4.5 Undetermined coefficients

Annihilator approach

An n^{th} order differential Eq. can be written as

$$a_n D^n y + a_{n-1} D^{n-1} y + \cdots + a_2 y + a_1 D y + a_0 y = g(x) \quad (1)$$

where $D^k y = \dfrac{d^k y}{dx^k}$

we can also write (1) as

$$Ly = g(x)$$

where L is a linear n^{th}-order differential or polynomial operator.

$$a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0.$$

Factoring operator.

When the coefficients a_i, $i = 0, 1, 2, \ldots, n$ are real constants, the linear differential operator can be factored whenever the characteristic polynomial $a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0$ factors.

For Example

If we have operator $D^2 + 5D + 6$ then we can factor it as

$$D^2 + 5D + 6 = (D+2)(D+3)$$
$$= (D+3)(D+2)$$

and we can write 2^{nd} order differential Equation as

$$(D^2 + 5D + 6)y = (D+3)(D+2)y = (D+2)(D+3)y.$$

So we have general property as

Factors of a linear differential operator with constant coefficient commute.

A differential Equation $y''+4y'+4y=0$ can be written as

$$(D^2+4D+4)y \Rightarrow (D+2)(D+2)y=0$$

$$(D+2)^2 y = 0.$$

Annihilator operator

If L is a linear differential operator with constant coefficient & f is sufficiently differentiable function such that

$$L(f(x))=0$$

then L is said to be an annihilator of a function.

* A constant function $y=K$ is annihilated by D as $DK=0$.
* The function $y=x$ is annihilated by D^2.
  As first derivative of x is 1 & 2^{nd} derivative is zero.

* The differential operator D^n annihilates each of the functions

$$1, x, x^2, \ldots, x^{n-1}.$$

* The function that are annihilated by a linear n^{th}-order differential operator L are simply those functions that can be obtained from the general solution of homogeneous differential equation $Ly=0$.

* The differential operator $(D-\alpha)^n$ annihilates each of the functions

$$e^{\alpha x}, xe^{\alpha x}, x^2 e^{\alpha x}, \ldots, x^{n-1}e^{\alpha x}.$$

Example

Find a differential operator that annihilates

I) $1-5x^2+8x^3$

As $D^4(x^3) = 0$

So
$$D^4(1 - 5x^2 + 8x^3) = 0$$

b) e^{-3x}

Here $\alpha = -3$, $n = 1$, so we have
$$(D+3)\,e^{-3x} = 0$$

c) $4e^{2x} - 10xe^{2x}$

Here $\alpha = 2$, $n = 2$
$$(D-2)^2(4e^{2x} - 10xe^{2x}) = 0$$

$*$ The differential operator $\left[D^2 - 2\alpha D + (\alpha^2 + \beta^2)\right]^n$ annihilates each of the functions

$$e^{\alpha x}\cos\beta x,\ xe^{\alpha x}\cos\beta x,\ x^2e^{\alpha x}\cos\beta x,\ \ldots,\ x^{n-1}e^{\alpha x}\cos\beta x$$

$$e^{\alpha x}\sin\beta x,\ xe^{\alpha x}\sin\beta x,\ x^2e^{\alpha x}\sin\beta x,\ \ldots,\ x^{n-1}e^{\alpha x}\sin\beta x$$

Example

Find a differential operator that annihilates
$$5e^{-x}\cos 2x - 9e^{-x}\sin 2x$$

Solution

By inspection of the functions $e^{-x}\cos 2x$, & $e^{-x}\sin 2x$, we get $\alpha = -1$, $\beta = 2$. So annihilator operator is
$$(D^2 + 2D + 5)(5e^{-x}\cos 2x - 9e^{-x}\sin 2x) = 0$$

$*$ When $\alpha = 0$ and $n = 1$, we get a special case
$$(D^2 + \beta^2)\begin{cases}\cos\beta x \\ \sin\beta x\end{cases} = 0$$

$*$ If L is a linear differential operator such that $L(y_1) = 0$ & $L(y_2) = 0$ then L will annihilate the linear combination as

$c_1 y_1(x) + c_2 y_2(x)$

* Let L_1 and L_2 are linear differential operator such that L_1 annihiate $y_1(x)$ & L_2 annihilate $y_2(x)$ but $L_1(y_2(x)) \neq 0$ & $L_2(y_1(x)) \neq 0$. Then the product of differential operator L_1L_2 annihilates the sum $c_1 y_1(x) + c_2 y_2(x)$.

$$L_1L_2(y_1+y_2) = L_1L_2(y_1) + L_1L_2(y_2) \qquad \because L_1L_2 = L_2L_1$$
$$= L_2(\underbrace{L_1(y_1)}_{=0}) + L_1(\underbrace{L_2(y_2)}_{=0})$$
$$= 0$$

* As D^2 annihilates $7-x$ & D^2+16 annihilates $\sin 4x$ then $D^2(D^2+16)$ will annihilate the linear combination $7-x+6\sin 4x$.

Example

Solve $y''+3y'+2y = 4x^2$

Solution

Solving the associated homogeneous Eq

$$y''+3y'+2y=0$$

So auxiliary Eq. is

$$m^2+3m+2=0$$
$$(m+1)(m+2)=0$$
$$m_1=-1 \quad \& \quad m_2=-2$$
$$y(x) = c_1 e^{-x} + c_2 e^{-2x}$$

Step 2

As $4x^2$ is annihilated by D^3, so applying D^3 on both sides

$$D^3(D^2+3D+2)y=0.$$

so auxiliary Eq. is

$$m^3(m^2+3m+2)=0$$

$$m^3(m+1)(m+2)=0$$

$m_1=0 \quad m_2=0 \quad m_3=0 \quad m_4=-1 \quad m_5=-2$

$$y(x)=c_1+c_2x+c_3x^2+\underbrace{c_4e^{-x}+c_5e^{-2x}}$$

As last two terms represent the ~~particular solution~~ complementary solution, so remaining terms can be considered as form of particular solution

$$y_p=A+Bx+Cx^2$$

$$y_p'=B+2Cx$$

$$y_p''=2C$$

Substituting in given DE.

$$y_p''+3y_p'+2y_p=4x^2$$

$$2C+3B+6Cx+2A+2Bx+2Cx^2=4x^2$$

comparing coefficients of x^2

$$2C=4 \Rightarrow C=2.$$

x: $\quad 2B+6C=0 \Rightarrow B=-6$

x^0: $\quad 2C+3B+2A=0$

$$3B=-2A-2C$$

$$A=7$$

so $y_p=7-6x+2x^2$

General solution is

$$y=c_1e^{-x}+c_2e^{-2x}+7-6x+2x^2.$$

Example

Solve $y'' - 3y' = 8e^{3x} + 4\sin x$.

Solution

Associated homogeneous Eq. is

$$y'' - 3y' = 0$$

auxiliary Eq. is

$$m^2 - 3m = 0 \Rightarrow m(m-3) = 0$$

$$\Rightarrow m = 0, \quad m = 3$$

$$y(x) = c_1 + c_2 e^{3x}$$

As $8e^{3x} + 4\sin x$ is annihilated by $(D-3)(D^2+1)$ so applying this operator on both sides

$$(D-3)(D^2+1)(D^2-3D)y = 0$$

Auxiliary Eq. is

$$(m-3)(m^2+1)(m^2-3m) = 0$$

$$\Rightarrow \quad m-3 = 0 \qquad m^2+1 = 0 \qquad m(m-3) = 0$$

$$m_1 = 3, \quad m_2 = i \quad m_3 = -i \quad m_4 = 0 \quad m_5 = 3$$

So

$$y(x) = c_1 + c_2 e^{3x} + c_3 x e^{3x} + c_4 \cos x + c_5 \sin x.$$

As $c_1 + c_2 e^{3x}$ are part of complementary solution so the form of y_p is

$$y_p = c_3 x e^{3x} + c_4 \cos x + c_5 \sin x.$$

$$y_p' = c_3(3xe^{3x} + e^{3x}) - c_4 \sin x + c_5 \cos x$$

$$y_p'' = c_3(9xe^{3x} + 3e^{3x} + 3e^{3x}) - c_4 \cancel{\sin}\cos x - c_5 \sin x$$

$$y_p'' = 9xe^{3x}c_3 + 6e^{3x}c_3 - c_4\cos x - c_5 \sin x.$$

using in given DE

$$9xe^{3x}c_3 + 6e^{3x}c_3 - c_4\cos x - c_5\sin x$$
$$-3\left(3xe^{3x}c_3 + e^{3x}c_3\right) = 8e^{3x} + 4\sin x$$
$$+3c_4\sin x - 3c_5\cos x.$$

comparing coefficient

e^{3x}

$$3c_3 = 8 \Rightarrow \quad c_3 = 8/3$$

$\cos x$

$$-c_4 - 3c_5 = 0$$

$\sin x$

$$-c_5 + 3c_4 = 4$$

Solving the equations, we get

$$c_4 = \frac{6}{5}, \quad c_5 = -2/5$$

So

$$y_p = \frac{8}{3}xe^{3x} + \frac{6}{5}\cos x - \frac{2}{5}\sin x.$$

$$y = c_1 + c_2e^{3x} + \frac{8}{3}xe^{3x} + \frac{6}{5}\cos x - \frac{2}{5}\sin x.$$

Example Solve $y'' + y = (x\cos x - \cos x)$

Here $(D^2+1)^2$ is the annihilator operator so applying on both sides.

$$(D^2+1)^2(D^2+1)y = 0.$$

Auxiliary Eq is

$$D = \pm i \qquad D = \pm i, \qquad D = \pm i$$

So

$$y(x) = c_1\cos x + c_2\sin x + c_3x\sin x + c_4x\cos x$$
$$+ c_5x^2\sin x + c_6x^2\cos x.$$

So

$$y_p = c_3x\sin x + c_4x\cos x + c_5x^2\sin x + c_6x^2\cos x.$$

Finding the derivatives y_p' & y_p'' & using in given DE.

$$y_p'' + y_p = 4Ex\cos x - 4Cx\sin x + (2B+2C)\cos x + (-2A+2E)\sin x$$

$$= x\cos x - \cos x.$$

Equating coefficients

$\Rightarrow \quad 4E = 1 \quad \Rightarrow \quad E = 1/4.$

$-4C = 0 \quad \Rightarrow \quad C = 0.$

$2B + 2C = -1 \quad \Rightarrow \quad B = -1/2$

$-2A + 2E = 0 \quad \Rightarrow \quad A = 1/4.$

So general solution is.

$$y = c_1\cos x + c_2\sin x + \frac{1}{4}x\cos x - \frac{1}{2}x\sin x + \frac{1}{4}x^2\sin x$$

Form of a particular Solution

Example Determine the form of a particular solution for

$$y'' - 2y' + y = 10e^{-2x}\cos x.$$

Solution Associated homogeneous Eq. is

$$y'' - 2y' + y = 0$$

Auxiliary Eq. is

$$m^2 - 2m + 1 = 0$$

$$(m-1)^2 = 0$$

$$m = 1 \quad m = 1.$$

So $\quad y(x) = c_1e^x + c_2xe^x$

Now $10e^{-2x}\cos x$ is annihilated by

$(D^2 + 4D + 5).$

Applying this operator on both sides.

$$(D^2+4D+5)(D^2-2D+1)y=0$$

Auxiliary Eq. is

$$(m^2+4m+5)(m^2-2m+1)=0$$

$$m^2+4m+5=0$$

$$m=\frac{-4\pm\sqrt{16-20}}{2}$$

$$=\frac{-4\pm 2i}{2}$$

$$m_3=-2+i \qquad m_4=-2-i$$

$$m^2-2m+1=0$$

$$m_1=1$$

$$m_2=1.$$

So

$$y(x)=c_1e^x+c_2xe^x+e^{-2x}(c_1\cos x+c_2\sin x)$$

So

$$y_p=Ae^{-2x}\cos x+Be^{-2x}\sin x.$$

Example Determine the form of particular solution

$$y'''-4y''+4y'=5x^2-6x+4x^2e^{2x}+3e^{5x}.$$

Solution Here

$$D^3(5x^2-6x)=0$$

$$(D-2)^3x^2e^{2x}=0$$

$$(D-5)e^{5x}=0$$

So annihilator operator is

$$D^3(D-2)^3(D-5)$$

Applying on both sides

$$D^3(D-2)^3(D-5)(D^3-4D^2+4D)y=0$$

Auxiliary Eq. is

$$m^3(m-2)^3(m-5)(m^3-4m^2+4m)=0$$

$$m^4(m-2)^3(m-5)(m^2-4m+4)=0$$

$$m^4(m-2)^3(m-5)(m-2)^2=0$$

$$m^4(m-2)^5(m-5)=0$$

So 10 roots are

$$0, 0, 0, 0, 2, 2, 2, 2, 2, 5.$$

So

$$y = c_1 + c_2x + c_3x^2 + c_4x^3 + c_5e^{2x} + c_6xe^{2x}$$
$$+ c_7x^2e^{2x} + c_8x^3e^{2x} + c_9x^4e^{2x} + c_{10}e^{5x}$$

Here complementary solution is

$$y_c = c_1 + c_5e^{2x} + c_6xe^{2x}.$$

So particular solution is

$$y_p = c_2x + c_3x^2 + c_4x^3 + c_7x^2e^{2x} + c_8x^3e^{2x} + c_9x^4e^{2x}$$
$$+ c_{10}e^{5x}.$$

Exercise 4.5

5) $y''' + 10y'' + 25y' = e^x$

Write the given DE in the form $Ly = g(x)$, where L is a linear differential operator with constant coefficient. If possible, factor L.

$$D^3y + 10D^2y + 25Dy = e^x$$

$$(D^3 + 10D^2 + 25D)y = e^x$$

$$D(D^2 + 10D + 25)y = e^x$$

$$D(D+5)^2 y = e^x.$$

10 $y^{(4)} - 8y'' + 16y = (x^3 - 2x)e^{4x}$

$$D^4y - 8D^2y + 16y = (x^3 - 2x)e^{4x}$$

$$(D^4 - 8D^2 + 16)y = (x^3 - 2x)e^{4x}$$

$$(D^2)^2 - 2(4)(D^2) + (4)^2 = (x^3 - 2x)e^{4x}$$

$$(D^2 - 4)^2 y = (x^3 - 2x)e^{4x}$$

$$(D^2 - 4)(D^2 - 4) = (x^3 - 2x)e^{4x}$$

$$(D+2)(D-2)(D+2)(D-2)y = (x^3 - 2x)e^{4x}.$$

$$(D-2)^2(D+2)^2 y = (x^3 - 2x)e^{4x}$$

Verify that the given differential operates annihilates the indicated functions.

12 $(2D-1)$ $\quad y = e^{x/2}$

$$(2D-1)y = 2De^{x/2} - e^{x/2} = 2 \times \frac{1}{2}e^{x/2} - e^{x/2} = e^{x/2} - e^{x/2}$$

$= 0$

14 $D^2 + 64$

$y = 2\cos 8x - 5\sin 8x$

$$(D^2+64)y = D^2(2\cos 8x - 5\sin 8x) + 64(2\cos 8x - 5\sin 8x)$$

$$= D(-16\sin 8x - 40\cos 8x) + 128\cos 8x - 320\sin 8x$$

$$= -\cancel{128}\cos 8x + \cancel{320}\sin 8x + \cancel{128}\cos 8x - \cancel{320}\sin 8x$$

$$= 0$$

Find a linear differential operator that annihilates the given function

22 $8x - \sin x + 10\cos 5x$

$8x$ is annihilated by D^2.

$-\sin x$ is annihilated by D^2+1.

$10\cos 5x$ is annihilated by D^2+25.

So annihilator operator is

$$D^2(D^2+1)(D^2+25).$$

such that

$$D^2(D^2+1)(D^2+25)(8x - \sin x + 10\cos 5x) = 0$$

26 $e^{-x}\sin x - e^{2x}\cos x$

$e^{-x}\sin x$ is annihilated by $(D^2+2D+1+1) = (D^2+2D+2)$ where $\alpha = 1$ & $\beta = 2$.

$e^{2x}\cos x$ is annihilated by $(D^2 - 4D + 1 + 2^2) = (D^2 - 4D + 5)$

So annihilator operator is

$$(D^2+2D+2)(D^2-4D+5)(e^{-x}\sin x + e^{2x}\cos x) = 0$$

Find linearly independent functions that are annihilated by the given differential operator.

30 $D^2 - 9D - 36$

$D^2 - 12D + 3D - 36 = D(D-12) + 3(D-12)$

$= (D+3)(D-12)$.

So it can annihilate the functions of the Type

e^{12x}, e^{-3x}

Solve the given differential Equation with undetermined coefficients.

42 $y'' - 2y' + y = x^3 + 4x$.

Associated homogeneous Eq. is

$y'' - 2y' + y = 0$

auxiliary Eq. is

$m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$m_1 = 1$ $\qquad$ $m_2 = 1$.

$y(x) = c_1 e^x + c_2 x e^x$

For particular solution, Annihilator operator for $x^3 + 4x$ is D^4. Given DE is

$(D^2 - 2D + 1)y = x^3 + 4x$.

Applying D^4 on both sides

$D^4(D^2 - 2D + 1)y = 0$

auxiliary Eq. is

$m^4(m^2 - 2m + 1) = 0$

$m_1=0 \quad m_2=0 \quad m_3=0 \quad m_4=0$

$m_5=1 \quad m_6=1$

$$y(x) = c_1 + c_2x + c_3x^2 + c_4x^3 + c_5e^x + c_6xe^x$$

So form of particular solution is

$$y_p(x) = A + Bx + Cx^2 + Dx^3$$

$$y_p' = B + 2Cx + 3Dx^2$$

$$y_p'' = 2C + 6Dx$$

Putting in given DE

$$(2C+6Dx) - 2(B+2Cx+3Dx^2) + (A+Bx+Cx^2+Dx^3) = x^3+4x$$

comparing coefficients

$\underline{x^3}$ $D=1$

x^2 $C-6D=0 \Rightarrow C=6$

x $6D-4C+B=4$

$B=22$

$\underline{x^0}$ $2C-2B+A=0$

$A=32$

So $y_p = 32+22x+6x^2+x^3$

So general solution is

$$y = y_c + y_p = c_1e^x + c_2xe^x + x^3 + 6x^2 + 22x + 32$$

S4. $y'' + y' + \frac{1}{4} y = e^{x}(\sin 3x - \cos 3x).$

Associated homogeneous Equation is

$$y'' + y' + \frac{y}{4} = 0$$

auxiliary Eq. is

$$m^2 + m + \frac{1}{4} = 0$$

$$\left(m + \frac{1}{2}\right)^2 = 0$$

$$m_1 = \frac{-1}{2} \qquad m_2 = -1/2.$$

$$y(x) = c_1 e^{-x/2} + c_2 x e^{-x/2}.$$

Annihilator operator is (for $\alpha = 1$ & $\beta = 3$)

$$D^2 - 2\alpha D + (\alpha^2 + \beta^2)$$

$$= D^2 - 2D + 10.$$

applying on both sides.

$$(D^2 - 2D + 10)(D^2 + D + \frac{4}{4})y = 0.$$

auxiliary Eq. is

$$(m^2 - 2\cancel{D}m + 10)(m^2 + m + \frac{1}{4}) = 0$$

$$m^2 - 2m + 10 = 0 \qquad (m + 1/2)^2 = 0$$

$$m = \frac{2 \pm \sqrt{4 - 40}}{2} \qquad m = \frac{-1}{2} \qquad m = -1/2.$$

$$= 1 \pm 3i$$

$$y(x) = c_1 e^{-x/2} + c_2 x e^{-x/2} + e^{x}(c_3 \cos 3x + c_4 \sin 3x).$$

So

$$y_p(x) = A \cos 3x \, e^{x} + B e^{x} \sin 3x.$$

$$y_p' = A e^{x} \cos 3x - 3 A e^{x} \sin 3x + B e^{x} \sin 3x + 3 B e^{x} \cos 3x.$$

$= (A+3B)\, e^x \cos 3x + (B-3A)\, e^x \sin 3x.$

$y_p'' = (A+3B)\, e^x \cos 3x - 3(A+3B) e^x \sin 3x$

$+ (B-3A)\, e^x \sin 3x + 3(B-3A)\, e^x \cos 3x.$

$= (6B-8A)\, e^x \cos 3x - (6A+8B)\, e^x \sin 3x.$

Inserting in given DE & comparing the coefficients of $\sin 3x$ & $\cos(3x)$ we get

$\left(9B - \frac{27}{4}A\right) = -1$

$-9A - \frac{27}{4}B = 1$

upon solving

$A = \frac{-4}{225}$ $\qquad B = -\frac{28}{225}$

So $y_p = \frac{-28}{225} e^x \sin 3x - \frac{4}{225} e^x \cos 3x.$

$y = y_c + y_p$

$= C_1 e^{-x/2} + C_2 x e^{-x/2} - \frac{28}{225} e^x \sin 3x$

$- \frac{4}{225} e^x \cos 3x.$

Q9 $\quad y'' + y = 8\cos 2x - 4\sin x \qquad y\left(\frac{\pi}{2}\right) = -1$

$y'\left(\frac{\pi}{2}\right) = 0.$

Associated homogeneous Eq. is

$y'' + y = 0$

$m^2 + 1 = 0 \qquad m = \pm i.$

$y(x) = c_1 \cos x + c_2 \sin x.$

* Annihilator ~~op~~ operator for $\cos 2x$ is (For $\alpha=0$ & $\beta=2$) $D^2 - 2\alpha D + (\alpha^2+\beta^2)$

$\Rightarrow \quad D^2+4$

& Annihilator operator for $\sin x$ for $\alpha=0$ & $\beta=1$ is D^2+1

So overall annihilator operator is $(D^2+4)(D^2+1)$

Applying on both sides of DE

$$\Rightarrow \quad (D^2+4)(D^2+1)(D^2+1)y = 0.$$

auxiliary Eq is

$$(m^2+4)(m^2+1)(m^2+1) = 0$$

$m_1 = -2i \qquad m_2 = 2i$

$m_3 = -i \qquad m_4 = i$

$m_5 = -i \qquad m_6 = i$

So

$$y(x) = \underbrace{c_1 \cos x + c_2 \sin x}_{y_c} + c_3 x \cos x + c_4 x \sin x + c_5 \cos 2x + c_6 \sin 2x.$$

So form of y_p is

$$y_p = c_3 x \cos x + c_4 x \sin x + c_5 \cos 2x + c_6 \sin 2x$$

or

$$y_p = Ax\cos x + Bx\sin x + C\cos 2x + E\sin 2x.$$

$$y_p' = A(\cos x - x\sin x) + B(\sin x + x\cos x) - 2C\sin 2x + 2E\cos 2x.$$

$$y_p'' = A(-\sin x - \sin x - x\cos x) + B(\cos x + \cos x - x\sin x) - 4C\cos 2x - 4E\sin 2x.$$

Putting in given DE & simplifying.

$$2B\cos x - 2A\sin x - 3C\cos 2x - 3D\sin 2x = 8\cos 2x - 4\sin x.$$

comparing coefficients

$\cos x \Rightarrow \quad 2B = 0 \quad \Rightarrow B = 0$

$\sin x \Rightarrow \quad -2A = -4 \quad \Rightarrow A = 2.$

$\cos 2x \quad -3C = 8 \quad \Rightarrow C = -8/3.$

$\sin 2x; \quad -3D = 0 \Rightarrow \quad D = 0$

So

$$y_p = -2x\cos x - \frac{8}{3}\cos 2x$$

&

$$y_c = c_1\cos x + c_2\sin x - 2x\cos x - \frac{8}{3}\cos 2x.$$

$y(\pi/2) = -1 \quad \Rightarrow$

$$-1 = c_2.$$

$$y'(x) = -c_1\sin x + c_2\cos x - 2\cos x + 2x\sin x + \frac{16}{3}\sin 2x.$$

$y'(\frac{\pi}{2}) = 0$

$\Rightarrow$

$$0 = -c_1 + \cancel{2}\times\frac{\pi}{\cancel{2}}\times\sin(\pi/2)$$

$$c_1 = \pi$$

~~$y(x) = -\pi\sin x - \cos x - 2\cos x + 2x\sin x$~~

$$y(x) = -\pi\cos x - \sin x - 2x\cos x - \frac{8}{3}\cos 2x.$$