Cauchy Euler Equation.

Definition

A linear DE of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = g(x)$$

where $a_n, a_{n-1}, \cdots, a_1, a_0$ are constant, is known as Cauchy Euler Equation.

Method of solution.

Consider the solution of the form $y = x^m$, where m is to be determined.

On substituting x^m, each term of cauchy Euler equation becomes a polynomial in m times x^m.

* For Example, substituting $y = x^m$ in

$$ax^2 \frac{d^2 y}{dx^2} + bx \frac{dy}{dx} + cy = 0$$

we get

$$am(m-1)x^m + bmx^m + cx^m = 0.$$

So the auxiliary Equation is

$$am(m-1) + bm + c = 0$$

or

$$am^2 + (b-a)m + c = 0 \qquad (1)$$

Case I, Distinct real roots.

Let m_1, m_2 denote the real roots of (1) such that $m_1 \neq m_2$, then $y_1 = x^{m_1}$ & $y_2 = x^{m_2}$ form a fundamental set of solutions. Hence general solution is

$$y = c_1 x^{m_1} + c_2 x^{m_2}$$

Case II Repeated real roots.

If the roots of Eq (1) are repeated i-e $m_1 = m_2$, then general solution is

$$y = c_1 x^{m_1} + c_2 x^{m_1} \ln x$$

Case III Conjugate complex roots.

If the roots of (1) are conjugate pairs $m_1 = \alpha + i\beta$, & $m_2 = \alpha - i\beta$ with $\alpha, \beta > 0$ are real, then general solution is

$$y = x^{\alpha}\left[c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)\right]$$

Example Solve $x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} - 4y = 0$.

Solution

Let $y = x^m$

$$\frac{dy}{dx} = m x^{m-1} \qquad \frac{d^2y}{dx^2} = m(m-1)x^{m-2}$$

$\Rightarrow \quad m(m-1)x^m - 2mx^m - 4x^m = 0$

$\Rightarrow \quad (m^2 - m - 2m - 4)x^m = 0$

$(m^2 - 3m - 4)x^m = 0$.

$\Rightarrow \quad m^2 - 3m - 4 = 0$

$m = -1 \qquad m = 4$

$\Rightarrow \quad y(x) = c_1 x^{-1} + c_2 x^4$

Example Solve $4x^2 \frac{d^2y}{dx^2} + 8x \frac{dy}{dx} + y = 0$

Solution

substitute $y = x^m$

$\Rightarrow \quad 4m(m-1)x^m + 8mx^m + x^m = 0$

$$(4m^2 - 4m + 8m + 1)\, x^m = 0$$

$$4m^2 + 4m + 1 = 0$$

$$\Rightarrow \quad (2m+1)^2 = 0 \quad \Rightarrow \quad m_1 = \frac{-1}{2}, \quad m_2 = -1/2$$

So general solution is

$$y = c_1 x^{-1/2} + c_2 x^{-1/2} \ln x.$$

* For higher order equations, if m_1 is root of multiplicity k, then it can be shown that

$$x^{m_1},\ x^{m_1} \ln x,\ x^{m_1} (\ln x)^2,\ \ldots,\ x^{m_1} (\ln x)^{k-1}$$

are k linearly independent solutions.

Example Solve $4x^2 y'' + 17y = 0 \qquad y(1) = -1,\ y'(1) = -1/2.$

Solution

Let $y = x^m$, so we get auxiliary Eq. as

$$4m(m-1) + 17 = 0$$

$$\Rightarrow \quad 4m^2 - 4m + 17 = 0.$$

using quadratic formula.

$$m_1 = \frac{1}{2} + 2i, \qquad m_2 = \frac{1}{2} - 2i$$

& general solution is

$$y = x^{1/2} \left(c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x) \right)$$

$$y'(x) = \frac{1}{2} x^{-1/2} \left(c_1 \cos(2\ln x) + c_2 \sin(2\ln x) \right)$$

$$+ x^{1/2} \left(-c_1 \sin(2\ln x) \times \frac{2}{x} + c_2 \cos(2\ln x) \times \frac{2}{x} \right)$$

$y(1) = -1$

$$\Rightarrow \quad -1 = 1 \left(c_1 \cos(0) + c_2 \sin 0 \right)$$

$$\Rightarrow \quad c_1 = -1.$$

$y'(1) = -\frac{1}{2}$

$$\Rightarrow \quad \frac{-1}{2} = \frac{1}{2}(1)(c_1 \cos 0 + c_2 \sin 0)$$
$$+ 1(-c_1 \sin 0 + 2c_2 \cos 0)$$

$$\frac{-1}{2} = \frac{1}{2} c_1 + 2c_2$$

$$\frac{-1}{2} = \frac{1}{2}(-1) + 2c_2 \qquad \because c_1 = -1.$$

$$\Rightarrow \quad c_2 = 0.$$

So

$$y(x) = -x^{1/2} \cos(2 \ln x).$$

Reduction to constant coefficients.

Any Cauchy-Euler Equation can always be written as linear DE with constant coefficient by substituting $x = e^t$.

Example

Solve $x^2 y'' - xy' + y = \ln x.$

Solution Let $x = e^t$ & $t = \ln x.$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{-1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2}$$

$$= \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

$$\Rightarrow \quad x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

Substituting in given differential Equation

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = t \qquad (1)$$

Auxiliary Eq. is (Associated homogeneous Eq.)

$$m^2 - 2m + 1 = 0$$

$$(m-1)^2 = 0$$

$$m_1 = 1 \qquad m_2 = 1$$

$$y(t) = c_1 e^t + c_2 t e^t$$

For particular solution, let $y_p = At + B$

$$y_p' = A$$

$$y_p'' = 0$$

$(1) \Rightarrow \quad 0 - 2A + At + B = t$

coefficient of t: $\Rightarrow A = 1$.

coefficient of t°

$\Rightarrow \quad -2A + B = 0 \quad \Rightarrow \quad B = 2$

$$y_p = t + 2$$

So general solution is

$$y = y_c + y_p$$

$$= c_1 e^t + c_2 t e^t + t + 2.$$

Exercise 4.7

10 $4x^2y'' + 4xy' - y = 0$

Let $y = x^m$

$\Rightarrow y' = mx^{m-1}$ $\qquad y'' = m(m-1)x^{m-2}$

using in given DE.

$$4m(m-1)x^m + 4mx^m - x^m = 0$$

$x^m \neq 0,$

$$(4m^2 - 4m + 4m - 1) = 0$$

$$4m^2 - 1 = 0$$

$$m = \pm \frac{1}{2}$$

$$y(x) = c_1 x^{1/2} + c_2 x^{-1/2}.$$

15 $x^3 y''' - 6y = 0$

$y = x^m$

$\Rightarrow y' = mx^{m-1}$, $\quad y'' = m(m-1)x^{m-2}$.

$$y''' = m(m-1)(m-2)x^{m-3}$$

$$= (m^2 - m)(m-2)x^{m-3}$$

$$= (m^3 - 3m^2 + 2m)x^{m-3}$$

using in given DE

$$\Rightarrow \quad y(m^3 - 3m^2 + 2m)x^m - 6x^m = 0$$

$$(m^3 - 3m^2 + 2m - 6) = 0$$

$$(m-3)(m^2+2) = 0$$

$\Rightarrow \quad m = 3, \quad m = \pm\sqrt{2}\, i$

$$y = c_1x^3 + c_2\cos(\sqrt{2}\ln x) + c_3\sin(\sqrt{2}\ln x).$$

18 $x^4 y^{(4)} + 6x^3 y''' + 9x^2 y'' + 3xy' + y = 0$

Let $y = x^m$, so auxiliary Eq. is.

$$m(m-1)(m-2)(m-3) + 6(m)(m-1)(m-2) + 9m(m-1) + 3m + 1 = 0$$

$$\Rightarrow (m^2-m)(m^2-5m+6) + 6m(m^2-3m+2) + 9m^2 - 9m + 3m + 1 = 0$$

$$\Rightarrow (m^4 - 6m^3 + 11m^2 - 6m) + (6m^3 - 18m^2 + 12m) + 9m^2 - 9m + 3m + 1 = 0$$

$$m^4 + 2m^2 + 1 = 0$$

$$(m^2+1)^2 = 0$$

$$m = \pm i \qquad\qquad m = \pm i$$

$$\Rightarrow y = c_1 \cos(\ln x) + c_2 \sin(\ln x) + c_3 \ln x \cos(\ln x) + c_4 \ln x \sin(\ln x).$$

28 $x^2 y'' - 3xy' + 4y = 0 \qquad y(1) = 5, \; y'(1) = 3.$

Auxiliary Eq..

$$m(m-1) - 3m + 4 = 0$$

$$m^2 - m - 3m + 4 = 0 \Rightarrow m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0$$

$$m_1 = 2 \qquad\qquad m_2 = 2.$$

$$y(x) = c_1 x^2 + c_2 \ln x . x^2 \qquad (1)$$

$$y'(x) = 2c_1 x + 2c_2 x \ln x + c_2 x. \qquad (2)$$

using $y(1) = 5$ in (1)

$$\Rightarrow 5 = c_1 (1)^2 + c_2 \ln(1). x^2 \qquad \because \ln(1) = 0$$

$$\Rightarrow c_1 = 5$$

using $y'(1) = 3$

$$\Rightarrow 3 = 2c_1 + c_2 \Rightarrow c_2 = 3 - 2 \times 5 = -7.$$

$y(x) = 5x^2 - 7x^2 \ln x$.

30 $x^2y'' - 5xy' + 8y = 8x^6 \qquad y(\frac{1}{2}) = 0 \quad y'(\frac{1}{2}) = 0$

Associated homogeneous Eq:

$$x^2y'' - 5xy' + 8y = 0$$

Auxiliary Eq. is

$$m(m-1) - 5m + 8 = 0$$
$$m^2 - 6m + 8 = 0$$
$$m = 4 \quad \& \quad m = 2$$

$\Rightarrow \quad y(x) = c_1x^2 + c_2x^4$.

For particular solution, using variation of parameters.

$$y_1 = x^2 \qquad y_2 = x^4.$$

$$W = \begin{vmatrix} x^2 & x^4 \\ 2x & 4x^3 \end{vmatrix} = 4x^5 - 2x^5 = 2x^5$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} = \begin{vmatrix} 0 & x^4 \\ 8x^6 & 4x^3 \end{vmatrix} = -8x^{10}$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} = \begin{vmatrix} x^2 & 0 \\ 2x & 8x^6 \end{vmatrix} = 8x^8$$

$$u_1' = \frac{W_1}{W} = \frac{-8x^{10}}{2x^5} = -4x^5$$

$\Rightarrow$ Integrating $\Rightarrow \quad u_1 = -\frac{2}{3}x^6$

$$u_2' = \frac{W_2}{W} = \frac{8x^8}{2x^5} = 4x^3$$

Integrating

$\Rightarrow \quad u_2 = -x^4$

So $\quad y_p = u_1y_1 + u_2y_2 = -\frac{2}{3}x^6 \times x^2 + (-x^4)(x^4)$

$$= -\frac{2}{3}x^8 - x^8 = -\frac{5}{3}x^8$$

so

$$y = y_c + y_p$$

$$= c_1 x^2 + c_2 x^4 - \frac{5}{3} x^8 \qquad (1)$$

$$y'(x) = 2c_1 x + 4c_2 x^3 - \frac{40}{3} x^7 \qquad (2)$$

using $y(\frac{1}{2}) = 0$ in (1)

$$\Rightarrow \quad 0 = \frac{c_1}{4} + \frac{c_2}{16} - \frac{5}{3} \times \cancel{\frac{8}{3}} \times \frac{1}{256}$$

$$4c_1 + c_2 = \frac{5}{3} \times \frac{1}{256} \times 16 = \frac{5}{48} \qquad (3)$$

using $y'(\frac{1}{2}) = 0$

$$\Rightarrow \quad c_1 + 4c_2 \times \frac{1}{8} - \frac{40}{3} \times \frac{1}{128} = 0$$

$$c_1 + \frac{c_2}{2} - \frac{5}{48} = 0$$

$$\Rightarrow \quad 2c_1 + c_2 = \frac{10}{48} \qquad (4)$$

(3) − (4)

$$\Rightarrow \quad 2c_1 = \frac{-5}{48}$$

$$c_1 = \frac{-5}{24}.$$

using in (4)

$$\Rightarrow \quad c_2 = \frac{10}{48} + \frac{10}{24} = 5/8.$$

32 $x^2 y'' - 9xy' + 25y = 0$

Let $x = e^t$

$$\Rightarrow \quad t = \ln x$$

$$\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx}$$

$$= \frac{1}{x}\frac{dy}{dt}$$

Similarly

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

using in given DE

$$\Rightarrow \quad \frac{d^2y}{dt^2} - \frac{dy}{dt} - 9\frac{dy}{dt} + 25y = 0$$

$$\frac{d^2y}{dt^2} - 10\frac{dy}{dt} + 25y = 0$$

Auxiliary Eq. is

$$m^2 - 10m + 25 = 0$$

$$m^2 - 5m - 5m + 25 = 0$$

$$(m-5)^2 = 0$$

$$m_1 = 5 \qquad m_2 = 5$$

$$y(t) = c_1 e^{5t} + c_2 t e^{5t}$$

As $\quad t = \ln x$

$$\Rightarrow \quad y(x) = c_1 e^{5\ln x} + c_2 \ln x \, e^{5\ln x}$$

$$= c_1 x^5 + c_2 \ln x \, x^5$$

36 $\quad x^3y''' - 3x^2y'' + 6xy' - 6y = 3 + \ln x^3$

Let $\quad x = e^t \Rightarrow t = \ln x.$

$$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

$$\frac{d^3y}{dx^3} = -\frac{2}{x^3}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + \frac{1}{x^2}\left(\frac{d^3y}{dt^2} \times \frac{1}{x} - \frac{1}{x}\frac{d^2y}{dt^2}\right)$$

$$\frac{d^3y}{dx^3} = \frac{1}{x^3}\left(-2\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + \frac{d^3y}{dt^3} - \frac{d^2y}{dt^2}\right)$$

$$\Rightarrow \quad x^3\frac{d^3y}{dt^3} = \frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}$$

So the DE converts into

$$\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 6\frac{dy}{dt} - 6$$
$$= 3 + \ln(e^t)^3$$

$$\frac{d^3y}{dt^3} - 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} - 6y = 3 + 3t \qquad (A)$$

Associated homogeneous Eq. is

$$\frac{d^3y}{dt^3} - 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} - 6y = 0$$

Auxiliary Eq. is

$$m^3 - 6m^2 + 11m - 6 = 0$$

So factors are

$$(m-1)(m-2)(m-3) = 0$$

$\Rightarrow \quad m_1 = 1 \quad m_2 = 2 \quad m_3 = 3.$

1	1	-6	11	-6
	—	1	-5	6
2	1	-5	6	0
	—	2	-6	
3	1	-3	0	
	—	3		
	1	0		

$$y(t) = c_1e^{t} + c_2e^{2t} + c_3e^{3t}$$
$$y(x) = c_1x + c_2x^2 + c_3x^3.$$

For particular solution

Let $\quad y_p = At + B$

$y_p' = A, \quad y_p'' = 0, \quad y_p''' = 0$

using in (A)

$\Rightarrow \quad 11A - 6At - 6B = 3 + 3t$

coefficients of $t \Rightarrow \quad -6A = 3 \quad \Rightarrow \quad A = -\frac{1}{2}$

coefficients of t°

$\Rightarrow \quad 11A - 6B = 3$

$\Rightarrow \quad 6B = 3 - 11A$

$= 3 - 11(-1/2) = 3 + \frac{11}{2} = \frac{17}{2}$

$B = \frac{17}{12}.$

$y_p = -\frac{1}{2}t + \frac{17}{12}.$

$y_p = -\frac{1}{2}\ln x + \frac{17}{12}$

∴ general solution is

$$y(x) = c_1 x + c_2 x^2 + c_3 x^3 - \frac{1}{2}\ln x + \frac{17}{12}.$$