1.2 Initial value problems

Initial value problem (IVP)

On some interval I, containing x_0, the problem

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)})$$

$$y(x_0) = y_0, \quad y'(x_0) = y_1; \ldots; y^{(n)}(x_0) = y_{n-1}$$

where $y_0, y_1, \ldots, y_{n-1}$ are arbitrarily specified real constants is called an initial-value problem.

* The values of $y(x)$ and its first $(n-1)$ derivatives at a single point x_0, $y(x_0) = y_0$, $y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}$ are called initial conditions.

First and Second order IVP

First order IVP

Solve
$$\frac{dy}{dx} = f(x, y)$$

Subject to
$$y(x_0) = y_0$$

Second order IVP

Solve
$$\frac{d^2y}{dx^2} = f(x, y, y')$$

Subject to $y(x_0) = y_0$, $y'(x_0) = y_1$

Example

As $y = ce^x$ is the solution of differential equation $y' = y$. If we impose an initial condition $y(0) = 3$ then

$$y = ce^x$$

$$y(0) = ce^0 = 3 \Rightarrow c = 3$$

then $y = 3e^x$ is the solution of IVP

$$y' = y, \quad y(0) = 3$$

that passes through $(0, 3)$

* Demanding that solution will pass through $(1,-2)$ leads to initial condition

$$y(1) = -2$$

$$y = ce^{x} \Rightarrow y(1) = ce = -2$$

$$\Rightarrow c = -2e^{-1}$$

So

$$y = -2e^{-1}e^{x} = -2e^{x-1}$$

is the solution of IVP

$$y' = y, \quad y(1) = -2.$$

Interval of definition of solution.

The differential equation $y' + 2xy^2 = 0$ has the solution $y = \dfrac{1}{x^2+c}$.

Imposing the initial condition $y(0) = -1$

$$\Rightarrow \quad y(0) = \frac{1}{0+c} = -1 \Rightarrow c = -1$$

$$y = \frac{1}{x^2-1}$$

* Considering as a function, the domain of $y = \dfrac{1}{x^2-1}$ is the set of real numbers except $x=-1$, & $x=1$.

* considering as solution of the differential equation, $y' = -2xy^2$ the interval I of definition of $y = \dfrac{1}{x^2-1}$ could be taken to be any interval on which y is defined and differentiable, so interval can be any of $(-\infty,-1)$, $(-1,1)$ & $(1,\infty)$.

* Considering as solution of IVP $y' + 2xy^2 = 0$, $y(0) = -1$, the interval of definition of $y = \dfrac{1}{x^2-1}$ could be taken to be the interval over which $y(x)$ is defined, differentiable and contains the point $x=0$. So in this case $(-1,1)$ is the interval for solution of IVP.

Example 2 Solution of $x'' + 16x = 0$ is $x = c_1 \cos 4t + c_2 \sin 4t$

Find the solution of IVP is

$$x'' + 16x = 0 \qquad x\left(\frac{\pi}{2}\right) = -2, \quad x'\left(\frac{\pi}{2}\right) = 1.$$

Solution

$$y \neq \quad x = c_1 \cos 4t + c_2 \sin 4t$$

$$x(\pi/2) = c_1 \cos\left(4 \times \frac{\pi}{2}\right) + c_2 \sin 4\left(\frac{\pi}{2}\right)$$

$$-2 = c_1 \cos 2\pi + c_2 \sin(2\pi)$$

$$\Rightarrow \quad c_1 = -2.$$

$$\Rightarrow \quad x = -2 \cos 4t + c_2 \sin 4t$$

Finding derivative

$$x'(t) = +8 \sin 4t + 4c_2 \cos 4t$$

$$x'(\pi/2) = 8 \sin(2\pi) + 4c_2 \cos(2\pi)$$

$$1 = 4c_2$$

$$\Rightarrow \quad c_2 = 1/4.$$

So $\quad x = -2 \cos 4t + \frac{1}{4} \sin 4t$

is the solution of given IVP.

An IVP can have several solution

For example the IVP

$$\frac{dy}{dx} = xy^{1/2} \qquad y(0) = 0$$

has at least two solutions $y = 0$ & $y = \frac{x^4}{16}$.

Existence of Unique Solution

Let R be a rectangular region in the xy-plane defined by $a \le x \le b$, $c \le y \le d$ that contains the point (x_0, y_0) in its interior. If $f(x,y)$ & $\frac{\partial f}{\partial y}$ are continuous on R, then there exist some interval $I_0: (x_0 - h, x_0 + h)$, $h > 0$, contained in $[a,b]$ and a unique function $y(x)$ defined on I_0, that is solution of the IVP.

Example

$$y' = y \qquad y(0) = 3$$

has the unique solution $y = 3e^{x}$ as $f(x,y) = y$ and $\frac{\partial f}{\partial y} = 1$ are continuous throughout the entire xy-plane.

* $y' = y \qquad y(1) = -2$

It also has continuous function $f(x,y) = y$, also $\frac{\partial f}{\partial y} = 1$. So it has unique solution $y = -2e^{x-1}$.

Exercise 1.2.

2) $y = \frac{1}{1+c_1 e^{-x}}$ is one parameter family of solutions of first order DE $y' = y - y^2$. Find the solution of first order IVP consisting of differential Eq. and the given initial condition.

$y(-1) = 2$.

$$y = \frac{1}{1+c_1 e^{-x}}$$

$$y(-1) = \frac{1}{1+c_1 e} \Rightarrow 2 = \frac{1}{1+c_1 e}$$

$$2 + 2c_1 e = 1.$$
$$2c_1 e = -1 \Rightarrow c_1 = \frac{-1}{2e}.$$

So

$$y = \frac{1}{1 - \frac{1}{2e} e^{-x}} = \frac{1}{1 - \frac{1}{2} e^{-x-1}}$$

9) $x = c_1 \cos t + c_2 \sin t$ is two-parameter of solution of $x'' + x = 0$

$x(\pi/6) = 1/2$, $x'(\pi/6) = 0$

$$x = c_1 \cos t + c_2 \sin t$$

* $x(\pi/6) = c_1 \cos(\pi/6) + c_2 \sin(\pi/6) = 1/2.$

$$\frac{\sqrt{3}}{2} c_1 + \frac{1}{2} c_2 = 1/2.$$

$$\sqrt{3} c_1 + c_2 = 1. \qquad (1)$$

Now $x' = -c_1 \sin t + c_2 \cos t$

$$x'(\pi/6) = -c_1 \sin\frac{\pi}{6} + c_2 \cos \pi/6 = 0$$

$$-\frac{c_1}{2} + \frac{\sqrt{3}}{2} c_2 = 0 \Rightarrow c_1 = \sqrt{3} c_2 \quad (2)$$

using (2) in (1)

$$3c_2 + c_2 = 1 \Rightarrow c_2 = 1/4.$$

From 2

$$c_1 = \frac{\sqrt{3}}{4}.$$

16 Determine by inspection at least two solution of given first order IVP.

$$xy' = 2y \qquad y(0) = 0$$

we'll try different functions satisfying both the differential Equation and initial condition.

1) Let $y = 0$.

$\Rightarrow y' = 0$

So

$$xy' = 2y \Rightarrow 0 = 0 \qquad y(0) = 0$$

2) $y = x^2$

$y' = 2x.$

$$xy' = 2y \Rightarrow x \times 2x = 2x^2 \Rightarrow 2x^2 = 2x^2.$$

$$y(x) = x^2 \Rightarrow y(0) = 0$$

So $y = 0$ & $y = x^2$ are two solutions of IVP

Determine whether Theorem 1.2.1 guarantees that the differential Equation $y' = \sqrt{y^2 - 9}$ possesses a unique solution through the given point.

27 $(2, -3)$

$$f(x, y) = \sqrt{y^2 - 9}$$

function is not continuous on $(-3, 3)$ because function will give imaginary values

$$\frac{\partial f}{\partial y} = \frac{2y}{2\sqrt{y^2-9}}.$$

$\frac{\partial f}{\partial y}$ is not continuous on $-3 \leq y \leq 3$.

So $\frac{\partial f}{\partial y}$ is not continuous at $(2,-3)$.

So theorem 1.2.1 does not guarantee that differential equation possesses a unique solution at $(2,-3)$.

31 (a) $y = \frac{-1}{x+c}$ is solution of $y' = y^2$ (verify)

$$y' = \frac{+1}{(x+c)^2}.$$

$$y' = y^2 \Rightarrow \frac{1}{(x+c)^2} = \left(\frac{-1}{x+c}\right)^2 = \frac{1}{(x+c)^2}.$$

So $y = \frac{-1}{x+c}$ is the solution of given eq.

(b) $y(0) = 1$.

$$y(0) = 1 \Rightarrow 1 = -\frac{1}{c} \Rightarrow c = -1$$

$$y = \frac{-1}{x-1}.$$

Interval is $(-\infty, 1)$

$y(0) = -1$

$$\Rightarrow y = \frac{-1}{x+c} \Rightarrow y(0) = \frac{-1}{c} = -1 \Rightarrow c = 1.$$

$$\Rightarrow y = \frac{-1}{x+1}$$

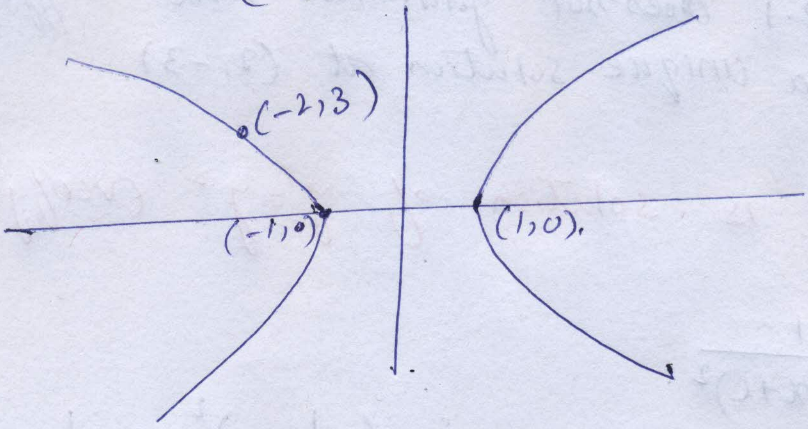
Largest interval is $(-1, \infty)$.

33 (b)

$$3x^2 - y^2 = 3.$$

$$x^2 - \frac{y^2}{3} = 0$$

$$\frac{x^2}{(1)^2} - \frac{y^2}{(\sqrt{3})^2} = 0.$$



Explicit solutions

$$3x^2 - y^2 = 3$$

$$\Rightarrow y^2 = 3(x^2-1)$$

$$y = \pm\sqrt{3(x^2-1)}$$

$$y = \sqrt{3}\sqrt{x^2-1} \quad , \quad y = -\sqrt{3}\sqrt{x^2-1}$$

(c) Checking the point $(-2,3)$.

when $x = -2 \Rightarrow y = \sqrt{3}(\sqrt{4-1}) = 3.$

other solution

$$x = -2 \Rightarrow y = -\sqrt{3}(\sqrt{3}) = -3$$

So $(-2,3)$ lies on $y = \sqrt{3}\sqrt{x^2-1}$