Separable variables

Separable Equation

A first order differential Equation of the form

$$\frac{dy}{dx} = g(x)h(y)$$

is said to be separable or to have separable variables.

Example

$$\frac{dy}{dx} = y^2 x e^{3x+4y} \qquad \text{Separable}$$

$$\frac{dy}{dx} = y + \sin x \qquad \text{Non-separable.}$$

As in first equation

$$f(x,y) = y^2 x e^{3x+4y}$$

$$= y^2 x e^{3x} e^{4y} = (xe^{3x})(y^2 e^{4y})$$

* Consider the differential Eq.

$$\frac{dy}{dx} = g(x)h(y)$$

$$\frac{1}{h(y)}\frac{dy}{dx} = g(x) \qquad \text{let } P(y) = \frac{1}{h(y)}$$

$$\int P(y)\,dy = \int g(x)\,dx.$$

$$H(y) = G(x) + c$$

where $H(y)$ & $G(x)$ are antiderivatives of $P(y) = \frac{1}{h(y)}$ & $g(x)$ respectively.

Example

Solve $(1+x)dy - y\,dx = 0$.

Solution

$$(1+x)dy = y\,dx$$

converting into separable Eq.

$$\frac{dy}{y} = \frac{dx}{1+x}.$$

$$\int \frac{dy}{y} = \int \frac{dx}{1+x}$$

$$\ln|y| = \ln|(1+x)| + c_1$$

$$\ln y = \ln|(1+x)| + \ln e^{c_1}$$

$$\ln y = \ln\left((1+x)\, e^{c_1}\right)$$

$$y = e^{c_1} \cancel{\ln} (1+x)$$

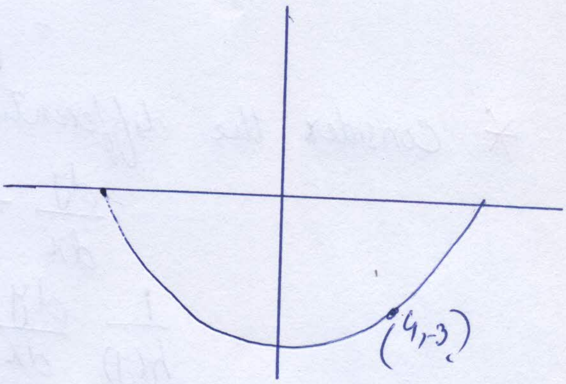
relabelling e^{c_1} as c

$$y = c\,(1+x).$$

Example Solve the initial value problem

$$\frac{dy}{dx} = -\frac{x}{y}, \quad y(4) = -3.$$

Solution



rewriting the Equation as

$$y\,dy = -x\,dx.$$

Integrating both sides.

$$\int y\,dy = -\int x\,dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + c_1$$

$$\frac{x^2}{2} + \frac{y^2}{2} = c_1 \quad \text{or} \quad x^2 + y^2 = c^2 \qquad 2c_1 = c^2$$

For $x = 4,\ y = -3$

$$\Rightarrow 16 + 9 = c^2 \Rightarrow c^2 = 25$$

So solution of IVP is

$$x^2 + y^2 = 25$$

converting the implicit solution into explicit solutions.

$$y = \pm\sqrt{25 - x^2}$$

the point $(4, -3)$ lies on the solution $y = -\sqrt{25 - x^2}$ which is a lower semicircle with domain $-5 < x < 5$.

Losing a solution.

Solve $\frac{dy}{dx} = y^2 - 4.$

Solution.

$$\frac{dy}{y^2-4} = dx.$$

$$dy\left(\frac{1}{(y+2)(y-2)}\right) = dx.$$

using partial fraction of $\frac{1}{(y+2)(y-2)}$

$$\frac{1}{(y+2)(y-2)} = \frac{A}{y+2} + \frac{B}{y-2}$$

$$1 = A(y-2) + B(y+2)$$

For $y-2=0 \Rightarrow y=2$

$\Rightarrow B = 1/4.$

For $y+2=0 \Rightarrow y=-2 \Rightarrow A = -1/4.$

So

$$\frac{1}{(y+2)(y-2)} = \frac{-1}{4(y+2)} + \frac{1}{4(y-2)}.$$

So we get

$$\int\left(\frac{-1}{4(y+2)} + \frac{1}{4(y-2)}\right) dy = \int dx.$$

$$-\frac{1}{4}\ln|y+2| + \frac{1}{4}\ln|y-2| = x + \ln c.$$

$$\ln\left|\frac{y-2}{y+2}\right| = 4x + 4\ln c$$

$$\ln\left|\frac{y-2}{y+2}\right| = \ln e^{4x+\ln c_1}.$$

$$\frac{y-2}{y+2} = e^{4x} \cdot c_1.$$

rearranging

$$\Rightarrow \quad y = 2\,\frac{(1+ce^{4x})}{1-ce^{4x}}. \qquad (A)$$

Now looking as $f(y) = y^2 - 4$.

$y = 2$ & $y = -2$ are Equilibrium solution for the given differential Eq. which cannot be obtained from solution (A) for any choice of parameter c.

* The solutions $y=2$ & $y=-2$ were lost at the time of solution procedure because

$$\frac{dy}{y^2-4} = dx$$

was undefined at these points.

Such kind of solutions are called Singular Solutions.

Example Solve $(e^{2y} - y)\cos x \frac{dy}{dx} = e^{y}\sin 2x \quad y(0)=0.$

Solution

$$\frac{e^{2y}-y}{e^{y}}\,dy = \frac{\sin 2x}{\cos x}\,dx.$$

$$(e^{y} - ye^{-y})\,dy = \frac{2\sin x\cancel{\cos x}}{\cancel{\cos x}}\,dx$$

Integrating both sides.

$$\int (e^{y} - ye^{-y})\,dy = 2\int \sin x\,dx.$$

$$e^{y} - \left[-ye^{-y} - \int -e^{-y}\,dy\right] = -2\cos x + c.$$

$$e^{y} + ye^{y} - \frac{e^{-y}}{-1} = -2\cos x + c.$$

$$e^{y} + ye^{y} + e^{-y} = -2\cos x + c.$$

$y(0) = 0$

$\Rightarrow \quad e^0 + 0 + e^0 = -2\cos(0) + c$

$2 = -2 + c \Rightarrow \quad c = 4.$

$\Rightarrow$ So solution is

$$e^y + ye^{-y} + e^{-y} = 4 - 2\cos x.$$

Solutions defined by integrals.

Solve $\dfrac{dy}{dx} = e^{-x^2}$, $y(3) = 5$

Solution. The functions $g(x) = e^{-x^2}$ is continuous on $(-\infty, \infty)$ but its antiderivative is not an elementary function using t as dummy variable

$$\int_3^x \frac{dy}{dt}\,dt = \int_3^x e^{-t^2}\,dt.$$

$$y(t)\Big|_3^x = \int_3^x e^{-t^2}\,dt.$$

$$y(x) - y(3) = \int_3^x e^{-t^2}\,dt$$

$$y(x) = y(3) + \int_3^x e^{-t^2}\,dt$$

$$= 5 + \int_3^{\cancel{x}\;3} e^{-t^2}\,dt.$$

when antiderivative of a function is not directly known we can represent the solution in terms of integral.

$\int_3^x e^{-t^2}\,dt$, $\int \sin(x^2)\,dx$ are called Non-elementary integral.

Exercise 2.2

Solve the given differential Eq. ~~for~~ by separation of variables

6) $\frac{dy}{dx} + 2xy^2 = 0$

$$\frac{dy}{dx} = -2xy^2 \Rightarrow \frac{1}{y^2}\frac{dy}{dx} = -2x$$

Integrating both sides

$$\Rightarrow \int y^{-2}\,dy = -\int 2x\,dx$$

$$\frac{y^{-1}}{-1} = -x^2 + c$$

$$\frac{1}{y} = x^2 + c$$

$$y = \frac{1}{x^2+c}$$

12 $\sin 3x\,dx + 2y\cos^3(3x)\,dy = 0$

$$2y\cos^3(3x)\,dy = -\sin 3x\,dx$$

$$2y\,dy = \frac{-\sin 3x}{\cos^3 3x}\,dx$$

$$2y\,dy = \cos^{-3}(3x)(-\sin 3x)\,dx$$

Integrating both sides

$$\Rightarrow y^2 = \frac{1}{3}\,\frac{\cos^{-2}(3x)}{-2} + c$$

$$y^2 = -\frac{1}{6}\cos^{-2}(3x) + c \qquad c \in \mathbb{R}$$

Implicit solution

20 $\frac{dy}{dx} = \frac{xy+2y-x-2}{xy-3y+x-3}$

$$\frac{dy}{dx} = \frac{y(x+2)-1(x+2)}{y(x-3)+1(x-3)} = \frac{(x+2)(y-1)}{(x-3)(y+1)}$$

$$\frac{y+1}{y-1}\,dy = \frac{x+2}{x-3}\,dx$$

$$\frac{y-1+1+1}{y-1}\,dy = \frac{x-3+3+2}{x-3}\,dx$$

$$\frac{y-1+2}{y-1}\,dy = \frac{(x-3)+5}{x-3}\,dx$$

$$\left(1+\frac{2}{y-1}\right)dy = \left(1+\frac{5}{x-3}\right)dx$$

Integrating both sides

$$\Rightarrow \quad y + 2\ln(y-1) = x + 5\ln(x-3) + c$$

$$\ln e^{y} + \ln(y-1)^2 = \ln e^{x} + \ln(x-3)^5 + \ln e^{c}$$

$$\Rightarrow \quad e^{y}(y-1)^2 = e^{x}e^{c}(x-3)^5$$

Find the explicit solution of IVP

27 $\sqrt{1-y^2}\,dx - \sqrt{1-x^2}\,dy = 0 \qquad y(0) = \frac{\sqrt{3}}{2}$

separating variables

$$\Rightarrow \quad \frac{1}{\sqrt{1-y^2}}\,dy = \frac{1}{\sqrt{1-x^2}}\,dx$$

Integrating both sides

$$\int \frac{1}{\sqrt{1-y^2}}\,dy = \int \frac{1}{\sqrt{1-x^2}}\,dx$$

$$\sin^{-1} y = \sin^{-1} x + c$$

$y(0) = \frac{\sqrt{3}}{2}$

$$\Rightarrow \quad \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \sin^{-1}(0) + c$$

$$\frac{\pi}{3} = c$$

$$\sin^{-1}(y) = \sin^{-1}x + \pi/3.$$

$$y = \sin\left(\frac{\pi}{3} + \sin^{-1}x\right).$$

$$y = \sin\frac{\pi}{3}\cos(\sin^{-1}x) + \cos(\pi/3)\sin(\sin^{-1}x)$$

$$= \frac{\sqrt{3}}{2}\cos(\sin^{-1}x) + \frac{1}{2}x.$$

32 Find a solution of

$$x\frac{dy}{dx} = y^2 - y$$

that passes through

(C) $\left(\frac{1}{2}, \frac{1}{2}\right)$

$$x\frac{dy}{dx} = y(y-1)$$

$$\frac{dy}{y(y-1)} = \frac{dx}{x}$$

$$\frac{1}{y(y-1)} = -\frac{1}{y} + \frac{1}{y-1}$$ (By partial fraction)

$$\int\left(\frac{-1}{y} + \frac{1}{y-1}\right)dy = \int\frac{dx}{x}.$$

$$-\ln y + \ln(y-1) = \ln x + \ln c$$

$$\ln\left(\frac{y-1}{y}\right) = \ln(cx)$$

$$\frac{y-1}{y} = cx.$$

$$y - 1 = cxy.$$

$$y - cxy = 1.$$

$$y = \frac{1}{1-cx}.$$

Initial condition is $y\left(\frac{1}{2}\right) = 1/2$

$$\Rightarrow \frac{1}{2} = \frac{1}{1 - \frac{1}{2}x} \Rightarrow \frac{1}{2} = \frac{1}{\frac{2-c}{2}}$$

$$\frac{1}{2} = \frac{2}{2-c}$$

$$\Rightarrow 2 - c = 4.$$

$$c = -2$$

So

$$y = \frac{1}{1+2x}.$$