4.2. Reduction of order

Reduction of order

consider the 2^{nd} order homogeneous differential Equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

Let y_1 define the non-trivial solution of (1). we want to find 2^{nd} solution y_2 such that y_1 & y_2 are linearly independent on I.

* The second solution will be of the form

$$y_2(x) = u(x)\,y_1(x).$$

The function $u(x)$ can be found by substituting $y_2(x) = u(x)y_1(x)$. Into given differential Equation.

This method is called reduction of order.

Example Given that $y_1 = e^x$ is a solution of $y'' - y = 0$ on the interval $(-\infty, \infty)$, use reduction of order to find a second solution y_2.

Solution If $y = u(x)y_1(x) = u(x)e^x$

$$y' = ue^x + u'(x)e^x$$

$$y'' = ue^x + u'(x)e^x + u'(x)e^x + u''(x)e^x$$

$$= u''(x)e^x + 2u'(x)e^x + u(x)e^x$$

so

$$y'' - y = 0$$

$$\Rightarrow u''(x)e^x + 2u'(x)e^x + \cancel{u(x)e^x} - \cancel{u(x)e^x} = 0$$

$$\left(u''(x) + 2u'(x)\right)e^x = 0$$

As $e^x \neq 0$ so

$$u''(x) + 2u'(x) = 0$$

$$w = u'(x), \quad w' = u''(x)$$

$$w' + 2w = 0$$

$$\frac{dw}{dx} = -2w$$

$$\frac{dw}{w} = -2\,dx$$

$$\ln w = -2x + c_1$$

$$\Rightarrow \quad w = c_1 e^{-2x}$$

Putting $w = u'$

$$u' = c_1 e^{-2x}$$

Integrating

$$u(x) = -\frac{c_1}{2} e^{-2x} + c_2$$

So

$$y(x) = u(x)\, y_1(x)$$

$$= -\frac{c_1}{2} e^{-x} + c_2 e^{x}$$

Let $c_2 = 0$ & $c_1 = -2$

$$y_2(x) = e^{-x}$$

which is the desired solution.

General case

If $y_1(x)$ is a one non-trivial solution of 2^{nd} order homogeneous differential Eq. then the second solution is

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}\, dx$$

Example The function $y = x^2$ is a solution of $x^2 y'' - 3xy' + 4y = 0$. Find the general solution of differential Equation on the interval $(0, \infty)$.

Solution Standard form of differential Equation is

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0$$

So

$$y_2(x) = y_1(x)\int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx$$

$$= x^2 \int \frac{e^{3\int \frac{dx}{x}}}{x^4}\,dx$$

$$= x^2 \int \frac{x^3}{x^4}\,dx = x^2 \int \frac{1}{x}\,dx = x^2 \ln x.$$

So general solution is

$$y(x) = c_1 x^2 + c_2 x^2 \ln x$$

Exercise 4.2.

4) $y'' + 9y = 0 \qquad y_1 = \sin 3x$

Finding reduction of order method

$$y_2(x) = \sin 3x \int \frac{e^{-\int \cancel{0} dx}}{(\sin 3x)^2}\, dx$$

$$= \sin 3x \int \frac{K}{\sin^2 3x}\, dx$$

$$= \sin 3x \int K \csc^2 3x \, dx$$

$$= K \sin 3x \left(\frac{\cot 3x}{3}\right)$$

$$= \frac{K}{3} \sin 3x \frac{\cos 3x}{\sin 3x} = \frac{K}{3} \cos 3x.$$

So $y_2 = \cos 3x$

& general solution is

$$y = c_1 \cos 3x + c_2 \sin 3x$$

8. $6y'' + y' - y = 0 \qquad y_1 = e^{x/3}$

converting into standard form.

$$y'' + \frac{1}{6} y' - \frac{1}{6} y = 0$$

So $p(x) = \frac{1}{6}$.

$$y_2 = e^{x/3} \int \frac{e^{-\int \frac{1}{6} dx}}{e^{2x/3}}\, dx = e^{x/3} \int \frac{e^{-x/6}}{e^{2x/3}}\, dx$$

$$= e^{x/3} \int e^{-5x/6}\, dx = e^{x/3} \cdot \frac{e^{-5x/6}}{-5/6}$$

$$= \frac{-6}{5} e^{-x/2}$$

So

$y_2 = e^{-x/2}$

general Solution is

$y(x) = c_1 e^{x/3} + c_2 e^{-x/2}$

13 $x^2 y'' - xy' + 2y = 0$ $\quad y_1 = x\sin(\ln x)$

converting into standard form

$y'' - \frac{1}{x} y' + \frac{2y}{x^2} = 0$

Here $p(x) = -1/x$

$y_2 = y_1(x) \int \frac{e^{-\int p(x)dx}}{y_1(x)^2} dx$

$= y_1(x) \int \frac{e^{\int 1/x \, dx}}{(x\sin(\ln x))^2} dx$

$= x\sin(\ln x) \int \frac{\cancel{x}}{x^{\cancel{2}} \sin^2(\ln x)} dx$

In integral, $u = \ln x$

$du = \frac{1}{x} dx$

$= x\sin(\ln x) \int \frac{1}{\sin^2 u} du$

$= x\sin(\ln x) \int \csc^2 u \, du$

$= x\sin(\ln x)(-\cot(u))$

$= -x \cancel{\sin(\ln x)} \frac{\cos(\ln x)}{\cancel{\sin(\ln x)}}$

$= -x\cos(\ln x)$

So $y_2 = x\cos(\ln x)$

$y(x) = c_1 x\sin(\ln x) + c_2 x\cos(\ln x)$

16 $(1-x^2)\ y''+2xy'=0$ $\qquad y_1 = 1$

$$y''+\frac{2x}{1-x^2}y'=0$$

$$p(x)=\frac{2x}{1-x^2}$$

$$e^{\int\frac{2x}{1-x^2}dx} = e^{-\ln(1-x^2)} = e^{\ln(1-x^2)^{-1}} = (1-x^2)^{\cancel{-1}}$$

So

$$y_2 = y_1(x)\int\frac{e^{-\int p(x)\,dx}}{y_1(x)^2}\,dx \neq y_1$$

$$= 1.\int 1-x^2\,dx = x-\frac{x^3}{3}$$

$$y(x) = c_1y_1+c_2y_2$$

$$= c_1+c_2\left(x-\frac{x^3}{3}\right).$$