So
$$T = 70 + 230e^{k\cdot}$$

Now using $T(3) = 200$

$$\Rightarrow \quad 200 = 70 + 230e^{3k}$$

$$130 = 230e^{3k}$$

$$\Rightarrow \quad e^{3t} = \frac{13}{23} \quad \Rightarrow \quad 3t = \ln\left(\frac{13}{23}\right).$$

$$k = \frac{\ln(13/23)}{3} = -0.19018$$

Thus
$$T(t) = 70 + 230e^{-0.19018t}$$

Looking at this solution, we get $\lim_{t\to\infty} T(t) = 70$ which says that cake will be at room temperature after a reasonably long time. It'll take approximately Half an hour to reach at room temperature.

$T(t)$	t (min)
75°	20.1
74°	21.3
73°	22.8
72°	24.9
71°	28.6
70.5°	32.3

Series Circuits

The linear differential Equation for the current $i(t)$ through a series circuit containing only a resistor and inductor is

$$L\frac{di}{dt} + Ri = E(t) \qquad (I)$$

where
L: Inductance
R: resistance

* The current $i(t)$ is called response of the system

If Voltage drop across capacitor is $q(t)/c$, then DE for the current $i(t)$ through a series circuit containing a resistor and a capacitor is

$$Ri + \frac{q}{c} = E(t).$$

As $i = \frac{dq}{dt}$, so we get

$$R\frac{dq}{dt} + \frac{q}{c} = E(t)$$

Example A 12 volt battery is connected to a series circuit in which inductance is $\frac{1}{2}$ henry, and resistance is 10 ohms. Determine the current i, if initial current is zero.

Solution From (I),

$$\frac{1}{2}\frac{di}{dt} + 10i = 12 \qquad i(0)=0$$

$$\frac{di}{dt} + 20i = 24.$$

$$I.F = e^{\int 20dt} = e^{20t}$$

$$\Rightarrow \frac{d}{dt}\left(e^{20t}i\right) = 24e^{20t}$$

Integrating

$$\Rightarrow e^{20t}i = \frac{24}{20}e^{20t} + c$$

$$\Rightarrow i(t) = \frac{6}{5} + ce^{-20t}$$

using I.C.

$$i(0) = \frac{6}{5} + ce^{0}$$

$$0 = \frac{6}{5} + c \Rightarrow c = -6/5.$$

So response is

$$i(t) = \frac{6}{5} - \frac{6}{5}e^{-20t}$$

we can write general solution of I is

$$i(t) = \frac{E_0}{R} + ce^{(-R/L)t}$$

MIXTURES

The mixing of two fluids sometimes gives rise to a linear first-order differential equation.

$$\frac{dA}{dt} = (input\ rate\ of\ salt) - (output\ rate\ of\ salt)$$

$$\frac{dA}{dt} = R_{in} - R_{out} \tag{1}$$

Example

Consider a large tank containing 300 gallons of a brine solution. Salt is entering and leaving the tank; a brine solution was being pumped into the tank at the rate of 3 gal/min; it mixed with the solution there, and then themixture was pumped out at the rate of 3 gal/min. The concentration of the salt in the inflow, or solution entering, was 2 lb/gal, so salt was entering the tank at the rate $R_{in} = 2\frac{lb}{gal}.\left(3\,\frac{\text{gal}}{\text{min}}\right) = 6\ \text{lb/min}$ and leaving the tank at the rate $Rout = \left(\frac{\text{A}}{300}\frac{\text{lb}}{\text{gal}}\right)\left(3\,\frac{\text{gal}}{\text{min}}\right) = \frac{\text{A}}{100}\,\text{lb/min}$. From this data and (1) we get equation

$$\frac{dA}{dt} + \frac{A}{100} = 6 \tag{2}$$

Let us pose the question: If 50 pounds of salt were dissolved initially in the 300 gallons, how much salt is in the tank after a long time?

Solution we have initial value problem

$$\frac{dA}{dt} + \frac{A}{100} = 6\,, \qquad A(0) = 50$$

Here the initial condition $A(0) = 50$ gives the amount of salt Not the initial amount of liquid in the tank.

Now $I.F = e^{\int \frac{1}{100} dt}$

$$= e^{t/100}$$

So Eq. becomes

$$\frac{d}{dt}\left(A e^{t/100}\right) = 6 e^{t/100}$$

Integrating

$$A e^{t/100} = 6\,\frac{e^{t/100}}{1/100} + c$$

$$A e^{t/100} = 600\, e^{t/100} + c$$

$$A = 600 + c e^{-t/100}$$

using I.C.

$$A(0) = 600 + ce^{0} = 50$$

$$c = 600 - 50$$

$$= -550$$

$$A(t) = 600 \cancel{+} -550e^{-t/100}$$

As $t \to \infty$ $\quad A(t) = 600$

Q4: The population of bacteria in a culture grows at a rate proportional to the number of bacteria present at time t. After 3 hours it is observed that 400 bacteria are present. After 10 hours 2000 bacteria are present. What was the initial number of bacteria?

Let population of bacteria in culture is x. Then we have

$$\frac{dx}{dt} = kx \qquad (1)$$

with conditions

$$x(3) = 400 \quad , \quad x(10) = 2000.$$

Solving Eq. (1)

$$\frac{dx}{x} = k\,dt$$

$$\ln x = kt + c_1$$

$$\ln x = \ln e^{kt + c_1}$$

$$x = ce^{kt} \qquad (2)$$

using initial condition, $x(3) = 400$

$$\Rightarrow \quad 400 = ce^{3k}$$

$$c = \frac{400}{e^{3k}}$$

using in (2)

$$\Rightarrow \quad x = \frac{400}{e^{3k}} e^{kt} = 400 e^{(t-3)k}$$

Now using 2^{nd} ~~initial~~ condition

$$x(10) = 400 e^{(10-3)k}$$

$$2000 = 400 e^{7k} \Rightarrow e^{7k} = 5$$

$$\Rightarrow \quad 7k = \ln 5$$

$$k = \frac{\ln 5}{7} = 0.23$$

So we have

$$x(t) = 400 e^{0.23(t-3)}$$

To find the initial number of bacteria, put $t = 0$

$$\Rightarrow \quad x(0) = 400 e^{0.23(-3)} = 201.$$

Q9: When a vertical beam of light passes through a transparent medium, the rate at which its intensity I decreases is proportional to I(t), where t represents the thickness of the medium (in feet). In clear seawater, the intensity 3 feet below the surface is 25% of the initial intensity I_0 of the incident beam. What is the intensity of the beam 15 feet below the surface?

As intensity of light decreases, proportional to $I(t)$

So we have

$$\frac{dI}{dt} = -kI \quad (1)$$

with conditions

$$I(t=0) = I_0 \quad (2)$$

$$I(t=3) = 25\% \, I_0 = \frac{25}{100} I_0 = 0.25 I_0 \quad (3)$$

we have to find $I(t=15)$

Solving Eq (1)

$$\Rightarrow \frac{dI}{I} = -k\,dt$$

$$\Rightarrow \ln I = -kt + \ln c$$

$$\ln I = +\ln e^{-kt} + \ln c$$

$$\Rightarrow I(t) = c e^{-kt}$$

using (2)

$$\Rightarrow I(0) = ce^{0} \Rightarrow I_0 = c$$

$$\Rightarrow I(t) = I_0 e^{-kt} \quad (4)$$

Now using. $I(3) = 0.25 I_0$

$$\Rightarrow 0.25 \cancel{I_0} = \cancel{I_0} e^{-k(3)} \Rightarrow e^{-3K} = 0.25$$

$$-3K = \ln(0.25) \Rightarrow K = -\frac{\ln(0.25)}{3}$$

$$= 0.462$$

using in (4) $\Rightarrow I(t) = I_0 e^{-0.462t}$

Now

$$I(15) = I_0 e^{-0.462 \times 15} = 0.00098 I_0 \ W/ft.$$

Q11: Archaeologists used pieces of burned wood, or charcoal, found at the site to date prehistoric paintings and drawings on walls and ceilings of a cave in Lascaux, France. Determine the approximate age of a piece of burned wood, if it was found that 85.5% of the C-14 found in living trees of the same type had decayed.

we have

$$\frac{dA}{dt} = KA$$

$A(0) = A_0$ & $A(\tfrac{1}{2}) = A(5600) = \frac{A_0}{2}$.

so $$A = A_0 e^{-\frac{\ln 2}{5600} t}$$

As 85.5% of A is decayed, we have $A = 14.5\%$.

At $A = 0.145 A_0$

$$\Rightarrow \quad 0.145 \cancel{A_0} = \cancel{A_0} e^{-\frac{\ln 2}{5600} t}$$

$$-\frac{\ln 2}{5600} t = \ln(0.145)$$

$$t = \frac{-5600 \ln(0.145)}{\ln 2}$$

$$= 15963 \text{ years}$$

Q16: Two large containers A and B of the same size are filled with different fluids. The fluids in containers A and B are maintained at 0° C and 100° C, respectively. A small metal bar, whose initial temperature is 100° C, is lowered into container A. After 1 minute the temperature of the bar is 90° C. After 2 minutes the bar is removed and instantly transferred to the other container. After 1 minute in container B the temperature of the bar rises 10°. How long, measured from the start of the entireprocess, will it take the bar to reach 99.9° C?

A small metal bar of Temperature 100°c is lowered into container A with temperature 0°c & then it is lowered into B with temperature 100°c

By Newton's Law of cooling

$$\frac{dT}{dt} = K(T - T_m)$$

Conditions

$T = T_0$ for container A at $t = 0 = 100°c$

$T\,(\text{at } t=1) = 90°$

$T = T_0$ for container B at ($t = 2$ in container A).

$T = (T_0 + 10)$ for container B at ($t = 1$ in container A).

Goal

we have to find the time when temperature of metal bar is 99.9°c.

For container A. ($T_m = 0°c$)

$$\frac{dT}{dt} = K_A (T - 0)$$

$$\frac{dT}{T} = K_A\, dt$$

$$\Rightarrow \ln T = K_A t + \ln c$$

$$T(t) = c e^{K_A t}$$

At $t = 0,\ T = 100$

$$\Rightarrow \quad c = 100.$$

$$T(t) = 100e^{k_A t}.$$

At $t = 1$, $T = 90^\circ$

$\Rightarrow \quad 90 = 100\, e^{k_A}$

$\Rightarrow \quad e^{k_A} = \dfrac{90}{100}$

$$k_A = \ln\left(\frac{90}{100}\right) = -0.105$$

Temperature of bar after $t = 2$.

$$T(t) = 100\, e^{-0.105 t}$$

$$T(2) = 100\, e^{-0.105 \times 2}$$

$$= 81^\circ c.$$

Now this is initial temperature of metal bar when it is transferred to B.

For container B $\quad T_m = 100^\circ c.$

$$\frac{dT}{dt} = k_B(T - 100)$$

$\Rightarrow \quad \dfrac{dT}{T-100} = k_B t \Rightarrow \quad T - 100 = ce^{k_B t}$

$\Rightarrow \quad T = 100 + ce^{k_B t}$

At $t = 0$, $T = 81^\circ$

$\Rightarrow \quad 81 = 100 + ce^0 \Rightarrow \quad c = -19.$

$\Rightarrow \quad T = -19e^{k_B t} + 100.$

Now at $t = 1$, $T = 81 + 10 = 91$

$\Rightarrow \quad 91 = -19e^{k_B} + 100$

$\Rightarrow \quad e^{k_B} = 9/19. \Rightarrow k_B = \ln(9/19) = -0.747$

So $\quad T(t) = -19e^{-0.747t} + 100.$

Finding t when $T = 99.9$

$\Rightarrow \quad 99.9 = -19e^{-0.747t} + 100.$

$$-19e^{-0.747t} = -0.1.$$

$$e^{-0.747t} = \frac{1}{190}$$

$$t = \frac{\ln(1/190)}{-0.747}$$

$t = 7.02$ minutes.

Now total time at which metal bar reaches $99.9°c$

$\Rightarrow$ Total time = Time in container A + Time in container B

$= 2 + 7.02$

$= 9.02$

Q23: A large tank is filled to capacity with 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped into the tank at a rate of 5 gal/min. The well mixed solution is pumped out at the same rate. Find the number A(t) of pounds of salt in the tank at time t.

Solution: initially, we have 500 gallons of pure water with no salt.

Input flow rate F_{in} of brine containing 2 pounds/gallon of salt is same as output flow rate F_{out} of well mixed fluid

$$\frac{dA}{dt} = \text{Input rate of salt} - \text{output rate of salt}$$

$$= F_{in} C_{in} - F_{out} C_{out}$$

A is amount of salt.

C is concentration of salt.

As $F_{in} = F_{out} = 5$ gal/min, $C_{in} = 2$ lb/gallon.

$$C_{out} = \frac{\text{Amount of salt}}{\text{Volume of tank}} = \frac{A}{500} \text{ lb/gallon}$$

So

$$\frac{dA}{dt} = 5 \times 2 - 5 \times \frac{A}{500}$$

$$= 10 - \frac{A}{100}$$

$$\int \frac{dA}{10 - A/100} = \int dt$$

$$-100 \int \frac{-dA}{1000 - A} = \int dt$$

$$-100 \ln|1000 - A| = t + C_1$$

$$\ln|1000 - A| = \frac{-t}{100} + C_2$$

$$1000 - A = \frac{-t}{100} + C_2$$

$$1000 - A = e^{c_2} e^{-t/100}$$

$$A(t) = 1000 - Ce^{-t/100}$$

Now using the initial condition that initially there is no salt i.e At $t=0$, $A=0$

$$\Rightarrow \quad 0 = 1000 - Ce^{0}$$

$$\Rightarrow \quad C = 1000$$

$$\Rightarrow \quad \boxed{A(t) = 1000 - 1000e^{-t/100}}$$

Q32: A 200-volt electromotive force is applied to an RC series circuit in which the resistance is 1000 ohms and the capacitance is 5×10^{-6} farad. Find the charge $q(t)$ on the capacitor if $i(0) = 0.4$. Determine the charge and current at $t = 0.005\ s$. Determine the charge as $t \to \infty$.

The differential Eq. for charge is.

$$R\frac{dq}{dt} + \frac{q}{C} = E(t)$$

we have to obtain charge $q(t)$ on capacitor, when $i(0) = 0.4$.

As $R = 1000\,\Omega$ $\quad C = 5 \times 10^{-6}$, $E = 200$

$$\Rightarrow \quad 1000\frac{dq}{dt} + \frac{q}{5\times10^{-6}} = 200$$

$$\Rightarrow \quad \frac{dq}{dt} + 200q = 0.2$$

$$\frac{dq}{dt} = 0.2 - 200q$$

Solving DE

$$\Rightarrow -\frac{1}{200}\int \frac{-200\,dq}{0.2 - 200q} = \int dt$$

$$-\frac{1}{200}\ln|0.2 - 200q| = t + \ln c$$

$$\ln|0.2-200q| = -200t + \ln c.$$

$$0.2-200q = ce^{-200t}$$

$$200q = 0.2 - ce^{-200t}$$

$$q = \frac{1}{1000} - Ke^{-200t}$$

As $i(t) = \frac{dq}{dt}$

$$i(t) = 200Ke^{-200t}.$$

As $i(0) = 0.4$

$$\Rightarrow \quad 0.4 = 200Ke^{0} \Rightarrow \quad K = \frac{1}{500}.$$

So

$$q(t) = \frac{1}{1000} - \frac{1}{500}e^{-200t}$$

&

$$i(t) = \frac{2}{5}e^{-200t}$$

To find charge after $t=0.005$

$$q(0.005) = \frac{1}{1000} - \frac{1}{500}e^{-200\times 0.005}$$

$$= 2.64\times 10^{-4} \quad C.$$

To find current $i(t)$ at $t=0.005$

$$i(0.005) = \frac{2}{5}e^{-200\times 0.005}$$

$$= 14.72\times 10^{-2} \text{ Amp.}$$

To find charge as $t\to\infty$.

$$q = \frac{1}{1000} - \frac{1}{500}e^{-200\times\infty}$$

$$= \frac{1}{1000}$$

Q42: A model that describes thepopulation of a fishery in which harvesting takes place at a constant rate is given by

$$\frac{dP}{dt} = kP - h,$$

where k and h are positive constants.

a. Solve the DE subject to $P(0) = P_0$.
b. Describe the behavior of the population P(t) for increasing time in the three cases $P_0 > \frac{h}{k}$, $P_0 = \frac{h}{k}$ and $0 < P_0 < \frac{h}{k}$
c. Use the results from part (b) to determine whether the fish population will ever go extinct in finite time, that is, whether there exists a time $T > 0$ such that $P(T) = 0$. If the population goes extinct, then find T.

Solution (a) Solving

$$\frac{dP}{dt} = kP - h$$

$$\Rightarrow \int \frac{dP}{kP-h} = \int dt$$

$$\frac{1}{k}\int \frac{k\,dP}{kP-h} = \int dt$$

$$\frac{1}{k}\ln|kP-h| = t + \ln c.$$

$$\ln|kP-h| = kt + \ln c.$$

$$kP - h = ce^{kt}$$

$$kP = h + ce^{kt}$$

$$P = \frac{h}{k} + ce^{kt}$$

using I.C. $P(0) = P_0$

$$\Rightarrow \quad P_0 = \frac{h}{k} + c \quad \Rightarrow \quad c = P_0 - \frac{h}{k}$$

So

$$P(t) = \frac{h}{k} + \left(P_0 - \frac{h}{k}\right)e^{kt} \qquad (A)$$

(b) For $P_0 > h/k$

R.H.S of Eq (A) is positive, so population of the fishery keep on increasing.

For $P_0 = \frac{h}{K}$

R.H.S of (A) becomes $\frac{h}{K}$, So population is constant.

For $0 < P_0 < h/K$.

R.H.S of (A) is -ve, so population of fishery decreases.

(C) Fish population will go extinct for $0 < P_0 < h/K$.

To find the time where population become extinct let we have

$$P_0 - \frac{h}{K} = p'$$

using $P = 0$ & $P_0 - \frac{h}{K} = p'$ in (A)

$$\Rightarrow \quad 0 = -p' e^{Kt_c} + \frac{h}{K}$$

$$p' e^{Kt_c} = \frac{h}{K} \Rightarrow e^{Kt_c} = \frac{h}{Kp'}$$

$$Kt_c = \ln\left(\frac{h}{Kp'}\right)$$

$$t_c = \frac{1}{K} \ln\left(\frac{h}{Kp'}\right)$$