14.1
Function of several variables.

Definition

Suppose D is a set on n-tuples of the real numbers $(x_1, x_2, \cdots, x_n)$ A real valued function f on D is a rule that assign a unique real number

$$w = f(x_1, x_2, \cdots, x_n)$$

to each element in D. The set D is called function's domain. The set of w-values taken on by f is the function's range. The symbol w is the dependent variable, of f, and f is said to be a function of n-independent variables x_1 to x_n. we also call x_j's the function's input variable and call w as the function's output variable.

As volume of a cylinder is

$$V = \pi r^2 h$$

So we can write

$$V = f(r, h).$$

Domains and Ranges.

* The domain of a function is assumed to be largest set for which the defining rule generates the real numbers.

i.e. Avoiding the values of input which leads to complex numbers & division by zero

i.e $f(x,y) = \sqrt{y - x^2}$

So y cannot be greater than x^2.

or $f(x,y) = \frac{1}{xy}$

xy cannot be zero.

* The range consist of set of output values

Example Function of two Variables

* $z = \sqrt{y - x^2}$

Domain $y \geq x^2$

Range $[0, \infty)$

* $z = 1/xy$

Domain $xy \neq 0$ Range $(-\infty, 0) \cup (0, \infty)$

* $z = \sin(xy)$

Domain Entire plane Range $[-1, 1]$

* Function of three variables

1) $w = \sqrt{x^2 + y^2 + z^2}$

Domain Entire plane Range $[0, \infty)$

2) $w = \dfrac{1}{x^2 + y^2 + z^2}$

Domain $(x, y, z) \neq (0, 0, 0)$ Range $(0, \infty)$

3) $w = xy \ln z$

Domain: $xy \ln z$ Range $(-\infty, \infty)$

Half plane $z > 0$

$y = x^2$

Bounded Region

A region in the plane is bounded if it lies inside the disk of finite radius.

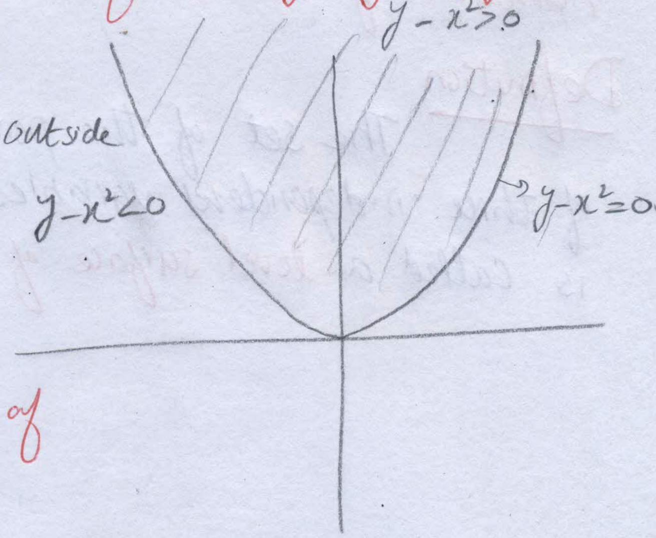
A region is unbounded if it is not bounded.

* Example of bounded set in the plane include ~~the~~ line segments, triangles, interior of triangles, rectangles, circle & disks.

* Example of unbounded sets in the plane include lines, coordinates axes, half-plane or plane itself

Example Describe the domain of the function $f(x,y)=\sqrt{y-x^2}$ (2)

Since f is defined only where $y-x^2 \geq 0$, the domain is the closed unbounded region shown in figure. The parabola $y=x^2$ is the boundary of the domain.



Graph, Level curves & contour function of two variables

Level curves

The set of points in the plane where a function $f(x,y)$ has a constant value $f(x,y)=c$ is called a **level curve** of f.

* The set of all points $(x,y,f(x,y))$ in space, for (x,y) in the domain of f is called the **graph of f**.

* The graph of f is also called the **surface** f $z=f(x,y)$.

Example Graph $f(x,y)=100-x^2-y^2$ & plot the level curves $f(x,y)=0$, $f(x,y)=51$ & $f(x,y)=75$ in the domain of f in the plane

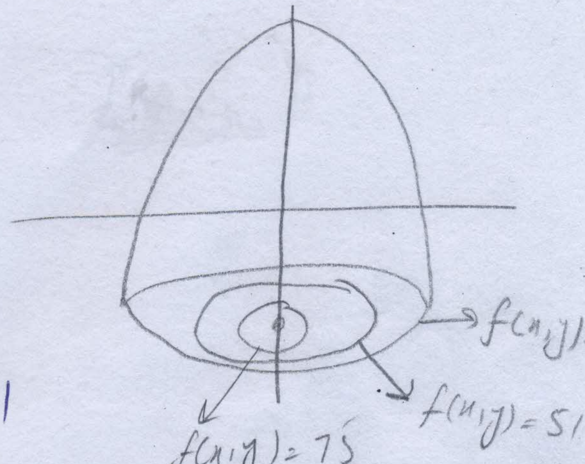
Solution The domain of f is the entire plane and range of f is the set of real numbers less than or equal to 100

The Level curve $f(x,y)=0$ is the set of pts in the xy-plane at which

$$f(x,y)=0$$

$$100-x^2-y^2=0 \Rightarrow x^2+y^2=100$$

which is circle of radius 10 centered at the origin



Similarly $f(x,y)=51 \Rightarrow 100-x^2-y^2=51$

$$\Rightarrow x^2+y^2=49$$

$$f(x,y)=100-x^2-y^2=75$$

$$\Rightarrow x^2+y^2=25$$

Function of three variables.

Definition

The set of the points (x,y,z) in space where a function of three independent variables has a constant value $f(x,y,z)=c$ is called a level surface of f.

Exercise 14.1.

Find the specific function values

4) $f(x,y,z) = \sqrt{49 - x^2 - y^2 - z^2}$

(a) $f(0,0,0)$

$$f(0,0,0) = \sqrt{49 - 0 - 0 - 0} = 7.$$

b) $f(2,-3,6)$

$$f(2,-3,6) = \sqrt{49 - 2^2 - (-3)^2 - 6^2} = \sqrt{49-49} = 0.$$

c) $f(-1,2,3)$

$$f(-1,2,3) = \sqrt{49 - ((-1)^2 + (2)^2 + (3)^2)}$$
$$= \sqrt{49-14} = \sqrt{35}$$

d) $f\left(\frac{4}{\sqrt{2}}, \frac{5}{\sqrt{2}}, \frac{6}{\sqrt{2}}\right)$

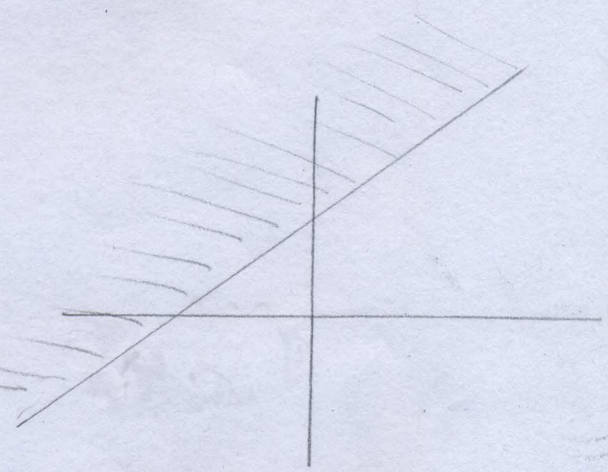
$$f\left(\frac{4}{\sqrt{2}}, \frac{5}{\sqrt{2}}, \frac{6}{\sqrt{2}}\right) = \sqrt{49 - \left(\frac{4}{\sqrt{2}}\right)^2 - \left(\frac{5}{\sqrt{2}}\right)^2 - \left(\frac{6}{\sqrt{2}}\right)^2}$$
$$= \sqrt{49 - \frac{77}{2}} = \sqrt{\frac{21}{2}}$$

Find & sketch the domain of each function.

5) $f(x,y) = \sqrt{y-x-2}$.

Domain: $y - x - 2 \geq 0$

$y \geq x + 2$.



10 $f(x,y) = \ln(xy + x - y \mp 1)$

Domain of $f(x,y) = \ln(xy+x-y\mp 1)$

$D = \{(x,y) \mid xy + x - y \mp 1 > 0\}$

$$xy + x - y \mp 1 = x(y+1) - 1(y+1)$$
$$= (x-1)(y+1)$$

For $(x-1)(y+1) > 0$

case I

$x - 1 > 0$ $\quad\quad$ $y + 1 > 0$

$x > 1$ $\quad\quad$ $y > -1$

CASE 2

$x-1<0 \qquad y+1<0$

$x<1 \qquad y<-1$

11 $f(x,y) = \sqrt{(x^2-4)(y^2-9)}$

Domain $(x^2-4)(y^2-9) \geq 0$

CASE I $(x^2-4)(y^2-9) \geq 0$

$\Rightarrow \quad x^2-4 \geq 0 \qquad y^2-9 \geq 0$

$x^2 \geq 4 \qquad y^2 \geq 9$

$x \geq 2, \; x \leq -2 \qquad y \geq 3 \quad y \leq -3$

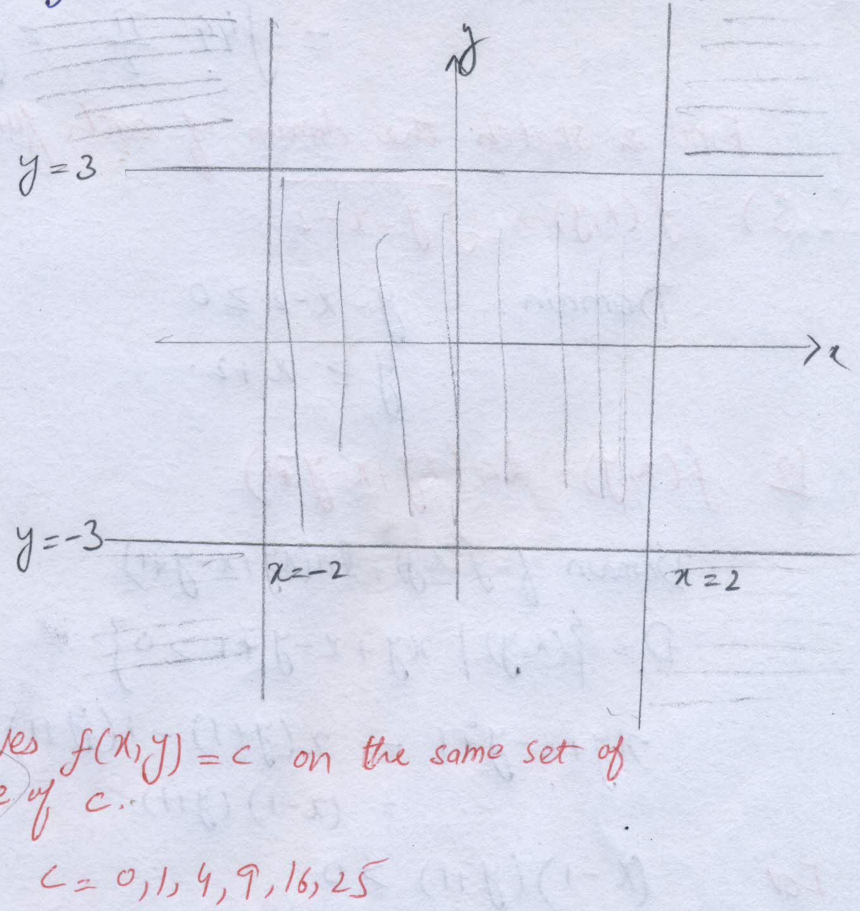
CASE II

$(x^2-4)(y^2-9) \geq 0$

$x^2-4 \leq 0 \qquad y^2-9 \leq 0$

$x^2 \leq 4 \qquad y^2 \leq 9$

$-2 \leq x \leq 2 \qquad -3 \leq y \leq 3$



Find & sketch the level curves $f(x,y)=c$ on the same set of coordinate axes for a given value of c.

14 $f(x,y) = x^2+y^2 \qquad c = 0, 1, 4, 9, 16, 25$

Level curves are

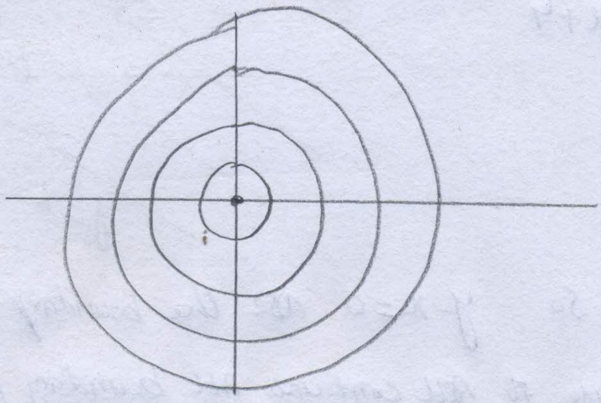
$x^2+y^2=0$ origin $(0,0)$

$x^2+y^2=1$ circle of radius 1

$x^2+y^2=9$ circle of radius 3

$x^2+y^2=16$ circle of radius 4

$x^2+y^2=25$ circle of radius 5



15 $f(x,y)=xy$ $c=-9,-4,-1,0,1,4,9.$

$\Rightarrow f(x,y)=c$

$\Rightarrow xy=c$

$xy=-9 \Rightarrow y=-9/x.$

$xy=-4$

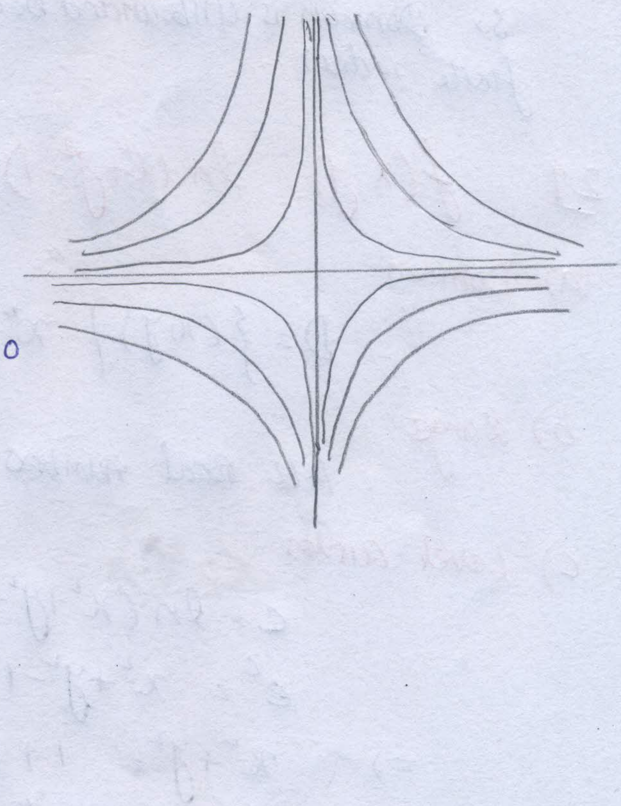
$xy=-1$

$xy=0 \Rightarrow$ either x or $y=0$

$xy=1$

$xy=4$

$xy=9.$



18 $f(x,y)=\sqrt{y-x}$

Domain $=\{(x,y) \mid y-x\geq 0\}$

Range $=[0,\infty)$

Level curves

$$f(x,y) = c$$

$$\Rightarrow \sqrt{y-x} = c \Rightarrow y - x = c^2$$

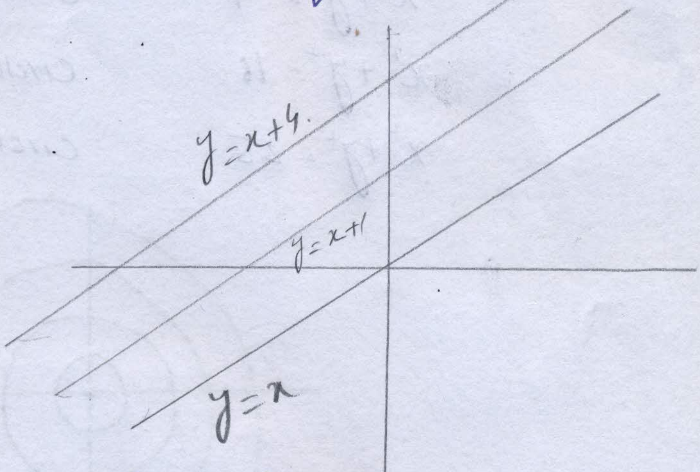
$$y = c^2 + x$$

The level curves are straight lines with slope 1 & y-intercept $(0, c^2)$

$c = 0 \Rightarrow y = x$

$c = 1 \Rightarrow y = x + 1$

$c = 2 \Rightarrow y = x + 4$



d Boundary pts

As domain is

$y - x \geq 0$ So $y - x = 0$ are the boundary pts.

e) Domain is closed because it ~~All~~ contains ALL boundary points.

f) $D = \{(x,y) \mid$ ~~x,~~ $y - x \geq 0\}$

So Domain is unbounded because it donot lie inside a disk of finite radius

29 $f(x,y) = \ln(x^2+y^2-1)$

(a) Domain

$$D = \{(x,y) \mid x^2+y^2 > 1\}$$

b) Range

ALL Real numbers.

c) Level curves

$$c = \ln(x^2+y^2-1)$$

$$e^c = x^2+y^2-1$$

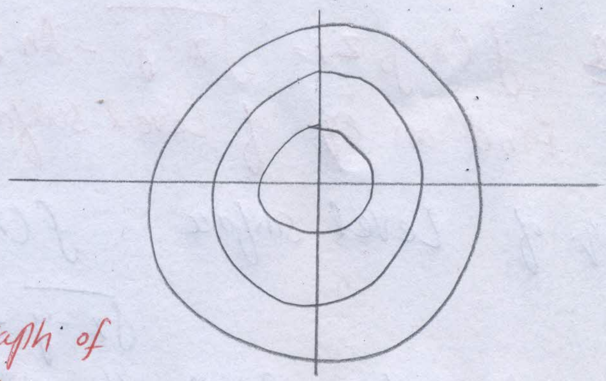
$$\Rightarrow x^2+y^2 = 1+e^c$$

So the equation represents the circle centered at origin & and radius $1+e^c$

For $c = 0$ $\quad x^2+y^2 = 1+e^0 = 2.$

$c=1 \qquad x^2+y^2=1+e$

$c=2 \qquad x^2+y^2=1+e^2.$



Finding level curves

Find an equation for and sketch the graph of the level curve of the function $f(x,y)$ that passes through the given point

51 $\quad f(x,y)=\sqrt{x+y^2-3} \qquad (3,-1).$

Level curve is

$f(x,y)=c$

$\Rightarrow \sqrt{x+y^2-3}=c \Rightarrow x+y^2-3=c^2$

As $(3,-1)$ passes through $(3,-1)$ so put $x=3,\ y=-1$

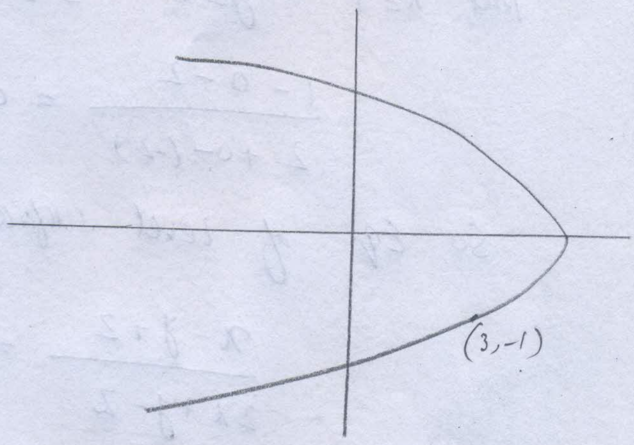
$\Rightarrow \sqrt{3+1-3}=c \Rightarrow c=1.$

So the level curve is

$\sqrt{x+y^2-3}=1 \Rightarrow x+y^2-3=1$

$x+y^2=4$

$x=4-y^2$



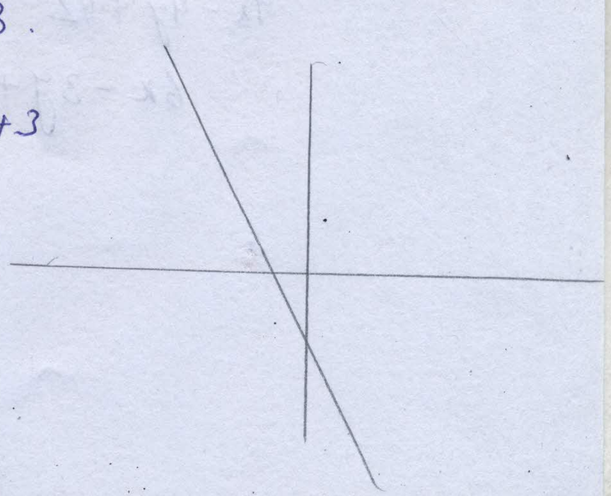
52 $\quad f(x,y)=\dfrac{2y-x}{x+y+1} \qquad (-1,1)$

$\dfrac{2y-x}{x+y+1}=c$

Put $x=-1,\ y=1 \Rightarrow c=\dfrac{2+1}{-1+1+1}=3.$

So $\dfrac{2y-x}{x+y+1}=3 \Rightarrow 2y-x=3x+3y+3$

$\Rightarrow 4x+y+3=0$



62 $f(x, y, z) = \sqrt{x-y} - \ln z \qquad (3, -1, 1)$

Find an Eq. of Level surface at given pt

Eq. of Level surface $\quad f(x, y, z) = c$

$$\sqrt{x-y} - \ln z = c.$$

As surface passes through $(3, -1, 1)$, $x = 3$, $y = -1$, $z = 1$.

$$\sqrt{3-(-1)} - \ln(1) = c$$

$$\sqrt{4} - 0 = c \Rightarrow c = 2.$$

$$\Rightarrow \sqrt{x-y} - \ln z = 2 \quad \text{Level surface.}$$

64 $g(x, y, z) = \dfrac{x-y+z}{2x+y-z} \qquad (1, 0, -2).$

$$g(x, y, z) = c$$

$$\Rightarrow \frac{x-y+z}{2x+y-z} = c$$

Put $x = 1 \quad y = 0 \quad z = -2$

$$\frac{1-0-2}{2+0-(-2)} = c \Rightarrow c = \frac{-1}{4}$$

So Eq. of Level surface

$$\frac{x-y+z}{2x+y-z} = -\frac{1}{4}$$

$$x-y+z = -\frac{1}{4}(2x+y-z)$$

$$4x-4y+4z = -2x-y+z$$

$$6x-3y+3z = 0$$