2.1: Solution curves without solutions

Direction fields

* A derivative $\frac{dy}{dx}$ of a differentiable function $y = y(x)$ gives the slopes of tangent lines at points on its graph.

Direction field (Definition)

If we have differential Equation

$$\frac{dy}{dx} = f(x,y)$$

If we systematically evaluate f over a rectangular grid of points in xy-plane and draw a line element at each (x,y), of grid with slope $f(x,y)$, then the collection of all these line elements is called a direction field or a slope field of the differential Equation $\frac{dy}{dx} = f(x,y)$.

* Visually, the direction field suggests the appearance or shape of family of solution curve of differential Equation and consequently it is possible to see the qualitative aspects of the solution.

Increasing / decreasing:

* Interpretation of the derivative $\frac{dy}{dx}$ as a function that gives slope plays the key role in the construction of direction field.

Another property of first derivative is given as

* If $\frac{dy}{dx} > 0$ for all x in an interval I, then a differentiable function $y = y(x)$ is increasing.

* If $\frac{dy}{dx} < 0$ for all x in an interval I, then a differentiable function $y = y(x)$ is decreasing.

This page plots a system of differential equations of the form dy/dx = f(x,y).

For a much more sophisticated direction field plotter, see the MATLAB plotter written by John C. Polking of Rice University.

dy/dt= 0.2*x*y

The direction field solver knows about trigonometric, logarithmic and exponential functions, but multiplication and evaluation must be entered explicitly (2*x and sin(x), not 2x and sin x).

The Display:

Minimum t: -5 Minimum y: -5 Arrow length: 15 ☐ Variable length arrows

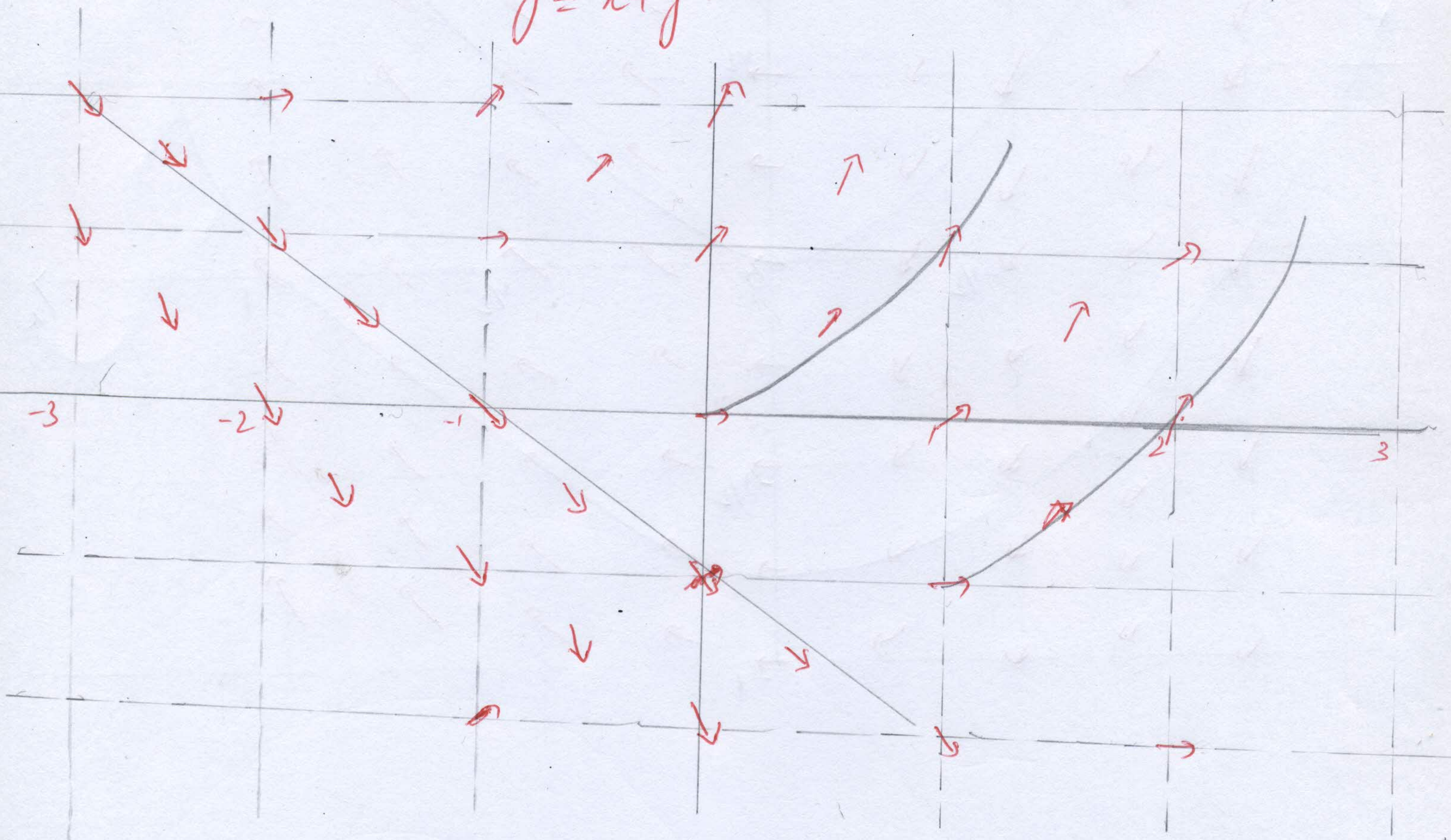
Maximum t: 5 Maximum y: 5 Number of arrows: 20

Graph Direction Field

$y = c e^{0.1x^2}$

Direction Field Example

$$y' = x + y.$$



* For $y'=0$, slope is zero i.e horizontant tangent along the line $y=x$.
* If the starting pt is $(1,0)$, $x+y=1$, so slope is $+1$.
* similarly for $(-1,0)$, $x+y=-1$, so slope is -1.

Example 2.

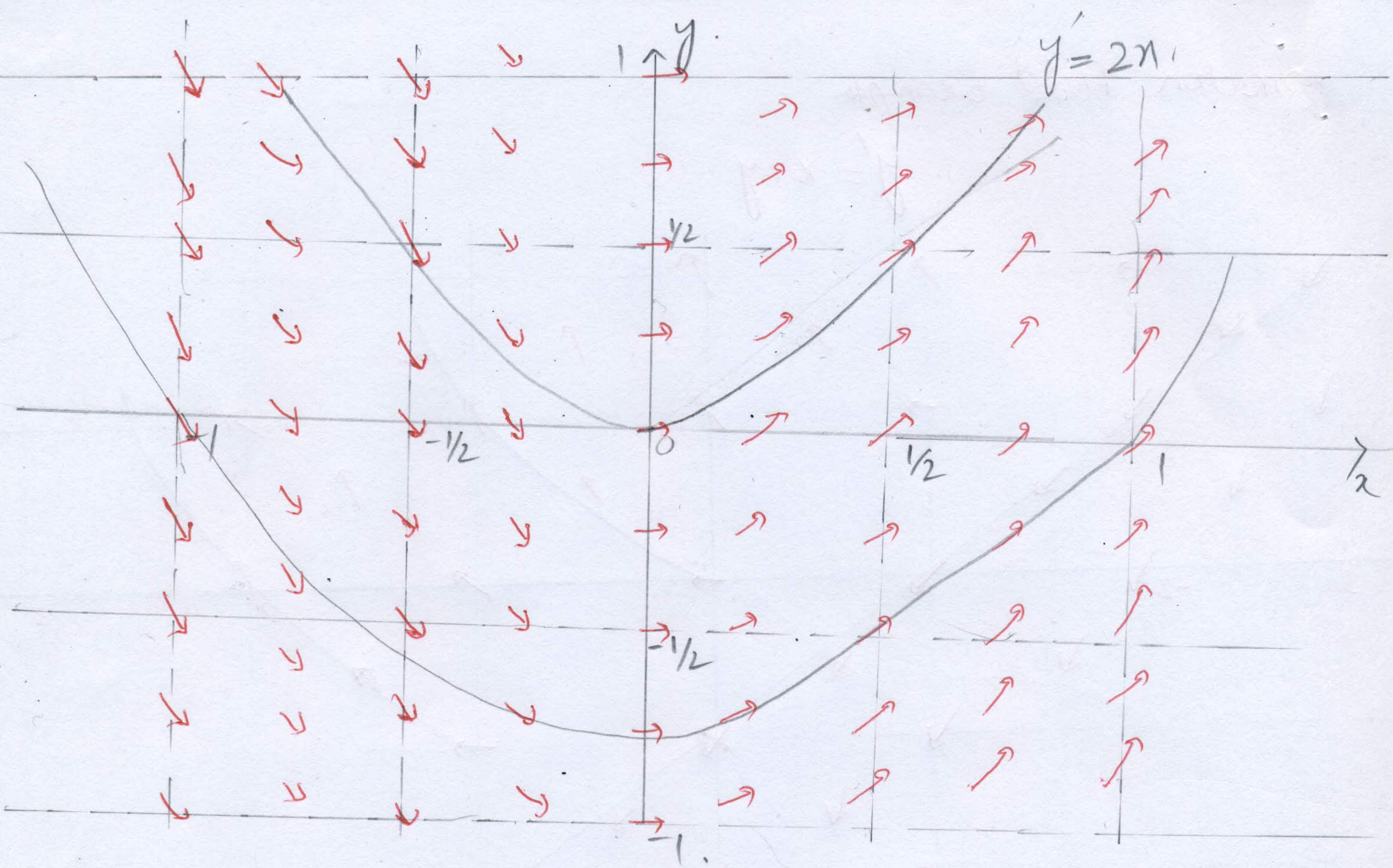
$$y' = 2x.$$

When $x=0$ slope $= 0$

when $x = 1/2$, slope $= 1$.

when $x=1$, slope $= 2$.

$x = -1$, slope $= -2$


y
y' = 2x
1
1/2
-1/2
-1
x
0
-1/2
-1
1/2
1

Autonomous First-order ODEs.

Autonomous first order DEs.

An ordinary differential equation in which the independent variable does not appear explicitly is said to be autonomous.

* If the symbol x denotes the independent variable, then an autonomous first order differential Equation can be written as $f(y, y') = 0$ or in normal form

$$\frac{dy}{dx} = f(y). \qquad (1)$$

Example

$$\frac{dy}{dx} = 1 + y^2 \quad \text{autonomous}$$

$$\frac{dy}{dx} = 0.2xy \quad \text{non-autonomous}.$$

Critical points

A real number c is a critical point of autonomous differential Equation, if it is zero of f that is $f(c) = 0$.

* A critical point is also called equilibrium point or stationary point.

As if we insert $y(x) = c$ in Eq(1), both sides of equation are zero. So we can say that

"If c is a critical point of Eq (1), then $y(x) = c$ is a constant solution of autonomous differential Equation".

* A constant solution $y(x) = c$ of (1) is called an equilibrium solution.
* Equilibrium points are only constant solutions of autonomous differential Equation.

Example

$$\frac{dP}{dt} = P(a-bP)$$

where a & b are constant, is an autonomous differential equation.

Here $f(P) = P(a-bP)$

For critical points,

$$f(P) = 0 \Rightarrow P(a-bP) = 0$$

so equilibrium points or Equilibrium solutions are

$$P = 0 \qquad P = \frac{a}{b}$$

Interval for $f(P)$ is $(-\infty, \infty)$.

Dividing the interval at critical points, we get subintervals at

$$(-\infty, 0), \ (0, \frac{a}{b}), \ (\frac{a}{b}, \infty)$$

At $(-\infty, 0)$

$$f(P) = bP(\frac{a}{b} - P)$$

is -ve i.e $\frac{dP}{dt} < 0$, so function is decreasing in the interval $(-\infty, 0)$

In $(0, \frac{a}{b})$

$f(P) = bP(\frac{a}{b} - P) > 0$ so $\frac{dP}{dt} > 0$ and function is increasing on $(0, \frac{a}{b})$

In $(\frac{a}{b}, \infty)$

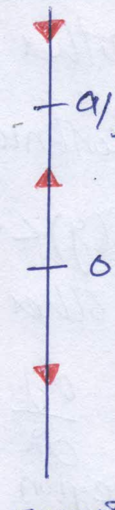
$f(P) = bP(\frac{a}{b} - P) < 0$ so $\frac{dP}{dt} < 0$ & function is decreasing on $(\frac{a}{b}, \infty)$.

* we can represent the behavior of function $f(P)$ on these intervals by arrows on vertical lines where upward arrow represents an increasing behavior and downward arrow represents ~~an~~ ~~the~~ decreasing behavior.

Such kind of representation is called one-dimensional phase portrait of differential Equation

$$\frac{dP}{dt} = P(a-bP)$$

or simply a phase portrait.



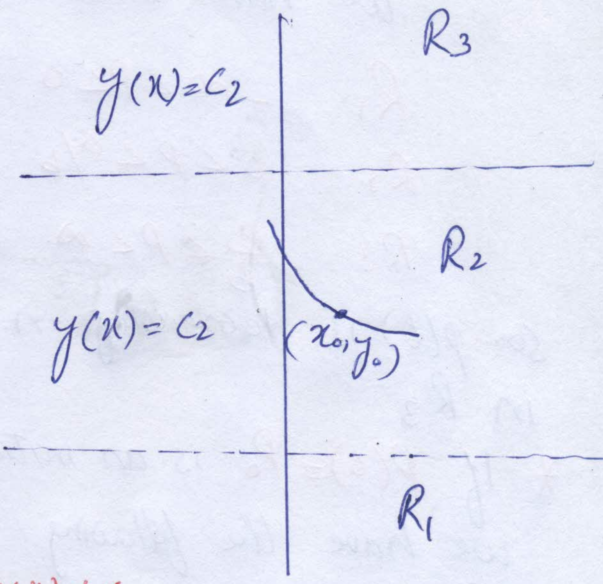
* The vertical line is called a phase line.

Solution curves

Without solving an autonomous differential equation, we can usually say a great deal about its solution curves.

As function f and its derivative are continuous function of y on some interval I of the y-axis, then by Theorem 1.2.1. there is only one solution curve of differential Equation passing through any point (x_0, y_0) of some region R.

* If Equation possesses two equilibrium points c_1 & c_2 such that $c_1 < c_2$ then graph of $y(x) = c_1$ & $y(x) = c_2$ are horizontal lines. and these lines partition the region into three subregions R_1, R_2 & R_3.



* we can draw some conclusion about non-constant solution $y(x)$ of differential equation.

* If (x_0, y_0) is in subregion R_i, $i=1,2,3$ and $y(x)$ is a solution whose graph passes through this point, then $y(x)$ remains in the subregion R_i for all x.

The solution curve in R_2 is bounded below by c_1 & above by c_2 i-e $c_1 < y(x) < c_2$. for all x.

The solution curve stays within R_2 for all x because graph of non-constant solution cannot cross the graph of either equilibrium solution $y(x) = c_1$ or $y(x) = c_2$.

* By continuity of f, $f(y)$ is either such that $f(y) > 0$ or $f(y) < 0$ for all x in a subregion R_i $i = 1, 2, 3$. In other words, $f(y)$ cannot change signs in a subregion.

* As $\frac{dy}{dx} = f(y(x))$ is either positive or negative in a subregion R_i, so $y(x)$ is either increasing or decreasing in the subregion R_i

* If $y(x)$ is bounded above by a critical pt., then the graph of of $y(x)$ must approach the graph of the equilibrium solution, $y(x) = c_1$ either as $x \to \infty$ as $x \to -\infty$.

* If $y(x)$ is bounded above and below by crictical pts i-e $c_1 \leq y(x) < c_2$ then the graph of $y(x)$ must approach the graph of equilibrium solutions $y(x) = c_1$ and $y(x) = c_2$ one as $x \to \infty$ and the other as $x \to -\infty$.

Example 3 (Revisited).

we have three regions

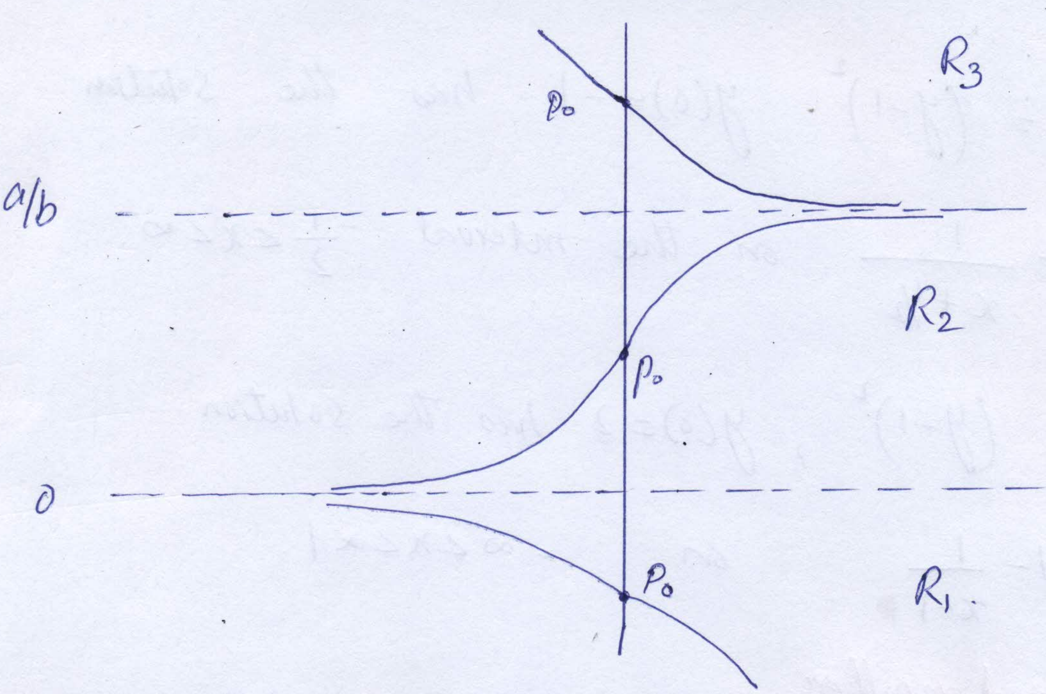
R_1: $-\infty < P < 0$

R_2: $0 < P < a/b$

R_3: $\frac{a}{b} < P < \infty$.

so $p(t)$ is decreasing in R_1, increasing in R_2 and decreasing in R_3.

* If $P(0) = P_0$ is an initial value, then in R_1, R_2 & R_3 we have the following.



Solution curves of autonomous DE.

$$\frac{dy}{dx} = (y-1)^2.$$

For critical pts: $(y-1)^2 = 0 \Rightarrow y = 1$

only on critical pt.

we have two subintervals $(-\infty, 1)$, $(1, \infty)$.

$f(y) = (y-1)^2$ is increasing on both $(-\infty, 1)$ and $(1, \infty)$

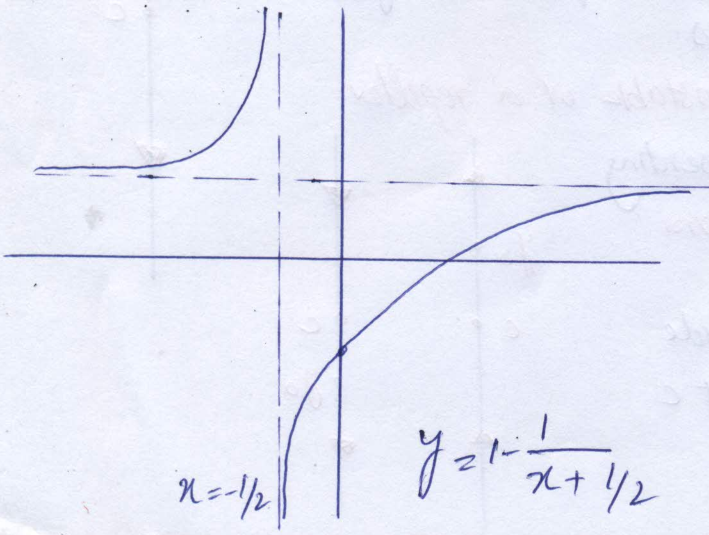
* For an initial condition $y(0) = y_0 < 1$, a solution $y(x)$ is increasing and bounded above by 1 so $y(x) \to 1$ as $x \to \infty$.

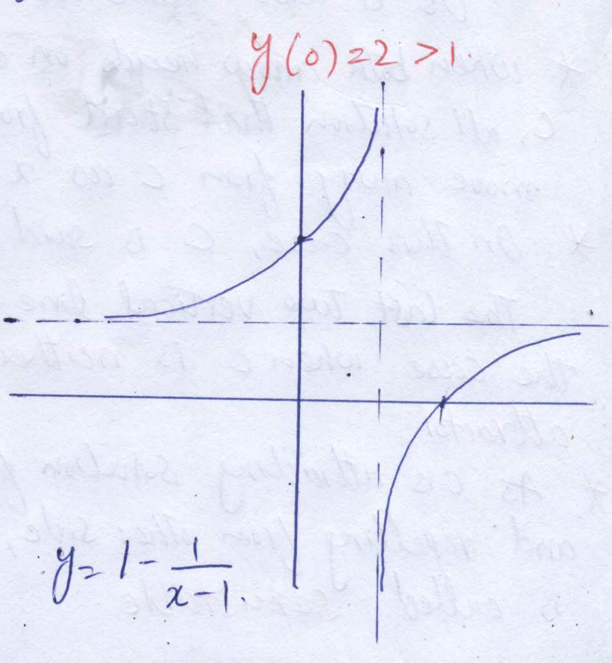
* For $y(0) = y_0 > 1$, a solution $y(x)$ is increasing and unbounded.

$y(x) = \frac{-1}{x-c} + 1$ one parameter family of solution for above Equation

For $y(0) = -1 < 1$.

$y(0) = 2 > 1$.





So $\dfrac{dy}{dx} = (y-1)^2 \quad y(0) = -1$ has the solution

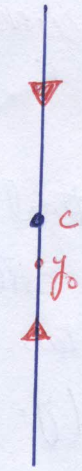
$$y = 1 - \frac{1}{x + 1/2} \quad \text{on the interval } -\frac{1}{2} < x < \infty.$$

and $\dfrac{dy}{dx} = (y-1)^2, \quad y(0) = 2$ has the solution

$$y = 1 - \frac{1}{x-1} \quad \text{on } -\infty < x < 1.$$

Attractor and repellers

Let $y(x)$ be non-constant solution of autonomous differential Eqs and c is the critical pt of DE.

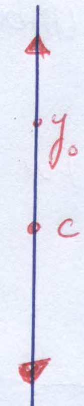


* when both arrowheads on either side of the dot labeled as c point towards c, all solutions $y(x)$ of DE that start from an initial point (x_0, y_0) sufficiently near c exhibit the asymptotic behavior

$$\lim_{x \to \infty} y(x) = c.$$

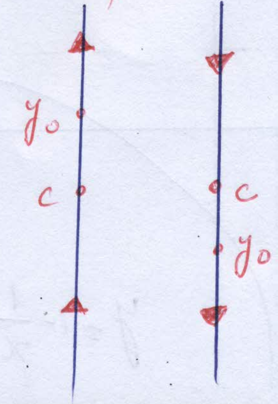
For this reason, the critical point c is said to be asymptotically stable.

It is also name as attractor.



* when both arrow heads on either side of c point away from c, all solution that start from an initial point (x_0, y_0) move away from c as x increases.
* In this case, c is said to be unstable or a repeller.

The last two vertical line are representing the case when c is neither repeller nor attractor



* As c is attracting solution from one side and repelling from other side, the point c is called semistable.