Past Exams - MAST 90014

Contents

1 2009 - Semester 1

1.1 Exam

The University of Melbourne
Semester 1, 2009

Department of Mathematics and Statistics
620-616 Optimisation for Industry

Question 1
[Total: 20 marks]

Organic Enterprise produces two competing honey products: OrganicBee and NaturalHive. The company wants to sell these products to two groups of customers: group 1 and group 2. Customers from each of the groups value each product differently. Group 1 customers value each unit of OrganicBee at V_{OB}^1 dollars and each unit of NaturalHive at V_{NH}^1, whereas group 2 customers place V_{OB}^2 dollars on each unit of OrganicBee and V_{NH}^2 dollars on each unit of NaturalHive. Group 1 customers will purchase OrganicBee if the difference between their perceived value and the actual price of the OrganicBee is not less than that of NaturalHive, and that the perceived value on OrganicBee is no smaller than its actual price. Group 2 customers will purchase NaturalHive if the difference between the perceived value and the actual price of the NaturalHive is not less than that of OrganicBee, and that the perceived value on NaturalHive is no smaller than its actual price. Group 1 has N_1 customers and group 2 has N_2 customers.

FORMULATE a linear program that will assist Organic Enterprise set prices for each product such that only group 1 customers purchase OrganicBee and only group 2 customers purchase NaturalHive and that revenues are maximised.

Now suppose $N_1 = 1000$ and $N_2 = 1500$, and the value each customer places on a unit of OrganicBee and NaturalHive is shown below:

	Group 1 Customer	Group 2 Customer
Value of OrganicBee to	\$10	\$12
Value of NativeHive to	\$8	\$15

What are the optimal prices for OrganicBee and NaturalHive that will maximise Organic Enterprise's revenues? SOLVE this graphically.

Question 2
[Total: 15 marks]

Dairy Daily (DD) Pty Ltd manufactures a range of composite dairy products. The set of composite dairy products, $\mathbf{P}$, consists of nutrient-contributing ingredients. Let the set of ingredients be $\mathbf{I}$ and the set of nutrients be $\mathbf{N}$. The amount of nutrient $n \in \mathbf{N}$, in grams, contained in each kilogram of composite dairy product $p \in \mathbf{P}$ must lie within the range $\left[\underline{R}_{pn}, \overline{R}_{pn}\right]$. Each kilogram of ingredient $i \in \mathbf{I}$ contains V_{in} grams of nutrient $n \in \mathbf{N}$. For the current production period, DD plans to manufacture at least $\underline{Q}_p$ but not more than $\overline{Q}_p$ kilograms of product $p \in \mathbf{P}$. DD wants to determine the ingredient mix for each of their product. You can assume no mass loss or nutrient loss during the mixing process. For example, mixing two kilograms of ingredients i and j yields four kilograms of final product, and the amount of nutrient n in the final product is $(V_{in} + V_{jn})$ grams. It costs C_i^U dollars for each kilogram of ingredient $i \in \mathbf{I}$ used. Ingredient $i \in \mathbf{I}$ is stored in cartons of size Q_i^C. For every carton of ingredient i used, a handling cost of C_i^F dollars per carton is incurred.

FORMULATE a mixed-integer program to provide DD with the minimal cost strategy to meet the current period's demand.

Question 3
[Total: 20 marks]

Consider a single-machine scheduling problem with job release date, where the objective is to minimise total completion time. In standard machine scheduling notation, this is a $1 \mid R_j \mid \sum C_j$ problem. For this problem, let P_j be the processing duration and R_j be the release date of job $j \in \{1 \ldots N\}$.

FORMULATE $1 \mid R_j \mid \sum C_j$ as a mixed-integer program.

The following rule, SPT*, can be used to schedule jobs for $1 \mid R_j \mid \sum C_j$:

> *Whenever a machine is freed, schedule the shortest job among those available for processing.*

This is an adaptation of the Shortest Processing Time (SPT) rule to $1 \mid R_j \mid \sum C_j$.

Consider the following problem instance for $1 \mid R_j \mid \sum C_j$.

j	P_j	R_j
1	1	1
2	2	1
3	3	1
4	4	1
5	5	0

Using the SPT* rule, DETERMINE the schedule for this problem instance.

SHOW, for this problem instance, that the schedule generated using SPT* rule is not optimal.

Question 4
[Total: 20 marks]

1. (14 marks) Consider the following IP:

$$\max \sum_{i=1}^{n} P_i x_i$$
$$s.t.$$
$$\sum_{i=1}^{n} W_i x_i \leq B$$
$$x_i \ integer, \quad i = 1, \ldots, n$$

where P_i is the profit from carrying out project i, W_i is the cost of carrying out project i, B is the available budget and x_i is the number of lots of project i carried out. Let $(n, B, P_1, P_2, W_1, W_2) = (2, 9, 2, 3, 3, 2)$. SKETCH the LP feasible region and INDICATE on the same sketch all feasible integer points. SOLVE this IP using the branch and bound method.

2. (6 marks) Figure 1 shows the result of solving a TSP modelled as an IP with variables $x_{ij} = 1$ iff city j is visited immediately after city i. WRITE down TWO different sub-tour breaking constraints for the subtours shown in Figure 1 that could be appended to the IP.

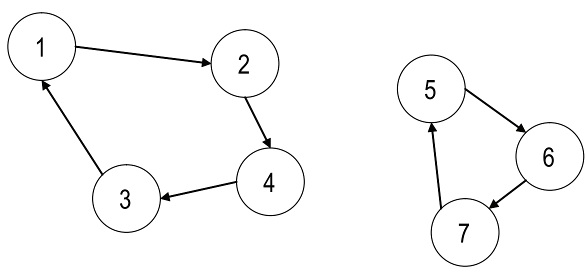


Figure 1:

Question 5
[Total: 25 marks]

Consider the following formulation for a single-product lot-sizing problem:

$$\min \sum_{t=1}^{T} (P_t y_t + H_t s_t + C_t x_t)$$
$$s.t.$$
$$s_1 = I^0 + y_1 - D_1$$
$$s_t = s_{t-1} + y_t - D_t, \quad t = 2, \ldots, T$$
$$y_t \le M x_t, \quad t = 1, \ldots, T$$
$$y_t, s_t \ge 0, \; x_t \in \{0,1\}, \quad t = 1, \ldots, T$$

where P_t is the cost of production in time period t, H_t is the cost of holding inventory at the end of time period t, C_t is the cost of production setup in time period t, D_t is the demand in time period t, I^0 is the opening stock at the beginning of time period 1 and M is a large number. Furthermore, the amount produced in time period t must not exceed L_t.

1. (5 marks) WRITE down an expression for M, such that the constraint $y_t \le M x_t$ is tight for all $t \in \{1, \ldots, T\}$.

2. (20 marks) Consider a two-period version of the above lot-sizing problem with uncertain demand in the second time period. The demand in the second time period can be low (D_2^L), medium (D_2^M) or high (D_2^H) with probability PR_L, PR_M and PR_H respectively. FORMULATE a two-stage stochastic mixed integer program with recourse to minimise the expected total cost of producing, holding and setting up the production for this two-period problem. WRITE down expressions for M for all big-M constraints.

END OF EXAMINATION

1.2 Solution

The University of Melbourne
Semester 1, 2009

Department of Mathematics and Statistics
620-616 Optimisation for Industry
SAMPLE SOLUTION

Question 1
[Total: 20 marks]

Organic Enterprise produces two competing honey products: OrganicBee and NaturalHive. The company wants to sell these products to two groups of customers: group 1 and group 2. Customers from each of the groups value each product differently. Group 1 customers value each unit of OrganicBee at V_{OB}^1 dollars and each unit of NaturalHive at V_{NH}^1, whereas group 2 customers place V_{OB}^2 dollars on each unit of OrganicBee and V_{NH}^2 dollars on each unit of NaturalHive. Group 1 customers will purchase OrganicBee if the difference between their perceived value and the actual price of the OrganicBee is not less than that of NaturalHive, and that the perceived value on OrganicBee is no smaller than its actual price. Group 2 customers will purchase NaturalHive if the difference between the perceived value and the actual price of the NaturalHive is not less than that of OrganicBee, and that the perceived value on NaturalHive is no smaller than its actual price. Group 1 has N_1 customers and group 2 has N_2 customers.

FORMULATE an LP that will assist Organic Enterprise set prices for each product such that only group 1 customers purchase OrganicBee and only group 2 customers purchase NaturalHive and that revenues are maximised.

SOLUTION:
Let p_{OB} and p_{NH} be the prices for OrganicBee and NaturalHive respectively.

$Revenue = N_1 p_{OB} + N_2 p_{NH}$

Group 1 will purchase OrganicBee if
$V_{OB}^1 - p_{OB} \geq V_{NH}^1 - p_{NH}$ and
$V_{OB}^1 \geq p_{OB}$

Group 2 will purchase NaturalHive if
$V_{NH}^2 - p_{NH} \geq V_{OB}^2 - p_{OB}$ and
$V_{NH}^2 \geq p_{NH}$

The LP is:

$\max N_1 p_{OB} + N_2 p_{NH}$
s.t.
$V^1_{OB} - p_{OB} \geq V^1_{NH} - p_{NH}$
$V^1_{OB} \geq p_{OB}$
$V^2_{NH} - p_{NH} \geq V^2_{OB} - p_{OB}$
$V^2_{NH} \geq p_{NH}$
$p_{OB}, p_{NH} \geq 0$
□

Now suppose $N_1 = 1000$ and $N_2 = 1500$, and the value each customer places on a unit of OrganicBee and NaturalHive is shown below:

	Group 1 Customer	Group 2 Customer
Value of OrganicBee to	\$10	\$12
Value of NativeHive to	\$8	\$15

What are the optimal prices for OrganicBee and NaturalHive that will maximise Organic Enterprise's revenues? SOLVE this graphically.

SOLUTION:
Substituting the values to the parameters, and rescaling objective function coefficients yields:

$\max 2p_{OB} + 3p_{NH}$
s.t.
$10 - p_{OB} \geq 8 - p_{NH}$
$10 \geq p_{OB}$
$15 - p_{NH} \geq 12 - p_{OB}$
$15 \geq p_{NH}$
$p_{OB}, p_{NH} \geq 0$

Solving this graphically, yields $p_{OB} = 10$ and $p_{NH} = 13$, giving a revenue of \$29,500. So, OrganicBee should be priced at \$10 per unit and NaturalHive at \$13 per unit.
□

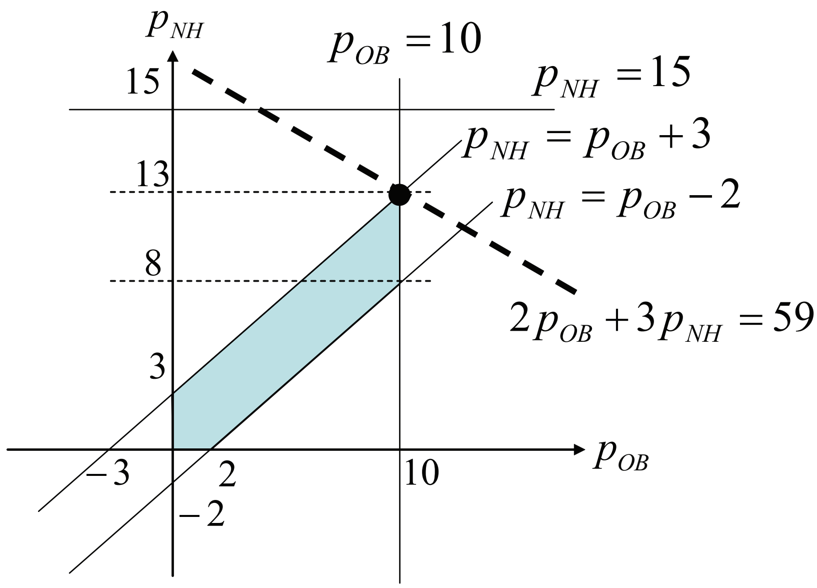


Figure 1:

Question 2
[Total: 15 marks]

Dairy Daily (DD) Pty Ltd manufactures a range of composite dairy products. The set of composite dairy products, $\mathbf{P}$, consists of nutrient-contributing ingredients. Let the set of ingredients be $\mathbf{I}$ and the set of nutrients be $\mathbf{N}$. The amount of nutrient $n \in \mathbf{N}$, in grams, contained in each kilogram of composite dairy product $p \in \mathbf{P}$ must lie within the range $[\underline{R}_{pn}, \overline{R}_{pn}]$. Each kilogram of ingredient $i \in \mathbf{I}$ contains V_{in} grams of nutrient $n \in \mathbf{N}$. For the current production period, DD plans to manufacture at least $\underline{Q}_p$ but not more than $\overline{Q}_p$ kilograms of product $p \in \mathbf{P}$. DD wants to determine the ingredient mix for each of their product. You can assume no mass loss or nutrient loss during the mixing process. For example, mixing two kilograms of ingredients i and j yields four kilograms of final product, and the amount of nutrient n in the final product is $(V_{in} + V_{jn})$ grams. It costs C_i^U dollars for each kilogram of ingredient $i \in \mathbf{I}$ used. Ingredient $i \in \mathbf{I}$ is stored in cartons of size Q_i^C. For every carton of ingredient i used, a handling cost of C_i^F dollars per carton is incurred.

FORMULATE a mixed-integer program to provide DD with the minimal cost strategy to meet the current period's demand.

SOLUTION:

Let

x_{pi} = number of kilograms of ingredient $i \in \mathbf{I}$ used in product $p \in \mathbf{P}$;

z_i = number of cartons of ingredient $i \in \mathbf{I}$ used.

MIP:

$\min \sum_{p \in \mathbf{P}} \sum_{i \in \mathbf{I}} C_i^U x_{pi} + \sum_{i \in \mathbf{I}} C_i^F z_i$

s.t.

$\underline{Q}_p \leq \sum_{i \in \mathbf{I}} x_{pi} \leq \overline{Q}_p, \quad \forall p \in \mathbf{P}$

$\underline{R}_{pn} \sum_{i \in \mathbf{I}} x_{pi} \leq \sum_{i \in \mathbf{I}} V_{in} x_{pi} \leq \overline{R}_{pn} \sum_{i \in \mathbf{I}} x_{pi}, \quad \forall p \in \mathbf{P}, n \in \mathbf{N}$

$\sum_{p \in \mathbf{P}} x_{pi} \leq Q_i^C z_i, \quad \forall i \in \mathbf{I}$

$x_{pi} \geq 0, \quad \forall p \in \mathbf{P}, i \in \mathbf{I}$

$z_i \ integer, \quad \forall i \in \mathbf{I}$

□

Question 3
[Total: 20 marks]

Consider a single-machine scheduling problem with job release date, where the objective is to minimise total completion time. In standard machine scheduling notation, this is a $1 \mid R_j \mid \sum C_j$ problem. For this problem, let P_j be the processing duration and R_j be the release date of job $j \in \{1 \ldots N\}$.

FORMULATE $1 \mid R_j \mid \sum C_j$ as a mixed-integer program.

SOLUTION:
Let
t_j be the start time of job in position j, for all $j = 1, \ldots, N$;
$x_{ij} = 1$ iff job i is in position j, for all $i = 1, \ldots, N$, $j = 1, \ldots, N$.

MIP:
min $\sum_{j=1}^{N} \left[\left(t_j + \sum_{i=1}^{N} P_i x_{ij} \right) \right]$
s.t.
$t_j \geq t_{j-1} + \sum_{i=1}^{N} P_i x_{i(j-1)}, \quad j = 2, \ldots, N$
$t_j \geq \sum_{i=1}^{N} R_i x_{ij}, \quad j = 1, \ldots, N$
$\sum_{i=1}^{N} x_{ij} = 1, \quad j = 1, \ldots, N$
$\sum_{j=1}^{N} x_{ij} = 1, \quad i = 1, \ldots, N$
$x_{ij} \in \{0, 1\}, \quad i = 1, \ldots, N,\ j = 1, \ldots, N$
$t_j \geq 0, \quad j = 1, \ldots, N$
$\square$

The following rule, SPT*, can be used to schedule jobs for $1 \mid R_j \mid \sum C_j$:

> *Whenever a machine is freed, schedule the shortest job among those available for processing.*

This is an adaptation of the Shortest Processing Time (SPT) rule to $1 \mid R_j \mid \sum C_j$.

Consider the following problem instance for $1 \mid R_j \mid \sum C_j$.

j	P_j	R_j
1	1	1
2	2	1
3	3	1
4	4	1
5	5	0

Using the SPT* rule, DETERMINE the schedule for this problem instance.

SOLUTION:
The SPT* schedule is shown below (Figure 2):

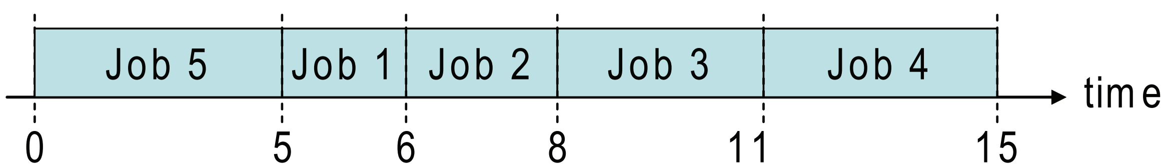


Figure 2:

Total completion time $= F^{SPT^*} = 5 + 6 + 8 + 11 + 15 = 45.$
□

SHOW, for this problem instance, that the schedule generated using SPT* rule is not optimal.

SOLUTION:
A better schedule can obtained by scheduling the jobs in SPT-order after time 1. The schedule is shown in Figure 3.

The total completion time is $2 + 4 + 7 + 11 + 16 = 40 < F^{SPT^*}$. Hence the schedule generated using SPT* rule is not optimal.
□

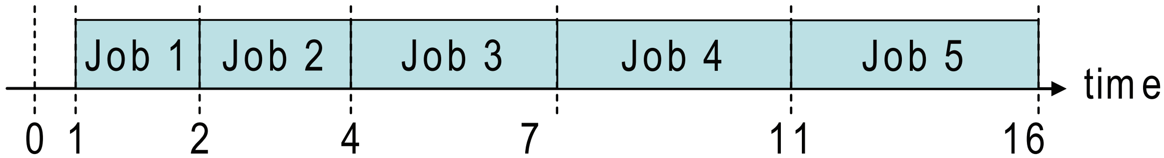


Figure 3:

Question 4
[Total: 20 marks]

1. (14 marks) Consider the following IP:

$$\max \sum_{i=1}^{n} P_i x_i$$
$$s.t.$$
$$\sum_{i=1}^{n} W_i x_i \leq B$$
$$x_i \ integer, \quad i = 1, \ldots, n$$

where P_i is the profit from carrying out project i, W_i is the cost of carrying out project i, B is the available budget and x_i is the number of lots of project i carried out. Let

$$(n, B, P_1, P_2, W_1, W_2) = (2, 9, 2, 3, 3, 2)\,.$$

SKETCH the LP feasible region and INDICATE on the same sketch all feasible integer points. SOLVE this IP using the branch and bound method.

SOLUTION:

The IP considered here is:

$$\max z = 2x_1 + 3x_2$$
$$s.t.$$
$$3x_1 + 2x_2 \leq 9$$
$$x_1, x_2 \ integer$$

The LP feasible region is shown in Figure 4.

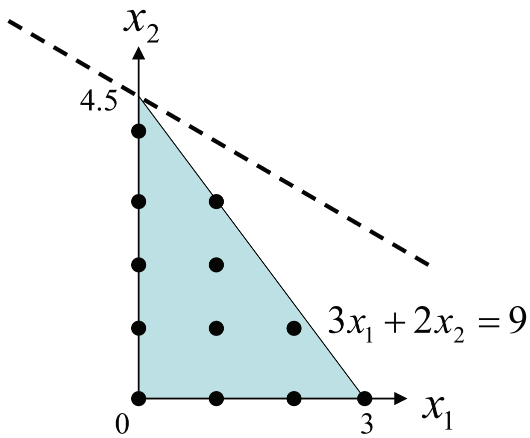


Figure 4:

The black dots in this Figure 4 are integer feasible points.

The following LP is solved at the root node (Node 0):

$$\max z = 2x_1 + 3x_2$$
$$s.t.$$
$$3x_1 + 2x_2 \leq 9$$

Solving this graphically yields the optimal solution $(z, x_1, x_2) = (13\frac{1}{2}, 0, 4\frac{1}{2})$. The branch and bound process is shown in Figure 5, where the rule depth-first search and to branch down on a variable first.

The optimal solution to the IP is $(z, x_1, x_2) = (12, 0, 4)$.

□

2. (6 marks) Consider a Travelling Salesperson Problem with seven cities. Figure 6 shows a result of solving an IP with variables $x_{ij} = 1$ iff city j is visited immediately after city i. WRITE down TWO different subtour breaking constraints for the subtours shown in Figure 6 that could be appended to the original IP.

SOLUTION:

Any two of the following subtour breaking constraints are acceptable:

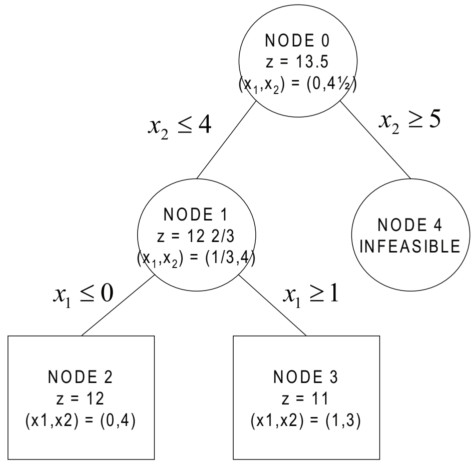


Figure 5:

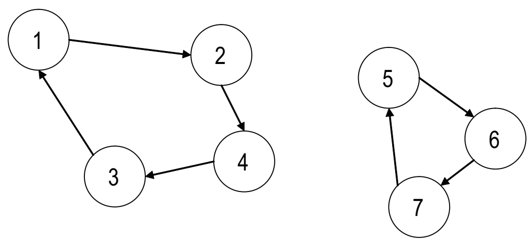


Figure 6:

$$\sum_{i\in\{1,2,3,4\}} \sum_{j\in\{1,2,3,4\}} x_{ij} \leq 3$$

$$\sum_{i\in\{5,6,7\}} \sum_{j\in\{5,6,7\}} x_{ij} \leq 2$$

$$\sum_{i\in\{1,2,3,4\}} \sum_{j\in\{5,6,7\}} x_{ij} \geq 1$$

□

Question 5
[Total: 25 marks]

Consider the following formulation for a single-product lot-sizing problem:

$$
\begin{aligned}
&\min \sum_{t=1}^{T} (P_t y_t + H_t s_t + C_t x_t) \\
&s.t. \\
&s_1 = I^0 + y_1 - D_1 \\
&s_t = s_{t-1} + y_t - D_t, \quad t = 2, \ldots, T \\
&y_t \leq M x_t, \quad t = 1, \ldots, T \\
&y_t, s_t \geq 0,\ x_t \in \{0, 1\}, \quad t = 1, \ldots, T
\end{aligned}
$$

where P_t is the cost of production in time period t, H_t is the cost of holding inventory at the end of time period t, C_t is the cost of production setup in time period t, D_t is the demand in time period t, I^0 is the opening stock at the beginning of time period 1 and M is a large number. Furthermore, the amount produced in time period t must not exceed L_t.

1. (5 marks) WRITE down an expression for M, such that the constraint $y_t \leq M x_t$ is tight for all $t \in \{1, \ldots, T\}$.

   SOLUTION:

   We know that for any time period t, we can at most produce at capacity L_t. So, we can set $M = L_t$. However, this can tighten further if demand is taken into consideration. Since no backlog is allowed, the amount produced in time period t can only be used to satisfy demands in time periods $k \in \{t, \ldots, T\}$. Therefore, we can let $M = \min\left(L_t, \sum_{k=t}^{T} D_k\right)$, resulting in the following constraint:

   $$y_t \leq \min\left(L_t, \sum_{k=t}^{T} D_k\right) x_t, \quad t = 1, \ldots, T$$

   □

2. (20 marks) Consider a two-period version of the above lot-sizing problem with uncertain demand in the second time period. The demand in the second time period

can be low (D_2^L), medium (D_2^M) or high (D_2^H) with probability PR_L, PR_M and PR_H respectively. FORMULATE a two-stage stochastic mixed integer program with recourse to minimise the expected total cost of producing, holding and setting up the production for this two-period problem. WRITE down expressions for M for all big-M constraints.

SOLUTION:

Let Stage 1 variables be:

y_1 be the production amount in time period 1;

s_1 be the stock amount at the end of time period 1;

$x_1 = 1$ iff setup is carried out in time period 1

We need to define recourse (Stage 2) variables for all scenarios in the second time period. Let

y_2^i be the production amount in time period 2, given scenario $i \in \{L, M, H\}$;

s_2^i be the stock amount at the end of time period 2, given scenario $i \in \{L, M, H\}$;

$x_2^i = 1$ iff setup is carried out in time period 2, given scenario $i \in \{L, M, H\}$.

The two-stage stochastic integer program is:

$$
\begin{aligned}
&\min P_1 y_1 + H_1 s_1 + C_1 x_1 + \sum_{i \in \{L,M,H\}} PR_i \left(P_2 y_2^i + H_2 s_2^i + C_2 x_2^i \right) \\
&s.t. \\
&s_1 = I^0 + y_1 - D_1 \\
&s_2^i = s_1 + y_2^i - D_2^i, \quad \forall i \in \{L, M, H\} \\
&y_1 \leq M_1 x_1, \quad t = 1, \dots, 2 \\
&y_2^i \leq M_2^i x_2^i, \quad \forall i \in \{L, M, H\} \\
& \\
&y_1 \geq 0,\ x_1 \in \{0, 1\} \\
&y_2^i \geq 0,\ x_2^i \in \{0, 1\}, \quad \forall i \in \{L, M, H\}
\end{aligned}
$$

where

$$M_1 = \min\left(L_1, D_1 + D_2^H\right)$$

since production in the first time period should account for the fact that a high demand scenario can occur;

$$M_2^i = \min\left(L_2, D_2^i\right)$$

$\square$

END OF SOLUTION

2 2010 - Semester 1

2.1 Exam

The University of Melbourne

Semester 1 Assessment 2010

Student Number: ....

Department of Mathematics and Statistics
620-616 Optimisation for Industry

Reading time: 15 minutes
Writing time: 120 minutes
This paper has 7 pages.

Identical Examination Papers:
N/A

Common Content Papers:
N/A

Authorised Materials:
The following items are authorised: Calculators.

Instructions to Invigilators:
No handouts are required.
The examination paper is to remain in the examination room.

Instructions to Students:
Answer all questions. Clearly define all parameters and variables in your answer, and state all assumptions.

Paper to be held by Baillieu Library:
No

Extra material required:

Graph paper ☐ Multiple Choice form ☐ Other (please specify) ☐

Question 1
[Total: 15 marks]

Consider a knapsack problem with N objects and a knapsack with weight limit L and (one-dimensional) space limit C. Let W_j, S_j and P_j be the weight, size and value of item $j \in \{1, \ldots, N\}$ respectively.

(a) (5 marks) FORMULATE the knapsack problem as an integer program, maximising the total value of items in the knapsack, such that the knapsack weight and space limits are not violated.

(b) (10 marks) Consider a 2-object knapsack problem with weight limit of 25 and space limit of 7, and the following item attributes:

Item	1	2
W_j	10	4
S_j	1	2
P_j	1	2

FIND the optimal solution to this knapsack instance using the branch and bound method for integer programs.

Question 2
[Total: 25 marks]

The *Wealthy Bank* is headquartered in Melbourne's Central Business District (CBD) and has N remote branches in Victoria. Let $\mathbf{M} = \{1, 2, \ldots, N\}$ be the set of all branches and $\mathbf{L} = \mathbf{M} \cup \{0\}$ be the set of all Bank locations (0 is the headquarters). All bank locations support cash deposit and withdrawal activities.

The Bank has to decide, on a daily basis, the movements of its hard cash (money notes) between its headquarters and branches. Let $\mathbf{D} = \{1, \ldots, T\}$ be the set of business days for which the Bank wants to plan its cash movements. It will cost the Bank $\$C_{ij}$ to transport any positive amount of hard cash between locations $i, j \in \mathbf{L}$. Cash movements can only be made once in a day, before the start of business hours of the day.

The amount of hard cash available at a Bank location $i \in \mathbf{L}$ before day $t = 1$ is $\$I_i^0$. The amount of cash deposits and withdrawals at location $i \in \mathbf{L}$ on day $t \in \mathbf{D}$ is $\$D_{it}$ and $\$W_{it}$ respectively. A cash deposit activity increases the amount of hard cash at a location, whereas a cash withdrawal activity decreases the amount of hard cash at a location. At any point during a business day, the amount of hard cash at location $i \in \mathbf{L}$ must not exceed $\$C_i^{max}$. At the end of a business day, the amount of hard cash at location $i \in \mathbf{L}$ must be at least $\$C_i^{min}$, otherwise the Bank will be penalised at a rate of $\$\alpha$ for every dollar below $\$C_i^{min}$.

The amount of hard cash at the end of a business day at the Bank's headquarters attracts a return of $\beta\%$ per day in interests. Suppose $\beta = 0.1\%$ and there is \$10,000 at the end of business day at its headquarters, the total interest earned is \$10. The amount of hard cash stored at the Bank's branches do not attract any interest returns. Assume the interest earned is not accessible immediately, i.e. it does not add to the pool of hard cash for circulation.

FORMULATE a mixed-integer linear program whose solution will minimise the total cost of moving hard cash for Wealthy Bank, where the total cost is given by

$$TotalCost = TotalTransportationCost + TotalPenalty - TotalInterestEarned$$

Clearly define all parameters and variables in your model, and state all assumptions. The value of M in any of your Big-M constraints must be clearly defined.

Question 3
[Total: 30 marks]

Consider the makespan minimisation problem on m identical parallel machines. In standard machine scheduling notation, this is a $Pm| |C_{max}$ problem. Let there be N jobs and P_j be the processing time for job $j \in \{1, \ldots, N\}$.

(a) (10 marks) FORMULATE $Pm| |C_{max}$ as a mixed integer linear program.

(b) (3 marks) STATE the lowerbound(s) for $Pm| |C_{max}$.

(c) (8 marks) SHOW for any list schedule,

$$\frac{C_{max}(LS)}{C_{max}(OPT)} \leq 2 - \frac{1}{m}$$

where $C_{max}(LS)$ and $C_{max}(OPT)$ are makespans of a list schedule and the optimal schedule respectively. You do not need to show the tightness of this bound.

(d) (9 marks) Consider 4 identical parallel machines and 9 jobs, whose processing times are given in the table below:

Jobs	1	2	3	4	5	6	7	8	9
P_j	7	7	6	6	5	5	4	4	4

SHOW that this is a worst case instance for the Longest Processing Time (LPT) heuristic on 4 identical parallel machines.

NOTES:

- When a machine is freed, the LPT heuristic schedules the longest job among those not yet processed to the machine.
- It is known that the worst case bound for the LPT heuristic is

$$\frac{C_{max}(LPT)}{C_{max}(OPT)} \leq \frac{4}{3} - \frac{1}{3m}$$

where $C_{max}(LPT)$ and $C_{max}(OPT)$ are makespans of LPT and optimal schedules respectively.

Question 4
[Total: 10 marks]

Consider the uncapacitated vehicle routing problem with V vehicles. Let $N = \{1, \ldots, n\}$ be the set of nodes and A be the set of all arcs for the problem. Node $0 \in N$ is the depot and the cost of going along arc (i, j) is C_{ij}. The arc-based formulation for this problem is as follows:

$$\min \sum_{(i,j)\in A} C_{ij} x_{ij}$$
$$s.t.$$
$$\sum_{i\in N} x_{ij} = 1, \qquad \forall j \in N \backslash \{0\}$$
$$\sum_{j\in N} x_{ij} = 1, \qquad \forall i \in N \backslash \{0\}$$
$$\sum_{j\in N} x_{0j} \leq V$$
$$\sum_{i\in S} \sum_{j\in \bar{S}} x_{ij} \geq 1, \qquad \forall S \subseteq N \backslash \{0\}, S \neq \emptyset$$
$$x_{ij} \in \{0, 1\}, \qquad \forall (i, j) \in A$$

where $x_{ij} = 1$ if arc (i, j) is used, 0 otherwise.

The path-based formulation is:

$$\min \sum_{r\in R} C_r z_r$$
$$s.t.$$
$$\sum_{r\in R} A_{ir} z_r = 1, \qquad \forall i \in N \qquad (dual : \alpha_i)$$
$$-\sum_{r\in R} z_r \geq -V, \qquad (dual : \beta)$$
$$z_r \in \{0, 1\}, \qquad r \in R$$

where
R is the set of feasible paths,
C_r is the cost of using path $r \in R$,
$A_{ir} = 1$ if node $i \in N$ is in path $r \in R$, 0 otherwise,
$z_r = 1$ if path $r \in R$ is used, 0 otherwise.

(a) (2 marks) WRITE down an expression of the reduced cost of variable z_r in terms of the parameters/variables in the path-based formulation.

(b) (6 marks) FORMULATE a linear program that generates a new column which could be appended to the path-based formulation.

(c) (2 marks) STATE the condition in which the new column generated by the linear program is appended to the path-based formulation.

Question 5
[Total: 20 marks]

The capacitated facility location (CFL) problem with deterministic demand can be formulated as follows:

$$\min \sum_{i\in L} F_i x_i + \sum_{i\in L}\sum_{j\in C} W_{ij} y_{ij}$$

$$s.t.$$

$$\sum_{i\in L} y_{ij} \geq D_j, \qquad \forall j \in C$$

$$\sum_{j\in C} y_{ij} \leq B_i x_i, \qquad \forall i \in L$$

$$y_{ij} \leq M x_i, \qquad \forall i \in L, j \in C$$

$$x_i \in \{0,1\}, \qquad \forall i \in L$$

$$y_{ij} \geq 0, \qquad \forall i \in L, j \in C$$

where
L is the set of all potential sites,
C is the set of all customers,
F_i is the fixed cost to setup facility at site $i \in L$,
W_{ij} is the unit cost to fulfill customer j's demand from site $i \in L$,
D_j is the demand for customer $j \in C$,
B_i is the capacity of site $i \in L$,
$x_i = 1$ if site $i \in L$ is used, 0 otherwise,
y_{ij} is the amount of customer j's demand fulfilled by site $i \in L$,
M is a sufficiently big number.

(a) (2 marks) WRITE down an expression for M such that the constraint $y_{ij} \leq Mx_i$ is tight for CFL.

(b) (18 marks) Now consider the stochastic version of CFL, where customer demands are uncertain. Let S be the set of all customer demand realisation scenarios, and let D_j^σ be customer j's demand given realisation scenario $\sigma \in S$. Each realisation scenario $\sigma \in S$ has realisation probability of P^σ, and $\sum_{\sigma\in S} P^\sigma = 1$. FORMULATE a stochastic integer linear program for the stochastic version of CFL. WRITE down expressions of M for all big-M constraints.

End of Examination

2.2 Solution

The University of Melbourne

Semester 1 Assessment 2010

SOLUTION

Department of Mathematics and Statistics

620-616 Optimisation for Industry

Question 1
[Total: 15 marks]

Consider a knapsack problem with N objects and a knapsack with weight limit L and (one-dimensional) space limit C. Let W_j, S_j and P_j be the weight, size and value of item $j \in \{1, \ldots, N\}$ respectively.

(a) (5 marks) FORMULATE the knapsack problem as an integer program, maximising the total value of items in the knapsack, such that the knapsack weight and space limits are not violated.

SOLUTION:

Let x_j be the number of item j in the knapsack.

(1 mark)

$$\max \sum_{j=1}^{N} P_j x_j$$
$$s.t.$$

(1.5 marks)

$$\sum_{j=1}^{N} W_j x_j \leq L$$

(1.5 marks)

$$\sum_{j=1}^{N} S_j x_j \leq C$$

(1 mark)

$$x_j \geq 0, int \qquad j = 1, \ldots, N$$

(b) (10 marks) Consider a 2-object knapsack problem with weight limit of 25 and space limit of 7, and the following item attributes:

Item	1	2
W_j	10	4
S_j	1	2
P_j	1	2

FIND the optimal solution to this knapsack instance using the branch and bound method for integer programs.

SOLUTION:

The integer program is:

$$\max z = x_1 + 2x_2$$
$$s.t.$$
$$10x_1 + 4x_2 \leq 25$$
$$x_1 + 2x_2 \leq 7$$
$$x_1, x_2 \geq 0, int$$

(2 marks)

The optimal solution is $(z, x_1, x_2) = (7, 1, 3)$. (See branch and bound tree in Figure 1)

(Max 8 marks - take 1 mark off each incorrect branching/bounding/LP-sol)

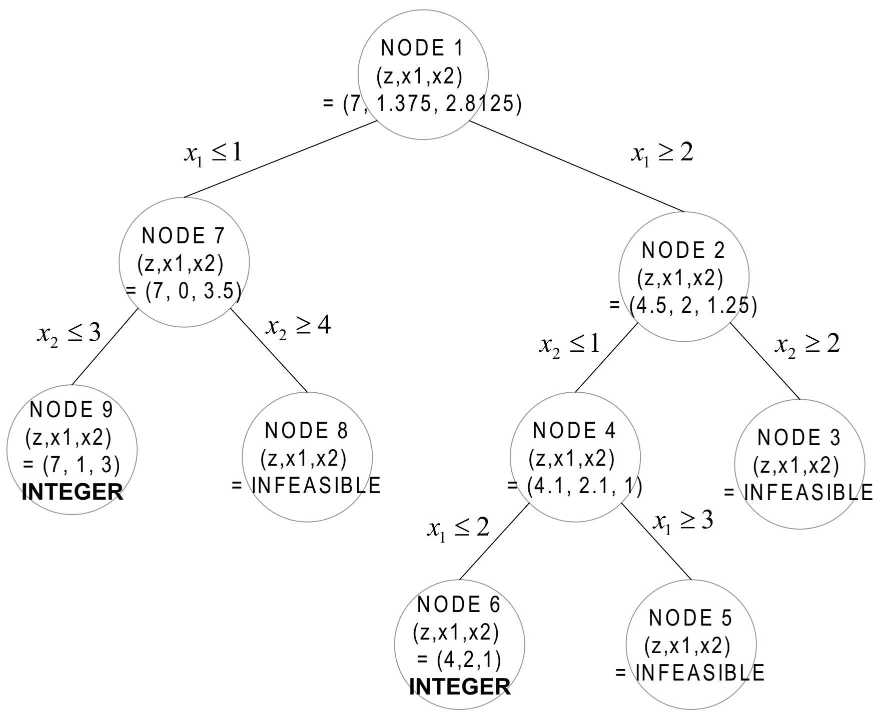


Figure 1:

Question 2
[Total: 25 marks]

The *Wealthy Bank* is headquartered in Melbourne's Central Business District (CBD) and has N remote branches in Victoria. Let $\mathbf{M} = \{1, 2, \ldots, N\}$ be the set of all branches and $\mathbf{L} = \mathbf{M} \cup \{0\}$ be the set of all Bank locations (0 is the headquarters). All bank locations support cash deposit and withdrawal activities.

The Bank has to decide, on a daily basis, the movements of its hard cash (money notes) between its headquarters and branches. Let $\mathbf{D} = \{1, \ldots, T\}$ be the set of business days for which the Bank wants to plan its cash movements. It will cost the Bank $\$C_{ij}$ to transport any positive amount of hard cash between locations $i, j \in \mathbf{L}$. Cash movements can only be made once in a day, before the start of business hours of the day.

The amount of hard cash available at a Bank location $i \in \mathbf{L}$ before day $t = 1$ is $\$I_i^0$. The amount of cash deposits and withdrawals at location $i \in \mathbf{L}$ on day $t \in \mathbf{D}$ is $\$D_{it}$ and $\$W_{it}$ respectively. A cash deposit activity increases the amount of hard cash at a location, whereas a cash withdrawal activity decreases the amount of hard cash at a location. At any point during a business day, the amount of hard cash at location $i \in \mathbf{L}$ must not exceed $\$C_i^{max}$. At the end of a business day, the amount of hard cash at location $i \in \mathbf{L}$ must be at least $\$C_i^{min}$, otherwise the Bank will be penalised at a rate of $\$\alpha$ for every dollar below $\$C_i^{min}$.

The amount of hard cash at the end of a business day at the Bank's headquarters attracts a return of $\beta\%$ per day in interests. Suppose $\beta = 0.1\%$ and there is \$10,000 at the end of business day at its headquarters, the total interest earned is \$10. The amount of hard cash stored at the Bank's branches do not attract any interest returns. Assume the interest earned is not accessible immediately, i.e. it does not add to the pool of hard cash for circulation.

FORMULATE a mixed-integer linear program whose solution will minimise the total cost of moving hard cash for Wealthy Bank, where the total cost is given by

$$TotalCost = TotalTransportationCost + TotalPenalty - TotalInterestEarned$$

Clearly define all parameters and variables in your model, and state all assumptions. The value of M in any of your Big-M constraints must be clearly defined.

SOLUTION:
Let
(3 marks)

x_{ijt} be the amount of hard cash transferred from $i \in \mathbf{L}$ to $j \in \mathbf{L}$ at the beginning of day $t \in \mathbf{D}$, $i \neq j$;

$z_{ijt} = 1$ if there is hard cash transferred from $i \in \mathbf{L}$ to $j \in \mathbf{L}$ on day $t \in \mathbf{D}$, 0 otherwise;

i_{kt} be the amount of hard cash available at $k \in \mathbf{L}$ at the end of day $t \in \mathbf{D}$;

s_{kt} be the amount of hard cash below C_k^{min};

The mixed-integer program is:

(3 marks)

$$\min \sum_{i\in\mathbf{L}} \sum_{j\in\mathbf{L}} \sum_{t\in\mathbf{D}} C_{ij} z_{ijt} + \sum_{i\in\mathbf{L}} \sum_{j\in\mathbf{L}} \alpha s_{kt} - \sum_{t\in\mathbf{D}} \beta i_{0t}$$

$$s.t.$$

(4 marks)

$$i_{kt} = i_{k(t-1)} + \sum_{i\in\mathbf{L}} x_{ikt} + D_{kt} - \sum_{i\in\mathbf{L}} x_{kit} - W_{kt}, \qquad \forall k \in \mathbf{L}, t \in \mathbf{D}, i_{k0} = I_k^0$$

(3 marks)

$$x_{ijt} \leq M_{ijt} z_{ijt}, \qquad \forall i, j \in \mathbf{L}, t \in \mathbf{D}$$

(4 marks)

$$i_{k(t-1)} + \sum_{i\in\mathbf{L}} x_{ikt} + D_{kt} \leq C_k^{max}, \qquad \forall k \in \mathbf{L}, t \in \mathbf{D}$$

(3 marks)

$$s_{kt} \geq C_k^{min} - i_{kt}, \qquad \forall k \in \mathbf{L}, t \in \mathbf{D}$$

(2 marks)

$$\mathbf{x}, \mathbf{i}, \mathbf{s} \geq 0, \; \mathbf{z} \in \{0, 1\}$$

(3 marks)

where $M_{ijt} = \min\left(C_i^{max}, C_j^{max}\right)$.

Question 3
[Total: 30 marks]

Consider the makespan minimisation problem on m identical parallel machines. In standard machine scheduling notation, this is a $Pm| \, |C_{max}$ problem. Let there be N jobs and P_j be the processing time for job $j \in \{1, \ldots, N\}$.

(a) (10 marks) FORMULATE $Pm| \, |C_{max}$ as a mixed integer linear program.

SOLUTION:

(1 mark)

Let

$x_{ijk} = 1$ if job i is in position j of machine k

t_{jk} be the start time of the job in position j of machine k

C_{max} be the makespan

The mixed integer program is:

(1 mark)

$$\min C_{max}$$

(1 mark)

$$C_{max} \geq t_{jk} + \sum_{j=1}^{N} P_i x_{ijk}$$

$$s.t.$$

(2 marks)

$$\sum_{j=1}^{N} \sum_{k=1}^{m} x_{ijk} = 1, \qquad i = 1, \ldots, N$$

(2 marks)

$$\sum_{i=1}^{N} x_{ijk} \leq 1, \qquad j = 1, \ldots, N, \; k = 1, \ldots, m$$

(2 marks)

$$t_{jk} \geq t_{(j-1)k} + \sum_{j=1}^{N} P_i x_{i(j-1)k}, \qquad j = 1, \ldots, N, \; k = 1, \ldots, m$$

(1 mark)

$$x_{ijk} \in \{0,1\}, \qquad i,j = 1,\ldots,N,\ k = 1,\ldots,m$$
$$t_{jk} \geq 0, \qquad j = 1,\ldots,N,\ k = 1,\ldots,m$$

(b) (3 marks) STATE the lowerbound(s) for $Pm| |C_{max}$.

SOLUTION:

$$C_{max}(OPT) \geq \max(P_{max}, \frac{\sum_{j=1}^{N} P_j}{m})$$

where $P_{max} = \max_{i=1}^{N}(P_j)$.

(c) (8 marks) SHOW for any list schedule,

$$\frac{C_{max}(LS)}{C_{max}(OPT)} \leq 2 - \frac{1}{m}$$

where $C_{max}(LS)$ and $C_{max}(OPT)$ are makespans of a list schedule and the optimal schedule respectively. You do not need to show the tightness of this bound.

SOLUTION:

(1 mark)

Let P_l be the processing time of the last job in a list schedule and t be the start time of this job. It follows that $C_{max}(LS) = t + P_l$.

(2 marks)

It is also clear that

$$t \leq \frac{\sum_{j=1:j\neq l}^{N} P_j}{m}$$

since jobs are not unnecessarily delayed.

(2 marks)

Therefore,

$$\begin{aligned}
C_{max}(LS) &\leq \frac{\sum_{j=1:j\neq l}^{N} P_j}{m} + P_l \\
&= \frac{\sum_{j=1:j\neq l}^{N} P_j}{m} + \frac{P_l - P_l}{m} + P_l \\
&= \frac{\sum_{j=1}^{N} P_j}{m} + \frac{m-1}{m} P_l
\end{aligned}$$

(1 mark)

Since

$$C_{max}(OPT) \geq P_{max} \geq P_l$$

and

$$C_{max}(OPT) \geq \frac{\sum_{j=1}^{N} P_j}{m}$$

it follows that

(2 marks)

$$\begin{aligned} C_{max}(LS) &\leq C_{max}(OPT) + \frac{m-1}{m} C_{max}(OPT) \\ &= \left(2 - \frac{1}{m}\right) C_{max}(OPT) \end{aligned}$$

Hence,

$$\frac{C_{max}(LS)}{C_{max}(OPT)} \leq 2 - \frac{1}{m}$$

(d) (9 marks) Consider 4 identical parallel machines and 9 jobs, whose processing times are given in the table below:

Jobs	1	2	3	4	5	6	7	8	9
P_j	7	7	6	6	5	5	4	4	4

SHOW that this is a worst case instance for the Longest Processing Time (LPT) heuristic on 4 identical parallel machines.

NOTES:

- When a machine is freed, the LPT heuristic schedules the longest job among those not yet processed to the machine.
- It is known that the worst case bound for the LPT heuristic is

$$\frac{C_{max}(LPT)}{C_{max}(OPT)} \leq \frac{4}{3} - \frac{1}{3m}$$

where $C_{max}(LPT)$ and $C_{max}(OPT)$ are makespans of LPT and optimal schedules respectively.

SOLUTION:

(3 marks)

Using the LPT heuristic, we get a makespan of 15. Letting J_i be the set of job scheduled on machine i, the LPT schedule is as follows:

$J_1 = \{1, 7, 9\}$

$J_2 = \{2, 8\}$

$J_3 = \{3, 5\}$

$J_4 = \{4, 6\}$

(4 marks)

The optimal schedule has makespan lowerbound of $\frac{\sum_{j=1}^{9} P_j}{4} = 12$. The schedule that achieves this is:

$J_1 = \{1, 5\}$

$J_2 = \{2, 6\}$

$J_3 = \{3, 4\}$

$J_4 = \{7, 8, 9\}$

(2 marks)

For $m = 4$, the performance of LPT is $\frac{C_{max}(LPT)}{C_{max}(OPT)} \leq \frac{4}{3} - \frac{1}{12} = \frac{15}{12}$. Hence this instance is a worst-case instance for LPT when $m = 4$.

Question 4
[Total: 10 marks]

Consider the uncapacitated vehicle routing problem with V vehicles. Let $N = \{0, 1, \ldots, n\}$ be the set of nodes and A be the set of all arcs for the problem. Node $0 \in N$ is the depot and the cost of going along arc (i, j) is C_{ij}. The arc-based formulation for this problem is as follows:

$$\min \sum_{(i,j)\in A} C_{ij} x_{ij}$$
$$s.t.$$
$$\sum_{i\in N} x_{ij} = 1, \qquad \forall j \in N\backslash\{0\}$$
$$\sum_{j\in N} x_{ij} = 1, \qquad \forall i \in N\backslash\{0\}$$
$$\sum_{j\in N} x_{0j} \leq V$$
$$\sum_{i\in S}\sum_{j\in \bar{S}} x_{ij} \geq 1, \qquad \forall S \subseteq N\backslash\{0\}, S \neq \emptyset$$
$$x_{ij} \in \{0, 1\}, \qquad \forall (i, j) \in A$$

where $x_{ij} = 1$ if arc (i, j) is used, 0 otherwise.

The path-based formulation is:

$$\min \sum_{r\in R} C_r z_r$$
$$s.t.$$
$$\sum_{r\in R} A_{ir} z_r = 1, \qquad \forall i \in N \qquad (dual : \alpha_i)$$
$$-\sum_{r\in R} z_r \geq -V, \qquad (dual : \beta)$$
$$z_r \in \{0, 1\}, \qquad r \in R$$

where
R is the set of feasible paths,
C_r is the cost of using path $r \in R$,
$A_{ir} = 1$ if node $i \in N$ is in path $r \in R$, 0 otherwise,
$z_r = 1$ if path $r \in R$ is used, 0 otherwise.

(a) (2 marks) WRITE down an expression of the reduced cost of variable z_r in terms of the parameters/variables in the path-based formulation.

SOLUTION:

$R.C.(z_r) = C_r - \sum_{i \in N} A_{ir}\alpha_i + \beta$

(b) (6 marks) FORMULATE a linear program that generates a new column which could be appended to the path-based formulation.

SOLUTION:

The subproblem is a shortest path problem. Introduce fictitious node $(n+1)$ as destination node, i.e. we wish to find the shortest path from node 0 to $(n+1)$.

Let $\bar{N} = N \cup \{n+1\}$, and $\bar{A} = A \cup \{(j, n+1) : j \in N \backslash \{0\}\}$.

Also let $x_{ij} = 1$ if arc $(i,j) \in \bar{A}$ is used, 0 otherwise.

The linear program is:

(2 marks)

$$\min \sum_{(i,j) \in \bar{A}} (C_{ij} - \alpha_i)\, x_{ij}$$

$$s.t.$$

(1 marks)

$$\sum_{i \in \bar{N}} x_{0i} = 1$$

(1 marks)

$$\sum_{i \in \bar{N}} x_{i(n+1)} = 1$$

(2 marks)

$$\sum_{i \in \bar{N}} x_{ij} = \sum_{i \in \bar{N}} x_{ji}, \qquad \forall j \in \bar{N} \backslash \{0, n+1\}$$

$$x_{ij} \geq 0, \forall (i,j) \in \bar{A}$$

(c) (2 marks) STATE the condition in which the new column generated by the linear program is appended to the path-based formulation.

SOLUTION:

Let w^* be the optimal value to the objective function for the shortest path subproblem defined earlier. Append column if $w^* + \beta < 0$.

Question 5
[Total: 20 marks]

The capacitated facility location (CFL) problem with deterministic demand can be formulated as follows:

$$\min \sum_{i \in L} F_i x_i + \sum_{i \in L} \sum_{j \in C} W_{ij} y_{ij}$$

$$s.t.$$

$$\sum_{i \in L} y_{ij} \geq D_j, \qquad \forall j \in C$$

$$\sum_{j \in C} y_{ij} \leq B_i x_i, \qquad \forall i \in L$$

$$y_{ij} \leq M x_i, \qquad \forall i \in L, j \in C$$

$$x_i \in \{0, 1\}, \qquad \forall i \in L$$

$$y_{ij} \geq 0, \qquad \forall i \in L, j \in C$$

where

L is the set of all potential sites,
C is the set of all customers,
F_i is the fixed cost to setup facility at site $i \in L$,
W_{ij} is the unit cost to fulfill customer j's demand from site $i \in L$,
D_j is the demand for customer $j \in C$,
B_i is the capacity of site $i \in L$,
$x_i = 1$ if site $i \in L$ is used, 0 otherwise,
y_{ij} is the amount of customer j's demand fulfilled by site $i \in L$,
M is a sufficiently big number.

(a) (2 marks) WRITE down an expression for M such that the constraint $y_{ij} \leq M x_i$ is tight for CFL.

SOLUTION:

$M = \min(B_i, D_j)$, since a site cannot supply more than its capacity, and a customer does not require more than its demand.

(b) (18 marks) Now consider the stochastic version of CFL, where customer demands are uncertain. Let S be the set of all customer demand realisation scenarios, and let D_j^σ be customer j's demand given realisation scenario $\sigma \in S$. Each realisation scenario $\sigma \in S$ has realisation probability of P^σ, and $\sum_{\sigma \in S} P^\sigma = 1$. FORMULATE a stochastic integer linear program for the stochastic version of CFL. WRITE down expressions of M for all big-M constraints.

SOLUTION:

First stage variables:

$x_i = 1$ if site i is used, 0 otherwise.

(2 marks)

Second stage variables:

y_{ij}^σ : the amount supplied to customer $j \in C$ from site $i \in L$, under scenario $\sigma \in S$

The stochastic integer program is:

(3 marks)

$$\min \sum_{i \in L} F_i x_i + \sum_{\sigma \in S} \sum_{i \in L} \sum_{j \in C} P^\sigma W_{ij} y_{ij}^\sigma$$

$$s.t.$$

(3 marks)

$$\sum_{i \in L} y_{ij}^\sigma \geq D_j^\sigma, \qquad \forall j \in C, \sigma \in S$$

(3 marks)

$$\sum_{j \in C} y_{ij}^\sigma \leq B_i x_i, \qquad \forall i \in L, \sigma \in S$$

(3 marks)

$$y_{ij}^\sigma \leq M_{ij\sigma} x_i, \qquad \forall i \in L, j \in C, \sigma \in S$$

(1 mark)

$$x_i \in \{0, 1\}, \qquad \forall i \in L$$

(1 mark)

$$y_{ij}^\sigma \geq 0, \qquad \forall i \in L, j \in C, \sigma \in S$$

(2 marks)

where $M_{ij\sigma} = \min\left(B_i, D_j^\sigma\right)$.

End of Examination

3 2011 - Semester 1

3.1 Exam

The University of Melbourne

Semester 1 Assessment 2011

Student Number: ....

Department of Mathematics and Statistics
MAST90014 Optimisation for Industry

Reading time: 15 minutes
Writing time: 120 minutes
This paper has 6 pages.

Identical Examination Papers:
N/A

Common Content Papers:
N/A

Authorised Materials:
The following items are authorised: Calculators.

Instructions to Invigilators:
No handouts are required.
The examination paper is to remain in the examination room.

Instructions to Students:
Answer all questions. Clearly define all parameters and variables in your answer, and state all assumptions.

Paper to be held by Baillieu Library:
No

Extra material required:

Graph paper ☐ Multiple Choice form ☐ Other (please specify) ☐

Question 1
[Total: 25 marks]

(a) Define all minimal knapsack cover inequalities for the following knapsack constraint:

$$4x_1 + 3x_2 + 5x_3 + 2x_4 + 6x_5 \leq 10$$

(b) Solve the following integer program using the LP-based branch-and-bound method:

$$\begin{aligned} \min \quad & 3x_1 + 2x_2 \\ s.t. \quad & \\ & 2x_1 + 2x_2 \geq 1 \\ & 4x_1 - x_2 \leq 6 \\ & x_1 + 6x_2 \leq 15 \\ & x_1, x_2 \geq 0, \ integer \end{aligned}$$

Question 2
[Total: 25 marks]

Bicycle-sharing schemes are gradually gaining its popularity worldwide, with the recent introduction of Australia's premier scheme - Melbourne Bike Share - in Melbourne, Australia. It has indeed serve well as a greener transport alternative to many already congested metropolitans. Bicycle relocation is one of the key maintenance activities required to guarantee the level of service provided by the bicycle-sharing scheme.

Consider a typical bicycle-sharing scheme with M bicycle stations located around Melbourne's Central Business District (CBD). A station's location can be described by coordinates (X_i, Y_i) for some station $i \in \{1, \ldots, M\}$ on a two-dimensional map of Melbourne's CBD. A station $i \in \{1, \ldots, M\}$ consists of S_i bicycle stands, where a maximum of S_i bicycles can be secured at that station at any given time. The distance between any two stations i and j can be described, in the simplest sense, by the Euclidean distance between the two stations, i.e.

$$D_{ij} = \sqrt{(X_i - X_j)^2 + (Y_i - Y_j)^2}$$

Let R_{iujv} be the expected number of bicycles picked up at station $i \in \{1, \ldots, M\}$ at the *beginning* of time period $u \in \{1, \ldots, T\}$ and dropped off at station $j \in \{1, \ldots, M\}$ at the *beginning* of time period $v \in \{1, \ldots, T\}$, where $i \neq j$, $u < v$, and T is the last day in the planning horizon considered for this problem, which must all be fulfilled. Let I_i^0 be the initial number of bicycles at station $i \in \{1, \ldots, M\}$ before the start of the planning horizon.

Let the decision variable y_{iujv} be the number of bicycles relocated from station i at the *end* of time period u to station j at the *beginning* of time period v, where $i \neq j$, $u < v$. The cost of relocating one bicycle per unit distance, if the relocation starts at the end of time period u and completes at the beginning of time period v, is $C^R_{(v-u+1)}$ dollars, where $u < v$. A fixed cost of C_i^F dollars is incurred if relocation is carried out from station i in any time period during the planning horizon.

Using the set of y-variables define earlier and other decision variables, FORMULATE a mixed-integer linear program that minimises the total cost of relocation over the planning horizon. You can ignore the transportation and routing aspects of this bicycle relocation problem. Clearly define values of big-M for any big-M constraints defined.

Question 3
[Total: 25 marks]

Consider the following machine scheduling problem: $P3| \, |C_{max}$ with the following data instance:

Table 1: Processing times for the 10 jobs to be scheduled.

Job i	p_i
1	4
2	5
3	10
4	3
5	3
6	5
7	4
8	1
9	1
10	9

Answer the following questions with the data instance shown in Table 1.

(a) What is the value of the lowerbound for this data instance?

(b) What is the makespan of the optimal schedule? DRAW a Gantt chart of the optimal schedule, and STATE the reason why the schedule is optimal.

(c) What is the makespan of the schedule created using Graham's (G) list scheduling rule? Assume that the jobs are presented in the order shown in Table 1. DRAW a Gantt chart of the schedule.

(d) What is the makespan of the schedule created using the Longest Processing Time (LPT) rule? DRAW a Gantt chart of the schedule.

(e) What is the makespan of the schedule created using the Multifit rule with 2 iterations (MF[2])? DRAW a Gantt chart of the schedule.

(f) Is the data instance in Table 1 a worst-case instance for any of the above heuristic rules? JUSTIFY your answer.

NOTES:

(i) The MF[k] rule is shown below:

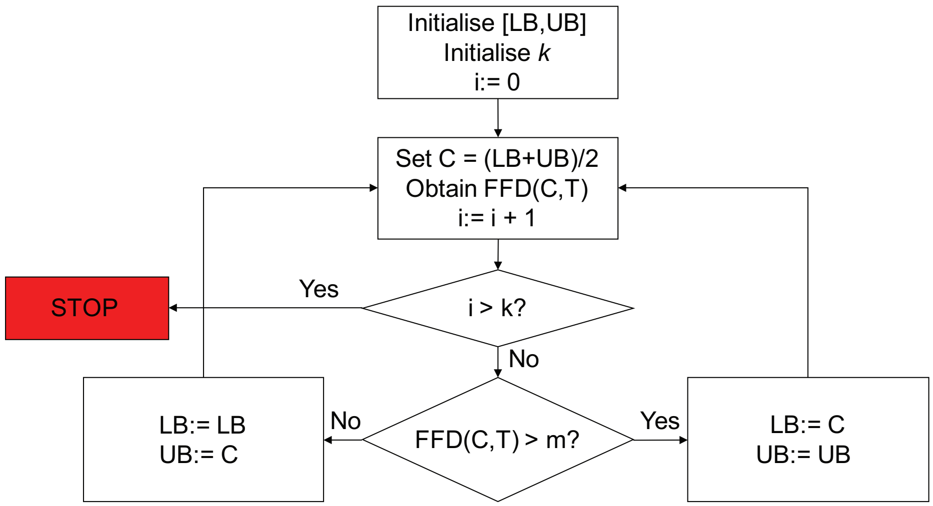


Figure 1: The MF[k] rule.

FFD(C,T) is the number of machines required when the first-fit decreasing rule is run on the set of jobs T and the capacity of the machines is C.

For initialisation of LB and UB, you may wish to use the following fact:

$$\frac{1}{m}\sum_{j=1}^{n} p_j \leq C_{max} \leq 2\max\left\{p_1, p_2, \ldots, p_n, \frac{1}{m}\sum_{j=1}^{n} p_j\right\}$$

(ii) Worst-case performances for some heuristic rules for $Pm|\ |C_{max}$ are:

$$\frac{C_{max}(G)}{C_{max}(OPT)} \leq 2 - \frac{1}{m}$$

$$\frac{C_{max}(LPT)}{C_{max}(OPT)} \leq \frac{4}{3} - \frac{1}{3m}$$

$$\frac{C_{max}(MF[k])}{C_{max}(OPT)} \leq 1.22 + 2^{-k}$$

where $C_{max}(OPT)$ is the optimal makespan.

Question 4
[Total: 25 marks]

(A Stochastic Knapsack Problem)
Let $\mathbf{S} = \{1, \ldots, N\}$ be the set of N items you can consider buying and bringing along with you on a business trip. You are only allowed to buy at most one of each item, and the items that you bring along can be sold during your business trip. An item $i \in \mathbf{S}$ weighs W_i kilograms and costs C_i dollars. You have a baggage allowance of B kilograms, but you are allowed to purchase additional baggage allowances at C^B dollars per kilogram.

Normally, the items can be sold at reasonable prices. An item i can be sold at R_i^N dollars.

However, on a *good* business trip, the items that were brought along can be sold at a better price. If you bring along item i, and the business trip is good, the can be sold at R_i^G dollars.

If the business trips turns out to be *bad*, an item i brought along with the trip can be only be sold at a lower than the normal price, i.e. R_i^B dollars.

The probabilities that the business trip is normal, good and bad are P^N, P^G and P^B respectively. You are not required to sell all items that were brought along with your trip.

(a) FORMULATE a two-stage stochastic mixed integer program with recourse that aims to maximise the expected profit for your business trip.

(b) Briefly DESCRIBE the key elements of the Benders Decomposition method when applied to this problem. Your description should include:

- The Master problem, its decisions variables, constraints and its objective function.
- The number of subproblems, their decision variables, constraints and objective functions.
- The mechanism in which the Master and subproblems "interact".

You are NOT required to show mathematical details of this decomposition. However, a mathematical description of this decomposition is also acceptable.

End of Examination

3.2 Solution

The University of Melbourne

Semester 1 Assessment 2011

SOLUTION

Department of Mathematics and Statistics
MAST90014 Optimisation for Industry

Question 1
[Total: 25 marks]

(a) Define all minimal knapsack cover inequalities for the following knapsack constraint:

$$4x_1 + 3x_2 + 5x_3 + 2x_4 + 6x_5 \leq 10$$

SOLUTION:

$x_3 + x_5 \leq 1$

$x_1 + x_2 + x_3 \leq 2$

$x_1 + x_2 + x_5 \leq 2$

$x_1 + x_3 + x_4 \leq 2$

$x_1 + x_4 + x_5 \leq 2$

$x_2 + x_4 + x_5 \leq 2$

[Marking style: ADD, THEN SUBTRACT]

[MAX: 11 marks]

[ADD: 1 mark for EACH correct 2-term cover]

[ADD: 2 marks for EACH correct 3-term cover]

[SUBTRACT: 0.5 mark for each wrong answer]

(b) Solve the following integer program using the LP-based branch-and-bound method:

$$\begin{array}{rl} \min & 3x_1 + 2x_2 \\ s.t. & \\ & 2x_1 + 2x_2 \geq 1 \\ & 4x_1 - x_2 \leq 6 \\ & x_1 + 6x_2 \leq 15 \\ & x_1, x_2 \geq 0, \ integer \end{array}$$

SOLUTION:

Branch-and-bound tree shown in Figure 1. Optimal solution is $(z, x_1, x_2) = (2, 0, 1)$.

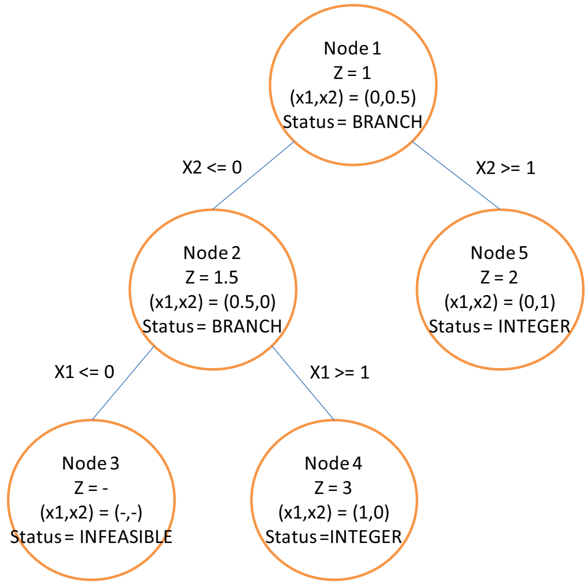


Figure 1:

[Marking style: SUBTRACT FROM MAX]

[MAX: 14 marks]

[SUBTRACT: 2 mark if no node number]

[SUBTRACT: 2 marks if value of LP-relaxation not shown on node]

[SUBTRACT: 2 marks if LP solution not shown on node]

[SUBTRACT: 2 marks if no branching information]

[SUBTRACT: 2 marks if infeasibility/integer-solution/cut-off not indicated]

[SUBTRACT: 4 marks wrong answer]

Question 2
[Total: 25 marks]

Bicycle-sharing schemes are gradually gaining its popularity worldwide, with the recent introduction of Australia's premier scheme - Melbourne Bike Share - in Melbourne, Australia. It has indeed serve well as a greener transport alternative to many already congested metropolitans. Bicycle relocation is one of the key maintenance activities required to guarantee the level of service provided by the bicycle-sharing scheme.

Consider a typical bicycle-sharing scheme with M bicycle stations located around Melbourne's Central Business District (CBD). A station's location can be described by coordinates (X_i, Y_i) for some station $i \in \{1, \ldots, M\}$ on a two-dimensional map of Melbourne's CBD. A station $i \in \{1, \ldots, M\}$ consists of S_i bicycle stands, where a maximum of S_i bicycles can be secured at that station at any given time. The distance between any two stations i and j can be described, in the simplest sense, by the Euclidean distance between the two stations, i.e.

$$D_{ij} = \sqrt{(X_i - X_j)^2 + (Y_i - Y_j)^2}$$

Let R_{iujv} be the expected number of bicycles picked up at station $i \in \{1, \ldots, M\}$ at the *beginning* of time period $u \in \{1, \ldots, T\}$ and dropped off at station $j \in \{1, \ldots, M\}$ at the *beginning* of time period $v \in \{1, \ldots, T\}$, where $i \neq j$, $u < v$, and T is the last day in the planning horizon considered for this problem, which must all be fulfilled. Let I_i^0 be the initial number of bicycles at station $i \in \{1, \ldots, M\}$ before the start of the planning horizon.

Let the decision variable y_{iujv} be the number of bicycles relocated from station i at the *end* of time period u to station j at the *beginning* of time period v, where $i \neq j$, $u < v$. The cost of relocating one bicycle per unit distance, if the relocation starts at the end of time period u and completes at the beginning of time period v, is $C^R_{(v-u+1)}$ dollars, where $u < v$. A fixed cost of C_i^F dollars is incurred if relocation is carried out from station i in any time period during the planning horizon.

Using the set of y-variables define earlier and other decision variables, FORMULATE a mixed-integer linear program that minimises the total cost of relocation over the planning horizon. You can ignore the transportation and routing aspects of this bicycle relocation problem. Clearly define values of big-M for any big-M constraints defined.

SOLUTION:

Decision variables:

s_{it} = the number of bicycles at station i at the end of time period t;

y_{iujv} = the number of bicycles relocated from station i at the beginning of time period u to station j at the beginning of time period v (already defined in question);

We assume here that the bicycles are moved from station i after all requests are fulfilled in time period u, and available for use at station j at the beginning of time period v.

x_{it} = 1 if relocation is carried out at station i in time period t, 0 otherwise;

The mixed-integer linear program is:

$$\min \sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{u=1}^{T}\sum_{v=1}^{T} C^R_{(v-u+1)} D_{ij} y_{iujv} + \sum_{i=1}^{M}\sum_{t=1}^{T} C^F_i x_{it} \tag{1}$$

s.t.

- Conservation of the number of bicycles at station i at the end of time period $t \geq 2$:

$$s_{it} = s_{i(t-1)} + \sum_{j=1}^{M}\sum_{u=1}^{T} R_{juit} - \sum_{j=1}^{M}\sum_{u=1}^{T} R_{itju} + \sum_{j=1}^{M}\sum_{u=1}^{T} y_{juit} - \sum_{j=1}^{M}\sum_{u=1}^{T} y_{itju}, \quad \forall i \in \{1, \ldots, M\}, t \in \{2, \ldots, T\} \tag{2}$$

- Conservation of the number of bicycles at station i at the end of time period $t = 1$:

$$s_{i1} = I^0_i + \sum_{j=1}^{M}\sum_{u=1}^{T} R_{jui1} - \sum_{j=1}^{M}\sum_{u=1}^{T} R_{i1ju} + \sum_{j=1}^{M}\sum_{u=1}^{T} y_{jui1} - \sum_{j=1}^{M}\sum_{u=1}^{T} y_{i1ju} \quad \forall i \in \{1, \ldots, M\} \tag{3}$$

- Capacity limit (maximum number of bicycle stands) at station i (assume worst-case):

$$s_{i(t-1)} + \sum_{j=1}^{M}\sum_{u=1}^{T} R_{juit} + \sum_{j=1}^{M}\sum_{u=1}^{T} y_{juit} \leq S_i, \quad \forall i \in \{1, \ldots, M\}, t \in \{1, \ldots, T\} \tag{4}$$

- Relocation indicator:

$$\sum_{j=1}^{M}\sum_{u=1}^{T} y_{itju} \leq M x_{it}, \quad \forall i \in \{1, \ldots, M\}, t \in \{1, \ldots, T\} \tag{5}$$

where $M = I^0_i + \sum_{j=1}^{M}\sum_{u=1}^{t} R_{juit} - \sum_{j=1}^{M}\sum_{u=1}^{t} R_{itju}$.

- Variable types:

$$s_{it} \geq 0, \quad \forall i \in \{1, \dots, M\}, t \in \{1, \dots, T\} \tag{6}$$

$$y_{iujv} \geq 0, \ integer, \quad \forall i, j \in \{1, \dots, M\}, u, v \in \{1, \dots, T\} \tag{7}$$

$$x_{it} \in \{0, 1\}, \quad \forall i \in \{1, \dots, M\}, t \in \{1, \dots, T\} \tag{8}$$

[Marking style: ADD, THEN SUBTRACT]
[MAX: 25 marks]
[ADD: 5 marks for OBJECTIVE function - partial marks can be awarded]
[ADD: 5 marks for EACH constraint, excluding variable type constraints - partial marks can be awarded]
[SUBTRACT: 1 mark if variable type constraints not defined]
[SUBTRACT: 2 marks if additional sets and parameters not defined]
[SUBTRACT: 2 marks for each decision variables not defined]
[SUBTRACT: 2 mark for incomplete "forall" or "sum" expressions]
[SUBTRACT: 1 mark if no assumptions]
[SUBTRACT: 2 mark for undefined big-M]

Question 3
[Total: 25 marks]

Consider the following machine scheduling problem: $P3| |C_{max}$ with the following data instance:

Table 1: Processing times for the 10 jobs to be scheduled.

Job i	p_i
1	4
2	5
3	10
4	3
5	3
6	5
7	4
8	1
9	1
10	9

Answer the following questions with the data instance shown in Table 1.

(a) What is the value of the lowerbound for this data instance?

SOLUTION:

Lowerbound $= \max\{\frac{\sum_{i=1}^{10} p_i}{3}, p_{max}\} = \max\{\frac{45}{3}, 10\} = 15.$

[Marking style: CORRECT ANSWER ONLY]

[MAX: 2 marks]

(b) What is the makespan of the optimal schedule? DRAW a Gantt chart of the optimal schedule, and STATE the reason why the schedule is optimal.

SOLUTION:

The optimal makespan is 15, with the optimal schedule shown in Figure 2. This is optimal since the lowerbound is 15.

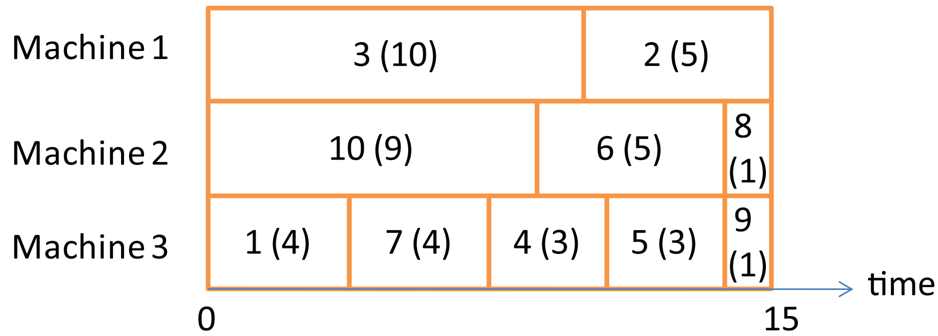


Figure 2: The optimal schedule.

[Marking style: CORRECT ANSWER ONLY]

[MAX: 6 marks]

[ADD: 3 marks for the optimal makespan]

[ADD: 2 marks drawing of GANTT chart]

[ADD: 1 mark for JUSTIFICATION]

(c) What is the makespan of the schedule created using Graham's (G) list scheduling rule? Assume that the jobs are presented in the order shown in Table 1. DRAW a Gantt chart of the schedule.

SOLUTION:

The G-schedule has makespan 21, with the schedule shown in Figure 3.

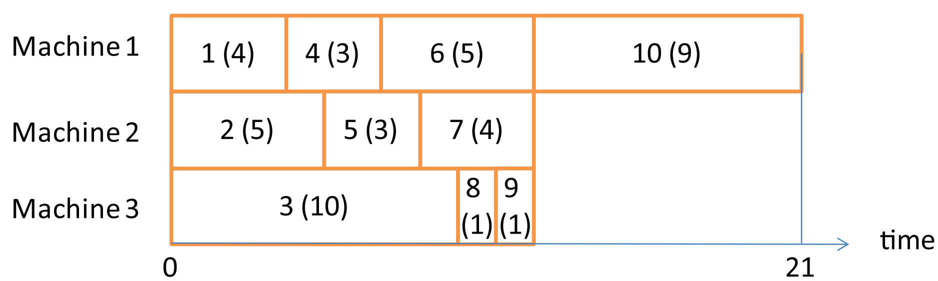


Figure 3: The G-schedule.

[Marking style: CORRECT ANSWER ONLY]

[MAX: 4 marks]

[ADD: 2 marks for correct application of heuristic and makespan given.]

[ADD: 2 marks drawing of GANTT chart]

(d) What is the makespan of the schedule created using the Longest Processing Time (LPT) rule? DRAW a Gantt chart of the schedule.

SOLUTION:

The LPT-schedule has makespan 16, with the schedule shown in Figure 4.

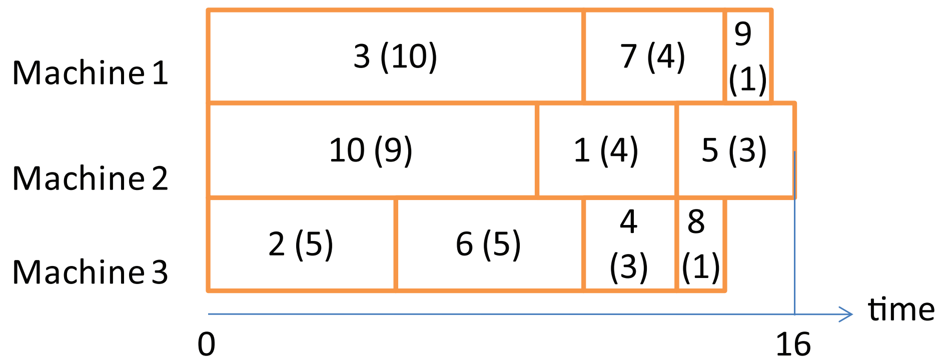


Figure 4: The LPT-schedule.

[Marking style: CORRECT ANSWER ONLY]

[MAX: 4 marks]

[ADD: 2 marks for correct application of heuristic and makespan given.]

[ADD: 2 marks drawing of GANTT chart]

(e) What is the makespan of the schedule created using the Multifit rule with 2 iterations (MF[2])? DRAW a Gantt chart of the schedule.

SOLUTION:

- Initialisation:
  $LB = 15$, $UB = 30$, $k = 2$, $i = 0$
- Iteration 1:
  $i = 1$, $C = \frac{30+15}{2} = 22.5$
  $FFD(22.5, T) = 3$, therefore set $UB = 22.5$.

- Iteration 2:
  $i = 2$, $C = \frac{22.5+15}{2} = 18.75$
  $FFD(18.75, T) = 3$, therefore set $UB = 18.75$
- Iteration 3:
  $i = 3$, $C = \frac{18.75+15}{2} = 16.875$
  $FFD(18.75, T) = 3$. STOP here since $i > k$.

So MF[2]-schedule has makespan 16, with the schedule shown in Figure 5.

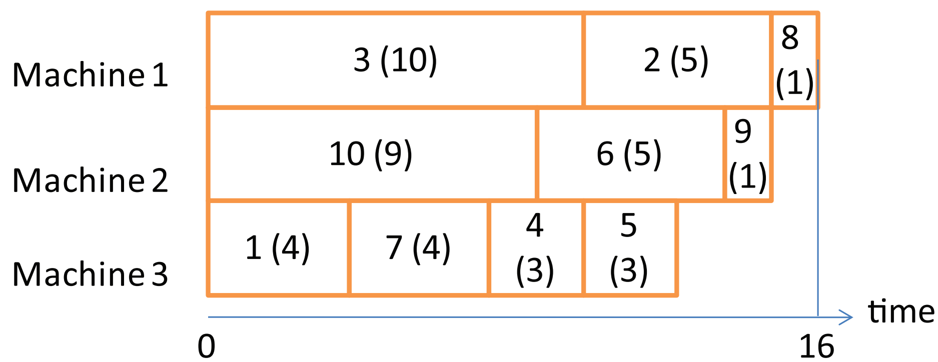


Figure 5: The MF[2]-schedule.

[Marking style: CORRECT ANSWER ONLY]

[MAX: 7 marks]

[ADD: 3 marks for correct application of heuristic]

[ADD: 2 marks for correct makespan given.]

[ADD: 2 marks drawing of GANTT chart]

(f) Is the data instance in Table 1 a worst-case instance for any of the above heuristic rules? JUSTIFY your answer.

SOLUTION:

For G-schedule:

$$\frac{C_{max}(G)}{C_{max}(OPT)} \leq 2 - \frac{1}{3} = \frac{5}{3}$$
$$\frac{C_{max}(G)}{C_{max(OPT)}} = \frac{21}{15} = \frac{7}{5} < \frac{5}{3}$$

Hence, not worst-case instance.

For LPT-schedule:

$$\frac{C_{max}(LPT)}{C_{max}(OPT)} \leq \frac{4}{3} - \frac{1}{3(3)} = \frac{11}{9}$$

$$\frac{C_{max}(LPT)}{C_{max(OPT)}} = \frac{16}{15} < \frac{11}{9}$$

Hence, not worst-case instance.

For MF[2]-schedule:

$$\frac{C_{max}(MF[k])}{C_{max}(OPT)} \leq 1.22 + 2^{-k} = 1.47$$

$$\frac{C_{max}(MF[2])}{C_{max(OPT)}} = \frac{16}{15} < 1.47$$

Hence, not worst-case instance.

[Marking style: CORRECT ANSWER ONLY]

[MAX: 2 marks]

NOTES:

(i) The MF[k] rule is shown below:

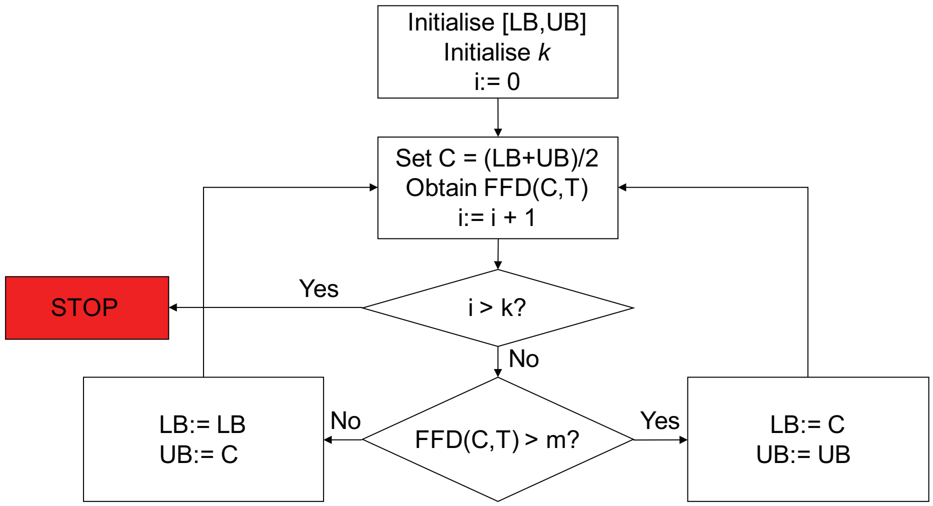


Figure 6: The MF[k] rule.

FFD(C,T) is the number of machines required when the first-fit decreasing rule is run on the set of jobs T and the capacity of the machines is C.

For initialisation of LB and UB, you may wish to use the following fact:

$$\frac{1}{m}\sum_{j=1}^{n} p_j \leq C_{max} \leq 2\max\left\{p_1, p_2, \ldots, p_n, \frac{1}{m}\sum_{j=1}^{n} p_j\right\}$$

(ii) Worst-case performances for some heuristic rules for $Pm|\ |C_{max}$ are:

$$\frac{C_{max}(G)}{C_{max}(OPT)} \leq 2 - \frac{1}{m}$$

$$\frac{C_{max}(LPT)}{C_{max}(OPT)} \leq \frac{4}{3} - \frac{1}{3m}$$

$$\frac{C_{max}(MF[k])}{C_{max}(OPT)} \leq 1.22 + 2^{-k}$$

where $C_{max}(OPT)$ is the optimal makespan.

Question 4
[Total: 25 marks]

(A Stochastic Knapsack Problem)
Let $\mathbf{S} = \{1, \dots, N\}$ be the set of N items you can consider buying and bringing along with you on a business trip. You are only allowed to buy at most one of each item, and the items that you bring along can be sold during your business trip. An item $i \in \mathbf{S}$ weighs W_i kilograms and costs C_i dollars. You have a baggage allowance of B kilograms, but you are allowed to purchase additional baggage allowances at C^B dollars per kilogram.

Normally, the items can be sold at reasonable prices. An item i can be sold at R_i^N dollars.

However, on a *good* business trip, the items that were brought along can be sold at a better price. If you bring along item i, and the business trip is good, the price for item i is R_i^G dollars.

If the business trips turns out to be *bad*, an item i brought along with the trip can be only be sold at a lower than the normal price, i.e. R_i^B dollars.

The probabilities that the business trip is normal, good and bad are P^N, P^G and P^B respectively. You are not required to sell all items that were brought along with your trip.

(a) FORMULATE a two-stage stochastic mixed integer program with recourse that aims to maximise the expected profit for your business trip.

SOLUTION:

Sets:

$\mathbf{S} = \{N, G, B\}$ = set of possible outcomes of the business trip, where $N = Normal$, $G = Good$ and $B = Bad$;

First stage decision variables:

$x_i = 1$ if item i is chosen to be brought along with the trip, 0 otherwise;
w = additional baggage allowance;

Second stage decision variables:

$y_{is} = 1$ if item i is sold on business trip when the outcome is $s \in \mathbf{S}$, 0 otherwise;

The two-stage stochastic mixed integer program is:

$$\max \sum_{i \in \mathbf{N}} \sum_{s \in \mathbf{S}} P^s R_i^B y_{is} - \sum_{i \in \mathbf{N}} C_i x_i - C^B w \tag{9}$$

s.t.

- Weight limit:
$$\sum_{i \in \mathbf{N}} W_i x_i \leq B + w \tag{10}$$

- An item can be sold only if it was brought along with the trip:
$$y_{is} \leq x_i, \quad \forall i \in \mathbf{N}, s \in \mathbf{S} \tag{11}$$

- Variable types:
$$x_i \in \{0, 1\}, \quad \forall i \in \mathbf{N} \tag{12}$$
$$y_{is} \in \{0, 1\}, \quad \forall i \in \mathbf{N}, s \in \mathbf{S} \tag{13}$$

[Marking style: ADD, THEN SUBTRACT]

[MAX: 15 marks]

[ADD: 5 marks for OBJECTIVE function - partial marks can be awarded]

[ADD: 5 marks for EACH constraint, excluding variable type constraints - partial marks can be awarded]

[SUBTRACT: 1 mark if variable type constraints not defined]

[SUBTRACT: 2 marks if additional sets and parameters not defined]

[SUBTRACT: 2 marks for each decision variables not defined]

[SUBTRACT: 2 mark for incomplete "forall" or "sum" expressions]

[SUBTRACT: 1 mark if no assumptions]

[SUBTRACT: 2 mark for undefined big-M]

(b) Briefly DESCRIBE the key elements of the Benders Decomposition method when applied to this problem. Your description should include:

- The Master problem, its decisions variables, constraints and its objective function.

- The number of subproblems, their decision variables, constraints and objective functions.
- The mechanism in which the Master and subproblems "interact".

You are NOT required to show mathematical details of this decomposition. However, a mathematical description of this decomposition is also acceptable.

SOLUTION:

The Master problem is:

$$\max z - \sum_{i \in \mathbf{N}} C_i x_i - C^B w \tag{14}$$

s.t.

- Weight limit:

$$\sum_{i \in \mathbf{N}} W_i x_i \leq B + w \tag{15}$$

- Benders optimality cuts, as a function of z and x_i variables. (defined later)
- Benders feasibility cuts as a function of x_i variables. (defined later)
- Variable types:

$$x_i \in \{0, 1\}, \quad \forall i \in \mathbf{N} \tag{16}$$

$$y_{is} \in \{0, 1\}, \quad \forall i \in \mathbf{N}, s \in \mathbf{S} \tag{17}$$

Let $\bar{z}$ and $\bar{x}_i$ be the Master problem's solution.

The OPTIMALITY subproblem for each $s \in \mathbf{S}$:

$$\max \sum_{i \in \mathbf{N}} \sum_{s \in \mathbf{S}} P^s R_i^B y_{is} \tag{18}$$

s.t.

- An item can be sold only if it was brought along with the trip:

$$y_{is} \leq \bar{x}_i, \quad \forall i \in \mathbf{N} \quad (dual: \ u_{is}) \tag{19}$$

  where u_{is} is the dual variable corresponding to this constraint.

Let w_{OPT} be the optimal value to this subproblem. If subproblem is feasible and $\bar{z} > w_{OPT}$, this subproblem will return $z \leq \sum_{i \in \mathbf{N}} \bar{u}_{is} x_i$ to the Master problem.

If OPTIMALITY subproblem is infeasible, we will have to solve the FEASIBILITY subproblem. The FEASIBILITY subproblem for each $s \in \mathbf{S}$:

$$\min \sum_{i \in \mathbf{N}} e_i \tag{20}$$

s.t.

- An item can be sold only if it was brought along with the trip:

$$y_{is} + e_i \leq \bar{x}_i, \quad \forall i \in \mathbf{N} \quad (dual: \ v_{is}) \tag{21}$$

  where v_{is} is the dual variable corresponding to this constraint.

This subproblem will return $\sum_{i \in \mathbf{N}} \bar{v}_{is} x_i \leq 0$ to the Master problem.

[Marking style: ESSAY]

[MAX: 10 marks]

[ADD: 3 marks for Master problem description]

[ADD: 3 marks for Subproblem description]

[ADD: 4 marks for Interaction]

End of Examination

4 2012 - Semester 1

4.1 Exam

The University of Melbourne

Semester 1 Assessment 2012

Student Number: . . . .

Department of Mathematics and Statistics
MAST90014 Optimisation for Industry

Reading time: 15 minutes
Writing time: 120 minutes
This paper has 8 pages.

Identical Examination Papers:
N/A

Common Content Papers:
N/A

Authorised Materials:
The following items are authorised: Calculators.

Instructions to Invigilators:
No handouts are required.
The examination paper is to remain in the examination room.

Instructions to Students:
Answer all questions. Clearly define all parameters and variables in your answer, and state all assumptions.

Paper to be held by Baillieu Library:
No

Extra material required:

Graph paper ☐ Multiple Choice form ☐ Other (please specify) ☐

Question 1
[Total: 25 marks]

In the local mX newspaper (available free-of-charge), you will find the the maze/puzzle — "A!maze" — in the Brainwave section. Samples of "A!maze"s are shown below (Figure 1):

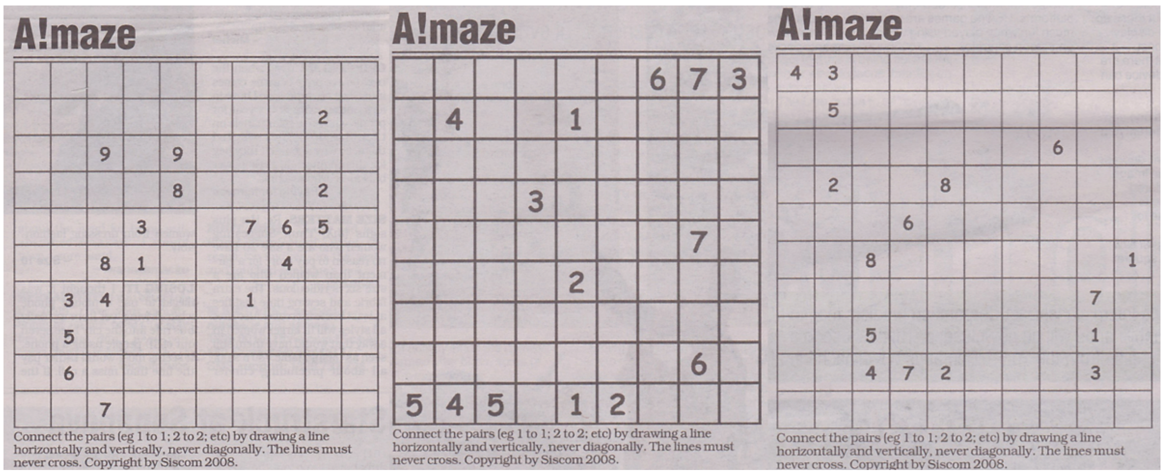


Figure 1: A!maze examples.

For each pair of the same numbers in A!maze, we aim to find a single path connecting the pair of numbers. We call the path connecting the pair of number n's an n-n path, e.g. the path connecting the pair 3's is a 3-3 path. An n-n path must only be built up of horizontal or vertical lines, i.e. diagonals are not allowed. This has to be done for all pairs of the same numbers, and the paths must not cross. Furthermore, the paths must not use any numbered cells. A solution for the 16th April 2012 A!maze is shown in Figure 2. The *length* of a n-n path is the number of cells, including its start and end cells, covered by the path minus one. For example, the 8-8 path shown Figure 2 has length of 4.

For the following questions, you may wish to use one of the sample "A!maze"s shown above to formulate your problem, *or* you may wish to formulate it based on a more general A!maze.

(a) FORMULATE an *arc-based* mixed-integer linear program such that the sum of all path lengths in the solution to an A!maze is minimised.

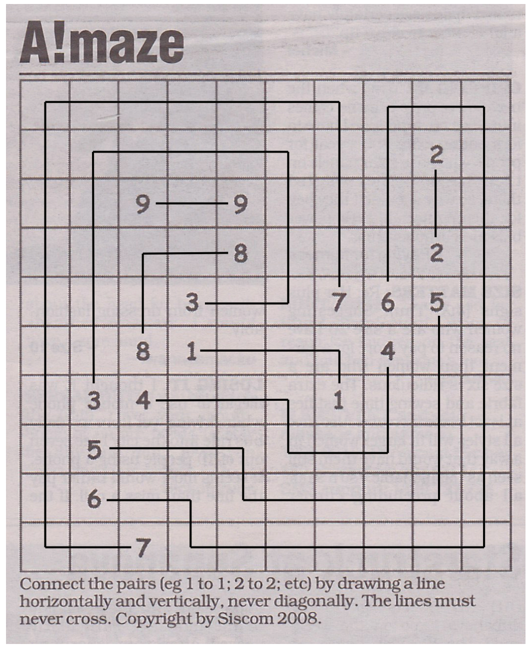


Figure 2: 16th April 2012 A!maze sample solution.

(b) FORMULATE a *path-based* mixed-integer linear program for the same problem in Part (a).

Question 2
[Total: 25 marks]

(a) DEFINE all minimal knapsack cover inequalities for the following knapsack constraint:

$$3x_1 + 5x_2 + 2x_3 + 7x_4 + 4x_5 \leq 9$$

where $x_1, x_2, x_3, x_4, x_5 \in \{0, 1\}$.

(b) SOLVE the following integer program using the LP-based branch-and-bound method:

$$\begin{aligned} \max \quad & 5x_1 + 2x_2 + 4x_3 \\ s.t. \quad & \\ & 5x_1 + 2x_2 + 3x_3 \leq 10 \\ & x_1, x_2 \geq 0, \ \textit{integer} \\ & x_3 \in \{0, 1\} \end{aligned}$$

Question 3
[Total: 25 marks]

Consider the following facility location problem with N candidate locations and M clients:

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{N} F_i x_i + \sum_{i=1}^{N}\sum_{j=1}^{M} C_{ij} y_{ij} \\
s.t. \quad & \\
& \sum_{i=1}^{N} y_{ij} \geq D_j, \qquad \forall j \in \{1, \dots, M\} \\
& \sum_{j=1}^{M} y_{ij} \leq B_i x_i, \qquad \forall i \in \{1, \dots, N\} \\
& y_{ij} \leq \min\{D_j, B_i\} x_i, \qquad \forall i \in \{1, \dots, N\}, j \in \{1, \dots, M\} \\
& x_i \in \{0, 1\}, \qquad \forall i \in \{1, \dots, N\} \\
& y_{ij} \geq 0, \qquad \forall i \in \{1, \dots, N\}, j \in \{1, \dots, M\}
\end{aligned}
$$

where
F_i is the fixed cost of setting up facility at location $i \in \{1, \dots, N\}$;
C_{ij} is the cost of serving client $j \in \{1, \dots, M\}$ from location $i \in \{1, \dots, N\}$;
D_j is the demand amount of client $j \in \{1, \dots, M\}$;
B_i is the capacity of facility $i \in \{1, \dots, N\}$;
$x_i = 1$ if facility $i \in \{1, \dots, N\}$ is used; $x_i = 0$ otherwise;
y_{ij} is the supply amount from facility $i \in \{1, \dots, N\}$ to client $j \in \{1, \dots, M\}$;

DESCRIBE how you solve the single facility supply location problem above using Benders Decomposition. In your solution, define:

- the master problem;
- the feasibility subproblem and corresponding constraint that will be appended to the master problem;
- the optimality subproblem, the corresponding constraint that will be appended to the master problem and the criteria that must be satisfied for this constraint to be added to the master problem; and
- the overall procedure.

Question 4
[Total: 25 marks]

Consider the problem of scheduling n jobs on a single machine with the aim of minimising total flowtime (sum of completion times), i.e. $1| |\sum C_i$.

(a) SHOW that the "Shortest Processing Time" (SPT) rule finds the optimal schedule for the $1| |\sum C_i$ problem.

(b) Now consider three data sets, namely Scenario 1, Scenario 2 and Scenario 3, with job processing times for each scenario outlined below:

Table 1: Processing times of jobs under Scenarios 1, 2 and 3.

Scenario	Job A	Job B	Job C
1	$P_{A1} = 10$	$P_{B1} = 9$	$P_{C1} = 6$
2	$P_{A2} = 7$	$P_{B2} = 8$	$P_{C2} = 5$
3	$P_{A3} = 8$	$P_{B3} = 10$	$P_{C3} = 7$

DETERMINE the optimal sequence for each of the scenarios above, and STATE their optimal values.

(c) Consider the following *stochastic version* of the $1| |\sum C_i$ problem: During the *planning phase*, we have to determine a sequence for the schedule. In the *execution phase*, one out of three possible scenarios could be realised. Scenario 1 can be realised with probability π_1, Scenario 2 with probability π_2 and Scenario 3 with probability π_3, and $\pi_1 + \pi_2 + \pi_3 = 1$. The processing times that is realised under each scenario is given in Table 1. The scenario tree for this problem is shown in Figure 3. FORMULATE a stochastic mixed-integer linear program that would determine the sequence which minimises the expected sum of completion times.

(d) Consider the case where $\pi_1 = \pi_2 = \pi_3$. What is the optimal sequence to the stochastic version of the $1| |\sum C_i$ problem, based on the data shown in Figure 3? What is the expected sum of completion times of the optimal sequence?
HINT: You may wish to determine the optimal sequence by complete enumeration.

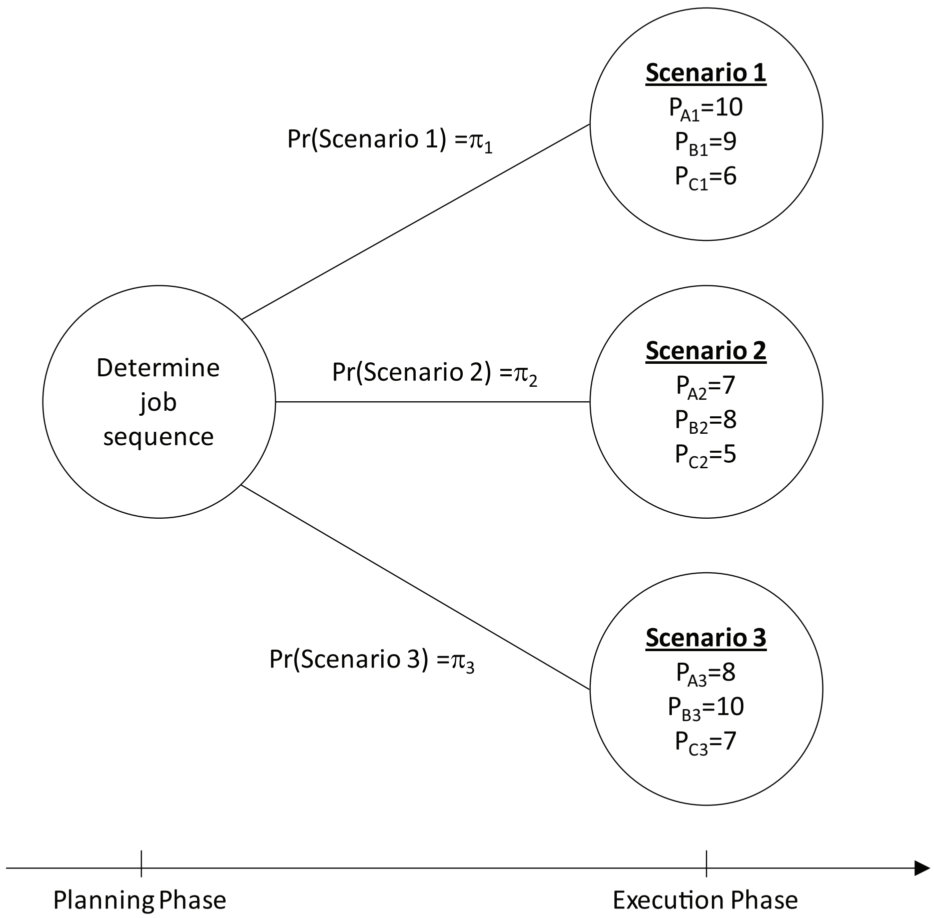


Figure 3: Scenario Tree

End of Examination

4.2 Solution

The University of Melbourne

Semester 1 Assessment 2012

SOLUTION

Question 1
[Total: 25 marks]

In the local mX newspaper (available free-of-charge), you will find the the maze/puzzle — "A!maze" — in the Brainwave section. Samples of "A!maze"s are shown below (Figure 1):

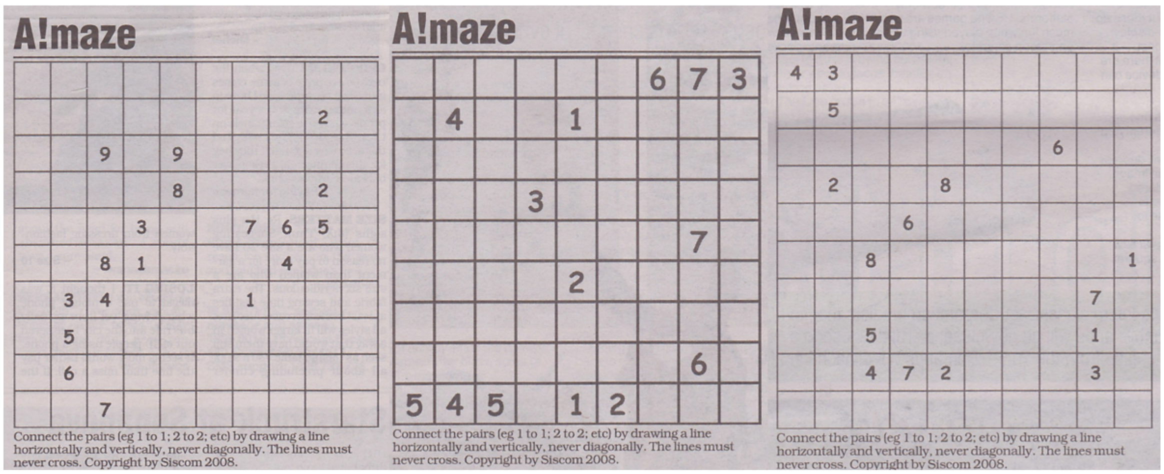


Figure 1: A!maze examples.

For each pair of the same numbers in A!maze, we aim to find a single path connecting the pair of numbers. We call the path connecting the pair of number n's an n-n path, e.g. the path connecting the pair 3's is a 3-3 path. An n-n path must only be built up of horizontal or vertical lines, i.e. diagonals are not allowed. This has to be done for all pairs of the same numbers, and the paths must not cross. Furthermore, the paths must not use any numbered cells. A solution for the 16th April 2012 A!maze is shown in Figure 2. The *length* of a n-n path is the number of cells, including its start and end cells, covered by the path minus one. For example, the 8-8 path shown Figure 2 has length of 4.

For the following questions, you may wish to use one of the sample "A!maze"s shown above to formulate your problem, *or* you may wish to formulate it based on a more general A!maze.

(a) FORMULATE an *arc-based* mixed-integer linear program such that the sum of all path lengths in the solution to an A!maze is minimised.

Answer:

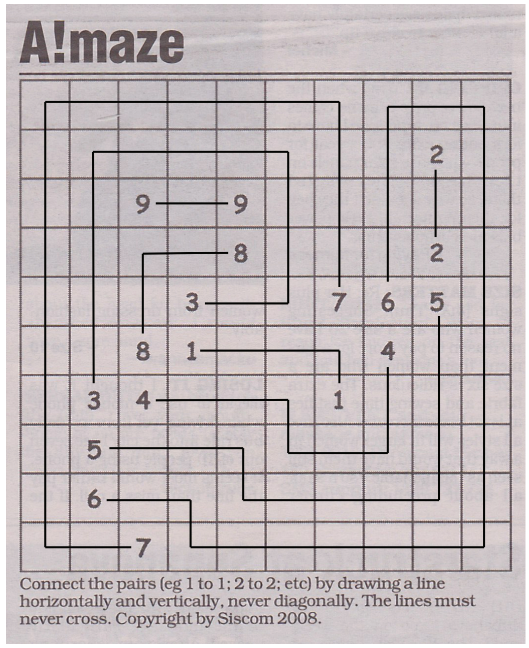


Figure 2: 16th April 2012 A!maze sample solution.

Label each cell as follows: the cell at the bottom left corner is $[1, 1]$, the cell immediately to its right is $[2, 1]$, the cell immediately on top of it is $[2, 1]$, and so forth. Assume in a more general A!maze, there are M columns along its width and N rows along its row. Let $\mathbf{C} = \{[a, b] : 1 \leq a \leq M, 1 \leq b \leq N\}$ be the set of all cells in A!maze, and let the parameter $T([a, b]) = 1$ if cell $[a, b] \in \mathbf{C}$ is a cell with a number, otherwise $T([a, b]) = 0$. Furthermore, let $\mathbf{C}' = \{i \in \mathbf{C} : T(i) = 0\}$. We define

$$\mathbf{N}_{[a,b]} = \{[v, b] \in \mathbf{C}' : v \in \{a-1, a+1\}\} \cup \{[a, v] \in \mathbf{C}' : v \in \{b-1, b+1\}\}$$

to be the set of neighbouring horizontal and vertical cells of cell $[a, b] \in \mathbf{C}'$.

Let $\mathbf{W}$ be the set of numbers in an A!maze. For an n-n path, let $S(n)$ and $E(n)$ be the start and end cells of the number $n \in \mathbf{W}$. Let

$$\begin{aligned} \mathbf{A}_n &= \left\{(S(n), i) : i \in \mathbf{N}_{S(n)}\right\} \cup \{(i, E(n)) : i \in \mathbf{C}',\ E(n) \in \mathbf{N}_i\} \\ &\quad \{(i, j) : i \in \mathbf{C}',\ j \in \mathbf{N}_i\} \end{aligned}$$

be the set of arcs for $n \in \mathbf{W}$, where a subset of it will form the n-n path.

Let $x_{ijn} = 1$ if arc $(i, j) \in \mathbf{A}_n$ is used on an n-n path; 0 otherwise. The length of an arc is 1, since the distance between any adjacent cells is 1. We also define t_{in} to be the "time" when cell $i \in \mathbf{C}'$ is used on an n-n path.

The arc-based integer linear program is formulated below:

$$\min \sum_{n \in \mathbf{W}} \sum_{(i,j) \in \mathbf{A}_n} x_{ijn} \tag{1}$$

s.t.

Must exit origin:

$$\sum_{j \in N_{S(n)}} x_{S(n),j,n} = 1, \qquad \forall n \in \mathbf{W} \tag{2}$$

Must enter destination:

$$\sum_{j \in \mathbf{C}' : (j,E(n)) \in \mathbf{A}_n} x_{j,E(n),n} = 1, \qquad \forall n \in \mathbf{W} \tag{3}$$

Flow in, flow out:

$$\sum_{j \in \mathbf{C}' : (i,j) \in \mathbf{A}_n} x_{ijn} = \sum_{j \in \mathbf{C}' : (j,i) \in \mathbf{A}_n} x_{jin}, \qquad \forall n \in \mathbf{W}, j \in \mathbf{C}' \tag{4}$$

Node visit "time":

$$t_{jn} \geq t_{in} + 1 - \left(|\mathbf{C}'| - |\mathbf{W}| + 1\right)\left(1 - x_{ijn}\right), \qquad \forall n \in \mathbf{W}, (i,j) \in \mathbf{A}_n \tag{5}$$

At most one arc going out of cell i:

$$\sum_{n \in \mathbf{W}} \sum_{j \in \mathbf{C}' : (i,j) \in \mathbf{A}_n} x_{ijn} \leq 1, \qquad \forall i \in \mathbf{C}' \tag{6}$$

Variable signs:

$$x_{ijn} \in \{0,1\}, \qquad \forall n \in \mathbf{W}, (i,j) \in \mathbf{A}_n \tag{7}$$

$$t_{in} \geq 0, \qquad \forall n \in \mathbf{W}, i \in \mathbf{C}' \tag{8}$$

[Marking style: ADD, THEN SUBTRACT]

[MAX: 18 marks]

[ADD: 3 marks for OBJECTIVE function - partial marks can be awarded]

[ADD: 3 marks for EACH constraint, excluding variable type constraints - partial marks can be awarded]

[SUBTRACT: 1 mark if variable type constraints not defined]

[SUBTRACT: 1 mark if additional sets and parameters not defined]

[SUBTRACT: 1 mark for *each* decision variable not defined]

[SUBTRACT: 1 mark for incomplete forall or sum expressions]

[SUBTRACT: 1 mark if no assumptions mentioned]

[SUBTRACT: 1 mark for undefined big-M]

(b) FORMULATE a *path-based* mixed-integer linear program for the same problem in Part (a).

Answer:

Let $\mathbf{P}_n$ be the set of all paths for $n \in \mathbf{W}$, C_p be the cost of path $p \in \mathbf{P}_n$, and $A_{ip} = 1$ if cell $i \in \mathbf{C}'$ is in path $p \in \mathbf{P}_n$, $A_{ip} = 0$ otherwise. Define $z_{pn} = 1$ if path $p \in \mathbf{P}_n$ is used for $n \in \mathbf{W}$, $z_{pn} = 0$ otherwise.

The path-based integer linear program is formulated as follows:

$$\min \sum_{n \in \mathbf{W}} \sum_{p \in \mathbf{P}_n} C_p z_{pn} \tag{9}$$

s.t.

At least one path chosen for each n (will be equal to 1, since objective is to minimise):

$$\sum_{p \in \mathbf{P}_n} z_{pn} \geq 1, \qquad \forall n \in \mathbf{W} \tag{10}$$

At most one path using cell i:

$$\sum_{n \in \mathbf{W}} \sum_{p \in \mathbf{P}_n} A_{ip} z_{pn} \leq 1, \qquad \forall i \in \mathbf{C}' \tag{11}$$

Variable sign:

$$z_{pn} \in \{0, 1\}, \qquad \forall n \in \mathbf{W}, p \in \mathbf{P}_n \tag{12}$$

[Marking style: ADD, THEN SUBTRACT]

[MAX: 7 marks]

[ADD: 2 marks for OBJECTIVE function - partial marks can be awarded]

[ADD: 2 marks for EACH constraint, excluding variable type constraints - partial marks can be awarded]

[SUBTRACT: 1 mark if variable type constraints not defined]

[SUBTRACT: 1 mark if additional sets and parameters not defined]

[SUBTRACT: 1 mark for *each* decision variable not defined]

[SUBTRACT: 1 mark for incomplete forall or sum expressions]

Question 2
[Total: 25 marks]

(a) DEFINE all minimal knapsack cover inequalities for the following knapsack constraint:

$$3x_1 + 5x_2 + 2x_3 + 7x_4 + 4x_5 \leq 9$$

where $x_1, x_2, x_3, x_4, x_5 \in \{0, 1\}$.

Answer:

$x_1 + x_4 \leq 1$

$x_2 + x_4 \leq 1$

$x_4 + x_5 \leq 1$

$x_1 + x_2 + x_3 \leq 2$

$x_1 + x_2 + x_5 \leq 2$

$x_2 + x_3 + x_5 \leq 2$

[Marking style: ADD]

[MAX: 9 marks]

[ADD: 1 mark for EACH correct 2-term minimal cover]

[ADD: 2 marks for EACH correct 3-term minimal cover]

(b) SOLVE the following integer program using the LP-based branch-and-bound method:

$$\begin{aligned} \max \quad & 5x_1 + 2x_2 + 4x_3 \\ s.t. \quad & \\ & 5x_1 + 2x_2 + 3x_3 \leq 10 \\ & x_1, x_2 \geq 0, \ integer \\ & x_3 \in \{0, 1\} \end{aligned}$$

Answer:

The branch-and-bound tree is shown in Figure 3. The optimal objective value is 11, and the optimal solution is $(x_1, x_2, x_3) = (1, 1, 1)$.

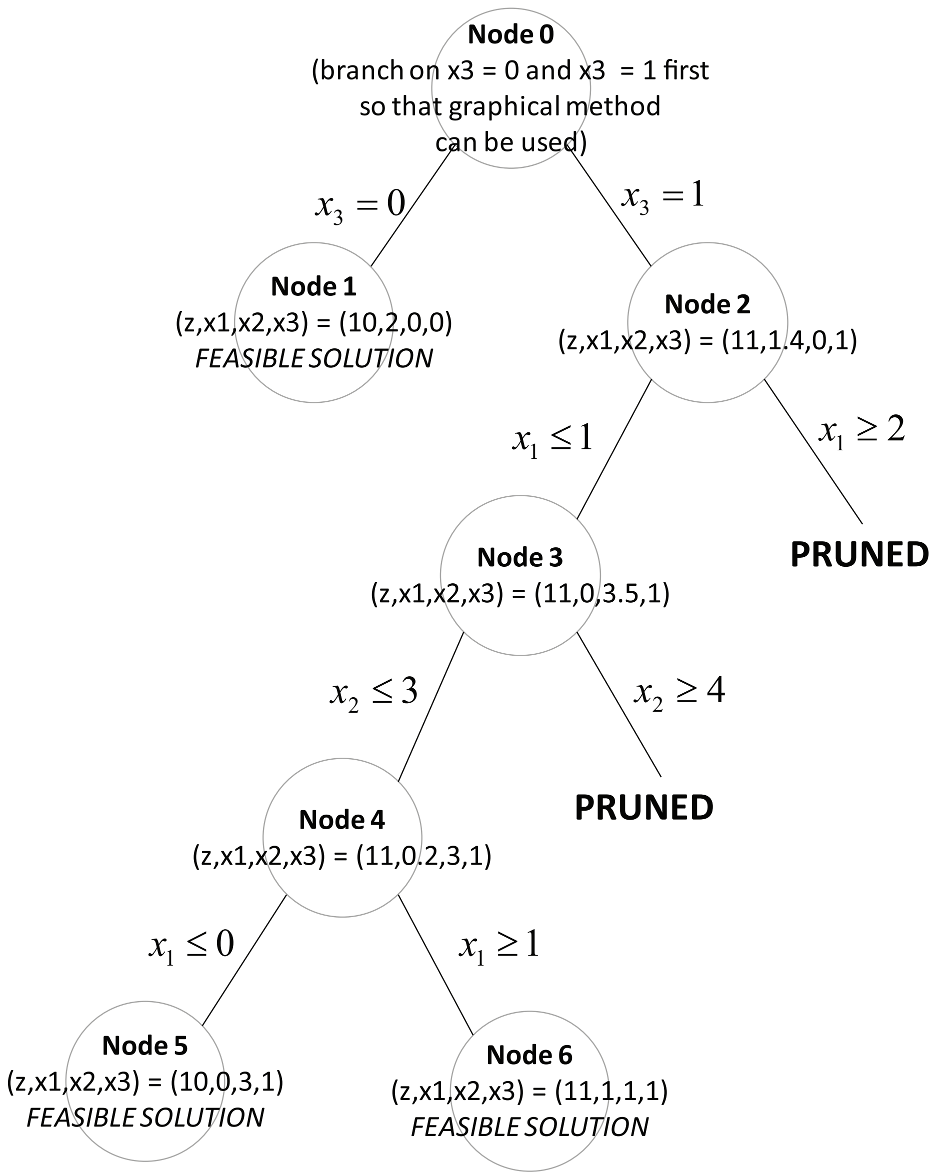


Figure 3: Branch-and-bound tree.

[Marking style: SUBTRACT FROM MAX]

[MAX: 16 marks]

[SUBTRACT: 3 marks if no node number]

[SUBTRACT: 3 marks if value of LP-relaxation not shown on node]

[SUBTRACT: 3 marks if LP solution not shown on node]

[SUBTRACT: 3 marks if no branching information]

[SUBTRACT: 3 marks if infeasibility/integer-solution/cut-off not indicated]

[SUBTRACT: 1 mark wrong answer]

Question 3

[Total: 25 marks]

Consider the following facility location problem with N candidate locations and M clients:

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{N} F_i x_i + \sum_{i=1}^{N}\sum_{j=1}^{M} C_{ij} y_{ij} \\
s.t. \quad & \\
& \sum_{i=1}^{N} y_{ij} \geq D_j, \qquad \forall j \in \{1,\dots,M\} \\
& \sum_{j=1}^{M} y_{ij} \leq B_i x_i, \qquad \forall i \in \{1,\dots,N\} \\
& y_{ij} \leq \min\{D_j, B_i\} x_i, \qquad \forall i \in \{1,\dots,N\}, j \in \{1,\dots,M\} \\
& x_i \in \{0,1\}, \qquad \forall i \in \{1,\dots,N\} \\
& y_{ij} \geq 0, \qquad \forall i \in \{1,\dots,N\}, j \in \{1,\dots,M\}
\end{aligned}
$$

where

F_i is the fixed cost of setting up facility at location $i \in \{1,\dots,N\}$;

C_{ij} is the cost of serving client $j \in \{1,\dots,M\}$ from location $i \in \{1,\dots,N\}$;

D_j is the demand amount of client $j \in \{1,\dots,M\}$;

B_i is the capacity of facility $i \in \{1,\dots,N\}$;

$x_i = 1$ if facility $i \in \{1,\dots,N\}$ is used; $x_i = 0$ otherwise;

y_{ij} is the supply amount from facility $i \in \{1,\dots,N\}$ to client $j \in \{1,\dots,M\}$;

DESCRIBE how you solve the single facility supply location problem above using Benders Decomposition. In your solution, define:

- the master problem;
- the feasibility subproblem and corresponding constraint that will be appended to the master problem;
- the optimality subproblem, the corresponding constraint that will be appended to the master problem and the criteria that must be satisfied for this constraint to be added to the master problem; and

- the overall procedure.

Answer:

Master Problem:

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{N} F_i x_i + \eta \\
s.t. \quad & \\
& \eta \geq g^k(\mathbf{x}), \qquad \forall k \in \mathbf{I} \\
& h^k(\mathbf{x}) \leq 0, \qquad \forall k \in \mathbf{I} \\
& x_i \in \{0,1\}, \qquad \forall i \in \{1,\ldots,N\} \\
& \eta \geq 0
\end{aligned}
$$

where η is a lower bound on the total service cost, $\mathbf{I}$ is the set of iterations, $g^k(\mathbf{x})$ is a function of the decision variables $\mathbf{x} = \{x_1, x_2, \ldots, x_N\}$ defined by the optimality subproblem at iteration $k \in \mathbf{I}$ and $h^k(\mathbf{x})$ is a function of the decision variables $\mathbf{x}$ defined by the feasibility subproblem at iteration $k \in \mathbf{I}$.

Let $\bar{\mathbf{x}}^k$ and $\bar{\eta}^k$ be the optimal solution to the Master Problem at iteration $k \in \mathbf{I}$.

Optimality Subproblem:

$$
\begin{aligned}
z^k = \min \quad & \sum_{i=1}^{N} \sum_{j=1}^{M} C_{ij} y_{ij} \\
s.t. \quad & \\
& \sum_{i=1}^{N} y_{ij} \geq D_j, \qquad \forall j \in \{1,\ldots,M\} \qquad (dual : u_j^k) \\
& -\sum_{j=1}^{M} y_{ij} \geq -B_i \bar{x}_i^k, \qquad \forall i \in \{1,\ldots,N\} \qquad (dual : v_i^k) \\
& -y_{ij} \geq -\min\{D_j, B_i\} \bar{x}_i^k, \qquad \forall i \in \{1,\ldots,N\}, j \in \{1,\ldots,M\} \qquad (dual : w_{ij}^k) \\
& y_{ij} \geq 0, \qquad \forall i \in \{1,\ldots,N\}, j \in \{1,\ldots,M\}
\end{aligned}
$$

Let $\bar{\mathbf{u}}^k$, $\bar{\mathbf{v}}^k$ and $\bar{\mathbf{w}}^k$ be the optimal dual solutions to the Optimality Subproblem in the kth iteration. If $z^k > \bar{\eta}^k$, then append the following constraint to the Master Problem:

$$
\eta \geq g^k(\mathbf{x}) = \sum_{j=1}^{M} \bar{u}_j^k D_j - \sum_{i=1}^{N} \bar{v}_i^k B_i x_i - \sum_{i=1}^{N} \sum_{j=1}^{M} \bar{w}_{ij}^k \min\{D_j, B_i\} x_i
$$

Feasibility Subproblem:

$$
\begin{aligned}
z_F^k \quad = \quad & \min \sum_{j=1}^{M} s_j^1 + \sum_{i=1}^{N} s_i^2 + \sum_{i=1}^{N} \sum_{j=1}^{M} s_{ij}^3 \\
& s.t. \\
& \sum_{i=1}^{N} y_{ij} + s_j^1 \geq D_j, \qquad \forall j \in \{1, \ldots, M\} \qquad (dual : a_j^k) \\
& -\sum_{j=1}^{M} y_{ij} + s_i^2 \geq -B_i \bar{x}_i^k, \qquad \forall i \in \{1, \ldots, N\} \qquad (dual : b_i^k) \\
& -y_{ij} + s_{ij}^3 \geq -\min\{D_j, B_i\} \bar{x}_i^k, \qquad \forall i \in \{1, \ldots, N\}, j \in \{1, \ldots, M\} \qquad (dual : c_{ij}^k) \\
& y_{ij} \geq 0, \qquad \forall i \in \{1, \ldots, N\}, j \in \{1, \ldots, M\} \\
& s_j^1 \geq 0, \qquad \forall j \in \{1, \ldots, M\} \\
& s_i^2 \geq 0, \qquad \forall i \in \{1, \ldots, N\} \\
& s_{ij}^3 \geq 0, \qquad \forall i \in \{1, \ldots, N\}, j \in \{1, \ldots, M\}
\end{aligned}
$$

where $\mathbf{s}^1$, $\mathbf{s}^2$ and $\mathbf{s}^3$ are the slack variables. Let $\bar{\mathbf{a}}^k$, $\bar{\mathbf{b}}^k$ and $\bar{\mathbf{c}}^k$ be the optimal dual solutions to the Feasibility Subproblem in the kth iteration. The following constraint will be appended to the Master Problem if the Feasibility Subproblem is invoked:

$$h^k(\mathbf{x}) = \sum_{j=1}^{M} \bar{a}_j^k D_j - \sum_{i=1}^{N} \bar{b}_i^k B_i x_i - \sum_{i=1}^{N} \sum_{j=1}^{M} \bar{c}_{ij}^k \min\{D_j, B_i\} x_i \leq 0$$

The general procedure:

(1) Set $k = 0$.

(2) Solve Master Problem.

(3) Solve Optimality Subproblem. If feasible, check criteria; otherwise go to Step b4. If criteria is true, append constraint and repeat Step b2; otherwise solution is optimal.

(4) Solve Feasibility Subproblem. Append constraint and repeat Step b2.

[Marking style: SUBTRACT FROM MAX]

Master problem:
[MAX: 8 marks]

Optimality subproblem:
[MAX: 8 marks]

Feasibility subproblem:
[MAX: 8 marks]

General procedure:
[MAX: 1 marks]
[SUBTRACT: 1 mark for *each* variable type constraints not defined]
[SUBTRACT: 1 mark for *each* set/parameter not defined]
[SUBTRACT: 1 mark for *each* decision variable not defined]
[SUBTRACT: 2 marks for any incomplete forall or sum expressions]
[SUBTRACT: 1 mark for wrong condition to append optimality cut]
[SUBTRACT: 2 marks for wrong optimality cut, partial subtraction allowed]
[SUBTRACT: 2 marks for wrong feasibility cut, partial subtraction allowed]
[SUBTRACT: 1 mark if procedure is *completely* wrong]

Question 4
[Total: 25 marks]

Consider the problem of scheduling n jobs on a single machine with the aim of minimising total flowtime (sum of completion times), i.e. $1| |\sum C_i$.

(a) SHOW that the "Shortest Processing Time" (SPT) rule finds the optimal schedule for the $1| |\sum C_i$ problem.

Answer:

We prove this by contradiction.

Let S be the optimal schedule, where its jobs are not in SPT order. In this schedule, we will find a pair of adjacent jobs i and j where $p_i > p_j$. Its sum of completion times, $TC = TC_1 + C_i + C_j + TC_2 = TC_1 + (t + p_i) + (t + p_i + p_j) + TC_2$, where t is the completion time of the job before i, TC_1 is the sum of completion times of jobs before time t and TC_2 is the sum of completion times of jobs after $(t + p_i + p_j)$.

Now consider a new schedule S' where jobs i and j are interchanged. The sum of completion times, $TC' = TC'_1 + C'_j + C'_i + TC'_2$. Clearly interchanging jobs i and j does not affect completion times of other jobs in the schedule, i.e. $TC'_1 = TC_1$ and $TC'_2 = TC_2$. In schedule S', $C'_j = t + p_j$ and $C'_i = t + p_i p_j$. Consequently,

$$\begin{aligned} TC' &= TC_1 + (t + p_j) + (t + p_i + p_j) + TC_2 \\ &< TC_1 + (t + p_i) + (t + p_i + p_j) + TC_2 = TC \end{aligned}$$

since $p_i > p_j$. Hence schedule S' has a smaller sum of completion times than schedule S — this contradicts the earlier assumption that schedule S is optimal.

The adjacent pairwise interchange is carried out until all pairs of adjacent jobs are in non-decreasing order.

[Marking style: ADD]

[MAX: 6 marks]

[ADD: 2 marks for defining non-SPT schedule]

[ADD: 2 marks for new schedule, as a result of adjacent pairwise interchange]

[ADD: 2 marks for showing new schedule has been objective than the non-SPT schedule]

Table 1: Processing times of jobs under Scenarios 1, 2 and 3.

Scenario	Job A	Job B	Job C
1	$P_{A1} = 10$	$P_{B1} = 9$	$P_{C1} = 6$
2	$P_{A2} = 7$	$P_{B2} = 8$	$P_{C2} = 5$
3	$P_{A3} = 8$	$P_{B3} = 10$	$P_{C3} = 7$

(b) Now consider three data sets, namely Scenario 1, Scenario 2 and Scenario 3, with job processing times for each scenario outlined below:

DETERMINE the optimal sequence for each of the scenarios above, and STATE their optimal values.

Answer:

Use SPT to schedule jobs for all scenario.

For Scenario 1, sequence "C-B-A" is optimal with sum of completion times equal 46.

For Scenario 2, sequence "C-A-B" is optimal with sum of completion times equal 37.

For Scenario 3, sequence "C-A-B" is optimal with sum of completion times equal 47.

[Marking style: ADD]

[MAX: 6 marks]

[ADD: 1 mark for each correct sequence for scenario]

[ADD: 1 mark for each correct objective value for scenario]

(c) Consider the following *stochastic version* of the $1||\sum C_i$ problem: During the *planning phase*, we have to determine a sequence for the schedule. In the *execution phase*, one out of three possible scenarios could be realised. Scenario 1 can be realised with probability π_1, Scenario 2 with probability π_2 and Scenario 3 with probability π_3, and $\pi_1 + \pi_2 + \pi_3 = 1$. The processing times that is realised under each scenario is given in Table 1. The scenario tree for this problem is shown in Figure 4. FORMULATE a stochastic mixed-integer linear program that would determine the sequence which minimises the expected sum of completion times.

Answer:

Let the set of job, $\mathbf{J} = \{A, B, C\}$, the set of positions $\mathbf{P} = \{1, 2, 3\}$ and the set of scenarios, $\mathbf{S} = \{1, 2, 3\}$.

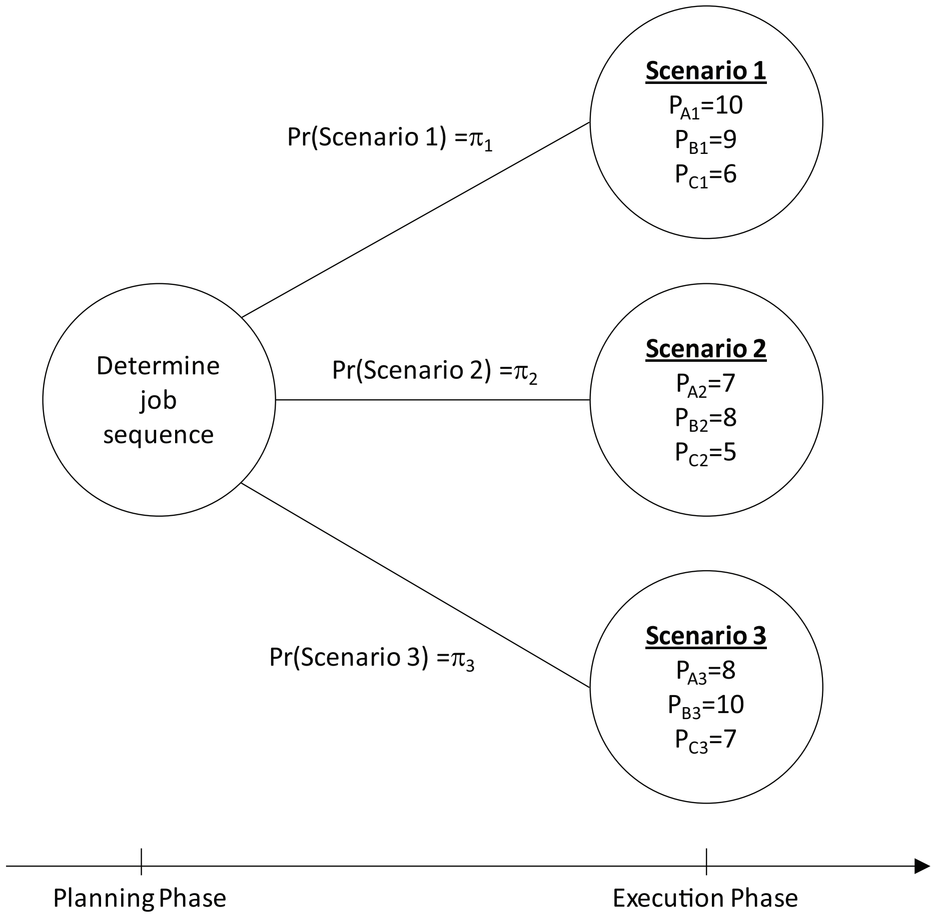


Figure 4: Scenario Tree

Let the decision variable $x_{ij} = 1$ if job $i \in \mathbf{J}$ is in position $j \in \mathbf{P}$, and $x_{ij} = 0$ otherwise. Also let c_{js} be the completion time of the job in position $j \in \mathbf{P}$ under scenario $s \in \mathbf{S}$.

The stochastic mixed-integer linear program is formulated as follows:

$$
\begin{aligned}
\min \quad & \sum_{s \in \mathbf{S}} \pi_s \left(\sum_{j \in \mathbf{J}} c_{js} \right) \\
s.t. \quad & \\
& \sum_{i \in \mathbf{J}} x_{ij} = 1, \qquad \forall j \in \mathbf{P} \\
& \sum_{j \in \mathbf{P}} x_{ij} = 1, \qquad \forall i \in \mathbf{J} \\
& c_{js} \geq c_{(j-1)s} + \sum_{i \in \mathbf{J}} P_{ij} x_{ij}, \qquad \forall j \in \mathbf{P}, s \in \mathbf{S} \\
& x_{ij} \in \{0, 1\}, \qquad \forall i \in \mathbf{J}, j \in \mathbf{P} \\
& c_{js} \geq 0, \qquad \forall j \in \mathbf{P}, s \in \mathbf{S}
\end{aligned}
$$

[Marking style: ADD, THEN SUBTRACT]

[MAX: 8 marks]

[ADD: 2 marks correct objective function, partial marks can be awarded]

[ADD: 2 marks *each* correct constraint excluding variable sign constraints, partial marks can be awarded]

[SUBTRACT: 1 mark if variable type constraints not defined]

[SUBTRACT: 1 mark if additional sets and parameters not defined]

[SUBTRACT: 1 mark for *each* decision variable not defined]

[SUBTRACT: 1 mark for incomplete forall or sum expressions]

(d) Consider the case where $\pi_1 = \pi_2 = \pi_3$. What is the optimal sequence to the stochastic version of the $1| |\sum C_i$ problem, based on the data shown in Figure 4? What is the expected sum of completion times of the optimal sequence?
HINT: You may wish to determine the optimal sequence by complete enumeration.

Answer:

We will find the optimal schedule by complete enumeration. There are six possible sequences to consider:

Sequence	Scenario 1 Obj	Scenario 2 Obj	Scenario 3 Obj	Expected Value
“A-B-C”	54	42	51	49
“A-C-B”	51	39	48	46
“B-A-C”	53	43	53	49.666...
“B-C-A”	49	41	52	47.333...
“C-A-B”	47	37	47	43.666...
“C-B-A”	46	38	49	44.333...

The optimal sequence is “3-1-2” with the smallest expected sum of completion times of 43.666 . . .

NOTE: equivalently, we could also provide proof of optimality of the Shortest Expected Processing Time (SEPT) rule, instead of complete enumeration.

[Marking style: SUBTRACT FROM MAX]

[MAX: 5 marks]

[SUBTRACT: 2 marks if incomplete enumeration]

[SUBTRACT: 2 marks for incorrect evaluation of any objective]

[SUBTRACT: 1 mark for incorrect optimal sequence]

End of Examination

5 2013 - Semester 1

5.1 Exam

The University of Melbourne

Semester 1 Assessment 2013

Student Number: ....

Department of Mathematics and Statistics
MAST90014 Optimisation for Industry

Reading time: 15 minutes
Writing time: 120 minutes
This paper has 6 pages.

Identical Examination Papers:
N/A

Common Content Papers:
N/A

Authorised Materials:
The following items are authorised: Calculators.

Instructions to Invigilators:
No handouts are required.
The examination paper is to remain in the examination room.

Instructions to Students:
Answer all questions. Clearly define all parameters and decision variables in your answer, and state all assumptions.

Paper to be held by Baillieu Library:
No

Extra material required:

Graph paper ☐ Multiple Choice form ☐ Other (please specify) ☐

Question 1
[Total: 30 marks]

The state of Victoria primarily relies on brown coal sources for electricity generation. Here, we wish to consider a simple variant of the problem of introducing and locating renewable energy resources, namely wind and solar farms, in the state of Victoria.

Let $\mathbf{L}$ be the set of candidate locations in Victoria and $\mathbf{R} = \{Wind, Solar\}$ be the set of renewable resource types. At most one renewable resource type is to be installed at any candidate location $l \in \mathbf{L}$. If some renewable resource type $r \in \mathbf{R}$ is installed at location $l \in \mathbf{L}$, the farm size must be at least S_{lr}^{MIN} units and at most S_{lr}^{MAX} units. A one-off cost, C_{lr}^{S}, is incurred when a farm with renewable resource type $r \in \mathbf{R}$ is setup at location $l \in \mathbf{L}$, and for each unit of the renewable resource type installed at the location, it costs C_{lr}^{U}.

Renewable resource locations and farm-size decisions are driven by *demand for electricity* and the *prospect/potential of a location* for renewable energy generation. For this simple problem variant, the renewable resources are not to be used to replace existing brown coal generation sources, but are instead used to cope with demand fluctuations beyond the base electricity supply generated by brown coal sources. Therefore, the "demand for electricity" is defined here as the amount of electricity to be supplied *solely* by the renewable resources.

Let the set of discrete, equally-sized, time periods be $\mathbf{T}$. Each time period may, for example, represent a 30-minute interval and the length of the planning horizon could be 5 days — in this case $\mathbf{T} = \{1, 2, 3, \ldots, 239, 240\}$. Let D_t be the demand for electricity at time $t \in \mathbf{T}$ and P_{lrt} be the amount of electricity that could (potentially) be generated at location $l \in \mathbf{L}$, using <u>one unit</u> of renewable resource type $r \in \mathbf{R}$ at time $t \in \mathbf{T}$.

The total renewable energy generation must be at least D_t for all $t \in \mathbf{T}$. There is a further requirement that the variability of the total renewable energy generation be "reasonable". To achieve a "reasonable" variability, any deviation from the average total generation for any given day is penalized as follows: one unit of positive deviation is penalized at C^P and one unit of negative deviation is penalized at C^M. For instance, consider a three-period example with total renewable generations G_1, G_2 and G_3 on time periods 1, 2, and 3, respectively. The average total generation is $A = \frac{G_1+G_2+G_3}{3}$ and suppose $G_1 < A$, $G_2 = A$ and $G_3 > A$. The total penalty for this example is given by $C^M(A - G_1) + 0 + C^P(G_3 - A)$.

FORMULATE a mixed-integer linear program (MILP) that will locate the renewable resources and determine their farm sizes such that the total farm establishment costs plus the total variability penalty is minimized, and demand for electricity is met.

HINT: The following set of decision variables may be required in your formulation:
$y_{lr} = 1$ if renewable resource type $r \in \mathbf{R}$ is installed at location $l \in \mathbf{L}$, $y_{lr} = 0$ otherwise.

Question 2
[Total: 20 marks]

(a) (6 marks) DEFINE all *minimal* knapsack cover inequalities for the following Knapsack Constraint:

$$2x_1 + 4x_2 + x_3 + 3x_4 \leq 5 \tag{1}$$

where $x_1, x_2, x_3, x_4 \in \{0, 1\}$.

(b) (14 marks) Consider the following binary program, where Knapsack Constraint (1) is one of the constraints:

$$\begin{aligned} \max \quad & 5x_1 + 2x_2 + 4x_3 + 2x_4 \\ s.t. \quad & \\ & 2x_1 + 4x_2 + x_3 + 3x_4 \leq 5 \\ & x_1 \geq 1 \\ & x_1, x_2, x_3, x_4 \in \{0, 1\} \end{aligned}$$

Using the *minimal* knapsack cover inequalities found in Part (a) and, if necessary, the LP-based branch-and-bound algorithm, FIND the optimal solution to the binary program and STATE its optimal objective value.

Question 3
[Total: 30 marks]

Consider the following compact formulation for the following variable-sized bin-packing problem (VSBP):

$$
\begin{aligned}
\min \quad & \sum_{j \in \mathbf{B}} C_j x_j \\
s.t. \quad & \\
& \sum_{j \in \mathbf{B}} y_{ij} = 1, \ \forall i \in \mathbf{I} \\
& \sum_{i \in \mathbf{I}} S_i y_{ij} \leq W_j x_j, \ \forall j \in \mathbf{B} \\
& x_j \in \{0, 1\}, \ \forall j \in \mathbf{B} \\
& y_{ij} \in \{0, 1\}, \ \forall i \in \mathbf{I}, j \in \mathbf{B}
\end{aligned}
$$

where
$\mathbf{B}$ is the set of bins;
$\mathbf{I}$ is the set of items;
C_j is the cost of using bin $j \in \mathbf{B}$;
W_j is the capacity of bin $j \in \mathbf{B}$;
S_i is the size of item $i \in \mathbf{I}$;
$x_j = 1$ if bin $j \in \mathbf{I}$ is used, $x_j = 0$ otherwise;
$y_{ij} = 1$ if item $i \in \mathbf{I}$ is packed into bin $j \in \mathbf{I}$, $y_{ij} = 0$ otherwise.

We define a packing pattern for some bin $j \in \mathbf{B}$ as a set of items that could be packed in the bin, and let $\mathbf{P}_j$ be the set of packing patterns[1] for this bin. For example consider a bin with size 3 and items a, b and c with sizes 1, 2 and 3, respectively. Feasible packing patterns for this bin are $\{a\}$, $\{b\}$, $\{c\}$ and $\{a, b\}$ — these patterns can form the set of packing patterns for this bin.

(please turn over)

[1]This set may not include all possible packing patterns.

(a) (10 marks) REFORMULATE the VSBP problem using the following set of *pattern-based* decision variables: $z_{pj} = 1$ if packing pattern $p \in \mathbf{P}_j$ is used for bin $j \in \mathbf{B}$, $z_{pj} = 0$ otherwise. Denote this formulation as "Pattern-based VSBP".

(b) (20 marks) DEFINE a column generation procedure (at the root node) for the LP-relaxation of the Pattern-based VSBP problem. Remember to include the following information in your solution:

- The column generation procedure.
- An expression of the reduced cost of the decision variable z_{pj}.
- One or more integer linear programs that generate new columns which could be appended to the LP-relaxation of the Pattern-based VSBP.
- The condition(s) when the generated column is appended to the LP-relaxation of the Pattern-based VSBP.
- Details of the new column generated, i.e. how set(s) and parameter(s) of the LP-relaxation of the Pattern-based VSBP are updated.
- The terminating condition(s) for the column generation procedure.

Question 4
[Total: 20 marks]

Provide short answers to the following questions:

(a) (2 marks) The worst-case performance of the greedy knapsack heuristic is 2. What does this mean?

(b) (2 marks) What is an advantage of the "depth first search" in a branch-and-bound algorithm?

(c) (2 marks) Consider a mixed-integer linear programming formulation with piecewise linear objective function. When is the problem easy, and when is the problem difficult?

(d) (2 marks) For routing problems, why would one typically prefer a path-based formulation over an arc-based formulation? Give ONE reason.

(e) (3 marks) Consider a Traveling Salesperson Problem (TSP) with 6 cities. When the "basic" formulation is solved, the solution found has two subtours: 1-2-5-1 and 3-4-6-3. Write down ONE constraint that would break the "1-2-5-1" subtour.

(f) (3 marks) Consider the binary decision variables x, y and z. Using *only* these variables, write down ONE linear constraint that would reflect the following statement: if $x = 1$ and $y = 1$, then $z = 1$.

(g) (3 marks) Let x be a binary variable and y be a continuous variable such that $0 \leq y \leq Q$. Using *only* these variables, write down constraints that would reflect the following statement: if $x = 1$, then $y = P$.

(h) (3 marks) Let x and y be two continuous decision variables such that $0 \leq x, y \leq Q$, and let z be a binary variable. Using *only* these variables, write down ONE linear constraint that would reflect the following statement: if $z = 1$, then $x \geq y$.

End of Examination

5.2 Solution

The University of Melbourne

Semester 1 Assessment 2013

Student Number: ....

Department of Mathematics and Statistics
MAST90014 Optimisation for Industry
SAMPLE SOLUTION AND MARKING SCHEME

Question 1

[Total: 30 marks]

The state of Victoria primarily relies on brown coal sources for electricity generation. Here, we wish to consider a simple variant of the problem of introducing and locating renewable energy resources, namely wind and solar farms, in the state of Victoria.

Let $\mathbf{L}$ be the set of candidate locations in Victoria and $\mathbf{R} = \{Wind, Solar\}$ be the set of renewable resource types. At most one renewable resource type is to be installed at any candidate location $l \in \mathbf{L}$. If some renewable resource type $r \in \mathbf{R}$ is installed at location $l \in \mathbf{L}$, the farm size must be at least S_{lr}^{MIN} units and at most S_{lr}^{MAX} units. A one-off cost, C_{lr}^{S}, is incurred when a farm with renewable resource type $r \in \mathbf{R}$ is setup at location $l \in \mathbf{L}$, and for each unit of the renewable resource type installed at the location, it costs C_{lr}^{U}.

Renewable resource locations and farm-size decisions are driven by *demand for electricity* and the *prospect/potential of a location* for renewable energy generation. For this simple problem variant, the renewable resources are not to be used to replace existing brown coal generation sources, but are instead used to cope with demand fluctuations beyond the base electricity supply generated by brown coal sources. Therefore, the "demand for electricity" is defined here as the amount of electricity to be supplied *solely* by the renewable resources.

Let the set of discrete, equally-sized, time periods be $\mathbf{T}$. Each time period may, for example, represent a 30-minute interval and the length of the planning horizon could be 5 days — in this case $\mathbf{T} = \{1, 2, 3, \ldots, 239, 240\}$. Let D_t be the demand for electricity at time $t \in \mathbf{T}$ and P_{lrt} be the amount of electricity that could (potentially) be generated at location $l \in \mathbf{L}$, using <u>one unit</u> of renewable resource type $r \in \mathbf{R}$ at time $t \in \mathbf{T}$.

The total renewable energy generation must be at least D_t for all $t \in \mathbf{T}$. There is a further requirement that the variability of the total renewable energy generation be "reasonable". To achieve a "reasonable" variability, any deviation from the average total generation for any given day is penalized as follows: one unit of positive deviation is penalized at C^P and one unit of negative deviation is penalized at C^M. For instance, consider a three-period example with total renewable generations G_1, G_2 and G_3 on time periods 1, 2, and 3, respectively. The average total generation is $A = \frac{G_1+G_2+G_3}{3}$ and suppose $G_1 < A$, $G_2 = A$ and $G_3 > A$. The total penalty for this example is given by $C^M(A - G_1) + 0 + C^P(G_3 - A)$.

FORMULATE a mixed-integer linear program (MILP) that will locate the renewable resources and determine their farm sizes such that the total farm establishment costs plus the total variability penalty is minimized, and demand for electricity is met.

HINT: The following set of decision variables may be required in your formulation:
$y_{lr} = 1$ if renewable resource type $r \in \mathbf{R}$ is installed at location $l \in \mathbf{L}$, $y_{lr} = 0$ otherwise.

Answer:

Define the following decision variables:
$y_{lr} = 1$ if renewable resource type $r \in \mathbf{R}$ is installed at location $l \in \mathbf{L}$, $y_{lr} = 0$ otherwise;
n_{lr} = number of units of renewable resource type $r \in \mathbf{R}$ installed at location $l \in \mathbf{L}$;
δ_t^+ = amount of electricity generated above average on day $t \in \mathbf{T}$;
δ_t^- = amount of electricity generated below average on day $t \in \mathbf{T}$;

The mixed-integer linear program is:

$$\min \sum_{l \in \mathbf{L}} \sum_{r \in \mathbf{R}} C_{lr}^S y_{lr} + \sum_{l \in \mathbf{L}} \sum_{r \in \mathbf{R}} C_{lr}^U n_{lr} + \sum_{t \in \mathbf{T}} \left(C^P \delta_t^+ + C^M \delta_t^- \right) \tag{1}$$

s.t.

- At most one renewable resource type installed at any location:

$$\sum_{r \in \mathbf{R}} y_{lr} \leq 1, \ \forall l \in \mathbf{L}; \tag{2}$$

- Maximum and minimum farm size:

$$S_{lr}^{MIN} y_{lr} \leq n_{lr} \leq S_{lr}^{MAX} y_{lr}, \ \forall l \in \mathbf{L}, r \in \mathbf{R}; \tag{3}$$

- Demand requirement:

$$\sum_{l \in \mathbf{L}} \sum_{r \in \mathbf{R}} P_{lrt} n_{lr} \geq D_t, \ \forall t \in \mathbf{T}; \tag{4}$$

- Deviation from average generation:

$$\delta_t^+ \geq \sum_{l \in \mathbf{L}} \sum_{r \in \mathbf{R}} P_{lrt} n_{lr} - A, \ \forall t \in \mathbf{T} \tag{5}$$

$$\delta_t^- \geq A - \sum_{l \in \mathbf{L}} \sum_{r \in \mathbf{R}} P_{lrt} n_{lr}, \ \forall t \in \mathbf{T} \tag{6}$$

  where $A = \frac{1}{|\mathbf{T}|} \left(\sum_{l \in \mathbf{L}} \sum_{r \in \mathbf{R}} \sum_{\theta \in \mathbf{T}} P_{lr\theta} n_{lr} \right)$;

- Variable signs:

$$y_{lr} \in \{0, 1\}, \ \forall l \in \mathbf{L}, r \in \mathbf{R} \tag{7}$$

$$n_{lr} \geq 0, \ integer, \ \forall l \in \mathbf{L}, r \in \mathbf{R} \tag{8}$$

$$\delta_t^+, \delta_t^- \geq 0, \ \forall t \in \mathbf{T} \tag{9}$$

Question 2
[Total: 20 marks]

(a) (6 marks) DEFINE all *minimal* knapsack cover inequalities for the following Knapsack Constraint:

$$2x_1 + 4x_2 + x_3 + 3x_4 \leq 5 \tag{10}$$

where $x_1, x_2, x_3, x_4 \in \{0, 1\}$.

Answer:

$$x_1 + x_2 \leq 1 \tag{11}$$

$$x_2 + x_4 \leq 1 \tag{12}$$

$$x_1 + x_3 + x_4 \leq 2 \tag{13}$$

(b) (14 marks) Consider the following binary program, where Knapsack Constraint (10) is one of the constraints:

$$\begin{aligned} \max \quad & 5x_1 + 2x_2 + 4x_3 + 2x_4 \\ s.t. \quad & \\ & 2x_1 + 4x_2 + x_3 + 3x_4 \leq 5 \\ & x_1 \geq 1 \\ & x_1, x_2, x_3, x_4 \in \{0, 1\} \end{aligned}$$

Using the *minimal* knapsack cover inequalities found in Part (a) and, if necessary, the LP-based branch-and-bound algorithm, FIND the optimal solution to the binary program and STATE its optimal objective value.

Answer:

Preprocessing:

- Since x_1 is binary, $x_1 \geq 1$ means that $x_1 = 1$.
- Knapsack Constraint (11) and $x_1 = 1$ implies $x_2 = 0$.

Preprocessing simplifies the original binary program:

$$\begin{aligned} \max \quad & 4x_3 + 2x_4 \\ s.t. \quad & \\ & x_3 + 3x_4 \leq 3 \\ & x_3 + x_4 \leq 1 \ (from\ Knapsack\ Constraint\ (13)) \\ & x_3, x_4 \in \{0, 1\} \end{aligned}$$

The feasible region and optimal (x_3, x_4) is shown in Figure 1.

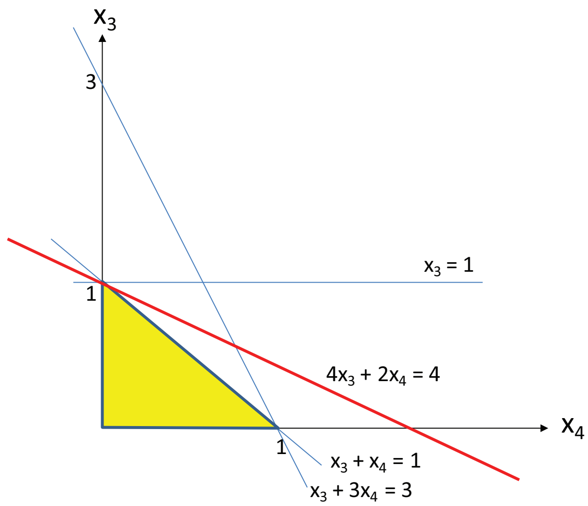


Figure 1:

So the optimal solution is $(x_1, x_2, x_3, x_4) = (1, 0, 1, 0)$.

The optimal objective value is 9.

Question 3
[Total: 30 marks]

Consider the following compact formulation for the following variable-sized bin-packing problem (VSBP):

$$\begin{aligned}
\min \quad & \sum_{j\in\mathbf{B}} C_j x_j \\
s.t. \quad & \\
& \sum_{j\in\mathbf{B}} y_{ij} = 1,\ \forall i \in \mathbf{I} \\
& \sum_{i\in\mathbf{I}} S_i y_{ij} \le W_j x_j,\ \forall j \in \mathbf{B} \\
& x_j \in \{0,1\},\ \forall j \in \mathbf{B} \\
& y_{ij} \in \{0,1\},\ \forall i \in \mathbf{I}, j \in \mathbf{B}
\end{aligned}$$

where
$\mathbf{B}$ is the set of bins;
$\mathbf{I}$ is the set of items;
C_j is the cost of using bin $j \in \mathbf{B}$;
W_j is the capacity of bin $j \in \mathbf{B}$;
S_i is the size of item $i \in \mathbf{I}$;
$x_j = 1$ if bin $j \in \mathbf{I}$ is used, $x_j = 0$ otherwise;
$y_{ij} = 1$ if item $i \in \mathbf{I}$ is packed into bin $j \in \mathbf{I}$, $y_{ij} = 0$ otherwise.

We define a packing pattern for some bin $j \in \mathbf{B}$ as a set of items that could be packed in the bin, and let $\mathbf{P}_j$ be the set of packing patterns[1] for this bin. For example consider a bin with size 3 and items a, b and c with sizes 1, 2 and 3, respectively. Feasible packing patterns for this bin are $\{a\}$, $\{b\}$, $\{c\}$ and $\{a, b\}$ — these patterns can form the set of packing patterns for this bin.

[1]This set may not include all possible packing patterns.

(a) (10 marks) REFORMULATE the VSBP problem using the following set of *pattern-based* decision variables: $z_{pj} = 1$ if packing pattern $p \in \mathbf{P}_j$ is used for bin $j \in \mathbf{B}$, $z_{pj} = 0$ otherwise. Denote this formulation as "Pattern-based VSBP".

Answer:

Let the parameter $A_{ipj} = 1$ if item $i \in \mathbf{I}$ is in pattern $p \in \mathbf{P}_j$ of bin $j \in \mathbf{B}$.

$$\begin{aligned}
\min \quad & \sum_{j \in \mathbf{B}} C_j \left(\sum_{p \in \mathbf{P}_j} z_{pj} \right) \\
s.t. \quad & \\
& -\sum_{p \in \mathbf{P}_j} z_{pj} \geq -1, \ \forall j \in \mathbf{B} \quad (dual: \beta_j) \\
& \sum_{j \in \mathbf{B}} \sum_{p \in \mathbf{P}_j} A_{ipj} z_{pj} = 1, \ \forall i \in \mathbf{I} \quad (dual: \mu_i) \\
& z_{pj} \in \{0, 1\}, \ \forall j \in \mathbf{B}, p \in \mathbf{P}_j
\end{aligned}$$

(b) (20 marks) DEFINE a column generation procedure (at the root node) for the LP-relaxation of the Pattern-based VSBP problem. Remember to include the following information in your solution:

- The column generation procedure.
- An expression of the reduced cost of the decision variable z_{pj}.
- One or more integer linear programs that generate new columns which could be appended to the LP-relaxation of the Pattern-based VSBP.
- The condition(s) when the generated column is appended to the LP-relaxation of the Pattern-based VSBP.
- Details of the new column generated, i.e. how set(s) and parameter(s) of the LP-relaxation of the Pattern-based VSBP are updated.
- The terminating condition(s) for the column generation procedure.

Answer:

The (root node) column generation procedure solves the LP-relaxation of the Pattern-based VSBP problem, then solves pricing problem(s) to find new columns. This is repeated until no more profitable columns are found.

The pricing problem's objective is to minimise the reduced cost of the column. Let β_j and μ_i be the dual variables — see Pattern-based VSBP. The reduced cost for z_{pj} is given by: $C_j + \beta_j^* - \sum_{i \in \mathbf{I}} \mu_i^* A_{ipj}$, where β_j^* and μ_i^* are optimal solutions to the corresponding dual variables.

The pricing problem is formulated for each bin $j \in \mathbf{B}$. This is a binary knapsack problem for each bin $j \in \mathbf{B}$, as shown below:

$$
\begin{aligned}
\max \quad & \sum_{i \in \mathbf{I}} \mu_i^* y_i \\
s.t. \quad & \\
& \sum_{i \in \mathbf{I}} S_i y_i \leq W_j, \ \forall j \in \mathbf{B} \\
& y_i \in \{0, 1\}, \ \forall i \in \mathbf{I}
\end{aligned}
$$

where $y_i = 1$ if item $i \in \mathbf{I}$ is packed into the bin, $y_i = 0$ otherwise.

Let z_j^* be the optimal value to this binary knapsack problem, and the optimal solution y_j^*.

If $C_j + \beta_j^* - z_j^* < 0$, then a new column is generated and appended to Pattern-based VSBP.

Let r be the index of the new column generated. The new column generated is $A_{irj} = y_i^*$, for all $i \in \mathbf{I}$, and append r to $\mathbf{P}_j$.

If $C_j + \beta_j^* - z_j^* \geq 0$, no new column is generated, and the (root node) column generation procedure is terminated.

Question 4
[Total: 20 marks]

Provide short answers to the following questions:

(a) (2 marks) The worst-case performance of the greedy knapsack heuristic is 2. What does this mean?

Answer: The objective value produced by the greedy knapsack heuristic is at most twice the optimal solution, for any data instance.

(b) (2 marks) What is an advantage of the "depth first search" in a branch-and-bound algorithm?

Answer: To get feasible solutions.

(c) (2 marks) Consider a mixed-integer linear programming formulation with piecewise linear objective function. When is the problem easy, and when is the problem difficult?

Answer: Easy when minimising/maximising a convex/concave piecewise linear function; Difficult when maximising/minimising a convex/concave piecewise linear function

(d) (2 marks) For routing problems, why would one typically prefer a path-based formulation over an arc-based formulation? Give ONE reason.

Answer: Path-based formulation breaks symmetry.

(e) (3 marks) Consider a Traveling Salesperson Problem (TSP) with 6 cities. When the "basic" formulation is solved, the solution found has two subtours: 1-2-5-1 and 3-4-6-3. Write down ONE constraint that would break the "1-2-5-1" subtour.

Answer: $\sum_{i\in\{1,2,5\}} \sum_{j\in\{1,2,5\}:j\neq i} x_{ij} \leq 2$

(f) (3 marks) Consider the binary decision variables x, y and z. Using *only* these variables, write down ONE linear constraint that would reflect the following statement: if $x = 1$ and $y = 1$, then $z = 1$.

Answer: $x + y - 1 \leq z$

(g) (3 marks) Let x be a binary variable and y be a continuous variable such that $0 \leq y \leq Q$. Using *only* these variables, write down constraints that would reflect the following statement: if $x = 1$, then $y = P$.

Answer: $Px \leq y \leq P + (Q - P)(1 - x)$

(h) (3 marks) Let x and y be two continuous decision variables such that $0 \leq x, y \leq Q$, and let z be a binary variable. Using *only* these variables, write down ONE linear constraint that would reflect the following statement: if $z = 1$, then $x \geq y$.

Answer: $x \geq y - Q(1 - z)$

End of Examination

6 2014 - Semester 1

6.1 Exam

The University of Melbourne

Semester 1 Assessment 2014

Student Number: . . . .

Department of Mathematics and Statistics
MAST90014 - Optimisation for Industry

Reading time: 15 minutes
Writing time: 120 minutes
This paper has 7 pages.

Identical Examination Papers:
N/A

Common Content Papers:
N/A

Authorised Materials:
The following items are authorised: Calculators.

Instructions to Invigilators:
No handouts are required.
The examination paper is to remain in the examination room.

Instructions to Students:
Answer all questions. Clearly define all parameters and decision variables in your answer, and state all assumptions.

Paper to be held by Baillieu Library:
No

Question 1
[Total: 30 marks]

A company produces the same product at two different factories (Factories A and B), and then the product has to be shipped to three consumer centers (C, D and E). The monthly production capacities of the factories A and B are 350 and 250 units, respectively. The demands for the next three months in the centers are expressed in the table below:

Center ↓ / Month →	1	2	3
C	100	120	100
D	200	180	150
E	250	220	300

The transportation costs (per unit) from a factory to a consumer center are expressed below:

Factory ↓ / Center →	C	D	E
A	\$5	\$7	\$9
B	\$10	\$7	\$3

The unitary production cost at factories A and B are \$10 and \$12, respectively. A maximum of 100 units of inventory can be held in each factory, at a cost of \$1 per unit of the product per period.

a) (15 marks) Write a linear program modelling the problem of supplying the demand while minimising the combined cost of production, transportation and inventory.

b) (8 marks) Modify your model to account for the fact that the transportation cost between a factory and a center in a period is given by a fixed amount of \$ 100 (that must be paid if there is transportation between the factory and the center) plus the cost of transportation per unit.

c) (7 marks) Modify your model to account for the fact that you can ship products from one factory to the other with a cost of \$1 per unit transported.

Attention: Write any assumptions that you make while modelling the problem.

Question 2
[Total: 20 marks]

Consider the binary knapsack problem formulated below:

$$\text{Max } 25x_1 + 20x_2 + 15x_3 + 41x_4 + 51x_5 \tag{1}$$

subject to:

$$3x_1 + 2x_2 + x_3 + 4x_4 + 5x_5 \leq 6 \tag{2}$$

$$x_i \in \{0,1\}, i = 1, \ldots, 6. \tag{3}$$

$$\tag{4}$$

and the following heuristic:

Require: N = array of items, w_i = weigth of item i, u_i = utility of item i, C = Knapsack Capacity

Order N in a non-ascending order of the ratios u_i/w_i (change the label of the items, so that the first item in N is now labelled as item 1, the second as item 2, etc.)
$W = 0$, $S = \{\}$
for $i = 1 \ldots 5$ **do**
  if $(w_i + W \leq C)$ **then**
    $W = W + w_i$
    $S = S + \{i\}$
  end if
end for
return S[1]

a) (5 marks) Explain the heuristic in a few words and give the solution it obtains for the problem presented earlier.

b) (5 marks) Explain how the heuristic can be modified to solve the linearly relaxed version of the problem, i.e., the version where instead of $x_i \in \{0,1\}$, $i = 1, \ldots, 5$, we have $0 \leq x_i \leq 1$, $i = 1, \ldots, 5$. Rewrite the pseudo-code.

c) (10 marks) Solve the problem with a LP-based Branch-and-Bound algorithm using the heuristic proposed in b).

[1] Please note that the solution is expressed in terms of the new labelling of items.

Question 3
[Total: 25 marks]

Consider the following cutting stock problem:
Assume you own a workshop where steel rods are cut into different pieces. The rods have length L and you have a demand for n smaller items, so that you need to cut the rods into these smaller pieces. Your goal is to minimise the number of rods you use in order to supply the demand. For each demand item $i = 1, \ldots, n$, you know the demand of the item, d_i, and the length of the item, l_i.
Below, we present a compact formulation using integer variables x_{ij} indicating the number of items i that are cut out of rod j and binary variables y_j indicating if rod j is used:

$$\text{Min} \sum_{j=1}^{h} y_j \tag{5}$$

subject to:

$$\sum_{j=1}^{h} x_{ij} = d_i \tag{6}$$

$$\sum_{i=1}^{n} l_i x_{ij} \leq L \tag{7}$$

$$\tag{8}$$

$$x_{ij} \geq 0 \text{ and integer, } y_j \in \{0, 1\} \tag{9}$$

a) (5 marks) What is a good value for parameter h ? Explain your choice. [You don't necessarily need to present a value, but rather mention a procedure for obtaining it.]

b) (5 marks) For technical reasons, you can not have both items 1 and 2 being cut out of the same rod unless item 3 is cut from the same rod. In other words: you are allowed to cut a number of items 1 and 2 from the same rod as long as at least one unit of item 3 is cut. Write these additional constraints in the tightest way you are able to. Use additional variables if necessary.

c) (15 marks) Define a cutting pattern as the number of each item that are cut out of a rod. For example, a pattern $[1, 3, 0, \ldots, 1]$ indicates that one unit of item 1 is cut out of the rod, 3 units of item 2, etc... Let $\mathbf{P}_j$ be the set of cutting patterns.

(*please turn over*)

REFORMULATE the cutting stock problem (including the constraints in item b)) using the following set of *pattern-based* decision variables: z_j is the number of rods that are cut with pattern $p \in \mathbf{P}_j$. Denote this formulation as "Pattern-based cutting stock problem (CSP)" .

DEFINE a column generation procedure (at the root node) for the LP-relaxation of the Pattern-based CSP. Remember to include the following information in your solution:

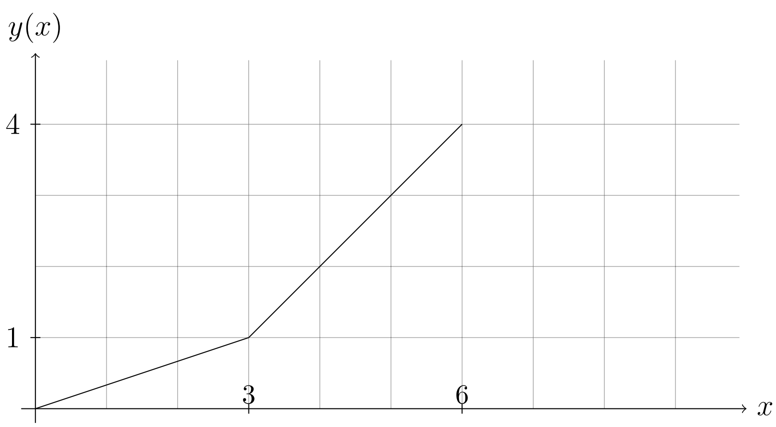
- An initial set of columns (mention if this is a good initial set and why).
- An expression of the reduced cost of the decision variable z_j.
- One or more integer linear programs that generate new columns which could be appended to the LP-relaxation of the Pattern-based CSP.
- The condition for a generated column to be appended to the LP-relaxation of the Pattern-based CSP.
- Details of the new column generated, i.e. how set(s) and parameter(s) of the LP-relaxation of the Pattern-based CSP are updated.
- The terminating condition(s) for the column generation procedure.

Question 4
[Total: 25 marks]

Provide short answers to the following questions:

(a) (3 marks) You developed a heuristic for the knapsack problem and run it for a set of n problems for which you knew the optimal solutions. For $n-1$ of the instances your heuristic found the optimal solutions but for one of them it found a solution of 75 when the optimal value was 100. What can you say about the worst-case performance of your heuristic ?

(b) (3 marks) Define the appropriate variables and write the following piecewise minimisation function $y(x)$ (and eventual constraints).



(c) (3 marks) Now suppose you want to maximise the function. Write your objective function (and eventual constraints).

(d) (3 marks) In the context of machine scheduling problem $1||\sum_{C_j}$, what is the shortest processing time rule ? Is the solution obtained with such a rule always optimal? Why?

(e) (3 marks) Consider a Traveling Salesperson Problem (TSP) with 6 cities with subtour elimination constraints relaxed. The solution obtained has subtours 1-2-5-1 and 3-4-6-3. Write at least two different subtour elimination constraint that can cut this solution out.

(please turn over)

(f) (3 marks) In the context of stochastic programming, what is the Estimated Value of Perfect Information and the Value of Stochastic Solution ? (Assume a maximisation problem.)

(g) (3 marks) Let x be a binary variable and y be a continuous variable such that $0 \leq y \leq Q$. Using *only* these variables, write down constraints that would reflect the following statement: if $x = 1$, then $y = P$.

(h) (4 marks) Let x and y be two continuous decision variables such that $0 \leq x, y \leq Q$, and let z be a binary variable. Using *only* these variables, write down ONE linear constraint that would reflect the following statement: if $z = 1$, then $x \geq y$.

End of Examination

6.2 Solution

The University of Melbourne

Semester 1 Assessment 2014

Student Number: ....

Department of Mathematics and Statistics
MAST90014 - Optimisation for Industry

Reading time: 15 minutes
Writing time: 120 minutes
This paper has 7 pages.

Identical Examination Papers:
N/A

Common Content Papers:
N/A

Authorised Materials:
The following items are authorised: Calculators.

Instructions to Invigilators:
No handouts are required.
The examination paper is to remain in the examination room.

Instructions to Students:
Answer all questions. Clearly define all parameters and decision variables in your answer, and state all assumptions.

Paper to be held by Baillieu Library:
No

Question 1

[Total: 30 marks]

A company produces the same product at two different factories (Factories A and B), and then the product has to be shipped to three consumer centers (C, D and E). The monthly production capacities of the factories A and B are 350 and 250 units, respectively. The demands for the next three months in the centers are expressed in the table below:

Center ↓ / Month →	1	2	3
C	100	120	100
D	200	180	150
E	250	220	300

The transportation costs (per unit) from a factory to a consumer center are expressed below:

Factory ↓ / Center →	C	D	E
A	\$5	\$7	\$9
B	\$10	\$7	\$3

The unitary production cost at factories A and B are \$10 and \$12, respectively. A maximum of 100 units of inventory can be held in each factory, at a cost of \$1 per unit of the product per period.

a) (15 marks) Write a linear program modelling the problem of supplying the demand while minimising the combined cost of production, transportation and inventory.

b) (8 marks) Modify your model to account for the fact that the transportation cost between a factory and a center in a period is given by a fixed amount of \$ 100 (that must be paid if there is transportation between the factory and the center) plus the cost of transportation per unit.

c) (7 marks) Modify your model to account for the fact that you can ship products from one factory to the other with a cost of \$1 per unit transported.

Attention: Write any assumptions that you make while modelling the problem.

Assessment Objectives:
General: verify if students can understand the description of a simplified (but still reasonably complex) industrial optimisation problem and model it with linear constraints and objective.

Specific:

a)

- (5 marks) Verify if students can properly define variables.
- (2 marks) Verify if students can write a simple objective function dealing with multiple terms (production, transportation and inventory).
- (2 marks) Verify if students can define mass conservation (inventory) constraints
- (2 marks) Verify if students can define demand constraints.
- (2 marks) Verify if students can define production capacity constraints.
- (2 marks) Verify if students can present the problem in a mathematical coherent way.

b)

- (4 marks) Verify if students can properly define new variables.
- (4 marks) Verify if students can define new constraints with setup costs.

c)

- (4 marks) Verify if students can properly define appropriate new variables.
- (3 marks) Verify if students can define new constraints for an existing problem.

Answer:
a) Let p_i and e_i be the production cost and production limit at factory $i = \{A, B\}$, respectively. Also, let d_{jt} be the demand at consumer center $j = \{C, D, E\}$ in period $t = \{1, 2, 3\}$ and c_{ij} be the unitary transportation cost between factory i and consumer j. Finally, let h be the cost of inventory per unit per period.
Define variables x_{it} as the amount of product produced in period t in factory i, z_{ijt} as the amount of product shipped from factory i to consumer center j in period t and I_{it} the amount of inventory at factory i at the end of period t ($I_{i0} = 0$). The production-transportation planning problem can be modeled as:

$$\text{Min} \sum_{t=1}^{3} \left(\sum_{i \in \{A,B\}} p_i x_{it} + \sum_{i \in \{A,B\}} \sum_{j \in \{C,D,E\}} c_{ij} z_{ijt} + \sum_{i \in \{A,B\}} h I_{it} \right) \tag{1}$$

subject to:

$$x_{it} \leq e_i, \qquad i \in \{A,B\}, t = 1, \ldots, 3, \tag{2}$$

$$\sum_{i \in \{A,B\}} z_{ijt} = d_{jt}, \qquad j \in \{C,D,E\}, t = 1, \ldots, 3 \tag{3}$$

$$x_{it} + I_{i,t-1} - \sum_{j \in \{C,D,E\}} z_{ijt} = I_{it}, \qquad i \in \{A,B\}, t = 1, \ldots, 3, \tag{4}$$

$$I_{it} \leq 100, \qquad i \in \{A,B\}, t = 1, \ldots, 3, \tag{5}$$

$$x_{it} \geq 0, I_{it} \geq 0, z_{ijt} \geq 0. \tag{6}$$

b) We need variables y_{ijt} indicating if there is flow between factory i and consumer center j in period t. The new term below is added to the objective function:

$$\sum_{t=1}^{3} \sum_{i \in \{A,B\}} \sum_{j \in \{C,D,E\}} 100 y_{ijt} \tag{7}$$

and the program has new constraints limiting the flow between arcs:

$$z_{ijt} \leq M_{jt} y_{ijt}, \qquad i \in \{A,B\}, j \in \{C,D,E\}, t = 1, \ldots, 3. \tag{8}$$

M is a sufficiently large value. We can use $M_{jt} = d_{jt}$, for instance.

c) We need variables to represent the flow between the factories. Let f_{kjt} be the flow of products from factory k to factory j in period $t = 1, \ldots, 3$; $k, j \in \{A,B\}$. The new flow conservation constraints read:

$$x_{it} + \sum_{j \in \{A,B\} - \{i\}} (f_{jit} - f_{ijt}) + I_{i,t-1} - \sum_{j \in \{C,D,E\}} z_{ijt} = I_{it}, \qquad i \in \{A,B\}, t = 1, \ldots, 3, \tag{9}$$

Question 2
[Total: 20 marks]

Consider the binary knapsack problem formulated below:

$$\text{Max } 25x_1 + 20x_2 + 15x_3 + 41x_4 + 51x_5 \tag{10}$$

subject to:

$$3x_1 + 2x_2 + x_3 + 4x_4 + 5x_5 \leq 6 \tag{11}$$

$$x_i \in \{0, 1\}, i = 1, \ldots, 6. \tag{12}$$

$$\tag{13}$$

and the following heuristic:

Require: N = array of items, w_i = weigth of item i, u_i = utility of item i, C = Knapsack Capacity

Order N in a non-ascending order of the ratios u_i/w_i (change the label of the items, so that the first item in N is now labelled as item 1, the second as item 2, etc.)

$W = 0$, $S = \{\}$

for $i = 1 \ldots 5$ **do**

  if $(w_i + W \leq C)$ **then**

    $W = W + w_i$

    $S = S + \{i\}$

  end if

end for

return S [1]

a) (5 marks) Explain the heuristic in a few words and give the solution it obtains for the problem presented earlier.

b) (5 marks) Explain how the heuristic can be modified to solve the linearly relaxed version of the problem, i.e., the version where instead of $x_i \in \{0, 1\}$, $i = 1, \ldots, 5$, we have $0 \leq x_i \leq 1$, $i = 1, \ldots, 5$. Rewrite the pseudo-code.

c) (10 marks) Solve the problem with a LP-based Branch-and-Bound algorithm using the heuristic proposed in b).

[1] Please note that the solution is expressed in terms of the new labelling of items.

Assessment Objectives:
General: verify if students understand a pseudo-code, modify it and relate it to a linear programming relaxation. Verify their understanding of the branch-and-bound algorithm.

Specific:

a)

- (3 marks) Verify if students can read and explain a pseudo-code.
- (2 marks) Verify if students can manually implement the solution specified by an algorithm.

b)

- (2 marks) Verify if students can properly modify a pseudo code.
- (3 marks) Verify if students can relate a heuristic algorithm for the knapsack problem to its linear relaxation.

c)

- Verify if students can solve a simple problem using the branch and bound algorithm:

  (2 marks) Verify if students can solve the root node.

  (2 marks) Verify if students can branch on appropriate variables.

  (2 marks) Verify if students can define appropriate branching constraints.

  (2 marks) Verify if students can identify primal bounds.

  (2 marks) Verify if students can prune non promising nodes.

Answer
a) The heuristics sort the items in a non-ascending order of u_i/w_i and tries to assign the items, in this order, to the knapsack. If the item fits in the knapsack, it is assigned. Otherwise, the heuristic moves on to the next item until all items have been examined.
In the example, the order of the items after the vector is sorted is $N = [3, 4, 5, 2, 1]$. Item 3 is assigned, yielding a residual capacity of 5. Then, item 4 is assigned, leaving a residual capacity of one. No other item can be assigned and the final solution is to select items 3 and 4 only.

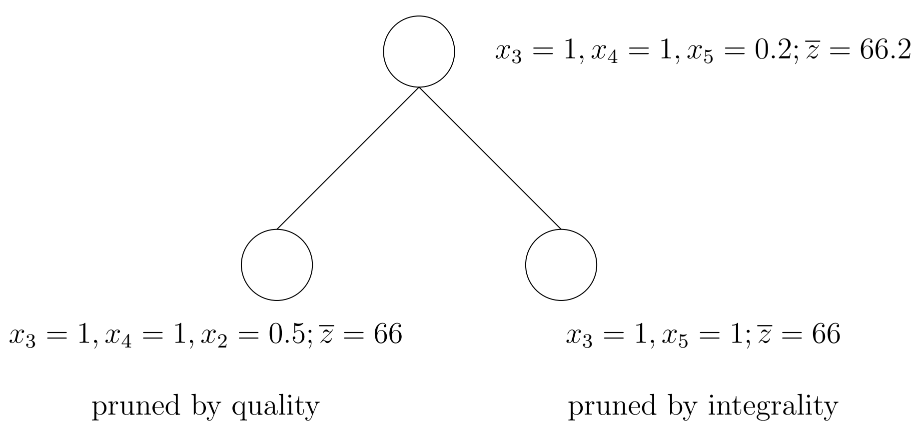
b) The needed modification is simply to take a fraction of the first item that does not fit. The new pseudo code reads:

Require: N = array of items, w_i = weigth of item i, u_i = utility of item i, C= Knapsack Capacity

```
Order N in a non-ascending order of u_i/w_i,
W = 0, S = [] (array containing the amount of each item taken) - Items are labeled in the same order as in the sorted array N.
for i = 1...5 do
    if (w_i + W ≤ C) then
        W = W + w_i
        S[i] = 1
    else
        S[i] = (C − W)/w_i
        Break
    end if
end for
return S
```

In the example above, the solution would be to select items 3 and 4 and 1/5 of item 5.

c) The solution at the root node is the one presented in b) and has value $z = 66.2 = 1 \times 15 + 1 \times 41 + 1/5 \times 51$. The branch and bound branches on variable x_5. On the left child node, x_5 is forced at zero, and the linear relaxation of the remaining problem obtains solution of $x_3 = 1, x_4 = 1, x_2 = 0.5$, with $z = 66$. In the right child node, x_5 is forced at one and the linear relaxation obtains an integer solution of $x_3 = 1, x_5 = 1$, with $z = 66$. Since this lower bound is $\geq$ the lower bound at the left child node, the left child node is pruned, after which there are no other nodes to explore, leaving us with the optimal solution found in the right child node.



Question 3
[Total: 25 marks]

Consider the following cutting stock problem:
Assume you own a workshop where steel rods are cut into different pieces. The rods have length L and you have a demand for n smaller items, so that you need to cut the rods into these smaller pieces. Your goal is to minimise the number of rods you use in order to supply the demand. For each demand item $i = 1, \dots, n$, you know the demand of the item, d_i, and the length of the item, l_i.
Below, we present a compact formulation using integer variables x_{ij} indicating the number of items i that are cut out of rod j and binary variables y_j indicating if rod j is used:

$$\text{Min} \sum_{j=1}^{h} y_j \tag{14}$$

subject to:

$$\sum_{j=1}^{h} x_{ij} = d_i \tag{15}$$

$$\sum_{i=1}^{n} l_i x_{ij} \leq L \tag{16}$$

$$\tag{17}$$

$$x_{ij} \geq 0 \text{ and integer, } y_j \in \{0, 1\} \tag{18}$$

a) (5 marks) What is a good value for parameter h ? Explain your choice. [You don't necessarily need to present a value, but rather mention a procedure for obtaining it.]

b) (5 marks) For technical reasons, you can not have both items 1 and 2 being cut out of the same rod unless item 3 is cut from the same rod. In other words: you are allowed to cut a number of items 1 and 2 from the same rod as long as at least one unit of item 3 is cut. Write these additional constraints in the tightest way you are able to. Use additional variables if necessary.

c) (15 marks) Define a cutting pattern as the number of each item that are cut out of a rod. For example, a pattern $[1, 3, 0, \dots, 1]$ indicates that one unit of item 1 is cut out of the rod, 3 units of item 2, etc... Let $\mathbf{P}_j$ be the set of cutting patterns.

(please turn over)

REFORMULATE the cutting stock problem (including the constraints in item b)) using the following set of *pattern-based* decision variables: z_j is the number of rods that are cut with pattern $p \in \mathbf{P}_j$. Denote this formulation as "Pattern-based cutting stock problem (CSP)" .

DEFINE a column generation procedure (at the root node) for the LP-relaxation of the Pattern-based CSP. Remember to include the following information in your solution:

- An initial set of columns (mention if this is a good initial set and why).
- An expression of the reduced cost of the decision variable z_j.
- One or more integer linear programs that generate new columns which could be appended to the LP-relaxation of the Pattern-based CSP.
- The condition for a generated column to be appended to the LP-relaxation of the Pattern-based CSP.
- Details of the new column generated, i.e. how set(s) and parameter(s) of the LP-relaxation of the Pattern-based CSP are updated.
- The terminating condition(s) for the column generation procedure.

Assessment Objectives:

General: Verify if students understand some details (like the importance of upper bounds on parameters) in a linear programming model and verify their ability to propose a column generation procedure.

Specific:

a)

- (3 marks) Verify if students understand the importance of tight bounds on some parameters.
- (2 marks) Verify if students understand that heuristics can be used within exact methods.

b)

- (2 marks) Verify if students can use appropriate new binary variables to impose logical constraints.

- (3 marks) Verify if students are aware of the importance of tight bounds on big M parameters.

c)

- Verify if students understand a simple column generation procedure:

  (2 marks) Verify if students can understand a pattern.

  (2 marks) Verify if students can propose a pattern-based formulation once they are given a pattern.

  (2 marks) Verify if students understand the importance of a initial feasible set of columns and propose them.

  (3 marks) Verify if students can propose a pricing objective function

  (3 marks) Verify if students can propose a pricing problem constraints

  (3 marks) Verify if students can sketch the column generation procedure

Answer

a) A good value for h is the optimal solution of problem (14) - (17). Since we don't know this value, we can use an upper bound (obtained with an heuristic, for example).

b) Let y_{1j} be a binary variable equal to one if and only if at least one unit of item $i = 1, 2$ is cut from rod j. We know that:

$$x_{ij} \leq \lfloor \frac{L}{l_i} \rfloor y_{ij}, \qquad i = 1, 2. \tag{19}$$

With these additional variables, we can write:

$$y_{1j} + y_{2j} \leq 1 + x_{3j}, \qquad j = 1, \ldots, h. \tag{20}$$

Then, if x_{3j} equals to zero (item 3 is not in rod j), only one type of item 1 or 2 can be cut.

c) Given a column $a_j = [a_{1j}, a_{2j}, \ldots, a_{nj}]$ where n is the number of items, we can write the problem as:

$$\text{Min} \sum_{j \in P_j} z_j \tag{21}$$

subject to:

$$\sum_{j \in P_j} a_{ij} z_j \geq d_i, \qquad i = 1, \ldots, n. \tag{22}$$

$$z_j \geq 0 \tag{23}$$

$$\tag{24}$$

A column generation procedure can be obtained by starting with a subset of the columns P_j. This subset must be able to generate a feasible solution and, therefore, a possible subset is any set of n linearly independent columns, for example, n columns of the form:

$$a_j = \begin{bmatrix} 0 \\ \vdots \\ \lfloor L/l_j \rfloor \\ \vdots \\ 0 \end{bmatrix}$$

Where the non-zero element is in position j for each one of the $j = 1, \ldots, n$ columns.
Let u_i be the dual variable associated with the i^{th} constraint in the master problem. We can look for new columns with negative reduced cost $\bar{c} = 1 - \sum_i a_i u_i$ by solving the following linear pricing program:

$$\text{Min } 1 - \sum_{i=1}^{n} a_i u_i \tag{25}$$

subject to:

$$\sum_{i=1}^{n} l_i a_i \leq L, \tag{26}$$

$$a_1 \leq \lfloor \frac{L}{l_1} \rfloor y_1, \qquad i = 1, 2. \tag{27}$$

$$a_2 \leq \lfloor \frac{L}{l_2} \rfloor y_2, \qquad i = 1, 2. \tag{28}$$

$$y_1 + y_2 \leq 1 + a_3, \qquad j = 1, \ldots, h. \tag{29}$$

$$a_i \geq 0 \text{ and integer}, y_1, y_2 \in \{0, 1\}. \tag{30}$$

If the solution value of (25)–(30) is negative, the obtained column $a = [a_1, a_2, \ldots, a_n]^t$ has a negative reduced cost and should be introduced in the master problem and the process reiterated. Otherwise, there is no promising column meaning that the last solution obtained by the master problem is optimal.

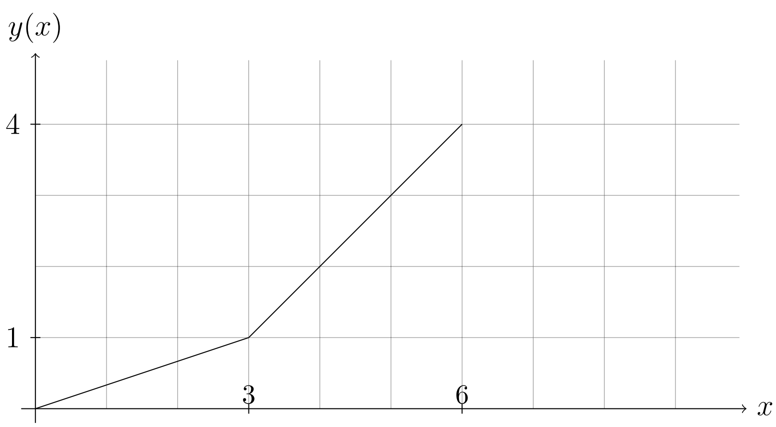
Question 4

[Total: 25 marks]

Provide short answers to the following questions:

(a) (3 marks) You developed a heuristic for the knapsack problem and run it for a set of n problems for which you knew the optimal solutions. For $n-1$ of the instances your heuristic found the optimal solutions but for one of them it found a solution of 75 when the optimal value was 100. What can you say about the worst-case performance of your heuristic ?

(b) (3 marks) Define the appropriate variables and write the following piecewise minimisation function $y(x)$ (and eventual constraints).



(c) (3 marks) Now suppose you want to maximise the function. Write your objective function (and eventual constraints).

(d) (3 marks) In the context of machine scheduling problem $1||\sum_{C_j}$, what is the shortest processing time rule ? Is the solution obtained with such a rule always optimal? Why?

(e) (3 marks) Consider a Traveling Salesperson Problem (TSP) with 6 cities with subtour elimination constraints relaxed. The solution obtained has subtours 1-2-5-1 and 3-4-6-3. Write at least two different subtour elimination constraint that can cut this solution out.

(please turn over)

(f) (3 marks) In the context of stochastic programming, what is the Estimated Value of Perfect Information and the Value of Stochastic Solution ? (Assume a maximisation problem.)

(g) (3 marks) Let x be a binary variable and y be a continuous variable such that $0 \leq y \leq Q$. Using *only* these variables, write down constraints that would reflect the following statement: if $x = 1$, then $y = P$.

(h) (4 marks) Let x and y be two continuous decision variables such that $0 \leq x, y \leq Q$, and let z be a binary variable. Using *only* these variables, write down ONE linear constraint that would reflect the following statement: if $z = 1$, then $x \geq y$.

Assessment Objectives:
General: Verify if students understand specific aspects of the course.

Specific:

a)

- (3 marks) Verify if students understand the idea of worst-case performance

b)

- (3 marks) Verify if students understand the piecewise linear functions (easy situation: no integer variables needed)

c)

- (3 marks) Verify if students understand the piecewise linear functions (hard situation: integer variables needed)

d)

- (1 mark) Verify if students understand machine scheduling notation
- (2 marks) and identify optimality of a simple heuristic.

e)

- (3 marks) Verify if students understand subtour elimination constraints.

f)

- (3 marks) Verify if students understand basic notions in stochastic programming.

g)

- (3 marks) Verify if students can write logical constraints with binary and continuous variables.

h)

- (4 marks) Verify if students can write complicate logical constraints.

Answer

a) The worst case performance is no larger than 0.75.

b) $Min \frac{1}{3}x_1 + x_2$, and $x_1 \leq 3, x_2 \leq 3, x_1 \geq 0, x_2 \geq 0$.

c) $Max \frac{1}{3}x_1 + x_2$, and $y_1 \leq \frac{1}{3}x_1, x_1 \leq 3, x_2 \leq 3y_1$.

d) The shortest processing time rule establishes that processes with shorter processing times should be processed first. This is optimal since any solution that does not comply with this rule can be improved since there is at least two neighboring processes in the wrong order (the one of a largest processing time coming first). This is easy to see since the completion times of all other processors remain the same, but the completion times of one of these two processes is reduced.
Indeed, let l_1 and l_2 be the processing times, $l_1 \leq l_2$ of processes 2 and 1, which are processed in this order. The completion time of 1 is $c_1 = c + l_2 + l_1$ and the completion time of 2 is $c_2 = c + l_2$ (where c is the initial processing time of the first of the tasks), yielding $c_1 + c_2 = 2c + 2l_2 + l_1$. Swapping the items, we get $c_1 + c_2 = 2c + 2l_1 + l_2$ which is smaller since $l_1 < l_2$ by hypothesis. (Note that the completion times of all other tasks remain the same).

e) Some subtour elimination constraints to cut this solution are:

$$\sum_{i,j \in \{1,2,5\}} x_{ij} \leq 2 \tag{31}$$

$$\sum_{i,j \in \{3,4,6\}} x_{ij} \leq 2 \tag{32}$$

$$\sum_{i \in \{1,2,5\}} \sum_{j \in \{3,4,6\}} x_{ij} \geq 1 \tag{33}$$

f) EVPI = Value of wait and see solution - value of average solution
VSS = Value of stochastic solution - value of average solution

g)
$y \leq Q + (P - Q)x$
$y \geq Px$

h) $y \leq x + Q(1 - z)$

End of Examination

7 2015 - Semester 1

7.1 Exam

THE UNIVERSITY OF MELBOURNE

Student ID

Semester 1 Assessment, 2015

School of Mathematics and Statistics

MAST90014 Optimisation for Industry

Writing time: 2 hours

Reading time: 15 minutes

This is NOT an open book exam

This paper consists of 5 pages (including this page)

Authorised materials:

- Non-programmable calculators are authorised.

Instructions to Students

- You must NOT remove this question paper at the conclusion of the examination.
- You should attempt all questions. Marks for individual questions are shown.
- Number the questions and question parts clearly, and start each question on a new page.
- Clearly define all parameters and variables in your answers, and state all assumptions.
- The total number of marks available is 40.

Instructions to Invigilators

- Students must NOT remove this question paper at the conclusion of the examination.

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Question 1 (8 marks)

A company is involved in the production of two items (1 and 2). The demand for Items 1 and 2 is known for the following five days and given by $d_{it}, i = 1, 2, t = 1, \ldots, 5$. The plant can produce only one type of item per day. The daily capacity of the plant is 400 units (of either Item 1 or 2). The plant has two storage rooms, A and B, that can keep up to 500 units of each item. Only one type of item can be kept at a storage room at a given time. Before the beginning of day 1, you have 300 units of Item 1 in storage room A and 300 units of Item 2 in storage room B. You predict that a high demand for item 1 will occur after day 5.

(a) Define appropriate variables and write a mixed-integer program that models the lot-sizing problem of the production of items 1 and 2 over the next five days, such that the amount of item 1 in storage at the end of day 5 is maximised, while meeting demands for items 1 and 2 in days 1-5.

(b) Modify your model in (a) to account for the fact that if a different item was produced in the previous day, a setup has to be made and the production capacity is reduced to 300 units in the current day. Assume that in day 0 (before the planning horizon started) the plant produced item 1.

(c) Modify your model in (a) to account for the fact that the two different products can be mixed in the storage rooms. However, if this happens for a given day, the capacity of the storage room must be reduced to 400 units in that day.

Question 2 (8 marks)

Consider the following problem:

$$\min_{x} \quad f(x_1, x_2, x_3, x_4) \;=\; (x_1 - 0.2)^2 + (x_2 - 1)^2 + (x_3 - 0.2)^2 + (x_4 - 0.8)^2$$

$$\text{subject to} \quad x_1, x_2, x_3, x_4 \in \{0, 1\}.$$

Note that a relaxed version of this problem in which constraints $x_i \in \{0, 1\}$ are substituted by $0 \leq x_i \leq 1$, $\forall i = 1, \ldots, 4$ can be solved by inspection. Using this relaxation to find lower bounds for your problem, solve it **with the Branch and Bound algorithm**. Use a depth-first search approach. If more than one variable is available for branching at a given node, branch on the variable with smallest index.

Number the nodes as you create them. The root node (node 1) of the Branch and Bound tree has been solved below:

1 $\boxed{(x_1, x_2, x_3, x_4) = (0.2, 1, 0.2, 0.8); \underline{z} = 0}$

Question 3 (8 marks)

Consider the mixed-integer problem below:

$$\text{Min } y_1 - y_2 + 5x_1 + 10x_2 \tag{1}$$

subject to:

$$x_1 \geq y_1 \tag{2}$$
$$x_2 \geq y_2 \tag{3}$$
$$y_1 + y_2 = 1 \tag{4}$$
$$y_1, y_2 \in \{0, 1\} \tag{5}$$
$$x_1, x_2 \geq 0. \tag{6}$$

In a Benders decomposition approach, let the y variables be in the Master problem and the x variables be in the subproblem.

(a) Write down the relaxed master problem.

(b) Write down the primal subproblem.

(c) Write down the dual subproblem.

(d) Perform one iteration of the Benders decomposition algorithm:

(d.1) Solve the relaxed Master problem and obtain a tentative y solution and a lower bound for the original problem.

(d.2) Using the tentative y solution, solve the dual subproblem (using the graphical method) and obtain a cut to be inserted in the master (if multiple optimal solutions are available, choose one that could possibly be obtained by the simplex algorithm). If the subproblem is feasible, give the obtained upper bound.

Question 4 (8 marks)

Assume you own a workshop where steel rods are cut into different pieces. The rods have length L and cost c. You have a demand for n smaller items, so that you need to cut the rods into these smaller pieces. Your goal is to minimise the cost of rods used in order to meet the demand. For each demand item $i = 1, \ldots, n$, you know the demand of the item, d_i, and the length of the item, l_i.

Define a cutting pattern as the number of times each item is cut out of a rod. For example, a pattern $[1, 3, 0, \ldots, 1]$ indicates that one unit of item 1 is cut out of the rod, 3 units of item 2, etc... Let $\mathbf{P}_j$ be the set of cutting patterns.

(a) Formulate the cutting stock problem using the following set of *pattern-based* decision variables: z_j is the number of rods that are cut with pattern $p \in \mathbf{P}_j$. Denote this formulation as "Pattern-based cutting stock problem (CSP)" .

(b) In a column generation procedure for the LP-relaxation of the Pattern-based CSP, define:

- b.1 An initial set of columns (mention if this is a good initial set and why).
- b.2 An expression for the reduced cost of the decision variable z_j.
- b.3 One or more integer linear programs that generate new columns which could be appended to the LP-relaxation of the Pattern-based CSP.
- b.4 The condition for a generated column to be appended to the LP-relaxation of the Pattern-based CSP.
- b.5 The terminating condition(s) for the column generation procedure.

(c) How could you adapt your procedure if now you have two different types of rods, with sizes L_1 and L_2 and with prices c_1 and c_2 (Your goal is still to minimise the total price of used rods) ?

Question 5 (8 marks)

Provide short answers to the following questions:

(a) Write down all **minimal** cover inequalities that you can obtain from the constraints:

$$3x_1 + 5x_2 + 7x_3 + 4x_4 \leq 10$$
$$x_1, x_2, x_3, x_4 \in \{0, 1\}$$

(b) A project manager desires to reduce the duration of her project by 25%. Using a software she found the critical path (there was only one critical path). She thought about the following four strategies:

- Strategy A: Reduce the duration of every task of the project by 25%.
- Strategy B: Reduce the duration of the longest task by 50%.
- Strategy C: Reduce the duration of all the tasks that belong to the critical path by 25%.
- Strategy D: Reduce the duration of all the tasks that do not belong to the critical path by 50%.

Which strategy or strategies will: i) NEVER achieve the desired goal; ii) ALWAYS achieve the desired goal ?

(c) Consider a Traveling Salesperson Problem (TSP) with 7 cities with subtour elimination constraints relaxed. The solution obtained has subtours 1-2-3-1 and 5-4-6-7-5. Let x_{ij} be a variable equal to one if and only if the edge (i, j) is in the solution. Write three different constraints that can, each one of them, eliminate these subtours.

(d) Given the problem:

$$\text{Max } x_1 + x_2 + x_3 \tag{7}$$

subject to:

$$ax_1 + bx_2 + cx_3 \leq 2 \tag{8}$$
$$x_1, x_2, x_3 \geq 0. \tag{9}$$

The parameters a, b and c are uncertain. You can predict 2 possible scenarios with equal probability:

- Scenario 1: $a = 1$, $b = 1$, $c = 1$.
- Scenario 2: $a = 0$, $b = 2$, $c = 1$.

Assume x_3 is a second-stage variable that can be decided after the uncertain information is revealed and model the associated stochastic program.

End of Exam—Total Available Marks = 40.

7.2 Solution

1/

Variables:

real X_{it} : ammount produced of item i at day t

binary Y_{it} : if there is production of product i at day t

real I_{ist} storage of product i in storage room s at the end of day t

binary Z_{ist} : if storage room is being used for item i at the end of day t.

a) $\text{Max} \quad I_{1A5} + I_{1B5}$

$$X_{it} + \sum_{s} I_{is,t-1} = d_{it} + \sum_{s} I_{ist} \quad , \quad \forall t, \forall i$$
(demand / flow conservation)

$$X_{it} \leq 400\, Y_{it} \quad , \quad \forall t, \forall i$$
(production capacity)

$$I_{ist} \leq 500\, Z_{ist} \quad , \quad \forall i, \forall s, \forall t$$
(storage capacity)

$$Y_{1t} + Y_{2t} = 1 \quad , \quad \forall t$$
(one product is made)

$$Z_{1st} + Z_{2st} = 1 \quad , \quad \forall t, \forall s$$
(one product is kept in storage at each day)

$I_{1A0} = 300 \qquad I_{2B0} = 300$ (initial conditions)

b) <u>Besides</u> constraints

$$X_{it} \leq 400\, Y_{it} \qquad \forall i, \forall t$$

we need constraints that reduce this capacity if product i is being produced now but has not been produced the ~~day~~ before:

$$X_{it} \leq 400\, Y_{it} - 100\,(Y_{it} - Y_{i,t-1})$$

with $Y_{10} = 1$

Note that we still need the original constraints otherwise we would allow the production of 100 unities of item i if item i is not setup for production now but was setup in the previous period.

c)

c)

Define binary variables p_{st} equal to one if both items are stored simultaneously in storage room s in period t.

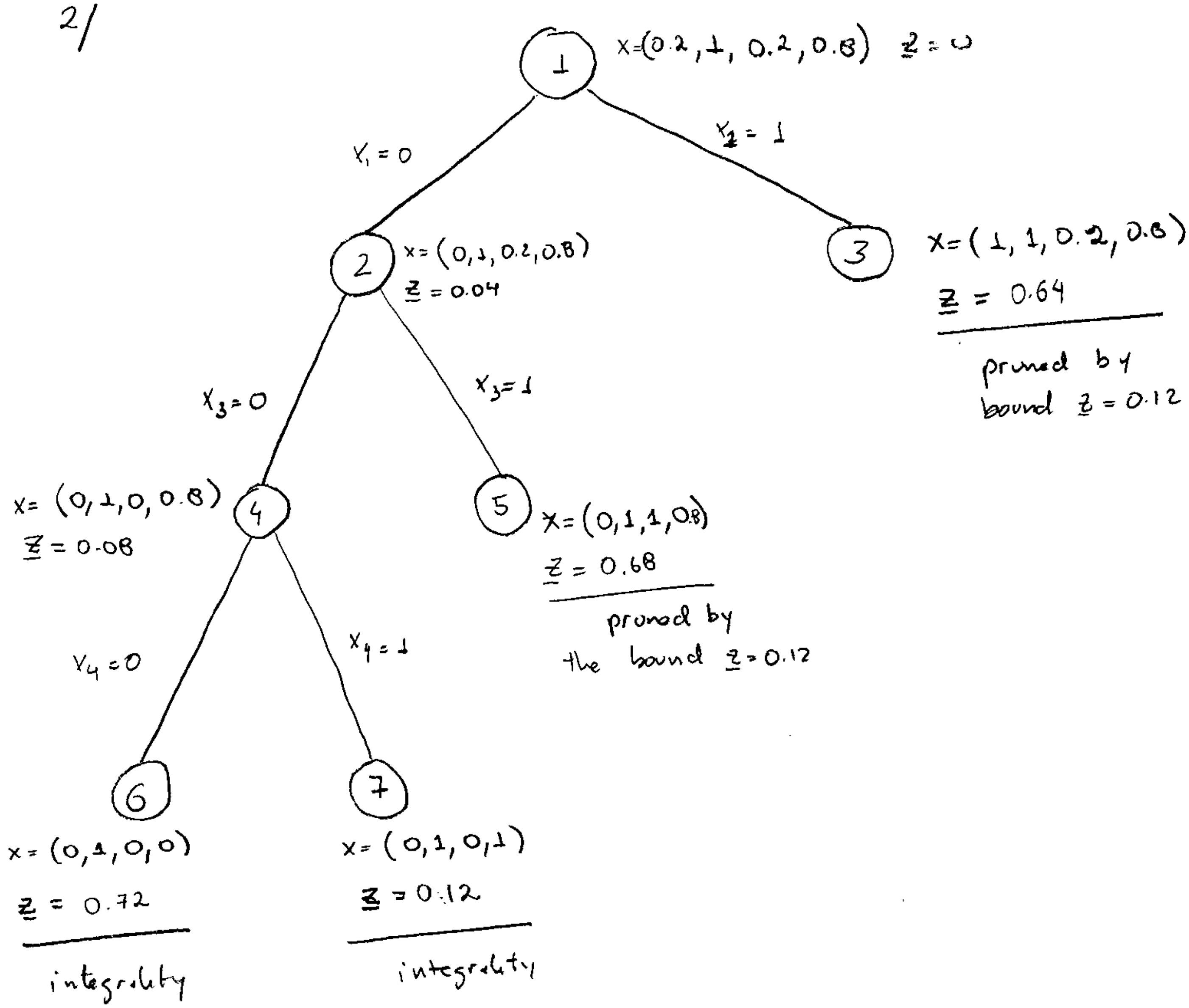
Constraint $z_{1st} + z_{2st} = 1$ is deleted

We add constraints

$$\sum_i I_{ist} \leq 500 - 100\, p_{st}$$

$$p_{st} \geq z_{1st} + z_{2st} - 1$$

2/



Solution : $\boxed{x = (0, 1, 0, 1)}$ $z^* = 0.12$

found on node 7

3/

a)

$$\text{Min } \quad y_1 - y_2 + s$$

s.t.

$$y_1 + y_2 = 1$$

$$y_1, y_2 \in \{0,1\}$$

$$s \geq 0$$

b)

$$\text{Min } \quad 5x_1 + 10x_2$$

s.t. $\quad x_1 \geq \bar{y}_1 \quad (u_1)$

$\quad x_2 \geq \bar{y}_2 \quad (u_2)$

$\quad x_1, x_2 \geq 0$

c)

$$\text{Max } \quad \bar{y}_1 u_1 + \bar{y}_2 \cdot u_2$$

s.t. $\quad u_1 \leq 5$

$\quad u_2 \leq 10$

d)

d.1 → by inspection, the solution of the first relaxed master problem is $y_1 = 0$, $y_2 = 1$, $s = 0$

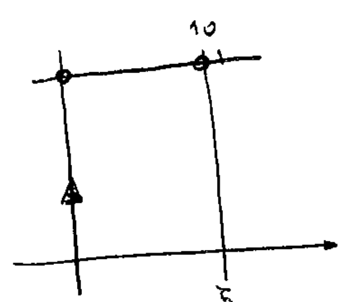
$LB = -1$.

d.2. → The resulting dual subproblem is

$$\text{Max } \quad u_2$$

s.t. $\quad u_1 \leq 5$

$\quad u_2 \leq 10$



the simplex algorithm will find one of the two extreme points. Using $(0,10)$: $(z = 10)$

cut: $\boxed{s \geq 10y_2}$ $\quad$ $\boxed{UB = -1 + 10 = 9}$

4/

a) $$\text{Min} \sum_j x_j \cdot c_j$$

$$\sum_{j \in P_j} a_{ij} \cdot z_j \geq d_i \quad , \forall_i$$

$$x_j \in \mathbb{Z}$$

where z_j = # times patern j is used

a_{ij} = i^{th} element of pattern j.

b) ① Initial set of columns

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \cdots \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \cdots \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

This is a set that can be improved using the maximum numbers of unities of a single item:

For example, for the first pattern → $\begin{bmatrix} \lfloor L/l_1 \rfloor \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

② Let v_i be the dual variable associated with demand constraint for item i

$$\bar{c}_j = c_j - \sum_i v_i \cdot a_{ij}$$

3

$$\text{Min } c_j - \sum_i a_{ij} \cdot v_i$$

s.t.

$$\sum a_{ij} \cdot l_i \leq L$$

$$a_{ij} \in \mathbb{Z}.$$

4 the solution of the problem in 3 must be negative.

5 we stop when the solution of 3 does not result negative.

Procedure:

1 - Solve the master problem and get dual var.

2 - Solve the subproblem 3

if the solution value < 0
add column to master and go to 1

else
the last master solution is optimal.

c)

We now have two subproblems that can generate new columns:

$$\text{Min } c_1 - \sum_i a_{ij} v_i$$

st.

$$\sum_i a_{ij} l_i \leq L_1$$

$$a_{ij} \in \mathbb{Z}$$

and

$$\text{Min } c_2 - \sum_i a_{ij} v_i$$

$$\sum_i a_{ij} l_i \leq L_2$$

$$a_{ij} \in \mathbb{Z}$$

Columns generated by each of these problems enter the master with the appropriate cost c_1 or c_2.

The algorithm only stops if both problems return a positive solution.
non-negative.

d)

$$\text{Max} \quad \sum_{s} \tfrac{1}{2}(x_1 + x_2 + x_{3s})$$

s.t.

$$x_1 + x_2 + x_{31} \leq 2$$

$$2x_2 + x_{32} \leq 2$$

$$x_1, x_2, x_{31}, x_{32} \geq 0$$

5/ (a)

$$x_1 + x_2 + x_4 \leq 2$$

$$x_1 + x_3 + x_4 \leq 2$$

$$x_2 + x_3 \leq 1$$

$$x_3 + x_4 \leq 1$$

(b)

i) strategy D

ii) strategy A

c) $$\sum_{i,j \in \{1,2,3\}} x_{ij} \leq 2$$

$$\sum_{i,j \in \{4,5,6,7\}} x_{ij} \leq 3$$

$$\sum_{i \in \{1,2,3\}} \sum_{j \in \{4,5,6,7\}} x_{ij} \geq 2$$

8 2016 - Semester 1

8.1 Exam

THE UNIVERSITY OF MELBOURNE

Student ID

Semester 1 Assessment, 2016

School of Mathematics and Statistics

MAST90014 Optimisation for Industry

Writing time: 2 hours

Reading time: 15 minutes

This is NOT an open book exam

This paper consists of 5 pages (including this page)

Authorised materials:

- University approved calculators.

Instructions to Students

- You must NOT remove this question paper at the conclusion of the examination.
- You should attempt all questions. Marks for individual questions are shown.
- The total number of marks available is 50.

Instructions to Invigilators

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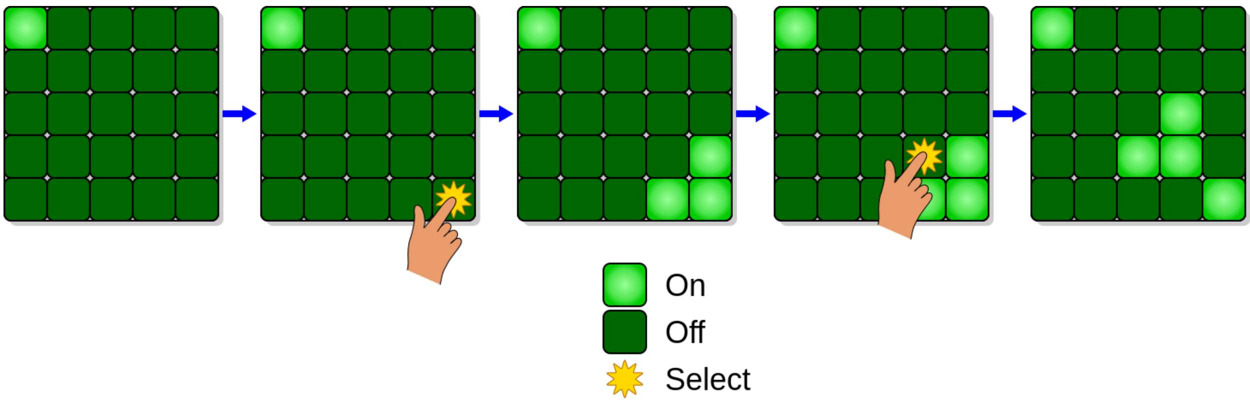
Question 1 (10 marks)

"Fiver is a puzzle game in which you need to change the state of all counters from 'off' to 'on' with the minimum number of clicks. Clicking on any counter will not only change the state of that counter from being 'off' to 'on' (or vice versa), but also change the state of its 4 neighboring counters (those immediately above, below, to the left and right of the counter that you clicked)."

The picture below, from Wikipedia, exemplifies some possible moves of the game (in this example, the counter on position (1,1) was already 'on').



(a) Propose constraints to model the fact that only odd positive integer numbers can be attributed to an integer variable z. You can propose new variables if necessary.

(b) What is the final state of an initially 'off' counter if x is the number of times it has been clicked and y_1, y_2, y_3 and y_4 are the number of times each one of its neighbours have been clicked?

(c) Propose decision variables for the fiver game in a board of size 5×5.

(d) Write a mixed integer program to model the game.

Question 2 (10 marks)

The linear relaxation for the binary Knapsack problem can be obtained by following the algorithm below:

Algorithm 1 Algorithm for solving the linear relaxation of the binary Knapsack problem

Given: n ▷ number of items
Given: u_i ▷ utility of each item $i = 1, \dots, n$
Given: w_i ▷ weigth of each item $i = 1, \dots, n$
Given: $\overline{W}$ ▷ Knapsack capacity
$P = [1, 2, \dots, n]$ ▷ Unordered vector of indexes
$O = Order(P, \frac{u_i}{w_i})$. ▷ O receives vector P ordered by descending value of $\frac{u_i}{w_i}$
$W = 0$; $S = []$; $i = 1$
$j = O(i)$
while $W < \overline{W}$ **do**
  if $W + w_j \leq \overline{W}$ **then**
    $S = [S, \{j, 1\}]$ ▷ Operation $V = [V, \{a, b\}]$ adds $\{a, b\}$ to the end of vector V.
    $W = W + w_j$
    $i = i + 1$
    $j = O(i)$
  else
    $S = [S, \{j, \frac{\overline{W} - W}{w_j}\}]$
Output: S. ▷ Each element of S, $\{a, b\}$, indicates that item with index a has value b in the relaxed solution.

(a) Consider the Knapsack problem with capacity $\overline{W} = 15$ and the following data for the items.

i	u_i	w_i	$\frac{u_i}{w_i}$
1	10	3	3.33
2	10	4	2.50
3	18	9	2.00
4	11	6	1.83

Solve this Knapsack problem using the Branch and Bound algorithm:

(i) Use depth-first node selection rule, opening left child nodes first.

(ii) Number the nodes as you explore them.

(iii) Indicate the reason(s) for pruning a node.

(iv) Present the values of the relaxed solution optimal variables and bounds at each explored node.

Note that you will have to use a modified version of the algorithm to account for branching constraints. You do **NOT** need to detail the execution of the algorithm.

(b) Write all Knapsack cover inequalities for the problem presented in (b). Which of these are **minimal** Knapsack cover inequalities ?

Question 3 (10 marks)

Uncapacitated Vehicle Routing Problem:

Consider a set of nodes $N = \{0, 1, \ldots, n\}$ where node 0 is a depot and the other nodes are clients that need to be visited by vehicles. There are V vehicles available. Each client needs to be visited by a single vehicle and the vehicles must start and end their routes at the depot. Each vehicle must serve a single route. The goal of the problem is to minimise the combined cost of the chosen routes. We assume a complete graph (each node is connected to each other node) and the cost of traversing an arc (i, j) is given by c_{ij}.

The following is an incomplete formulation of the problem that uses variables x_{ij}, where $x_{ij} = 1$ iff arc (i, j) is traversed by a vehicle.

$$\text{Min} \sum_{i \in N, j \in N} c_{ij} x_{ij} \tag{1}$$

subject to:

$$\sum_{i \in N} x_{ij} = 1, \qquad \forall j \in N \backslash \{0\} \tag{2}$$

$$\sum_{j \in N} x_{ij} = 1, \qquad \forall i \in N \backslash \{0\} \tag{3}$$

$$x_{ij} \in \{0, 1\} \qquad \forall i, j \in N \tag{4}$$

(a) Complete the formulation and explain the role of each additional constraint.

(b) A column-based formulation for this problem uses columns $A_r = [a_{ir}]$, where a_{ir} is equal to 1 if node i is visited in route r. Assume R is the set of routes and C_r is the cost of route $r \in R$.

(i) Use variables z_r, where $z_r = 1$ iff route A_r is used and write the formulation (for each constraint indicate the associated dual variable).

(ii) What is an initial set of columns that you could use that would ensure a feasible initial solution for the linear relaxation of the model described in (ii) ?

(iii) Write down an expression for the reduced cost of variable z_r.

(iv) In a branch and price algorithm, what would be the difficulty of branching in fractional variables z_r ?

Question 4 (20 marks)

Consider the following optimisation problem:

$$\text{Min } -3y_1 + 4x_1 - 2x_2 \tag{5}$$

subject to:

$$x_1 - x_2 - y_1 - 1 \geq 0 \tag{6}$$
$$x_1, x_2 \geq 0 \tag{7}$$
$$y_1 \in \{0, 1\} \tag{8}$$

(a) Write the Benders relaxed master, subproblem and dual subproblem.

(b) Solve the problem using the Benders decomposition algorithm. Use the relaxed master and the dual subproblem. Indicate the bounds at each iteration.

(c) Use a Lagrangian multiplier u and write the Lagrangian reformulation of this problem.

(d) What are the bounds that you find by using Lagrangian multipliers with the values below. Indicate for each bound if it is a lower bound or an upper bound on the problem.

(i) $u = 1$

(i) $u = 2$

(i) $u = 4$

(i) $u = 6$

End of Exam—Total Available Marks = 50.

8.2 Solution

Question 1

(a)

$y \in \mathbb{Z}$

$y \geq 1$

$x = 2y - 1$

(b) Its state is 'on' if $x + \sum_{i=1}^{4} y_i$ is odd and 'off' if it is even.

(c) x_{ij} is an integer non-negative variable indicating the number of times counter on position (i,j) has been clicked.

(d) $$\text{Min} \sum_{i=1}^{5} \sum_{j=1}^{5} x_{ij}$$

(e)

$$x_{ij} + x_{i-1,j} + x_{i+1,j} + x_{i,j+1} + x_{i,j-1} = 2y_{ij} - 1 \quad , \quad \forall i,j = 1 \dots 5 \qquad (1)$$

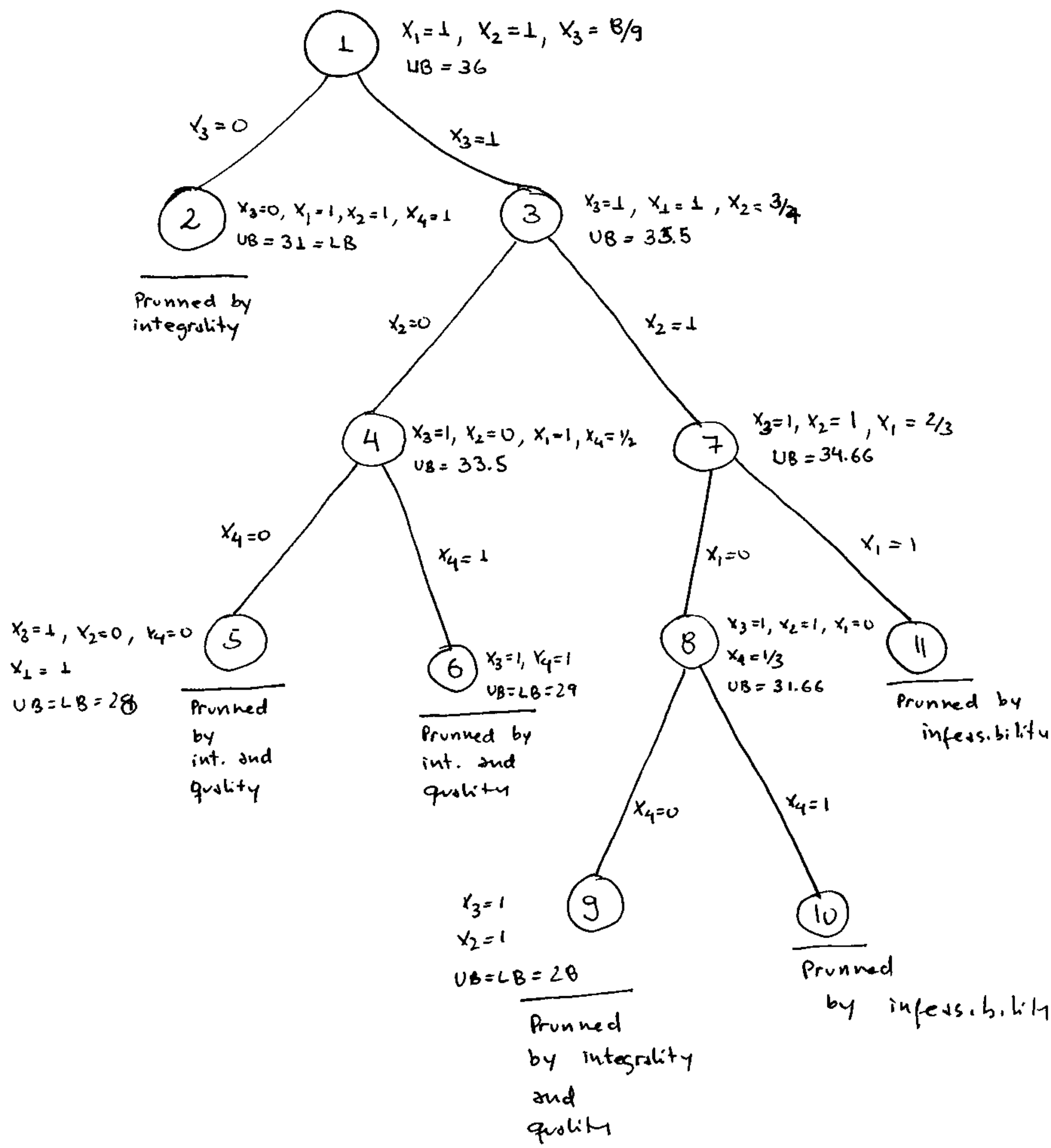
$$y_{ij} \geq 1 \qquad \forall i,j = 1 \dots 5 \qquad (2)$$

$$x_{ij}, y_{ij} \in \mathbb{Z}^+ \qquad \forall i,j = 1 \dots 5 \qquad (3)$$

In constraints (1) a variable x_{st} assumes value 0 if s or t are outside the boundaries of the board, i.e. if

$$s \leq 0 \text{ or } s \geq 6 \text{ or } t \leq 0 \text{ or } t \geq 6 .$$

Question 2



(b)

$$x_1 + x_2 + x_3 + x_4 \le 3$$

$$x_1 + x_2 + x_3 \le 2 \quad \text{(Minimal)}$$

$$x_2 + x_3 + x_4 \le 2 \quad \text{(Minimal)}$$

Question 3

(a)

- Limit on vehicles $\sum_{j \in N \setminus \{0\}} x_{0j} \leq V$

- Subtour elimination $\sum_{i \in S} \sum_{j \in S} x_{ij} \leq |S| - 1$, $S \subseteq N \setminus \{0\}$, $S \neq \emptyset$

(b)

(i) $$\text{Min} \sum_{r \in R} c_r z_r$$

s.t.

$$\sum_{r \in R} a_{ir} \cdot z_r = 1 \quad , \quad \forall i \in N \setminus \{0\} \qquad (\alpha_i)$$

$$-\sum_{r \in R} z_r \geq -V \qquad (\beta)$$

$$z_r \in \{0, 1\}$$

(ii) A single route visiting all nodes, for example.

(iii) $$\bar{c}_r = c_r - \sum_{i \in N \setminus \{0\}} a_{ir} \cdot \alpha_i + \beta$$

(iv) Imposing in the subproblem that the obtained route must be different from previous ones that have been eliminated by branching constraints.

Question 4

(a)

M

$$\text{Min } -3y_1 + z$$
$$\text{s.t. } \quad z \geq -M$$
$$y_1 \in \{0,1\}$$

S

$$\text{Min } 4x_1 - 2x_2$$
$$\text{s.t. } \quad x_1 - x_2 \geq (\bar{y}_1 + 1)$$
$$x_1 + x_2 \geq 0$$

DS

$$\text{Max } (\bar{y}_1 + 1)\alpha$$
$$\alpha \leq 4$$
$$-\alpha \leq -2$$

(b)

Iter 1

Master

$$\text{Min } -3y_1 + z$$
$$z \geq -M \quad \longrightarrow \quad y_1 = 1, \; z = -M, \; \boxed{LB = -M-3}$$
$$y_1 \in \{0,1\}$$

$\Longrightarrow$ DS

$$\text{Max } 2\alpha$$
$$\alpha \geq 2$$
$$\alpha \leq 4$$

$\longrightarrow$ $\alpha = 4$, $\boxed{UB = -3 + 8 = 5}$

cut

$$z \geq (y_1 + 1)4$$

Iter 2

Master

$$\text{Min } -3y_1 + z \quad \longrightarrow \quad y_1 = 0, \; z = 4, \; \boxed{LB = 4}$$
$$z \geq -M$$
$$z \geq (y_1 + 1)4$$

$\Longrightarrow$ DS

$$\text{Max } \alpha$$
$$\alpha \geq 2$$
$$\alpha \leq 4$$

$\longrightarrow$ $\alpha = 4$, $\boxed{UB = 0 + 4 = 4}$

Since UB = LB we have obtained the optimal solution.

(c)

$$\text{Min} \quad -3y_1 + 4x_1 - 2x_2 + \alpha(1 + y_1 - x_1 + x_2)$$

s.t. $\quad x_1, x_2 \geq 0$

$y_1 \in \{0,1\}$

$\alpha \geq 0$

(d) Rewriting:

$$\text{Min} \quad (-3+\alpha)y_1 + (4-\alpha)x_1 + (-2+\alpha)x_2 + \alpha$$

s.t. $\quad x_1, x_2 \geq 0$

$y_1 \in \{0,1\}$

$\alpha \geq 0$

(i) for $\alpha = 1$, $y_1 = 1$, $x_1 = 0$, $x_2 = \infty$, $\Rightarrow$ $LB = -\infty$

(ii) for $\alpha = 2$, $y_1 = 1$, $x_1 = 0$ $\Rightarrow$ $LB = 1$

(iii) for $\alpha = 4$, $y_1 = 0$, $x_2 = 0$ $\Rightarrow$ $LB = 4$

(iv) for $\alpha = 6$, $y_1 = 0$, $x_1 = \infty$, $x_2 = 0$ $\Rightarrow$ $LB = -\infty$

9 2017 - Semester 1

9.1 Exam

THE UNIVERSITY OF MELBOURNE

Student Number

Semester 1 Assessment, 2017

School of Mathematics and Statistics

MAST90014 Optimisation for Industry

Writing time: 2 hours

Reading time: 15 minutes

This is an OPEN BOOK exam

This paper consists of 6 pages (including this page)

Authorised Materials

- Mobile phones, smart watches and internet or communication devices are forbidden.
- Students can bring one A4 sheet with handwritten notes to the exam.

Instructions to Students

- You must NOT remove this question paper at the conclusion of the examination.
- You should attempt all questions. Marks for individual questions are shown.
- The total number of marks available is 100.

Instructions to Invigilators

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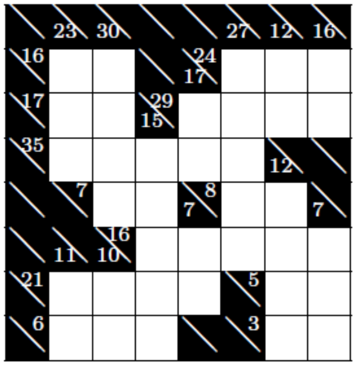
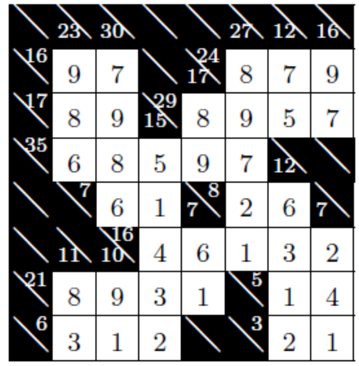
Question 1 (20 marks) *Kakuro* is a puzzle similar to Sudoku or KenKen (that you modelled and solved in Assignment 1). An example (extracted from the "Another math Programming Consultant" blog) is presented below (at the left you can see a puzzle and at the right, the same puzzle with a feasible solution):





The rules are:

1. Each (white) cell must be filled with a value $1\dots,9$
2. Each (white) cell belongs to a horizontal and/or a vertical block of contiguous cells.
3. The values of the cells in each block add up to a given constant. These constants are to the left (or top) of the block. A notation $a\backslash b$ is used: a is the sum for the block below and b is the sum for the block to the right.
4. The values in each block must be unique (i.e. no duplicates).

Define variables for this problem and explain their meaning. Using these variables, model the Kakuro puzzle as a Linear Integer Program.

Notes:

- *Explain the meaning of the objective function and each one of the constraints you define.*
- *If you define sets or other auxiliary data structures to represent the data, clearly indicate their meanings.*

Question 2 (20 marks) Provide brief answers to the following questions (do not use more than 50 words for each item).

(a) Define the TSP problem and comment on the size of the problems that we are able to solve with mixed-integer programming techniques.

(b) What are totally unimodular matrices ? What is the relation of totally unimodularity and mixed-integer programming ?

(c) Give 5 examples of practical problems that are typically solved with mixed-integer programming techniques.

(d) In the context of mixed-integer programming, what are primal bounds and dual bounds ?

(e) Explain the cutting plane algorithms of Chvàtal-Gomory using words.

Question 3 (20 marks) Consider the mixed-integer problem below:

$$\text{Min } 6x_1 + 4x_2 + 10x_3$$

$$\text{s.t}$$

$$\begin{aligned} 4x_1 + 2x_2 + 4x_3 &\geq 11 \\ x_3 &\geq 1 \\ x_1, x_2, x_3 &\in \{0, 1, 2, 3\} \end{aligned}$$

(a) Describe a simple procedure to obtain the solution to this problem when the integrality constraints are relaxed.

(b) Solve the original problem (with the integrality constraints) using Branch-and-Bound with depth-first search.

Notes:

- *Number the nodes as you explore them: only give a node a number after you solved the linear relaxed problem in that node.*
- *For each node you solve, list the value of the variables and the obtained bound (if any).*
- *Indicate the reason for pruning nodes.*
- *Highlight the node where you found the optimal solution.*

Question 4 (20 marks) Consider the following single-dimensional cutting-stock problem: you want to cut items from stocks (for example, sheets of paper of different dimensions from a large roll of paper). Let N be the set of items you want to cut, and L be the length of the stocks. Each item $i \in N$ has a dimension $l_i < L$ and a demand d_i.

(a) Present an integer model with a polynomial (with respect to the size of the input) number of variables.

(b) Present a model with an exponential (with respect to the size of the input) number of variables.

(c) What is the difficulty of solving your model in (a) using branch-and-bound ?

(d) Suppose you want to solve the relaxed version (with the integrality constraints relaxed) of your model using the column generation technique. Describe what would be your relaxed master problem (including information on the initial columms) and what would be the subproblem you would need to solve in order to obtain new columns for your master problem.

(e) Now assume that you want to extend your algorithm in (d) in order to obtain integer solutions. Explain how you could proceed.

Question 5 (20 marks) Consider the following mixed-integer program:

$$\text{Min } y + x$$

s.t

$$\begin{aligned} x &\geq 2 \\ -3y + x &\leq 0 \\ x \geq 0, y \in \{0, 1\} & \end{aligned}$$

Consider the classical Benders reformulation/decomposition technique.

(a) Write the Benders subproblem.

(b) Write the dual of the subproblem.

(c) Draw the feasible region of the dual problem.

(d) List all extreme points and rays associated with the dual subproblem.

(e) Write the full Benders reformulation of the problem.

End of Exam—Total Available Marks = 100

9.2 Solution

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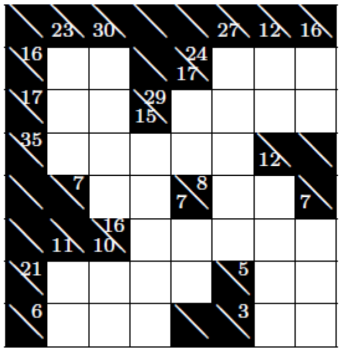

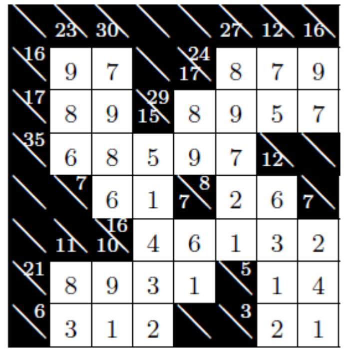

The rules are:

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4. The values in each block must be unique (i.e. no duplicates).

Define variables and explain their meaning. Using these variables, model the Kakuro puzzle as a Linear Integer Program.

Note: If you define sets or other auxiliary data structures to represent the data, clearly indicate their meanings.

Let : I be the set of white cells. ($I = \{(i,j) \mid$ cell (i,j) is white)

B be the set of blocks. ($B = \{b_1, b_2, \ldots\}$)

v_b be the value to be added in block $b \in B$

B_b be the set of cells in block $b \in B$

Variables : $x_{ijk} = 1$ if cell (i,j) has value k, $(i,j) \in I$, $k \in \{1, \ldots 9\}$

v_{ij} (auxiliar) : value at cell (i,j)

Model:

$$\text{Max} \quad 0$$

s.t.

$$\sum_{k=1}^{9} x_{ijk} = 1 \quad , \quad \forall (i,j) \in I$$ [each white cell has a single value]

$$v_{ij} = \sum_{k=1}^{9} k \cdot x_{ijk} \quad , \quad \forall (i,j) \in I$$ [relation between x and v variables]

$$\sum_{(i,j) \in B_b} v_{ij} = v_b \quad , \quad \forall b \in B$$ [values in each block add up to the correct value]

$$\sum_{(i,j) \in B_b} x_{ijk} \leq 1 \quad , \quad \forall b \in B, \; k \in \{1, \dots, 9\}$$ [values are unique in each block]

$$x_{ijk} \in \{0,1\} \quad , \quad (i,j) \in B, \; k \in \{1, \dots, 9\}$$

$$v_{ij} \in \{1, \dots, 9\}$$

Question 2 (20 marks) Provide brief answers to the following questions (try not to use more than 100 words for each item).

(a) Define the TSP problem and comment on the size of the problems that we are able to solve with mixed-integer programming techniques.

(b) What are totally unimodular matrices ? What is the relation of totally unimodularity and mixed-integer programming ?

(c) Give 5 examples of practical problems that are tipically solved with mixed-integer programming techniques.

(d) In the context of mixed-integer programming, what are primal bounds and dual bounds ?

(e) Explain the cutting plane algorithms of Chvàtal-Gomory.

a) The TSP consists of finding an Helmitonian cycle in a graph with minimum length. Using state-of-the art solvers like Concorde, we can solve problems with some tens of thousand nodes (cities).

b) It is a matrix A such that each subsquare matrix of A has determinant 0, -1 or +1. MIP with resource constraints that are unimodular matrices can be solved with the simplex algorithm

c) Network design problems
crew scheduling
production planning
(supply chain)
timetabling
revenue management
vehicle routing

d) • A primal bound is the value of a feasible solution.
• A dual bound is the value of an optimal solution for a relaxed version of the problem.

e) It consists in solving the problem without the integrality constraints, finding a separation inequality and repeating until the solution of the relaxed problem is integer.

Question 3 (20 marks) Consider the mixed-integer problem below:

$$\text{Min } \overset{6}{\cancel{3}}x_1 + \overset{4}{\cancel{2}}x_2 + \overset{10}{\cancel{4}}x_3$$

s.t

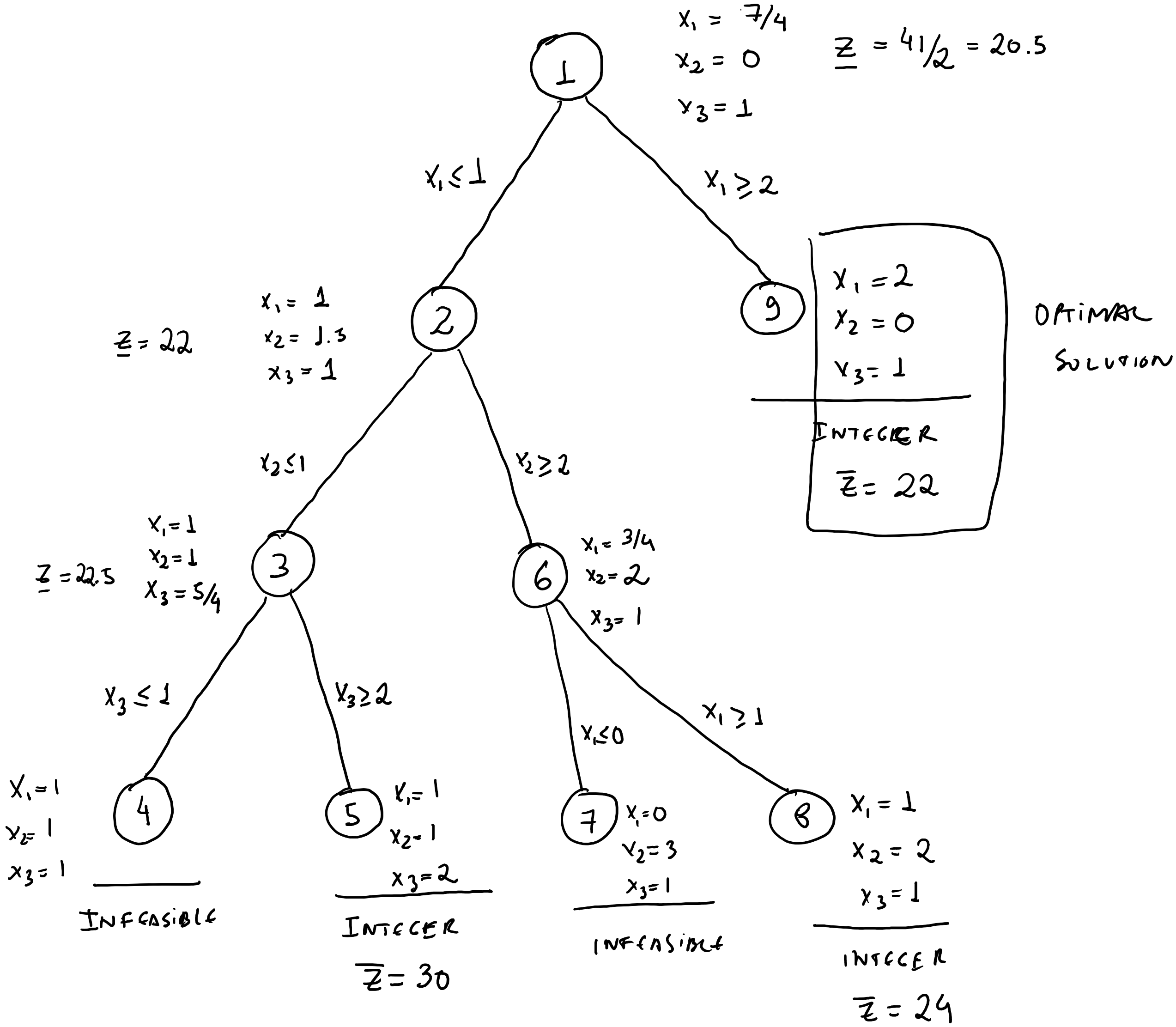
$$\begin{aligned} \overset{4}{\cancel{2}}x_1 + \overset{2}{}x_2 + \overset{4}{\cancel{2}}x_3 &\geq \cancel{10}\ 11 \\ x_3 &\geq 1 \\ x_1, x_2, x_3 &\in \cancel{\mathbb{Z}^+}\ \{0, 1, 2, 3\} \end{aligned}$$

(a) Describe a procedure to obtain the solution to this problem when the integrality constraints are relaxed.

(b) Solve the original problem (with the integrality constraints) using Branch-and-Bound.

a)

- since $\boxed{\underset{\text{item 3}}{(4/10)} < \underset{\text{item 2}}{(2/4)} < \underset{\text{item 1}}{(4/6)}}$, rank items as 1, 2, 3.
- fill in the minimum required for bounding constraints (such as $x_3 \geq 1$)
- fill the maximum of items 1, 2, 3 (in this order) until the knapsack constraint is respected.


1
x1 = 7/4
x2 = 0
x3 = 1
Z = 41/2 = 20.5
x1 ≤ 1
x1 ≥ 2
2
x1 = 1
x2 = 1.3
x3 = 1
Z = 22
9
x1 = 2
x2 = 0
x3 = 1
INTEGER
Z = 22
OPTIMAL SOLUTION
x2 ≤ 1
x2 ≥ 2
3
x1 = 1
x2 = 1
x3 = 5/4
Z = 22.5
6
x1 = 3/4
x2 = 2
x3 = 1
x3 ≤ 1
x3 ≥ 2
x1 ≤ 0
x1 ≥ 1
4
x1 = 1
x2 = 1
x3 = 1
INFEASIBLE
5
x1 = 1
x2 = 1
x3 = 2
INTEGER
Z = 30
7
x1 = 0
x2 = 3
x3 = 1
INFEASIBLE
8
x1 = 1
x2 = 2
x3 = 1
INTEGER
Z = 24

Question 4 (20 marks) Consider the following single-dimensional cutting-stock problem: you want to cut items from stocks (for example, sheets of paper of different dimensions from a large roll of paper). Let N be the set of items you want to cut, and L be the length of the stocks. Each item $i \in N$ has a dimension $l_i < L$ and a demand d_i.

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(c) What is the difficulty of solving your model in (a) using branch-and-bound ?

(d) Suppose you want to solve the relaxed version (with the integrality constraints relaxed) of your model using the column generation technique. Describe what would be your relaxed master problem (including information on the initial columms) and what would be the subproblem you would need to solve in order to obtain new columns for your master problem.

(e) Now assume that you want to extend your algorithm in (d) in order to obtain integer solutions. Explain how you could proceed.

a) $\text{Min} \sum_{j=1}^{|N|} y_j$

s.t.

$\sum_{j=1}^{|N|} x_{ij} \geq d_i \quad , \quad i \in N$

$\sum_{i \in N} l_i \cdot x_{ij} \leq L \quad , \quad j = 1 \dots |N|$

$x_{ij} \in \{0,1\} \quad , \quad i \in N, \ j = 1, \dots, |N|$

$y_j \in \{0,1\} \quad , \quad j = 1 \dots |N|$

(polynomial)

b) $\text{Min} \sum_{i=1}^{P} x_p$

$\sum_{p \in P} a_{ip} x_p \geq d_i \quad , \quad i \in N$

$x_p \in \mathbb{Z}^+$

c) Symmetry will slow the convergence of the branch-and-bound algorithm.

d)

$$\text{Min} \sum_{p \in P_R} x_p \qquad \Bigg| \qquad \text{Min } 1 - a^T u$$

$$\sum_{p \in P_r} a_{ip} x_p \geq d_i, \qquad i \in N \quad (u_i) \qquad \Bigg| \qquad \text{s.t.} \quad \sum_{i \in N} a_i \leq L$$

$$x_p \in \mathbb{R}^+ \qquad \qquad \qquad \qquad a_i \in \mathbb{Z}^+$$

(Master)

associated dual variables

P_r is the set of initial columns. It can be, for example, the set of simple patterns containing a single item.

e) You would need to branch-and-price.

Question 5 (20 marks) Consider the following mixed-integer program:

$$\text{Min } y + x$$

s.t

$$\begin{aligned} x &\geq 2 \\ -3y + x &\leq 0 \\ x \geq 0, y &\in \{0, 1\} \end{aligned}$$

Consider the classical Benders reformulation/decomposition technique.

(a) Write the Benders subproblem.

(b) Write the dual of the subproblem.

(c) List all extreme points and rays associated with the dual subproblem.

(d) Write the full Benders reformulation of the problem.

d) Min x

s.t.

$x \geq 2 \quad (v_1)$

$-x \geq -3y \quad (v_2)$

$x \geq 0$

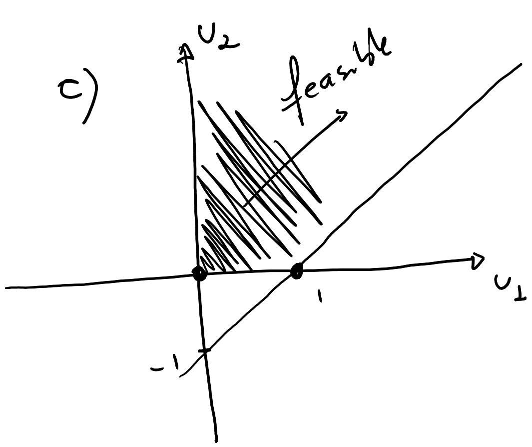
associated dual variables

b) Max $2v_1 - 3\bar{y}v_2$

s.t. $v_1 - v_2 \leq 1$

$v_1, v_2 \geq 0$

c)



d) Extreme points: $\overset{P_1}{(0,0)}, \overset{P_2}{(1,0)}$

Extreme rays: $\underset{R_1}{(0,1)}, \underset{R_2}{(1,1)}$

e) Min $y + z$

$z \geq 0 \quad (P_1)$

$z \geq 2 \quad (P_2)$

$-3y \leq 0 \quad (R_1)$

$2 - 3y \leq 0 \quad (R_2)$

$y \in \{0, 1\}$

$x \in \mathbb{R}^+$.

End of Exam—Total Available Marks = 100

10 2018 - Semester 1

10.1 Exam

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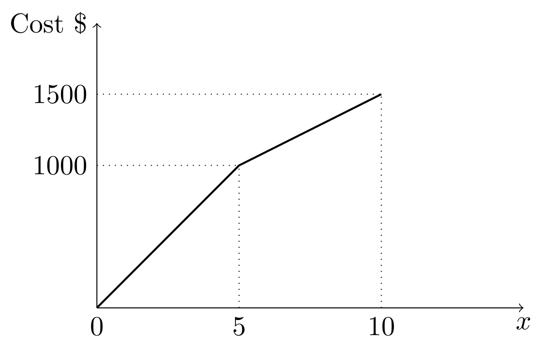
Question 1 (8 marks) Consider the Orienteering Problem (OP) defined as follows: a set of N vertices i is given, each with a score S_i. The starting point (vertex 1) and the end point (vertex n) are fixed. The time t_{ij} needed to travel from vertex i to j is known for all vertices. Not all vertices can be visited since the available time is limited to a given time budget T_{max}. The goal of the OP is to determine a path, with duration limited by T_{max}, that visits some of the vertices, in order to maximise the total collected score. Each vertex can be visited at most once.

Propose a (Mixed) Integer Linear Programming formulation for this problem. Clearly define all variables. Briefly explain the meaning of all constraints and the objective function.

Question 2 (4 marks) A company produces a single type of item and wants to plan its production for the next two months. The company can produce at most 10 units of the item per month and the monthly production cost depends on the number of units x produced as expressed in the graph below:



Assume that the demand for the next two months is given by $d_1 \leq 10$ and $d_2 \leq 10$ units, respectively. The initial inventory of products is zero. Propose a (Mixed) Integer Linear Programming formulation for the problem of production planning with the goal of supplying all the demand while minimising the production costs. Clearly define all variables. Explain the meaning of all constraints and the objective function.

Question 3 (6 marks) Consider the optimisation problem:

$$\text{Max } 2x_1 + x_2$$

s.t.

$$\begin{aligned} x_1 + x_2 &\leq 3 \\ 2x_1 &\leq b \\ x_1, x_2 \geq 0 &\text{ and integers} \end{aligned}$$

a) Write an optimisation problem to determine which values of b make the second constraint redundant. Describe how you could use the results of the problem to take the appropriate conclusions.

b) Consider $b = 5$ and solve the linear relaxation of the problem (either with the simplex algorithm or with the graphical method). What is the optimal partition $[B|N]$?

c) Write the problem constraints ($x_B = B^{-1}b - B^{-1}Nx_N$) using the optimal partition $[B|N]$ found above.

d) Write a Gomory cut. Express the cut in terms of the original variables.

Question 4 (4 marks) Consider the generalised assignment problem below.

$$\text{Max } \sum_{j=1}^{n} \sum_{i=1}^{m} c_{ij} x_{ij}$$

s.t.

$$\sum_{j=1}^{n} x_{ij} \leq 1, \; i = 1, \ldots, m \tag{1}$$

$$\sum_{i=1}^{m} a_{ij} x_{ij} \leq b_j, \; j = 1, \ldots, n \tag{2}$$

$$x_{ij} \in \{0, 1\}, i = 1, \ldots, m, j = 1, \ldots, n$$

In the context of Lagrangian duality:

a) Dualise constraints (1) and (2) and write the Lagrangian relaxation problem for a given set of dual variables.

b) Give a rule to solve the problem in a) by inspection.

c) Write the Lagrangian dual optimisation problem, that will give you the best Lagrangian bound.

Question 5 (8 marks) (Adapted from "Integer Programming", by Wolsey) Consider the following one-dimmensional cutting-stock problem: Suppose that 15 pieces of length 32cm, 35 of length 20cm, 17 of length 15cm and 42 of length 11cm must be cut from sheets of length 104cm. For operational reasons, no more than 5 pieces of the same length can be cut out of the same sheet.

a) Write a compact formulation (i.e., a formulation with a polynomial number of constraints and variables). Define any additional parameters you use.

b) Write a pattern-based formulation.

c) Use your answer in b) to propose a column generation approach:

   i) write the master problem,

   ii) write the pricing subproblem,

   iii) write an initial set of columns to be used in the relaxed master problem.

Question 6 (10 marks) Consider the fixed-charge network design problem described as follows. Let $G(N, E)$ be a complete graph with $N = \{1, \ldots, n\}$ the set of nodes and E the set of edges. You want to send one unit of flow from each node in $\{1, \ldots, n-1\}$ to node n. The cost of sending one unit of flow through edge $(i, j) \in E$ is given by c_{ij}, while the cost of 'building' an edge is given by f_{ij}. Flow can only be sent through an edge if the edge is built.

a) Write a mixed-integer linear program to minimise the cost of sending all the required flows from origins to destination.

b) In the context of the Benders decomposition approach and for the model obtained in a):

   i) write the relaxed master problem,

   ii) write the subproblem,

   iii) write the general form of Benders optimality and feasibility cuts.

End of Exam—Total Available Marks = 40

10.2 Solution

MAST 90014. Solutions, exam 2018.

1) Let $x_{ij} = 1$ if arc (ij) is traversed, 0 otherwise.

The problem reads:

$$\text{Max } S_1 + \sum_{j\in N}\sum_{i\in N} x_{ji} \cdot S_i \qquad \text{(max score obtained)}$$

s.t.

$$\sum_{j\in N} x_{1j} = 1 \qquad \text{(path starts at node 1)}$$

$$\sum_{j\in N} x_{j|N|} = 1 \qquad \text{(path ends at node } |N|\text{)}$$

$$\sum_{j\in N} x_{ji} = \sum_{j\in N} x_{ij}, \quad i = 2 \ldots |N|-1 \qquad \text{(flow conservation)}$$

$$\sum_{i\in N}\sum_{j\in N} t_{ij} x_{ij} \le T_{max} \qquad \text{(time budget)}$$

$$\sum_{i,j \in S} x_{ij} \le |S| - 1, \quad S \subseteq N \qquad \text{(subtour elimination)}$$

$$x_{ij} \in \{0,1\}, \quad i,j \in N \qquad \text{(scope of variables)}$$

2) I_t = inventory at the end of period t

Let x_t be the number of units produced in period t

z_t the production cost of month t

λ_1^t, λ_2^t and y_1^t, y_2^t be auxiliary variables for the linearisation by parts.

The problem reads:

$$
\begin{aligned}
&\text{Min } z_t \\
&\text{s.t.} \\
&x_1 = d_1 + I_1 \\
&x_2 + I_1 = d_2 \\
&x_t = 5\lambda_1^t + 10\lambda_2^t \quad , \quad t = 1,2 \\
&z_t = 1000 \cdot \lambda_1^t + 1500 \cdot \lambda_2^t \; , \quad t = 1,2 \\
&x_t, z_t, I_t \geq 0 \qquad t = 1,2 \\
&\lambda_t^i \geq 0 \; , \qquad t = 1,2 \; ; \quad i = 1,2 \\
&y_t^i \in \{0,1\} \; . \qquad t = 1,2 \; , \quad i = 1,2
\end{aligned}
$$

3)

a) $z = \text{Max} \quad 2x_1$

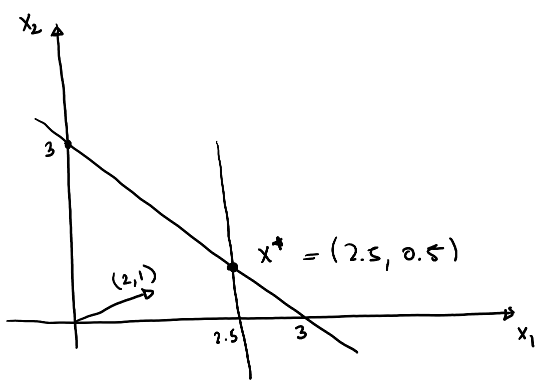
s.t.

$$x_1 + x_2 \leq 3$$

$$x_1, x_2 \geq 0 \quad \text{and integers}$$

The second constraint is redundant for any $b \geq z$.

b)



$$\text{Max} \quad 2x_1 + x_2$$

$$x_1 + x_2 + s_1 = 3$$

$$2x_1 + s_2 = 5$$

$$x_1, x_2, s_1, s_2 \geq 0 \quad \text{and integers}$$

Optimal partition:

$$B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \quad N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 0 & 1/2 \\ 1 & -1/2 \end{bmatrix}$$

c) writing the problem as:

$$Bx_B + Nx_N = b$$

$$x_B = B^{-1}b - B^{-1}N x_N$$

$$x_B = \begin{bmatrix} 0 & 1/2 \\ 1 & -1/2 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 0 & 1/2 \\ 1 & -1/2 \end{bmatrix}\begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$$

$x_1 = 2.5 - 1/2\, s_2 \longrightarrow x_1 + 1/2\, s_2 = 2.5 \Rightarrow$ Gomory cut $0.5 \cdot s_2 \geq 0.5$

$x_2 = 0.5 - s_1 + 1/2\, s_2$

Since $S_2 = 5 - 2x_1$

$\Rightarrow$ Gomory cut:

$$0.5(5 - 2x_1) \geq 0.5$$

$$2.5 - x_1 \geq 0.5$$

$$x_1 \leq 2$$

4)

a)

$$\text{Max} \sum_{j=1}^{n} \sum_{i=1}^{m} c_{ij} x_{ij} + \sum_{i=1}^{m} v_i \left(1 - \sum_{j=1}^{n} x_{ij}\right) + \sum_{j=1}^{n} v_j \left(b_j - \sum_{i=1}^{m} a_{ij} x_{ij}\right)$$

$$\text{Max} \sum_{j=1}^{n} \sum_{i=1}^{m} (c_{ij} - v_i - v_j a_{ij}) x_{ij} + \sum_{i=1}^{m} v_i + \sum_{j=1}^{n} v_j b_j$$

b) for each i, j, if

$c_{ij} - u_i - v_j a_{ij} > 0$, put the associated x_{ij} at One, otherwise, put the $x_{ij} = 0$

c)

$$W_{LD} = \min_{\substack{u_i \geq 0 \\ v_j \geq 0}} \left\{ \text{Max} \sum_{j=1}^{n} \sum_{i=1}^{m} (c_{ij} - v_i - v_j a_{ij}) x_{ij} + \sum_{i=1}^{m} v_i + \sum_{j=1}^{n} v_j b_j \right\}$$

5)

a) Define variables

$y_j = 1$ if sheet j is cut, 0 otherwise

$x_{ij} =$ number of items of type i cut from sheet j

$$\text{Min} \sum_{j=1}^{h} y_j$$

s.t.

$$\sum_{i=1}^{4} l_i \cdot x_{ij} \leq 104 \cdot y_j \quad , \quad \forall j$$

$$\sum_{j} x_{ij} \geq d_i \quad , \quad \forall i$$

$$x_{ij} \leq 5 \quad , \quad \forall i, \forall j$$

$$x_{ij} \in \mathbb{Z}_+$$

$$y_j \in \{0,1\}$$

(where h is any upper bound on the solution)

b)

$$\text{Min} \sum_{p=1}^{P} x_p$$

s.t.

$$\sum_{i} a_{ip} x_p \geq d_i$$

$$x_p \geq 0$$

(where $\underline{a_{ip}}$ is the number of items i cut from pattern p and $\underline{d_i}$ is the demand of item i)

c) Associating dual variables u_i with the constraints of the master, the pricing subproblem reads:

$$\text{Min} \quad 1 - u_i \cdot a_i$$

$$\text{s.t.} \quad a_i \leq 5 \qquad \forall i$$

$$\sum_i l_i a_i \leq 104$$

$$a_i \in \mathbb{Z}^+$$

Any feasible set of columns will be enough to start.

For example, a feasible solution.

6)

$$\text{Min} \quad \sum_i \sum_j \left(c_{ij} x_{ij} + f_{ij} y_{ij} \right)$$

$$\sum_j x_{ij} - \sum_j x_{ji} = 1 \quad , \quad i = 1, \dots, n-1$$

$$\sum_{j=1}^{n-1} x_{jn} = n-1$$

$$x_{ij} \leq M y_{ij}$$

$$x_{ij} \geq 0 \quad , \quad y_{ij} \in \{0,1\}$$

Master:

$$\text{Min } \sum_i \sum_j f_{ij} y_{ij} + z$$

s.t.

$$z \geq -\infty$$

$$y_{ij} \in \{0,1\}$$

Subproblem

$$\text{Min } \sum_i \sum_j c_{ij} x_{ij}$$

(u_i) $$\sum_j x_{ij} - \sum_j x_{ji} = 1 \quad , \quad i = 1, \dots, n-1$$

(v) $$\sum_{j=1}^{n-1} x_{jn} = n-1$$

(β_{ij}) $$x_{ij} \leq M \overline{y_{ij}}$$

cuts:

$$z \geq \sum_i u_i^p + (n-1) v + \sum_i \sum_j \beta_{ij} \cdot (M \cdot y_{ij})$$

for each extreme point

$$\sum_i u_i^p + (n-1) v + \sum_i \sum_j \beta_{ij} \cdot (M \cdot y_{ij}) \leq 0$$

for each extreme ray.

11 2019 - Semester 1

11.1 Exam and solution

THE UNIVERSITY OF MELBOURNE

Student Number	

Semester 1 Assessment, 2019

School of Mathematics and Statistics

MAST90014 Optimisation for Industry

Writing time: 2 hours

Reading time: 15 minutes

This is NOT an open book exam

This paper consists of 8 pages (including this page)

Authorised Materials

- Mobile phones, smart watches and internet or communication devices are forbidden.
- A4 double sided hand written notes is allowed.
- Casio FX-82 calculator allowed

Instructions to Students

- You must NOT remove this question paper at the conclusion of the examination.
- You should attempt all questions.
- There are 4 questions with marks as shown. The total number of marks available is 100.

Instructions to Invigilators

- Students must NOT remove this question paper at the conclusion of the examination.

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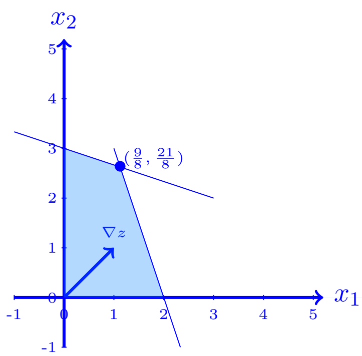
Question 1 (40 marks) For all items below, consider the simple linear program:

$$z = \max\ x_1 + x_2 \tag{1}$$
s.t.
$$3x_1 + x_2 \leq 6, \tag{2}$$
$$x_1 + 3x_2 \leq 9, \tag{3}$$
$$x_1, x_2 \geq 0. \tag{4}$$

a) Solve the problem using the graphical method and indicate the variable and objective function values associated with the optimal solution.

The feasible space is sketched below:



The optimal point is given by $(\frac{9}{8}, \frac{21}{8})$ with objective function $z(\frac{9}{8}, \frac{21}{8}) = \frac{30}{8} = \frac{15}{4}$.

b) Put the problem in standard form with a maximisation objective and equality constraints.

Standard form:

$$z = \max\ x_1 + x_2 \tag{5}$$
s.t.
$$3x_1 + x_2 + x_3 = 6, \tag{6}$$
$$x_1 + 3x_2 + x_4 = 9, \tag{7}$$
$$x_1, x_2, x_3, x_4 \geq 0. \tag{8}$$

c) Obtain the basic matrix B and the non-basic matrix N associated with the optimal solution obtained in a).

In the optimal solution, x_1 and x_2 are in the basis and x_3 and x_4 are out of the basis.

$$B = \begin{bmatrix} a_1 & a_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

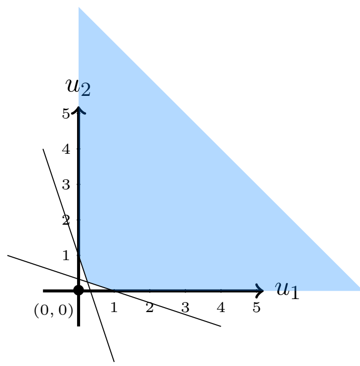
$$N = \begin{bmatrix} a_3 & a_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

d) Obtain the dual of problem (1)–(4) using dual variables u_1 and u_2 associated with constraints (2) and (3), respectively.

The dual is given problem is given by:

$$d = \min\ 6u_1 + 9u_2 \tag{9}$$
s.t.
$$3u_1 + u_2 \geq 1, \tag{10}$$
$$u_1 + 3u_2 \geq 1, \tag{11}$$
$$u_1, u_2 \geq 0. \tag{12}$$

e) Solve the dual problem using the graphical method (provide values for the variables and for the objective function). The sketch of the dual feasible space is presented below.



The optimal point is $(u_1, u_2) = (\frac{1}{4}, \frac{1}{4})$ with $d(\frac{1}{4}, \frac{1}{4}) = \frac{15}{4}$

f) Using the optimal solution values of the dual problem, explain what would be the (local) expected change in the objective function value if constraint (2) is modified to $3x_1+x_2 \leq 5$.

The dual variable associated with that constraint gives the desired answer. If the right hand side is reduced in one unity, the expected local change in value is $-u_1 = -1/4$.

g) Consider a new objective function $z = \max\ x_1$. Is the solution obtained in a) still optimal. Use the reduced costs of the simplex algorithm in order to justify your answer.

The reduced costs for x_3 and x_4 (the variables out of the basis at the optimal solution) are given by:

$$\hat{c}_3 = c_3 - c_B^T B^{-1} a_3$$

and

$$\hat{c}_4 = c_4 - c_B^T B^{-1} a_4.$$

with $c_B^T = \begin{bmatrix}1 & 0\end{bmatrix}$ (already considering the new objective function), $B = \begin{bmatrix}3 & 1\\1 & 3\end{bmatrix} \Rightarrow B^{-1} = \begin{bmatrix}\frac{3}{8} & -\frac{1}{8}\\-\frac{1}{8} & \frac{3}{8}\end{bmatrix}$

$$\hat{c}_3 = 0 - \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\frac{3}{8} & -\frac{1}{8}\\-\frac{1}{8} & \frac{3}{8}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = -\frac{3}{8}$$

and

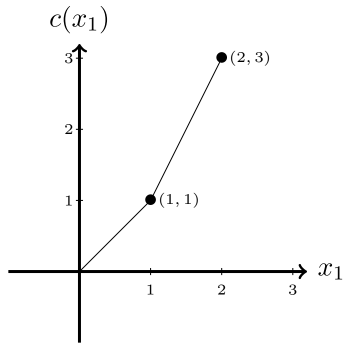
$$\hat{c}_4 = 0 - \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\frac{3}{8} & -\frac{1}{8}\\-\frac{1}{8} & \frac{3}{8}\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{8}.$$

The solution is not optimal since we are maximising and $\hat{c}_4 > 0$.

h) Consider again constraints (2)–(4) and the following objective function:

$$\max\ c(x_1) + x_2,$$

with $c(x_1)$ being the function described below:



Model the problem using only linear constraints.

Define continuous variables c, λ_1, λ_2 and λ_3 and binary variables y_1 and y_2. The model reads:

$$z = \max\ c + x_2 \tag{13}$$
s.t.
$$3x_1 + x_2 \leq 6, \tag{14}$$
$$x_1 + 3x_2 \leq 9, \tag{15}$$
$$x_1 + 3x_2 \leq 9, \tag{16}$$
$$\lambda_1 \leq y_1 \tag{17}$$
$$\lambda_2 \leq y_1 + y_2 \tag{18}$$
$$\lambda_3 \leq y_2 \tag{19}$$
$$y_1 + y_2 \leq 1 \tag{20}$$
$$x_1 = \lambda_2 + 2\lambda_3 \tag{21}$$
$$c = \lambda_2 + 3\lambda_3 \tag{22}$$
$$x_1, x_2, \lambda_1, \lambda_2, \lambda_3, c \geq 0 \tag{23}$$
$$y_1, y_2 \in \{0, 1\} \tag{24}$$

i) Out of the four following algorithms: Simplex, Branch-and-Bound, Column Generation and Benders decomposition, which one(s) could be used to solve the problem modelled in item h) to optimality? Briefly justify.

You must use an algorithm for mixed-integer programs (Branch and Bound or Benders decomposition). The simplex and the column generation can only be applied to linear programs.

Question 2 (20 marks) Consider a facility location problem with a set of potential facilities to be open, each one with a given opening cost and a capacity. Also consider a set of customers with fixed demands and a service cost c_{pf} associated with delivering one unit of demand from facility f to customer p. The goal is to select a subset of the potential facilities such that the demand of each customer is satisfied while the total combined cost (opening facilities and providing service to the customers) is minimized. In the following questions, use the notation below:

F: set of facilities.

P: set of customers.

u_f: capacity (in terms of units of demand) of facility $f \in F$.

o_f: fixed cost of opening facility $f \in F$.

d_p: units of demand of customer $p \in P$.

c_{pf}: cost of serving one unit of demand of customer $p \in P$ by facility $f \in F$.

a) Consider that each customer can only be served by a single facility and model the problem as a mixed-integer program.

b) Modify your answer in a) in order to consider that the demand of a customer can be served by multiple facilities.

c) Assume that the demand of each node $p \in P$ is uncertain. Location decisions must be taken before the uncertainty is reveallled. The uncertainty is modelled via a set of equiprobable scenarios S. The demand of node $p \in P$ in scenario $s \in S$ is given by d_{ps}. Propose a stochastic mixed integer programming model for the problem of serving the stochastic demand with minimum cost.

Question 3 (20 marks) Consider the simple mixed-integer program below:

$$
\begin{aligned}
& z = \max\ x_1 + x_2 + 2y_1 + 2y_2 && (25)\\
& \text{s.t.} \\
& 3x_1 + x_2 + 5y_1 \leq 6, && (26)\\
& x_1 + 3x_2 + 6y_2 \leq 9, && (27)\\
& x_1, x_2 \geq 0. && (28)\\
& y_1, y_2 \in \{0, 1\}. && (29)
\end{aligned}
$$

a) Write the Benders reformulation of the problem containing all feasibility and optimality cuts.

b) In a Benders decomposition algorithm framework starting with a master problem with no cuts, indicate which cuts will be generated (and in which order) until the method converges.

Question 4 (20 marks) Consider the following compact formulation for the bin-packing problem with bins of different sizes and costs (Variable-Sized Bin Packing Problem - VSBP)

$$
\begin{aligned}
& z = \min \sum_{j \in B} c_j x_j && (30)\\
& \text{s.t.} \\
& \sum_{j \in B} y_{ij} = 1, \quad \forall i \in I, && (31)\\
& \sum_{i \in I} s_i y_{ij} \leq w_j x_j, \quad \forall j \in B, && (32)\\
& x_j \in \{0, 1\}, \quad \forall j \in B, && (33)\\
& y_{ij} \in \{0, 1\}, \quad \forall i \in I, j \in B. && (34)
\end{aligned}
$$

where:

B: set of bins,

I: set of items to be packed,

c_j: cost of using bin $j \in B$,

w_j: capacity of bin $j \in B$,

s_i: weight of item $i \in I$,

a) Explain the meaning of the variables y_i and x_{ij}.

b) Explain the meaning of constraints (31) and (32).

c) We define a packing pattern for some bin $j \in B$ as a set of items that could be packed in the bin, and P_j be an initial (incomplete) set of packing patterns for this bin $j \in B$. For example consider a bin with size 3 and items a, b and c with sizes 1, 2 and 3, respectively. Two possible feasible packing patterns for this bin are:

$$p_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } p_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

In the first pattern, p_1, one unit of a and one unit of b are cut and in the second patter, p_2, one unit of item c is cut.

c1) Reformulate the VSBP problem using the following set of pattern-based decision variables: $z_{pj} = 1$ if packing pattern $p \in P_j$ is used for bin $j \in B$, $z_{pj} = 0$, otherwise. Denote this formulation as a pattern based VSBP.

c2) Define a column generation procedure (at the root node) for the LP-relaxation of the Pattern-based VSBP problem. Remember to include the following information in your solution:

- An expression of the reduced cost of the decision variable z_{pj}.

- One or more integer linear programs that generate new columns which could be appended to the LP-relaxation of the Pattern-based VSBP.

- The condition(s) when the generated column is appended to the LP-relaxation of the Pattern-based VSBP.

End of Exam—Total Available Marks = 100

12 2020 - Semester 1

12.1 Exam

THE UNIVERSITY OF MELBOURNE

Student Number

Semester 1 Assessment, 2020

School of Mathematics and Statistics

MAST90014 Optimisation for Industry

This exam consists of 13 pages (including this page)

Authorised materials: printed one-sided copy of the Exam or the Masked Exam made available earlier (or an offline electronic PDF reader), one double-sided A4 handwritten sheet of notes, and blank A4 paper

Instructions to Students

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  (b) write your answers on the Masked Exam PDF if you were able to print it single-sided before the exam day.

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- Assemble all the exam pages (or template pages) in correct page number order and the correct way up, and add any extra pages with additional working at the end.
- Scan your exam submission to a single PDF file with a mobile phone or a scanner. Scan from directly above to avoid any excessive keystone effect. Check that all pages are clearly readable and cropped to the A4 borders of the original page. Poorly scanned submissions may be impossible to mark.
- Upload the PDF file via the Canvas Assignments menu and submit the PDF to the GradeScope tool by first selecting your PDF file and then clicking on Upload PDF.
- Confirm with your Zoom supervisor that you have GradeScope confirmation of submission before leaving Zoom supervision.
- You should attempt all questions.
- There are 6 questions with marks as shown. The total number of marks available is 100.

Question 1 (25 marks)

Consider the following distribution problem:

> "There are several commodities, represented by set $I = \{1, \dots \bar{i}\}$, produced at several plants, $P = \{1, \dots \overline{p}\}$, with known production capacities $\overline{k}_p$. The commodities should be delivered to customers $C = \{1, \dots, \overline{c}\}$ which have specific demands for each commodity given by b_{ic}. This demand is satisfied by shipping via regional distribution centers (DCs), $D = \{1, \dots, \bar{d}\}$, with each customer being assigned exclusively to a single distribution center. There are lower as well as upper bounds on the allowable total annual throughput of each $d \in D$, given by $\underline{m}_d$ and $\overline{m}_d$, respectively. The possible locations for the DCs are given, but the particular sites to be used are to be selected so as to result in the least total distribution cost. The DC costs are expressed as fixed charges (imposed for the sites actually used), f_d, plus a cost t_d per item transported through DC $d \in D$. Transportation costs are linear and equal to v_{pdc} for each unit of commodity transported from plant p to customer c through distribution center d.
>
> Thus the problem is to determine which DC sites to use, what load to have at each DC selected site, what customer zones should be served by each DC, and what the pattern of transportation flows should be for all commodities. This is to be done so as to meet the given demands at minimum total distribution cost subject to the plant capacity and DC throughput constraints."

Use continuous variables $x_{ipdc} \geq 0$ to indicate the number of commodities of type i transported from plant p through distribution center d to supply demand of consumer c. Any other variables and parameters you use must be clearly defined.

Model the problem as a mixed-integer program. Explain the meaning of each constraint and of each term in your objective function.

Question 2 (20 marks)

Consider the following linear programanming model:

$$
\begin{array}{llcl}
\max & x_1 & & \\
\text{s.t.} & & & \\
& x_1 + x_2 - x_3 + x_4 & = & 10 \\
& x_1 + 2x_2 + x_3 + x_5 & = & 20 \\
& -x_1 - x_2 + x_3 + x_6 & = & 20
\end{array}
$$

In this problem, consider a basic partition with variables x_1, x_5 and x_6 (in this order).

a) What are the values of components $x_1, \ldots, x_6$ at the extreme point associated with the given partition ?

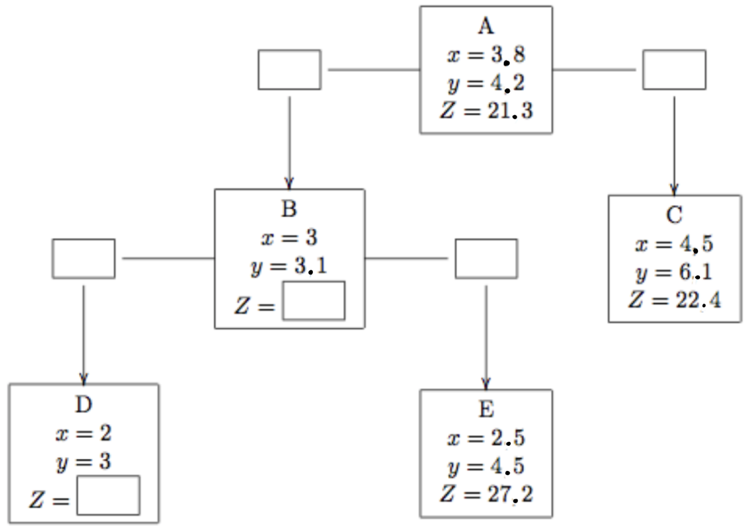
b) Compute the reduced costs associated with the non-basic variables of the given partition and explain why this solution is not optimal. Which variable should enter the basis ?

c) Obtain a new basis by performing one iteration of the simplex algorithm starting at the basis proposed in (a). Who leaves the basis ? What are the new basic variables and their values?

d) The basis in (c) is optimal. What are the dual values associated with the constraints ? Use these dual values to estimate the change in the objective function if the vector b is changed to $b = \begin{bmatrix} 10.1 \\ 20.1 \\ 10.2 \end{bmatrix}$

Question 3 (10 marks)

Consider the following branch-and-bound tree for a problem on integer variables x and y, in which Z indicates the optimal solution value for the linear programming relaxation at the node.



a) Fill in the blanks in the figure with the following options:
$y \leq 3$, $x \geq 4$, $y \geq 4$, $x \leq 3$, 23, 25.1

b) Is this a minimisation or maximisation problem? Explain.

c) What is the tighest lower bound you can infer on the optimal solution ?

d) What is the tighest upper bound you can infer on the optimal solution ?

e) Which node(s) are still open, i.e., which nodes can still contain the optimal solution ? Explain.

Question 4 (10 marks)

A student has obtained the following master and dual subproblem when solving a mixed integer programming problem with the Benders decomposition algorithm:

Master:

$$
\begin{array}{llcccccc}
\max & 5y_1 & - & 2y_2 & + & 9y_3 & & \\
st & y_1 & & & & & \leq & 5 \\
 & & & y_2 & & & \leq & 5 \\
 & & & & & y_3 & \leq & 5 \\
 & y_1 & & y_2 & & y_3 & \in & \mathcal{Z}^+
\end{array}
$$

Dual subproblem:

$$
\begin{array}{llclcc}
\min & (-2 - 5\bar{y}_1 + 3\bar{y}_2 - 7\bar{y}_3)u_1 & + & (10 - 4\bar{y}_1 - 2\bar{y}_2 - 4\bar{y}_3)u_2 & & \\
st & 2u_1 & + & 3u_2 & \geq & 2 \\
 & 3u_1 & - & u_2 & \geq & -3 \\
 & 6u_1 & + & 3u_2 & \geq & 4 \\
 & u_1, & & u_2 & \in & \mathcal{R}^+
\end{array}
$$

a) Write the original problem for which the student obtained this master problem and dual subproblem.

b) Apply one iteration of the Benders decomposition algorithm.

Question 5 (11 marks)

Consider the cutting stock (paper mill, as Alison Harcourt called it) problem with the following data:
Original rolls length are $L = 25m$. The cost of a roll is AUD100. The Items to be produced with their lenghts and demands are given below:

item 1) length $(l_1) = 3$m , demand $(d_1) = 13$

item 2) length $(l_2) = 5$m , demand $(d_2) = 15$

item 3) length $(l_3) = 7$m , demand $(d_3) = 10$

item 4) length $(l_4) = 8$m , demand $(d_4) = 5$

item 5) length $(l_5) = 11$m , demand $(d_5) = 3$

For technical reasons, item 3 can not be produced with item 4 and the number of item types cut from the same roll can not exceed 3.

a) Propose a pattern-based formulation. Don't forget to define the meaning of your pattern.

b) Propose an initial set of patterns to be used in a column generation approach.

c) Define dual variables and propose a pricing subproblem for your column generation approach:

d) Suppose rolls that cut at least one unit of item 5 need to go through a special procedure that increases the cost of a roll by 10AUD. What needs to be modified in your column generation approach?

Question 6 (24 marks)

Answer the questions below:

a) Consider a computer with 8 parallel processors. It is possible that a branch-and-cut algorithm will run faster for a given problem if it uses only one of the processors instead of the eight processors available. True or false ? Justify your answer.

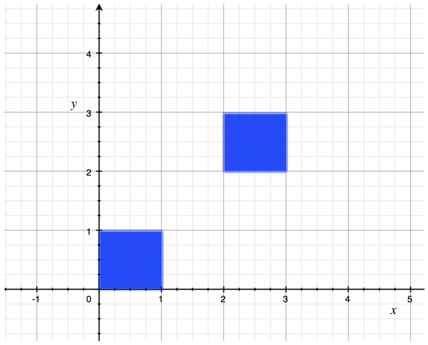
b) Explain what is strong branching in the context of the branch-and-bound algorithm.

c) What does it mean to say that "Dantzig-Wolfe decomposition" is a dual procedure to Benders decomposition ?

d) Consider a bin-packing formulation with binary variables x_{ij} with $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots b\}$ equal to one if item i is packed in bin j. Explain what would be the impact of the following constraints on items 1 and 2:

$$\sum_{j=1}^{b} j x_{1j} < \sum_{j=1}^{b} j x_{2j}$$

e) Model the feasible space given by the two dark areas in the figure below.



f) What is the depth-first strategy in the branch-and-bound algorithm and why is it can be benefitial to initialise the algorithm with this strategy ?

g) Draw the relation between variables x and y given by the constraints:

$x = \lambda_2 + 2\lambda_3,$
$y = 2\lambda_2 + 3\lambda_3,$
$\sum_{i=1}^{3} \lambda_i = 1$
$\lambda_1 \leq y_1$
$\lambda_2 \leq y_1 + y_2$
$\lambda_3 \leq y_2$
$y_1 + y_2 \leq 1$
$\lambda_1, \lambda_2, \lambda_3 \geq 0$
$y_1, y_2 \in \{0, 1\}.$

End of Exam—Total Available Marks = 100

12.2 Solution

THE UNIVERSITY OF MELBOURNE

Student Number

Semester 1 Assessment, 2020

School of Mathematics and Statistics

MAST90014 Optimisation for Industry

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> Thus the problem is to determine which DC sites to use, what load to have at each DC selected site, what customer zones should be served by each DC, and what the pattern of transportation flows should be for all commodities. This is to be done so as to meet the given demands at minimum total distribution cost subject to the plant capacity and DC throughput constraints."

Use continuous variables $x_{ipdc} \geq 0$ to indicate the number of commodities of type i transported from plant p through distribution center d to supply demand of consumer c. Any other variables and parameters you use must be clearly defined.

Model the problem as a mixed-integer program. Explain the meaning of each constraint and of each term in your objective function.

Let

- y_{dc} be binary variables equal to one if consumer c is served by distribution center d, and to zero, otherwise;
- z_d be binary variables equal to one if distribution center c is used for at least one consumer.

The production-distribution problem reads as:

$$\begin{array}{lrcll}
\min & \multicolumn{4}{l}{\sum_{i\in I}\sum_{p\in P}\sum_{d\in D}\sum_{c\in C} v_{pdc}x_{ipdc} + \sum_{d\in D} f_d z_d + \sum_{d\in D}\sum_{c\in c} t_d b_{ic} y_{dc}} \\
\text{s.t.} & & & & \\
& \displaystyle\sum_{i\in I}\sum_{d\in D}\sum_{c\in C} x_{ipdc} & \leq & \overline{k}_p, & p\in P, \\
& \displaystyle\sum_{p\in P}\sum_{d\in D} x_{ipdc} & = & b_{ic}, & i\in I, c\in C, \\
& \displaystyle\sum_{d\in D} y_{dc} & = & 1, & c\in C, \\
& \underline{m}_d z_d \leq \displaystyle\sum_{ic} b_{ic} y_{dc} & \leq & \overline{m}_d z_d, & c\in C, \\
& x_{ipdc} & \geq & 0, & i\in I, p\in P, d\in D, c\in C, \\
& y_{dc} & \in & \{0,1\}, & d\in D, c\in C, \\
& z_d & \in & \{0,1\}, & d\in D.
\end{array}$$

The terms in the objective function correspond to the transportation costs, the fixed-charge cost of openning distribution center c and the variable cost for using distribution center c, respectively.

- The first set of constraints model plant capacities.
- The second set of constraints model demand supply.
- The third set of constraints model that consumers must be served by a single DC.
- The fourth set of constraints relate variables z_d to y_{dc} and model the fact that the distribution center must be activated to be used (i.e., its fixed charge must be paid) and limit the flow at the DC to be between its lower and upper bounds.
- The remaining constraints define the scope of the variables.

Question 2 (20 marks)

Consider the following linear programanming model:

$$\begin{array}{lrcrcrcrcrcrcl} \max & x_1 & & & & & & & & & & & & \\ \text{s.t.} & & & & & & & & & & & & & \\ & x_1 & + & x_2 & - & x_3 & + & x_4 & & & & & = & 10 \\ & x_1 & + & 2x_2 & + & x_3 & & & + & x_5 & & & = & 20 \\ & -x_1 & - & x_2 & + & x_3 & & & & & + & x_6 & = & 20 \end{array}$$

In this problem, consider a basic partition with variables x_1, x_5 and x_6 (in this order).

a) What are the values of components $x_1, \ldots, x_6$ at the extreme point associated with the given partition ?

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, b = \begin{bmatrix} 10 \\ 20 \\ 10 \end{bmatrix}$$

The current variables are given by:

$$x_B = B^{-1}b = \begin{bmatrix} 10 \\ 10 \\ 30 \end{bmatrix}, x_N = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

b) Compute the reduced costs associated with the non-basic variables of the given partition and explain why this solution is not optimal. Which variable should enter the basis ?

$C_N^t = [0 \quad 0 \quad 0], C_B^t = [1 \quad 0 \quad 0],$

reduced cost expression: $\hat{c}_N^t = c_N^t - c_B^t B^{-1} N$

$$\hat{c}_N = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

and therefore the solution is not optimal since we are maximising and not all reduced costs are negative. The second non-basis variable (x_3) should enter the basis.

c) Obtain a new basis by performing one iteration of the simplex algorithm starting at the basis proposed in (a). Who leaves the basis ? What is the new basic variables and their values?

change vector: $y = B^{-1}a_3 = [-1 \quad 2 \quad 0]$,
current basis (from (a)): $B^{-1}b = [10 \quad 10 \quad 20]$

As we increase x_3 only the second basic variable, x_5, decreases (by two units for each unit of x_3). Since it is currently at 10, it will reach zero at $x_3 = 5$. The other basic variables have values $x_1 = 10 + 5 = 15$ and $x_6 = 20$.

d) The basis in (c) is optimal. What are the dual values associated with the constraints ? Use these values duals to estimate the change in the objective function if the vector b is changed to $b = \begin{bmatrix} 10.1 \\ 20.1 \\ 10.2 \end{bmatrix}$

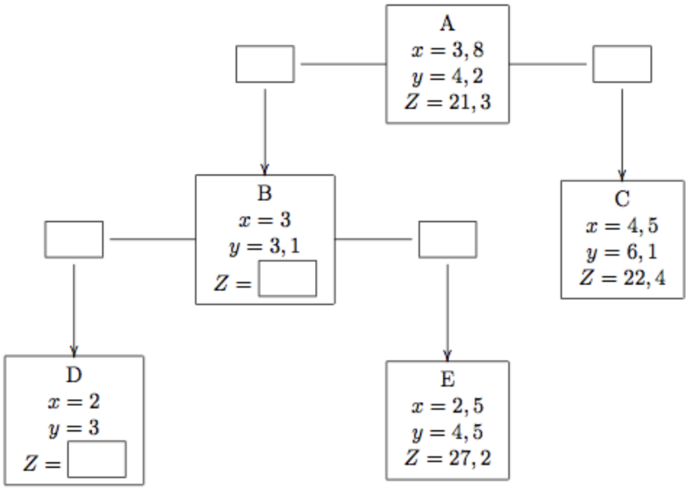
The new basis is given by the columns of x_1, x_3 and x_6.

$$B = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \text{ and } B^{-1} = \begin{bmatrix} 0.5 & 0.5 & -0.0 \\ -0.5 & 0.5 & -0.0 \\ 1.0 & 0.0 & 1.0 \end{bmatrix}$$

$u = c_B^t B^{-1} = [.5 \quad .5 \quad 0]$
with a change in b of $\Delta b' = [.1 \quad .1 \quad .2]$, the expected change in the objective function will be $u'\Delta b = 0.1$.

Question 3 (10 marks)

Consider the following branch-and-bound tree for a problem on integer variables x and y, in which Z indicates the optimal solution value for the linear programming relaxation at the node.



a) Fill in the blanks in the figure with the following options:
$y \leq 3$, $x \geq 4$, $y \geq 4$, $x \leq 3$, 23, 25.1

b) Is this a minimisation or maximisation problem? Explain.

Minimisation. The value of the LP increases as we add the branching constraints.

c) What is the tighest lower bound you can infer on the optimal solution ?

The tightest lower bound is given in node C: 22.4.

d) What is the tighest upper bound you can infer on the optimal solution ?

The tightest upper bound is given in node D: 25.1.

e) Which node(s) are still open, i.e., which nodes can still contain the optimal solution ? Explain.

Node C. It is the only one promising less than the current lower bound of 25.1

Question 4 (10 marks)

A student has obtained the following master and dual subproblem when solving a mixed integer programming problem with the Benders decomposition algorithm:

Master:

$$\begin{array}{llllllll} \max & 5y_1 & - & 2y_2 & + & 9y_3 & & \\ st & y_1 & & & & & \leq & 5 \\ & & & y_2 & & & \leq & 5 \\ & & & & & y_3 & \leq & 5 \\ & y_1 & & y_2 & & y_3 & \in & \mathcal{Z}^+ \end{array}$$

Dual subproblem:

$$\begin{array}{llllll} \min & (-2 - 5\bar{y}_1 + 3\bar{y}_2 - 7\bar{y}_3)u_1 & + & (10 - 4\bar{y}_1 - 2\bar{y}_2 - 4\bar{y}_3)u_2 & & \\ st & 2u_1 & + & 3u_2 & \geq & 2 \\ & 3u_1 & - & u_2 & \geq & -3 \\ & 6u_1 & + & 3u_2 & \geq & 4 \\ & u_1, & & u_2 & \in & \mathcal{R}^+ \end{array}$$

a) Write the original problem for which the student obtained this master problem and dual subproblem.

The primal subproblem is given by:

$$\begin{array}{llllllll} \max & 2x_1 & - & 3x_2 & + & 4x_3 & & \\ & 2x_1 & + & 3x_2 & + & 6x_3 & \leq & -2 - 5\bar{y}_1 + 3\bar{y}_2 - 7\bar{y}_3 \\ & 3x_1 & - & x_2 & + & 3x_3 & \leq & 10 - 4\bar{y}_1 - 2\bar{y}_2 - 4\bar{y}_3 \\ & x_1, & & x_2, & & x_3 & \in & \mathcal{R}^+ \end{array}$$

The original problem is the primal subproblem without the fixing of the y variables and with the addition of the constraints (on y variables only) from the master problem.

$$\begin{array}{llllllllllllll} \max & 2x_1 & - & 3x_2 & + & 4x_3 & + & 5y_1 & - & 2y_2 & + & 9y_3 & & \\ & 2x_1 & + & 3x_2 & + & 6x_3 & - & 5y_1 & + & 3y_2 & - & 7y_3 & \leq & 2 \\ & 3x_1 & - & x_2 & + & 3x_3 & - & 4y_1 & - & 2y_2 & - & 4y_3 & \leq & 10 \\ & & & & & & & y_1 & & & & & \leq & 5 \\ & & & & & & & & & y_2 & & & \leq & 5 \\ & & & & & & & & & & & y_3 & \leq & 5 \\ & x_1, & & x_2, & & x_3 & & & & & & & \in & \mathcal{R}^+ \\ & & & & & & & y_1, & & y_2, & & y_3 & \in & \mathcal{Z}^+ \end{array}$$

b) Apply one iteration of the Benders decomposition algorithm.

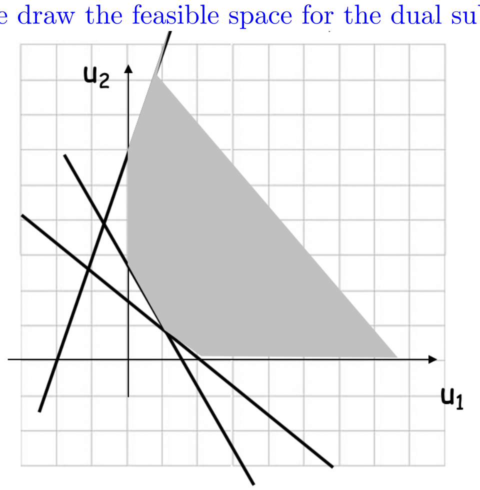
By inspection, we can solve the original master and obtain the optimal solution on y variables:

$$y = [5 \quad 0 \quad 5]',$$

which provides a first UB of 70 and yields the following objective function for the dual subproblem:

$$\min\ (-2 - 5\bar{y}_1 + 3\bar{y}_2 - 7\bar{y}_3)u_1 + (10 - 4\bar{y}_1 - 2\bar{y}_2 - 4\bar{y}_3)u_2 = \min\ -62u_1 - 30u_2$$

We draw the feasible space for the dual subproblem:



we can find the optimal solution of the first dual subproblem for this objective function as the unbounded ray:

$$(u_1, u_2) = (1, 0)$$

which provides no lower bound but yields the first Benders cut to be added to the master as:

$$-5y_1 + 3y_2 - 7y_3 - 2 \geq 0$$

Question 5 (11 marks)

Consider the cutting stock (paper mill, as Alison Harcour called it) problem with the following data:
Original rolls length are $L = 25m$. The cost of a roll is AUD100. The Items to be produced with their lenghts and demands are given below:

item 1) length (l_1) = 3m , demand (d_1) = 13

item 2) length (l_2) = 5m , demand (d_2) = 15

item 3) length (l_3) = 7m , demand (d_3) = 10

item 4) length (l_4) = 8m , demand (d_4) = 5

item 5) length (l_5) = 11m , demand (d_5) = 3

For technical reasons, item 3 can not be produced with item 4 and the number of item types cut from the same roll can not exceed 3.

a) Propose a pattern-based formulation. Don't forget to define the meaning of your pattern (column).

A pattern can be represented by a vector with 5 rows, with the value a_{ip} in each row $i = \{1, \dots 5\}$ representing the number of items of type i to be cut in that pattern.
The model reads:

$$z = \min \sum_{p \in P} 100 x_p$$

s.t.

$$\sum_{p \in P} a_{ip} x_p \geq d_i \quad \forall i = 1, \dots, 5$$

$$x_p \in \mathbb{Z}_+$$

b) Propose an initial set of patterns to be used in a column generation approach.

A initial set of patterns need to yield the master problem feasible. We can use, for example, 5 naive patterns that cut one unit of each item only.

c) Define dual variables and propose a pricing subproblem for your column generation approach:

Let $u = [u_1, \ldots, u_5]'$ be the vector of dual variables associated with the model in a). Using integer variables a_i as the value in the i^{th} column, binary variables y_i equal to one if item i is cut from the pattern, the model reads:

$$z = \max\ 100 - \sum_{i=1}^{5} u_i a_i$$

s.t.

$$\sum_{l_i} a_i \leq L,$$

$$a_i \leq M y_i, \quad \forall i = 1, \ldots, 5,$$

$$y_1 + y_4 \leq 1$$

$$\sum_{i=1}^{5} y_i \leq 3$$

$$a_i \in \mathcal{Z}^+$$

$$y_i \in \{0, 1\}$$

d) Consider that rolls that cut at least one unit of item 5 need to go through a special procedure that increases the cost of the roll in 10AUD. What needs to be modified in your column generation approach?

The cost of rolls that cut 5 are increased by 10 in the master problem. The subproblem objective function also needs to be modified to include this fact in the pricing process:
The model reads:

$$z = \max\ 100 + 10y_5 - \sum_{i=1}^{5} u_i a_i$$

s.t.

$$\sum_{l_i} a_i \leq L,$$

$$y_1 + y_4 \leq 1$$

$$\sum_{i=1}^{5} y_i \leq 3$$

$$a_i \in \mathcal{Z}^+$$

$$y_i \in \{0, 1\}$$

Question 6 (24 marks)

Answer the questions below:

a) Consider a computer with 8 paralel processors. It is possible that a branch-and-cut algorithm will run faster for a given problem if it uses only one of the processors instead of the eight processors available. True or false ? Justify your answer.

True. It might happen that by using a single core the branch-and-cut eventually explores a smaller tree (for example, by finding a feasible solution early on the exploration).

b) Explain what is strong branching in the context of the branch-and-bound algorithm.

Strong branching is a variable selection technique that evaluates the impact of branching on each of the available fractional variables of a node before deciding on which one to branch. It selects the variable which generates the largest impact on the bounds.

c) What does it mean to say that "Dantzig-Wolfe decomposition" is a dual procedure to Benders decomposition ?

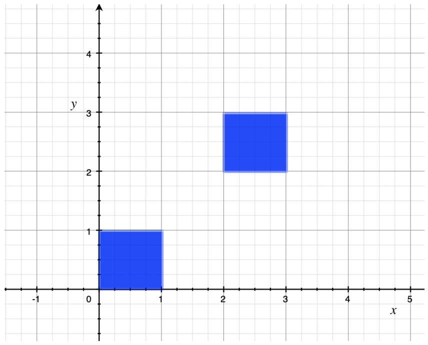
Dantzig-Wolfe rewrites the original model with an exponential number of columns (variables) and iteratively generate these columns. Benders rewrites the model as a problem with an exponential number of rows (constraints) and iteratively generates these rows. Since the dual of a problem 'converts' constraints to variables (and vice-versa), it makes sense to think of the processes as duals to each other.

d) Consider a bin-packing formulation with binary variables x_{ij} with $i \in \{1, \dots, n\}$ and $j \in \{1, \dots b\}$ equal to one if item i is packed in bin j. Explain what would be the impact of the following constraints on items 1 and 2:

$$\sum_{j=1}^{b} j x_{1j} < \sum_{j=1}^{b} j x_{2j}$$

The constraints ensure that item 1 will be packed in a bin with index smaller than the bin in which item 2 is packed. (Precedence constraint between 1 and 2).

e) Model the feasible space given by the two dark areas in the figure below.



Let $y_1 \in \{0, 1\}$ be a variable equal to one if the feasible point is in the square that touches the origin and to zero otherwise.

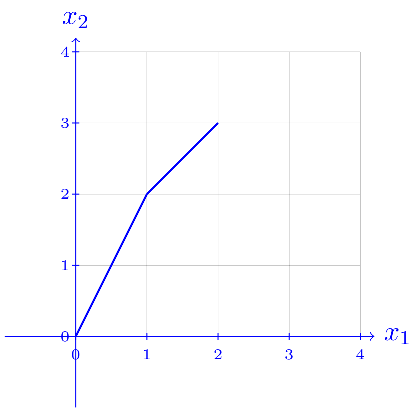
$x \geq 0,$
$y \geq 0,$
$x \leq 1 + M(1 - y_1),$
$y \leq 1 + M(1 - y_1),$
$x \geq 2 - My_1,$
$y \geq 2 - My_1,$
$x \leq 3 + My_1,$
$y \leq 3 + My_1,$
$y \in \{0, 1\}.$

f) What is the depth-first strategy in the branch-and-bound algorithm and why is it can be benefitial to initialise the algorithm with this strategy ?

Depth-first is a node selection strategy in the branch-and-bound algorithm that uses a depth-first search strategy, exploring a (always left or always right) child node of the last explored node and backtracking only when the node is pruned. It may be useful to initialise the branch-and-bound algorithm with this strategy as it is more prone to finding feasible solutions (and therefore, allow pruning to start).

g) Draw the relation between variables x and y given by the constraints:

$$
\begin{aligned}
&x = \lambda_2 + 2\lambda_3,\\
&y = 2\lambda_2 + 3\lambda_3,\\
&\textstyle\sum_{i=1}^{3} \lambda_i = 1\\
&\lambda_1 \leq y_1\\
&\lambda_2 \leq y_1 + y_2\\
&\lambda_3 \leq y_2\\
&y_1 + y_2 \leq 1\\
&\lambda_1, \lambda_2, \lambda_3 \geq 0\\
&y_1, y_2 \in \{0, 1\}.
\end{aligned}
$$



End of Exam—Total Available Marks = 100

13 2021 - Semester 1

13.1 Exam

THE UNIVERSITY OF MELBOURNE

Student number

Semester 1 Assessment, 2021

School of Mathematics and Statistics

MAST90014 Optimisation for Industry

Reading time: 30 minutes — Writing time: 2 hours — Upload time: 30 minutes

This exam consists of 16 pages (including this page)

Permitted Materials

- This exam and/or an offline electronic PDF reader, one or more copies of the masked exam template made available earlier, blank loose-leaf paper and a Casio FX-82 calculator.
- One double sided A4 page of notes (handwritten or printed).

Instructions to Students

- If you have a printer, print the exam one-sided. If using an electronic PDF reader to read the exam, it must be disconnected from the internet. Its screen must be visible in Zoom. No mathematical or other software on the device may be used. No file other than the exam paper may be viewed.
- Ask the supervisor if you want to use the device running Zoom.

Writing

- There are 7 questions with marks as shown. The total number of marks available is 100.
- Write your answers in the boxes provided on the exam that you have printed or the masked exam template that has been previously made available. If you need more space, you can use blank paper. Note this in the answer box, so the marker knows. The extra pages can be added to the end of the exam to scan.
- If you have been unable to print the exam and do not have the masked template write your answers on A4 paper. The first page should contain only your student number, the subject code and the subject name. Write on one side of each sheet only. Start each question on a new page and include the question number at the top of each page.

Scanning

- Put the pages in number order and the correct way up. Add any extra pages to the end. Use a scanning app to scan all pages to PDF. Scan directly from above. Crop pages to A4. Make sure that you upload the correct PDF file and that your PDF file is readable.

Submitting

- **You must submit while in the Zoom room.** No submissions will be accepted after you have left the Zoom room.
- Go to the Gradescope window. Choose the Canvas assignment for this exam. Submit your file. Wait for Gradescope email confirming your submission. Tell your supervisor when you have received it.

Question 1 (20 marks)

A company is developing its marketing plans for next year's new products. For three of these products, the decision has been made to purchase a total of five TV spots for commercials on national television networks. Now they need to decide how to allocate the five spots to these three products, with a maximum of three spots (and a minimum of zero) for each product.
The table below shows the estimated impact of allocating zero, one, two, or three spots to each product. This impact is measured in terms of the profit (in units of millions of dollars) from the additional sales that would result from the spots, considering also the cost of producing the commercial and purchasing the spots. The objective is to allocate five spots to the products so as to maximize the total profit.

Number of TV Spots	Profit		
	Product		
	1	**2**	**3**
0	0	0	0
1	1	0	−2
2	3	2	−1
3	3	3	6

(a) Clearly define your variables and model this optimisation problem **using only linear constraints**. Explain the meaning of all your constraints.

(b) The marketing team informs you that you can delay the decision on the number of spots to be assigned to products 2 and 3 until after the campaign on product 1 is launched. This will allow them to better estimate the value of the campaigns for these products. They predict three equiprobable scenarios for the additional profit associated with these last campaigns:

Number of TV Spots assigned to product 2	Profit		
	Scenario		
	1	**2**	**3**
0	0	0	0
0	1	3	0
2	2	3	0
3	4	4	2

Number of TV Spots assigned to product 3	Profit		
	Scenario		
	1	**2**	**3**
0	0	0	0
1	-2	0	1
2	-1	2	2
3	6	8	3

Modify your problem in (a) to incorporate this information and maximise the expected value of the three campaigns. You can use p_{ijs} (= the profit of campaign $i \in \{2, 3\}$ when j spots are used for that campaign in scenario s) to represent the data in the two tables above.

Question 2 (15 marks)

In an assembly line, a product is assembled while moving through a set of ordered work stations. At each station, some of the tasks needed for the manufacturing of the product are executed. One version of this problem, known as *the simple assembly line balancing problem of type-2* is modelled below:

$$
\begin{aligned}
&\min z = C \\
&\text{s.t.} \\
&\qquad \textstyle\sum_{s \in S} x_{is} = 1, && i \in I, && (1)\\
&\qquad \textstyle\sum_{i \in I} t_i x_{is} \le C, && s \in S, && (2)\\
&\qquad \textstyle\sum_{s \in S} s x_{is} \le \sum_{s \in S} s x_{js}, && (i,j) \in P, && (3)\\
&\qquad x_{is} \in \{0,1\}, C \ge 0 && && (4)
\end{aligned}
$$

where

I: set of tasks, $I = \{1, \ldots, n\}$,
S: set of ordered stations, $S = \{1, 2, \ldots, m\}$,
P: set of precedences, $P = \{(i,j) | i, j \in I, i \text{ must precede task } j\}$,
t_i: execution time of task i,

and x_{is} are binary variables equal to 1 if task i is assigned to station s.

(a) Explain the meaning of variable C and of each constraint.

(b) When balancing assembly lines in sheltered work centres with workers with disabilities, each worker might take a different amount of time to execute each task. In this situation, the execution time of a task is given by:

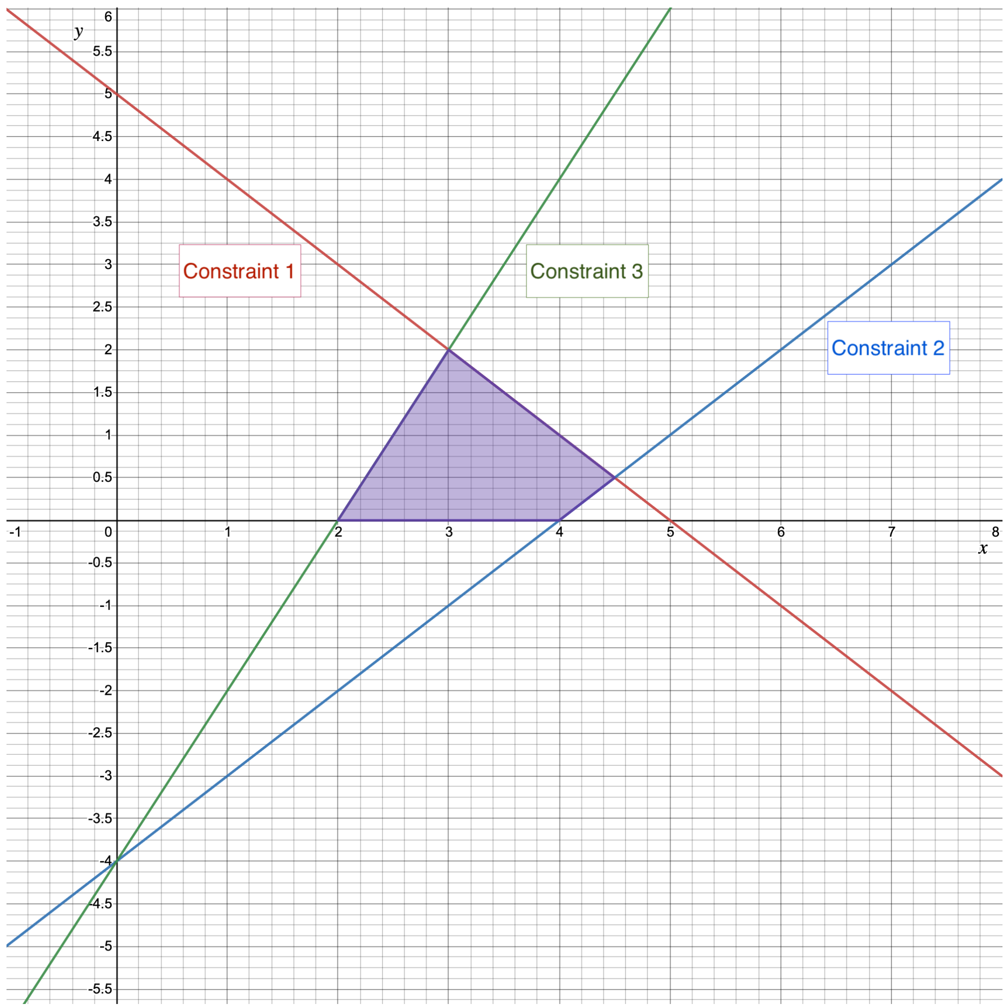
t_{iw}: execution time of task i when assigned to worker w.

Workers can not be considered homogeneous anymore and, therefore, in addition to deciding on the assignment of tasks to workers, one also needs to decide on the assignment of workers to stations. Consider a set of workers W (with the same cardinality as the set of stations S) and modify the model in (a) to represent the assembly line problem in this context. **Your model needs to remain linear**.

- Hint: In addition to a continuous variable C, use binary variables x_{isw} (equal to 1 if task i is assigned to station s and worker w) and y_{sw} (equal to 1 if worker w is assigned to station s) and modify the model in (a) to consider this situation. Explain each of your constraints and objective function.

Question 3 (15 marks)

The dark area in the figure below indicates the feasible region defined by three constraints of a linear program:



(a) The feasible region is defined in terms of variables x and y, as shown in the axes. Use slack variables s_1, s_2 and s_3 associated to constraints 1, 2 and 3, respectively, and write the problem in standard form (that is, as equalities). Do not forget to define the scope of all variables in the problem.

(b) The origin is clearly not feasible for the region above. Show this algebraically, using the equations you defined in (a).

(c) Explain how you can obtain an initial feasible solution using the phase 1 method of the simplex algorithm. Set up (but do not solve) the phase-1 optimisation problem.

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(d) Consider an objective function $\max x + y$, and compute the reduced costs for the non-basic variables at $(x, y) = (4, 0)$. Interpret the values you obtain by making explicit the contribution of each basic and non-basic variable.

(e) Still considering an objective function of $\max x+y$, carry out one iteration of the simplex algorithm, starting at the same point as you used in (d): $(x, y) = (4, 0)$. What are the new basic and non-basic components and matrices?

(f) Compute the reduced costs for the non-basic variables of the new extreme point you found in (e). Use the values to determine if the new point is the unique optimal solution of the problem, one of multiple optimal solutions or a suboptimal solution.

Question 4 (10 marks)

Consider the optimisation problem:

$$\begin{gathered} z = \max x_1 + x_2 \\ -2x_1 + 2x_2 + s_1 = 3 \\ 7x_1 + 3x_2 + s_2 = 22 \\ x_1, x_2, s_1, s_2 \in Z_+ \end{gathered}$$

The optimal solution of the linear relaxation is $(x_1, x_2, s_1, s_2) = (1.75, 3.25, 0, 0)$. Use this information to obtain a Gomory-cut for the integer problem. Write this cut in terms of the variables x_1 and x_2 and show that it cuts the optimal solution of the relaxed problem.

Hint: you might need one of the inverses below:

- $\begin{bmatrix} -2 & 1 \\ 7 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0.15 \\ 1 & 0.28 \end{bmatrix}$; $\begin{bmatrix} -2 & 2 \\ 7 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -0.15 & 0.1 \\ 0.35 & 0.1 \end{bmatrix}$; $\begin{bmatrix} -2 & 0 \\ 7 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -0.5 & 0 \\ 3.5 & 1 \end{bmatrix}$.
- $\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0.33 \\ 1 & -0.66 \end{bmatrix}$; $\begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0.5 & 0 \\ -1.5 & 1 \end{bmatrix}$; $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Question 5 (10 marks)

Consider an optimisation problem with constraints:

$$Ax + Gy \leq b$$
$$x \in \{0,1\}^n, y \in \mathcal{R}^m$$

One of the constraints in the system of inequalities $Ax + Gy \leq b$ is given by:

$$x_1 + 2x_2 + y_1 + y_{17} \leq 10 \qquad (*)$$

Let $A'x + G'y \leq b'$ be the original system without this constraint.

(a) Write a pre-processing optimisation problem to check if constraint (*) is redundant to the problem. Explain how to interpret the result.

(b) Write a pre-processing optimisation problem to check if constraint (*) makes the problem infeasible. Explain how to interpret the result.

Question 6 (15 marks)

Consider the problem:

$$
\begin{array}{rl}
\max & 16x_1 + 10x_2 + 4x_4 \\
\text{s.t.} & 8x_1 + 2x_2 + x_3 + 4x_4 \leq 10, \\
& x_1 + x_2 \leq 1 \\
& x_3 + x_4 \leq 1 \\
& x_1, x_2, x_3, x_4 \in \{0, 1\}.
\end{array}
$$

(a) Obtain the Lagrangian relaxation by dualising the first constraint using a Lagrangian multiplier λ.

(b) Write the Lagrangian dual problem.

(c) Start with a Lagrangian multiplier $\lambda = 0.5$ and a carry out two iterations (that is, obtain two bounds) of the subgradient method using a step size of 0.25. What is the best bound you obtain?

Question 7 (15 marks)

Consider the problem:

$$
\begin{aligned}
&\max x_1 + x_2 - y_1 - y_2 \\
\text{s.t.}\quad & \\
& x_1 - 3y_1 \leq -1 \\
& x_1 + 2x_2 - y_2 \leq 8 \\
& x_1, x_2 \geq 0, y_1, y_2 \in \{0, 1\}
\end{aligned}
$$

and answer the questions below in the context of the Benders decomposition reformulation and algorithm.

(a) Write an initial relaxed master problem.

(b) Write the primal subproblem.

(c) Write the dual subproblem.

(d) Obtain all extreme points and extreme rays for the problem obtained in (c)

(e) Use your answer in (d) to write the full Benders reformulation of the problem.

(f) If you started the problem with the initial relaxed master problem in (a), indicate which would be the first two cuts added to the master problem when solving the problem with the Benders decomposition algorithm. Explain your reasoning.

End of Exam — Total Available Marks = 100

13.2 Solution

THE UNIVERSITY OF MELBOURNE

Student number	

Semester 1 Assessment, 2021

School of Mathematics and Statistics

MAST90014 Optimisation for Industry

Reading time: 30 minutes — Writing time: 2 hours — Upload time: 30 minutes

This exam consists of 16 pages (including this page)

Permitted Materials

- This exam and/or an offline electronic PDF reader, one or more copies of the masked exam template made available earlier, blank loose-leaf paper and a Casio FX-82 calculator.
- One double sided A4 page of notes (handwritten or printed).

Instructions to Students

- If you have a printer, print the exam one-sided. If using an electronic PDF reader to read the exam, it must be disconnected from the internet. Its screen must be visible in Zoom. No mathematical or other software on the device may be used. No file other than the exam paper may be viewed.
- Ask the supervisor if you want to use the device running Zoom.

Writing

- There are 7 questions with marks as shown. The total number of marks available is 100.
- Write your answers in the boxes provided on the exam that you have printed or the masked exam template that has been previously made available. If you need more space, you can use blank paper. Note this in the answer box, so the marker knows. The extra pages can be added to the end of the exam to scan.
- If you have been unable to print the exam and do not have the masked template write your answers on A4 paper. The first page should contain only your student number, the subject code and the subject name. Write on one side of each sheet only. Start each question on a new page and include the question number at the top of each page.

Scanning

- Put the pages in number order and the correct way up. Add any extra pages to the end. Use a scanning app to scan all pages to PDF. Scan directly from above. Crop pages to A4. Make sure that you upload the correct PDF file and that your PDF file is readable.

Submitting

- **You must submit while in the Zoom room.** No submissions will be accepted after you have left the Zoom room.
- Go to the Gradescope window. Choose the Canvas assignment for this exam. Submit your file. Wait for Gradescope email confirming your submission. Tell your supervisor when you have received it.

Question 1 (20 marks)

A company is developing its marketing plans for next year's new products. For three of these products, the decision has been made to purchase a total of five TV spots for commercials on national television networks. Now they need to decide how to allocate the five spots to these three products, with a maximum of three spots (and a minimum of zero) for each product.
The table below shows the estimated impact of allocating zero, one, two, or three spots to each product. This impact is measured in terms of the profit (in units of millions of dollars) from the additional sales that would result from the spots, considering also the cost of producing the commercial and purchasing the spots. The objective is to allocate five spots to the products so as to maximize the total profit.

Number of TV Spots	Profit		
	Product		
	1	**2**	**3**
0	0	0	0
1	1	0	−2
2	3	2	−1
3	3	3	6

(a) Clearly define your variables and model this optimisation problem **using only linear constraints**. Explain the meaning of all your constraints.

Let y_{ij} be a binary variable equal to 1 if j spots are assigned to product i.

The model reads:

$$\text{Maximize} \quad Z = y_{11} + 3y_{12} + 3y_{13} + 2y_{22} + 3y_{23} - y_{31} - 2y_{32} + 6y_{33},$$

subject to

$$\begin{aligned} y_{11} + y_{12} + y_{13} &\leq 1, \\ y_{21} + y_{22} + y_{23} &\leq 1, \\ y_{31} + y_{32} + y_{33} &\leq 1, \\ y_{11} + 2y_{12} + 3y_{13} + y_{21} + 2y_{22} + 3y_{23} + y_{31} + 2y_{32} + 3y_{33} &= 5 \\ y_{ij} &\in \{0, 1\} \end{aligned}$$

- The first three constraints limit the number of spots chosen for each one of the products: note that the constraint being $\leq$ enables the option of no spot to be chosen for a given product.
- The last constraint forces the total number of spots to 5.

(b) The marketing team informs you that you can delay the decision on the number of spots to be assigned to products 2 and 3 until after the campaign on product 1 is launched. This will allow them to better estimate the value of the campaigns for these products. They predict three equiprobable scenarios for the additional profit associated with these last campaigns:

Number of TV Spots assigned to product 2	Profit		
	Scenario		
	1	**2**	**3**
0	0	0	0
0	1	3	0
2	2	3	0
3	4	4	2

Number of TV Spots assigned to product 3	Profit		
	Scenario		
	1	**2**	**3**
0	0	0	0
1	-2	0	1
2	-1	2	2
3	6	8	3

Modify your problem in (a) to incorporate this information and maximise the expected value of the three campaigns. You can use p_{ijs} (= the profit of campaign $i \in \{2,3\}$ when j spots are used for that campaign in scenario s) to represent the data in the two tables above.

Variables y_{2j} and y_{3j} receive an additional index, becoming y_{2js} (equal to 1 if j spots are assigned to campaign 2 in scenarion s) and y_{3js} (equal to 1 if j spots are assigned to campaign 3 in scenarion s).

The model reads:

$$\text{Maximize} \quad Z = y_{11} + 3y_{12} + 3y_{13} + \frac{1}{3}\sum_{s=1}^{3}\sum_{i=2}^{3}\sum_{j=1}^{3} p_{ijs}y_{ijs},$$

subject to

$$\begin{aligned}
y_{11} + y_{12} + y_{13} &\leq 1, && \\
y_{21s} + y_{22s} + y_{23s} &\leq 1, && s \in S, \\
y_{31s} + y_{32s} + y_{33s} &\leq 1, && s \in S, \\
y_{11} + 2y_{12} + 3y_{13} + y_{21s} + 2y_{22s} + 3y_{23s} + y_{31s} + 2y_{32s} + 3y_{33s} &= 5, && s \in S, \\
y_{1j}, y_{2js}, y_{3js} &\in \{0,1\}. &&
\end{aligned}$$

- The first three constraints limit the number of spots chosen for product 1.
- The following two sets of constraints limit the number of spots chosen for products 2 and 3, respectively, in each one of the scenarios,
- The last set of constraints forces the total number of spots to 5 in all scenarios.

Question 2 (15 marks)

In an assembly line, a product is assembled while moving through a set of ordered work stations. At each station, some of the tasks needed for the manufacturing of the product are executed. One version of this problem, known as *the simple assembly line balancing problem of type-2* is modelled below:

$$\begin{aligned}
&\min z = C \\
&\text{s.t.} \\
&\quad \textstyle\sum_{s\in S} x_{is} = 1, && i \in I, && (1)\\
&\quad \textstyle\sum_{i\in I} t_i x_{is} \le C, && s \in S, && (2)\\
&\quad \textstyle\sum_{s\in S} s x_{is} \le \sum_{s\in S} s x_{js}, && (i,j) \in P, && (3)\\
&\quad x_{is} \in \{0,1\}, C \ge 0 && && (4)
\end{aligned}$$

where

I: set of tasks, $I = \{1, \ldots, n\}$,
S: set of ordered stations, $S = \{1, 2, \ldots, m\}$,
P: set of precedences, $P = \{(i,j) | i, j \in I, i \text{ must precede task } j\}$,
t_i: execution time of task i,

and x_{is} are binary variables equal to 1 if task i is assigned to station s.

(a) Explain the meaning of variable C and of each constraint.

- C is the load of the most loaded station (bottleneck or cycle time).
- Constraints (1) enforce that each task must be assigned to one station.
- Constraints (2) enforce that the load of each station is bounded by the cycle time.
- Constraints (3) establish that for each pair of tasks with a precedence relation (i,j), a task i can only be executed in the same station or in a earlier station than the station to which task j is assigned.
- Constraints (4) limit the scope of the variables.

(b) When balancing assembly lines in sheltered work centres with workers with disabilities, each worker might take a different amount of time to execute each task. In this situation, the execution time of a task is given by:

t_{iw}: execution time of task i when assigned to worker w.

Workers can not be considered homogeneous anymore and, therefore, in addition to deciding on the assignment of tasks to workers, one also needs to decide on the assignment of workers to stations. Consider a set of workers W (with the same cardinality as the set of stations S) and modify the model in (a) to represent the assembly line problem in this context. **Your model needs to remain linear**.

- Hint: In addition to a continuous variable C, use binary variables x_{isw} (equal to 1 if task i is assigned to station s and worker w) and y_{sw} (equal to 1 if worker w is assigned to station s) and modify the model in (a) to consider this situation. Explain each of your constraints and objective function.

The model reads:

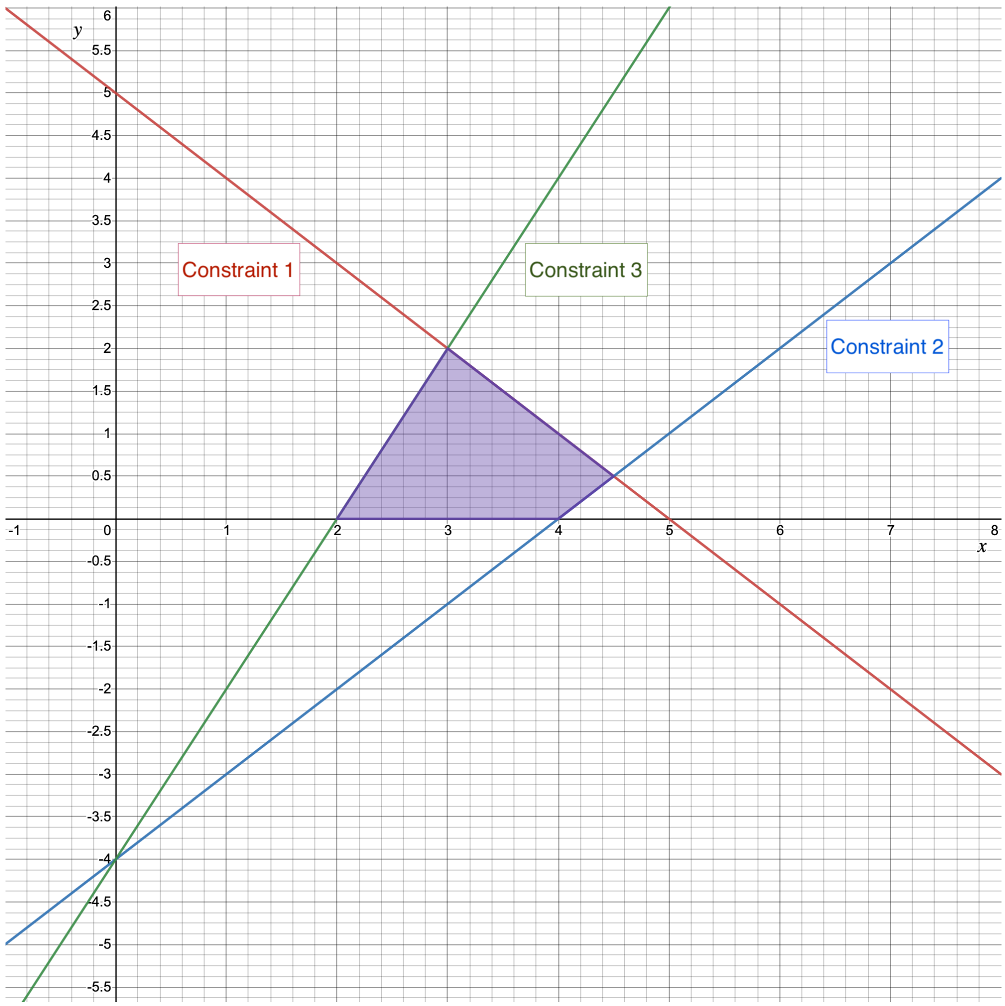
$$
\begin{aligned}
&\min z = C\\
&\text{s.t.}\\
&\quad \textstyle\sum_{s\in S}\sum_{w\in W} x_{isw} = 1, \quad i \in I\\
&\quad x_{isw} \le y_{sw}, \quad i \in I, s \in S, w \in W\\
&\quad \textstyle\sum_{w\in W} y_{sw} = 1, \quad s \in S\\
&\quad \textstyle\sum_{s\in S} y_{sw} = 1, \quad w \in W\\
&\quad \textstyle\sum_{i\in I} t_{iw} x_{isw} \le C, \quad s \in S, w \in W.\\
&\quad \textstyle\sum_{s\in S}\sum_{w\in W} s x_{isw} \le \sum_{s\in S}\sum_{w\in W} s x_{jsw}, \quad (i,j) \in P,\\
&\quad x_{isw} \in \{0,1\}, C \ge 0
\end{aligned}
$$

The constraints define, respectively:

- Each task is assigned,
- A task can only be assigned to a worker in a station if the worker is also assigned to that station,
- Each station has a worker,
- Each worker is assigned to a station,
- Cycle time is respected at all stations,
- precedence constraints are respected,
- escope of variables.

Question 3 (15 marks)

The dark area in the figure below indicates the feasible region defined by three constraints of a linear program:



(a) The feasible region is defined in terms of variables x and y, as shown in the axes. Use slack variables s_1, s_2 and s_3 associated to constraints 1, 2 and 3, respectively, and write the problem in standard form (that is, as equalities). Do not forget to define the scope of all variables in the problem.

constraint 1: $x + y + s_1 = 5$
constraint 2: $x - y + s_2 = 4$
constraint 3: $2x - y - s_3 = 4$
$x, y, s_1, s_2, s_3 \geq 0.$

(b) The origin is clearly not feasible for the region above. Show this algebraically, using the equations you defined in (a).

Substituting (x, y) for $(0, 0)$ in the equations, we obtain $s_3 = -4$. As this is negative, it indicates that constraint 3 is being violated.

(c) Explain how you can obtain an initial feasible solution using the phase 1 method of the simplex algorithm. Set up (but do not solve) the phase-1 optimisation problem.

Add an artificial variable a in order to obtain a feasible initial basis. Then, solve the problem:

$$
\begin{aligned}
&\min z = a \\
&\text{s.t.} \\
&\qquad x + y + s_1 = 5 \\
&\qquad x - y + s_2 = 4 \\
&\qquad 2x - y - s_3 + a = 4 \\
&\qquad x, y, s_1, s_2, s_3, a \geq 0
\end{aligned}
$$

(d) Consider an objective function $\max x + y$, and compute the reduced costs for the non-basic variables at $(x, y) = (4, 0)$. Interpret the values you obtain by making explicit the contribution of each basic and non-basic variable.

$X_B = (x, s_1, s_3)$

$X_N = (y, s_2)$

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & -1 \end{bmatrix}$$

$$N = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ -1 & 0 \end{bmatrix}$$

$\hat{c}_N = [\hat{c}_y, \hat{c}_{s_2}] = c_N^t - c_B^t B^{-1} N = (2, -1)$

Interpretation:

- Introducing the first non-basic variable (y) has a gain of 2, as for each unit of y introduced, one unit of x is added to the basis. Both these variables have coefficient 1 in the objective function and therefore, the combined gain is 2. All other variables have no coefficient in the objective function, so they do not contribute anything.
- Introducing the second non-basic variable (s_2) has a gain of -1, as for each unit of s_2 introduced, one unit of x is reduced from the the basis while y remains at zero. Variable x has a coefficient of 1 in the objective function and therefore, the combined gain is -1.

(e) Still considering an objective function of $\max x+y$, carry out one iteration of the simplex algorithm, starting at the same point as you used in (d): $(x, y) = (4, 0)$. What are the new basic and non-basic components and matrices?

It's advantageous to add y to the basis. Let the entering variable be y,

$$x_B = \begin{bmatrix} x \\ s_1 \\ s_3 \end{bmatrix} = B^{-1}b = \begin{bmatrix} 4 \\ 1 \\ 4 \end{bmatrix}, y = B^{-1}a_y = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

The only basic variable which has its value reduced when y is added is s_1. It reaches zero when $y = 1/2$.

The new basis and non basis components are: $X_B = (x, y, s_3)$ and $X_N = (s_1, s_2)$

and the new basic and non-basic matrices are:

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & -1 \end{bmatrix}, N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

(f) Compute the reduced costs for the non-basic variables of the new extreme point you found in (e). Use the values to determine if the new point is the unique optimal solution of the problem, one of multiple optimal solutions or a suboptimal solution.

$X_B = (x, y, s_3)$ and $X_N = (s_1, s_2)$

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & -1 \end{bmatrix}, N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$\hat{c}_N = [\hat{c}_{s_1}, \hat{c}_{s_2}] = c_N^t - c_B^t B^{-1} N = (-1, 0)$

Interpretation: the fact that all reduced costs are $\leq$ show that the point is optimal. As $\hat{c}_{s_2} = 0$ (and we can actually enter with this variable in the basis) shows that the optimal is not unique.

Question 4 (10 marks)

Consider the optimisation problem:

$$\begin{aligned} z &= \max x_1 + x_2 \\ -2x_1 + 2x_2 + s_1 &= 3 \\ 7x_1 + 3x_2 + s_2 &= 22 \\ x_1, x_2, s_1, s_2 &\in Z_+ \end{aligned}$$

The optimal solution of the linear relaxation is $(x_1, x_2, s_1, s_2) = (1.75, 3.25, 0, 0)$. Use this information to obtain a Gomory-cut for the integer problem. Write this cut in terms of the variables x_1 and x_2 and show that it cuts the optimal solution of the relaxed problem.

Hint: you might need one of the inverses below:

- $\begin{bmatrix} -2 & 1 \\ 7 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0.15 \\ 1 & 0.28 \end{bmatrix}$; $\begin{bmatrix} -2 & 2 \\ 7 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -0.15 & 0.1 \\ 0.35 & 0.1 \end{bmatrix}$; $\begin{bmatrix} -2 & 0 \\ 7 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -0.5 & 0 \\ 3.5 & 1 \end{bmatrix}$.
- $\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0.33 \\ 1 & -0.66 \end{bmatrix}$; $\begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0.5 & 0 \\ -1.5 & 1 \end{bmatrix}$; $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

The basic components are $x_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and the non basic components are $x_N = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$. Therefore:

$B = \begin{bmatrix} -2 & 2 \\ 7 & 3 \end{bmatrix}$, $N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Using the general solution of the system:

$x_B + B^{-1}Nx_n = B^{-1}b$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -0.15 & 0.1 \\ 0.35 & 0.1 \end{bmatrix} = \begin{bmatrix} 1.75 \\ 3.25 \end{bmatrix}$

We can use, for example, the second row to write the Gomory cut: $0.35s_1 + 0.1s_2 \geq 0.25$

Using $s_1 = 3 + 2x_1 - 2x_2$ and $s_2 = 22 - 7x_1 - 3x_2$

we obtain:

$0.35(3 + 2x_1 - 2x_2) + 0.1(22 - 7x_1 - 3x_2) \geq 0.25$

$-x_2 \geq -3 \Rightarrow x_2 \leq 3$

Which clearly cuts the original solution with $x_2 = 3.25$.

Page 11 of 16 — add any extra pages after page 16 — Page 11 of 16

Question 5 (10 marks)

Consider an optimisation problem with constraints:

$$Ax + Gy \leq b$$
$$x \in \{0,1\}^n, y \in \mathcal{R}^m$$

One of the constraints in the system of inequalities $Ax + Gy \leq b$ is given by:

$$x_1 + 2x_2 + y_1 + y_{17} \leq 10 \qquad (*)$$

Let $A'x + G'y \leq b'$ be the original system without this constraint.

(a) Write a pre-processing optimisation problem to check if constraint (*) is redundant to the problem. Explain how to interpret the result.

Solve the problem

$$\max x_1 + 2x_2 + y_1 + y_{17}$$
s.t.
$$A'x + G'y \leq b'$$
$$x \in \{0,1\}^n, y \in \mathcal{R}^m$$

If the result of this problem is smaller than 10, it means that the constraint is redundant as it is already implied by the other constraints in the problem.

(b) Write a pre-processing optimisation problem to check if constraint (*) makes the problem infeasible. Explain how to interpret the result.

Solve the problem

$$\min x_1 + 2x_2 + y_1 + y_{17}$$
s.t.
$$A'x + G'y \leq b'$$
$$x \in \{0,1\}^n, y \in \mathcal{R}^m$$

If the result of this problem is greater than 10, it means that the constraint will make the problem infeasible, as in the presence of other constraints there are no values of x_1, x_2, y_1 and $y_1 7$ that will satisfy the constraint.

Question 6 (15 marks)

Consider the problem:

$$\begin{aligned} \max \quad & 16x_1 + 10x_2 + 4x_4 \\ \text{s.t.} \quad & 8x_1 + 2x_2 + x_3 + 4x_4 \le 10, \\ & x_1 + x_2 \le 1 \\ & x_3 + x_4 \le 1 \\ & x_1, x_2, x_3, x_4 \in \{0, 1\}. \end{aligned}$$

(a) Obtain the Lagrangian relaxation by dualising the first constraint using a Lagrangian multiplier λ.

$$\begin{aligned} \text{maximize} \quad & (16-8\lambda)x_1 + (10-2\lambda)x_2 + (0-\lambda)x_3 + (4-4\lambda)x_4 + 10\lambda \\ \text{s.t.} \quad & x_1 + x_2 \le 1 \\ & x_3 + x_4 \le 1 \\ & x_1, x_2, x_3, x_4 \in \{0, 1\}. \end{aligned}$$

(b) Write the Lagrangian dual problem.

$$\begin{aligned} \min_{\lambda \ge 0} \max_x \quad & (16-8\lambda)x_1 + (10-2\lambda)x_2 + (0-\lambda)x_3 + (4-4\lambda)x_4 + 10\lambda \\ \text{s.t.} \quad & x_1 + x_2 \le 1 \\ & x_3 + x_4 \le 1 \\ & x_1, x_2, x_3, x_4 \in \{0, 1\}. \end{aligned}$$

(c) Start with a Lagrangian multiplier $\lambda = 0.5$ and a carry out two iterations (that is, obtain two bounds) of the subgradient method using a step size of 0.25. What is the best bound you obtain?

Iteration 1:

For $\lambda = 0.5$, the Lagrangian relaxation reads:

$$\begin{array}{rl} \text{maximize} & 12x_1 + 9x_2 - 0.5x_3 + 2x_4 + 10\lambda \\ \text{s.t.} & x_1 + x_2 \leq 1 \\ & x_3 + x_4 \leq 1 \\ & x_1, x_2, x_3, x_4 \in \{0, 1\}. \end{array}$$

which solution can be obtained by inspection as $x_1 = 1, x_2 = 0, x_3 = 0, x_4 = 1$, which gives a bound of 19. The new λ is obtained as

$$\lambda_1 = \lambda_0 - 0.25(10 - 8x_1 - 2x_2 - x_3 - 4x_4) = 1$$

Iteration 2:

The new Lagrangian relaxation reads:

$$\begin{array}{rl} \text{maximize} & 8x_1 + 8x_2 - x_3 + 0x_4 + 10\lambda \\ \text{s.t.} & x_1 + x_2 \leq 1 \\ & x_3 + x_4 \leq 1 \\ & x_1, x_2, x_3, x_4 \in \{0, 1\}. \end{array}$$

which solution can be obtained by inspection as $x_1 = 1, x_2 = 0, x_3 = 0, x_4 = 0$, which gives a bound of 18.

Page 14 of 16 — add any extra pages after page 16 — Page 14 of 16

Question 7 (15 marks)

Consider the problem:

$$
\begin{aligned}
&\max x_1 + x_2 - y_1 - y_2 \\
\text{s.t.}\quad & \\
&x_1 - 3y_1 \leq -1 \\
&x_1 + 2x_2 - y_2 \leq 8 \\
&x_1, x_2 \geq 0, y_1, y_2 \in \{0, 1\}
\end{aligned}
$$

and answer the questions below in the context of the Benders decomposition reformulation and algorithm.

(a) Write an initial relaxed master problem.

$$
\begin{aligned}
&\max -y_1 - y_2 + z \\
&\text{s.t.} \\
&z \leq M, \\
&y_1, y_2 \in \{0, 1\},
\end{aligned}
$$

(b) Write the primal subproblem.

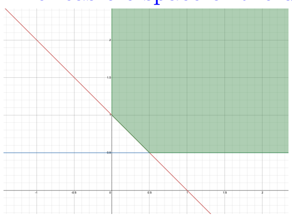
$$
\begin{aligned}
&\max x_1 + x_2 \\
&\text{s.t.} \\
&x_1 \leq 3\overline{y}_1 - 1 \\
&x_1 + 2x_2 \leq 8 + \overline{y}_2 \\
&x_1, x_2 \geq 0.
\end{aligned}
$$

(c) Write the dual subproblem.

$$
\begin{aligned}
&\min (3\overline{y}_1 - 1)u_1 + (8 + \overline{y}_2)u_2 \\
&\text{s.t.} \\
&u_1 + u_2 \geq 1 \\
&2u_2 \geq 1 \\
&u_1, u_2 \geq 0.
\end{aligned}
$$

(d) Obtain all extreme points and extreme rays for the problem obtained in (c)

The feasible space of the dual in the plane (u_1, u_2) is:



in which we identify rays $(1,0)$ and $(0,1)$, and extreme points $(.5,.5)$ and $(0,1)$.

(e) Use your answer in (d) to write the full Benders reformulation of the problem.

$$
\begin{aligned}
&\max -y_1 - y_2 + z \\
&\text{s.t.} \\
&z \le 1.5y_1 + 0.5y_2 + 3.5, \quad \text{extreme point } (0.5, 0.5) \\
&z \le y_2 + 8, \quad \text{extreme point } (0,1) \\
&3y_1 \ge 1, \quad \text{extreme ray } (1,0) \\
&y_2 \ge -8, \quad \text{extreme ray } (0,1) \\
&y_1, y_2 \in \{0,1\},
\end{aligned}
$$

(f) If you started the problem with the initial relaxed master problem in (a), indicate which would be the first two cuts added to the master problem when solving the problem with the Benders decomposition algorithm. Explain your reasoning.

The first solution of the restricted master would be (0,0) and generate the dual subproblem objective function

$$-u_1 + 8u_2$$

yielding the ray (1,0) and the associated cut $3y_1 \ge 1$.

The second solution would be (1,0) and generate the dual subproblem objective of

$$2u_1 + 8u_2$$

yielding the extreme point (.5,.5) and the associated cut $z \le 1.5y_1 + 0.5y_2 + 3.5$.

End of Exam — Total Available Marks = 100

14 2022 - Semester 1

14.1 Exam

THE UNIVERSITY OF MELBOURNE

Student number

Semester 1 Assessment, 2022

School of Mathematics and Statistics

MAST90014 Optimisation for Industry

Reading time: 30 minutes — Writing time: 2 hours — Upload time: 30 minutes

This exam consists of 7 pages (including this page)

Permitted Materials

- This exam and/or an offline electronic PDF reader, blank loose-leaf paper and a Casio FX-82 calculator.
- One double sided A4 page of notes (handwritten or printed).

Instructions to Students

- If you have a printer, print the exam. If using an electronic PDF reader to read the exam, it must be disconnected from the internet. Its screen must be visible in Zoom. No mathematical or other software on the device may be used. No file other than the exam paper may be viewed.
- Ask the supervisor if you want to use the device running Zoom.

Writing

- Write your answers on A4 paper. Page 1 should only have your student number, the subject code and the subject name. Write on one side of each sheet only. Each question should be on a new page. The question number must be written at the top of each page.

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- Go to the Gradescope window. Choose the Canvas assignment for this exam. Submit your file. Wait for Gradescope email confirming your submission. Tell your supervisor when you have received it.

Question 1

Each month, a company produces three types of items ($I = \{1, 2, 3\}$). The items are produced in one of the two plants the company owns ($P = \{1, 2\}$), and then transported to one of its three warehouses ($W = \{1, 2, 3\}$). From the warehouses, the items are sent to the final consumers ($C = \{1, 2, 3, 4, 5, 6\}$).

Transportation between plants and warehouses and between warehouses and consumers is made using vehicles that can carry K kg. Items are indivisible. The products arriving at a warehouse $w \in W$ are unloaded from the incoming trucks and reloaded in outgoing trucks. Each truck leaving a plant travels to a single warehouse and each truck leaving a warehouse travels to a single consumer location.

The following parameters are given:

- c_{ip} : the cost to produce a unit of item $i \in I$ in plant $p \in P$.
- b_i : weight of item $i \in I$ (kg).
- d_{ic} : demand of consumer $c \in C$ for product $i \in I$.
- r_{pw} : cost to be paid for each trip of a truck between plant $p \in P$ and warehouse $w \in W$.
- r_{wc} : cost to be paid for each trip of a truck between warehouse $w \in W$ and consumer $c \in C$.

(i) Clearly define decision variables and model a mixed-integer program that minimises the total production and transportation costs. Explain any additional parameters you use in your answer.

(ii) Discuss the strength of the linear relaxation of your model in (i) with respect to the value of the parameter $\alpha = \frac{\max_{i \in I} b_i}{K}$.

Answer:

- x_{ipwv} : amount of item i produced in plant p and transported to warehouse w in vehicle $v \in V_1$.
- x_{iwcv} : amount of item i transported from warehouse w to consumer c in vehicle $v \in V_2$.
- y^1_v : binary variable equal to 1 if vehicle $v \in V_1$ is used for transportation between plant p and warehouse w.
- y^2_{wc} : binary variable equal to 1 if vehicle $v \in V_2$ is used for transportation between warehouse w and consumer c.

The model reads:

$$\text{Minimize} \quad z = \sum_i \sum_p \sum_w \sum_{v \in V_1} c_{ip} x_{ipwv} + \sum_{v \in V_1} r_{pw} y^1_v + \sum_{v \in V_2} r_{pw} y^2_v$$

subject to

$$\sum_{p}\sum_{v\in V_1} x_{ipwv} = \sum_{c}\sum_{v\in V_2} x_{iwcv}, \qquad w\in W, i\in I$$
$$\sum_{w}\sum_{v\in V_2} x_{iwcv} = d_{ic}, \qquad i\in I, c\in C,$$
$$\sum_{i\in I} b_i x_{ipwv} \le K * y_v^1, \qquad p\in P, w\in W, v\in V_1,$$
$$\sum_{i\in I} b_i x_{iwcv} \le K * y_v^2, \qquad w\in W, c\in C, v\in V_2,$$
$$y, x\in \mathbb{Z}^+.$$

V_1 and V_2 are the set of vehicles available in the first and second echelon. We assume they are enough to transport the load (a loose upper bound is given by the number of items to be transported).

(ii) As α increases, the packing problem becomes more important and the linear relaxation becomes weaker.

Question 2

Model the following problems as Mixed Integer Programs.

(i) A set of n jobs must be carried out on a single machine that can do only one job at a time. Each job j takes p_j hours to complete. Given job weights w_j for $j = 1,\ldots n$, in what order should the jobs be carried out so as to minimize the weighted sum of their start times? Define variables and formulate this scheduling problem as a mixed integer program. Provide a possible value for any big-M parameter you use in your answer, if any.

(ii) Let $f(x)$ and $g_i(x), i = 1,\ldots,m$ be linear functions on $x\in\mathbb{R}^n$. Let L_i be a lower bound on $g_i(x), i = 1,\ldots,m$ and propose a Mixed Integer Programming model for the problem of minimising $f(x)$ such that at least k of the constraints $g_i(x)\ge 0, i = 1,\ldots,m$ are satisfied.

Question 3

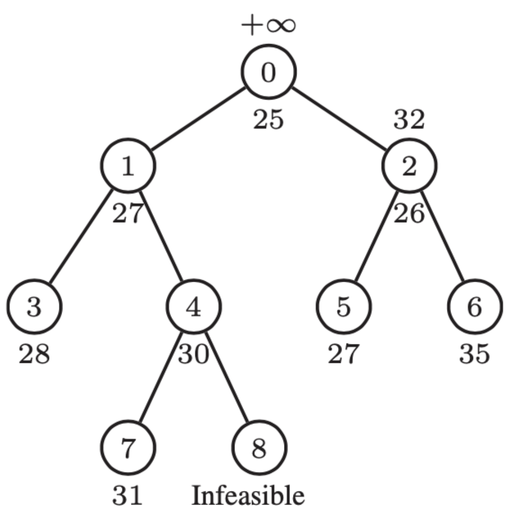
Formulate the following as Mixed Integer Programs.

(i) $u = \min\{y_1, y_2\}$, assuming that $0 \leq y_j \leq C$ for $j = 1, 2$.

(ii) $z = \max a^t x$ subject to the set $X \setminus \{x^*\}$ where $X = \{x \in \{0,1\}^n : Ax \leq b\}$ and $x^* \in X$. (Hint, Let I_0 be the set of indices for which $x_i^* = 0$ and I_1 be the set of indices for which $x_i^* = 1$).

Question 4

Consider the partial branch-and-bound tree for a mixed-integer model in the figure below. Each node of the tree is numbered. The value below each node is the optimal solution value of the relaxation of the problem at the node, with the exception of node 8, for which the relaxation was infeasible. The value on top of node 2 indicates the value of a feasible solution found via a feasibility pump heuristic applied at that node.



Answer the following questions:

i) Is the problem a maximisation, minimisation or it is impossible to tell with the given information? Justify your answer.

ii) What are the tightest possible lower and upper bounds on the optimal solution of the problem?

iii) Which nodes can be prunned and which nodes need to be investigated further?

Question 5

Given a 0-1 knapsack set in the form $X = \left\{x \in \{0,1\}^n : \sum_{j=1}^{n} a_j x_j \leq b\right\}$ with $a_j \geq 0$ for all j, answer TRUE, FALSE or 'I don't know' for each of the following statements.
The notation $\bar{x}_i$ is used to denote the complement of binary variable x_i.

i) $X = \emptyset$ if $b < 0$.

ii) $X = \{0,1\}^n$ if $\sum_{j=1}^{n} a_j \leq b$. In other words, the constraint is redundant.

iii) If $a_k > b$ for some $k \in \{1, \ldots, n\}$, then the problem is infeasible.

iv) If $a_k > b$ for some $k \in \{1, \ldots, n\}$, then $x_k = 0$ in all feasible solutions.

v) If $a_j = b$ for all $j \in \{1, \ldots, n\}$, then $\sum_{j=1}^{n} x_j \leq 1$ is a valid inequality for the problem.

vi) If $a_j + a_k > b$ for some $j, k \in \{1, \ldots n\}$, with $j \neq k$, then $x_j + x_k \leq 1$ in all feasible solutions.

vii) If $x_j + x_k \leq 1$ and $x_j + \bar{x}_k \leq 1$, for some $j, k \in \{1, \ldots n\}$, with $j \neq k$, then $x_j = 0$.

viii) Given weigths $w_j > 0$ for each item j, lower and upper bounds for the problem are given by 0 and $\sum_{j=1}^{n} w_j$, respectively.

ix) If $x_j + x_k \leq 1$ and $\bar{x}_j + \bar{x}_k \leq 1$, for some $j, k \in \{1, \ldots n\}$, with $j \neq k$, then $x_j + x_k = 1$.

x) If $x_j + \bar{x}_k \leq 1$ and $\bar{x}_j + \bar{x}_k \leq 1$, for some $j, k \in \{1, \ldots n\}$, with $j \neq k$, then $x_k = 1$.

Note: You do **not** need to justify your answers. One point will be awarded for each correct answer and one point deducted for each incorrect answer. Points will not be awarded or deducted for the items you answer 'I don't know'.

Question 6

Consider a linear maximisation problem on integer variables x_1, x_2. Answer the following questions:

(i) Is it possible for the problem to have no feasible solution given that its linear relaxation has an optimal solution? If yes, draw the feasible space of such a problem and indicate the gradient of the objective function. If it is not possible, explain why.

(ii) Is it possible for the problem to have a single optimal solution while its linear relaxation has multiple optimal solutions? If yes, draw the feasible space of such a problem and indicate the gradient of the objective function. If it is not possible, explain why.

(iii) Is it possible for the problem to have an objective function value larger than the value of the optimal solution value of its linear relaxation ? If yes, draw the feasible space of such a problem and indicate the gradient of the objective function. If it is not possible, explain why.

(iv) Is it possible for the problem to have a feasible solution given that its linear relaxation has no feasible solution? If yes, draw the feasible space of such a problem and indicate the gradient of the objective function. If it is not possible, explain why.

Question 7

Prove that:

(i) The space defined by a set of linear equality constraints on variables $x \in \mathbb{R}^n$ is convex.

(ii) Given a primal-dual linear programming pair, the value of the objective function for the minimisation problem is always larger than or equal to the value of the objective function for the maximisation problem.

Question 8

In the context of Benders decomposition, consider the Mixed Integer Program below:

$$\min y_1 + y_2 - x_1$$

s.t.

$$y_1 - x_1 \geq 2$$
$$y_2 + x_1 \geq 3$$
$$(y_1, y_2) \in \mathbb{Z}_+^2, x \geq 0$$

(i) Write the relaxed master problem.

(ii) Write the primal subproblem.

(iii) Write the dual suproblem and identify all extreme ray(s) and extreme point(s).

(iv) Write the full Benders reformulation.

Question 9

Consider the uncapacitated vehicle routing problem with n clients. The goal is to visit all clients while minimising the total cost of the routes (measured as the total distance travelled). Any number of vehicles can be used. A column generation is proposed with columns in the format:

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_i \\ \vdots \\ a_n \end{bmatrix}$$

where parameters $a_i, i = 1, \ldots, n$ is binary and indicates if customer i is visited in the route associated with the column ($a_i = 1$) or not ($a_i = 0$).

(i) Which problem do you need to solve to obtain the cost of a route associated with a column?

(ii) Let P be the set of all possible columns. Consider binary variables x_p indicating if column $p \in P$ is used and write a mixed integer program for the problem. Clearly explain the meaning of any parameters you define.

(iii) Propose a set of columns that can be used to start the solution of the problem using column generation.

End of Exam

14.2 Solution

THE UNIVERSITY OF MELBOURNE

Student number

Semester 1 Assessment, 2022

School of Mathematics and Statistics

MAST90014 Optimisation for Industry

Reading time: 30 minutes — Writing time: 2 hours — Upload time: 30 minutes

This exam consists of 10 pages (including this page)

Permitted Materials

- This exam and/or an offline electronic PDF reader, blank loose-leaf paper and a Casio FX-82 calculator.
- One double sided A4 page of notes (handwritten or printed).

Instructions to Students

- If you have a printer, print the exam. If using an electronic PDF reader to read the exam, it must be disconnected from the internet. Its screen must be visible in Zoom. No mathematical or other software on the device may be used. No file other than the exam paper may be viewed.
- Ask the supervisor if you want to use the device running Zoom.

Writing

- Write your answers on A4 paper. Page 1 should only have your student number, the subject code and the subject name. Write on one side of each sheet only. Each question should be on a new page. The question number must be written at the top of each page.

Scanning

- Put the pages in question order and all the same way up. Use a scanning app to scan all pages to PDF. Scan directly from above. Crop pages to A4. Make sure that you upload the correct PDF file and that your PDF file is readable.

Submitting

- **You must submit while in the Zoom room.** No submissions will be accepted after you have left the Zoom room.
- Go to the Gradescope window. Choose the Canvas assignment for this exam. Submit your file. Wait for Gradescope email confirming your submission. Tell your supervisor when you have received it.

Question 1

Each month, a company produces three types of items ($I = \{1, 2, 3\}$). The items are produced in one of the two plants the company owns ($P = \{1, 2\}$), and then transported to one of its three warehouses ($W = \{1, 2, 3\}$). From the warehouses, the items are sent to the final consumers ($C = \{1, 2, 3, 4, 5, 6\}$).

Transportation between plants and warehouses and between warehouses and consumers is made using vehicles that can carry K kg. Items are indivisible. The products arriving at a warehouse $w \in W$ are unloaded from the incoming trucks and reloaded in outgoing trucks. Each truck leaving a plant travels to a single warehouse and each truck leaving a warehouse travels to a single consumer location.

The following parameters are given:

- c_{ip} : the cost to produce a unit of item $i \in I$ in plant $p \in P$.
- b_i : weight of item $i \in I$ (kg).
- d_{ic} : demand of consumer $c \in C$ for product $i \in I$.
- r_{pw} : cost to be paid for each trip of a truck between plant $p \in P$ and warehouse $w \in W$.
- r_{wc} : cost to be paid for each trip of a truck between warehouse $w \in W$ and consumer $c \in C$.

(i) Clearly define decision variables and model a mixed-integer program that minimises the total production and transportation costs. Explain any additional parameters you use in your answer.

(ii) Discuss the strength of the linear relaxation of your model in (i) with respect to the value of the parameter $\alpha = \frac{\max_{i \in I} b_i}{K}$.

Answer:

- x_{ipwv} : amount of item i produced in plant p and transported to warehouse w in vehicle $v \in V_1$.
- x_{iwcv} : amount of item i transported from warehouse w to consumer c in vehicle $v \in V_2$.
- y^1_v : binary variable equal to 1 if vehicle $v \in V_1$ is used for transportation between plant p and warehouse w.
- y^2_{wc} : binary variable equal to 1 if vehicle $v \in V_2$ is used for transportation between warehouse w and consumer c.

The model reads:

$$\text{Minimize} \qquad z = \sum_i \sum_p \sum_w \sum_{v \in V_1} c_{ip} x_{ipwv} + \sum_{v \in V_1} r_{pw} y^1_v + \sum_{v \in V_2} r_{pw} y^2_v$$

subject to

$$\sum_{p}\sum_{v\in V_1} x_{ipwv} = \sum_{c}\sum_{v\in V_2} x_{iwcv}, \qquad w\in W, i\in I$$

$$\sum_{w}\sum_{v\in V_2} x_{iwcv} = d_{ic}, \qquad i\in I, c\in C,$$

$$\sum_{i\in I} b_i x_{ipwv} \le K * y^1_v, \qquad p\in P, w\in W, v\in V_1,$$

$$\sum_{i\in I} b_i x_{iwcv} \le K * y^2_v, \qquad w\in W, c\in C, v\in V_2,$$

$$y, x \in \mathbb{Z}^+.$$

V_1 and V_2 are the set of vehicles available in the first and second echelon. We assume they are enough to transport the load (a loose upper bound is given by the number of items to be transported).

(ii) As α increases, the packing problem becomes more important and the linear relaxation becomes weaker.

Question 2

Model the following problems as Mixed Integer Programs.

(i) A set of n jobs must be carried out on a single machine that can do only one job at a time. Each job j takes p_j hours to complete. Given job weights w_j for $j = 1, \ldots n$, in what order should the jobs be carried out so as to minimize the weighted sum of their start times? Define variables and formulate this scheduling problem as a mixed integer program. Provide a possible value for any big-M parameter you use in your answer, if any.

(ii) Let $f(x)$ and $g_i(x), i = 1, \ldots, m$ be linear functions on $x \in \mathbb{R}^n$. Let L_i be a lower bound on $g_i(x), i = 1, \ldots, m$ and propose a Mixed Integer Programming model for the problem of minimising $f(x)$ such that at least k of the constraints $g_i(x) \ge 0, i = 1, \ldots, m$ are satisfied.

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Answer (i):

- y_{ij} : binary variable, equal to 1 if job i preceedes job j, 0 otherwise.
- t_j : start time of job j.

$$\text{Minimize} \quad z = \sum_j w_j t_j$$

subject to

$$\begin{aligned} t_j \geq t_i + p_i - M(1 - y_{ij}), &\qquad i, j \in 1, \ldots, n, \\ y_{ij} + y_{ji} = 1, &\qquad i, j \in 1, \ldots, n, \\ &\qquad y \in \{0, 1\}. \\ &\qquad t \in \mathcal{R}_+. \end{aligned}$$

The big-M value can be defined as $\sum_j p_j$.

Answer (ii):

Let δ_i be binary variables indicating if the constraint is satisfied. The new constraints read:

$$g_i(x) \geq (1 - \delta_i) L_i, \quad i = 1, \ldots, m$$

$$\sum_{i=1}^{m} \delta_i \geq k$$

Question 3

Formulate the following as Mixed Integer Programs.

(i) $u = \min\{y_1, y_2\}$, assuming that $0 \leq y_j \leq C$ for $j = 1, 2$.

(ii) $z = \max a^t x$ subject to the set $X \setminus \{x^*\}$ where $X = \{x \in \{0,1\}^n : Ax \leq b\}$ and $x^* \in X$. (Hint, Let I_0 be the set of indices for which $x_i^* = 0$ and I_1 be the set of indices for which $x_i^* = 1$).

Answer (i):

$$\text{Minimize} \quad u$$

subject to

$$\begin{aligned} u &\geq y_1 - Cz, \\ u &\geq y_2 - C(1-z), \\ y_1, y_2 &\in \mathbb{R}_+, \\ z &\in \{0,1\} \end{aligned}$$

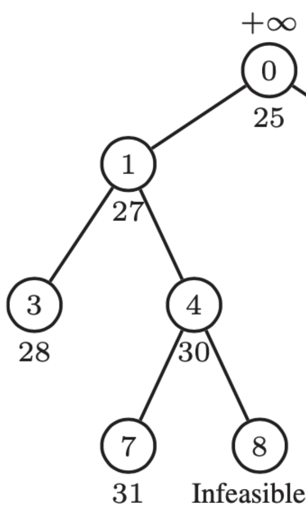
Answer (ii):

ii)

$$X \setminus \{x^*\} = \{x \in \{0,1\}^n : Ax \leq b, \sum_{i \in I_0} x_i + \sum_{i \in I_1} 1 - x_i \geq 1\}$$

Question 4

Consider the partial branch-and-bound tree for a mixed-integer model in the figure below. Each node of the tree is numbered. The value below each node is the optimal solution value of the relaxation of the problem at the node, with the exception of node 8, for which the relaxation was infeasible. The value on top of node 2 indicates the value of a feasible solution found via a feasibility pump heuristic applied at that node.



Answer the following questions:

i) Is the problem a maximisation, minimisation or it is impossible to tell with the given information? Justify your answer.

ii) What are the tightest possible lower and upper bounds on the optimal solution of the problem?

iii) Which nodes can be prunned and which nodes need to be investigated further?

Answer:

i) Minimisation, as the value of the relaxed solution increases as we go down the tree and add branching constraints.

ii) The only upper bound is 32. The tightest lower bound is 27.

iii) Nodes 3,7 and 5 need to be investigated further. Nodes 6 and 8 can be prunned.

Question 5

Given a 0-1 knapsack set in the form $X = \{x \in \{0,1\}^n : \sum_{j=1}^n a_j x_j \leq b\Big\}$ with $a_j \geq 0$ for all j, answer TRUE, FALSE or 'I don't know' for each of the following statements.

i) $X = \emptyset$ if $b < 0$.

ii) $X = \{0,1\}^n$ if $\sum_{j=1}^n a_j \leq b$. In other words, the constraint is redundant.

iii) If $a_k > b$, then the problem is infeasible.

iv) If $a_k > b$, then $x_k = 0$ in all feasible solutions.

v) If $a_j = b$ for all $j = 1, \ldots, n$, then $\sum_{j=1}^n x_j \leq 1$ is a valid inequality for the problem.

vi) If $a_j + a_k > b$ with $j \neq k$, then $x_j + x_k \leq 1$ in all feasible solutions.

vii) If $x_j + x_k \leq 1$ and $x_j + \bar{x}_k \leq 1$, then $x_j = 0$.

viii) Given weigths $w_j > 0$ for each item j, lower and upper bounds for the problem are given by 0 and $\sum_{j=1}^n w_j$, respectively.

ix) If $x_j + x_k \leq 1$ and $\bar{x}_j + \bar{x}_k \leq 1$, then $x_j + x_k = 1$.

x) If $x_j + \bar{x}_k \leq 1$ and $\bar{x}_j + \bar{x}_k \leq 1$, then $x_k = 1$.

Note: You do **not** need to justify your answers. One point will be awarded for each correct answer and one point deducted for each incorrect answer. Points will not be awarded or deducted for the items you answer 'I don't know'.

Answer:
All statements are true except for (iii)

Question 6

Consider a linear maximisation problem on integer variables x_1, x_2. Answer the following questions:

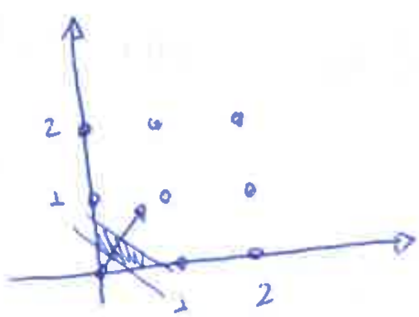
(i) Is it possible for the problem to have no feasible solution given that its linear relaxation has an optimal solution? If yes, draw the feasible space of such a problem and indicate the gradient of the objective function. If it is not possible, explain why.

(ii) Is it possible for the problem to have a single optimal solution while its linear relaxation has multiple optimal solutions? If yes, draw the feasible space of such a problem and indicate the gradient of the objective function. If it is not possible, explain why.

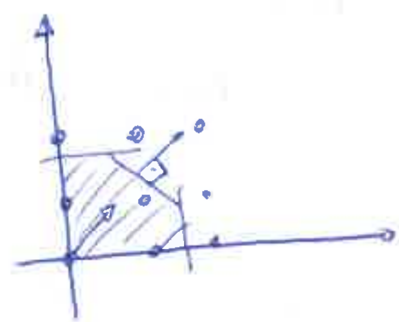
(iii) Is it possible for the problem to have an objective function value larger than the value of the optimal solution value of its linear relaxation ? If yes, draw the feasible space of such a problem and indicate the gradient of the objective function. If it is not possible, explain why.

(iv) Is it possible for the problem to have a feasible solution given that its linear relaxation has no feasible solution? If yes, draw the feasible space of such a problem and indicate the gradient of the objective function. If it is not possible, explain why.

(i) yes



(ii) Yes



(iii) No, by definition of a relaxation

(iv) No, by definition of a relaxation

Question 7

Prove that:

(i) The space defined by a set of equality constraints on variables $x \in \mathbb{R}^n$ is convex.

(ii) Given a primal-dual linear programming pair, the value of the objective function for the minimisation problem is always larger than or equal to the value of the objective function for the maximisation problem.

(i) a set of equality constraints can be written as $Ax = b$, let $X = \{x : Ax = b\}$

if $x_1 \in X$ and $x_2 \in X$

$$x_3 = \alpha x_1 + (1-\alpha) x_2 \qquad \text{with } 0 \leq \alpha \leq 1$$

then

$$Ax_3 = \alpha A x_1 + (1-\alpha) A x_2 = \alpha b + (1-\alpha) b = b$$

(ii) Given a primal dual pair:

~~min $c^T x$ s.t. $Ax \geq b$, $x \geq 0$~~ ~~max $b^T u$ s.t. $u^T A \leq c^T$~~, $u \geq 0$

max $c^T x$ s.t. $Ax \leq b$, $x \geq 0$

min $b^T u$ s.t. $u^T A \geq c^T$, $u \geq 0$

then:

~~$c^T x$~~ ~~$b^T u$~~

$$c^T x \leq (u^T A) x \leq u^T b = b^T u$$

Question 8

In the context of Benders decomposition, consider the Mixed Integer Program below:

$$\min y_1 + y_2 - x_1$$

s.t.

$$y_1 - x_1 \geq 2$$
$$y_2 + x_1 \geq 3$$
$$(y_1, y_2) \in \mathbb{Z}^2_+, x \geq 0$$

(i) Write the relaxed master problem.

(ii) Write the primal subproblem.

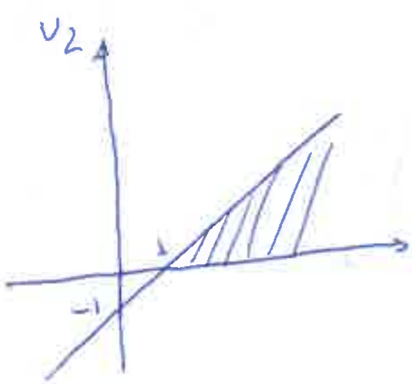
(iii) Write the dual suproblem and identify all extreme ray(s) and extreme point(s).

(iv) Write the full Benders reformulation.

(i) Min $y_1 + y_2 + z$
s.t
$z \geq -M$

(ii) Min $-x_1$
s.t. $-x_1 \geq 2 - \bar{y}_1$
$x_1 \geq 3 - \bar{y}_2$
$x_1 \in \mathbb{R}^+$

(iii) Max $(2-\bar{y}_1)v_1 + (3-\bar{y}_2)v_2$
s.t. $-v_1 + v_2 \leq -1$



rays: $(1,0)$
$(1,1)$

ext. points $(1,0)$

(iv) Min $y_1 + y_2 + z$
$z \geq 2 - y_1$
$2 - y_1 \leq 0$
$5 - y_1 - y_2 \leq 0$
z free, $y_1, y_2 \in \mathbb{Z}_+$

Question 9

Consider the uncapacitated vehicle routing problem with n clients. The goal is to visit all clients while minimising the total cost of the routes (measured as the total distance travelled). Any number of vehicles can be used. A column generation is proposed with columns in the format:

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_i \\ \vdots \\ a_n \end{bmatrix}$$

where parameters $a_i, i = 1, \ldots, n$ is binary and indicates if customer i is visited in the route associated with the column ($a_i = 1$) or not ($a_i = 0$).

(i) Which problem do you need to solve to obtain the cost of a route associated with a column?

(ii) Let P be the set of all possible columns. Consider binary variables x_p indicating if column $p \in P$ is used and write a mixed integer program for the problem. Clearly explain the meaning of any parameters you define.

(iii) Propose a set of columns that can be used to start the solution of the problem using column generation.

(i) TSP

(ii) • Let $a_{ip} = 1$ if customer i is visited by the route associated with column p

• Let c_p be the cost of column p

End of Exam

$$\min \sum_{p \in P} c_p x_p$$

s.t.

$$\sum a_{ip} x_p = 1, \quad i = 1, \ldots, n$$

$$x_p \in \{0, 1\}$$

(iii) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

15 2023 - Semester 1

15.1 Exam

Student number

Semester 1 Assessment, 2023

School of Mathematics and Statistics

MAST90014 Optimisation for Industry

Reading time: 30 minutes — Writing time: 2 hours

This exam consists of 20 pages (including this page) with 9 questions and 100 total marks

Permitted Materials

- One double sided A4 page of notes (handwritten or printed).
- A Casio FX-82 calculator is permitted.
- Any mobile phones or internet-enabled devices brought into the exam room must be **turned off** and placed on the floor under your table.

Instructions to Students

- Unless stated otherwise, always make sure that your models for each modeling problem rely solely on linear constraints and a linear objective function.
- Explain all your working.
- Write your answers in the boxes provided on the exam. There is extra space you can use for answers to any question commencing on page 18. If you still do not have enough space, tick the box near the bottom of page 20 and request a booklet from an invigilator—include the question number at the top of each page in the booklet.
- You must NOT remove this question paper, or any booklets provided to you, at the conclusion of the examination.

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Question 1 (20 marks)

Let P, W, A_w and K be the set of plants, warehouses, areas of an warehouse $w \in W$, and commodities involved in a production/transportation problem. Also, let set $T = \{1, \ldots, 4\}$ be the set of planning periods in the problem. The following parameters are provided:

l_{pw} : distance from plant $p \in P$ to warehouse $w \in W$,

l_{wc} : distance from warehouse $w \in W$ to client $c \in C$,

d_{ck} : demand of client $c \in C$ for product $k \in K$ (at each period of the planning horizon),

f_p: capacity (in standard units) of plant $p \in P$ (at each period of the planning horizon),

f_{wa}: capacity of area $a \in A_w$ of warehouse $w \in W$.

Define the following variables:

x^1_{pwakt}: amount of commodity k produced at plant p and transported to area a of warehouse w at period t,

x^2_{wackt}: amount of commodity k transported fom area a of warehouse w to client c at period t,

I_{wakt}: amount of commodity k stored at area a of warehouse w at the end of period t. (Assume $I_{wak0} = 0$).

Now, consider the following mixed-integer programming formulation which models a multi-period production and transportation problem.

$$\begin{aligned} \min \quad & \sum_{p \in P} \sum_{w \in W} \sum_{a \in A} \sum_{k \in K} \sum_{t \in T} l_{pw} \cdot x^1_{pwakt} \\ & + \sum_{w \in W} \sum_{a \in A} \sum_{c \in C} \sum_{k \in K} \sum_{t \in T} l_{wc} \cdot x^2_{wackt} \end{aligned}$$

s.t.

$$\sum_{w \in W} \sum_{a \in A_w} \sum_{k \in K} x^1_{pwakt} \leq f_p, \qquad p \in P, t \in T, \tag{1}$$

$$\sum_{w \in W} \sum_{a \in A_w} x^2_{wackt} \geq d_{ck}, \qquad c \in C, k \in K, t \in T, \tag{2}$$

$$\sum_{p \in P} x^1_{pwakt} + I_{wak(t-1)} - \sum_{c \in C} x^2_{wackt} = I_{wakt}, \qquad w \in W, a \in A_w, k \in K, t \in T, \tag{3}$$

(a) Briefly explain the meaning of the objective function and each of the constraints.

(b) Assume that for each product k produced in plant p on period t you need a setup process. A different setup must be made once for each type of product produced in the period. The setup costs depend on the product and on the plant it is produced and is given by st_{pk}. Also, the setup takes time and consumes sc_{pk} units of standard capacity of plant p when producing commodity k.

Introduce or modify variables, constraints and/or the objective function to consider the following addition to the model. Properly define any additional parameters you use and explain the meaning of each constraint and/or objective function.

(c) Assume that if two different products are stored in consecutive periods in the same area of the warehouse, a deep clean of the warehouse needs to be conducted to avoid cross-contamination. The cost for this cleaning process depends on the products and is given by $cl_{kl}, k, l \in K$.

Introduce or modify variables, constraints and/or the objective function to consider the following addition to the model. Properly define any additional parameters you use and explain the meaning of each constraint and/or objective function.

(d) Assume that the company wants to work with backlogs. That is, it might not supply all demand for a commodity at a given period. However, it has to pay a backlog cost given by b_k per unity of commodity that is delivered late per period it is late. All demand must have been supplied by the end of the last period.

Introduce or modify variables, constraints and/or the objective function to consider the following addition to the model. Properly define any additional parameters you use and explain the meaning of each constraint and/or objective function.

Question 2 (12 marks)

Sudoku is a logic-based number puzzle game that involves filling a 9x9 grid with digits from 1 to 9. The grid is divided into 9 smaller 3x3 grids called regions, and each of the regions must contain all the digits from 1 to 9. The game starts with some of the cells (i, j) in the grid already filled in with a number $k = 1 \leq n \leq 9$. The goal is to fill in the remaining cells such that each row, column, and region must contain all the digits from 1 to 9 without any repetition. The figure below shows a Sudoku puzzle (on the left) and a valid solution (on the right).

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

(a) Define parameters and variables, and model Sudoku as an optimisation problem on binary variables. Explain.

(b) Let the following be a solution for a Sudoku puzzle: $P = \{(i,j,k) \mid \text{cell (i,j) contains number } k\}$. Add a constraint to your model in (a) so that it will find an alternative solution (if one exists) or make the problem infeasible (in case an alternative solution does not exist).

Question 3 (8 marks)

The constraints below are part of a feasible linear program. Let u_1 and u_2 be the dual values associated with the constraints.

$$
\begin{aligned}
&2x_1 + 5x_2 + 3x_3 \leq 10, && (u_1)\\
&4x_1 + ax_2 + 6x_3 \leq 21, && (u_2)\\
&x_1, x_2, x_3 \geq 0.
\end{aligned}
$$

(a) What is the value of u_2 when $a = 10$? Explain.

(b) What is the value of u_2 when $a = 9$? Explain.

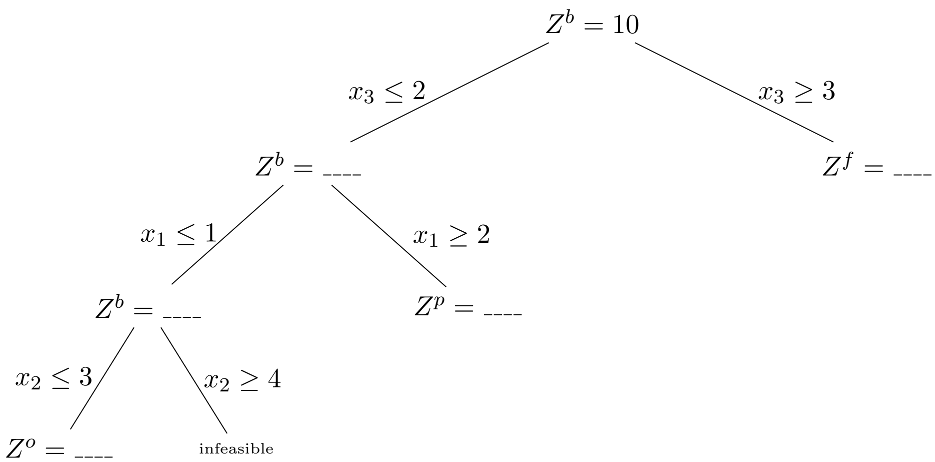
Question 4 (5 marks)

We consider a **minimisation** problem on integer variables x_1, x_2, x_3. A partial branch-and-bound tree for the problem is presented below. In the figure, we use the notation:

> Z^b: optimal value of the linear problem at this node. In this partial tree, this node has already been branched.
>
> Z^f: optimal value of the linear problem at this node. In this partial tree, this node is feasible and has not been pruned.
>
> Z^o: optimal value of the linear problem at this node. In this partial tree, this node has all variables integer.
>
> Z^p: optimal value of the linear problem at this node. In this partial tree, this node has been pruned by quality.

Fill the spots marked as ____ with possible values for the optimal solution of the linear problem at each node.

Note: you can ignore the answer box and present your answer directly in the figure provided below.



Question 5 (10 marks)

Consider the following optimization problem:

$$\begin{array}{ll} \text{maximize} & f(x) = 2x_1 + 5x_2 \\ \text{s.t.} & x_1 + 2x_2 \leq 5.3, \\ & x_2 \leq 2.5, \\ & x_1, x_2 \geq 0, \text{and integer.} \end{array}$$

(a) Solve the problem using the branch and bound method. You must use the best-bound rule for node selection and the most fractional rule for variable selection. Number the nodes as you explore them and indicate the optimal solution and optimal solution value at each node. Also indicate infeasible nodes, nodes pruned by integrality and nodes pruned by quality.

Question 6 (10 marks)

Consider the following optimisation problem:

$$\begin{array}{ll} \text{maximize} & 2x + 3y \\ \text{s.t.} & x + 2y \leq 3, \\ & 4x + 5y \leq 10, \\ & x, y \geq 0, \text{and integer.} \end{array}$$

(a) Write the problem with equality constraints using slack variables s_1 and s_2.

(b) Which are the basic variables at the optimal solution? Explain your working.

(c) Write the basic and non-basic matrices at the optimal solution.

(d) Create one Chvatal-Gomory cut and write it in terms of the original variables of the problem.

Question 7 (5 marks)

Consider the following optimisation problem:

$$\begin{aligned} &\text{Maximise } x_1 + x_2 \\ &\text{Subject to} \\ &2x_1 + x_2 \leq 4, \\ &x_1 + 3x_2 \leq 3, \\ &x_1, x_2 \in \mathbb{Z}_0^+. \end{aligned}$$

Apply one iteration of the Feasibility Pump method after the solution on the root node is obtained. Does it obtain a feasible solution? You can use a figure and a simple geometric analysis to explain the steps executed.

Question 8 (15 marks)

You are responsible for planning the production of a farm and must decide what to plant in each of its plots (also deciding the size of the plots) this coming year. The plots have the same productivities and you want to minimise the total area you use, knowing that you have to meet a given demand for each crop. For technical reasons (easiness of care), each plot has a single crop at a given time. For each crop i you have the following information:

d_i: annual demand of crop i;

t_i: growing time of crop i (in months);

p_i: annual productivity of crop i per unity of area (m^2);

f_i: botanical family of crop i.

You do not want to plant two crops of the same botanical family in the same plot more than twice in the year, since that would increase the incidence of plagues.

(a) Model the problem of deciding the crops to grow in order to minimise the needed area while respecting the demands. Use integer variables x_{ij} to indicate how many times crop i is planted in plot j and continuous variables y_j to indicate the area (in m^2) of plot j. Define any new parameters you use. **For this particular question, it is acceptable to have a non-linear model**.

Explain the meaning of the objective function and each constraint.

(b) Present a pattern-based formulation (Explain what is a column in your case).

Question 9 (15 marks) Consider the following mixed-integer programming model.

$$\text{Minimise } z = 4x_1 + 3x_2 + 2y_1 + 5y_2$$

Subject to:

$$\begin{aligned} 2x_1 + x_2 + y_1 &\leq 8, \\ x_1 + 3x_2 + 2y_1 + 3y_2 &\leq 20, \\ x_1, x_2 &\geq 0, \\ y_1, y_2 &\in \{0, 1\}. \end{aligned}$$

In the context of Benders decomposition: assume that integer variables x_1 and x_2 are kept in the master problem and the continuous variables are relegated to the subproblem.

(a) Write the master problem.

(b) Write the subproblem.

(c) Write the dual subproblem.

(d) Draw the feasible space of the dual problem on the plane (u_1, u_2) and list all its extreme points.

(e) Write the Benders cut associated with the master solution $y_1 = 0$, $y_2 = 0$.

Additional writing space for any question commences on the next page

Additional answer space for any question—submit this page even if blank

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End of Exam — Total Available Marks = 100

You must tick this box if you have used extra booklets ☐

15.2 Solution

THE UNIVERSITY OF MELBOURNE

Student number

Semester 1 Assessment, 2023

School of Mathematics and Statistics

MAST90014 Optimisation for Industry

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l_{wc} : distance from warehouse $w \in W$ to client $c \in C$,

d_{ck} : demand of client $c \in C$ for product $k \in K$ (at each period of the planning horizon),

f_p: capacity (in standard units) of plant $p \in P$ (at each period of the planning horizon),

f_{wa}: capacity of area $a \in A_w$ of warehouse $w \in W$.

Define the following variables:

x^1_{pwakt}: amount of commodity k produced at plant p and transported to area a of warehouse w at period t,

x^2_{wackt}: amount of commodity k transported fom area a of warehouse w to client c at period t,

I_{wakt}: amount of commodity k stored at area a of warehouse w at the end of period t. (Assume $I_{wak0} = 0$).

Now, consider the following mixed-integer programming formulation which models a multi-period production and transportation problem.

$$\begin{aligned} \min \quad & \sum_{p \in P} \sum_{w \in W} \sum_{a \in A} \sum_{k \in K} \sum_{t \in T} l_{pw} \cdot x^1_{pwakt} \\ & + \sum_{w \in W} \sum_{a \in A} \sum_{c \in C} \sum_{k \in K} \sum_{t \in T} l_{wc} \cdot x^2_{wackt} \end{aligned}$$

s.t.

$$\sum_{w \in W} \sum_{a \in A_w} \sum_{k \in K} x^1_{pwakt} \leq f_p, \qquad p \in P, t \in T, \tag{1}$$

$$\sum_{w \in W} \sum_{a \in A_w} x^2_{wackt} \geq d_{ck}, \qquad c \in C, k \in K, t \in T, \tag{2}$$

$$\sum_{p \in P} x^1_{pwakt} + I_{wak(t-1)} - \sum_{c \in C} x^2_{wackt} = I_{wakt}, \qquad w \in W, a \in A_w, k \in K, t \in T, \tag{3}$$

(a) Briefly explain the meaning of the objective function and each of the constraints.

The objective function minimises the transportation (combined transported distance) costs between plants and warehouses (first tem) and between warehouses and clients (second term).

Constraints (1) state that the capacity of each plant at each period must be respected.

Constraints (2) state that the demand of client c for commodity k must be delivered at each period t.

Constraints (3) impose the balance of flow for each commodity k at each area of the warehouse a is respected at each period of the planning horizon, t.

Note that in equation (3), for $t = 1$, we need to define $I_{wak(0)} = 0$.

(b) Assume that for each product k produced in plant p on period t you need a setup process. A different setup must be made once for each type of product produced in the period. The setup costs depend on the product and on the plant it is produced and is given by st_{pk}. Also, the setup takes time and consumes sc_{pk} units of standard capacity of plant p when producing commodity k.

Introduce or modify variables, constraints and/or the objective function to consider the following addition to the model. Properly define any additional parameters you use and explain the meaning of each constraint and/or objective function.

The following new variables are needed:

- q_{pkt}, binary, equal to 1 if plant p produces product k in period t.

The following new constraints are needed:

$$\sum_{w \in W} \sum_{a \in A} x_{pwakt} \leq (f_p - sc_{pk}) q_{pkt}, \qquad p \in P, k \in K, t \in T.$$

$$\sum_{w \in W} \sum_{a \in A} \sum_{k \in K} x_{pwakt} + sc_{pk} \cdot q_{pkt} \leq f_p, \qquad p \in P, t \in T.$$

The first constraints ensure that a product is only produced if the setup cost is paid. The second constraints ensure that the capacity at the plants are respected.

The new objective has the additional term:

$$\sum_{p \in P} \sum_{w \in W} \sum_{t \in T} st_{pk} \cdot q_{pkt}$$

(c) Assume that if two different products are stored in consecutive periods in the same area of the warehouse, a deep clean of the warehouse needs to be conducted to avoid cross-contamination. The cost for this cleaning process depends on the products and is given by $cl_{kl}, k, l \in K$.

Introduce or modify variables, constraints and/or the objective function to consider the following addition to the model. Properly define any additional parameters you use and explain the meaning of each constraint and/or objective function.

The following new variables are needed:

- y_{wakt}, binary, equal to 1 if area a of warehouse w contains product k at the end of period t.
- z_{waklt}, binary, equal to 1 if area a of warehouse w contains product k at the end of period $t-1$ and product $l \neq k$ at the end of period t.

The following new constraints are needed:

$$\begin{aligned} I_{wakt} &\leq My_{wakt}, && w \in W, a \in A, k \in K, t \in T. \\ z_{waklt} &\geq y_{wak(t-1)} + y_{walt} - 1, && w \in W, a \in A, k, l \in K, k \neq l, t \in T. \end{aligned}$$

The constraints link the new variables to the previous binary variables y, ensuring their meaning as described above. Note that for constraints with $t = 0$, we define $y_{wak0} = 0$.
The new objective has the additional term:

$$\sum_{w \in W} \sum_{a \in A} \sum_{k \in K} \sum_{l \in K, l \neq k} \sum_{t \in T} cl_{kl} \cdot z_{waklt}$$

(d) Assume that the company wants to work with backlogs. That is, it might not supply all demand for a commodity at a given period. However, it has to pay a backlog cost given by b_k per unity of commodity that is delivered late per period it is late. All demand must have been supplied by the end of the last period.

Introduce or modify variables, constraints and/or the objective function to consider the following addition to the model. Properly define any additional parameters you use and explain the meaning of each constraint and/or objective function.

The following new variables are needed:

- b_{kct}, continuous, the backlog of product k at client c at the end of period t, $b_{kc(-1)} = 0$

Constraints 2 are modified to:

$$\sum_{w \in W} \sum_{a \in A_w} x^2_{wackt} - b_{kc(t-1)} = d_{ck} + b_{kct}, \quad c \in C, k \in K, t \in T,$$

The new objective has the additional term:

$$\sum_{c \in C} \sum_{k \in K} \sum_{t \in T} b_k \cdot b_{kct}$$

Question 2 (12 marks)

Sudoku is a logic-based number puzzle game that involves filling a 9x9 grid with digits from 1 to 9. The grid is divided into 9 smaller 3x3 grids called regions, and each of the regions must contain all the digits from 1 to 9. The game starts with some of the cells (i, j) in the grid already filled in with a number $k = 1 \leq n \leq 9$. The goal is to fill in the remaining cells such that each row, column, and region must contain all the digits from 1 to 9 without any repetition. The figure below shows a Sudoku puzzle (on the left) and a valid solution (on the right).

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

(a) Define parameters and variables, and model Sudoku as an optimisation problem on binary variables. Explain.

Let x_{ijk} be a binary decision variable that takes the value 1 if the number k is assigned to cell (i,j), and 0 otherwise.

For a standard 9x9 Sudoku puzzle, i,j,k all range from 1 to 9.
Sudoku is a feasibity problem, no objective function needs to be defined. The constraints read:

$$\sum_{k=1}^{9} x_{ijk} = 1, \qquad i, j \in \{1, 2, ..., 9\}^2, \tag{4}$$

$$\sum_{i=1}^{9} x_{ijk} = 1, \qquad j, k \in \{1, 2, ..., 9\}^2, \tag{5}$$

$$\sum_{i=3p+1}^{3p+3} \sum_{j=3q+1}^{3q+3} x_{ijk} = 1, \qquad p, q, k \in \{0, 1, 2\}. \tag{6}$$

Each cell must have exactly one number assigned to it, each row must have exactly one occurrence of each number, each column must have exactly one occurrence of each number, and each region (3x3 sub-grid) must have exactly one occurrence of each number.

For pre-filled cells, we impose that the corresponding variables x_{ijk} must be equal to 1.

(b) Let the following be a solution for a Sudoku puzzle: $P = \{(i, j, k) \mid \text{cell (i,j) contains number } k\}$. Add a constraint to your model in (a) so that it will find an alternative solution (if one exists) or make the problem infeasible (in case an alternative solution does not exist).

$$\sum_{(i,j,k)\in P} x_{ijk} \leq 80.$$

This constraint will ensure that the original solution is not repeated.

Question 3 (8 marks)

The constraints below are part of a feasible linear program. Let u_1 and u_2 be the dual values associated with the constraints.

$$\begin{aligned} &2x_1 + 5x_2 + 3x_3 \leq 10, && (u_1)\\ &4x_1 + ax_2 + 6x_3 \leq 21, && (u_2)\\ &x_1, x_2, x_3 \geq 0. \end{aligned}$$

(a) What is the value of u_2 when $a = 10$? Explain.

$u_2 = 0$.

For no value $a \leq 10$, the second constraint will be active in the presence of the first constraint. Non-active constraints always have the associated dual variable equal to 0.

(b) What is the value of u_2 when $a = 9$? Explain.

$u_2 = 0$.

For no value $a \leq 10$, the second constraint will be active in the presence of the first constraint. Non-active constraints always have the associated dual variable equal to 0.

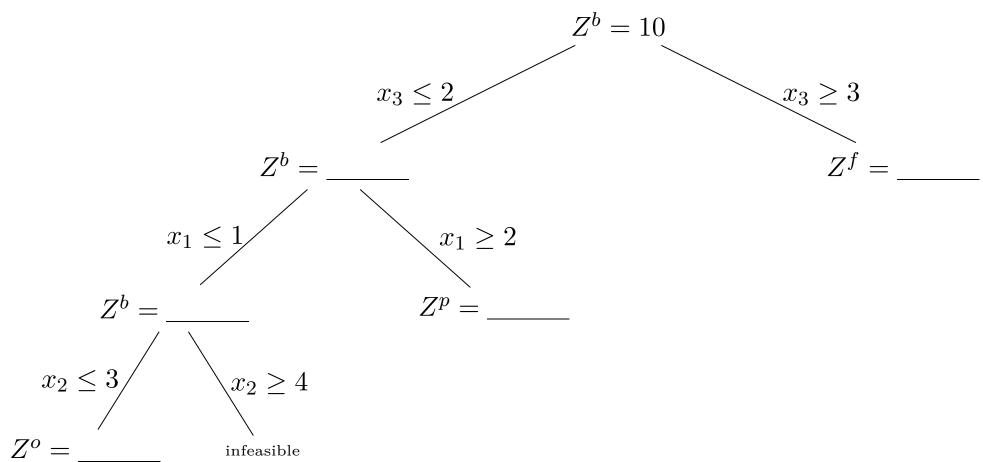
Question 4 (5 marks)

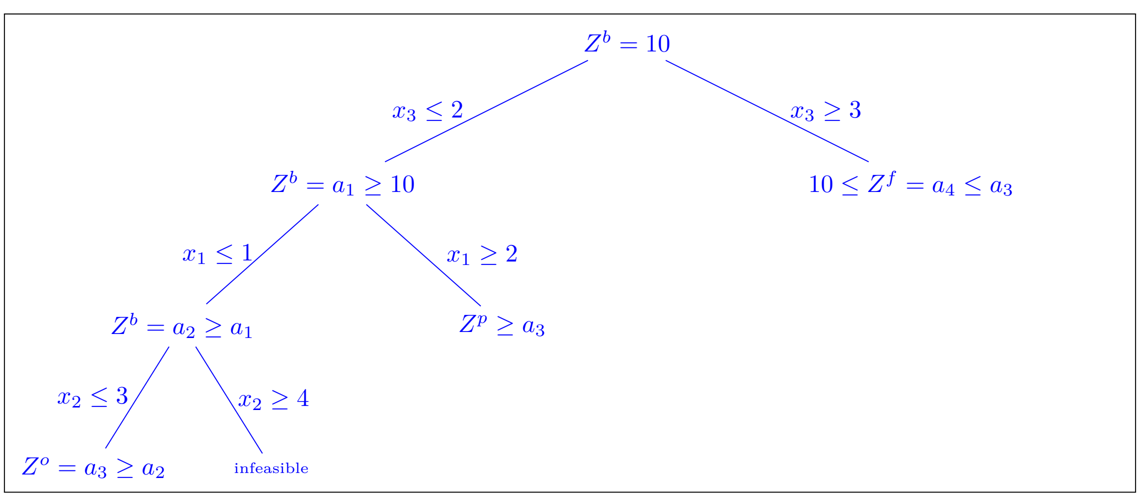
We consider a **minimisation** problem on integer variables x_1, x_2, x_3. A partial branch-and-bound tree for the problem is presented below. In the figure, we use the notation:

> Z^b: optimal value of the linear problem at this node. In this partial tree, this node has already been branched.
>
> Z^f: optimal value of the linear problem at this node. In this partial tree, this node is feasible and has not been pruned.
>
> Z^o: optimal value of the linear problem at this node. In this partial tree, this node has all variables integer.
>
> Z^p: optimal value of the linear problem at this node. In this partial tree, this node has been pruned by quality.

Fill the spots marked as ______ with possible values for the optimal solution of the linear problem at each node.

Note: you can ignore the answer box and present your answer directly in the figure provided below.



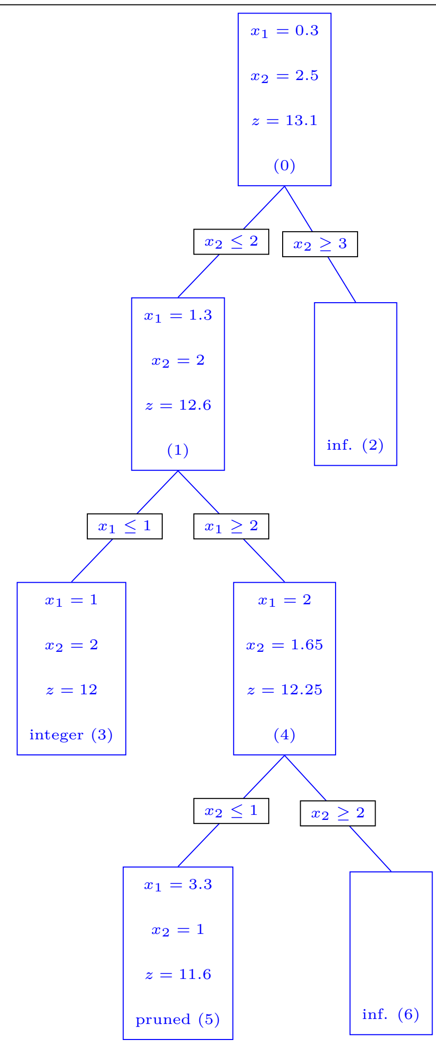


Question 5 (10 marks)

Consider the following optimization problem:

$$\begin{array}{ll} \text{maximize} & f(x) = 2x_1 + 5x_2 \\ \text{s.t.} & x_1 + 2x_2 \leq 5.3, \\ & x_2 \leq 2.5, \\ & x_1, x_2 \geq 0, \text{and integer.} \end{array}$$

(a) Solve the problem using the branch and bound method. You must use the best-bound rule for node selection and the most fractional rule for variable selection. Number the nodes as you explore them and indicate the optimal solution and optimal solution value at each node. Also indicate infeasible nodes, nodes pruned by integrality and nodes pruned by quality.



Note that we could also prune node (4) by quality as we know the objective function must be integer.

Question 6 (10 marks)

Consider the following optimisation problem:

$$\begin{aligned} \text{maximize} \quad & 2x + 3y \\ \text{s.t.} \quad & x + 2y \le 3, \\ & 4x + 5y \le 10, \\ & x, y \ge 0, \text{and integer.} \end{aligned}$$

(a) Write the problem with equality constraints using slack variables s_1 and s_2.

$$\begin{aligned} \text{maximize} \quad & 2x + 3y \\ \text{s.t.} \quad & x + 2y + s_1 = 3, \\ & 4x + 5y + s_2 = 10, \\ & x, y \ge 0, \text{and integer.} \end{aligned}$$

(b) Which are the basic variables at the optimal solution? Explain your working.

Using the graphical method, we can see that the basic variables are x and y and the optimal solution is $x = 5/3$ and $y = 2/3$.

(c) Write the basic and non-basic matrices at the optimal solution.

$B = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}, N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(d) Create one Chvatal-Gomory cut and write it in terms of the original variables of the problem.

The system reads:

$x_B = B^{-1}b - B^{-1}Nx_N$

$x_B = \begin{bmatrix} -\frac{5}{3} & \frac{2}{3} \\ \frac{4}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 3 \\ 10 \end{bmatrix} - \begin{bmatrix} -\frac{5}{3} & \frac{2}{3} \\ \frac{4}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_N$

$x_B = \begin{bmatrix} \frac{5}{3} \\ \frac{2}{3} \end{bmatrix} + \begin{bmatrix} -\frac{5}{3} & \frac{2}{3} \\ \frac{4}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$

from the first row, we can read the Chvatal-Gomory cut:

$\frac{1}{3}s_1 + \frac{2}{3}s_2 \geq \frac{2}{3}$

which written in the original variables read:

$\frac{1}{3}(3 - x - 2y) + \frac{2}{3}(10 - 4x - 5y) \geq \frac{2}{3}$

$-3x + -4y \geq -7$

$3x + 4y \leq 7$

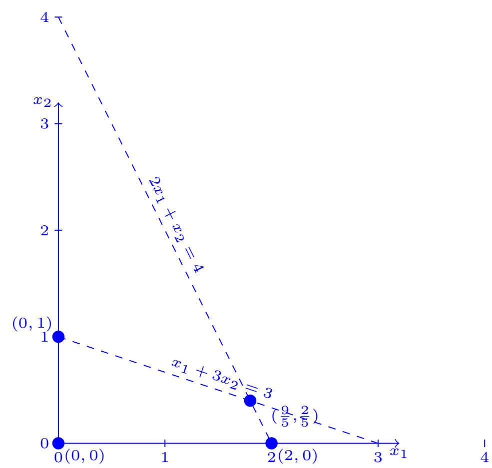
Question 7 (5 marks)

Consider the following optimisation problem:

$$\begin{aligned} &\text{Maximise } x_1 + x_2 \\ &\text{Subject to} \\ &\quad 2x_1 + x_2 \leq 4, \\ &\quad x_1 + 3x_2 \leq 3, \\ &\quad x_1, x_2 \in \mathbb{Z}_0^+. \end{aligned}$$

Apply one iteration of the Feasibility Pump method after the solution on the root node is obtained. Does it obtain a feasible solution? You can use a figure and a simple geometric analysis to explain the steps executed.

The feasible solution of the problem is depicted below:



The optimal linear solution is $(x, y) = (\frac{9}{5}, \frac{2}{5})$. In the FP method, it is then rounded to $(2, 1)$. As the closest solution to point $(2, 1)$ is still $(\frac{9}{5}, \frac{2}{5})$, the FP method will return to the original fractional solution and do not provide a feasible integer solution.

Question 8 (15 marks)

You are responsible for planning the production of a farm and must decide what to plant in each of its plots (also deciding the size of the plots) this coming year. The plots have the same productivities and you want to minimise the total area you use, knowing that you have to meet a given demand for each crop. For technical reasons (easiness of care), each plot has a single crop at a given time. For each crop i you have the following information:

d_i: annual demand of crop i;

t_i: growing time of crop i (in months);

p_i: annual productivity of crop i per unity of area (m^2);

f_i: botanical family of crop i.

You do not want to plant two crops of the same botanical family in the same plot more than twice in the year, since that would increase the incidence of plagues.

(a) Model the problem of deciding the crops to grow in order to minimise the needed area while respecting the demands. Use integer variables x_{ij} to indicate how many times crop i is planted in plot j and continuous variables y_j to indicate the area (in m^2) of plot j. Define any new parameters you use. **For this particular question, it is acceptable to have a non-linear model**.

Explain the meaning of the objective function and each constraint.

a) With the suggested variables, we can write:

$$\text{Min } \sum_{j=1}^{m} y_j \tag{7}$$

s.t.:

$$\sum_{i=1}^{n} t_i x_{ij} \leq 12, \qquad j = 1, \ldots, m, \tag{8}$$

$$\sum_{j=1}^{m} p_i x_{ij} y_j \geq d_i, \qquad i = 1, \ldots, n, \tag{9}$$

$$x_{ij} \in \{0, 1, 2\}, y_j \geq 0. \tag{10}$$

The problem minimises the total area. Constraints (8) ensure that there is time available to plant all crops that are assigned to a plot while constraints (9) ensure that the total production meets the demand. Constraints (10) define the scope of the variables. Parameter n is the number of crops and parameter m is an upper bound on the number of plots.

We also need a constraint to avoid two crops of different families at the same plot. This can be obtained by introducing an indicator binary variable if a crop of family f is at the plot and limiting the sum of all these binary variables to one.

(b) Present a pattern-based formulation (Explain what is a column in your case).

A column refers to a plot and indicates how many times each crop is planted in that plot in the year. For example, a column:

$$\left[\begin{array}{cccccccc} 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]^t$$

indicates that crop 1 (crisp head lettuce, in the example) is planted twice in the year while crop 5 (tomato) is planted once. Note that this column respects both the planting time constraints (total planting time: 10 months) and the condition on not having more than two crops of the same family being planted in the same crop.
Given all feasible columns j:

$$a_j = \left[\begin{array}{cccccccc} a_{j1} & a_{j2} & a_{j3} & a_{j4} & a_{j5} & a_{j6} & a_{j7} & a_{j8} \end{array}\right]^t$$

We can write the optimisation problem as:

$$\text{Min} \sum_{j=1}^{m} y_j \tag{11}$$

s.t.:

$$\sum_{j=1}^{m} p_i a_{ij} y_j \geq d_i, \qquad i = 1, \ldots, n, \tag{12}$$

Note that this is no longer nonlinear (since the decision on the crops to plant in a given plot is already made and expressed by the many columns). Also note that we no longer need to worry about the time constraints (it is already implicitly respected if we construct only the appropriate columns).

Question 9 (15 marks) Consider the following mixed-integer programming model.

$$\text{Minimise } z = 4x_1 + 3x_2 + 2y_1 + 5y_2$$

Subject to:

$$\begin{aligned} 2x_1 + x_2 + y_1 &\leq 8, \\ x_1 + 3x_2 + 2y_1 + 3y_2 &\leq 20, \\ x_1, x_2 &\geq 0, \\ y_1, y_2 &\in \{0, 1\}. \end{aligned}$$

In the context of Benders decomposition: assume that integer variables x_1 and x_2 are kept in the master problem and the continuous variables are relegated to the subproblem.

(a) Write the master problem.

$$\begin{aligned} &\text{Minimise } z = 2y_1 + 5y_2 + z \\ &\text{Subject to} \\ &\qquad z \geq -M, \\ &\qquad y_1, y_2 \in \{0, 1\}. \end{aligned}$$

(b) Write the subproblem.

$$\begin{aligned} &\text{Minimise } 4x_1 + 3x_2 \\ &\text{Subject to} \\ &\quad 2x_1 + x_2 \leq 8 - \overline{y}_1, \\ &\quad x_1 + 3x_2 \leq 20 - 2\overline{y}_1 - 3\overline{y}_2, \\ &\quad x_1, x_2 \geq 0. \end{aligned}$$

or

$$\begin{aligned} &\text{Minimise } 4x_1 + 3x_2 \\ &\text{Subject to} \\ &\quad -2x_1 - x_2 \geq \overline{y}_1 - 8, \\ &\quad -x_1 - 3x_2 \geq 2\overline{y}_1 + 3\overline{y}_2 - 20, \\ &\quad x_1, x_2 \geq 0. \end{aligned}$$

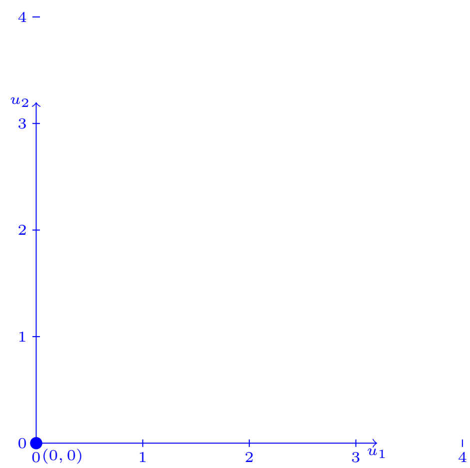
(c) Write the dual subproblem.

$$\begin{aligned}&\text{Maximise } (\overline{y}_1 - 8)u_1 + (2\overline{y}_1 + 3\overline{y}_2 - 20)u_2\\&\text{Subject to}\\&-2u_1 - u_2 \le 4,\\&-u_1 - 3u_2 \le 3,\\&u_1 + u_2 \ge 0.\end{aligned}$$

$=$

$$\begin{aligned}&\text{Maximise } (\overline{y}_1 - 8)u_1 + (2\overline{y}_1 + 3\overline{y}_2 - 20)u_2\\&\text{Subject to}\\&2u_1 + u_2 \ge -4,\\&u_1 + 3u_2 \ge -3,\\&u_1 + u_2 \ge 0.\end{aligned}$$

(d) Draw the feasible space of the dual problem on the plane (u_1, u_2) and list all its extreme points.



(e) Write the Benders cut associated with the master solution $y_1 = 0$, $y_2 = 0$.

For $y_1 = 0$, $y_2 = 0$ the dual objective function reads: Max $-8u_1 - 20u_2$. This yields and optimal dual point $(u_1, u_2) = (0, 0)$, yielding a Benders optimality cut:

$$z \ge 0$$

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End of Exam — Total Available Marks = 100

You must tick this box if you have used extra booklets ☐