4.1 Preliminary Theory - Linear Equations

Initial-Value and Boundary-Value Problem

For a linear differential equation, an nth order initial value problem is

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x) y = g(x).$$

Subject to

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \ \ldots, \ y^{(n-1)}(x_0) = y_{n-1}. \qquad (1)$$

Existence and uniqueness of solution — Theorem 4.1.1

Let $a_n(x), a_{n-1}(x), \ldots, a_0(x)$ & $g(x)$ be continuous on interval I & let $a_n(x) \neq 0$ for every x in this interval. If $x = x_0$ is any point in this interval, then a solution $y(x)$ of the initial value problem given in Eq. (1) exists on the interval and is unique.

Example

The initial value problem

$$3y''' + 5y'' - y' + 7y = 0 \qquad y(1) = 0, \ y'(1) = 0, \ y''(1) = 0$$

Possesses a trivial solution $y = 0$. Because given Equation is third order linear differential Equation with constant coefficient, it fullfills the condition of above-mention theorem 4.1.1. So $y = 0$ is the only solution on an interval containing $x = 1$

Boundary value problem.

* It is the type of the problem which consist of solution of differential Eq. of order 2 or greater in which dependent variable y & its derivative are specified at different pts.

* A problem such as

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Subject to:

$$y(a) = y_0\,, \quad y(b) = y_1$$

is called a boundary-value problem (BVP).

* The prescribed values $y(a) = y_0$ & $y(b) = y_1$ are called boundary conditions.

* A solution of the foregoing problem is a function satisfying the differential Eq. on some interval I, containing a & b whose graph passes through the two points (a, y_0) & (b, y_1).

* Other pairs of boundary conditions:

For second order differential Equation, other pair of boundary conditions could be

$$y'(a) = y_0\,, \qquad y(b) = y_1$$

$$y(a) = y_0\,, \qquad y'(b) = y_1$$

$$y'(a) = y_0\,, \qquad y'(b) = y_1.$$

where y_0 & y_1 are arbitrary constants

Homogeneous Equations

A linear n^{th}-order differential equation of the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0y = 0.$$

is said to homogeneous.

whereas an equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

where $g(x)$ is not identically zero is called nonhomogeneous.

Examples

$$2y'' + 3y' - 5y = 0$$ second order linear homogeneous equation

$$x^3y''' + 6y' + 10y = e^x$$

Non-homogeneous linear third order DE.

* To solve a non-homogeneous linear equation, we must be able to solve associated homogeneous equation

Differential operator

The derivative can be denoted as

$$\frac{dy}{dx} = Dy.$$

The symbol D is called differential operator because it transforms a differentiable function into another function.

* Higher order derivatives can be expressed in terms of D as

$$\frac{d^2y}{dx^2} = D^2y, \quad \frac{d^ny}{dx^n} = D^ny.$$

The polynomial function involving D such as $D+3$, $5x^3D^3 - 6x^2D^2 + 4xD + 9$ are also differential operator.

* An n^{th} order differential operator, or polynomial operator, can be defined as

$$L = a_n(x) D^n + a_{n-1}(x) D^{n-1} + \cdots + a_1(x) D + a_0(x)$$

* An n^{th} order differential operator is a linear operator i-e

$$L[\alpha f(x) + \beta g(x)] = \alpha L(f(x)) + \beta L(g(x))$$

Differential Equation

Any differential

$$y'' + 5y' + 6y = 5x - 3$$

can be written in terms of D as

$$(D^2 + 5D + 6) y = 5x - 3$$

$$Ly = 5x - 3$$

* Linear n^{th} order homogeneous & non-homogeneous Equations can be expressed as

$$Ly = 0 \quad \& \quad Ly = g(x).$$

Superposition principle - Homogeneous Equation.

Let $y_1, y_2, \ldots, y_k$ be solutions of homogeneous n^{th} order differential Equation on the interval I, then the linear combination

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x)$$

where $c_1, c_2, \ldots c_k$ are arbitrary constants, is also a solution on the interval.

* A constant multiple $y = c_1 y_1(x)$ of a solution $y_1(x)$ of a homogeneous linear DE is also a solution
* A homogeneous linear DE always possesses the trivial solution

Example

If $y_1 = x^2$ & $y_2 = x^2 \ln x$ both are solution of homogeneous linear Equation $x^3 y''' - 2xy' + 4y = 0$ on the interval $(0, \infty)$; By superposition principle, linear combination

$$y = c_1 x^2 + c_2 x^2 \ln x$$

is also a solution of Equation on the interval.

Linear dependence and Linear Independence

* A set of functions $f_1(x), f_2(x), \cdots, f_n(x)$ is said to be **linearly dependent** on an interval I, if there exist constants, $c_1, c_2, \cdots, c_n$, not all zero such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for every x in the interval.

* If the set of functions is not linearly dependent on the interval, then it is said to be **linearly independent**.

Wronskian

Suppose each of the functions $f_1(x), f_2(x), \cdots, f_n(x)$ possesses atleast $(n-1)$ derivatives. The determinant

$$W(f_1, f_2, \cdots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & & & \\ f_1^{(n-1)} & f_2^{(n-1)} & & f_n^{(n-1)} \end{vmatrix}$$

Where prime denotes derivative is called the Wronskian of the function.

Theorem 4.1.3 Criterion for linearly Independent Solution.

Let $y_1, y_2, \cdots, y_n$ be n Solutions of homogeneous linear n^{th}-order differential equation on interval I. The Set of solutions is linearly independent on I, if and only if $W(y_1, y_2, \cdots, y_n) \neq 0$ for every x in the interval.

* Fundamental set of solutions.

Let $y_1, y_2, \cdots y_n$ be set of n linearly independent Solutions of homogeneous n^{th} order differential Equation on an interval I. Then this set of solution is called Fundamental set of solution

* Existence of Fundamental set

There exists a fundamental set of solutions for homogeneous n^{th} order differential equation on an interval I.

* General Solution - Homogeneous Equations

Let $y_1, y_2, \cdots y_n$ be a fundamental set of solutions of homogeneous linear n^{th}-order differential equation on interval I. Then the general solution of the Equation on the interval is

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x).$$

Where c_i, $i = 1, 2, \cdots n$ are arbitrary constants.

Example The function $y_1 = e^x$, $y_2 = e^{2x}$ & $y_3 = e^{3x}$ satisfy the third-order equation

Then the general solution of the equation on the interval is

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p$$

where c_i, $i = 1, 2, \ldots, n$ are arbitrary constant

* Complementary Function

The general solution of non-homogeneous Equation is sum of two functions:

$$y = y_c(x) + y_p(x).$$

where y_c is called complementary function for non-homogeneous linear differential Equation.

y = complementary function + any complementary solution

* Example

We have $y_p = \frac{-11}{12} - \frac{x}{2}$ is particular solution of of non-homogeneous equation

$$y''' - 6y'' + 11y' - 6y = 3x \qquad (1)$$

solution of associated homogeneous Equation is

$$y_c = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$$

So general solution of Eq. (1) is

$$y = y_c + y_p$$

$$= c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x} + - \frac{11}{12} - \frac{x}{2}.$$

* Superposition Principle- Nonhomogeneous Equations.

Let $y_{p_1}, y_{p_2}, \ldots, y_{p_k}$ be k particular solutions of the nonhomogeneous linear nth order differential.

$y''' - 6y'' + 11y' - 6y = 0$. Since

$$W(e^{x}, e^{2x}, e^{3x}) = \begin{vmatrix} e^{x} & e^{2x} & e^{3x} \\ e^{x} & 2e^{2x} & 3e^{3x} \\ e^{x} & 4e^{2x} & 9e^{3x} \end{vmatrix}$$

$$= 2e^{6x} \neq 0$$

for every real value of x, the functions, y_1, y_2, & y_3 form a fundamental set of solution on $(-\infty, \infty)$. So general solution is

$$y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$$

* Non-homogeneous Equations:

A function y_p free of arbitrary parameters that satisfies the non-homogeneous equation

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x) y = g(x)$$

is called particular solution or particular integral.

Example

$y_p = 3$ is particular solution of non-homogeneous equation

$$y'' + 9y = 27.$$

Theorem 4.1.6.

General solution - Non-homogeneous Equation

Let y_p be any particular solution of non-homogeneous linear n^{th}-order differential equation on an interval I and let $y_1, y_2, \ldots, y_n$ be a fundamental set of solution of the associated homogeneous differential equation I.

equations on an interval I corresponding, in turns, to K distinct functions $g_1, g_2, \dots, g_K$. That is, suppose y_{p_i} denotes a particular solution of the corresponding differential equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g_i(x)$$

where $i = 1, 2, \dots, K$. Then

$$y_p = y_{p_1}(x) + y_{p_2}(x) + \cdots + y_{p_K}(x)$$

is particular solution of

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y$$
$$= g_1(x) + g_2(x) + \cdots + g_K(x)$$

Superposition – Nonhomogeneous DE

we have

$y_{p_1} = -4x^2$ as particular solution of $y'' - 3y' + 4y = -16x^2 + 24x - 8$

$y_{p_2} = e^{2x}$ as particular solution of $y'' - 3y' + 4y = 2e^{2x}$.

$y_{p_3} = xe^x$ as particular solution of $y'' - 3y' + 4y = 2xe^x - e^x$

then by superposition principal,

$$y = y_{p_1} + y_{p_2} + y_{p_3}$$
$$= -4x^2 + e^{2x} + xe^x$$

is solution of

$$y'' - 3y' + 4y = \underbrace{-16x^2 + 24x - 8}_{g_1(x)} + \underbrace{2e^{2x}}_{g_2(x)} + \underbrace{2xe^x - e^x}_{g_3(x)}$$

Exercise 4.1.

Find a member of the family that is a solution of initial value problem

2) $y = c_1 e^{4x} + c_2 e^{-x}$ $\quad (-\infty, \infty)$.

$y'' - 3y' - 4y = 0$ $\quad y(0) = 1,\ y'(0) = 2$.

we have the family of the solution

$$y(x) = c_1 e^{4x} + c_2 e^{-x} \qquad (1)$$

Taking derivative

$$y'(x) = 4c_1 e^{4x} - c_2 e^{-x} \qquad (2)$$

using first condition in Eq. (1)

$$\Rightarrow \quad y(0) = c_1 e^{0} + c_2 e^{0}$$

$$1 = c_1 + c_2 \qquad (3)$$

using 2^{nd} condition in Eq. (2)

$$y'(0) = 4c_1 e^{0} \cancel{+} - c_2 e^{0}$$

$$2 = 4c_1 - c_2 \qquad (4)$$

Solving (3) & (4)

$\Rightarrow$ (3) + (4)

$$\Rightarrow \quad \begin{aligned} c_1 + \cancel{c_2} &= 1 \\ 4c_1 \cancel{- c_2} &= 2 \\ \hline 5c_1 &= 3 \end{aligned} \Rightarrow c_1 = 3/5$$

using in (3)

$$\Rightarrow \quad c_2 = 1 - c_1 = 1 - \frac{3}{5} = 2/5$$

so solution of IVP is

$$y(x) = \frac{3}{5} e^{4x} + \frac{2}{5} e^{-x}.$$

Q7 Given that $x(t) = c_1 \cos\omega t + c_2 \sin\omega t$ is the general solution of $x'' + \omega^2 x = 0$ on the interval $(-\infty, \infty)$, show that solution satisfying the initial conditions

$$x(0) = x_0 \quad \& \quad x'(0) = x_1$$

is given by

$$x(t) = x_0 \cos\omega t + \frac{x_1}{\omega} \sin\omega t.$$

Solution

using $x(0) = x_0$

$$\Rightarrow x(0) = \cancel{x_0}\ c_1 \cos 0 + c_2 \sin 0$$

$$x_0 = c_1 \quad \Rightarrow \quad c_1 = x_0$$

Now finding the derivative

$$x'(t) = -c_1 \omega \sin\omega t + c_2 \omega \cos\omega t.$$

using condition $x'(0) = x_1$

$$\Rightarrow \quad x_1 = -c_1 \omega \sin 0 + c_2 \omega \cos 0$$

$$\Rightarrow \quad c_2 = \frac{x_1}{\omega}.$$

So given solution is

$$x(t) = x_0 \cos\omega t + \frac{x_1}{\omega} \sin\omega t.$$

Find an interval centered about $x = 0$ for which the given initial value problem has unique solution.

$$y'' + (\tan x) y = e^x \qquad y(0) = 1 \quad y'(0) = 0$$

Here $a_2(x) = 1$ $\quad a_1(x) = 0$ $\quad a_0(x) = \tan x.$

$$g(x) = e^x.$$

Now the functions $a_2(x)$ & $\cancel{a_0(x)}$ $g(x)$ are continuous on $(-\infty, \infty)$. However $a_0(x)$ is continuous on

$$\ldots, \left(-\frac{5\pi}{2}, -\frac{3\pi}{2}\right), \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \ldots$$

out of all these intervals, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is the interval centered about zero.

So the interval for which the given IVP has a unique solution is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

~~Q8~~

Q12 Use $y = c_1 + c_2 x^2$ to find a solution of $xy'' - y' = 0$ that satisfies the boundary conditions $y(0)=1, y'(1)=6$.

Solution

$$y = c_1 + c_2 x^2$$

using $y(0) = 1$

$$\Rightarrow \quad y(0) = c_1 + c_2 \times 0$$

$$1 = c_1 \quad \Rightarrow \quad c_1 = 1.$$

Finding derivative

$$y'(x) = 2c_2 x$$

using $y'(1) = 6$

$$\Rightarrow \quad 6 = 2c_2 \Rightarrow \quad c_2 = 3.$$

So solution is

$$y = 1 + 3x^2.$$

Q16 Determine whether the given set of function is linearly independent on the interval $(-\infty, \infty)$.

$$f_1(x) = 0, \quad f_2(x) = x, \quad f_3(x) = e^x.$$

Finding the wronskian

$$W(f_1, f_2, f_3) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_1'(x) & f_2'(x) & f_3'(x) \\ f_1''(x) & f_2''(x) & f_3''(x) \end{vmatrix}$$

$$= \begin{vmatrix} 0 & x & e^x \\ 0 & 1 & e^x \\ 0 & 0 & e^x \end{vmatrix}$$

Q.7 Given that $x(t) = c_1 \cos\omega t + c_2 \sin\omega t$ is the general solution of $x'' + \omega^2 x = 0$ on the interval $(-\infty, \infty)$, show that solution satisfying the initial conditions

$$x(0) = x_0 \quad \& \quad x'(0) = x_1$$

is given by

$$x(t) = x_0 \cos\omega t + \frac{x_1}{\omega} \sin\omega t.$$

Solution

using $x(0) = x_0$

$$\Rightarrow x(0) = c_1 \cos 0 + c_2 \sin 0$$

$$x_0 = c_1 \quad \Rightarrow \quad c_1 = x_0$$

Now finding the derivative

$$x'(t) = -c_1 \omega \sin\omega t + c_2 \omega \cos\omega t.$$

using condition $x'(0) = x_1$

$$\Rightarrow \quad x_1 = -c_1 \omega \sin 0 + c_2 \omega \cos 0$$

$$\Rightarrow \quad c_2 = \frac{x_1}{\omega}.$$

So given solution is

$$x(t) = x_0 \cos\omega t + \frac{x_1}{\omega} \sin\omega t.$$

Find an interval centered about $x = 0$ for which the given initial value problem has unique solution.

$$y'' + (\tan x)\, y = e^x \qquad y(0) = 1 \quad y'(0) = 0$$

Here $a_2(x) = 1 \qquad a_1(x) = 0 \qquad a_0(x) = \tan x.$

$$g(x) = e^x.$$

Now the functions $a_2(x)$ & ~~$a_0(x)$~~ $g(x)$ are continuous on $(-\infty, \infty)$. However $a_0(x)$ is continuous on

$$\ldots, \left(-\frac{5\pi}{2}, -\frac{3\pi}{2}\right), \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \ldots$$

out of all these intervals, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is the interval centered about zero.

So the interval for which the given IVP has a unique solution is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

~~Q8~~

Q12 Use $y = c_1 + c_2 x^2$ to find a solution of $xy'' - y' = 0$ that satisfies the boundary conditions $y(0) = 1$, $y'(1) = 6$.

Solution

$$y = c_1 + c_2 x^2$$

using $y(0) = 1$

$$\Rightarrow \quad y(0) = c_1 + c_2 \times 0$$

$$1 = c_1 \quad \Rightarrow \quad c_1 = 1.$$

Finding derivative

$$y'(x) = 2c_2 x$$

using $y'(1) = 6$

$$\Rightarrow \quad 6 = 2c_2 \quad \Rightarrow \quad c_2 = 3.$$

So solution is

$$y = 1 + 3x^2.$$

Q16. Determine whether the given set of function is linearly independent on the interval $(-\infty, \infty)$.

$$f_1(x) = 0, \quad f_2(x) = x, \quad f_3(x) = e^x.$$

Finding the wronskian

$$W(f_1, f_2, f_3) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_1'(x) & f_2'(x) & f_3'(x) \\ f_1''(x) & f_2''(x) & f_3''(x) \end{vmatrix}$$

$$= \begin{vmatrix} 0 & x & e^x \\ 0 & 1 & e^x \\ 0 & 0 & e^x \end{vmatrix}$$

$W(f_1, f_2, f_3) = 0$

As wronskian is zero, so this set of equation is linearly dependent.

20 $f_1(x) = 2+x$ $\quad$ $f_2(x) = 2+|x|$

Here we can break $f_2(x)$ into two function

$f_2(x) = 2+x$ $\quad$ $f_3(x) = 2-x$

Calculating wronskian.

$$W(f_1, f_2, f_3) = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}$$

$$= \begin{vmatrix} 2+x & 2+x & 2-x \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{vmatrix}$$

Finding determinant, we get

$w(f_1, f_2, f_3) = 0$

Since wronskian is zero, so function is linearly dependent

22 $f_1(x) = e^{+x}$ $\quad$ $f_2(x) = e^{-x}$ $\quad$ $f_3(x) = \sinh x$.

Finding the wronskian.

$$w(f_1, f_2, f_3) = \begin{vmatrix} e^{+x} & e^{-x} & \sinh x \\ e^{x} & -e^{-x} & \cosh x \\ e^{x} & e^{-x} & \sinh x \end{vmatrix}$$

$$= e^{x}\begin{vmatrix} -e^{-x} & \cosh x \\ e^{-x} & \sinh x \end{vmatrix} - e^{-x}\begin{vmatrix} e^{x} & \cosh x \\ e^{x} & \sinh x \end{vmatrix} + \sinh x\begin{vmatrix} e^{x} & -e^{-x} \\ e^{x} & e^{-x} \end{vmatrix}$$

$= e^{x}(-e^{-x}\sinh x - e^{-x}\cosh x) - e^{-x}(e^{x}\sinh x - e^{x}\cosh x)$
$+ \sinh x(1-(-1))$

$= -\sinh x - \cancel{\cosh x} - \sinh x + \cancel{\cosh x} + 2\sinh x$

$= -2\sinh x + 2\sinh x = 0$

So set of functions is linearly dependent.

Verify that the given function form a fundamental set of solutions of the differential equation on the indicated interval. Find the general solution.

26 $4y'' - 4y' + y = 0$ $\quad e^{x/2}, xe^{x/2}$ $\quad (-\infty, \infty)$

we have to check $y_1 = e^{x/2}$ & $y_2 = xe^{x/2}$ as a solution of given differential Eq.

$y_1 = e^{x/2}$.

$y_1 = e^{x/2}$, $y_1' = \frac{1}{2}e^{x/2}$, $y_1'' = \frac{1}{4}e^{x/2}$.

using in given DE

$4y'' - 4y' + y = 0$

$\cancel{4} \times \frac{1}{\cancel{4}} e^{x/2} - \cancel{4}^{2} \times \frac{1}{\cancel{2}} e^{x/2} + e^{x/2} = 0$

$e^{x/2} - 2e^{x/2} + e^{x/2} = 0$

$\Rightarrow \quad 0 = 0$

$y_2 = xe^{x/2}$

$y_2' = xe^{x/2} \times \frac{1}{2} + e^{x/2} = \frac{1}{2}xe^{x/2} + e^{x/2}$

$y_2'' = \frac{1}{2}\left(\frac{1}{2}xe^{x/2} + e^{x/2}\right) + \frac{1}{2}e^{x/2}$

$= \frac{1}{4}xe^{x/2} + \frac{1}{2}e^{x/2} + \frac{1}{2}e^{x/2}$

$= \frac{1}{4}xe^{x/2} + e^{x/2}$

using in given DE.

$\Rightarrow 4\times\frac{1}{4}xe^{x/2}+4e^{x/2}-4\times\frac{1}{2}xe^{x/2}-4e^{x/2}+xe^{x/2}$

$= xe^{x/2}+4e^{x/2}-2xe^{x/2}-4e^{x/2}+xe^{x/2}$

$=0$

For a set of function to be fundamental set of solution, the set should be linearly independent. Finding wronskian.

w.r.t.

$$W(y_1,y_2)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{x/2} & xe^{x/2} \\ \frac{1}{2}e^{x/2} & e^{x/2}+\frac{1}{2}xe^{x/2} \end{vmatrix}$$

$$= e^{x/2}\left(e^{x/2}+\frac{1}{2}xe^{x/2}\right) - xe^{x/2}\times\frac{1}{2}e^{x/2}$$

$$= e^{x}+\frac{1}{2}xe^{x}-\frac{1}{2}e^{x} = e^{x} \neq 0.$$

So given functions are linearly independent and can form a fundamental set of equation.

General Solution is

$$y = c_1e^{x/2}+c_2xe^{x/2}.$$

32 verify that the given two parameter family of solution is the general solution of non-homogeneous differential Eq. on the indicated interval.

$$y''+y = \sec x.$$

$$y = c_1\cos x + c_2\sin x + x\sin x + \cos x\ln(\cos x) \quad \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$$

Solution

First of all we'll check that $y_1=\cos x$, $y_2=\sin x$ are solutions of associated homogeneous equation

$y_1 = \cos x$.

$$y_1' = -\sin x \qquad y_1'' = -\cos x.$$

so $y_1'' + y_1 = 0 \Rightarrow -\cos x + \cos x = 0 \Rightarrow 0 = 0$

$y_2 = \sin x$.

$$y_2' = \cos x \qquad y_2'' = -\sin x$$

$\Rightarrow \quad y_2'' + y_2' = \sin x - \sin x = 0$

Now checking whether y_1 & y_2 are linearly independent or not.

$$W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x. & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1 \neq 0.$$

So given set of function is linearly independent.

* Now verifying the particular solution

$$y_P = x\sin x + \cos x \ln(\cos x)$$

$$y_P' = x\cos x + \sin x + \cancel{\cos x} \times \frac{1}{\cancel{\cos} x} \times -\sin x$$
$$+ \ln(\cos x)(-\sin x)$$
$$= x\cos x + \cancel{\sin x} - \cancel{\sin x} - \sin x \ln(\cos x)$$
$$= x\cos x - \sin x \ln \cos x.$$

$$y_P'' = -x\sin x + \cos x - \cos x \ln \cos x$$
$$- \sin x \times \frac{1}{\cos x} \times -\sin x$$
$$= -x\sin x + \cos x - \cos x(\ln \cos x) + \frac{\sin^2 x}{\cos x}.$$

Now using these values in

$$y_P'' + y_P = \sec x$$

$$\Rightarrow -x\cancel{\sin x} + \cos x - \cos x(\cancel{\ln \cos x}) + \frac{\sin^2 x}{\cos x} + \cancel{x}\sin x + \cancel{\cos x}\ln\cos x = \sec x$$

$$\frac{\cos^2 x + \sin^2 x}{\cos x} = \sec x \Rightarrow \sec x = \sec x.$$

So

$y = c_1 \cos x + c_2 \sin x + x \sin x + \cos x \ln(\cos x)$ is general solution of given DE on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

35 (a) verify that $y_{P_1} = 3e^{2x}$ & $y_{P_2} = x^2 + 3x$ are respectively particular solution of

$$y'' - 6y' + 5y = -9e^{2x} \quad (1)$$

$$y'' - 6y' + 5y = 5x^2 + 3x - 16 \quad (2)$$

Solution

$$y_{P_1} = 3e^{2x}, \quad y'_{P_1} = 6e^{2x}, \quad y''_{P_1} = 12e^{2x}.$$

using these values in (1)

$$\Rightarrow 12e^{2x} - 6 \times 6e^{2x} + 5 \times 3e^{2x} = -9e^{2x}$$

$$12e^{2x} - 36e^{2x} + 15e^{2x} = -9e^{2x}$$

$$-9e^{2x} = -9e^{2x}$$

* $y_{P_2} = x^2 + 3x$

$$y'_{P_2} = 2x + 3 \qquad y''_{P_2} = 2.$$

using in (2)

$$\Rightarrow 2 - 6(2x+3) + 5(x^2+3x) = 5x^2 + 3x - 16$$

$$2 - 12x - 18 + 5x^2 + 15x = 5x^2 + 3x - 16.$$

$$5x^2 + 3x - 16 = 5x^2 + 3x - 16.$$

b) Use part (a) to find particular solution of

$$y'' - 6y' + 5y = 5x^2 + 3x - 16 - 9e^{2x}$$

$$y'' - 6y' + 5y = -10x^2 - 6x + 32 + e^{2x}.$$

$$y'' \cancel{+} - 6y' + 5y = \underbrace{5x^2 + 3x - 16}_{g_2(x)} + \underbrace{(-9e^{2x})}_{g_1(x)}$$

By superposition principle:

$$y = y_{p_1} + y_{p_2} = 3e^{2x} + x^2 + 3x$$

is solution of given DE (particular solution).

$$y'' - 6y' + 5y = -10x^2 - 6x + 32 + e^{2x} \qquad (A)$$

$$= -2(5x^2 - 3x + 16) + \frac{-1}{9}(-9e^{2x})$$

$$= -2\, g_2(x) - \frac{1}{9} g_1(x)$$

By superposition principle:

$$y = \frac{-1}{9} y_{p_1} - 2 y_{p_2}$$

$$= \frac{-1}{3} e^{2x} - 2(x^2 + 3x)$$

$$= -\frac{1}{3} e^{2x} - 2x^2 - 6x$$

is the particular solution of (A).