# 2.3 Linear Equations

## Linear Equation

A first order equation of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \qquad (1)$$

is said to be linear Equation in the dependent variable $y$.

## Homogeneous Equation

If $g(x)$ in Eq (1) in zero i-e $g(x)=0$, Eq. (1) is said to be homogeneous.

* If $g(x) \neq 0$, then Eq (1) is said to be non-homogeneous.

## Standard form

Dividing Eq (1) by leading coefficient i-e by $a_1(x)$, we get

$$\frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}y = \frac{g(x)}{a_1(x)}$$

$$\frac{dy}{dx} + P(x)y = f(x) \qquad (2)$$

which is named as Standard form of linear Eq which is more useful one.

The Solution of Eq (2) exists on the interval where both $P(x)$ & $f(x)$ are continuous.

## Solution of non-homogeneous linear Eq

The Solution of Eq. (2) is the sum of two solutions

$$y = y_c + y_p$$

* The solution $y_c$ is a solution of the associated homogeneous equation

$$\frac{dy}{dx} + P(x)y = 0$$

* $y_p$ is a particular Solution of nonhomogeneous equation.

using $y(x) = y_c + y_p$ in (2)

$$\frac{d}{dx}(y_c + y_p) + P(x)(y_c + y_p)$$

$$= \underbrace{\frac{dy_c}{dx} + P(x)\,y_c}_{} + \underbrace{\frac{dy_p}{dx} + P(x)\,y_p}_{}$$

$$= 0 + f(x)$$

* considering the homogeneous Equation

$$\frac{dy}{dx} + P(x)y = 0$$

$$\frac{dy}{y} = -\int P(x)\,dx$$

$$\ln y = -\int P(x)\,dx + \ln c.$$

$$y_c = c\,e^{-\int P(x)\,dx}$$

$$= c y_1$$

where $y_1 = e^{-\int P(x)\,dx}$.

## Finding particular Solution

Particular Solution can be found using the method named as variation of parameter.

* The particular Solution is of the form

$$y_p = u(x)\,y_1(x)$$

$$= u(x)\,e^{-\int P(x)\,dx}$$

using $y_p = u y_1$ in Eq 2

$$\Rightarrow u\frac{dy_1}{dx} + y_1\frac{du}{dx} + P(x)\,u y_1 = f(x)$$

$$\Downarrow$$

$$u\left(\frac{dy_1}{dx} + P(x)y_1\right) + y_1\frac{du}{dx} = f(x)$$

$\because$ as $y_1$ is the solution of homogeneous differential Equation

$$\Rightarrow \quad y_1\frac{du}{dx} = f(x)$$

$$du = \frac{f(x)}{y_1}\,dx.$$

$$u = \int \frac{f(x)}{e^{-\int P(x)dx}}\,dx = \int e^{\int P(x)dx} f(x)\,dx.$$

So

$$y_p = e^{-\int P(x)dx} \int e^{\int P(x)dx} f(x)\,dx$$

overall solution is

$$y = y_c + y_p$$

$$= Ce^{-\int P(x)dx} + e^{-\int P(x)dx}\int e^{\int P(x)dx} f(x)\,dx.$$

Solving a linear first-order Equation

1) Put the Equation of the form (1) in to standard form (2).
2) From the standard form, identify $P(x)$, and then find the integrating factor $e^{\int P(x)dx}$.
3) Multiply the standard form of equation by the integrating factor. The left-hand side of the resulting equation is automatically the derivative of I.F. & $y$.

$$\frac{d}{dx}\left[e^{\int P(x)dx}y\right] = e^{\int P(x)dx}f(x).$$

4) Integrate both sides of this last Equation

## Example

Solve $\frac{dy}{dx} - 3y = 0$.

Eq. is already in standard form

$$\frac{dy}{dx} + P(x)y = f(x)$$

with $P(x) = -3$.

I.F. $e^{\int -3\,dx} = e^{-3x}$.

xing with $e^{-3x}$

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = 0$$

$$\frac{d}{dx}\left(e^{-3x}y\right) = 0$$

Integrating both sides

$$e^{-3x}y = c \Rightarrow y = ce^{3x}.$$

## Example

Solve $\frac{dy}{dx} - 3y = 6$.

### Solution

Again Integrating factor: $e^{-3x}$

xing with $e^{-3x}$

$$\Rightarrow e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = 6e^{-3x}$$

$$\frac{d}{dx}\left(e^{-3x}y\right) = 6e^{-3x}$$

$$d(e^{-3x}y) = 6e^{-3x}\,dx$$

Integrating both sides

$$e^{-3x}y = 6\,\frac{e^{-3x}}{-3} + c$$

$$y = -2 + ce^{3x}$$

$$= y_p + y_c.$$

## General Solution

As $y = y_c + y_p$ is a one parameter family of Solution of Eq. (2) and every solution of (2) defined on I is a member of this family.

* There fore this Solution is called general solution.

As
$$\frac{dy}{dx} = -P(x)y + f(x)$$
$$F(x,y) = -P(x)y + f(x)$$
$$\frac{\partial F}{\partial y} = -P(x)$$

So continuity of $F$ & $\frac{\partial F}{\partial y}$ on I guarantee that there is one and only one solution of IVP

$$\frac{dy}{dx} + P(x)y = f(x) \qquad y(x_0) = y_0$$

defined on Some interval $I_0$ containing $x_0$.

**Example** Solve $x\frac{dy}{dx} - 4y = x^6 e^x$.

**Solution** Converting into standard form i-e dividing by $x$.

$$\frac{dy}{dx} - 4\frac{y}{x} = x^5 e^x \qquad (A)$$

$$I.F = e^{-\int 4/x\, dx} = e^{-4\ln x} = e^{\ln x^{-4}} = x^{-4}$$

xing (A) by $x^{-4}$

$$x^{-4}\frac{dy}{dx} - 4x^{-5}y = xe^x$$
$$\frac{d}{dx}(x^{-4}y) = xe^x$$
$$d(x^{-4}y) = xe^x dx$$

Integrating both sides

$$x^{-4}y = xe^x - e^x + c.$$
$$y = x^5e^x - x^4e^x + cx^4$$

Example Find the general solution of
$$(x^2-9)\frac{dy}{dx} + xy = 0$$

Solution

Writing the differential Equation in the standard form
$$\frac{dy}{dx} + \frac{x}{x^2-9}\,y = 0 \qquad (A)$$

Identifying $P(x) = \frac{x}{x^2-9}$. Although $P(x)$ is continuous on $(-\infty,-3)$, $(-3,3)$ and $(3,\infty)$, we shall solve the equation on the first and 3rd interval.

Finding I.F
$$e^{\int \frac{x}{x^2-9}dx} = e^{\frac{1}{2}\int \frac{2x}{x^2-9}dx}$$
$$= e^{\frac{1}{2}\ln|x^2-9|} = e^{\ln|x^2-9|^{1/2}}$$
$$= \sqrt{x^2-9}$$

xing (A) with I.F.
$$\sqrt{x^2-9}\,\frac{dy}{dx} + \frac{x}{x^2-9}\,y = 0$$
$$\frac{d}{dx}\left(\sqrt{x^2-9}\;y\right) = 0$$

Integrating factor
$$\sqrt{x^2-9}\;y = c$$
$$y = \frac{c}{\sqrt{x^2-9}}$$

Here $x=3$ & $x=-3$ are singular pts. of the equation and every function in the general solution is discontinuous at these pts.

Example → Solve $\frac{dy}{dx}+y=x$ $\quad y(0)=4$.

Solution →

$$p(x)=1 \qquad I.F.\ e^{\int 1 dx}=e^{x}$$

$$e^{x}\frac{dy}{dx}+e^{x}y=xe^{x}$$

$$\frac{d}{dx}(e^{x}y)=xe^{x}$$

Integrating

$$e^{x}y=xe^{x}-e^{x}+c$$

$$y=x-1+ce^{-x}$$

$$y(0)=4 \Rightarrow 4=0-1+c$$

$$\Rightarrow c=5$$

$$y=x-1+5e^{-x} \qquad -\infty<x<\infty$$

$$y=y_p+y_c$$

For general complementary solution

$$y_c=ce^{-x}$$

* For increasing $x$, graph of all members of family of solution become closer to particular solution.

* So contribution of $y_c=ce^{-x}$ to the values of solution become negligible for increasing values of $x$.

* $y=ce^{-x}$ is a transient term as $y_c\to 0$ as $x\to\infty$.

**Example** Solve $\dfrac{dy}{dx} + y = f(x)$, $y(0) = 0$, where

$$f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$$

**Solution** For $0 \le x \le 1$, $f(x) = 1$, so DE becomes

$$\frac{dy}{dx} + y = 1,$$

I.F. $e^{\int 1\,dx} = e^{x}$

$$\Rightarrow \quad e^{x}\frac{dy}{dx} + e^{x}y = e^{x}$$

$$\frac{d}{dx}(e^{x}y) = e^{x}$$

Integrating $\quad e^{x}y = e^{x} + c$

$$y = 1 + ce^{-x}$$

using $y(0) = 0 \Rightarrow 0 = 1 + c \Rightarrow c = -1$

So $\quad y = 1 - e^{-x} \qquad 0 \le x \le 1$

* For $x > 1$, $f(x) = 0$

$$\frac{dy}{dx} + y = 0$$

So $\quad y = c_2 e^{-x}$

From previous solution

$$y(1) = 1 - e^{-1}$$

using this as an initial condition

$$y(1) = c_2 e^{-1} = 1 - e^{-1}$$

$$\Rightarrow \quad c_2 = e - 1$$

So

$$y(x) = (e-1)e^{-x}$$

$$y(x) = \begin{cases} 1-e^{-x} & 0 \le x \le 1. \\ (e-1)e^{-x} & x > 1. \end{cases}$$

Function defined by integrals

Two special type of non-elementary integrals are

error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$$

complementary error function

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\,dt$$

we have a known result that

$$\int_0^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}.$$

As

$$\int_0^\infty e^{-t^2}\,dt = \int_0^x e^{-t^2}\,dt + \int_x^\infty e^{-t^2}\,dt$$

$$\frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2}\,dt = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt + \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt$$

$$\frac{2}{\sqrt{\pi}} \times \frac{\sqrt{\pi}}{2} = \operatorname{erf}(x) + \operatorname{erfc}(x)$$

$$\Rightarrow \quad \operatorname{erf}(x) + \operatorname{erfc}(x) = 1.$$

Example

Solve IVP $\frac{dy}{dx} - 2xy = 2, \quad y(0) = 1.$

$$\frac{dy}{dx} - 2xy = 2$$

$$I.F = e^{\int -2x\,dx} = e^{-x^2}$$

xing with $e^{-x^2}$

$$e^{-x^2}\frac{dy}{dx} - 2xe^{-x^2}y = 2e^{-x^2}$$

$$\frac{d}{dx}\left(e^{-x^2}y\right) = 2e^{-x^2}$$

Integrating

$$\Rightarrow \quad e^{-x^2}y = 2\int_0^x e^{-t^2}\,dt + c$$

$$y = 2e^{x^2}\int_0^x e^{-t^2}\,dt + ce^{x^2}$$

using $y(0) = 1$

$$\Rightarrow \quad 1 = c$$

So

$$y = 2e^{x^2} \times \frac{\sqrt{\pi}}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt + e^{x^2}$$

$$= e^{x^2}\left[\sqrt{\pi}\,\mathrm{erf}(x) + 1\right].$$

# Exercise 2.3.

Find the general solution of the given differential equation. Give the largest interval $I$ over which the general solution is defined. Determine whether there are any transient terms in general solution

4) $3\frac{dy}{dx} + 12y = 4.$

converting into standard form

$\div$ing by 3

$$\frac{dy}{dx} + 4y = 4/3. \qquad \text{(A)}$$

Identifying the I.F. $e^{\int 4\,dx} = e^{4x}$

$\times$ing $e^{4x}$ with Eq (A)

$$e^{4x}\frac{dy}{dx} + 4e^{4x}y = 4/3\, e^{4x}$$

$$\frac{d}{dx}(e^{4x}y) = \frac{4}{3}e^{4x}.$$

$$\int d(e^{4x}y) = \int \frac{4}{3}e^{4x}\,dx$$

$$e^{4x}y = \frac{\cancel{4}}{3}\frac{e^{4x}}{\cancel{4}} + c.$$

$$y = \frac{1}{3} + ce^{-4x}.$$

Interval: $(-\infty, \infty)$

Transient term: $e^{-4x}.$

9) $x\frac{dy}{dx} - y = x^2\sin x.$

converting into Standard form i-e- ~~Multiplying w~~

Dividing with $x$

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = x\sin x \qquad \text{(A)}$$

Identifying the I.F. $e^{-\int \frac{1}{x} dx} = e^{-\ln x} = x^{-1}$

xing (A) with $\frac{1}{x}$

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2} y = \sin x.$$

$$\frac{d}{dx}\left(\frac{1}{x} y\right) = \sin x.$$

Integrating both sides

$$\Rightarrow \quad \frac{1}{x} y = -\cos x + c.$$

$$y = -x\cos x + cx.$$

Interval ~~(0,∞)~~ $(0, \infty)$

21 $\frac{dr}{d\theta} + r\sec\theta = \cos\theta.$

Identifying the Integrating factor as Equation B already in standard form.

$$I.F. = e^{\int \sec\theta \, d\theta} \quad \cancel{\int t} \cancel{t/e}$$

$$= e^{\ln(\sec\theta + \tan\theta)}$$

$$= \sec\theta + \tan\theta.$$

xing the given DE with I.F

$$\Rightarrow (\sec\theta + \tan\theta)\frac{dr}{d\theta} + r\sec\theta(\sec\theta + \tan\theta) = \cos\theta(\sec\theta + \tan\theta)$$

$$\frac{d}{d\theta}\left(r(\sec\theta + \tan\theta)\right) = 1 + \sin\theta.$$

Integrating

$$\Rightarrow \quad r(\sec\theta + \tan\theta) = \theta - \cos\theta + c.$$

$$r = \frac{\theta - \cos\theta + c}{\sec\theta + \tan\theta}.$$

Interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$

28 $\dfrac{dT}{dt} = K(T - T_m)$ $\qquad T(0) = T_0.$

Rearranging the equation

$$\frac{dT}{dt} - kT = -kT_m$$

$$I.F. \quad e^{-\int k\,dt} = e^{-kt}$$

xing with $e^{-kT}$

$$\Rightarrow \quad e^{-kt}\frac{dT}{dt} - ke^{-kt}T = -ke^{-kt}T_m$$

$$\int \frac{d}{dt}\left(e^{-kt}T\right) = \frac{-ke^{-kt}}{-k}T_m + C$$

$$e^{-kt}T = e^{-kt}T_m + C$$

$$T = T_m + Ce^{-kt}$$

Interval $\quad (-\infty, \infty).$

using IC $T(0) = T_0$

$$\Rightarrow \quad C = T_0 - T_m$$

So

$$T(t) = T_m + (T_0 - T_m)e^{-kt}$$

31 $\dfrac{dy}{dx} + 2y = f(x), \quad y(0) = 0$ where

$$f(x) = \begin{cases} 1 & 0 \le x \le 3 \\ 0 & x > 3. \end{cases}$$

For $0 \le x \le 3$, $f(x) = 1$

$$\frac{dy}{dx} + 2y = 1$$

$$I.F = e^{\int 2\,dx} = e^{2x}$$

$$\Rightarrow \quad e^{2x}\frac{dy}{dx} + 2e^{2x}y = e^{2x}.$$

$$\frac{d}{dx}(e^{2x}y) = e^{2x}$$

$$e^{2x}y = \frac{e^{2x}}{2} + c$$

$$y = \frac{1}{2} + ce^{-2x}$$

$y(0) = 0 \Rightarrow \quad 0 = \frac{1}{2} + c \Rightarrow c = -1/2$

$\Rightarrow \quad y = \frac{1}{2} - \frac{1}{2}e^{-2x} \quad \Rightarrow \quad y(3) = \frac{1}{2} - \frac{1}{2}e^{-6}$

For $x > 3$

$$\frac{dy}{dx} + 2y = 0$$

$$\text{I.F.} = e^{\int 2\,dx} = e^{2x}$$

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = 0$$

$$\frac{d}{dx}(e^{2x}y) = 0 \quad \Rightarrow \quad e^{2x}y = c$$

$$\Rightarrow \quad c = ye^{+2x}$$

$$y = ce^{-2x}$$

$$y(3) = ce^{-6} = \frac{1}{2} - \frac{1}{2}e^{-6}$$

$$c = \frac{1}{2}e^{6} - \frac{1}{2}$$

$$y = \frac{1}{2}(e^{6} - 1)e^{-2x}$$

$$y(x) = \begin{cases} \frac{1}{2} - \frac{e^{-2x}}{2} & 0 \le x \le 3 \\ \frac{1}{2}e^{6-2x} - \frac{e^{-2x}}{2} & x > 3 \end{cases}$$

36 Consider the IVP $y' + e^{x}y = f(x)$, $y(0) = 1$. Express the solution of IVP for $x > 0$ as a non-elementary integral when $f(x) = 1$. What is the solution when $f(x) = 0$? when $f(x) = e^{x}$.

we have

$$y' + e^x y = f(x)$$

$$\text{I.F. } e^{\int e^x dx} = e^{e^x}$$

$$\Rightarrow \quad e^{e^x} y' + e^{e^x} e^x y = e^{e^x} f(x)$$

$$\frac{d}{dx}\left(e^{e^x} y\right) = e^{e^x} f(x)$$

Integrating

$$\Rightarrow \quad e^{e^x} y = \int_0^x e^{e^t} f(t)\, dt + c$$

$$y = e^{-e^x} \int_0^x e^{e^t} f(t)\, dt + c\, e^{-e^x}$$

using $y(0)=1$

$$1 = e^{-e^0} \int_0^0 e^{e^t} f(t)\, dt + c e^{-1}$$

$$c = e.$$

$$y(x) = e^{-e^x} \int_0^x e^{e^t} f(t)\, dt + e^{1-e^x}$$

For $f(t)=1$

$$y(t) = e^{-e^x} \int_0^x e^{e^t}\, dt + e^{1-e^x}.$$

For $f(t)=0$

$$y(t) = e^{1-e^x}.$$

For $f(t) = e^t$.

$$y(t) = e^{-e^x} \int_0^x e^t e^{e^t}\, dt + e^{1-e^x}$$

$$= e^{-e^x} \left[ e^{e^t} \right]_0^x + e^{1-e^x}$$

$$= e^{-e^x} \left[ e^{e^x} - e \right] + e^{1-e^x}$$

$$= e^{1-e^x} + e^{1-e^x} = 2e^{1-e^x}.$$