University of Minnesota: Twin Cities

CE 8351: Analytical Modeling in Civil Engineering

Project 2: Elliptical, Impermeable Element

Part I: Setup

Christopher Bulkley-Logston

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0 Given Formulae

0.1 Plane Projections

This assignment guides the modeling of an aquifer with uniform flow containing an impermeable, elliptic object in addition to a well of some discharge. As presented in section 8.7 of *Analytical Groundwater Mechanics* (draft - 2016) by Otto D.L. Strack of the University of Minnesota, this process is best carried out using the method of images.

As discussed previously, unwanted computational phenomena can be avoided by projecting the physical z plane into that of a dimensionless Z such that the corresponding Ω function is an asymptotic expansion. This is carried out for this assigment's case as follows:

$$Z = \frac{z - \frac{1}{2}(z_2 + z_1)}{\frac{1}{2}(z_2 - z_1)} \tag{1}$$

Where:

- z is complex, physical location
- z_1 is the ellipse's 1^{st} focal point
- z_2 is the ellipse's 2^{nd} focal point

This $z \to Z$ transformation facilitates computations when using elements such as wells, line sinks and objects that are circular in the physical plane. However, given the elliptical shape of the the key element at hand, yet another transformation is carried out into a χ plane where elliptical elements are mapped as circles.

$$\chi = \left(Z + \sqrt{Z+1}\sqrt{Z-1}\right)\nu \tag{2}$$

Where:

- ν is a parameter accomodating the shape of the ellipse, calculated using:

$$\nu = A - B \tag{3}$$

Where:

- A is the major principal axis of the ellipse in the Z plane
- B is the minor principal axis of the ellipse in the Z plane

Equation 3 is pulled from a set of 3 equations which are used in transforming ellipses in the Z plan to circles in the χ plane. The other two are as follows:

$$A + B = \frac{1}{\nu} \tag{4}$$

$$A^2 - B^2 = 1 \tag{5}$$

Thus, ν can be expressed entirely in terms of A using eqs. 3 and 5 as follows:

$$B^2 = A^2 - 1 \to B = \sqrt{A^2 - 1} \to \nu = A - \sqrt{A^2 - 1} \tag{6}$$

0.2 Element Formulae

Using these projections, the text provides the following expressions for the complex potential resulting from both uniform flow and a well in the system in which this elliptical element is placed.

The complex potential resulting from the **well** is given as follows:

$$\Omega(\chi)_{\text{uf}} = \frac{Q_0 L}{4\nu}\left[\chi e^{i(\alpha-\beta)} + \frac{1}{\chi}e^{-i(\alpha-\beta)}\right] \tag{7}$$

Where:

- Q_0 is the magnitude of uniform flow (scalar - $\frac{\text{area}}{\text{time}}$)
- L is the length between each of the ellipse's foci, calculated using eq. 8.
- α is the angle of orientation of the ellipse, ccw against positive horizontal (in radians).
- β is the angle of uniform flow, ccw against positive horizontal (in radians).

Conceptually, L is the length of the hypotenuse of a right triangle of which z_1 and z_2 are the non-90° bearing points. However, for coding purposes, it is equated as follows:

$$L = \sqrt{(z_2 - z_1)\left(\,\overline{z_2 - z_1}\,\right)} \tag{8}$$

Also, the complex potential for **uniform flow** is given as follows:

$$\Omega(\chi)_{\text{well}} = \frac{Q_w}{2\pi}\ln\left[\frac{(\chi-\chi_w)\left(\chi - \frac{1}{\overline{\chi}_w}\right)}{\chi}\right] \tag{9}$$

Where:

- Q_w is the well discharge (volume)

Thus, the total governing function for complex potential in this field is, using the χ plane projection is:

$$\Omega_{\text{pre-calibrated}} = \Omega_{\text{uf}} + \Omega_{\text{well}} = \frac{Q_0 L}{4\nu}\left[\chi e^{i(\alpha-\beta)} + \frac{1}{\chi}e^{-i(\alpha-\beta)}\right] + \frac{Q_w}{2\pi}\ln\left[\frac{(\chi-\chi_w)\left(\chi - \frac{1}{\overline{\chi}_w}\right)}{\chi}\right] \tag{10}$$

Also, this model will need to be *calibrated* according to known conditions at some distant reference point. Thus, a constant C is introduced:

$$\Omega_{\text{total}} = \Omega_{\text{uf}} + \Omega_{\text{well}} + C = \frac{Q_0 L}{4\nu}\left[\chi e^{i(\alpha-\beta)} + \frac{1}{\chi}e^{-i(\alpha-\beta)}\right] + \frac{Q_w}{2\pi}\ln\left[\frac{(\chi-\chi_w)\left(\chi - \frac{1}{\overline{\chi}_w}\right)}{\chi}\right] + C \tag{11}$$

Finally, it should be noted, that even after several plane projections, eq. 11 is still ultimately a function of physical z.

$$\Omega(\chi) = \Omega(\chi(Z,\nu)) = \Omega(\chi(Z(z,z_1,z_2),\nu)) \tag{12}$$

1 Calculating Unknown α, ν and C From Knowns

It may be the case that the following properties are known:

- z_1
- z_2
- z_w
- Q_0
- Q_w
- β
- z_0: the location of known piezometric head
- ϕ_0: the head at that location
- k: hydraulic conductivity of aquifer
- ρ: the ratio of the principle major and minor axes in the physical z plane

That is:

$$\rho = \frac{2a}{2b} = \frac{a}{b} \tag{13}$$

Where a is the semi-major principle axis and b is the semi-minor principle axis in the z plane.

In this case, both α and ν need to be calculated before eq. 10 can be used. The former is calculated by applying basic trigonometry to a triange whose hypotenuse is formed by the foci:

$$\boxed{\alpha = \tan^{-1}\left(\frac{\Re(z_2) - \Re(z_1)}{\Im(z_2) - \Im(z_1)}\right)} \tag{14}$$

The latter, ν can be expressed as a function of z_1, z_2 and ρ by observing that the the inter-foci distance L in the z plane is transformed to a constant 2 in that of Z. Thus, the principle axis in each plane can be scaled accordingly:

$$A = \frac{2a}{L} \tag{15}$$

If a is known, eq. 15 can be used to generate an A value, which, in combination with α from eq. 14, satisfies all the unknowns required to calculate eq. 10. Therefore, an expression for a in terms of z_1, z_2 and L is derived as follows:

When an ellipse is centered about the origin, the following expression can be used to calculate f, the distance between either of its focal points and the origin.

$$f = \sqrt{a^2 - b^2} \tag{16}$$

In this case, that length is half of L, the inter-focal point length:

$$\frac{L}{2} = \sqrt{a^2 - b^2} \tag{17}$$

b can be isolated in eq. 13 and substituted into eq. 17:

$$b = \frac{a}{\rho} \rightarrow \frac{L}{2} = \sqrt{a^2 - \left(\frac{a}{\rho}\right)^2}$$

With this, a is isolated first by squaring both sides:

$$\left(\frac{L}{2}\right)^2 = a^2 - \left(\frac{a}{\rho}\right)^2 = a^2 - \frac{a^2}{\rho^2}$$

pulling out a on the left:

$$\left(\frac{L}{2}\right)^2 = a^2\left(1 - \frac{1}{\rho^2}\right)$$

simplifying the right:

$$\frac{L^2}{4} = a^2\left(1 - \frac{1}{\rho^2}\right)$$

dividing both sides by the parenthetical term:

$$\frac{L^2}{4\left(1 - \frac{1}{\rho^2}\right)} = a^2$$

and taking the root of both sides:

$$\sqrt{\frac{L^2}{4\left(1 - \frac{1}{\rho^2}\right)}} = a$$

Then, swapping sides and simplifying:

$$a = \frac{L}{2\sqrt{1 - \frac{1}{\rho^2}}}$$

substituting this into eq. 15 and simplifying:

$$A = \frac{2\left(\frac{L}{2\sqrt{1-\frac{1}{\rho^2}}}\right)}{L} = \frac{1}{\sqrt{1 - \frac{1}{\rho^2}}}$$

and substituting this into eq. 6:

$$\boxed{\nu = \frac{1}{\sqrt{1 - \frac{1}{\rho^2}}} - \sqrt{\left(\frac{1}{\sqrt{1 - \frac{1}{\rho^2}}}\right)^2 - 1}}$$

With equations available for $\nu = f(\rho)$ and $\alpha = f(z_1, z_2)$, eq. 10 can be used to plot an uncalibrated model. However, C must be calculated for the Ω_{total} model to be fully developed. This is done with the known piezometric head (ϕ_0) at the reference location (z_0).

Assuming an unconfined aquifer:

$$\Phi_0 = \frac{1}{2}k\phi_0^2 \tag{18}$$

This is equivalent to the real component of the total model function at that location:

$$\Phi_0 = \Re\left[\Omega_{\text{total}}(z_0)\right] = \Re\left[\Omega_{\text{uf}} + \Omega_{\text{well}} + C\right] \tag{19}$$

Given that C is only real, it can be isolated as an unknown as follows:

$$C = \Phi_0 - \Re\left[\Omega_{\text{total}}(z_0)\right] = \Phi_0 - \Re\left[\Omega_{\text{uf}} + \Omega_{\text{well}}\right] \tag{20}$$

Presenting the full expression:

$$C = \Phi_0 - \Re\left[\frac{Q_0 L}{4\nu}\left(\chi e^{i(\alpha-\beta)} + \frac{1}{\chi}e^{-i(\alpha-\beta)}\right) + \frac{Q_w}{2\pi}\ln\left(\frac{(\chi - \chi_w)\left(\chi - \frac{1}{\overline{\chi}_w}\right)}{\chi}\right)\right] \tag{21}$$

With C calculated along with α and ν, eq. 11 can be used to plot the flow net shown in the following sections.

2 Flow Net for Well Only

For the first test of the formulae presented and developed in sections 0 and 1, an aquifer with a well with no uniform flow is considered. To create this, the following parameters are set as given.

- $z_1 = -800 - 700i$
- $z_2 = -400 - 200i$
- $z_w = 700 + 700i$
- $z_0 = 1000 + 0i$
- $\phi_0 = 28m$
- $k = 10\frac{m^3}{day}$
- $Q_w = 200\frac{m^3}{day}$
- $Q_0 = 0\frac{m^3}{day}$
- $\beta = 0°$
- $\rho = 1.5$
- aquifer is unconfined

These values are used to generate figures 1 and 2 to the right. As shown in the head contours (bottom), the developed model of Ω_{total} generates a head at the reference point equal to the given value passed into the calculation process. This is also checked in lines 26 - 30 of the Matlab® Code, as shown in appendix section A.1.

With these parameters, the following values are calculated:

- $\alpha = 0.8961\ rad$
- $\nu = 0.4472$
- $C = 3897.1\frac{m^3}{\text{day}}$

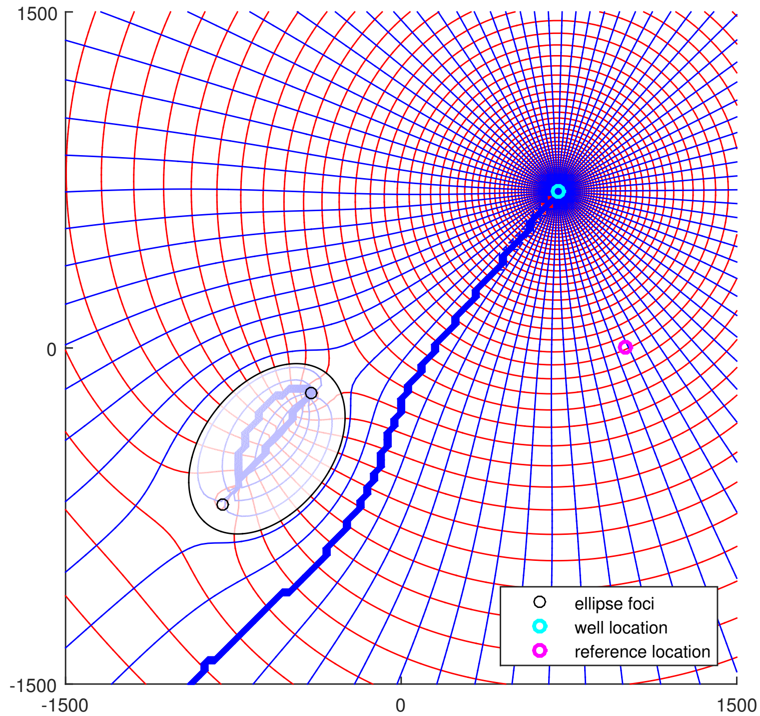

Figure 1: Flow net for well flow only

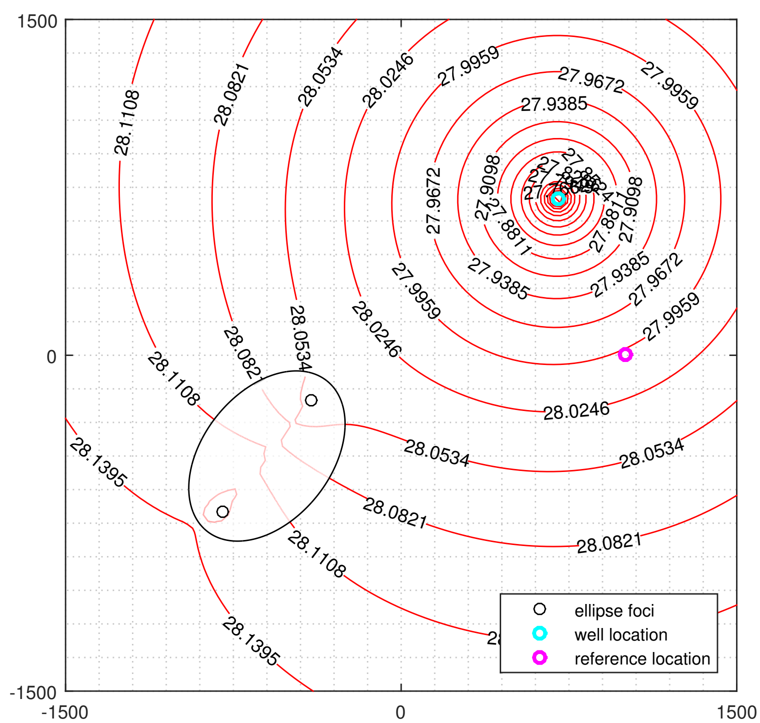

Figure 2: Head contours for well flow only

3 Flow Net for Well and Uniform Flow

Using almost all the same given values as section 2, the amount and direction of uniform flow is also set as given:

- $\beta = 30^\circ$ (0.5236rad.)
- $Q_{uf} = -0.4 \frac{m^2}{day}$

As shown in the figures 3 and 4 to the right, uniform flow has a dramatic effect. It can again be seen from the head contours that the reference head constraint is again satisfied.

Finally, with uniform flow added, the following value is recalculated:

- $C = 4567.9 \frac{m^3}{\text{day}}$

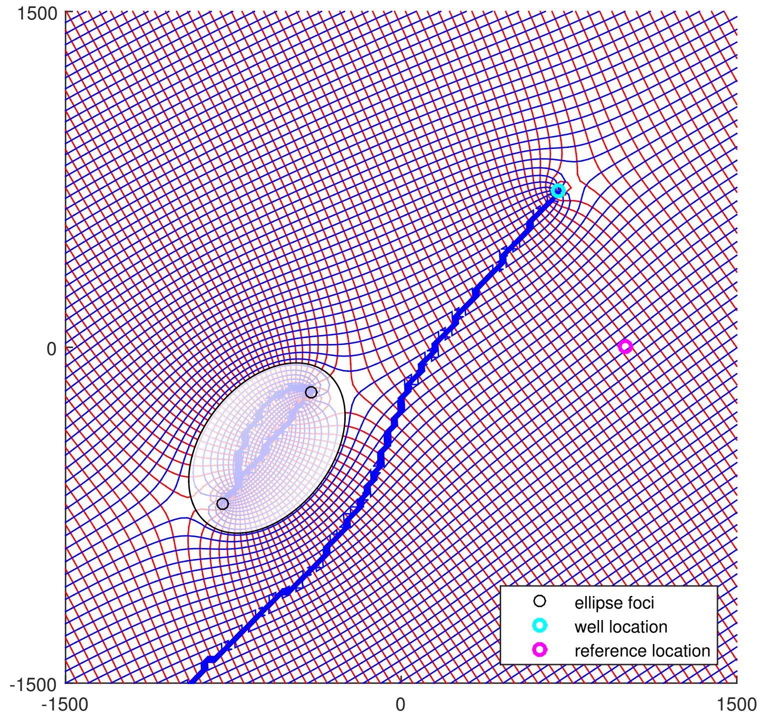

Figure 3: Flow net for well and uniform flow

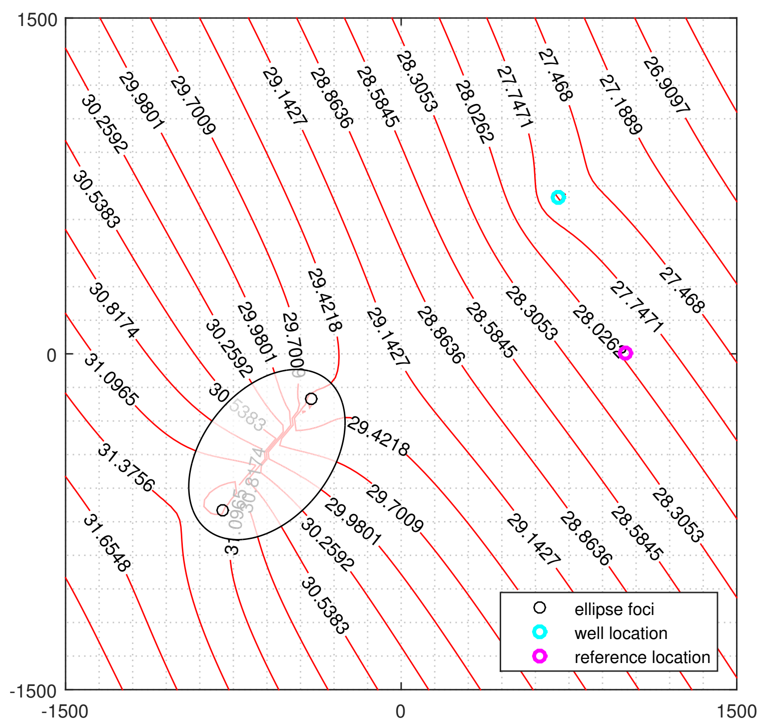

Figure 4: Head contours for well and uniform flow

A Appendix

A.1 Master Script

```
%%%% CEGE 8351 Project 2 Part I Master Script %%%%

clc
close all
clear all

k = 10;                                   %hydraulic conductivity [m/day]
Q0 = -0.4;                                         %uniform flow [m^2/day]
Qw = 200;                                       %well discharge [m^3/day]
zw = complex(700,700);                        %well location [complex - m]
z1 = complex(-800,-700);                   %ellipse 1st focus [complex - m]
z2 = complex(-400,-200);                   %ellipse 2nd focus [complex - m]
z0 = complex(1000,0);                 %reference location [complex - m]
phi0 = 28;                                            %reference head [m]
Phi0= big_Phi_of_little_phi(phi0,k);          %reference discharge potential

roe = 1.5;             %ratio of major and minor principle axes (in z plane)
beta = 30*(pi/180);                       %uniform flow orientation [rad]

C = 0;
C = Phi0 -  real(Omega_total(z0,z1,z2,roe,beta,Q0,Qw,zw,C));
error = 1e-10;

%%% reference head and discharge potential check

assert(real(Omega_total(z0,z1,z2,roe,beta,Q0,Qw,zw,C))-Phi0 < error,...
      'reference head not matched');    %check reference discharge potential

assert(little_phi_of_big_Phi(real(Omega_total(z0,z1,z2,roe,beta,Q0,...
Qw,zw,C)),k)-phi0 < error,'reference head not matched');
                                        %check reference discharge potential
%%% flow nets and head contours

wind = 1500;
figure
ContourMe_flow_net(-wind, wind, 100, -wind, ...
      wind, 100,@(z)Omega_total(z,z1,z2,roe,beta,Q0,Qw,zw,C),50);
hold on
Ellipse_aoverb(roe,z1,z2,1000);
lwell = plot(real(zw),imag(zw),'co','Linewidth',2);
lfoc = plot([real(z1) real(z2)],[imag(z1) imag(z2)],'ko');
lref = plot(real(z0),imag(z0),'mo','Linewidth',2);
ax = gca;
ax.XTick = [-wind 0 wind];
ax.YTick = [-wind 0 wind];
hold  off
legend([lfoc,lwell,lref],{'ellipse foci','well location',...
'reference location'},'Location','Eastoutside');
axis([-wind wind -wind wind]);
print('103','-depsc','-r300')

figure
plot(real(z0),imag(z0),'mo','Linewidth',2);
hold on; grid minor
plot(real(zw),imag(zw),'co','Linewidth',2);
ContourMe_flow_net_phi(-wind, wind, 100, -wind, ...
      wind, 100,@(z)Omega_total(z,z1,z2,roe,beta,Q0,Qw,zw,C),20,k);
Ellipse_aoverb(roe,z1,z2,1000);
lwell = plot(real(zw),imag(zw),'co','Linewidth',2);
lfoc = plot([real(z1) real(z2)],[imag(z1) imag(z2)],'ko');
lref = plot(real(z0),imag(z0),'mo','Linewidth',2);
ax = gca;
ax.XTick = [-wind 0 wind];
ax.YTick = [-wind 0 wind];
hold  off
legend([lfoc,lwell,lref],{'ellipse foci','well location',...
'reference location'},'Location','Eastoutside');
axis([-wind wind -wind wind]);
print('104','-depsc','-r300')
```

A.2 Function Scripts

A.2.1 $\mathbf{Z = f\left(z, z_1, z_2\right)}$)

```
function [ Z ] = bigZ(z,z1,z2)
Z = (z - 0.5*(z2 + z1))/(0.5*(z2-z1));
end
```

A.2.2 $\chi = \mathbf{f}\left(\mathbf{Z}, \nu\right)$

```
function [chi] = chi_of_Z_and_nu(z,z1,z2,nu)
Z = bigZ(z,z1,z2);
chi = (Z + sqrt(Z-1)*sqrt(Z+1))*nu;
end
```

A.2.3 $\nu = \mathbf{f}(\rho)$

```
function [nu] = func_nu(roe)
A = 1/sqrt(1 - roe^(-2));
nu = A - sqrt(A^2 - 1);
end
```

A.2.4 $\alpha = \mathbf{f}(\mathbf{z_1}, \mathbf{z_2})$

```
function [alpha] = alpha(z1,z2)
rise = imag(z2) - imag(z1);
run = real(z2) - real(z1);
alpha = atan(rise/run);
end
```

A.2.5 $\mathbf{\Omega_{uf}}$

```
function [ Omega ] = Omega_uf_of_chi(z,z1,z2,roe,beta,Q0)
aalpha = alpha(z1,z2);
nu = func_nu(roe);
chi = chi_of_Z_and_nu(z,z1,z2,nu);
L = sqrt((z2-z1)*conj(z2-z1));
Omega = ((Q0*L)/(4*nu))*(chi*exp(1i*(aalpha-beta))+ ...
                                    (1/chi)*(exp(-1i*(aalpha-beta))));
end
```

A.2.6 $\mathbf{\Omega_{well}}$

```
function [ Omega ] = Omega_well_of_chi(z,z1,z2,roe,Qw,zw)
nu = func_nu(roe);
chi = chi_of_Z_and_nu(z,z1,z2,nu);
chiw = chi_of_Z_and_nu(zw,z1,z2,nu);
Omega = (Qw/(2*pi))*log(((chi - chiw)*(chi - 1/conj(chiw)))/(chi));
end
```

A.2.7 $\mathbf{\Omega_{total}}$

```
function [ Omega] = Omega_total(z,z1,z2,roe,beta,Q0,Qw,zw,C)
Omega_uf = Omega_uf_of_chi(z,z1,z2,roe,beta,Q0);
Omega_well = Omega_well_of_chi(z,z1,z2,roe,Qw,zw);
Omega = Omega_uf+Omega_well + C;
end
```

A.2.8 $\mathbf{\Phi} = f(\phi, k)$

```
function [ Phi] = big_Phi_of_little_phi(phi,k)
Phi = 0.5*k*phi^2;
end
```

A.2.9 $\phi = f(\Phi, k)$

```
function [phi] = little_phi_of_big_Phi(Phi,k)
phi = sqrt(2*Phi/k);
end
```

A.2.10 $\alpha = \mathbf{f}(\mathbf{z_1}, \mathbf{z_2})$

```
function [alpha] = alpha(z1,z2)
rise = imag(z2) - imag(z1);
run = real(z2) - real(z1);
alpha = atan(rise/run);
end
```

A.3 Ploting Function Files

```
function [Grid] = ContourMe_flow_net(xfrom, xto, Nx, yfrom, yto, Ny, func,
       nint)

Grid = zeros(Ny,Nx);

X = linspace(xfrom, xto, Nx);
Y = linspace(yfrom, yto, Ny);

for row = 1:Ny
    for col = 1:Nx
        Grid(row,col) = func( complex( X(col), Y(row) ) );
    end
end

Bmax=max(imag(Grid));
Bmin=min(imag(Grid));
Cmax=max(Bmax);
Cmin=min(Bmin);
D=Cmax-Cmin;
del=D/nint;
Bmax=max(real(Grid));
Bmin=min(real(Grid));
Cmax=max(Bmax);
Cmin=min(Bmin);
D=Cmax-Cmin;
nintr=round(D/del);

hold
contour(X, Y,real(Grid),nintr,'r');
contour(X, Y,imag(Grid),nint,'b');
axis square
axis equal
```

A.3.1 Head Contour

```
function [Grid] = ContourMe_flow_net_phi(xfrom, xto, Nx, yfrom, yto, Ny, func
       ,nint,k)

Grid = zeros(Ny,Nx);

X = linspace(xfrom, xto, Nx);
Y = linspace(yfrom, yto, Ny);

for row = 1:Ny
    for col = 1:Nx
        Grid(row,col) = func( complex( X(col), Y(row) ) );
    end
end

for row = 1:Ny
    for col = 1:Nx
        Grid(row,col) = sqrt(2*real(Grid(row,col))/k);
    end
end

[x,y] = contour(X, Y,Grid,nint,'r');
clabel(x,y)
```

A.3.2 Ellipse Boundary

```
function [out] = Ellipse_aoverb( ratio,z1,z2,npoints )
% Ratio the major principal axis dividded by the minor one
% z1 and z2 are the foci
t=0;
arg=(ratio-1)/(ratio+1);
nu=sqrt(arg);
n=npoints;
delt=2*pi/n;
x=zeros(1,n);
y=zeros(1,n);
for i=1:n
    t=t+delt;
    chi=cos(t)+1i*sin(t);
    Z=0.5*(chi/nu+nu/chi);
    z=0.5*(z2-z1)*Z+0.5*(z1+z2);
    x(i)=real(z);
    y(i)=imag(z);
end
out = fill(x,y,'w');
axis equal;
hold on;
xe=zeros(1,2);
ye=zeros(1,2);
xe(1,1)=real(z1);
xe(1,2)=real(z2);
ye(1,1)=imag(z1);
ye(1,2)=imag(z2);
%plot(xe,ye,'k');
end
```