2.5 Solution by Substitution

Homogeneous Equation.

If a function f possesses the property $f(tx, ty) = t^{\alpha} f(x,y)$ for some real number α, the f is said to be a homogeneous function of degree α.

Example

$$f(x,y) = x^3 + y^3$$

$$f(tx, ty) = t^3x^3 + t^3y^3 = t^3 f(x,y)$$

which is homogeneous function of degree 3.

* A first order DE in differential form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is said to be homogeneous if both coefficient function M and N are homogeneous of the same degree e.g.

$$M(tx, ty) = t^{\alpha} M(x,y)$$
$$N(tx, ty) = t^{\alpha} N(x,y)$$

* If M and N are homogeneous functions of degree α, we can also write

$$M(x,y) = x^{\alpha} M(1,u)$$
$$N(x,y) = x^{\alpha} N(1,u)$$

where $u = y/x$

and

$$M(x,y) = y^{\alpha} M(v,1)$$
$$N(x,y) = y^{\alpha} N(v,1)$$

where $v = x/y$

Example

Solve $(x^2+y^2)\,dx + (x^2-xy)\,dy = 0$. (1)

Solution

$M(x,y) = x^2+y^2$

$M(tx,ty) = t^2x^2+t^2y^2$

$= t^2(x^2+y^2)$

$= t^2 M(x,y)$

$N(x,y) = x^2-xy$

$N(tx,ty) = t^2x^2 - t^2xy$

$= t^2(x^2-xy)$

$= t^2 N(x,y)$

So $M(x,y)$ & $N(x,y)$ are homogeneous of degree 2.

Let $y = ux$

so $dy = u\,dx + x\,du$

substituting in (1)

$$(x^2+y^2)\,dx + (x^2-xy)\,dy = 0$$

$$(x^2+u^2x^2)\,dx + (x^2-ux^2)(u\,dx+x\,du) = 0$$

$$x^2dx + \cancel{u^2x^2\,dx} + x^2u\,dx + x^3du - \cancel{u^2x^2\,dx} - ux^3\,du = 0$$

$$x^2(1+u)\,dx + x^3(1-u)\,du = 0$$

$$\cancel{x^2}(1+u)\,dx = -x^{\cancel{3}}(1-u)\,du$$

$$\frac{-dx}{x} = \frac{1-u}{1+u}\,du$$

$$= \frac{1-1-u+1}{1+u}\,du$$

$$-\frac{dx}{x} = \frac{2-(1+u)}{1+u}\,du$$

$$\frac{dx}{x} = \left(-\frac{2}{1+u}+1\right)du$$

$$\ln x = -2\ln|1+u| + u + \ln c$$

$$u = \ln x + 2\ln|1+u| + \ln c$$

$$u = \ln\left|\frac{x(1+u)^2}{c}\right|$$

put $u = y/x$

$$\frac{y}{x} = \ln\left|\frac{x(1+y/x)^2}{c}\right|$$

$$= \ln\left|\frac{x \times \frac{1}{x^2}(x+y)^2}{c}\right|$$

$$= \ln\left|\frac{(x+y)^2}{cx}\right|$$

$$\Rightarrow \quad (x+y)^2 = cx\, e^{y/x}.$$

~~Bernoulli~~

Reduction to separation of variables.

A differential equation of the form

$$\frac{dy}{dx} = f(Ax+By+C)$$

can always be reduced to an equation with separable variable by substituting

$$u = Ax+By+C \qquad B \neq 0.$$

Example

$$\frac{dy}{dx} = (-2x+y)^2 - 7 \qquad y(0)=0$$

Solution

Let $u = -2x+y$

$$\frac{du}{dx} = 2 + dy/dx$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{du}{dx} - 2$$

So Eq becomes

$$\frac{du}{dx}+2=u^2-7.$$

$$\frac{du}{dx}=u^2-\cancel{8}9.$$

$$\frac{du}{u^2-9}=dx. \qquad (1)$$

using partial fraction for $\frac{1}{u^2-9}$

$$\frac{1}{(u+3)(u-3)}=\frac{A}{u+3}+\frac{B}{u-3}.$$

$$1=A(u-3)+B(u+3)$$

For $u=3, \quad B=1/6$

$u=-3, \quad A=-1/6$

so $$\frac{1}{u^2-9}=\frac{-1}{6(u+3)}+\frac{1}{6(u-3)}$$

So we get from (1)

$$\cancel{du}\left(\frac{-1}{6(u+3)}+\frac{1}{6(u-3)}\right)du=dx$$

Integrating

$$\frac{-1}{6}\ln(u+3)+\frac{1}{6}\ln(u-3)=x+c$$

$$-\ln(u+3)^{\frac{1}{6}}+\ln(u-3)^{\frac{1}{6}}=x+C.$$

$$\frac{1}{6}\ln\left|\frac{u-3}{u+3}\right|^{6}=x+c.$$

$$\ln\left|\frac{u-3}{u+3}\right|=6x+6c.$$

$$\frac{u-3}{u+3}=e^{6x+6c}=c_1e^{6x}.$$

resubstituting $u=-2x+y$

$$u=\frac{3(1+ce^{6x})}{1-ce^{6x}} \Rightarrow -2x+y=\frac{3(1+ce^{6x})}{1-ce^{6x}}$$

$$y=2x+\frac{3(1+ce^{6x})}{1-ce^{6x}}.$$

Bernoulli Equation

The differential equation

$$\frac{dy}{dx} + P(x)y = f(x)y^n \qquad (A)$$

where n is any real number, is called Bernoulli Equation.

* For $n=0$, & $n=1$, Equation (A) is linear.
* For $n \neq 0$ & $n \neq 1$, the substitution ~~y~~ $u = y^{1-n}$ reduces any equation of the form (A) into a linear equation.

Example

Solve $x\frac{dy}{dx} + y = x^2y^2$.

Solution

writing in the standard form

$$\Rightarrow \quad \frac{dy}{dx} + \frac{y}{x} = xy^2 \qquad (1)$$

Here $n=2$, so we can substitute $u = y^{1-n} = y^{1-2} = y^{-1}$.

$$\Rightarrow y = 1/u.$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$= -\frac{1}{u^2}\frac{du}{dx}$$

So (1) $\Rightarrow$

$$-\frac{1}{u^2}\frac{du}{dx} + \frac{1}{ux} = \frac{x}{u^2}$$

$$\Rightarrow \quad \frac{du}{dx} - \frac{u}{x} = -x.$$

Now Finding the I.F $\Rightarrow$ $e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$

$$\Rightarrow \quad x^{-1}\frac{du}{dx} - ux^{-2} = -1.$$

$$\frac{d}{dx}(ux^{-1}) = -1.$$

Integrating

$$\Rightarrow \quad x^{-1}u = -x + c$$

$$\Rightarrow \quad u = -x^2 + cx.$$

Put $u = \frac{1}{y}$

$$\Rightarrow \quad \frac{1}{y} = -x^2 + cx.$$

$$y = \frac{1}{cx - x^2}$$

Exercise 2.5

Solve the given DE by using appropriate substitution

5) $(y^2+yx)dx - x^2 dy = 0$

Let $y = ux$ $\rightarrow (1)$

$dy = u\,dx + x\,du$

$\Rightarrow (u^2x^2 + ux^2)\,dx - x^2(u\,dx + x\,du) = 0$

$u^2x^2\,dx + \cancel{ux^2\,dx} - \cancel{x^2u\,dx} - x^3\,du = 0$

$x^{\cancel{3}}\,du = u^2\cancel{x^2}\,dx$

$u^{-2}\,du = \frac{dx}{x}$

Integrating

$\Rightarrow \frac{u^{-1}}{-1} = \ln x + c$

$-\frac{1}{u} = \ln x + c$

from (1), $u = y/x$

$\Rightarrow -\frac{x}{y} = \ln x + c$

$\Rightarrow \frac{y}{x} = \frac{-1}{c+\ln x} = \frac{1}{c-\ln x}$

$y = \frac{x}{c-\ln x}$

10 $x\frac{dy}{dx} = y + \sqrt{x^2-y^2}$ $x>0$

$y = ux$

$\frac{dy}{dx} = x\frac{du}{dx} + u$

$\Rightarrow ux + x \times x\frac{du}{dx} = ux + \sqrt{x^2 - u^2x^2}$

$\cancel{ux} + x^2\frac{du}{dx} = \cancel{ux} + x\sqrt{1-u^2}$

$$\frac{du}{\sqrt{1-u^2}} = \frac{dx}{x}$$

Integrating

$\Rightarrow \sin^{-1} u = \ln x + c.$

$\sin^{-1}(y/x) = \ln x + c.$

$y = x \sin(\ln x + c).$

14 Solve the given IVP.

$$y\,dx + x(\ln x - \ln y - 1)\,dy = 0 \qquad y(1) = e$$

Solution Let $x = vy$.

$dx = v\,dy + y\,dv$

$\Rightarrow y(v\,dy + y\,dv) + vy(\ln(vy) - \ln y - 1)\,dy = 0$

$\cancel{vy\,dy} + y^2\,dv + vy\ln vy\,dy - vy\ln y\,dy - \cancel{vy\,dy} = 0$

$dy(vy(\ln vy - \ln y)) = -y^2\,dv$

$dy\left(vy\ln\left(\frac{vy}{y}\right)\right) = -y^2\,dv$

$vy\ln v\,dy = -y^2\,dv$

$$-\frac{1}{y}\,dy = \frac{dv}{v\ln v}$$

Integrating

$$\Rightarrow -\int \frac{1}{y}\,dy = \int \frac{dv}{v\ln v}$$

Let $u = \ln v \Rightarrow du = \frac{1}{v}\,dv$

$$\Rightarrow -\int \frac{1}{y}\,dy = \int \frac{du}{u}$$

$-\ln y = \ln u + \ln c$

$\ln c = \ln u + \ln y$

$\ln c = \ln uy$

$\Rightarrow \quad c = uy$

$c = y \ln(v)$ Back substituting

$= y \ln(x/y)$

$c = y(\ln x - \ln y)$

using initial condition $y(1) = e$

$\Rightarrow \quad c = 1(\ln \cancel{1} - \ln \cancel{e})$

$c = -e$

$\Rightarrow$ Solution is

$$y \ln(x/y) = -e.$$

Solve the given DE by using appropriate substitution.

20 $\quad 3(1+t^2)\dfrac{dy}{dt} = 2ty(y^3-1).$ Q1

$$3(1+t^2)\frac{dy}{dt} + 2ty = 2ty^4$$

$\div$ing both sides by y^4

$$\frac{3(1+t^2)}{y^4}\frac{dy}{dt} + \frac{2t}{y^3} = 2t \qquad (1).$$

Let $\quad u = \dfrac{1}{y^3}, \quad \dfrac{du}{dy} = \dfrac{-3}{y^4}$

Now

$$\frac{du}{dt} = \frac{du}{dy}\frac{dy}{dt} = \frac{-3}{y^4}\frac{dy}{dt}$$

$$\Rightarrow \frac{3}{y^{4}}\frac{dy}{dt} = -\frac{du}{dt}$$

Substituting in (1)

$$\Rightarrow \quad -(1+t^2)\frac{du}{dt} + 2tu = 2t.$$

$$-(1+t^2)\frac{du}{dt} = 2t(1-u)$$

$$\Rightarrow \quad -\frac{du}{1-u} = \frac{2t}{1+t^2}\,dt$$

$$\Rightarrow \quad \ln|1-u| = \ln|1+t^2| + \ln c.$$

$$1-u = c(1+t^2)$$

$$\Rightarrow \quad u = 1-c(1+t^2)$$

$$\frac{1}{y^3} = 1-c(1+t^2)$$

$$\Rightarrow \quad y = \frac{1}{(1-c(1+t^2))^{1/3}}$$

<u>22</u> $y^{1/2}\dfrac{dy}{dx} + y^{3/2} = 1 \qquad y(0)=4.$

Let $u = y^{3/2}$

$$\frac{du}{dy} = \frac{3}{2}y^{1/2}.$$

now

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$$

$$= \frac{2}{3}y^{-1/2}\frac{du}{dx}$$

$$y^{1/2}\frac{dy}{dx} = \frac{2}{3}\frac{du}{dx}$$

put in given DE

$$\frac{2}{3}\frac{du}{dx} + u = 1$$

$$\Rightarrow \quad \frac{2}{3}\frac{du}{dx} = 1-u.$$

$$\Rightarrow \quad \frac{du}{dx} + \frac{3}{2}u = 3/2.$$

I. F. $e^{\int \frac{3}{2} dx} = e^{3/2 x}$

$\Rightarrow e^{\frac{3}{2}x} \frac{du}{dx} + \frac{3}{2} e^{\frac{3}{2}x} u = \frac{3}{2} e^{3/2 x}$

$\frac{d}{dx}\left(e^{(3/2)x} u \right) = \frac{3}{2} e^{(3/2)x}$

Integrating

$\Rightarrow e^{\frac{3}{2}x} u = \frac{\cancel{3}}{2} \frac{e^{(3/2)x}}{\cancel{3/2}} + c$

$u = 1 + c e^{-\frac{3}{2}x}$

$y^{3/2} = 1 + c e^{(-3/2)x}$

using I.C. $y(0) = 4$

$(4)^{3/2} = 1 + c e^{0}$

$8 = 1 + c \Rightarrow c = 7$

$\Rightarrow y^{3/2} = 1 + 7 e^{-\frac{3}{2}x}$

24 $\frac{dy}{dx} = \frac{1-x-y}{x+y}$

Let $u = x+y \Rightarrow \frac{du}{dx} = 1 + \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{du}{dx} - 1$

$\Rightarrow \frac{du}{dx} - \cancel{1} = \frac{1-u}{u} = \frac{1}{u} - \cancel{1}$

$u\,du = dx$

Integrating

$\Rightarrow \frac{u^2}{2} = x + c \Rightarrow u^2 = 2x + c$

$(x+y)^2 = 2x + c$

27 $\frac{dy}{dx} = 2 + \sqrt{y-2x+3}$

Let $u = y - 2x + 3 \rightarrow \frac{du}{dx} = \frac{dy}{dx} - 2$

$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + 2$

$\Rightarrow \cancel{2} + \frac{du}{dx} = \cancel{2} + \sqrt{u}$

$u^{-1/2}\, du = dx$

Integrating

$\frac{u^{1/2}}{1/2} = x + c \Rightarrow u^{1/2} = \frac{1}{2}x + c$

$\sqrt{y-2x+3} = \frac{x}{2} + c$

29 $\frac{dy}{dx} = \cos(x+y)$ $\qquad y(0) = \pi/4$.

Let $u = x + y$.

$\frac{du}{dx} = 1 + \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{du}{dx} - 1$

$\Rightarrow \frac{du}{dx} - 1 = \cos u$.

$\Rightarrow \frac{du}{dx} = \cos u + 1$.

$\frac{1}{1+\cos u}\, du = dx$.

$\frac{1}{1+\cos u} \cdot \frac{(1-\cos u)}{1-\cos u}\, du = dx \Rightarrow \frac{1-\cos u}{1-\cos^2 u}\, du = dx$.

$\Rightarrow \frac{1-\cos u}{\sin^2 u}\, du = dx \Rightarrow (\operatorname{cosec}^2 u - \cot u \csc u)\, du = dx$.

Integrating

$-\cot u + \csc u = x + c$.

$\Rightarrow -\cot(x+y) + \csc(x+y) = x + c$.

using I.C

$\Rightarrow -\cot(\pi/4) + \csc(\pi/4) = c \Rightarrow c = -1 + \sqrt{2}$

$\Rightarrow -\cot(x+y) + \csc(x+y) = x - 1 + \sqrt{2}$.