Introduction to differential Equations

Differential Equation

A differential equation is an equation that contains one or more derivatives of an unknown function.

A differential equation is an ordinary differential equation (ODE) if it involves an unknown function of only one variable.

Example

$$y' = y + x$$

is a differential equation. The unknown function is y depending on the variable x and derivative of y is involved in the equation.

$$y^{a} = 5 + x$$

is not a differential equation. There is a function y in the equation but derivative is not involved.

Where differential Equations arise

In physical and many real life problems, we want to study the relations b/w changing quantities. To apply the mathematical Method to such a problem, we need to formulate the problem using mathematical concepts and construct a mathematical model to describe it

The process of developing a mathematical model is called Mathematical Modeling.

Example Population Growth and Decay.

The number P of member of population (People in a given country, bacteria in a labortary culture etc) at any given time t can be modeled by Differential Equations. In most models, It is assumed that the differential equations takes the form

$$P'(t) = a(P)\, P(t)$$

where a is a continuous function of population $P(t)$ that represent the relative rate of change per unit time, known as growth rate.

Newton's Law of cooling:

According to Newton's law of cooling, the temperature of a body changes at a rate directly proportional to the difference in the temperatures b/w the temperature of the body and the temperature of its surrounding. If T_m is the temperature of the surrounding and $T = T(t)$ is the temperature of the body at time t, then

$$T'(t) = -K\,(T(t) - T_m(t))$$

where K is a positive constant and -ve sign indicates that heat transfer from hot to cold object.

Exponential Growth

The exponential grown can be represented by

$$\frac{dy}{dt} = Ky \qquad K > 0.$$

* Also the mathematical description or mathematical model of experiments, observation or theories may be a differential equation

Section 1.1

Differential Equations with Boundary Value Problems
by Dennis G. Zill and Michael R. Curren
7th Edition

Definition: Differential Equation.

An equation containing the derivative of one or more dependent variables, with respect to one or more independent variables, is said to be a differential equation.

Classification

A differential equation can be classified on the basis of type, order and linearity.

Classification by Types.

~~OD~~ ~~o~~ordinary Differential Equations (ODE).

If an equation involves the only ordinary derivatives of one or more dependent variables with respect to only one independent variable, it is said to be ordinary differential equation.

Example

$$\frac{dy}{dx} = x\sin x$$

$$\frac{dx}{dt} + \frac{dy}{dt} = 2x + y \rightarrow \text{More than one dependent variables}$$

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0$$

* Partial Differential Equations (PDEs)

An equation involving partial derivative of ~~two~~ one or more dependent variable with respect to two or more independent variables is called a partial differential equation

Notations

* Ordinary derivatives

Leibniz notation $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots$

Prime notation. $y', y'', y''', \ldots$

Actually prime notation is used to denote only first three derivatives. The forth derivative is written as $y^{(4)}$ instead of y''''.

Newton's dot notation.

It is used to denote the time derivative.

$$\frac{d^2s}{dt^2} = -32 \text{ can be written as } \ddot{s} = -32.$$

* Partial derivatives.

Partial derivatives are usually denoted by subscript notation indicating the independent variables.

e.g.

$$u_{xx} = u_{tt} - 2u_t.$$

* Classification by order.

The order of a differential equation is the order of the highest derivative in the Equation.

e.g.

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x$$

is a second order differential Equation.

* An nth order ordinary differential equation in one dependent variable can be written in general form as

$$F(x, y, y', \ldots, y^{(n)}) = 0. \quad (1)$$

* The normal form of equation (1) is written as

$$\frac{d^ny}{dx^n} = f(x, y, y', \ldots, y^{(n-1)})$$

Normal form of $y'' - y' + 6y = 0$ is $y'' = y' - 6y$.

Classification by Linearity.

An n^{th} order ordinary differential equation is said to be **linear** if F is linear in $y, y', \ldots, y^{(n)}$. a
An n^{th} order ODE is linear when it is of the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x). \quad (A)$$

The characteristic properties of linear ODEs are

* The dependent variable y and all of its derivatives $y', y'', \ldots, y^{(n)}$ are of the first degree, i-e- power of each term involving y is (1)
* The coefficients $a_0, a_1, \ldots a_n$ of $y, y', y'' \ldots y^{(n)}$ depends at most on the independent variable x.

Example

$$(y-x)\,dx + 4x\,dy = 0$$

Linear first order ordinary differential equation.

$$y'' - 2y' + y = 0$$

Second order linear ordinary differential equation.

$$\frac{d^3y}{dx^3} + x\frac{dy}{dx} - 5y = e^x$$

3^{rd}-order linear ordinary differential equation

* Nonlinear ODE

A nonlinear ordinary differential equation is the one which fails to be linear.

* Non-linear function of dependent variable and its derivative such as $\sin y$, e^y etc. cannot appear in linear Equation

$$(1-y)y' + 2y = e^x, \quad \frac{d^2y}{dx^2} + \sin y = 0, \quad \frac{d^4y}{dx^4} + y^2 = 0$$

Non linear terms.

Solution of an ODE.

Any function ϕ defined on an interval I, and possessing atleast n derivatives that are continuous on I, which when substituted into an n^{th} order differential equation reduces the equation to an identity, is said to be a solution of the equation on the interval.

Interval of defintion

The interval I over which the function ϕ is the solution of differential equation is called interval of definition, interval of existence, the interval of validity, or the domain of the solution.

* Interval of definition can be an open interval, (a,b), a closed interval $[a,b]$, an infinite interval (a,∞) and so on.

* Verification of the Solution

Verify that the indicated function is a solution of given differential Eq. on $(-\infty,\infty)$.

a) $\dfrac{dy}{dx} = xy^{1/2}$ $\quad y = \dfrac{x^4}{16}$

Left hand side

$$\frac{dy}{dx} = \frac{4x^3}{16} = \frac{x^3}{4}$$

Right hand side

$$xy^{1/2} = x \times \left(\frac{x^4}{16}\right)^{1/2} = x \times \frac{x^2}{4} = \frac{x^3}{4}.$$

$\Rightarrow$ L.H.S = R.H.S.

So $y = \dfrac{x^4}{16}$ is the solution of given differential Eq.

b) $y'' - 2y' + y = 0$ $\quad y = xe^x$

Left hand side:

$$y' = xe^x + e^x$$

$$y'' = xe^x + e^x + e^x = xe^x + 2e^x$$

$$\Rightarrow \quad y'' - 2y' + y = \cancel{x}e^x + \cancel{2}e^x - 2xe^x \cancel{2}e^x + \cancel{xe^x}$$

$$= 0 = R.H.S.$$

Trivial Solution

A solution of differential equation that is identically zero on an interval I is said to be a trivial solution.

* Solution curve

The graph of solution φ of an ODE is called a solution curve.

* There may be difference b/w the graph of the function φ and the graph of the solution φ.

(Domain of the given function & Domain of solution may be different)

Function versus function

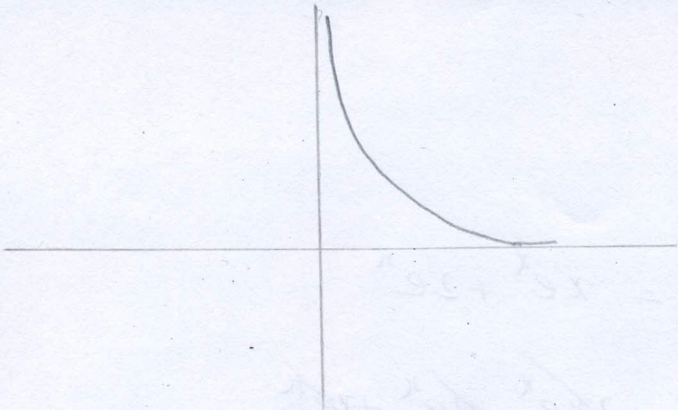
The domain of $y = \frac{1}{x}$ as a function is all real numbers except 0 i-e $(-\infty, 0) \cup (0, \infty)$.

* $y = \frac{1}{x}$ is also the solution of the Eq. $xy' + y = 0$.

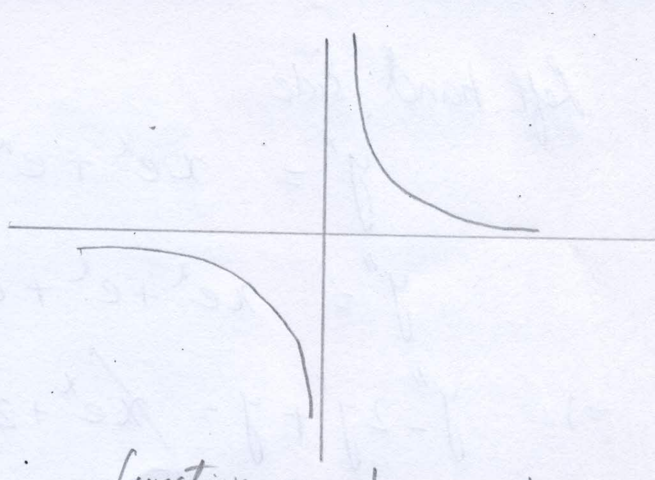
But when we say that $y = \frac{1}{x}$ is a solution of this DE, we mean that it is a function defined on the interval I, on which it is differentiable and satisfies the differential equation.

So $y = \frac{1}{x}$ is solution of DE on any interval that does not contain 0.

So I here can be either $(-\infty, 0)$ or $(0, \infty)$



Solution $y = \frac{1}{x}$ $(0, \infty)$



function $y = \frac{1}{x}$, $x \neq 0$.

Explicit and Implicit solution

A Solution in which dependent variable is expressed Solely in terms of independent variable and constants is said to be an explicit solution.

Implicit Solution

A relation $G(x, y) = 0$ is said to be an implicit Solution of an ordinary differential Equation on an interval I, provided that there exist at least one function ϕ that Satisfies the relation as well as the differential equation on I.

* Verification of an Implicit Solution

The relation $x^2 + y^2 = 25$ is an implicit solution of

$$\frac{dy}{dx} = -\frac{x}{y}$$

on the open interval $(-5, 5)$.

$$x^2 + y^2 = 25$$

By implicit differentiation

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

Two explicit solutions $y = \pm\sqrt{25 - x^2}$ satisfy the relation $x^2 + y^2 = 25$ and the differential Eq. $\frac{dy}{dx} = -\frac{x}{y}$ on the interval $(-5, 5)$.

Families of the solutions

* When we solve a differential equation of the form $f(x, y, y') = 0$, we obtain a solution containing a single arbitrary parameter or constant.

One parameter family of solutions

A solution containing an arbitrary constant represents a set $G(x, y, c) = 0$ of the solutions called one-parameter family of the solutions.

* When we solve an n^{th} order ODE, we get a n-parameter family of the solutions $G(x, y, c_1, c_2, \ldots c_n) = 0$.

* It means that a single differential equation can possess infinitely many solutions for unlimited choices of the solutions.

* Particular solution.

A solution of differential Equation that is free of arbitrary constants is called particular solution.

Example

The functions $x = c_1 \cos 4t$ and $x = c_2 \sin 4t$, where c_1 and c_2 are arbitrary constants and or parameters, are both solution of linear differential Equation $x'' + 16x = 0$

$$x = c_1 \cos 4t$$
$$x' = -4c_1 \sin 4t$$
$$x'' = -16c_1 \cos 4t$$

$$\Rightarrow \quad x'' + 16x = -16c_1 \cos 4t + 16c_1 \cos 4t$$
$$= 0$$

Also, for

$$x = c_1 \sin 4t$$
$$x' = 4c_1 \cos 4t$$

$$x'' = -16 c_2 \sin 4t$$

$$x'' + 16x = -16c_2 \sin 4t + 16 c_2 \sin 4t$$
$$= 0$$

* Also linear combination of the solutions, or the two parameter family

$$x = c_1 \cos 4t + c_2 \sin 4t$$

is also a solution of differential equation

System of differential Equations.

A system of differential equations is two or more equations involving the derivative of two or more unknown functions of a single independent variable

If x and y denote dependent variables and t denotes the independent variable, then a system of two first order differential equations is

$$\frac{dx}{dt} = f(t, x, y)$$

$$\frac{dy}{dt} = g(t, x, y)$$

* A solution of such system is pair of differentiable functions $x = \phi_1(t)$, $y = \phi_2(t)$ defined on the common interval I that satisfies each equation of system on this interval.

Exercise 1.1

State the order of given differential equation. Determine whether the equation is linear or non-linear by using the definitions:

3) $t^5 y^{(4)} - t^3 y'' + 6y = 0$.

Fourth order Linear differential Equation.

7) $\sin\theta\, y''' - \cos\theta\, y' = 0$

Third order linear differential Equation

8) $\ddot{x} - \left(1 - \frac{\dot{x}^2}{3}\right)\dot{x} + x = 0$

Second order non-linear.

10. $u\,dv + (v + uv - ue^u)\,du$.

Check whether Eq. is linear or nonlinear in the given variable.

(1) in v.

Eq is linear in v.

(2) in u.

Equation is nonlinear in u.

Verify that the indicated function is an explicit solution of the given differential Equations. Assume the appropriate interval of definition of each solution.

14 $y'' + y = \tan x$ $\quad y = -\cos x \ln(\sec x + \tan x)$

Verifying the solution

$$y' = -\cos x \times \frac{1}{\sec x + \tan x}(\sec x \tan x + \sec^2 x) + \sin x \ln(\sec x + \tan x)$$

$$= -\cos x \times \sec x + \sin x \ln(\sec x + \tan x)$$

$$= -1 + \sin x \ln(\sec x + \tan x).$$

$$y'' = \sin x \times \frac{1}{(\sec x + \tan x)} (\sec x \tan x + \sec^2 x)$$

$$+ \cos x \ln(\sec x + \tan x)$$

$$= \tan x + \cos x \ln(\sec x + \tan x)$$

$$y'' + y = \tan x + \cos x \cancel{\ln(\sec x + \tan x)}$$
$$- \cos x \cancel{\ln(\sec x + \tan x)}$$

$$= \tan x = R.H.S.$$

15 $(y-x)y' = y - x + 8$, $\quad y = x + 4\sqrt{x+2}$

$$y = x + 4\sqrt{x+2}$$

$$y' = 1 + 4 \times \frac{1}{2\sqrt{x+2}} = 1 + \frac{2}{\sqrt{x+2}}.$$

using in given differential equation

$$\overset{y-x)}{(\cancel{x} + 4\sqrt{x+2} - \cancel{x})} \left(1 + \frac{2}{\sqrt{x+2}}\right) = x + 4\sqrt{x+2} - x + 8$$

$$(4\sqrt{x+2})\left(1 + \frac{2}{\sqrt{x+2}}\right) = 4\sqrt{x+2} + 8$$

$$4\sqrt{x+2} + 8 = 4\sqrt{x+2} + 8$$

Domain of function $\quad y = x + 4\sqrt{x+2}$

$$[-2, \infty)$$

Domain as a solution

As y' is not defined at $x = -2$

$$\Rightarrow (-2, \infty)$$

18 $2y' = y^3 \cos x \quad (\quad y = (1 - \sin x)^{-1/2}.$

$$y' = \frac{-1}{2}(1-\sin x)^{-3/2}(-\cos x)$$

$$= \frac{\cos x}{2}(1-\sin x)^{-3/2}.$$

$$2y' = y^3 \cos x$$

$$\cos x (1-\sin x)^{-3/2} = (1-\sin x)^{-3/2} \cos x$$

so given function is the solution of differential Equation

* Domain of y as a function

$$x \in \left(\frac{-3\pi}{2} + 2n\pi, \frac{\pi}{2} + 2n\pi\right)$$

Domain as a solution

$$\left(-\frac{3\pi}{2}, \frac{\pi}{2}\right)$$

20 $2xy\,dx + (x^2 - y)\,dy = 0$ $\quad -2x^2y + y^2 = 1.$

we have

$$-2x^2y + y^2 = 1$$

Diff. w.r.t x

$$-4xy - 2x^2 \frac{dy}{dx} + 2yy' = 0$$

$$(-2x^2 + 2y)\frac{dy}{dx} = 4xy \Rightarrow \frac{dy}{dx} = \frac{-2xy}{x^2 - y} \quad (1)$$

Given differential eq. is

$$2xy\,dx + (x^2 - y)\,dy = 0$$

$$(x^2 - y)\,dy = -2xy.dx.$$

$$\frac{dy}{dx} = \frac{-2xy}{x^2 - y}. \quad (2)$$

from (1) & (2)

$$\frac{-2xy}{x^2 - y} = \frac{-2xy}{x^2 - y}.$$

To find the explicit solution

$$y^2 - 2x^2y - 1 = 0$$

using quadratic formula.

$$y = \frac{2x^2 \pm \sqrt{4x^2+4}}{2} = x^2 \pm \sqrt{x^2+1}$$

where interval of definition is $(-\infty, \infty)$.

22 $\frac{dy}{dx} + 2xy = 1$ $\quad y = e^{-x^2}\int_0^x e^{t^2}\,dt + c_1 e^{-x^2}$

$$\frac{dy}{dx} = e^{-x^2}(-2x)\int_0^x e^{t^2}\,dt + \underbrace{e^{-x^2}e^{x^2}}_{\text{by fundamental theorem of calculus}} - 2xe^{-x^2}c_1$$

$$\frac{dy}{dx} + 2xy = -2xe^{-x^2}\cancel{\int_0^x e^{t^2}\,dt} + 1 - 2x\cancel{e^{-x^2}c_1}$$

$$+ 2xe^{-x^2}\cancel{\int_0^x e^{t^2}\,dt} + \cancel{2c_1 x e^{-x^2}}$$

$$= 1.$$

function y is defined for all $x \in R$.

25 Verify that

$$y = \begin{cases} -x^2 & x < 0 \\ x^2 & x \geq 0 \end{cases}$$

is a solution of differential equation $xy' - 2y = 0$ on $(-\infty, \infty)$

$y = x^2 \qquad x \geq 0$

$y' = 2x$.

$xy' - 2y = 2x^2 - 2x^2 = 0$.

* $y = -x^2 \qquad x < 0$

$y' = -2x$

$xy' - 2y = -2x^2 - 2(-x^2)$

$= -2x^2 + 2x^2 = 0$

Find values of m so that the function $y = e^{mx}$ is the solution of differential Eq.

29 $y'' - 5y' + 6y = 0$

$y = e^{mx}$

$y' = m e^{mx} \Rightarrow y'' = m^2 e^{mx}$

$y'' - 5y' + 6y = 0$

$m^2 e^{mx} - 5m e^{mx} + 6e^{mx} = 0$

$(m^2 - 5m + 6) e^{mx} = 0.$

$e^{mx} \neq 0$ $m^2 - 5m + 6 = 0$

$\Rightarrow m = 2$ or $m = 3.$

31 Find values of m so that the function $y = x^m$ is the solution of the given differential Eq.

$xy'' + 2y' = 0.$

$y = x^m$

$y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$

$xy'' + 2y' = 0$

$m(m-1)x^{m-1} + 2m x^{m-1} = 0$

$(m^2 - m + 2m) x^{m-1} = 0.$

$(m^2 + m) x^{m-1} = 0$ $x^{m-1} \neq 0$

$m(m+1) = 0$

$m = 0$ & $m = -1.$

38 verify that the indicated pair of function is a solution of given system of differential Eq. on $(-\infty, \infty)$.

$$\frac{d^2x}{dt^2} = 4y + e^t \quad (1)$$

$$\frac{d^2y}{dt^2} = 4x - e^t \quad (2)$$

$x = \cos 2t + \sin 2t + \frac{1}{5} e^t$

$y = -\cos 2t - \sin 2t - \frac{e^t}{5}$

Observe that $y = -x$.

Now $x' = -2\sin 2t + 2\cos 2t + \frac{e^t}{5}$

$x'' = -4\cos 2t - 4\sin 2t + \frac{e^t}{5}$

Also

$y' = 2\sin 2t - 2\cos 2t - \frac{e^t}{5}$

$y'' = 4\cos 2t + 4\sin 2t - \frac{e^t}{5}$

From (1) & (2)

$-4\cos 2t - 4\sin 2t + \frac{e^t}{5} = -4\cos 2t - 4\sin 2t - \frac{4}{5}e^t + e^t$

$0 = 0$.

Also:

$4\cos 2t + 4\sin 2t - \frac{e^t}{5} = 4(\cos 2t) + 4\sin 2t + \frac{4}{5}e^t - e^t$

$0 = 0$