Operation on power series

1) The series multiplication for power series:

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ converges absolutely for $|x|<R$ and

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0$$
$$= \sum_{k=0}^{n} a_k b_{n-k}$$

then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x|<R$

$$\sum_{n=0}^{\infty} a_n x^n * \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} c_n x^n$$

Example

$$\sum_{n=0}^{\infty} x^n, \quad \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{(n+1)}$$

$$= (1 + x + x^2 + \cdots)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right)$$

$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right) + \left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} + \cdots\right)$$
$$+ \left(x^3 - \frac{x^4}{2} + \frac{x^5}{3} + \cdots\right) + \cdots$$

$$= x + \left(-\frac{1}{2} + 1\right)x^2 + \left(\frac{1}{3} - \frac{1}{2} + 1\right)x^3 + \cdots$$

$$= x + \frac{x^2}{2} + \frac{5x^3}{6} + \cdots$$

Theorem If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x|<R$, then $\sum_{n=0}^{\infty} a_n (f(x))^n$ converges for every continuous function f on $|f(x)|<R$

* Since $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ converges absolutely for $|x|<1$ then

$$\frac{1}{1-4x^2} = \sum_{n=0}^{\infty} (4x^2)^n \text{ converges absolutely for } |4x^2| < 1$$

$$\Rightarrow \quad |4x^2| < 1 \quad \Rightarrow \quad |x^2| < 1/4$$

$$|x| < 1/2$$

Term by Term Differentiation.

If $\sum c_n (x-a)^n$ has radius of convergence $R > 0$, it defines a function

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \text{ on the interval } a-R < x < a+R$$

The function f has derivative of all the orders inside the interval, and we obtain the derivatives by differentiation of the original series term by term

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2}$$

and so on. Each of the series converges at every point of the interval $a-R < x < a+R$.

Example 4 Find series of $f'(x)$ & $f''(x)$ if

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \dots$$

$$= \sum_{n=0}^{\infty} x^n \qquad -1 < x < 1$$

Solution we differentiate the power series on the right term by term

$$f'(x) = \frac{-1}{(1-x)^2} \times -1 = 1 + 2x + 3x^2 + \dots + nx^{n-1} + \dots$$

$$= \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} \qquad -1 < x < 1.$$

$$f''(x) = \frac{-2}{(1-x)^3} \times -1 = \sum_{n=2}^{\infty} n(n-1) x^{n-2} \qquad -1 < x < 1$$

* Term by Term differentiation might not work for other kinds of series. For example, the trigonometric series

$$\sum_{n=1}^{\infty} \frac{\sin(n!\,x)}{n^2}$$

converges for all x. But if we differentiate term by term, we get the series

$$\sum_{n=1}^{\infty} \frac{n!\cos(n!\,x)}{n^2}$$

which diverges for all x.

Term by Term Integration Theorem

Suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

converges for $a-R < x < a+R$. $(R>0)$ Then

$$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

converges for $a-R<x<a+R$. and

$$\int f(x)\,dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$$

for $a-R < x < a+R$.

Example Identify the function

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots \qquad -1 \le x \le 1.$$

Solution

Diff. the original series term by term

$$f'(x) = 1 - x^2 + x^4 + - x^6 + - \qquad -1<x<1$$

$$f'(x) = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}$$

Integrating both sides

$$\int f'(x)\,dx = \int \frac{dx}{1+x^2} = \tan^{-1}x + c$$

$$\Rightarrow \quad f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots = \tan^{-1}x + c. \qquad -1<x<1$$

Example

The series

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$$

converges on the open interval $-1<t<1$. Therefore

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots \Big|_0^x$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

or

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n!} \qquad -1<x<1.$$

Exercise 10.7.

(a) Find the series radius & interval of convergence. For what values of x does the series converge (b) absolutely, (c) conditionally.

1) $\sum_{n=0}^{\infty} x^n$

(a) This is geometric series with $r=x$.

It will converge for $|r|=|x|<1$

So the interval of convergence is $x \in (-1,1)$

(b) The absolute ~~conv~~ series is $\sum_{n=0}^{\infty} |x|$.

So common ratio for geometric series is $|r|=|x|$.

It will converge only for $|r|=|x|<1$.

Series converges absolutely when $x \in (-1,1)$.

(c) A series is said to be conditionally convergent when it is ~~both~~ convergent but not absolutely convergent.

* As here interval of convergence of series and absolute series are same so series will never converges conditionally.

8) $\sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{n}$.

Applying ratio test

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}(x+2)^{\cancel{n+1}}}{n+1} \times \frac{n}{(-1)^n (x+\cancel{2)^n}}\right|$$

$$= \lim_{n\to\infty} \frac{n}{n+1} |x+2|$$

$$= |x+2| \lim_{n\to\infty} \frac{1}{1+1/n} = |x+2|.$$

The series converges absolutely for $|x+2|<1$

$$-1 < x+2 < 1.$$

As an absolutely convergent series is also convergent.
So radius of convergence is 1

$$-1 < x+2 < 1 \Rightarrow \quad -3 < x < 1.$$

So series converges for $-3 < x < 1$.

To test the convergence at $x = -3$ & $x = 1$.

At $x = -3$

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{(-1)^n (-3+2)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

which is not convergent.
So at $x = -3$, the series is not convergent as well as not absolutely convergent.

At $x = 1$

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{(-1)^n (-1+2)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

which gives a convergent series i.e $\lim_{n\to\infty} \frac{(-1)^n}{n} \to 0$.
but as $\lim_{n\to\infty} \frac{1}{n}$ is divergent. Therefore for $x = -1$, the series is convergent but not absolutely convergent.

So radius of convergence $= 1$

Interval of convergence $= -3 < x \leq -1$.

Interval of absolute convergence : $-3 < x < -1$.

value of x for which series converges ~~absolutely~~ conditionally $\Rightarrow x = -1$

33 $\sum_{n=1}^{\infty} \frac{1}{2.4.6 \cdots (2n)} x^n$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{2^n (1.2.3 \cdots n)} x^n = \sum_{n=1}^{\infty} \frac{x^n}{2^n n!}$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}/(2^{n+1})(n+1)!}{x^n/2^n(n!)}\right|$$

$$= \lim_{n\to\infty} \left|\frac{x^{n+1} \; 2^n \; n!}{x^n \; 2^{n+1} \; (n+1)!}\right|$$

$$= \lim_{n\to\infty} \left|\frac{x}{2(n+1)}\right| = |x| \lim_{n\to\infty} \frac{1}{2n+2}$$

$$= 0 < 1 \qquad \forall \; x \in R.$$

So series converges absolutely for all x. So converges for all x.

* Radius of convergence $= \infty$.

The series is convergent & absolute convergent for all x.

36 $\sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n})(x-3)^n$

$\Rightarrow \sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \times \sqrt{n+1} + \sqrt{n} \quad (x-3)^n$

$\Rightarrow = \sum_{n=1}^{\infty} \frac{\cancel{n}+1 - \cancel{n}}{\sqrt{n+1} + \sqrt{n}} (x-3)^n$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} (x-3)^n.$

Applying the ratio Test:

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{\sqrt{n+2}+\sqrt{n+1}} \times \frac{\sqrt{n+1}+\sqrt{n}}{(x-3)^n}\right|$$

$$= \lim_{n\to\infty} (x-3)^{\cancel{n}+1} \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+2}+\sqrt{n+1}}$$

$$= |(x \mp 3)| \lim_{n\to\infty} \frac{\sqrt{1+1/n}+\sqrt{1/n}}{\sqrt{1+2/n}+\sqrt{1+1/n}}$$

$$= |x \mp 3| \lim_{n} \frac{1+1}{1+1} = |x \mp 3|$$

So series converges for

$$|x-3| < 1$$

$$\Rightarrow -1 < x-3 < 1 \Rightarrow 2 < x < 4.$$

Now at $x = 2$

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}+\sqrt{n}}$$

This series is conditionally convergent.

At $x=4$

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

which is always divergent.

* So radius of convergence is 1 & interval of convergence is $-2 \le x < 4$.
* Absolute convergence at $2 < x < 4$.
* conditionally convergent at $x = 2$.

Find the radius of convergence

37 $\sum_{n=1}^{\infty} \frac{n!}{3\cdot 6\cdot 9 \cdots 3n} x^n$

$$\sum_{n=1}^{\infty} \frac{n!}{3^n(1.2.3 \cdots n)} x^n = \sum_{n=1}^{\infty} \frac{\cancel{n!}}{3^n(\cancel{n!})} x^n$$

$$a_n = \frac{x^n}{3^n} \qquad a_{n+1} = \frac{x^{n+1}}{3^{n+1}}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}/3^{n+1}}{x^n/3^n} \right| = \lim_{n\to\infty} \left| \frac{x}{3} \right|$$

The series converges for $L<1$.

$$\left|\frac{x}{3}\right|<1 \Rightarrow |x|<3$$

40 $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2} x^n$

By root test

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left(\left(\frac{n}{n+1}\right)^{n^2} x^n\right)^{1/n}$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n |x|$$

Let

$$y = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$$

$$\ln y = \lim_{n\to\infty} \ln\left(\frac{n}{n+1}\right)^n$$

$$= \lim_{n\to\infty} n \ln\left(\frac{n}{n+1}\right)$$

$$\ln y = \lim_{n\to\infty} \frac{\ln\left(\frac{n}{n+1}\right)}{1/n}$$

$$\ln y = \lim_{n\to\infty} \frac{\ln\left(\frac{1}{1+1/n}\right)}{1/n}$$

$t = 1/n$

$$\ln y = \lim_{n\to\infty} \frac{\ln\left(\frac{1}{1+t}\right)}{t}$$

$$= \lim_{t\to 0} \frac{\ln(1/1+t)}{t} = \lim_{t\to 0} \frac{\ln 1 - \ln(1+t)}{t}$$

$$= \lim_{t\to 0} \frac{-\ln(1+t)}{t}$$

Applying L'Hopital rule:

$$\ln y = \lim_{t\to 0} \frac{-1/1+t}{1} = -1$$

$$\Rightarrow y = e^{-1} \Rightarrow y = 1/e.$$

$$\Rightarrow L = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n |x| = \frac{|x|}{e}$$

For convergence

$$|L| < 1$$

$$\frac{|x|}{e} < 1 \Rightarrow |x| < e$$

Find the series's interval of convergence & within this interval, find the sum of series as function of x.

42 $\sum_{n=0}^{\infty} (e^x - 4)^n$

As $\sum_{n=0}^{\infty} x^n$ converges when $|x| < 1$

By theorem 20, given series converges if

$|f(x)| = |e^x - 4| < 1$

$$-1 < e^x - 4 < 1$$

$$3 < e^x < 5 \Rightarrow \quad \ln 3 < x < \ln 5$$

$\sum_{n=0}^{\infty} (e^x - 4)^n$ is a geometrical series with initial term 1 &

common ratio $e^x - 4$.

Sum of geometric Series.

$$S = \frac{a}{1-r}$$

$$= \frac{1}{1-(e^x-4)} = \frac{1}{-e^x+5} \qquad \ln 3 < x < \ln 5.$$

48 $\sum_{n=0}^{\infty} \left(\frac{x^2-1}{2}\right)^n$

$$f(x) = \frac{x^2-1}{2}$$

$$|f(x)| = \left|\frac{x^2-1}{2}\right| < 1$$

$$|x^2-1| < 2 \qquad \Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}$$

$$-2 < x^2 - 1 < 2 \Rightarrow \quad -1 < x^2 < 3$$

$$\Rightarrow -\sqrt{3} < x < \sqrt{3}$$

So Series has first term 1 & common ratio $\frac{x^2-1}{2}$

$$S = \frac{1}{1-\frac{x^2-1}{2}} = \frac{1}{\frac{2-x^2+1}{2}} = \frac{2}{3-x^2}$$

So

$$\sum_{n=0}^{\infty} \left(\frac{x^2-1}{2}\right)^n = \frac{2}{3-x^2} \qquad -\sqrt{3} < x < \sqrt{3}.$$