4.6 Variation of Parameter.

To adapt the method of variation of parameter, to a linear second order differential Equation,

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

put the equation into a standard form. (by dividing with $a_2(x)$)

$$y'' + \frac{a_1(x)}{a_2(x)}y' + \frac{a_0(x)}{a_2(x)}y = \frac{g(x)}{a_2(x)}$$

$$y'' + P(x)y' + q(x)y = f(x). \qquad (1)$$

Let y_1 & y_2 be solutions of associated homogeneous equation (Linearly independent solution) then let particular solution be of the form

$$y_p = u_1(x)\,y_1(x) + u_2(x)\,y_2(x).$$

$$y_p' = u_1'(x)y_1(x) + u_1(x)y_1'(x) + u_2'(x)y_2(x) + u_2(x)y_2'(x)$$

$$y_p'' = u_1''(x)y_1(x) + u_1'(x)y_1'(x) + u_1'(x)y_1'(x) + u_1(x)y_1''(x) + u_2''(x)y_2'(x) + u_2'(x)y_2'(x) + u_2'(x)y_2'(x) + u_2(x)y_2''(x)$$

After using these values in (1) & using the fact that

$$y_1'' + Py_1' + qy_1 = 0$$

&

$$y_2'' + Py_2' + qy_2 = 0.$$

we get

$$y_p'' + P(x)y_p' + q(x)y_p = \frac{d}{dx}(y_1u_1' + y_2u_2') + P(y_1u_1' + y_2u_2') + y_1'u_1' + y_2'u_2' = f(x).$$

From here we get two Equations

$$y_1 u_1' + y_2 u_2' = 0$$

$$y_1' u_1' + y_2' u_2' = f(x).$$

After solving for u_1' & u_2', we get

$$u_1' = \frac{W_1}{W} = \frac{-y_2 f(x)}{W}$$

$$u_2' = \frac{W_2}{W} = \frac{+y_1 f(x)}{W}$$

where

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

Example

$$y'' - 4y' + 4y = (x+1)e^{2x}$$

Solution Associated homogeneous Equation is

$$y'' - 4y' + 4y = 0$$

writing the auxiliary Eq.

$$\Rightarrow \quad m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0$$

$$\Rightarrow \quad m_1 = 2 \qquad m_2 = 2$$

and complementary solution is

$$y(x) = c_1 e^{2x} + c_2 x e^{2x}.$$

So $y_1 = e^{2x}$ $\qquad y_2 = xe^{2x}$.

Finding Wronskian

$$W(e^{2x}, xe^{2x}) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix}$$

$$= e^{2x}(2xe^{2x} + e^{2x}) - 2xe^{2x} \cdot e^{2x}$$

$$= 2xe^{4x} + e^{4x} - 2xe^{4x}$$

$$= e^{4x}.$$

Identifying $f(x) = (x+1)e^{2x}$

$$W_1 = \begin{vmatrix} 0 & xe^{2x} \\ (x+1)e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix}$$

$$= -x(x+1)e^{4x} = -(x^2+x)e^{4x}.$$

$$W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (x+1)e^{2x} \end{vmatrix} = (x+1)e^{4x}.$$

Now $u_1' = \dfrac{W_1}{W} = \dfrac{-(x^2+x)\cancel{e^{4x}}}{\cancel{e^{4x}}} = -x^2 - x$.

Integrating

$$\Rightarrow \quad u_1 = -\frac{x^3}{3} - \frac{x^2}{2}.$$

$$u_2' = \frac{W_2}{W} = \frac{(x+1)\cancel{e^{4x}}}{\cancel{e^{4x}}} = x+1$$

Integrating

$$u_2 = \frac{x^2}{2} + x.$$

So

$$y_p = u_1 y_1 + u_2 y_2 = \left(-\frac{x^3}{3} - \frac{x^2}{2}\right)e^{2x} + \left(\frac{x^2}{2} + x\right)xe^{2x}.$$

$$y_p = \frac{1}{6}x^3e^{2x} + \frac{1}{2}x^2e^{2x}$$

$$y = y_c + y_p.$$

$$= c_1e^{2x} + c_2xe^{2x} + \frac{1}{6}x^3e^{2x} + \frac{1}{2}x^2e^{2x}.$$

Example

Solve $4y'' + 36y = \csc 3x$..

Solution

converting into standard form

$$y'' + 9y = \frac{1}{4}\csc 3x.$$

Associated homogeneous Equation is

$$y'' + 9y = 0$$

Auxiliary Eq. is

$$m^2 + 9 = 0 \quad \Rightarrow \quad m = \pm 3i$$

So

$$y(x) = c_1\cos 3x + c_2\sin 3x.$$

Identifying $y_1 = \cos 3x$ & $y_2 = \sin 3x$.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix}$$

$$= 3\cos^2 3x + 3\sin^2 3x = 3(\cos^2 3x + \sin^2 3x) = 3.$$

$$W_1 = \begin{vmatrix} 0 & \sin 3x \\ \csc 3x & 3\cos 3x \end{vmatrix} = -\sin 3x \csc 3x = 1.$$

$$W_2 = \begin{vmatrix} \cos 3x & 0 \\ -3\sin 3x & \csc 3x \end{vmatrix} = \cos 3x \cdot \csc 3x$$

$$= \frac{\cos 3x}{\sin 3x}.$$

So $u_1' = \frac{W_1}{W} = \frac{-1}{12}$

Integrating

$$u_1 = \frac{-x}{12}.$$

* $u_2' = \frac{W_2}{W} = \frac{1}{12}\,\frac{\cos 3x}{\sin 3x}.$

Integrating

$$u_2 = \frac{1}{12} \times \frac{1}{3} \ln|\sin 3x| = \frac{1}{36} \ln|\sin 3x|.$$

So

$$y_p = u_1 y_1 + u_2 y_2$$

$$= -\frac{x}{12}\cos 3x + \frac{1}{36}\sin 3x \ln|\sin 3x|.$$

and

$$y = y_c + y_p$$

$$= c_1 \cos 3x + c_2 \sin 3x - \frac{x}{12}\cos 3x + \frac{\sin 3x}{36}\ln|\sin 3x|.$$

Solve $y'' - y = \frac{1}{x}$.

Solution Associated homogeneous Equation

$$y'' - y = 0$$

$\Rightarrow$ $m^2 - 1 = 0$ $\Rightarrow$ $m = \pm 1$.

$y(x) = c_1 e^x + c_2 e^{-x}$.

$y_1 = e^x$ $\qquad$ $y_2 = e^{-x}$.

$$W(e^x, e^{-x}) = \begin{vmatrix} e^x & e^{-x} \\ e^{+x} & -e^{-x} \end{vmatrix} = -e^x e^{-x} - e^x e^{-x}$$

$$= -e^0 - e^0 = -2$$

using $f(x) = 1/x$

$$W_1 = \begin{vmatrix} 0 & e^{-x} \\ 1/x & -e^{-x} \end{vmatrix} = \frac{-e^{-x}}{x},$$

$$W_2 = \begin{vmatrix} e^x & 0 \\ e^x & 1/x \end{vmatrix} = \frac{e^x}{x}$$

So

$$u_1' = \frac{W_1}{W} \qquad\qquad u_2' = \frac{W_2}{W}$$

$$u_1' = -\frac{e^{-x}}{x} \Big/ -2 \qquad\qquad u_2' = \frac{e^x}{x} \Big/ 2$$

$$= \frac{1}{2} \cdot \frac{e^{-x}}{x} \qquad\qquad = \frac{e^x}{2x}$$

Integrating $\qquad\qquad$ Integrating

$$u_1 = \frac{1}{2} \int_{x_0}^{x} \frac{e^{-t}}{t}\, dt \qquad\qquad u_2 = \frac{1}{2} \int_{x_0}^{x} \frac{e^{t}}{t}\, dt$$

So

$$y_p = u_1 y_1 + u_2 y_2$$

$$= e^x \times \frac{1}{2} \int_{x_0}^{x} \frac{e^{-t}}{t}\, dt + \frac{e^{-x}}{2} \int_{x_0}^{x} \frac{e^{t}}{t}\, dt$$

&

$$y = y_c + y_p$$

Higher order Equations

For higher order, Let suppose for a third order DE.

$$y_p = u_1 y_1 + u_2 y_2 + u_3 y_3$$

where $u_1' = \frac{W_1}{W}$, $u_2' = \frac{W_2}{W}$, $u_3' = \frac{W_3}{W}$

and

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ f(x) & y_2'' & y_3'' \end{vmatrix}.$$

and similarly we can write W_2 & W_3.

Exercise 4.6.

8 $y'' - y = \sinh 2x$.

using the formula that

$$\sinh 2x = \frac{e^{2x} \cancel{+} - e^{-2x}}{2}$$

Associated homogeneous Eq. is

$$y'' - y = 0$$

auxiliary Eq. is $m^2 - 1 = 0 \Rightarrow m = \pm 1$,

$$y(x) = c_1 e^{x} + c_2 e^{-x}.$$

So $y_1(x) = e^{x}$. $y_2 = e^{-x}$.

$$w(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix}$$

$$= -e^{0} - e^{0} = -2.$$

$$W_1 = \begin{vmatrix} 0 & e^{x} \\ \frac{1}{2}e^{2x} - \frac{1}{2}e^{-2x} & -e^{-x} \end{vmatrix}$$

$$= -\left[e^{-x}\left(\frac{1}{2}e^{2x} - \frac{1}{2}e^{-2x}\right)\right]$$

$$= \frac{1}{2}\left(e^{-3x} - e^{x}\right).$$

$$W_2 = \begin{vmatrix} e^{x} & 0 \\ e^{x} & \frac{e^{2x}}{2} - \frac{e^{-2x}}{2} \end{vmatrix} = \frac{1}{2}\left(e^{3x} - e^{-x}\right)$$

Now

$$u_1' = \frac{w_1}{w} = \frac{1}{2}\,\frac{e^{-3x} - e^{-x}}{-2} = -\frac{1}{4}e^{-3x} + \frac{1}{4}e^{-x},$$

$$u_1 = \cancel{+} - \frac{e^{-3x}}{-12} + \frac{e^{-x}}{4} = \frac{1}{12}e^{-3x} + \frac{1}{4}e^{-x}.$$

$$u_2' = \frac{W_2}{W}$$

$$= \frac{\frac{1}{2}(e^{3x} - e^{-x})}{-2} = \frac{1}{4}(e^{-x} - e^{3x})$$

Integrating

$$u_2 = -\frac{e^{-x}}{4} - \frac{e^{3x}}{12}$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \left(\frac{1}{4}e^{x} + \frac{1}{12}e^{-3x}\right)e^{x} + \left(-\frac{1}{4}e^{-x} - \frac{1}{12}e^{3x}\right)e^{-x}$$

$$= \frac{1}{4}e^{2x} + \frac{1}{12}e^{-2x} - \frac{1}{4}e^{-2x} - \frac{1}{12}e^{2x}$$

$$= \frac{1}{6}e^{2x} \cancel{+} - \frac{1}{6}e^{-2x}$$

$$= \frac{1}{3}\left(\frac{e^{2x} - e^{-2x}}{2}\right) = \frac{1}{3}\sinh(2x)$$

$$y = y_c + y_p$$

$$= c_1 e^{x} + c_2 e^{-x} + \frac{1}{3}\sinh 2x$$

12 $y'' \cancel{+} - 2y' + y = \frac{e^x}{1+x^2}$

Associated homogeneous Eq. is

$$y'' - 2y' + y = 0$$

auxiliary Eq. is

$$m^2 - 2m + 1 = 0$$

$$(m-1)^2 = 0$$

$m = 1$ $\quad\quad m = 1$

$$y(x) = c_1 e^{x} + c_2 x e^{x}$$

$y_1 = e^x \qquad y_2 = xe^x$

$$W(y_1, y_2) = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix}$$

$$= xe^{2x} + e^{2x} - xe^{2x} = e^{2x}$$

Here $f(x) = \dfrac{e^x}{1+x^2}$

$$W_1 = \begin{vmatrix} 0 & xe^x \\ \dfrac{e^x}{1+x^2} & xe^x + e^x \end{vmatrix}$$

$$= \frac{xe^{2x}}{1+x^2}$$

$$W_2 = \begin{vmatrix} e^x & 0 \\ e^x & \dfrac{e^x}{1+x^2} \end{vmatrix}$$

$$= \frac{e^{2x}}{1+x^2}$$

So $u_1' = \dfrac{W_1}{W} = \dfrac{xe^{2x}/1+x^2}{e^{2x}} = \dfrac{x}{1+x^2}$

Integrating

$$u_1 = \frac{1}{2}\int \frac{2x}{1+x^2}\,dx = \frac{1}{2}\ln(1+x^2)$$

$$u_2' = \frac{W_2}{W} = \frac{e^{2x}/1+x^2}{e^{2x}} = \frac{1}{1+x^2}$$

Integrating

$$u_2 = \tan^{-1}x$$

So $y_p = u_1 y_1 + u_2 y_2 = \dfrac{e^x}{2}\ln(1+x^2) + xe^x \tan^{-1}x$

$$y = c_1 e^x + c_2 xe^x + \frac{e^x}{2}\ln(1+x^2) + xe^x \tan^{-1}x.$$

Solve each DE by variation of parameter subject to I.C. $y(0) = 1$, $y'(0) = 0$.

22 $y'' - 4y' + 4y = (12x^2 - 6x)e^{2x}$

Associated homogeneous Eq is

$$y'' - 4y' + 4y = 0$$

auxiliary Eq is

$$m^2 - 4m + 4 = 0 \quad \Rightarrow \quad (m-2)^2 = 0$$

$$m_1 = 2 \qquad m_2 = 2.$$

$$y(x) = c_1 e^{2x} + c_2 x e^{2x}$$

So $y_1(x) = e^{2x}$. $\quad y_2 = xe^{2x}$.

$$W = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix}$$

$$= e^{4x} + 2xe^{4x} - 2xe^{4x} = e^{4x}$$

$$W_1 = \begin{vmatrix} 0 & xe^{2x} \\ (12x^2 - 6x)e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix}$$

$$= (-12x^3 + 6x^2)e^{4x}$$

$$W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (12x^2 - 6x)e^{2x} \end{vmatrix}$$

$$= (12x^2 - 6x)e^{4x}$$

$$u_1' = \frac{W_1}{W} = \frac{(-12x^3 + 6x^2)\cancel{e^{4x}}}{\cancel{e^{4x}}}$$

Integrating

$$u_1 = -3x^4 + 2x^3$$

$$u_2' = \frac{W_2}{W} = \frac{(12x^2-6x)\cancel{e^{4x}}}{\cancel{e^{4x}}}$$

Integrating

$$u_2 = 4x^3 - 3x^2$$

So $y_p = (-3x^4+2x^3)e^{2x} + (4x^3-3x^2)xe^{2x}$

$$= x^4e^{2x} - x^3e^{2x}$$

$$y = y_c + y_p$$

$$= c_1e^{2x} + c_2xe^{2x} + x^4e^{2x} - x^3e^{2x}$$

using $y(0)=1$

$\Rightarrow \quad 1 = c_1$

Taking derivative

$$y'(x) = 2c_1e^{2x} + c_2(e^{2x}+2xe^{2x}) + 4x^3e^{2x} + 2x^4e^{2x} - 3x^2e^{2x} - 2x^3e^{2x}.$$

using

$y'(0) = 0$

$\Rightarrow \quad 2c_1 + c_2 = 0 \quad \Rightarrow \quad c_2 = -2$

Final solution is

$$y(x) = e^{2x} - 2xe^{2x} + x^4e^{2x} - x^3e^{2x}.$$

Find the general solution of given non-homogeneous Eq.

Sol. 24 $\quad x^2y'' + xy' + y = \sec(\ln x)$

$y_1 = \cos(\ln x) \qquad y_2 = \sin(\ln x).$

As $y_1 = \cos(\ln x)$ & $y_2 = \sin(\ln x)$ are L.I. function so complementary solutions is

$$y(x) = c_1 \cos(\ln x) + c_2 \sin(\ln x).$$

Now

$$W = \begin{vmatrix} \cos(\ln x) & \sin(\ln x) \\ -\frac{\sin(\ln x)}{x} & \frac{\cos(\ln x)}{x} \end{vmatrix}$$

$$= \frac{1}{x}\cos^2(\ln x) + \frac{1}{x}\sin^2(\ln x)$$

$$= \frac{1}{x}.$$

$$W_1 = \begin{vmatrix} 0 & \sin(\ln x) \\ \frac{\sec(\ln x)}{x^2} & \frac{\cos(\ln x)}{x} \end{vmatrix}$$

$$= -\sin(\ln x) . \frac{\sec(\ln x)}{x^2}$$

$$= -\frac{\tan(\ln x)}{x^2}$$

$$W_2 = \begin{vmatrix} \cos(\ln x) & 0 \\ -\frac{\sin(\ln x)}{x} & \frac{\sec(\ln x)}{x^2} \end{vmatrix}$$

$$= \frac{\cos(\ln x)\sec(\ln x)}{x^2} = \frac{1}{x^2}$$

$$u_1' = \frac{W_1}{W} = \frac{-\tan(\ln x)/x^2}{1/x} = \frac{-\tan(\ln x)}{x}$$

Integrating

$$u_1 = -\int \frac{\tan(\ln x)}{x}\,dx$$

Let $u = \ln x \Rightarrow du = \frac{1}{x}\,dx$

$$\Rightarrow \quad u_1 = -\int \tan u\,du = -\int \frac{\sin u}{\cos u}\,du$$

$$= \ln(\cos u) = \ln(\cos(\ln x))$$

$$u_2' = \frac{W_2}{W}$$

$$= \frac{1/x^2}{1/x} = 1/x$$

Integrating

$$u_2 = \ln x.$$

So

$$y_p = \cos(\ln x)\ln(\cos(\ln x)) + \sin(\ln x)\ln x$$

& General solution is

$$y = y_c + y_p.$$