4.3

Homogeneous Linear Equations with constant Coefficients:

Auxiliary Equation

Consider the second order equation

$$ay'' + by' + cy = 0 \tag{1}$$

Where $a, b,$ & c are constant.

Consider the solution of the form $y = e^{mx}$ then

$$y' = me^{mx}, \quad y'' = m^2 e^{mx}.$$

Substituting in (1)

$$\Rightarrow am^2 e^{mx} + bme^{mx} + ce^{mx} = 0$$

$$(am^2 + bm + c)\, e^{mx} = 0$$

As $e^{mx} \neq 0 \ \forall\, x$ so we can find m from

$$am^2 + bm + c = 0 \tag{3}$$

Where Eq. (3) is called auxiliary Equation of the differential Equation.

Since Eq. (3) is a quadratic, it has two roots

$$m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

There will be three forms of general solution of Eq. (1) corresponding to three cases.

* m_1 & m_2 real and distinct $(b^2 - 4ac > 0)$
* m_1 & m_2 are real & equal $(b^2 - 4ac = 0)$
* m_1 & m_2 are conjugate complex number $(b^2 - 4ac < 0)$

Homogeneous Linear Equations with constant Coefficients:

Auxiliary Equation

Consider the second order equation

$$ay'' + by' + cy = 0 \qquad (1)$$

Where $a, b, \&\ c$ are constant.
Consider the solution of the form $y = e^{mx}$ then

$$y' = me^{mx}, \quad y'' = m^2 e^{mx}.$$

Substituting in (1)

$$\Rightarrow am^2 e^{mx} + bme^{mx} + ce^{mx} = 0$$

$$(am^2 + bm + c) e^{mx} = 0$$

As $e^{mx} \neq 0 \ \forall x$ so we can find m from

$$am^2 + bm + c = 0 \qquad (3)$$

Where Eq. (3) is called auxiliary equation of the differential Equation.
Since Eq. (3) is a quadratic, it has two roots

$$m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

There will be three forms of general solution of Eq. (1) corresponding to three cases.

* m_1 & m_2 real and distinct $(b^2 - 4ac > 0)$
* m_1 & m_2 are real & equal $(b^2 - 4ac = 0)$
* m_1 & m_2 are conjugate complex number $(b^2 - 4ac < 0)$

Case I Distinct real roots.

If auxiliary equation has two unequal real roots m_1 & m_2, we get two solutions, $y_1 = e^{m_1 x}$ & $y_2 = e^{m_2 x}$. These functions are linearly independent, so they form fundamental set. So the general solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

Case II Repeated Real Roots.

When $m_1 = m_2$, we have an exponential solution $y_1 = e^{m_1 x}$ and other will be $y_2 = x e^{m_1 x}$. So general solution is

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$$

Case: III Conjugate complex roots

If m_1 & m_2 are complex, then we write $m_1 = \alpha + i\beta$ & $m_2 = \alpha - i\beta$, where α & $\beta > 0$ are real and $i^2 = -1$. So general solution is

$$y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}$$

$$= c_1 e^{\alpha x} e^{i\beta x} + c_2 e^{\alpha x} e^{-i\beta x}.$$

using Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$\Rightarrow$

we can convert the complex exponents as

$$y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$$

Example: Solve the following DE.

(a) $2y'' - 5y' - 3y = 0$

writing the auxiliary Eq.

$2m^2 - 5m - 3 = 0$

$2m - 6m + m - 3 = 0 \Rightarrow 2m(m-3) + 1(m-3) = 0$

$\Rightarrow (2m+1)(m-3) = 0$

$m_1 = -\frac{1}{2}, \quad m_2 = 3$

So general solution is

$y = c_1 e^{-x/2} + c_2 e^{3x}$

b) $y'' - 10y' + 25 = 0$

writing auxiliary Eq

$m^2 - 10m + 25 = 0$

$\Rightarrow (m-5)^2 = 0 \Rightarrow m_1 = 5, \quad m_2 = 5$

So general solution is

$y = c_1 e^{5x} + c_2 x e^{5x}$

(c) $y'' + 4y' + 7y = 0$

$m^2 + 4m + 7 = 0$

$m = \frac{-4 \pm \sqrt{16 - 28}}{2} = \frac{4 \pm \sqrt{12}\, i}{2} = 2 \pm \sqrt{3}\, i$

So $m_1 = 2 + \sqrt{3}\, i \qquad m_2 = 2 - \sqrt{3}\, i$

General solution is

$y(x) = e^{2x}(c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x)$

Example

Solve $4y'' + 4y' + 17y = 0 \quad y(0) = -1, \; y'(0) = 2$

Finding auxiliary Eq.

$4m^2 + 4m + 17 = 0.$

$$m = \frac{-4 \pm \sqrt{16 - 272}}{8} = \frac{-4 \pm \sqrt{256}\, i}{8}$$

$$= \frac{-4 \pm 16i}{8} = -\frac{1}{2} \pm 2i$$

$$m_1 = -\frac{1}{2} + 2i \qquad m_2 = -\frac{1}{2} - 2i$$

So general solution is

$$y(x) = e^{-x/2}(c_1 \cos 2x + c_2 \sin 2x) \qquad (1)$$

using $y(0) = -1$

$$\Rightarrow \quad -1 = e^{0}(c_1 + c_2 \times 0) \Rightarrow c_1 = -1.$$

Diff (1) w.r.t

$$y'(x) = -\frac{1}{2} e^{-x/2}(c_1 \cos 2x + c_2 \sin 2x)$$
$$+ e^{-x/2}(-2c_1 \sin 2x + 2c_2 \cos 2x)$$

$y'(0) = 2$

$$\Rightarrow \quad 2 = -\frac{1}{2} e^{0}(c_1 + 0) + e^{0}(-2c_1 \times 0 + 2c_2)$$

using $c_1 = -1$

$$2 = -\frac{1}{2} \times -1 + 2c_2$$

$$\Rightarrow \quad 2c_2 = 2 - 1/2 \Rightarrow \quad c_2 = 3/4.$$

So solution of IVP is

$$y(x) = e^{-x/2}\left(-\cos 2x + \frac{3}{4} \sin 2x\right)$$

Two Important Equations

(1) $$y'' + k^2 y = 0$$

Auxiliary Equation

$$m^2 + k^2 = 0 \Rightarrow m = \pm ik$$

So general solution with $\alpha = 0$ & $\beta = k$ is.

$$y(x) = c_1 \cos kx + c_2 \sin kx.$$

(2) $$y'' - k^2 y = 0$$

Auxilliary Eq. is: $m^2 - k^2 = 0$

$$m^2 = k^2 \Rightarrow m = \pm k.$$

So

$$y(x) = c_1 e^{kx} + c_2 e^{-kx}.$$

For $c_1 = c_2 = \frac{1}{2}$

$$y(x) = \frac{1}{2} e^{kx} + \frac{1}{2} e^{-kx} = \cosh kx.$$

For $c_1 = \frac{1}{2}$, $c_2 = -\frac{1}{2}$

$$y(x) = \frac{1}{2} e^{kx} - \frac{1}{2} e^{-kx} = \sinh kx$$

~~So~~ As $\sinh kx$ & $\cosh kx$ are linearly independent, so we can write alternative form of general solution

$$y(x) = c_1 \cosh kx + c_2 \sinh kx.$$

Higher Order Equations

To solve an n^{th}-order differential equation with constant coefficient, auxiliary equation will be an n^{th} degree polynomial equation

$$a_n m^n + a_{n-1} m^{n-1} + \dots + a_2 m^2 + a_1 m + a_0 = 0 \quad (1)$$

If all the roots of (1) are real & distinct, then general solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$$

* Roots of an auxiliary equation of degree greater than two may occur in many combinitions

For example

A fifth degree equation could have five distinct roots,

or three distinct real or two complex roots.

or one real and four complex roots.

or five real roots but two of them are equal.

* When m_1 is a root of multiplicity K of n^{th} degree auxiliary Equation

"Multiplicity of K means K roots are equal to m_1"

So general solution must contain the linear combination

$$c_1 e^{m_1 x} + c_2 x e^{m_2 x} + c_3 x^2 e^{m_2 x} + \cdots + c_K x^{K-1} e^{m_1 x}.$$

* Complex roots of an auxiliary equation always appear in conjugate pairs.

Example

Solve $y''' + 3y'' - 4y = 0$

Solution

Auxiliary Eq. is

$$m^3 + 3m^2 - 4 = 0 \qquad (1)$$

As $m_1 = 1$ is the root of (1) so one factor will by $m-1$. Dividing by $m-1$

$$\begin{array}{r|l} & m^2 + 4m + 4 \\ m-1 & m^3 + 3m^2 - 4 \\ & \pm m^3 \mp m^2 \\ \hline & 4m^2 - 4 \\ & \pm 4m^2 \mp 4m \\ \hline & 4m - 4 \\ & 4m - 4 \\ \hline & 0 \end{array}$$

$$\Rightarrow m^3 + 3m^2 - 4 = (m-1)(m^2 + 4m + 4)$$

$$= (m-1)(m+2)^2.$$

So $m_1 = 1, \quad m_2 = m_3 = -2$

$$\Rightarrow \quad y(x) = c_1 e^{x} + c_2 e^{-2x} + c_3 x e^{-2x}$$

Example

Solve $$\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = 0$$

Solution Auxiliary Equation is

$$m^4 + 2m^2 + 1 = 0$$

$$(m^2+1)^2 = 0.$$

$m^2 + 1 = 0$ $\qquad\qquad$ $m^2 + 1 = 0$

$m = \pm i$ $\qquad\qquad$ $m = \pm i$

So $\quad m_1 = i \quad m_2 = -i \quad m_3 = i \quad m_4 = -i$

So general solution is

$$y(x) = c_1 e^{ix} + c_2 x e^{ix} + c_3 e^{-ix} + c_4 x e^{-ix}.$$

Now

$c_1 e^{ix} + c_3 e^{-ix}$ can be rewritten as

$c_1 \cos x + c_3 \sin x.$

& $x(c_2 e^{ix} + c_4 e^{-ix})$ can be rewritten as

$x(c_2 \cos x + c_4 \sin x).$

$$\Rightarrow \quad y = c_1 \cos x + c_3 \sin x + c_2 x \cos x + c_4 x \sin x.$$

* For repeated complex roots, $m_1 = \alpha + i\beta$ is of ~~complex~~ multiplicity k and its conjugate $m_2 = \alpha - i\beta$ is also of multiplicity k, we can write the general solution as linear combination of

$$e^{\alpha x}\cos\beta x,\ x e^{\alpha x}\cos\beta x,\ x^2 e^{\alpha x}\cos\beta x, \dots, x^{k-1} e^{\alpha x}\cos\beta x$$

$$e^{\alpha}\sin\beta x,\ x e^{\alpha x}\sin\beta x,\ x^2 e^{\alpha x}\sin\beta x, \dots, x^{k-1} e^{\alpha x}\sin\beta x$$

Exercise 4.3

4) $y'' - 3y' + 2y = 0$

Let $y = e^{mx}$

So $y' = me^{mx}$ $\quad$ $y'' = m^2 e^{mx}$

& given DE is

$$m^2 e^{mx} - 3me^{mx} + 2e^{mx} = 0$$

So auxiliary Eq. is

$$m^2 - 3m + 2 = 0$$

$$m^2 - 2m - m + 2 = 0 \quad \Rightarrow \quad m(m-2) - 1(m-2) = 0$$

$$(m-1)(m-2) = 0$$

$$m_1 = 1 \qquad m_2 = 2$$

$$y(x) = c_1 e^{x} + c_2 e^{2x}$$

10. $3y'' + y = 0$

For $y = e^{mx}$, $y' = me^{mx}$, $y'' = m^2 e^{mx}$

& we get

$$3m^2 e^{mx} + e^{mx} = 0$$

& auxiliary Eq. is

$$(3m^2 + 1) = 0 \quad \Rightarrow \quad m = \pm \frac{i}{\sqrt{3}}$$

So solution is

$$y(x) = c_1 \cos\left(\frac{1}{\sqrt{3}} x\right) + c_2 \sin\left(\frac{1}{\sqrt{3}} x\right)$$

Find the general solution of higher order DE

19. $\dfrac{d^3u}{dt^3} + \dfrac{d^2u}{dt^2} - 2u = 0$

Auxiliary Equation is

$$m^3 + m^2 - 2 = 0$$

$$\begin{array}{c|ccc|c} 1 & 1 & 1 & 0 & -2 \\ & - & 1 & 2 & 2 \\ \hline & 1 & 2 & 2 & 0 \end{array}$$

So we get

$$(m-1)(m^2 + 2m + 2) = 0$$

$$m^2+2m+2=0$$

$$m = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

So

$$m_1 = 1, \quad m_2 = -1+i, \quad m_3 = -1-i$$

$$y(x) = c_1 e^{t} + c_2 e^{-t}(\cos t) + c_3 e^{-t} \sin t$$

27 $$\frac{d^5u}{dr^2} + 5\frac{d^4u}{dr^4} - 2\frac{d^3u}{dr^3} - 10\frac{d^2u}{dr^2} + \frac{du}{dr} + 5u = 0$$

Let $u = e^{mr}$

$$\frac{du}{dr} = me^{mr}, \quad \frac{d^2u}{dr^2} = m^2 e^{mr}, \quad \frac{d^3u}{dr^3} = m^3 e^{mr}$$

$$\frac{d^4u}{dr^4} = m^4 e^{mr}, \quad \frac{d^5u}{dr^5} = m^5 e^{mr}$$

So auxiliary Eq. is

$$m^5 + 5m^4 - 2m^3 - 10m^2 + m + 5 = 0$$

Using synthetic division

-5	1	5	-2	-10	1	5
	$-$	-5	0	10	$-$	-5
	1	0	-2	0	1	0

So ~~remaing~~ remaining Eq. are

$$(m+5)(m^4 - 2m^2 + 1) = 0$$

$$(m+5)(m^2-1)^2 = 0$$

$$m_1 = -5 \qquad m = \pm 1 \qquad m = \pm 1$$

$$m_1 = -5 \quad m_2 = 1 \quad m_3 = -1 \quad m_4 = 1 \quad m_5 = -1$$

So

$$u(r) = c_1 e^{-5r} + c_2 e^{r} + c_3 e^{-r} + c_4 r e^{r} + c_5 r e^{-r}$$

30. $\frac{d^2y}{d\theta^2} + y = 0$ $\qquad y(\frac{\pi}{3}) = 0$, $y'(\frac{\pi}{3}) = 2$.

Let $y = e^{m\theta}$

$y' = me^{m\theta} \Rightarrow y'' = m^2 e^{m\theta}$

Auxiliary eq. is

$m^2 + 1 = 0 \Rightarrow m = \pm i$

$y(\theta) = c_1 \cos\theta + c_2 \sin\theta$.

$y'(\theta) = -c_1 \sin\theta + c_2 \cos\theta$.

using $y(\frac{\pi}{3}) = 0$

$\Rightarrow c_1 \cos\frac{\pi}{3} + c_2 \sin\frac{\pi}{3} = 0$

$c_1 \times \frac{1}{2} + c_2 \frac{\sqrt{3}}{2} = 0$

$c_1 + \sqrt{3} c_2 = 0 \qquad (1).$

using $y'(\pi/3) = 2$

$\Rightarrow -c_1 \times \frac{\sqrt{3}}{2} + \frac{c_2}{2} = 2$

$\Rightarrow -\sqrt{3} c_1 + c_2 = 4. \qquad (2)$

from (1) $c_1 = -\sqrt{3} c_2$.

using in (2)

$\Rightarrow +3c_2 + c_2 = 4$

$4c_2 = 4 \Rightarrow c_2 = 1$

& $c_1 = -\sqrt{3}$

So

$y(\theta) = -\sqrt{3}\cos\theta + \sin\theta$

Solve the given boundary value problem.

37 $y'' - 10y' + 25y = 0 \qquad y(0) = 1, \; y(1) = 0$

Solution

Auxiliary Eq. is

$$m^2 - 10m + 25 = 0$$

$$(m-5)^2 = 0$$

$$m_1 = 5 \qquad m_2 = 5$$

$$y(x) = c_1 e^{5x} + c_2 x e^{5x}.$$

using $y(0) = 1$

$\Rightarrow \quad 1 = c_1$

using

$y(1) = 0$

$\Rightarrow \quad 0 = c_1 e^5 + c_2 e^5$

$\Rightarrow \quad c_2 = -c_1 \Rightarrow c_2 = -1.$

$\Rightarrow \quad y(x) = e^{5x} - x e^{5x}.$

40 $y'' - 2y' + 2y = 0 \qquad y(0) = 1, \quad y(\pi) = 1.$

Auxiliary Eq. is

$$m^2 - 2m + 2 = 0$$

$$m = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

$$y(x) = e^x (c_1 \cos x + c_2 \sin x).$$

$y(0) = 1 \Rightarrow 1 = c_1$

$y(\pi) = 1 \Rightarrow 1 = e^{\pi} c_1 \Rightarrow c_1 = \frac{1}{e^{\pi}}$

As we have both values of c_1 & no value of c_2

So boundary value problem has no solutions