleq \arccos y$

Thus

$$\tan\ell = \frac{\sin\ell}{\cos\ell} = \frac{\ell - \ell^3 \cdot \frac{1}{6} + \ell^5 \cdot \frac{1}{120} - \cdots}{1 - \ell^2 \cdot \frac{1}{2} + \ell^4 \cdot \frac{1}{24} - \cdots} = \ell + \ell^3 \cdot \frac{1}{3} + \ell^5 \cdot \frac{2}{15} + \cdots$$

$$\cot\ell = \frac{1}{\tan\ell} = \frac{1}{\ell + \ell^3 \cdot \frac{1}{3} + \ell^5 \cdot \frac{2}{15} + \cdots} = \frac{1}{\ell} - \ell \cdot \frac{1}{3} - \ell^3 \cdot \frac{1}{45} + \cdots$$

- Hyperbolic logarithm and exponential

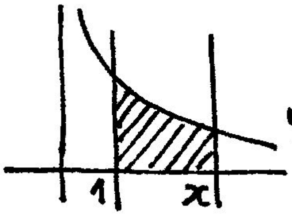

Fig.2

For the hyperbola in Fig.2, the area function is $\int_1^x \frac{1}{x}\Delta x = \log x$ (base e), thus $\int \frac{1}{x}\Delta x = \log x + c$

On the other hand:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \qquad \boxed{= 1 + x(1 + x + x^2 + \cdots)}$$

$$\therefore \frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + \cdots$$

$$\int \frac{1}{1+x}\Delta x = x - x^2 \cdot \frac{1}{2} + x^3 \cdot \frac{1}{3} - x^4 \cdot \frac{1}{4} + \cdots$$

$= \log(1+x) + c$ (holds for all values of x, including 0)

$= \log(1+x)$

This is the first time that we could actually obtain the value of logarithm of a given number by simply performing $+, -, \times$ on it! $\log(\frac{3}{2}) = \log(1 + \frac{1}{2}) = \frac{1}{2} - (\frac{1}{2})^2 \cdot \frac{1}{2} + (\frac{1}{2})^3 \cdot \frac{1}{3} - (\frac{1}{2})^4 \cdot \frac{1}{4} + \cdots$

Regarding $\log(1+x)$ as a whole variable y:

$$y = x - x^2 \cdot \frac{1}{2} + x^3 \cdot \frac{1}{3} - x^4 \cdot \frac{1}{4} + \cdots$$

solve x in terms of y: $x = 0 + y + y^2 \cdot \frac{1}{2} + y^3 \cdot \frac{1}{6} + y^4 \cdot \frac{1}{24} + \cdots$

$= e^y - 1$ (where e denotes the base of the hyperbolic logarithm)

Thus $e^y = 1 + y + y^2 \cdot \frac{1}{2} + y^3 \cdot \frac{1}{6} + y^4 \cdot \frac{1}{24} + \cdots$

$e^u = +, -, \times$ on u

$e^1 = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots$ (evaluation of the base e of hyperbolic logarithm by $+, -, \times$)

- ## The triumph

We achieved expressing the frequently-encountered transcendental quantities sin, cos, arcsin, arccos, log, e^u as infinite polynomials. This is the first time that we could access these transcendental quantities in such a unified - and simple way —— just performing the basic algebraic operations $+, -, \times$. (Before the infinite polynomial expressions, people had to invent dedicated and specific methods to takle these quantities)

Under the enlightenment of the infinite polynomial expressions of these transcendental quantities, we could make many new discoveries, for example: $\frac{\pi}{6} = \arcsin\frac{1}{2} = \frac{1}{2} + \left(\frac{1}{2}\right)^3 \cdot \frac{1}{6} + \left(\frac{1}{2}\right)^5 \cdot \frac{3}{40} + \cdots$

(which is the first time that π could be evaluated by simply performing $+, -, \times, \div$ on integers!);

and solve some problems which could hardly be solved in the past, for example, the Kepler's equation, by which people could predict the location of a planet at a given time.

- ## The Kepler's equation

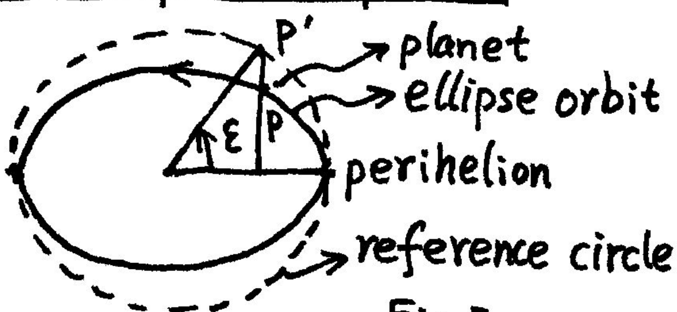

Fig.3

A traditional way of locating a planet is finding the angular position ε of the reference point P' (of the planet P) on the reference circle (Fig.3).

Starting from the perihelion, the angle ε varies from 0 as the time t elapses, they are entangled in the Kepler's equation:

$$\frac{(\varepsilon - e\cdot\sin\varepsilon)\cdot\frac{1}{2}}{\pi} = \frac{t}{T},$$

(where e is the eccentricity of the ellipse orbit, T is the period of the planet.)

which could be derived from the equal-time-equal-area law (See Appendix III).

So if we want to know the location of Mercury when $\frac{1}{4}$ orbit period of time passes from the perihelion, then we have to solve the equation:

$$\frac{(\varepsilon - (0.205635\cdots)\sin\varepsilon)\times\frac{1}{2}}{3.1415926\cdots} = \frac{1}{4} \quad (\ddagger)$$

As we have expressed the transcendental quantity $\sin\varepsilon$ as an infinite polynomial of ε, the equation $(\ddagger)$ thus becomes an equation of polynomials, and could be solved by Newton's series method.

Notice that, before the infinite polynomial expression of sin was discovered, we have no good method to solve the equation $(\ddagger)$, because only the table of specific values of 'sin' is at our disposal, lacking a unified way of computing $\sin x$ for an arbitrary given value of x.

Appendix I. $\ell = x + x^3\cdot\frac{1}{6} + x^5\cdot\frac{3}{40} + x^7\cdot\frac{5}{112} + \cdots$, solve x in terms of ℓ.

set $f(x) = -\ell + x + x^3\cdot\frac{1}{6} + x^5\cdot\frac{3}{40} + \cdots$ we want to find x such that $f(x)=0$.

when $\ell=0$, we have $f(0)=0$ (when $\ell=0$, the equation $f(x)=0$ becomes an ordinary equation with definite constant coefficients, and $x=0$ is a solution)

when ℓ is a tiny variable, we have $f(0)\sim\ell$, where $\sim$ signifies "as small as".

By adjusting 0 to $0+\delta$, we hope to make $f(0+\delta)$ smaller:

$$f(0+\delta) = f(0) + \delta\cdot f'(0) + \cdots \qquad \left(f'(x) = 1 + x^2\cdot\frac{1}{2} + x^4\cdot\frac{3}{8} + \cdots\right)$$
$$= \underbrace{-\ell + \delta\cdot 1} + \cdots$$

if δ is such that "$\smile$" $=0$ (i.e. $\delta=\ell$), then $\delta\sim\ell$ and $f(\underset{0+\ell}{0+\delta})\sim\ell^2$

$$f(0+\ell+\delta) = f(0+\ell) + \delta\cdot f'(0+\ell) + \cdots$$
$$= \underbrace{\ell^3\cdot\frac{1}{6} + \delta\cdot 1} + \cdots$$

if δ is such that "$\smile$" $=0$ (i.e. $\delta = -\ell^3\cdot\frac{1}{6}$), then $\delta\sim\ell^3$, and thus $f(\underset{0+\ell-\ell^3\cdot\frac{1}{6}}{0+\ell+\delta})\sim\ell^3$

$$f(0+\ell-\ell^3\tfrac{1}{6}+\delta) = f(0+\ell-\ell^3\tfrac{1}{6}) + \delta\cdot f'(0+\ell-\ell^3\cdot\tfrac{1}{6}) + \cdots$$
$$\vdots$$

Thus the limit object $0+\ell-\ell^3\frac{1}{6}+\cdots$ is the solution for $f(x)=0$.

Appendix II. $y = 1 - \ell^2\cdot\frac{1}{2} + \ell^4\cdot\frac{1}{24} - \ell^6\cdot\frac{1}{720} + \cdots$, solve ℓ in terms of y.

Set $f(\ell) = -y + 1 - \ell^2\cdot\frac{1}{2} + \ell^4\cdot\frac{1}{24} - \ell^6\cdot\frac{1}{720} + \cdots = -y + \cos\ell$,

we want to find ℓ such that $f(\ell) = 0$.

when $y = 0$, we have $f(\frac{\pi}{2}) = 0$ (when $y=0$, the equation $f(\ell)=0$ becomes $0 + \cos\ell = 0$, and has a solution $\ell = \frac{\pi}{2}$)

when y is a tiny variable, we have $f(\frac{\pi}{2}) \sim y$.

By adjusting $\frac{\pi}{2}$ to $\frac{\pi}{2} + \delta$, we hope to make $f(\frac{\pi}{2} + \delta)$ smaller:

$$f(\tfrac{\pi}{2} + \delta) = f(\tfrac{\pi}{2}) + \delta\cdot f'(\tfrac{\pi}{2}) + \cdots \quad \left(f'(\ell) = 0 - \ell + \ell^3\cdot\tfrac{1}{6} - \ell^5\cdot\tfrac{1}{120} + \cdots = -\sin\ell \right)$$
$$= \underline{-y + \delta\cdot(-1)} + \cdots$$

if δ is such that "$\sim\sim$" $= 0$ (i.e. $\delta = -y$), then $\delta \sim y$ and $f(\frac{\pi}{2} + \delta) \sim y^2$

$\frac{\pi}{2} - y$

$$f(\tfrac{\pi}{2} - y + \delta) = f(\tfrac{\pi}{2} - y) + \delta\cdot f'(\tfrac{\pi}{2} - y) + \cdots \quad \left(f(\tfrac{\pi}{2} - y) = -y + \cos(\tfrac{\pi}{2} - y) = -y + \sin y = \cdots \right)$$
$$= \underline{y^3\cdot\tfrac{1}{6} + \delta\cdot(-1)} + \cdots$$

if δ is such that "$\sim\sim$" $= 0$ (i.e. $\delta = y^3\cdot\frac{1}{6}$), then $\delta \sim y^3$ and $f(\frac{\pi}{2} - y + \delta) \sim y^3$

$$f(\tfrac{\pi}{2} - y + y^3\cdot\tfrac{1}{6} + \delta) = f(\tfrac{\pi}{2} - y + y^3\cdot\tfrac{1}{6}) + \delta\cdot f'(\tfrac{\pi}{2} - y + y^3\cdot\tfrac{1}{6}) + \cdots \qquad \tfrac{\pi}{2} - y + y^3\cdot\tfrac{1}{6}$$
$$\vdots$$

Thus the limit object $\frac{\pi}{2} - y + y^3\cdot\frac{1}{6} - \cdots$ is the solution for $f(\ell) = 0$.

Appendix III. The Kepler's equation, how comes it?

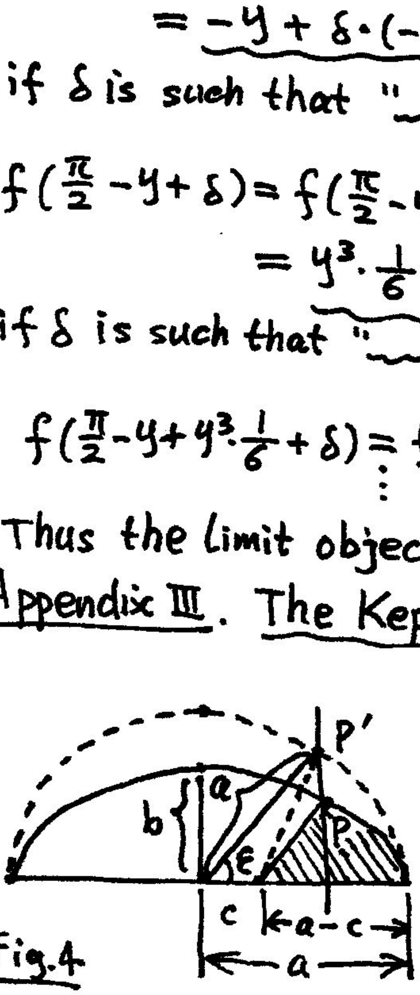

Fig.4

By the equal-area-equal-time law of the planet motion, we have (Fig.4):

$$\frac{\text{shaded area}}{\text{ellipse area}} = \frac{t}{T} \quad \begin{matrix} \rightsquigarrow \text{time elapsed} \\ \rightsquigarrow \text{orbit period} \end{matrix}$$

where: shaded area = the area with base $a-c$ and apex P, denoted by $(a-c)P$.

$= $ the area of $(a-c)P'$ compressed by the factor $\frac{b}{a}$.

$= (\text{sector } \varepsilon \text{ of the big circle} - \text{triangle } cP')\cdot\frac{b}{a}$.

$= (\pi aa\cdot\frac{\varepsilon}{2\pi} - c\cdot a\sin\varepsilon\cdot\frac{1}{2})\cdot\frac{b}{a}$ (notice that $c = a\cdot e$)

$= (\varepsilon - e\cdot\sin\varepsilon)\cdot\frac{1}{2}ab$

Thus $\dfrac{(\varepsilon - e\cdot\sin\varepsilon)\cdot\frac{1}{2}\cancel{ab}}{\pi\cancel{ab}} = \dfrac{t}{T}$.