CHAPTER 12

Boundary value problem in rectangular coordinates

Linear partial differential equation

Let u denote the dependent variable and let x & y denote the independent variables, then the general form of a linear second-order partial differential equation is given

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G \quad (1)$$

where the coefficients $A, B, C, \dots, G$ are functions of x & y.

* When $G(x,y)=0$, Eq. (1) is said to be homogeneous. otherwise it is non-homogeneous.

Examples

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad \text{Homogeneous Equation}$$

$$\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial y}=xy \qquad \text{Non-homogeneous Equation.}$$

Solution of PDE

A solution of a linear partial differential equation (1) is a function $u(x,y)$ of two independent variables that possesses all partial derivatives occuring in the equation and that satisfies the equation in some region of xy-plane.

Separation of variables

In method of separation of variables, we consider a particular solution of the form of product of function of x and function of y.

$$u(x,y) = X(x)\,Y(y).$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = X'Y, \qquad \frac{\partial u}{\partial y} = X(x)\,Y'(y)$$

$$\frac{\partial^2 u}{\partial x^2} = X''Y, \qquad \frac{\partial^2 u}{\partial y^2} = XY''$$

Example

Find the product solution of

$$\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial u}{\partial y}$$

Solution Substituting $u(x,y) = X(x)\,Y(y)$

$$\Rightarrow \quad X''(x)\,Y(y) = 4X(x)\,Y'(y)$$

$$\Rightarrow \quad \frac{X''}{4X} = \frac{Y'}{Y} = -\lambda$$

$$\Rightarrow \quad \frac{X''}{4X} = -\lambda \qquad \frac{Y'}{Y} = -\lambda$$

$$X'' + 4\lambda X = 0, \qquad Y' + \lambda Y = 0.$$

Three cases of λ.

Case 1 $\lambda = 0$

$$\Rightarrow \quad X'' = 0 \qquad Y' = 0$$

$$\Rightarrow \quad X = c_1 + c_2 x \qquad Y = c_3$$

$$u = XY = (c_1 + c_2 x)\,c_3$$

$$= c_1 c_3 + c_2 c_3 x = A_1 + B_1 x.$$

Case II

$\lambda = -\alpha^2 < 0$

$$\Rightarrow X'' - 4\alpha^2 X = 0 \qquad Y' - \alpha^2 Y = 0.$$

$$m^2 - 4\alpha^2 = 0$$

$$m = \pm 2\alpha$$

$$X(x) = c_1 . e^{2\alpha x} + c_2 e^{-2\alpha x}$$

or

$$X(x) = c_1 \cosh(2\alpha x) + c_2 \sinh(2\alpha x). \qquad (A)$$

$$Y' - \alpha^2 Y = 0$$

$$\Rightarrow \frac{dY}{dy} = +\alpha^2 Y \Rightarrow \frac{dY}{Y} = +\alpha^2 dy.$$

$$\Rightarrow \ln Y = +\alpha^2 y + c_3$$

$$Y = c_3 e^{+\alpha^2 y} \qquad (B)$$

Case III

$\lambda = \alpha^2 > 0$

$$X'' + 4\alpha^2 X = 0 \qquad Y' + \alpha^2 Y = 0$$

$$m^2 + 4\alpha^2 = 0$$

$$m = \pm 2i\alpha.$$

$$\Rightarrow X(x) = c_1 \cos(2\alpha x) + c_2 \sin(2\alpha x) \qquad (C)$$

$$Y' = -\alpha^2 Y$$

$$\frac{dY}{dy} = -\alpha^2 Y \Rightarrow \frac{dY}{Y} = -\alpha^2 dy.$$

$$Y = c_3 e^{-\alpha^2 y}. \qquad (D)$$

From (C) & (D),

$$\begin{aligned} u(x,y) &= X(x)\,Y(y) \\ &= \left(c_1 \cos(2\alpha x) + c_2 \sin(2\alpha x)\right) c_3 e^{-\alpha^2 y} \\ &= A_3 e^{-\alpha^2 y} \cos(2\alpha x) + A_4 e^{-\alpha^2 y} \sin(2\alpha x) \end{aligned}$$

Solution of Case II from (A) & (B)

$$u(x,y) = X(x)\,Y(y)$$

$$= \left(c_1 \cosh(2\alpha x) + c_2 \sinh(2\alpha x)\right) c_3 e^{\alpha^2 y}$$

$$= A_2 e^{\alpha^2 y} \cosh(2\alpha x) + B_2 e^{\alpha^2 y} \sinh(2\alpha x).$$

Superposition principle

If $u_1, u_2, \cdots, u_k$ are solutions of a homogeneous linear partial differential equation, then the linear combination

$$u = c_1 u_1 + c_2 u_2 + c_3 u_3 + \cdots + c_k u_k$$

where the c_i, $i = 1, 2, \cdots, k$ are constants, is also a solution

Classification of Equations

The linear second-order partial differential equation

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu = 0$$

where A, B, C, D, E and F are real constants, is said to be

hyperbolic if $B^2 - 4AC > 0$

Parabolic if $B^2 - 4AC = 0$

Elliptic if $B^2 - 4AC < 0$.

Example Classify the following equation

a) $3 \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial y}$

$$3\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = 0$$

$\Rightarrow A = 3 \quad B = 0 \quad C = 0$

$\Rightarrow \quad B^2 - 4AC = 0 - 0 = 0$

So equation is parabolic

b) $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$$

$A = 1, \; B = 0, \; C = -1.$

$B^2 - 4AC = 0 + 4 = +4 > 0.$

So equation is hyperbolic.

c) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

$A = 1 \quad B = 0 \quad C = 1.$

$\Rightarrow \quad B^2 - 4AC = -4 < 0.$

So equation is elliptic.

Exercise 12.1

Use separation of variables to find, if possible, product solutions for given partial differential Equation.

(3) $u_x + u_y = u$.

Let $u(x,y) = X(x)\,Y(y)$

put in given DE

$\Rightarrow \quad X'(x)\,Y(y) + X(x)\,Y'(y) = X(x)\,Y(y)$

$\Rightarrow$ $\div$ing by $X(x)\,Y(y)$

$$\Rightarrow \quad \frac{X'(x)}{X(x)} + \frac{Y'(y)}{Y(y)} = 1$$

$$\frac{X'}{X} = 1 - \frac{Y'}{Y} = -\lambda$$

$$\Rightarrow \quad X' = -\lambda X \quad \Rightarrow \quad X' + \lambda X = 0 \qquad (1)$$

$$1 - \frac{Y'}{Y} = -\lambda \quad \Rightarrow \quad \frac{Y'}{Y} = 1 + \lambda \qquad (2)$$

Solving (1) by separation of variable.

$$\Rightarrow \quad \frac{dX}{X} = -\lambda\,dx$$

$$\Rightarrow \quad \ln X = -\lambda x + \ln c_1$$

$$\Rightarrow \quad X = c_1 e^{-\lambda x}.$$

$$(2) \Rightarrow \quad \frac{dY}{Y} = (1+\lambda)\,dy$$

$$\ln Y = \cancel{c_4}\,(1+\lambda)\,y + \ln c$$

$$Y = c_2 e^{(1+\lambda)y}.$$

So

$$u(x,t) = X(x)\ Y(y)$$
$$= c_1 e^{-\lambda x} \cdot c_2 e^{(1+\lambda)y}$$
$$= k e^{-\lambda x + (1+\lambda)y}.$$

8 $y \dfrac{\partial^2 u}{\partial x \partial y} + u = 0$

$$u(x,\cancel{t}y) = X(x)\ Y(y)$$

$\Rightarrow$ $$\frac{\partial u}{\partial y} = X(x)\ Y'(y)$$
$$\frac{\partial^2 u}{\partial x \partial y} = X'(x)\ Y'(y)$$

$\Rightarrow$ $y\ X'(x)\ Y'(y) + X(x) Y(y) = 0$ $\quad$ $\div$ing by $X(x)\ Y(y)$

$\Rightarrow$ $y\ \dfrac{X'(x)}{X(x)} * \dfrac{Y'(y)}{Y(y)} + 1 = 0$

$$\frac{X'(x)}{X(x)} = -\frac{1}{y}\ \frac{Y(y)}{Y'(y)} = -\lambda$$

$\Rightarrow$ $X'(x) = -\lambda X(x)$

$X(x) = c_1 e^{-\lambda \cancel{x}}$

$\Rightarrow$ $-\dfrac{1}{y}\ \dfrac{Y(y)}{Y'(y)} = -\lambda$

$\Rightarrow$ $-Y(y) = -\lambda y\ Y'(y)$

$$Y'(y) - \frac{1}{\lambda y} Y(y) = 0$$

Integrating factor

$$\Rightarrow \quad e^{\int -\frac{1}{\lambda y} dy} = e^{-\frac{1}{\lambda} \int \frac{1}{y} dy}$$

$$= e^{-\frac{1}{\lambda} \ln y} \Rightarrow e^{\ln y^{-1/\lambda}}$$

$$= y^{-1/\lambda}$$

$$\Rightarrow \quad \frac{d}{dy}\left(y^{-1/\lambda} Y\right) = 0 \quad \text{after solving}$$

$$Y = c_2 y^{1/\lambda}$$

So

$$u(x,y) = X(x)\, Y(y)$$

$$= c_1 e^{-\lambda x} \cdot c_2 y^{1/\lambda}$$

$$= k e^{-\lambda x} y^{1/\lambda}$$

12 $$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + 2k \frac{\partial u}{\partial t} \qquad k > 0$$

$$u(x,t) = X(x)\, T(t)$$

$$\Rightarrow \quad a^2 X'' T(t) = X(x) T''(t) + 2k X(x) T'(t)$$

÷ing by $X(x) T(t)$

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{a^2 T(t)} + \frac{2k}{a^2} \frac{T'(t)}{T(t)} = -\lambda$$

1

$$X''(x) + \lambda X(x) = 0$$

$$T''(t) + 2k T'(t) + \lambda a^2 T(t) = 0$$

$\lambda = 0$

$$X'' = 0$$

$$X = c_1 x + m_1$$

$$T'' + 2KT' = 0$$

$$m^2 + 2Km = 0 \qquad m(m+2k) = 0$$

$$m = 0 \qquad m = -2K$$

$$T(t) = c_2 + c_3 e^{-2Kt}$$

$$u(x,t) = X(x)\, T(t)$$

$$= (c_1 x + m_1)(c_2 + c_3 e^{-2Kt})$$

~~$\lambda = \alpha^2$~~

$\lambda = \alpha^2 > 0$

$$X'' + \alpha^2 X = 0$$

$$m^2 + \alpha^2 = 0 \qquad \Rightarrow m = \pm i\alpha$$

$$X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x.$$

$$T'' + 2KT' + a^2\alpha^2 T = 0$$

auxiliary Equation

$$\Rightarrow \quad m^2 + 2Km + a^2\alpha^2 = 0$$

$$m = \frac{-2K \pm \sqrt{4K^2 - 4\alpha^2 a^2}}{2}$$

$$= -K \pm \sqrt{K^2 - a^2\alpha^2}$$

$$= -K \pm n \qquad \text{where } n = \sqrt{K^2 - a^2\alpha^2}.$$

$$T(t) = c_3 e^{(-K+n)t} + c_4 e^{-(K+n)t}$$

$$u(x,t) = X(x)\, T(t)$$

$$= (c_1 \cos \alpha x + c_2 \sin \alpha x)(c_3 e^{(-K+n)t} + c_4 e^{-(K+n)t})$$

b. $\lambda = -\alpha^2$

$$X'' - \alpha^2 X = 0$$

$$\Rightarrow \quad m^2 - \alpha^2 = 0 \quad \Rightarrow \quad m = \pm \alpha .$$

$$X(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$$

$$T'' + 2KT' + \alpha^2 a^2 T = 0$$

$$\Rightarrow \quad m^2 + 2Km + \alpha^2 a^2 = 0$$

$$m = -K \pm \sqrt{K^2 + \alpha^2 a^2}$$

$$= -K \pm \omega \qquad \omega = \sqrt{K^2 + \alpha^2 a^2}$$

$$T(t) = c_3 e^{(-K+\omega)t} + c_4 e^{-(K+\omega)t}$$

so

$$u(x,t) = (c_1 e^{\alpha x} + c_2 e^{-\alpha x})(c_3 e^{(-K+\omega)t} + c_4 e^{-(K+\omega)t})$$

16 $a^2 u_{xx} - g = u_{tt}$

$$u(x,t) = X(x)\,T(t)$$

$$\Rightarrow \quad a^2 X''(x)\,T(t) - g = X\,T''(t)$$

$\div$ing by $X(x)\,T(t)$

$$\Rightarrow \quad a^2 \frac{X''(x)}{X} - \frac{g}{X\,T(t)} = \frac{T''(t)}{T(t)}$$

so equation is not separable.

Classify the given partial differential equation as hyperbolic, parabolic or elliptic

18 $3 \dfrac{d^2 u}{dx^2} + 5 \dfrac{\partial^2 u}{\partial x \partial x} + \dfrac{\partial^2 u}{\partial y^2} = 0$

Here $A = 3$

$B = 5$

$C = 1$

So

$$B^2 - 4AC = 25 - 4(3)(1)$$
$$= 25 - 12$$
$$= 13 > 0$$

So given Eq. is hyperbolic.