Ch 9.2: Support Vector Classifier

Lecture 28 - CMSE 381

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Wed, Nov 15, 2023

Last time:

- 9.1 Maximal Margin Classifier

This lecture:

- 9.2 Support Vector Classifier

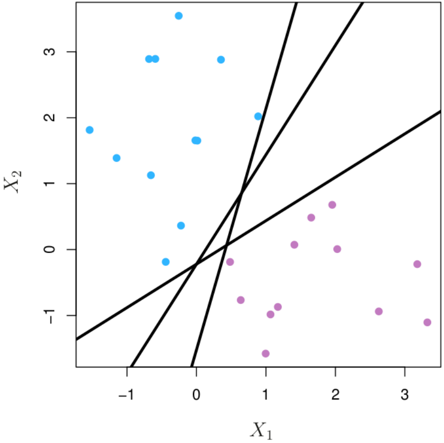
Announcements:

- HW #7....

Section 1

Last time

Separating Hyperplane



Require that for every data point:

$$\beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip} > 0 \text{ if } y_i = 1$$

$$\beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip} < 0 \text{ if } y_i = -1$$

Equivalently

Require that for every data point

$$y_i(\beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip}) > 0$$

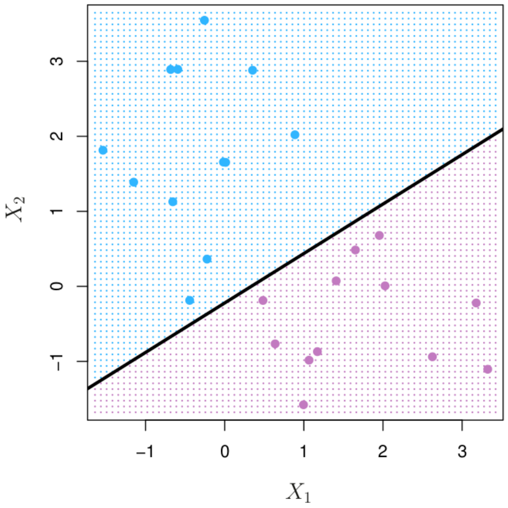
Separating hyperplane becomes a classifier

If you have a separating hyperplane:

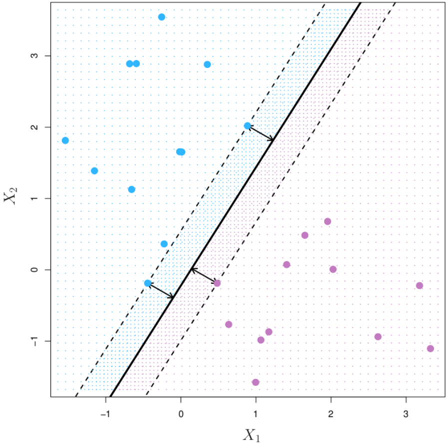
- Check

$$f(x^*) = \beta_0 + \beta_1 x_1^* + \beta_2 x_2^* + \cdots + \beta_p x_p^*$$

- If positive, assign $\hat{y} = 1$
- If negative, assign $\hat{y} = -1$



Maximal margin classifier



- For a hyperplane, the *margin* is the smallest distance from any data point to the hyperplane.
- Observations that are closest are called *support vectors*.
- The *maximal margin hyperplane* is the hyperplane with the largest margin
- The classifier built from this hyperplane is the *maximal margin classifier*.

Mathematical Formulation

$$\underset{\beta_0,\beta_1,\ldots,\beta_p,M}{\text{maximize}}\; M$$

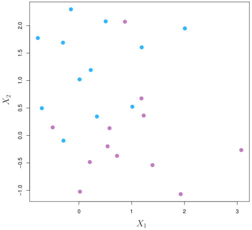
$$\text{subject to } \sum_{j=1}^{p} \beta_j^2 = 1,$$

$$y_i(\beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip}) \geq M \;\; \forall\, i = 1, \ldots, n$$

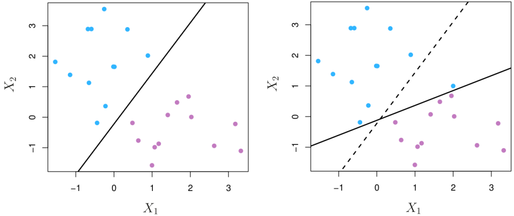
- Unit normal requirement, we can always write a given hyperplane this way.
- Last eq forces points to be on the correct side of the hyperplane in order for $M > 0$
- The product is the distance from the point to the hyperplane
- Making M as big as possible is the maximal margin hyperplane

Problems

Might be no separating hyperplane



Sensitivity to new points



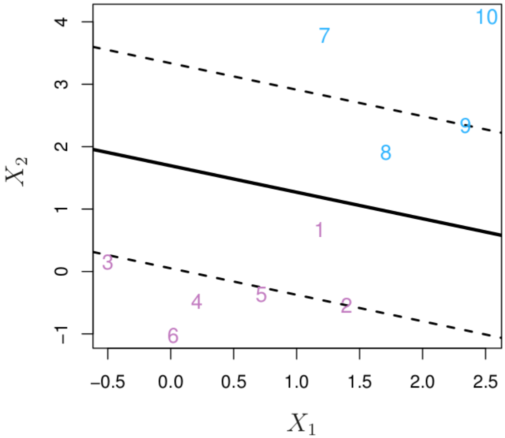
Section 2

Support Vector Classifier

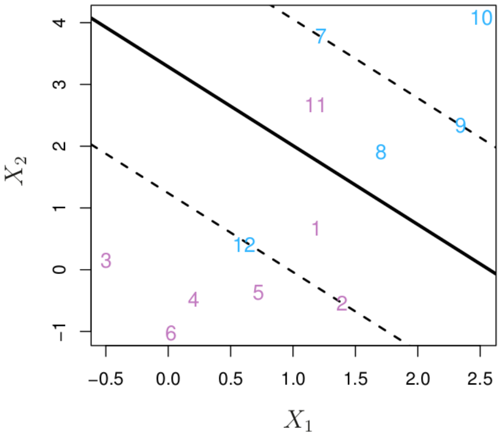
Basic idea

- Be ok with having a classifier that isn't quite perfect
- Aim for greater robustness to individual observations
- Better classification of most of the training observations
- Result is a support vector classifier
- Soft margin classifier

Soft margin



Some points on wrong side of margin



Some points on wrong side of hyperplane (Misclassified)

Mathematical Formulation of SVC

$$\underset{\beta_0,\beta_1,\ldots,\beta_p,\epsilon_1,\ldots,\epsilon_n,\,M}{\text{maximize}} \quad M$$

$$\text{subject to } \sum_{j=1}^{p} \beta_j^2 = 1,$$

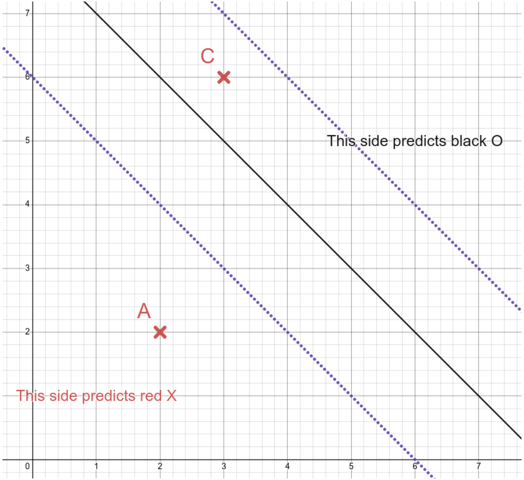
$$y_i(\beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip}) \geq M(1 - \epsilon_i),$$

$$\epsilon_i \geq 0, \quad \sum_{i=1}^{n} \epsilon_i \leq C,$$

- C is nonnegative tuning parameter
- M is the width of the margin
- $\varepsilon_1, \cdots, \varepsilon_n$ are slack variables allowing observations to go to the other side

Find positive ε's that will satisfy this

Fix $M = \sqrt{2}$ $\quad y_i(\beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip}) \geq M(1 - \varepsilon_i)$

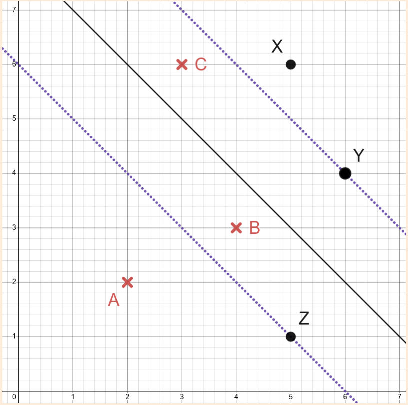


- Point out M is distance from center hyperplane to noted margin.
- A is on the correct side of the hyperplane.
  - Left side is already bigger than M, so set $\varepsilon_i = 0$.
- C is on the wrong side of the hyperplane for its label.
  - Left side of the equation is negative. What ε satisfies this?
  - $-\frac{\sqrt{2}}{2} = \sqrt{2}(1 - \varepsilon)$
  - $\varepsilon = 3/2$

What is ε?

$$\text{Fix } M = \sqrt{2} \qquad y_i(\beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip}) \geq M(1 - \varepsilon_i)$$

Fill in the table so that the inequality is satisfied.

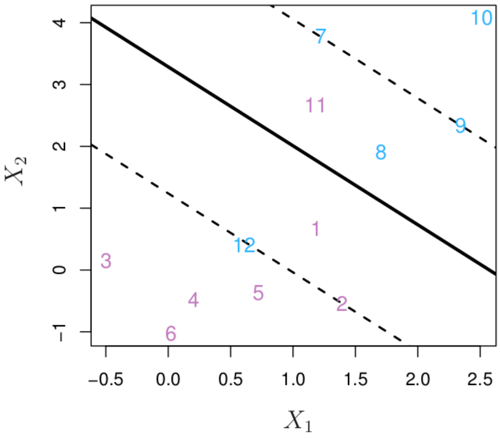


Point	Left Side	ε_i	$M(1-\varepsilon_i)$
A	$2\sqrt{2}$	0	$\sqrt{2}$
B	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	$\sqrt{2}(1-1/2) = \frac{\sqrt{2}}{2}$
C	$-\frac{\sqrt{2}}{2}$	1.5	$-\frac{\sqrt{2}}{2}$
X	$\frac{3\sqrt{2}}{2}$	0	$\sqrt{2}$
Y	$\sqrt{2}$	0	$\sqrt{2}$
Z	$-\sqrt{2}$	2	$\sqrt{2}(1-2) = -\sqrt{2}$

To use for later slide: Total ε is 4

What is ε?

- If $\varepsilon_i = 0$, then on correct side of margin
- If $\varepsilon_i > 0$ then on the wrong side of margin (Violated margin)
- If $\varepsilon_i > 1$ then on the wrong side of hyperplane



What is C?

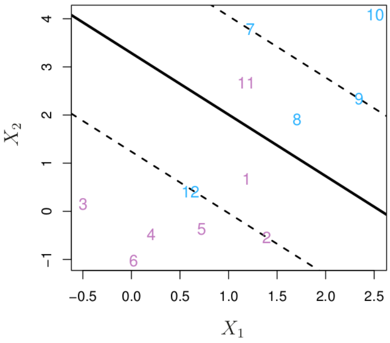
$$\underset{\beta_0,\beta_1,\ldots,\beta_p,\epsilon_1,\ldots,\epsilon_n,\,M}{\text{maximize}} \quad M$$

$$\text{subject to } \sum_{j=1}^{p} \beta_j^2 = 1,$$

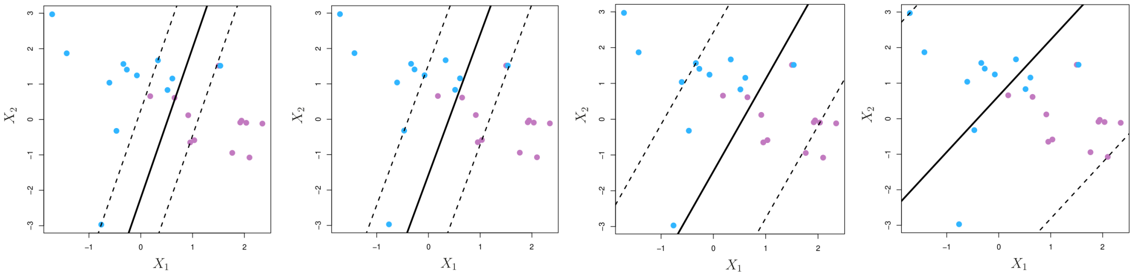
$$y_i(\beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip}) \geq M(1 - \epsilon_i),$$

$$\epsilon_i \geq 0, \quad \sum_{i=1}^{n} \epsilon_i \leq C,$$

- Bounds sum of ε_i, so controls number & severity of violating margin (budget)
- $C = 0$ means no violations allowed
- $C > 0$ means at most C observations can be on wrong side of hyperplane
- In previous example, our total of ε was 4, so would be a valid hyperplane for C at most 4.



Examples messing with C



Increasing C $\rightarrow$

- For increasing C, we have more flexibility, so more points allowed violate margin/hyperplane
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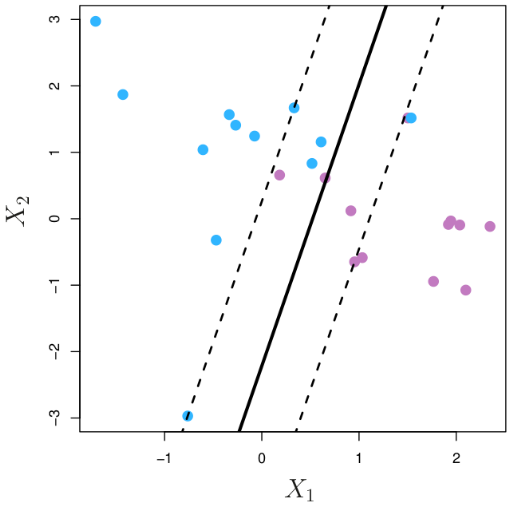
What affects the hyperplane?



- Only observations on the margin or violating the margin affect the hyperplane
- These observations are called support vectors
- Changing other points positions doesn't affect hyperplane found
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Coding

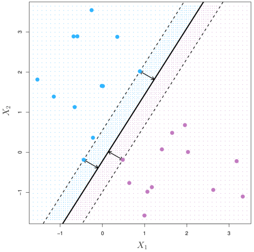
TL;DR

Maximal Margin Classifier

$$\underset{\beta_0,\beta_1,\ldots,\beta_p,M}{\text{maximize}} \; M$$

$$\text{subject to } \sum_{j=1}^{p} \beta_j^2 = 1,$$

$$y_i(\beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip}) \geq M \;\; \forall\, i = 1, \ldots, n$$



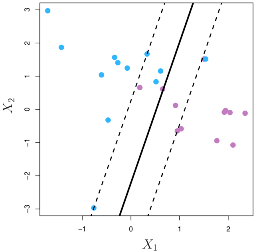
Support Vector Classifier

$$\underset{\beta_0,\beta_1,\ldots,\beta_p,\epsilon_1,\ldots,\epsilon_n,\, M}{\text{maximize}} \quad M$$

$$\text{subject to } \sum_{j=1}^{p} \beta_j^2 = 1,$$

$$y_i(\beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip}) \geq M(1 - \epsilon_i),$$

$$\epsilon_i \geq 0, \quad \sum_{i=1}^{n} \epsilon_i \leq C,$$



Next time

Status	Lec #	Date			Reading	Homeworks
		Mon	Oct 23	No class - Fall break		
		Wed	Oct 25	**Midterm #2**		
Done	20	Fri	Oct 27	Dimension Reduction	6.3	
Done	21	Mon	Oct 30	More dimension reduction; High dimensions	6.4	
Done	22	Wed	Nov 1	Polynomial & Step Functions	7.1,7.2	
Pushed	23	Fri	Nov 3	Step Functions; Basis functions; Start Splines	7.2 - 7.4	
	24	Mon	Nov 6	Regression Splines	7.4	~~HW #6 Due~~
	25	Wed	Nov 8	Decision Trees	8.1	HW #6 Due
	26	Fri	Nov 10	Random Forests	8.2.1, 8.2.2	
	27	Mon	Nov 13	Maximal Margin Classifier	9.1	
	28	Wed	Nov 15	SVC	9.2	
	29	Fri	Nov 17	SVM	9.3, 9.4	
	30	Mon	Nov 20	Single layer NN	10.1	
	31	Wed	Nov 22	Virtual: Project office hours		
		Fri	Nov 24	No class - Thanksgiving		
		Mon	Nov 27	**Review**		
		Wed	Nov 29	**Midterm #3**		
	32	Fri	Dec 1	Multi Layer NN	10.2	
	33	Mon	Dec 4	CNN	10.3	
	34	Wed	Dec 6	Unsupervised Learning & Clustering	12.1, 12.4	
	35	Fri	Dec 8	Virtual: Project office hours		Project due