Ch 9.3-4: Support Vector Machine

Lecture 29 - CMSE 381

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Fri, Nov 17, 2023

Announcements

Announcements:

- HW #7 due Monday

Lec #	Date			Reading	Homeworks
20	Fri	Oct 27	Dimension Reduction	6.3	
21	Mon	Oct 30	More dimension reduction; High dimensions	6.4	
22	Wed	Nov 1	Polynomial & Step Functions	7.1,7.2	
23	Fri	Nov 3	Step Functions; Basis functions; Start Splines	7.2 - 7.4	
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28	Wed	Nov 15	SVC	9.2	
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35	Fri	Dec 8	Virtual: Project office hours		Project due

Last time:

- 9.2 Support Vector Classifier

This lecture:

- 9.3 Support Vector Machine

Section 1

Last Time

Classification Setup

Data matrix:

$$X = \begin{pmatrix} - & x_1^T & - \\ - & x_2^T & - \\ & \vdots & \\ - & x_n^T & - \end{pmatrix}_{n \times p}$$

Observations in one of two classes,
$y_i \in \{-1, 1\}$

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$$x_1 = \begin{pmatrix} x_{11} \\ \vdots \\ x_{1p} \end{pmatrix}, \cdots, x_n = \begin{pmatrix} x_{n1} \\ \vdots \\ x_{np} \end{pmatrix}$$

Separate out a test observation

$$x^* = (x_1^* \ \cdots \ x_p^*)^T$$

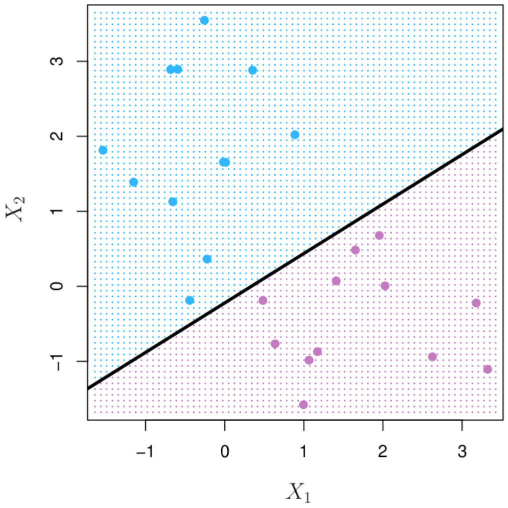
Hyperplane becomes a classifier

If you have a separating hyperplane:

- Check

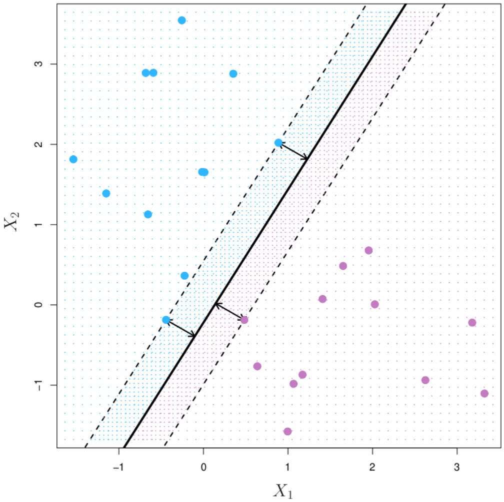
$$f(x^*) = \beta_0 + \beta_1 x_1^* + \beta_2 x_2^* + \cdots + \beta_p x_p^*$$

- If positive, assign $\hat{y} = 1$
- If negative, assign $\hat{y} = -1$



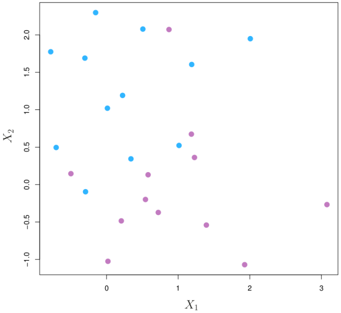
How do we pick? Old version

Maximal margin classifier

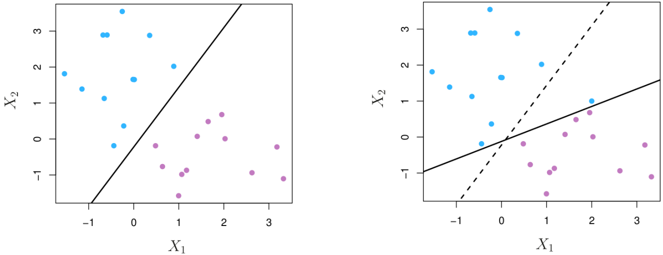


- For a hyperplane, the *margin* is the smallest distance from any data point to the hyperplane.
- Observations that are closest are called *support vectors*.
- The *maximal margin hyperplane* is the hyperplane with the largest margin
- The classifier built from this hyperplane is the *maximal margin classifier*.

Issues

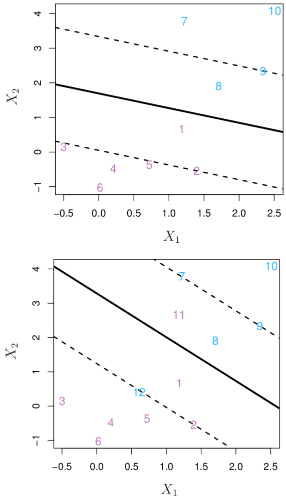


No separating hyperplane exists



Choice of hyperplane is sensitive to new points

Support Vector Classifier



$$\underset{\beta_0,\beta_1,\ldots,\beta_p,\epsilon_1,\ldots,\epsilon_n,\,M}{\text{maximize}} \quad M$$

$$\text{subject to} \quad \sum_{j=1}^{p} \beta_j^2 = 1,$$

$$y_i(\beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip}) \geq M(1 - \epsilon_i),$$

$$\epsilon_i \geq 0, \quad \sum_{i=1}^{n} \epsilon_i \leq C,$$

Two formulations side by side

Support Vector Classifier

$$\underset{\beta_0,\beta_1,\ldots,\beta_p,\epsilon_1,\ldots,\epsilon_n,\,M}{\text{maximize}} \quad M$$

$$\text{subject to} \quad \sum_{j=1}^{p} \beta_j^2 = 1,$$

$$y_i(\beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip}) \geq M(1 - \epsilon_i),$$

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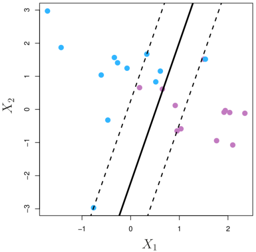
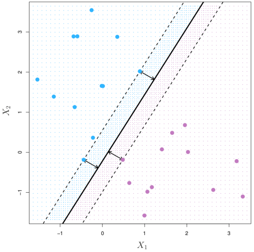


Maximal Margin Classifier

$$\underset{\beta_0,\beta_1,\ldots,\beta_p,M}{\text{maximize}} \; M$$

$$\text{subject to} \; \sum_{j=1}^{p} \beta_j^2 = 1,$$

$$y_i(\beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip}) \geq M \quad \forall \; i = 1, \ldots, n$$



So many variables

$$\underset{\beta_0,\beta_1,\ldots,\beta_p,\epsilon_1,\ldots,\epsilon_n,\,M}{\text{maximize}} \quad M$$

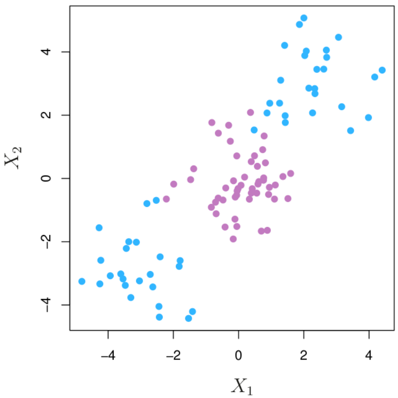
$$\text{subject to} \quad \sum_{j=1}^{p} \beta_j^2 = 1,$$

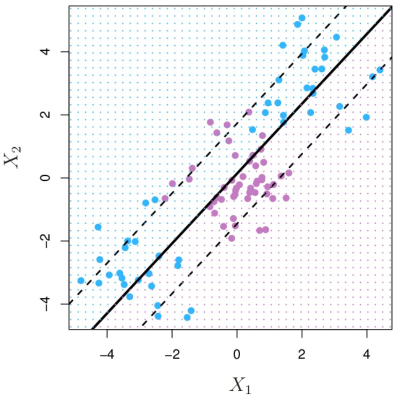
$$y_i(\beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip}) \geq M(1 - \epsilon_i),$$

$$\epsilon_i \geq 0, \quad \sum_{i=1}^{n} \epsilon_i \leq C,$$

- C is nonnegative tuning parameter
- M is the width of the margin
- $\varepsilon_1, \cdots, \varepsilon_n$ are slack variables allowing observations to go to the other side

Limiting factor of SVC





Section 2

Support Vector Machine

Example of using more features

Want $2p$ features:

$$X_1, X_1^2, X_2, X_2^2, \cdots, X_p, X_p^2$$

Optimization becomes:

$$\underset{\beta_0,\beta_{11},\beta_{12},\ldots,\beta_{p1},\beta_{p2},\epsilon_1,\ldots,\epsilon_n,\, M}{\text{maximize}} \quad M$$

$$\text{subject to } y_i \left(\beta_0 + \sum_{j=1}^{p} \beta_{j1} x_{ij} + \sum_{j=1}^{p} \beta_{j2} x_{ij}^2 \right) \geq M(1 - \epsilon_i),$$

$$\sum_{i=1}^{n} \epsilon_i \leq C, \quad \epsilon_i \geq 0, \quad \sum_{j=1}^{p} \sum_{k=1}^{2} \beta_{jk}^2 = 1.$$

Kernels

Inner products

$$\langle a, b \rangle = \sum_{i=1}^{r} a_i b_i$$

Quick computations

What are the following?

- $\langle (1,1),(0,3) \rangle$
- $\langle (1,1),(3,2) \rangle$
- $\langle (2,3),(0,3) \rangle$
- $\langle (2,3),(3,2) \rangle$

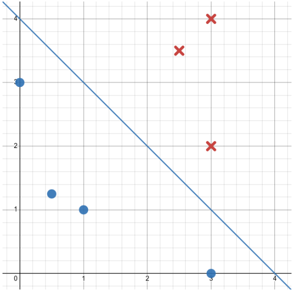
SVC via inner products

$$f(x) = \beta_0 + \sum_{i=1}^{n} \alpha_i \langle x, x_i \rangle$$

Example

$$-2\sqrt{2} + \frac{\sqrt{2}}{2}X_1 + \frac{\sqrt{2}}{2}X_2 = 0$$

$$-2\sqrt{2} + \frac{\sqrt{2}}{18}\langle (X_1, X_2), (0,3)\rangle + \frac{\sqrt{2}}{6}\langle (X_1, X_2), (3,2)\rangle = 0$$



What are the α_is?

Data point	α_i
$(3,4)$	
$(2.5,3.5)$	
$(3,2)$	
$(3,0)$	
$(0,3)$	
$(1,1)$	
$(0.5,1.25)$	

What α_i's are needed to write the hyperplane

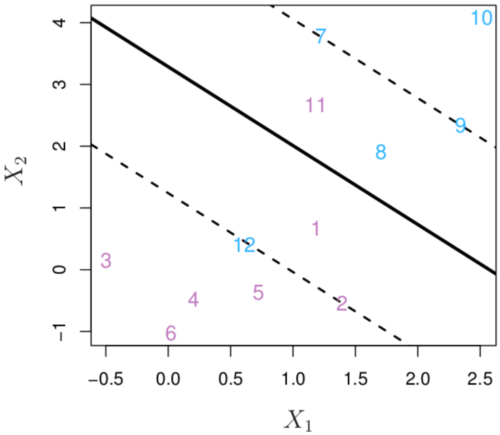
$$-2\sqrt{2}+\frac{\sqrt{2}}{18}\langle(X_1,X_2),(0,3)\rangle+\frac{\sqrt{2}}{6}\langle(X_1,X_2),(3,2)\rangle$$

of the previous page in the form

$$f(x)=\beta_0+\sum_{i\in\mathcal{S}}\alpha_i\langle x,x_i\rangle?$$

SVC via inner products of support vectors

$$f(x) = \beta_0 + \sum_{i \in \mathcal{S}} \alpha_i \langle x, x_i \rangle$$



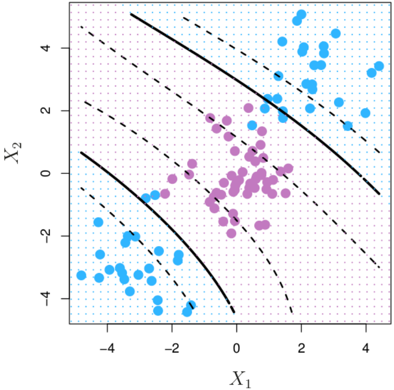
The kernel

$$K(x_i, x_i')$$

$$f(x) = \beta_0 + \sum_{i \in \mathcal{S}} \alpha_i K(x, x_i)$$

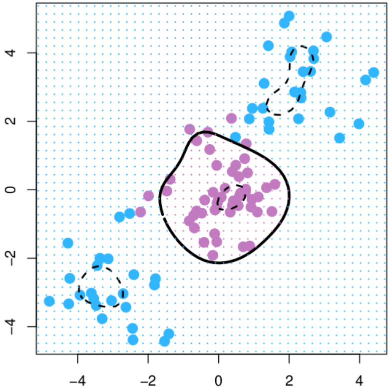
A polynomial kernel

$$K(x_i, x_{i'}) = \left(1 + \sum_{j=1}^{p} x_{ij} x_{i'j}\right)^d$$



A radial kernel

$$K(x_i, x_i') = \exp\left(-\gamma \sum_{j=1}^{p} (x_{ij} - x_{i'j})^2\right)$$



Support Vector Machine

$$f(x) = \beta_0 + \sum_{i \in \mathcal{S}} \alpha_i K(x, x_i)$$

Coding

Section 3

SVM with more than two classes

One-Vs-One Classification

Also called all-pairs

One-Vs-All Classification

TL;DR

$$f(x) = \beta_0 + \sum_{i \in \mathcal{S}} \alpha_i K(x, x_i)$$

Kernels

- Linear

$$K(x_i, x_{i'}) = \sum_{j=1}^{p} x_{ij} x_{i'j}$$

- Polynomial

$$K(x_i, x_{i'}) = \left(1 + \sum_{j=1}^{p} x_{ij} x_{i'j}\right)^d$$

- Radial

$$K(x_i, x_i') = \exp\left(-\gamma \sum_{j=1}^{p} (x_{ij} - x_{i'j})^2\right)$$

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