10.7 Power Series

Power series

A power series about $x=0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots \qquad (1)$$

A power series about $x=a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

in which the center a & the coefficients $c_0, c_1, \cdots c_n, \cdots$ (2) are constant.

Example 1

Let $c_0, c_1, \cdots c_n, \cdots$ in (1) are equal to 1, we get the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \qquad (3)$$

→ First term of this series is 1.

→ Ratio of geometric series is x.

Series (3) converges to $\frac{1}{1-x}$ for $|x|<1$.

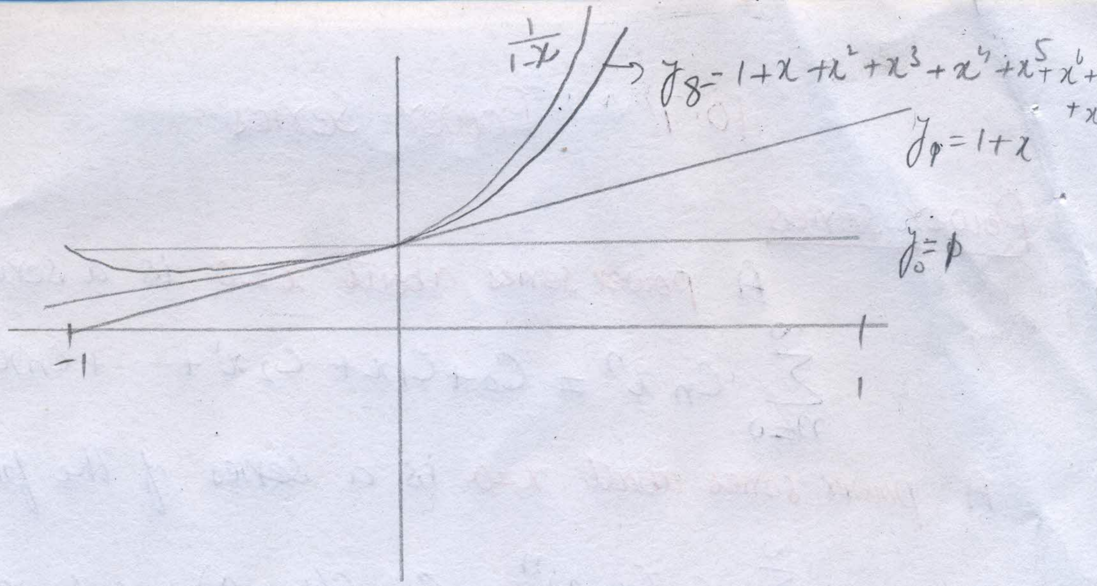
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \qquad |x|<1 \quad (4)$$

i-e $-1<x<1$

* Now partial sum of the terms of the series on the right hand side of Eq (4) can be used as polynomial $P_n(x)$ which approximates the function on the left.

* For value of x very close to zero, only few terms of the series give a good approximation.

* for x close to 1 or -1, more terms are required.



Example The power series

$$1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 + \cdots + \left(\frac{-1}{2}\right)^n (x-2)^n + \cdots$$

It matches the Eq (2) with $a=2$ & $c_0 = 1,\ c_1 = -1/2,$ $c_2 = \frac{1}{4}, \cdots, c_n = \left(\frac{-1}{2}\right)^n$

* It is a geometric series with first term 1 & ratio $-\frac{x-2}{2}$.

* The series converges for $\left|\frac{x-2}{2}\right| < 1$ or

$$-1 < \frac{x-2}{2} < 1$$

$$-2 < x-2 < 2$$

$$\Rightarrow \quad 0 < x < 4.$$

The sum is

$$\frac{1}{1-r} = \frac{1}{1+\frac{x-2}{2}} = \frac{1}{\frac{\cancel{2} + x \mp \cancel{2}}{2}} = \frac{2}{x}$$

So

$$\frac{2}{x} = 1 - \frac{x-2}{2} + \frac{(x-2)^2}{4} + \cdots + \left(\frac{-1}{2}\right)^n (x-2)^n + \cdots \quad 0 < x < 4.$$

* So polynomial approximation of $f(x) = 2/x$ for value near x is

$$P_0(x) = 1$$

$$P_1(x) = 1 - \frac{1}{2}(x-2) = 2 - x/2$$

$$P_2(x) = 1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2$$

Example For what values of do the following power series converges.

(a) $$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

Applying the ratio test (For absolute convergence).

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(-1)^n x^{n+1}/n+1}{(-1)^{n-1} x^n/n}\right| = \left|\frac{n}{n+1} x\right|$$

$$= \frac{n}{n+1}|x|$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \frac{1/n}{1+1/n}|x| = |x|$$

* The series converges absolutely for $|x|<1$ and it diverges for $|x|>1$. because n^{th} term does not converges to zero.
* At $x=1$, we get

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \quad \text{(harmonic series)}$$

which converges.

* At $x=-1$, we get

$$1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \cdots$$

It diverges

* So series converges for $-1 < x \leq 1$ & diverges elsewhere.

(b) $$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots$$

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{2n+1}}{2n+1} \times \frac{2n-1}{x^{2n-1}}\right| = \frac{2n-1}{2n+1} x^2$$

$$\lim_{n\to\infty} \frac{2-1/n}{2+1/n} x^2 = x^2$$

* The series converges absolutely for $x^2<1$. It diverges for $x^2>1$ because the n^{th} term do not converge to zero

* At $x=1$, the series become $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$, which converges.

* At $x=-1$, again it is an alternating series $-1+\frac{1}{3}-\frac{1}{5}+\cdots$ So it again converges.

* Series converges for $-1\le x\le 1$ & diverges else where.

(c) $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}/(n+1)!}{x^n/n!}\right| = \left|\frac{x \; n!}{(n+1)!}\right|$$

$$= \left|\frac{x\,\cancel{n!}}{\cancel{n!}\,(n+1)}\right| = |x|\frac{1}{n+1}$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = |x| \lim_{n\to\infty}\frac{1}{n+1} = 0$$

for every x.

So the series converges absolutely for all x.

(d) $\sum_{n=0}^{\infty} n!\,x^n = 1+x+2!\,x^2+3!\,x^3+\cdots$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\;x^{n+1}}{n!\;\;x^n}\right|$$

$$= \lim_{n\to\infty}|(n+1)\,x|$$

$$= \infty \quad \text{unless } x=0.$$

The series diverges for all values of x except at $x=0$

Convergence theorem for power series.

If the power series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

converges at $x = c \neq 0$, then in converges absolutely for all x with $|x| < |c|$.

If the series diverges at $x = d$, then it diverges for all x with $|x| > |d|$.

* Radius of convergence of power series.

The convergence of the series $\sum c_n (x-a)^n$ is described by following three cases.

1) There is a positive number R such that the series diverges the series diverges for x with $|x-a| > R$ but converges absolutely for x with $|x-a| \leq R$. The series may or may not converges at either of the end points $x = a-R$ and $x = a+R$.
2) The series converges absolutely for every x. $(R = \infty)$.
3) The series converges at $x = a$ & diverges elsewhere $(R = 0)$

* How to test the power series for convergence.

1) Use the Ratio Test (or Root test) to find the intervals where the series converges absolutely. Ordinarily, this is an open interval

$$|x-a| < R \quad \text{or} \quad a-R < x < a+R$$

2) If the interval of absolute convergence is finite, test for convergence or divergence at each end point.
3) If the interval of absolute convergence is $a-R < x < a+R$ the series diverges for $|x-a| > R$ because nth term does not approach zero for those values of x.