3.1 Linear Models.

Growth and Decay

The phenomenon of either growth or decay can be modeled by initial value problem

$$\frac{dx}{dt} = kx \qquad x(t_0) = x_0$$

where k is constant of proportionality

Example A culture initially has P_0 number of bacteria. At $t=1$, the number of bacteria is measured to be $\frac{3}{2}P_0$. If the rate of growth is proportional to the number of bacteria $P(t)$ present at time t, determine the time necessary for the number of bacteria to triple.

Solution

using the differential Eq. from Growth model with $x(t)$ replaced by $P(t)$, and x_0 by P_0

$$\frac{dP}{dt} = kP \qquad P(0) = P_0. \qquad (1)$$

where the condition $P(1) = \frac{3}{2}P_0$ will be used to find the proportionality constant k.

Solving (1)

$$\frac{dP}{dt} - kP = 0$$

$$I.F = e^{-\int k\,dt} = e^{-kt}$$

$$\Rightarrow \quad e^{-kt}\frac{dP}{dt} - ke^{-kt}P = 0$$

$$\frac{d}{dt}\left(Pe^{-kt}\right) = 0$$

Integrating

$$Pe^{-kt} = c$$

$$\Rightarrow \quad P = ce^{+kt}$$

using initial condition $P(0) = P_0$

$$\Rightarrow \quad P(0) = c\,e^{0} = c$$

$$\Rightarrow \quad c = P_0$$

So we get $\quad P(t) = P_0 e^{+kt}$

At $t = 1$.

$$P(1) = P_0 e^{+k}$$

As given condition is $\quad P(1) = 3/2\, P_0$

$$\Rightarrow \frac{3}{2}\cancel{P_0} = \cancel{P_0}\, e^{k}$$

$$k = \ln(3/2) = 0.4055$$

$$\Rightarrow \quad P(t) = P_0\, e^{0.4055t}$$

Now to find the time when $P(t) = 3P_0$

$$\Rightarrow \quad 3\cancel{P_0} = \cancel{P_0}\, e^{0.4055t}$$

$$3 = e^{0.4055t}$$

$$0.4055t = \ln 3$$

$$t = \frac{\ln 3}{0.4055} \approx 2.71\,h$$

* Depending on sign of k, k is either growth constant $(k > 0)$ or a decay constant $k < 0$.

Example Half life of plutonium

A breeder reactor converts relatively stable uranium 238 into the isotope plutonium 239. After 15 years, it is determined that 0.043% of initial value A_0 of plutonium has been disintegrated. Find the half life of this isotope if the rate of disintegration is proportional to amount remaining.

Solution.

initial value problem for radioactive decay when $A(t)$ is representing the amount of plutonium is given by

$$\frac{dA}{dt} = KA \qquad A(0) = A_0$$

whose solution is

$$A(t) = A_0 e^{Kt} \qquad \to (1)$$

* if 0.043 percent of atoms of A_0 is disintegrated then remaining amount of A after 15 years is 99.957 % of A_0

$$\Rightarrow \quad A(15) = A_0 \times 0.99957.$$

using in (1) to find K

$$\Rightarrow \quad \cancel{A_0} \times 0.99957 = \cancel{A_0} e^{K \times 15}$$

$$e^{15K} = 0.99957$$

$$15K = \ln(0.99957)$$

$$K = \frac{\ln(0.99957)}{15} = -0.00002867.$$

$$\Rightarrow \quad K = -0.00002867.$$

$$A(t) = A_0 e^{-0.00002867t}$$

Half life is the corresponding value of time t at which $A(t) = \frac{A_0}{2}$

$$\Rightarrow \quad \frac{\cancel{A_0}}{2} = \cancel{A_0} e^{-0.00002867t}$$

$$-0.00002867t = \ln(1/2) \qquad \because \ln(1/2) = \ln(1) - \ln 2$$

$$\cancel{+}0.00002867t = \cancel{+}\ln 2$$

$$t = \frac{\ln 2}{0.00002867} = 24.180 \text{ yr.}$$

Age of Fossil

A fossilized bone is found to contain one thousand of the C-14 level found in living matter. Estimate the age of Fossil.

Solution

Note: Note that half-life of radioactive C-14 is approximately 5600 years.

Solution: Using solution of radioactive decay from previous example

$$A(t) = A_0 e^{kt} \qquad (1)$$

as half life of C-14 is 5600 so we can write

$$A(5600) = \frac{A_0}{2}$$

(1) $\Rightarrow$

$$\frac{\cancel{A_0}}{2} = \cancel{A_0} e^{5600k}$$

$$e^{5600k} = \frac{1}{2}$$

$$5600k = \ln(1/2) = -\ln 2$$

$$k = -\frac{\ln 2}{5600} = -0.00012378$$

so

$$A(t) = A_0 e^{-0.00012378}$$

To find the time when $A(t) = \frac{1}{1000} A_0$

$$\Rightarrow \frac{1}{1000} \cancel{A_0} = \cancel{A_0} e^{-0.00012378t}$$

$$e^{-0.00012378t} = \frac{1}{1000}$$

$$-0.00012378t = -\ln(1000)$$

$$\Rightarrow t = 55{,}800 \text{ yrs}$$

Newton's Law of cooling.

Newton's law of cooling and warming of an object is given by linear first-order differential equation

$$\frac{dT}{dt} = K(T - T_m) \qquad (1)$$

where

K: Constant of proportionality

$T(t)$: Temperature of object for $t > 0$.

T_m: Ambient temperature.

Example Cooling of cake

When a cake is removed from an oven, Its temperature is measured at 300° F. Three minutes later, its temperature is 200° F. How long will it take for the cake to cool off to a room temperature of 70° F.

Solution

comparing with Eq (1) $\Rightarrow$ $T_m = 70°F$.

So we have IVP

$$\frac{dT}{dt} = K(T - 70) \qquad T(0) = 300$$

and we'll determine K from the condition $T(3) = 200$.

Solving the IVP by separation of variable

$$\frac{dT}{T-70} = K\,dt$$

Integrating the Eq. gives yield

$$\ln|T - 70| = Kt + C_1$$

$$\Rightarrow \quad T - 70 = C_2 e^{Kt} \qquad e^{C_1} = C_2$$

$$\Rightarrow \quad T = 70 + C_2 e^{KT}$$

using the initial condition $T(0) = 300$

$$\Rightarrow \quad 300 = 70 + C_2$$

$$\Rightarrow \quad C_2 = 230$$

So

$$T = 70 + 230 e^{k \cdot}$$

Now using $T(3) = 200$

$$\Rightarrow \quad 200 = 70 + 230 e^{3k}$$

$$130 = 230 e^{3k}$$

$$\Rightarrow \quad e^{3t} = \frac{13}{23} \quad \Rightarrow \quad 3t = \ln\left(\frac{13}{23}\right).$$

$$k = \frac{\ln(13/23)}{3} = -0.19018$$

Thus

$$T(t) = 70 + 230 e^{-0.19018 t}$$

Looking at this solution, we get $\lim_{t\to\infty} T(t) = 70$ which says that cake will be at room temperature after a reasonably long time. It'll take approximately Half an hour to reach at room temperature.

$T(t)$	t (min)
75°	20.1
74°	21.3
73°	22.8
72°	24.9
71°	28.6
70.5°	32.3

Series Circuits

The linear differential Equation for the current $i(t)$ through a series circuit containing only a resistor and inductor is

$$L \frac{di}{dt} + Ri = E(t) \qquad (I)$$

where
L: Inductance
R: resistance

* The current $i(t)$ is called response of the system