Iteration methods for solving system of linear equations

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Homework assignment:

Make an iteration algorithm for finding solution to equation $Ax = b$ using the Jacobi method and the Gauss-Seidel method.

Where $\gamma \in \{0.5, 2, 5\}$

$$\mathbf{A} = \begin{pmatrix} \gamma & -1 & & & \\ -1 & \gamma & -1 & & \\ & -1 & \gamma & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & \gamma \end{pmatrix} \in \mathbb{R}^{20,20}, \quad \mathbf{b} = \begin{pmatrix} \gamma - 1 \\ \gamma - 2 \\ \gamma - 2 \\ \vdots \\ \gamma - 2 \\ \gamma - 2 \\ \gamma - 1 \end{pmatrix} \in \mathbb{R}^{20}$$

and the algorithm converges when the euclidean norm is smaller than 10^{-6}.

$$\frac{\|\mathbf{A}\tilde{\mathbf{x}} - \mathbf{b}\|_2}{\|\mathbf{b}\|_2} < 10^{-6}$$

Algorithm uses double precision (Float64) and was written using Julia.

Methods:

Jacobi method:
Jacobi method is relatively simple approach that uses the fact that in the original equation $Ax = b$, matrix can be decomposed as $A = R + D$, where D is a matrix of diagonal values from A and $R = A - D$.
Then we can iteratively find solution x using $x^{i+1} = D^{-1} (b - Rx^i)$.
Inspiration from [1].

Gauss-Seidel method:
In equation $Ax = b$, matrix A can be decomposed to $A = U + L$, where U is triangular upper matrix without diagonal and L is triangular lower matrix including diagonal. When a matrix is in upper triangular form, the solution of $Ux = b$ can be found quickly.
This means that equation can be rewritten as $(U + L)x = b$ and then iteratively calculated using $x^{i+1} = L^{-1} (b - Ux^i)$.
Inspiration from [2].

Results:

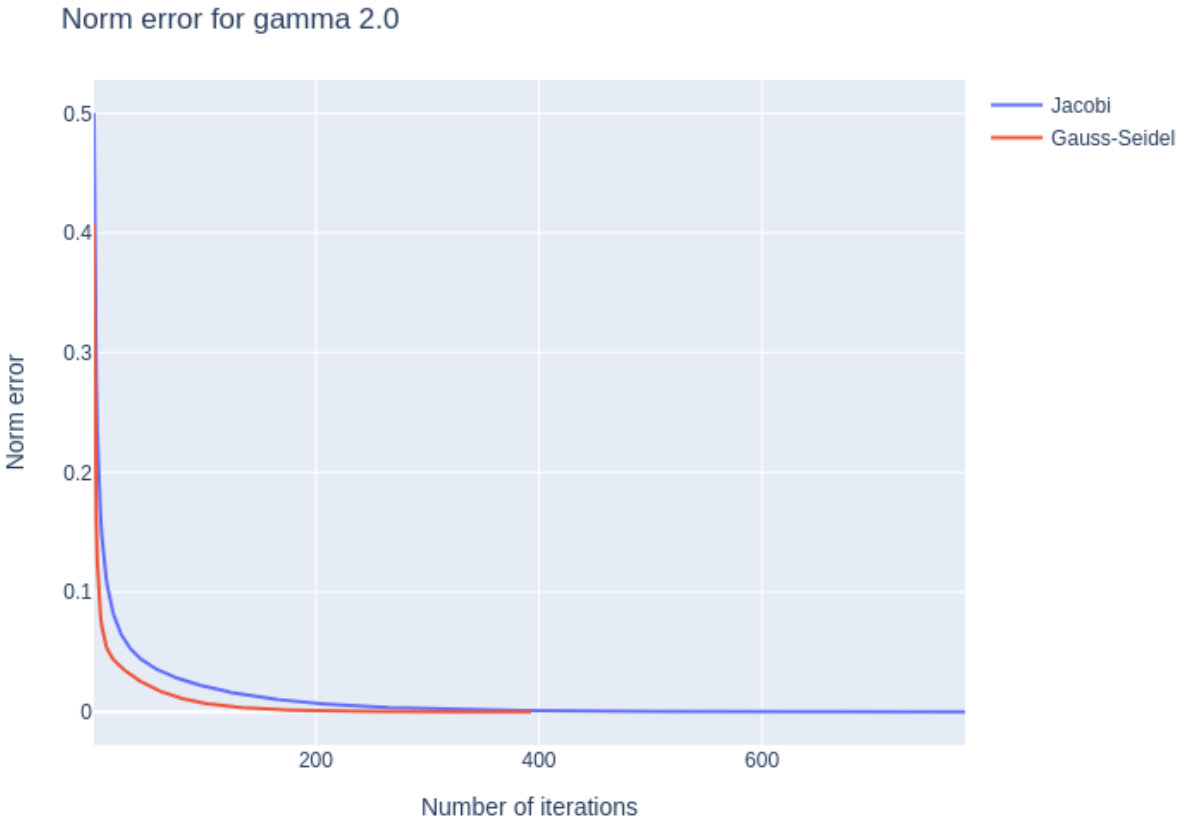
Jacobi method:

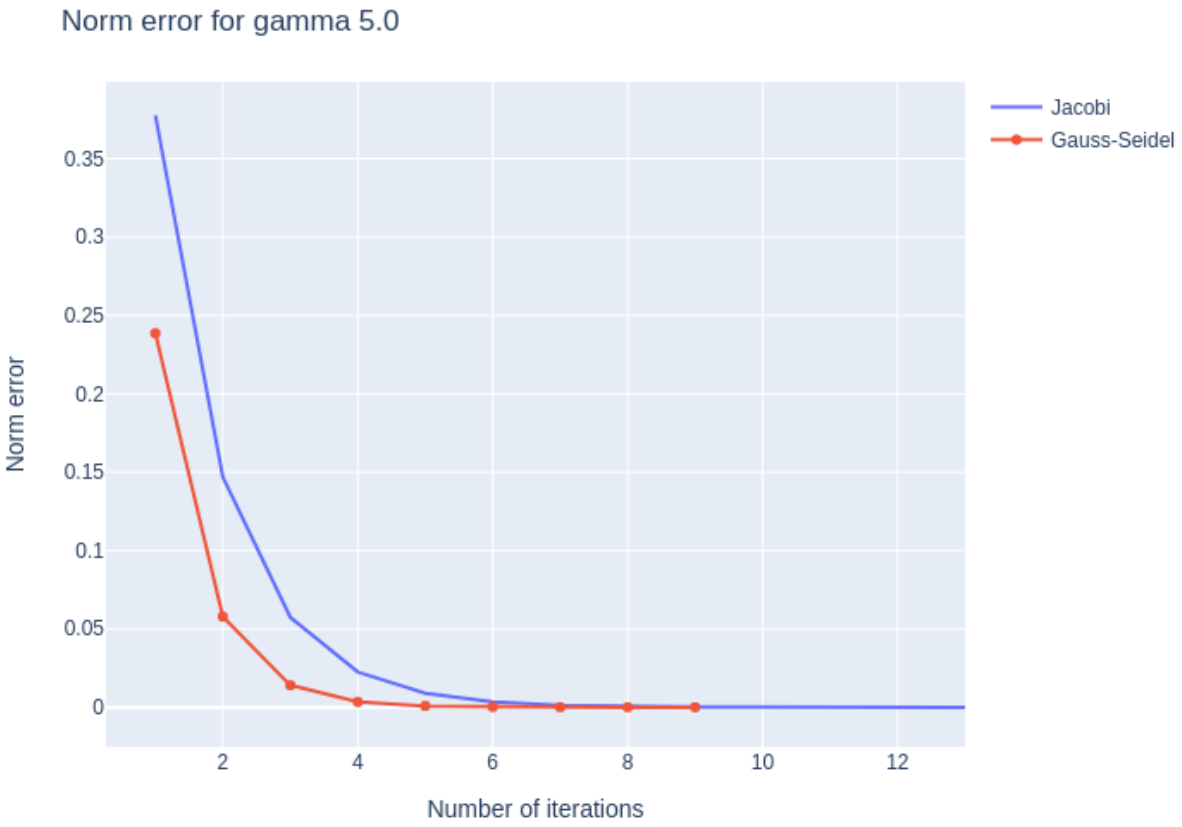
γ	Norm error	Iterations
0.5	Did not converge	Did not converge
2.0	9.968534524306131e-6	782
5.0	5.210627849163448e-6	13

Gauss-Seidel method:

γ	Norm error	Iterations
0.5	Did not converge	Did not converge
2.0	9.807646243636393e-6	393
5.0	2.879016358091445e-6	9

On the graphs below we can see how Gauss-Seidel method needs less iterations to calculate a comparably precise solution. Also, the precision in both methods improves exponentially slowly.





Manual for executing code:

1. In order to execute code you need to have *Julia* installed.
2. Then download in the same folder source files *main.jl* and *inputs.jl*. In *main.jl* at the very end you can find simple commands to execute code with comments explaining the meaning of variables. All functions have comments.
3. In *inputs.jl* you can find the definition of input matrices and variable dimension of the matrices. Default dimension is 20x20.

Materials used in this work:

[0] Handout MPI
https://courses.fit.cvut.cz/NI-MPI/latex/lectures/czech/mi-mpi-prednaska-handouts.pdf
[1] Jacobi iteration method
https://www.ucg.ac.me/skladiste/blog_10701/objava_23569/fajlovi/jacobi.pdf
[2] Gauss-Seidel Iteration Method
https://www.patnauniversity.ac.in/e-content/science/physics/MScPhy52.pdf