6.1 Solution about ordinary point.

Power series

An infinite series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots$$

is called power series centered at a.

* Convergence.

A power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ is convergent at a specified value of x, if its sequence of partial sums converges i.e $\lim_{N\to\infty} S_N(x) = \lim_{N\to\infty} \sum_{n=0}^{N} c_n (x-a)^n$ exists. If limit does not exist at x, the series is said to be divergent.

Interval of convergence

Every power series has an interval of convergence. The interval of convergence is set of all real numbers x for which the series converges.

Radius of convergence.

Every power series has a radius of convergence R.

* If $R>0$, then the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges for $|x-a|<R$ & diverges for $|x-a|>R$.
* If the series converges only at center "a" then $R=0$
* If series converges for all x, then we write $R=\infty$.

Absolute convergence

Within its interval of convergence, a power series converges absolutely. In other words, if x is a number in the interval of convergence and is not an end point of the interval then the series of absolute values

$$\sum_{n=0}^{\infty} |c_n (x-a)^n|$$

converges.

Ratio Test

convergence of a power series can often be determined by the ratio test. Suppose that $c_n \neq 0$, for all n, and that

$$\lim_{n\to\infty} |x-a| \left|\frac{c_{n+1}}{c_n}\right| = |x-a| \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = L$$

* If $L < 1$, the series converges absolutely.
* If $L > 1$, the series diverges.
* If $L = 1$, the test is inconclusive.

* A power series defines a function

A power series defines a function $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ whose domain is the interval of convergence of the series.

* If the radius of convergence $R > 0$, then f is continuous, differentiable and integrable on the interval $(a-R, a+R)$.
* $f'(x)$ and $\int f(x)\,dx$ can be found by term-by-term differentiation and integration.

Identity property

If $\sum_{n=0}^{\infty} c_n x^n = 0$, $R > 0$ for all number x in the interval of convergence, then $c_n = 0$ for all n,

* Analytic at a point

A function f is analytic at a point 'a', if it can be represented by a power series in $(x-a)$ with a positive or infinite radius of convergence.

* Arithematic of power series:

Power series can be combined through the operation of addition, multiplication and division.

* The procedure for power series are similar to those by which two polynomials are added, multiplied and divided.

Example

write $\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1}$ as a single power series whose general term involves x^k.

Solution

* Two add two series, it is neccessary that both summation indices start with the same number and the powers of x in each series be "in phase).

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1}$$

$$= 2.1.c_2 + \sum_{n=3}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1}$$

Series starts with x for $n=3$ | Series starts with x for $n=0$.

Now summation index should starts with same numbers and should have same exponent for n.

$$\Rightarrow 2c_2 + \sum_{n=3}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1}$$

replace $k = n-2$ $\quad$ replace $k = n+1$

$n = k+2$ $\quad$ $n = k-1$

$$\Rightarrow = 2c_2 + \sum_{k+2=3}^{\infty} (k+2)(k+1).c_{k+2} x^k + \sum_{k-1=0}^{\infty} c_{k-1} x^k$$

$$\Rightarrow = 2c_2 + \sum_{k=1}^{\infty} (k+1)(k+2) c_{k+2} x^k + \sum_{k=1}^{\infty} c_{k-1} x^k$$

Now we can add the series

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n-2} = 2c_2 + \sum_{k=1}^{\infty} \left[(k+1)(k+2) c_{k+1} + c_{k-1}\right] x^k$$

Power Series Solution.

Ordinary and singular pts.

Consider the 2^{nd} order differential equation

$$a_2(x) y'' + a_1(x) y' + a_0(x) y = 0$$

$\div$ing by $a_2(x)$

$$y'' + P(x) y' + Q(x) y = 0 \qquad (1)$$

* A point x_0 is said to be an ordinary point of the differential equation if both $P(x)$ & $Q(x)$ are analytic at x_0.
* A point that is not an ordinary point is said to be a singular point of the equation.

Example

$$y'' + e^x y' + (\sin x) y = 0$$

The pt $x = 0$ is the ordinary point of differential equation since both e^x & $\sin x$ are analytic at $x = 0$.

$$y'' + e^x y' + \ln x \, y = 0$$

Here $x = 0$ is a singular point because $\ln x$ is discontinuous at $x = 0$

* If $a_2(x)$, $a_1(x)$ and $a_0(x)$ are polynomials with no common factors, then both rational functions $P(x) = \frac{a_1(x)}{a_2(x)}$ and $Q(x) = \frac{a_0(x)}{a_2(x)}$ are analytic except when $a_2(x) = 0$.

So we can write

$x = x_0$ is ordinary point of (1) if $a_2(x_0) \neq 0$ whereas $x = x_0$ is a singular point if $a_2(x_0) = 0$.

e-g. $(x^2 - 1) y'' + 2x y' + 6y = 0$
has only two singular pts $x = \pm 1$.

Existance of Power Series Solution:

If $x = x_0$ is an ordinary point of differential equation, we can always find two linearly independent solutions in the form of power series centered at x_0, that is $y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$. A series solution converges at least on some interval defined by $|x - x_0| < R$, Where R is the distance from x_0 to the closest singular points.

Example Solve

$$y'' + xy = 0 \qquad (A)$$

Solution There is no finite singular point, so there are two power series solutions centered at 0, convergent for $|x| < \infty$.

Let
$$y = \sum_{n=0}^{\infty} c_n x^n \qquad (1)$$

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}.$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}. \qquad (2)$$

using (1) & (2) in (A)

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + x \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

$$2(2-1) c_2 + \sum_{n=3}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

$$n-2 = k \qquad\qquad n+1 = k$$
$$n = k+2 \qquad\qquad n = k-1.$$

$$\Rightarrow \quad 2c_2 + \sum_{k=1}^{\infty} (k+2)(k+1) c_{k+2} x^k + \sum_{k=1}^{\infty} c_{k-1} x^k = 0$$

$$2c_2 + \sum_{k=1}^{\infty} \left((k+1)(k+2) c_{k+2} + c_{k-1} \right) x^k = 0$$

Here all the coefficients should be zero individually.

$$2c_2 = 0$$

$$\Rightarrow \quad c_2 = 0$$

$$\boxed{(K+1)(K+2)\, C_{K+2} + C_{K-1} = 0} \qquad (3)$$

Eq. (3) is called recurrence relation

$$C_{K+2} = \frac{-C_{K-1}}{(K+1)(K+2)} \qquad K = 1, 2, 3, \ldots \qquad (4)$$

$K=1$ $\quad C_3 = \dfrac{-C_0}{2 \cdot 3}$

$K=2$ $\quad C_4 = \dfrac{-C_1}{3 \cdot 4}$

$K=3$ $\quad C_5 = \dfrac{-C_2}{4 \cdot 5}$

Since $C_2 = 0$

$\Rightarrow \quad C_5 = 0$

$K=4$ $\quad C_6 = \dfrac{-C_3}{5 \cdot 6} = \dfrac{+1}{2 \cdot 3 \cdot 5 \cdot 6} C_0$

$K=5$ $\quad C_7 = \dfrac{-C_4}{6 \cdot 7} = \dfrac{C_1}{3 \cdot 4 \cdot 6 \cdot 7}$

$K=6$ $\quad C_8 = \dfrac{-C_5}{7 \cdot 8} = 0 \quad$ since $C_5 = 0$

$K=7$ $\quad C_9 = \dfrac{-C_6}{8 \cdot 9} = \dfrac{-1}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} C_0$

$K=8$ $\quad C_{10} = \dfrac{-C_7}{9 \cdot 10} = \dfrac{1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10} C_1$

$K=9$ $\quad C_{11} = \dfrac{-C_8}{10 \cdot 11} = 0$

We have

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + c_7 x^7 + c_8 x^8 + c_9 x^9 + c_{10} x^{10} + c_{11} x^{11} + \cdots$$

$$= c_0 + c_1 x + 0 - \frac{c_0}{2.3} x^3 - \frac{c_1}{3.4} x^4 + \frac{c_0}{2.3.5.6} x^6 + \frac{c_1}{3.4.6.7} x^7 + 0 - \frac{c_0}{2.3.5.6.8.9} x^9 - \frac{c_1}{3.4.6.7.9.10} x^{10} + 0 + \cdots$$

$$= c_0\left(1 - \frac{x^3}{2.3} + \frac{x^6}{2.3.5.6} - \frac{x^9}{2.3.5.6.8.9} + \cdots\right) + c_1\left(x - \frac{x^4}{3.4} + \frac{x^7}{3.4.6.7} - \frac{x^{10}}{3.4.6.7.9.10} + \cdots\right)$$

$$= c_1 y_1 + c_2 y_2.$$

Now consider y_1

$$y_1 = 1 - \frac{x^3}{3.\ 1.2} + \frac{x^6}{3.6.\ 1.2.5} - \frac{x^9}{3.6.9\ \ 2.5.8} + \cdots$$

$$= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^{3k}}{2.3.\cdots(3k-1)3k}$$

$$y_2 = x - \frac{x^4}{3.4} + \frac{x^7}{4.7.\ 3.6} - \frac{x^{10}}{4.7.10.\ 3.6.9} + \cdots$$

$$= x + \sum_{k=1}^{\infty} \frac{(-1)^k x^{3k+1}}{3.4.\cdots(3k)(3k+1)}.$$

Example Solve $(x^2+1)y''+xy'-y=0$

Solution. $\because x^2+1=0 \Rightarrow x=\pm i$

So $x=\pm i$ are singular pts. & solution of differential Equation about $x=0$ will converges for $|x|<1$ (distance to nearest point).

Let $y=\sum_{n=0}^{\infty} c_n x^n$

$$y'=\sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$y''=\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

$$(x^2+1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^n + \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{\text{opening two terms}} + \underbrace{\sum_{n=1}^{\infty} n c_n x^n}_{\text{opening one term}} - \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{\text{opening two terms}} = 0$$

$$2(2-1)c_2x^0 + 3(3-1)c_3x^1 + 1\cancel{c_1x} + -c_0 - \cancel{c_1x} + \sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=4}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=2}^{\infty} n c_n x^n - \sum_{n=2}^{\infty} c_n x^n = 0$$

$k=n \quad\quad k=n-2 \quad\quad k=n \quad\quad k=n$

$$2c_2+6c_3x-c_0+\sum_{k=2}^{\infty} k(k-1)c_k x^k + \sum_{k=2}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=2}^{\infty} k c_k x^k - \sum_{k=2}^{\infty} c_k x^k = 0$$

$$2C_2 - C_0 + 6C_3 x + \sum_{K=2}^{\infty} \left(K(K-1) C_K + (K+2)(K+1) C_{K+2} - KC_K - C_K \right) x^K = 0$$

$$2C_2 - C_0 + 6C_3 x + \sum_{K=2}^{\infty} \left((K+1)(K-1) C_K + (K+1)(K+2) C_{K+2} \right) x^K = 0$$

Putting all the coefficients of power of x equal to zero

$$\Rightarrow \quad 2C_2 - C_0 = 0 \quad \Rightarrow \quad C_2 = \frac{1}{2} C_0$$

$$6C_3 = 0 \quad \Rightarrow \quad C_3 = 0.$$

$$(K+1)(K-1) C_K + (K+1)(K+2) C_{K+2} = 0$$

$$\cancel{(K+1)}(K+2) C_{K+2} = -\cancel{(K+1)}(K-1) C_K$$

$$\boxed{C_{K+2} = -\frac{(K-1)}{(K+2)} C_K} \qquad K = 2, 3, 4, .$$

K=2

$$C_4 = \frac{-1}{4} C_2 = \frac{-1}{4} \times \frac{1}{2} C_0$$

$$= \frac{-1}{2^2 \, 2!} C_0.$$

K=3

$$C_5 = \frac{-2}{5} \times C_3 = 0$$

K=4

$$C_6 = \frac{-3}{6} C_4 = \frac{1.3}{6.\, 2^2 . 2!} C_0$$

$$= \frac{1.3}{2^3 \, 3!} C_0$$

K=5.

$$C_7 = -\frac{4}{7} C_5 = 0.$$

K=6

$$C_8 = -\frac{5}{8} C_6 = -\frac{1.3.5 \, C_0}{4.2.\, 2^3 . 3!} = \frac{-1.3.5\, C_0}{2^4 \, 4!}$$

K=7

$$C_9 = -\frac{6}{9} C_7 = 0$$

K=8

$$C_{10} = \frac{-7}{10} = \frac{1.3.5.7 \, C_0}{2^5 \, 5!}$$

So

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + c_7 x^7 + c_8 x^8 + c_9 x^9 + c_{10} x^{10} + \cdots$$

$$= c_0 + \frac{c_0 x^2}{2} - \frac{c_0}{2^2\, 2!} x^4 + \frac{1.3}{2^3\, 3!} x^6 - \frac{1.3.5}{2^3\, 3!} x^8 + \frac{1.3.5.7}{2^5\, 5!} x^{10} - \cdots + c_1 x.$$

$$= c_0 \left[1 + \frac{x^2}{2} - \frac{x^4}{2^2\, 2!} + \frac{1.3}{2^3\, 3!} x^6 - \frac{1.3.5}{2^3\, 3!} x^8 + \frac{1.3.5.7}{2^5\, 5!} x^{10} \right] + \cdots + c_1 x$$

$$= c_0\, y_1(x) + y_2(x).$$

$$y_1(x) = 1 + \frac{x^2}{2} - \frac{x^4}{2^2\, 2!} + \frac{1.3}{2^3\, 3!} x^6 - \frac{1.3.5}{2^3\, 3!} x^8 + \frac{1.3.5.7}{2^5\, 5!} x^{10} + \cdots$$

$$= 1 + x^2 + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{1.3.5.7 \cdots (2n-3)}{2^n\, n!} x^{2n} \qquad |x| < 1.$$

$a_0 = -1$

$a_n = a_0 + (n-1)d$

$= -1 + 2(n-1)$

$= 2n-3$

Example

$$y'' - (1+x)y = 0$$

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}.$$

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^n - \sum_{n=0}^{\infty} c_n x^{n+1} = 0.$$

open series (one term) one term

$$2(2-1) c_2 + - c_0 + \sum_{\substack{n=3\\ n=k+2}}^{\infty} n(n-1) c_n x^{n-2} - \sum_{\substack{n=1\\ n=k}}^{\infty} c_n x^n - \sum_{\substack{n=0\\ n=k-1}}^{\infty} c_n x^{n+1} = 0.$$

$$2c_2 - c_0 + \sum_{k=1}^{\infty} (k+2)(k+1) c_{k+2} x^k - \sum_{k=1}^{\infty} c_k x^k - \sum_{k=1}^{\infty} c_{k-1} x^k = 0$$

$$2c_2 - c_0 + \sum_{k=1}^{\infty} \left((k+2)(k+1) c_{k+2} - c_k - c_{k-1} \right) x^k = 0$$

$$\Rightarrow \quad 2c_2 - c_0 = 0 \quad \Rightarrow \quad \boxed{c_2 = \frac{1}{2} c_0}$$

$$(k+1)(k+2) c_{k+2} = c_k + \cancel{c_{k+1}} c_{k-1}$$

$$\boxed{c_{k+2} = \frac{c_k + c_{k-1}}{(k+1)(k+2)}} \qquad k = 1, 2, 3 \ldots$$

Here all coefficients will include both c_0 & c_1.

For Simplicity: Let us first consider $c_0 \neq 0$ & $c_1 = 0$ and then $c_0 = 0$ & $c_1 \neq 0$

Case 1 $c_0 \neq 0 \quad c_1 = 0$

$k=1$
$$c_3 = \frac{c_1 + c_0}{2 \cdot 3} = \frac{c_0}{2 \cdot 3} = \frac{c_0}{6}$$

$k=2$
$$c_4 = \frac{c_2 + c_1}{3 \cdot 4} = \frac{c_2}{3 \cdot 4} = \frac{1}{2 \cdot 3 \cdot 4} c_0 = \frac{c_0}{24}.$$

$K=3$

$$c_5 = \frac{c_3 + c_2}{4 \cdot 5} = \frac{1}{4 \cdot 5}\left[\frac{c_0}{2 \cdot 3} + \frac{1}{2}c_0\right]$$

$$= \frac{c_0}{30}.$$

So

$$y_1(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$
$$= c_0 + 0 + \frac{1}{2}c_0 x^2 + \frac{1}{6}c_0 x^3 + \frac{1}{24}c_0 x^4 + \frac{1}{30}c_0 x^5 + \cdots$$
$$= c_0\left(1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{30} + \cdots\right)$$

For $c_0 = 0 \quad c_1 \neq 0$

$$c_0 = 0 \Rightarrow c_2 = \frac{1}{2}c_0 = 0.$$

$K=1$

$$c_3 = \frac{c_1 + c_0}{2 \cdot 3} = \frac{c_1}{6}$$

$$c_4 = \frac{c_2 + c_1}{3 \cdot 4} = \frac{c_1}{3 \cdot 4} = \frac{c_1}{12}.$$

$$c_5 = \frac{c_3 + c_2}{4 \cdot 5} = \frac{c_3}{4 \cdot 5} = \frac{c_1}{4 \cdot 5 \cdot 6} = \frac{c_1}{120}.$$

$$y_2(x) = c_1\left(x + \frac{x^3}{6} + \frac{x^4}{12} + \frac{x^5}{120} + \cdots\right)$$

General solution is

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$