10.8 Taylor & Maclaurin Series:

Series representation

Consider the power series about $x=a$

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

$$= a_0 + a_1 (x-a) + a_2 (x-a)^2 + \cdots + a_n (x-a)^n + \cdots$$

with a positive radius of convergence.

$$f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \cdots + n a_n (x-a)^{n-1} + \cdots \quad (1)$$

$$f''(x) = 2a_2 + 3.2.\, a_3 (x-a) + 3.4\, a_4 (x-a)^2 + \cdots \quad (2)$$

$$f'''(x) = 3.2.1\, a_3 + 3.4.2.\, a_4 (x-a) + \cdots \quad (3)$$

So general form is

$$f^{(n)}(x) = n!\, a_n + \text{a sum of Terms with } x-a \text{ as a factor.}$$

For $x=a$ (1) $\Rightarrow$

$\Rightarrow$ $f'(a) = a_1$

(2) $\Rightarrow$ $2a_2 = f''(a) \Rightarrow f''(a) = 2a_2$

(3) $\Rightarrow$ $f'''(a) = 3a_3$

$\Rightarrow$ $f^{(n)}(a) = n!\, a_n$

$\Rightarrow$ $a_n = \dfrac{f^{(n)}(a)}{n!}$

So the series become

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

Taylor & Maclaurin Series

Let f be a function with derivatives of all orders throughout some interval containing "a" as an interior point. Then the Taylor series generated by f at $x=a$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

The Maclaurin series of f is the Taylor series generated by f at $x=0$, or

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0)+xf'(0)+\frac{x^2}{2}f''(0)+\cdots+\frac{f^{(n)}(0)}{n!}+\cdots$$

Example

Find the Taylor series generated by $f(x)=\frac{1}{x}$ at $a=2$, where if anywhere, does the series converges for $1/x$.

Solution

We need to find $f(2), f'(2), f''(2)\ldots$, Taking derivatives, we get

$$f'(x) = -1/x^2 \Rightarrow f'(2) = -1/2^2$$

$$f''(x) = \frac{+2}{x^3} \Rightarrow \frac{f''(2)}{2} = \frac{1}{2^3}$$

$$f'''(x) = \frac{-2\times 3}{x^4} \Rightarrow \frac{f'''(2)}{3!} = -\frac{1}{2^4}$$

$$\vdots$$

$$f^n(x) = \frac{(-1)^n n!}{x^{n+1}} \Rightarrow \frac{f^n(x)}{n!} = \frac{(-1)^n}{2^{n+1}}$$

So the Taylor series is.

$$f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \cdots + \frac{f^{(n)}(2)}{n!} + \cdots$$

$$= \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} + \cdots + \frac{(-1)^n (x-2)^n}{2^{n+1}}$$

$$+ \cdots$$

* This the geometric series, with 1^{st} term as $1/2$ & common ratio $\frac{-(x-2)}{2}$ so it converges absolutely for $|x-2| < 2$.

It sum is

$$\frac{1/2}{1 + \frac{(x-2)}{2}} = \frac{1}{2+x-2} = \frac{1}{x}.$$

Taylor's Polynomial

Let f be a function with derivatives of order k for $k = 1, 2, \ldots N$, in some interval containing 'a' as an interior point. Then for any integer n from 0 through N, the Taylor polynomial of order n generated by f at $x=a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$
$$+ \frac{f^k(a)}{k!}(x-a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Example Find the Taylor series & Taylor polynomial generated by $f(x) = e^x$.

Solution: Since $f^{(n)}(x) = e^x$ & $f^{(n)}(0) = 1$ for every $n = 0, 1, 2, \ldots$ the Taylor series generated by f at $x=0$

$$f(0)+xf'(0)+\frac{x^2}{2}f''(0)+\cdots+\frac{f^{(n)}(0)}{n!}fx^n+\cdots$$
$$=1+x+\frac{x^2}{2}+\cdots+\frac{x^n}{n!}+\cdots$$
$$=\sum_{k=0}^{\infty}\frac{x^k}{k!}$$

* This is also the Maclaurin series for e^x.
* The Taylor polynomial of order n at $x=0$ is

$$P_n(x)=1+x+\frac{x^2}{2}+\cdots+\frac{x^n}{n!}$$

Example. Find the Taylor Series & polynomials generated by $f(x)=\cos x$ for all $x=0$.

Solution

The cosine & its derivatives are.

$f(x)=\cos x$ $\qquad$ $f'(x)=-\sin x$

$f''(x)=-\cos x$ $\qquad$ $f'''(x)=\sin x$

$\vdots$ $\qquad$ $\vdots$

$f^{(2n)}=(-1)^n\cos x$ $\qquad$ $f^{(2n+1)}=(-1)^{n+1}\sin x.$

$\Rightarrow$ $f^{(2n)}(0)=(-1)^n$ $\qquad$ $f^{2n+1}(0)=(-1)^{n+1}(0)=0$

The Taylor Series generated by f at $x=0$ is

$$f(0)+xf'(0)+\frac{f''(0)}{2}x^2+\frac{f'''(0)}{3!}x^3+\cdots+\frac{f^{(n)}(0)}{n!}x^n$$
$$=1+0-\frac{x^2}{2}+0+\frac{x^4}{4!}+\cdots+(-1)^n\frac{x^{2n}}{(2n)!}$$
$$=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}$$

This is also a Maclaurin series for $\cos x$.

* As Taylor series of $\cos x$ contains only even powers, so it is varifying the $\cos x$ is an even function.

Because $f^{(2n+1)}(0) = 0$ So the Taylor polynomials of order $2n$ & $2n+1$ are identical

$$P_{2n}(x) = P_{2n+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!}$$

Exercise 10.8

Find the Taylor polynomials of order 0, 1, 2 & 3 generated by f at a.

1) $f(x) = e^{2x}$ $\quad a=0$

$f'(x) = 2e^{2x}$ $\qquad f'(0) = 2e^{0} = 2$

$f''(x) = 4e^{2x}$ $\qquad f''(0) = 4e^{0} = 4$

$f'''(x) = 8e^{2x}$ $\qquad f'''(0) = 8e^{0} = 8$

$\vdots$

General form of Taylor's polynomial

$$P_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

so

$P_0(x) = f(0) = 1$

$P_1(x) = f(0) + xf'(0) = 1 + 2x.$

$P_2(x) = f(0) + xf'(0) + \frac{x^2}{2}f''(0) + \cdots$

$= 1 + 2x + 2x^2$

$P_3(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0)$

$= 1 + 2x + 2x^2 + \frac{4}{3}x^3.$

6) $f(x) = \frac{1}{x+2}$ $\quad a=0$

$f(x) = \frac{1}{x+2}$ $\qquad f(0) = 1/2$

$f'(x) = \frac{-1}{(x+2)^2}$ $\qquad f'(0) = -1/4$

$f''(x) = \frac{2}{(x+2)^3}$ $\qquad f''(0) = \frac{2}{2^3} = 1/4$

$f'''(x) = \frac{-6}{(x+2)^4}$ $\qquad f'''(0) = 6/2^4 = -3/8$

$P_0(x) = f(0)$

$= 1/2$

$P_1(x) = f(0) + \frac{f'(0)}{1!}(x-0)$

$= \frac{1}{2} - \frac{1}{4}x$

$P_2(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2$

$= \frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2$

$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2}(x-0)^2 + \frac{f'''(0)}{6}(x-0)^3$

$= \frac{1}{2} - \frac{1}{4}x + \frac{x^2}{8} - \frac{x^3}{16}$

8) $f(x) = \tan x \qquad a = \pi/4$

$f'(x) = \sec^2 x$

$f''(x) = 2\sec x \times \sec x \tan x = 2\sec^2 x \tan x$

$f'''(x) = 2(\sec^2 x \cdot \sec^2 x + \tan x \times 2\sec x \cdot \sec x \tan x)$

$= 2(\sec^4 x + 2\sec^2 x \tan^2 x)$

$f(\pi/4) = \tan(\pi/4) = 1$

$f'(\pi/4) = \sec^2(\pi/4) = (\sqrt{2})^2 = 2$

$f''(\pi/4) = 2\sec^2 \pi/4 \tan \pi/4$

$= 4$

$f'''(\pi/4) = 2(4 + 2\times 2\times 1) = 2\times 8 = 16.$

$P_0(x) = 1$

$P_1(x) = 1 + 2(x - \pi/4)$

$P_2(x) = 1 + 2x + \frac{4}{2}x^2 = 1 + 2x + 2x^2$

$$P_2(x) = 1 + 2(x - \pi/4) + 2(x - \pi/4)^2$$

$$P_3(x) = 1 + 2(x - \pi/4) + 2(x - \pi/4)^2 + \frac{16}{6}(x - \pi/4)^3.$$

10 $f(x) = \sqrt{1-x}$ $\quad a = 0$

$$f(x) = \sqrt{1-x}$$

$$f'(x) = \frac{1}{2\sqrt{1-x}}(-1) = -\frac{1}{2}(1-x)^{-1/2}$$

$$f''(x) = \frac{1}{4}(1-x)^{-3/2} \times -1$$

$$= -\frac{1}{4}(1-x)^{-3/2}$$

$$f'''(x) = -\frac{3}{8}(1-x)^{-5/2}(-1)$$

2) $f(0) = 1$

$f'(0) = -1/2$

$f''(0) = -1/4$

$f'''(0) = -3/8$

$$P_0(x) = f(0) = 1$$

$$P_1(x) = f(0) + x f'(0) = 1 - x/2$$

$$P_2(x) = f(0) + x f'(0) + \frac{x^2}{2} f''(0)$$

$$= 1 - \frac{x}{2} - \frac{x^2}{8}$$

$$P_3(x) = f(0) + x f'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{6} f'''(0)$$

$$= 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16}$$

Find the Maclaurin series of the function

14 $\dfrac{2+x}{1-x}.$

$$f(x) = \frac{2+x}{1-x}$$

$$f'(x) = \frac{(1-x)(1) - (2+x)(-1)}{(1-x)^2}$$

$$= \frac{1-\cancel{x}+2+\cancel{x}}{(1-x)^2} = \frac{3}{(1-x)^2}$$

$$f''(x) = \frac{-6}{(1-x)^3} \times -1 = \frac{6}{(1-x)^3}$$

$$f'''(x) = \frac{-18}{(1-x)^4} \times -1 = \frac{18}{(1-x)^4}$$

$f(0) = 2$

$f'(0) = 3 \cdot 1$

$f''(0) = 6 = 3 \cdot 2!$

$f'''(0) = 18 = 3 \cdot 3 \cdot 2 = 3 \cdot 3!$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$= 2 + 3x + \frac{3 \cdot \cancel{2!}}{\cancel{2!}} \cancel{f''(0)} x^2 + \frac{3 \cdot \cancel{3!}}{\cancel{3!}} \cancel{f'''(0)} x^3 + \cdots$$

$$= 2 + 3x + 3x^2 + 3x^3 + \cdots$$

$$= 2 + 3(x + x^2 + x^3 + \cdots)$$

$$= 2 + 3 \sum_{n=1}^{\infty} x^n$$

19 $\cosh x = \dfrac{e^x + e^{-x}}{2}$

$$e^x = \sum_n$$