5.1 Linear Models: Initial Value Problem.

Spring / Mass Systems: Free undamped motion

* Newton's Second Law.

When a mass is attached to a spring, it stretches the spring by an amount s and attains a position at which its weight is balanced by the restoring force ks.

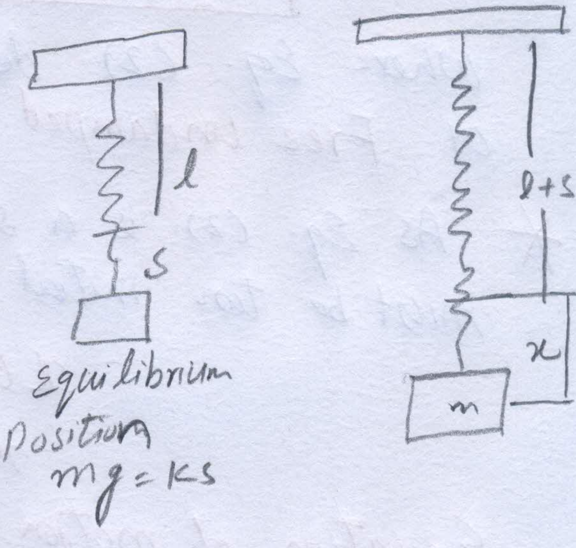
* weight is defined by mg.

* The condition of equilibrium is

$$mg = ks$$

or

$$mg - ks = 0$$



* If the mass is displaced by an amount of x from its equilibrium position, the restoring force of spring is $k(x+s)$.

* In the absence of any retarding force on the system and assuming that mass vibrates free of other external forces, we can equate the Newton's second law with the net or resultant force of restoring force & weight.

$$m\frac{d^2x}{dt^2} = -k(s+x) + mg \qquad (1)$$

where -ve sign with restoring force is the indication that it is in the opposite direction of motion

As $mg - ks = 0$

So Eq. (1) becomes

$$m\frac{d^2x}{dt^2} = -Kx$$

÷ing by m

$$\Rightarrow \frac{d^2x}{dt^2} = -\frac{K}{m}x$$

$$\boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0} \qquad \omega^2 = K/m, \quad (2)$$

Where Eq. (2) describes simple harmonic motion or Free undamped motion.

* As Eq. (2) is a second order differential Eq. So there must be two initial conditions with general form

$$x(0) = x_0$$
$$x'(0) = x_1$$

Equation of motion

To solve Eq. (2), let auxiliary Eq. is

$$m^2 + \omega^2 = 0.$$
$$m = \pm i\omega$$

So

$$x(t) = c_1 \cos\omega t + c_2 \sin\omega t$$

Finally, when the initial condition are used to determine the constants c_1 & c_2 in (3), we say that the resulting particular solution or response is the equation of motion.

Example A mass weighing 2 pounds stretches a spring 6 inches. At $t=0$, the mass is released from a point 8 inches below the equilibrium position with an upward velocity of $4/3$ ft/s. Determine the Eq. of motion

Solution

In the engineering system of units, inches must be converted into feet

$$6\,in = \frac{1}{2}\,ft.$$

$$8\,in = \frac{2}{3}\,ft$$

Calculating mass as ~~wtro~~ $m = \frac{W}{g}$

$$m = \frac{2}{32} = \frac{1}{16}\text{ slug}$$

From Hook's Law

$$W = Ks$$

$$2 = K \times \frac{1}{2} \Rightarrow \quad K = 4$$

As we have

$$\frac{d^2x}{dt^2} + w^2 x = 0$$

$$w^2 = K/m = \frac{4}{1/16} = 64.$$

So differential Eq. is

$$\frac{d^2x}{dt^2} + 64x = 0.$$

Auxiliary Eq. is

$$m^2 + 64 = 0$$

$$m = \pm 8i$$

$$x(t) = c_1 \cos 8t + c_2 \sin 8t \qquad (1)$$

Initial conditions are

$$x(0) = 8\text{ inches} \qquad x'(0) = -\frac{4}{3}\,ft$$

$$x(0) = \frac{2}{3}\,ft, \qquad x'(0) = -\frac{4}{3}$$

using $x(0) = \frac{2}{3}$ in (1)

$$\Rightarrow \quad \frac{2}{3} = c_1.$$

Taking derivative

$$x'(t) = -c_1 \times 8 \sin 8t + 8c_2 \cos 8t$$

$$x'(0) = -4/3$$

$$\Rightarrow -\frac{4}{3} = +8c_2 \Rightarrow c_2 = -1/6$$

$$x(t) = \frac{2}{3}\cos 8t - \frac{1}{6}\sin 8t$$

Alternative Solution

considering the general solution

$$x(t) = c_1 \cos\omega t + c_2 \sin\omega t \qquad (A)$$

Let $c_1 = A\sin\varphi$ (1^*) $\qquad c_2 = A\cos\varphi$. (2^*)

$$\Rightarrow \frac{c_1}{A} = \sin\varphi \quad (1) \qquad \frac{c_2}{A} = \cos\varphi \quad (2).$$

$(1) \div (2)$

$$\Rightarrow \frac{\sin\varphi}{\cos\varphi} = \frac{c_1/A}{c_2/A} = \frac{c_1}{c_2}$$

$$\tan\varphi = \frac{c_1}{c_2} \Rightarrow \varphi = \tan^{-1}\left(\frac{c_1}{c_2}\right). \qquad (3)$$

Squaring & adding (1^*) & (2^*)

$$c_1^2 + c_2^2 = A^2\sin^2\varphi + A^2\cos^2\varphi$$

$$\Rightarrow A^2 = c_1^2 + c_2^2$$

$$A = \sqrt{c_1^2 + c_2^2}. \qquad (4)$$

using (1) & (2) in (A)

$$\Rightarrow x(t) = A\sin\varphi\cos\omega t + A\cos\varphi\sin\omega t$$

$$x(t) = A\sin(\omega t + \varphi). \qquad (5).$$

From above data, using values of c_1, c_2 & ω from previous example.

$$\omega = 8$$

$$c_1 = \frac{2}{3} \qquad c_2 = -\frac{1}{6}$$

So

$$A_1 = \sqrt{c_1^2 + c_2^2} = \sqrt{\frac{4}{9} + \frac{1}{36}} = 0.69\,ft.$$

$$\phi = \tan^{-1}\left(\frac{c_1}{c_2}\right) = \tan^{-1}\left(\frac{2/3}{-1/6}\right)$$

$$= \tan^{-1}\left(\frac{-12}{3}\right) = \tan^{-1}(-4) = -1.326 \text{ rad.}$$

(4th quadrant)

Bringing ϕ in 2nd quadrant.

$$\phi = \pi - 1.326 = 1.816.$$

using these values in Eq (5)

$$x(t) = 0.69 \sin(8t + 1.816).$$

§5.1.2 Spring/Mass System:

- Free damped system.

Differential Eq. for free damped motion

In mechanics, damping force acting on a body are considered to be proportional to a power of instantaneous velocity.

* Here it is assumed that damping force is constant multiple of $\frac{dx}{dt}$.

When there is not external force, Newton's second law of motion is

$$m\frac{d^2x}{dt^2} = -Kx - \beta\frac{dx}{dt}$$

$\div$ing by m

$$\Rightarrow \frac{d^2x}{dt^2} = -\frac{K}{m}x - \frac{\beta}{m}\frac{dx}{dt}$$

Let $\frac{K}{m} = \omega^2$, $\frac{\beta}{m} = 2\lambda$

$$\Rightarrow \frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0 \qquad (1)$$

Auxiliary Eq. B

$$m^2 + 2\lambda m + \omega^2 = 0.$$

corresponding roots are

$$m = \frac{-2\lambda \pm \sqrt{4\lambda^2 - 4\omega^2}}{2} = \frac{-2\lambda \pm 2\sqrt{\lambda^2-\omega^2}}{2}$$

$$m = -\lambda \pm \sqrt{\lambda^2-\omega^2}$$

So

$$m_1 = -\lambda + \sqrt{\lambda^2-\omega^2} \qquad m_2 = -\lambda - \sqrt{\lambda^2-\omega^2}$$

Now we have three cases, depending on the sign of $\lambda^2-\omega^2$.

Case I $\lambda^2 - \omega^2 > 0$.

In this case, the system is said to be overdamped because damping coefficient β is larger compared to spring constant.

Corresponding solution is

$$x(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t}$$
$$= c_1 e^{-\lambda t} e^{(\sqrt{\lambda^2-\omega^2})t} + c_2 e^{-\lambda t} e^{-\sqrt{\lambda^2-\omega^2}\, t}$$

$$x(t) = e^{-\lambda t}\left(c_1 e^{\sqrt{\lambda^2-\omega^2}\,t} + c_2 e^{-\sqrt{\lambda^2-\omega^2}\,t}\right)$$

Case II $\lambda^2 - \omega^2 = 0$.

The system is said to be critically damped in this case because a slight decrease in damping force will result in oscillatory motion. In this case

$m_1 = -\lambda$ & $m_2 = -\lambda$

(Equal & real roots).

So solution is

$$x(t) = e^{-\lambda t}(c_1 + c_2 t).$$

Case III $\lambda^2 - \omega^2 < 0$.

In this case, system is said to be underdamped. Since damping coefficient is smaller as compared to spring constant. So we have two complex roots.

$m_1 = -\lambda + \sqrt{\omega^2-\lambda^2}\, i$ $\qquad$ $m_2 = -\lambda + \sqrt{\omega^2-\lambda^2}\, i$

So solution is

$$x(t) = e^{-\lambda t}\left(c_1 \cos\sqrt{\omega^2-\lambda^2}\, t + c_2 \sin\sqrt{\omega^2-\lambda^2}\, t\right)$$

* The factor $e^{-\lambda t}$ is included in each solution and named as damping factor when $\lambda > 0$.

Example

Consider the initial value problem.

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 4x = 0 \qquad x(0) = 1$$

$$x'(0) = 1.$$

Comparing with Eq. (1)

$$2\lambda = 5 \qquad \omega^2 = 4.$$

$$\lambda = 2.5 \qquad \omega^2 = 4.$$

$\Rightarrow \lambda^2 - \omega^2 = 4.25 - 4 = 0.25 > 0.$

So it is a case of overdamped motion

Finding the auxiliary Eq.

$$m^2 + 5m + 4 = 0$$

$$m^2 + 4m + m + 4 = 0$$

$$m(m+4) + 1(m+4) = 0 \qquad (m+1)(m+4) = 0$$

$$m = -1, \qquad m = -4.$$

$\Rightarrow \quad x(t) = c_1 e^{-t} + c_2 e^{-4t} \qquad (1)$

using $x(0) = 1$

$\Rightarrow \quad c_1 + c_2 = 1. \qquad$ (a)

Finding derivative of (1)

$\Rightarrow \quad x'(t) = -c_1 e^{-t} - 4c_2 e^{-4t}$

using $x'(0) = 1$

$\Rightarrow \quad -c_1 - 4c_2 = 1. \qquad$ (b)

(a) + (b) gives

$$-3c_2 = 2 \Rightarrow c_2 = -2/3.$$

using in (a)

$\Rightarrow \quad c_1 = 1 - c_2 = 1 - (-2/3) = 1 + 2/3 = 5/3.$

$$x(t) = \frac{5}{3} e^{-t} + -\frac{2}{3} e^{-4t}$$

Example A mass weighing 8 pounds stretches a spring 2 feet. Assuming that the damping force numerically equals to 2 times the instantaneous velocity acting on the system. determine the equation of motion if mass is initially released from equilibrium position with an upward velocity of 3 ft/s.

Solution From Hook's Law

$$W = KS$$

$$8 = 2K \Rightarrow \quad K = 4.$$

$$W = mg \Rightarrow \quad m = \frac{W}{g} = \frac{8}{32} = 1/4.$$

$$\beta = 2.$$

So

$$2\lambda = \frac{\beta}{m} = \frac{2}{1/4} = 8$$

$$w^2 = \frac{K}{m} = \frac{4}{1/4} = 16$$

So differential Eq. is

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + w^2 x = 0$$

$$\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 16x = 0 \qquad (1)$$

Initial conditions $\quad x(0) = 0, \quad x'(0) = -3.$

auxiliary Eq. is

$$m^2 + 8m + 16 = 0$$

$$(m+4)^2 = 0$$

$\Rightarrow \quad m_1 = m_2 = -4.$ (Critically damped system)

$$x(t) = c_1 e^{-4t} + c_2 t e^{-4t} \qquad (2)$$

$$x'(t) = -4c_1 e^{-4t} + c_2 (e^{-4t} - 4te^{-4t}) \qquad (3)$$

using $x(0) = 0$ (in (2))

$\Rightarrow \quad c_1 = 0$

using $x'(0) = -3$ (in (3))

$\Rightarrow \quad -3 = -4c_1 + c_2.$

$$c_2 = -3 - 4c_1 = -3.$$

using $c_1 = 0$ & $c_2 = -3$ in (2)

$\Rightarrow \quad x(t) = -3te^{-4t}$

Example A mass weighing 16 pounds is attached to 5-feet-long spring. At equilibrium, the spring measures 8.2 feet. If the mass is initially released from rest at a pt 2 feet above equilibrium, find displacement $x(t)$ if it is further known that the surrounding medium offer resistance numerically equal to instantaneous velocity.

Solution The stretch in spring after spring is attached is

$$S = 8.2 - 5 = 3.2.$$

By Hook's Law

$$W = KS$$

$$16 = 3.2 \times K \Rightarrow K = 5 \ lb/ft$$

also $W = mg \Rightarrow \quad m = \frac{W}{g} = \frac{16}{32} = \frac{1}{2}$ slug

So $\quad 2\lambda = \frac{\beta}{m} \quad , \quad \omega^2 = \frac{k}{m} = \frac{5}{1/2} = 10$

$\beta = 1.$

$\Rightarrow \quad 2\lambda = \frac{1}{1/2} = 2.$

We have DE

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 10x = 0$$

Initial conditions are $\quad x(0) = -2 \; , \; x'(0) = 0$

auxiliary Eq is

$$m^2 + 2m + 10 = 0$$

$$\Rightarrow \quad m = \frac{-2 \pm \sqrt{4 - 4\times 10}}{2} = \frac{-2 \pm 6i}{2}$$

$$\Rightarrow \quad m_1 = -1 + 3i \qquad\qquad m_2 = -1 - 3i$$

So it is the case when system is underdamped.

Solution is

$$x(t) = e^{-t}(c_1\cos 3t + c_2 \sin 3t) \qquad (A)$$

$$x'(t) = -e^{-t}(c_1\cos 3t + c_2\sin 3t) + e^{-t}(-3c_1\sin 3t + 3c_2\cos 3t) \qquad (B)$$

using $x(0) = -2$ in A

$\Rightarrow \quad -2 = c_1 \quad \Rightarrow \quad \boxed{c_1 = -2}$

using $x'(0) = 0$ in B

$\Rightarrow \quad 0 = -c_1 + 3c_2 \Rightarrow c_2 = -2/3$

so $\quad x(t) = e^{-t}(-2\cos 3t - 2/3 \sin 3t)$