FUZZY SET THEORY

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Classical Set Theory

- Define: An element having certain property belongs to a set or not.
- Set 'A' contained in an universal space 'X'
- We can state whether 'x' is or is not an element of set 'A'
- These sets are called crisp sets

Classical Set Theory

Classical set theory enumerates all its elements using

$A = \{ a_1, a_2, a_3, a_4, \ldots a_n \}$

If the elements a_i $(i = 1, 2, 3, \ldots n)$ of a set **A** are subset of universal set **X**, then set **A** can be represented for all elements $x \in X$ by its **characteristic function**

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{otherwise} \end{cases}$$

Classical Set Theory

$$\left.\begin{array}{l} A : X \rightarrow [0, 1] \\ A(x) = 1, \ x \text{ is a member of } A \\ A(x) = 0, \ x \text{ is not a member of } A \end{array}\right\} \quad \textbf{Eq.(1)}$$

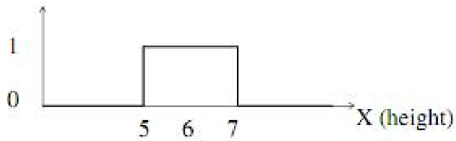
Alternatively, the set **A** can be represented for all elements $x \in X$ by its characteristic function $\mu_A(x)$ defined as

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{otherwise} \end{cases} \qquad \textbf{Eq.(2)}$$

Member function of Crisp Set

Crisp set: membership of element X of set A is defined by

$$\mu_A(x) = \begin{cases} 0, \text{ if } x \notin A, \\ 1, \text{ if } x \in A. \end{cases}$$



Example: Set of heights from 5 to 7 feet

Fuzzy sets

Crisp set theory is not capable of representing descriptions and classifications in many cases; In fact, Crisp set does not provide adequate representation for most cases.

The proposition of Fuzzy Sets are motivated by the need to capture and represent real world data with uncertainty due to imprecise measurement.

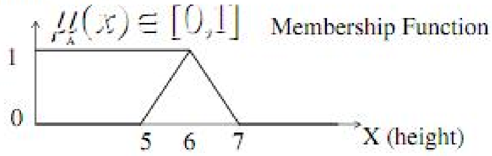
Fuzzy sets

- $\mu_A(x)$ is also called as Characteristic Function of the set 'A'
- For crisp sets this can have only two values
  0 OR 1
- Real world problems can not be characterised by such function eg tall person , worm temperature
- The characteristic function can be generalised
  Which will be called as 'membership function'

Fuzzy sets

Fuzzy set: Contain objects that satisfy imprecise properties of membership

Example : The set of heights in the region around 6 feet



Fuzzy sets

Name	Height, cm	Degree of Membership	
		Crisp	*Fuzzy*
Chris	208	1	1.00
Mark	205	1	1.00
John	198	1	0.98
Tom	181	1	0.82
David	179	0	0.78
Mike	172	0	0.24
Bob	167	0	0.15
Steven	158	0	0.06
Bill	155	0	0.01
Peter	152	0	0.00

Fuzzy sets

- Many decision-making and problem-solving tasks are too complex to be understood quantitatively, however, **people succeed by using knowledge that is imprecise rather than precise.**
- Fuzzy set theory resembles human reasoning in its use of approximate information and uncertainty to generate decisions.
- It was specifically designed to mathematically represent uncertainty and vagueness and provide formalized tools for dealing with the imprecision intrinsic to many problems.

Fuzzy sets

- Since knowledge can be expressed in a more natural way by using fuzzy sets, many engineering and decision problems can be greatly simplified.

Fuzzy sets

A fuzzy set **A** is written as a set of pairs **{x, A(x)}** as

A = {{x , A(x)}} , x in the set X

where **x** is an element of the universal space **X**, and

A(x) is the value of the function **A** for this element.

The value **A(x)** is the **membership grade** of the element **x** in a fuzzy set **A**.

Define a fuzzy set " small " for numbers up to 12.

Fuzzy sets

Example : Set **SMALL** in set **X** consisting of natural numbers ≤ **to 12**.

Assume: SMALL(1) = 1, SMALL(2) = 1, SMALL(3) = 0.9, SMALL(4) = 0.6,
SMALL(5) = 0.4, SMALL(6) = 0.3, SMALL(7) = 0.2, SMALL(8) = 0.1,
SMALL(u) = 0 for u >= 9.

Then, following the notations described in the definition above :

Set SMALL = {{1, 1 }, {2, 1 }, {3, 0.9}, {4, 0.6}, {5, 0.4}, {6, 0.3}, {7, 0.2}, {8, 0.1}, {9, 0 }, {10, 0 }, {11, 0}, {12, 0}}

Note that a fuzzy set can be defined precisely by associating with each **x** , its grade of membership in **SMALL**.

Graphical interpretation of fuzzy set small

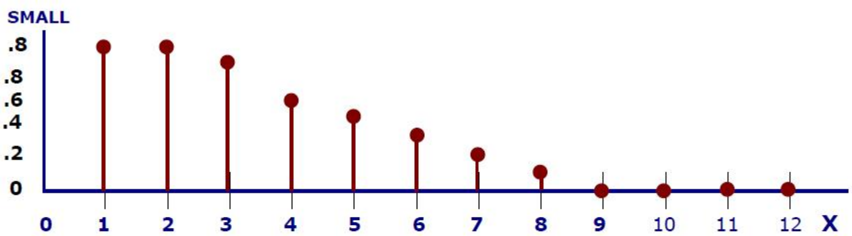


Fig Graphic Interpretation of Fuzzy Sets SMALL

Fuzzy sets

- Non-Crisp Representation to represent the notion of a tall person.

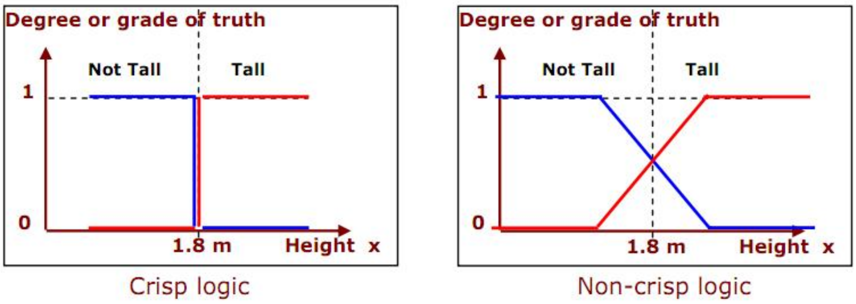


Fig. 1 Set Representation – Degree or grade of truth

Representation of fuzzy sets

$$A = \mu_1/x_1 + \mu_2/x_2 + \mu_3/x_3 + \ldots$$

The symbol / here does not denote division, nor does the symbol + denote summation. The summation symbol is used to connect the terms and thus it means a union of single-term subsets.

Example

Let the values of temperature in °C under consideration be

$T = \{0, 5, 10, 15, 20, 25, 30, 35, 40\}$.

Then, the term *hot* can be defined by a fuzzy set as follows

HOT = {(0,0), (5,0.1), (10,0.3), (15,0.5), (20,0.6), (25,0.7), (30,0.8), (35,0.9), (40,1.0)}.

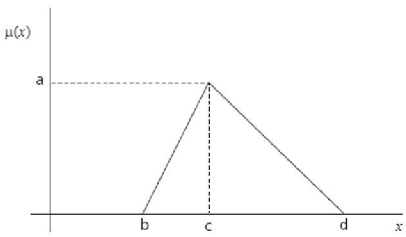
This fuzzy set reflects the point of view that 0 °C is not hot at all, 5, 10, and 15 °C are somewhat hot, and 40 °C is indeed hot. Another person could have defined the set differently.

Example

Let $X = \{x_1, x_2, x_3, x_4\}$. One can define a fuzzy set as:

$A = 0.8/x_1 + 0.4/x_2 + 0.1/x_3 + 0.9/x_4$.

Triangular MF

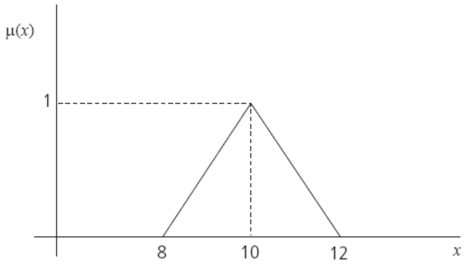


$\mu(x) = a\,(b-x)/(b-c)$; b x c

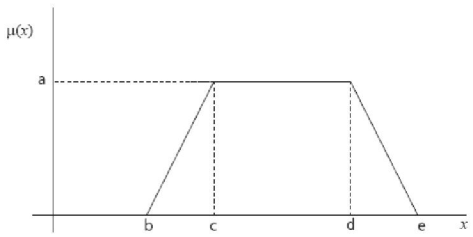
$= a\,(d-x)/(d-c)$; c x d

$= 0$; otherwise

Vague expression Around ten



Trapezoidal MF



$\mu(x)$ = a (b-x)/(b-c) ; b x c

= a ; c x d

= a (e-x)/(e-d) ; d x e

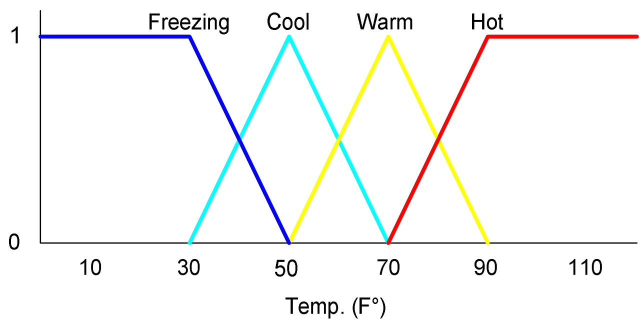
= 0 ; otherwise

Fuzzy Logic to represent Linguistic variables

- Fuzzy logic:
  - A way to represent variation or imprecision in logic
  - A way to make use of natural language in logic
  - Approximate reasoning
- Humans say things like "If it is sunny and warm today, I will drive fast"
- Linguistic variables:
  - Temp: {freezing, cool, warm, hot}
  - Cloud Cover: {overcast, partly cloudy, sunny}
  - Speed: {slow, fast}

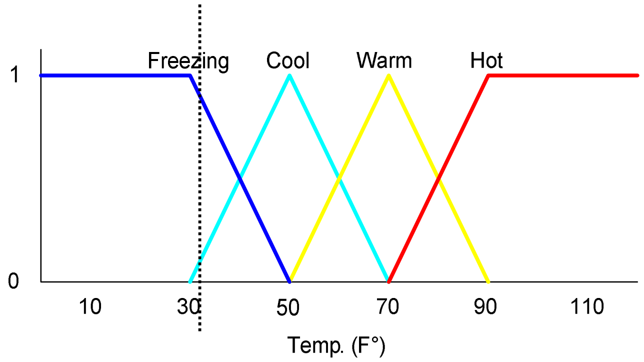
Membership Functions

- Temp: {Freezing, Cool, Warm, Hot}
- Degree of Truth or "Membership"



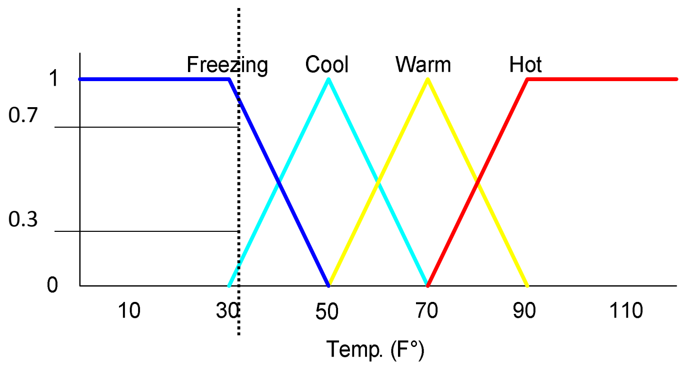
Membership Functions

- How cool is 36 F° ?



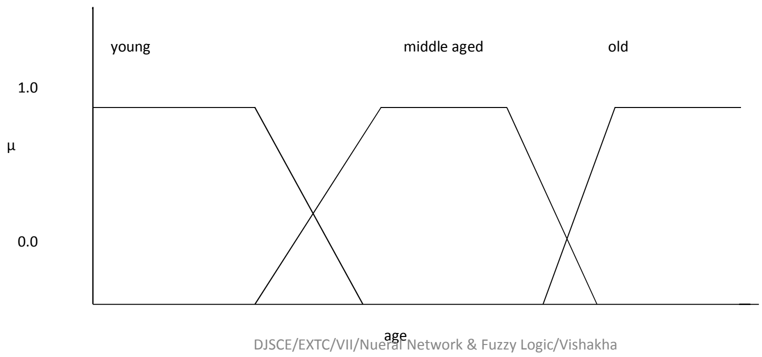
Membership Functions

- How cool is 36 F° ?
- It is 30% Cool and 70% Freezing



Membership Functions

Define membership functions “Young”, ”Middle aged” and “Old”



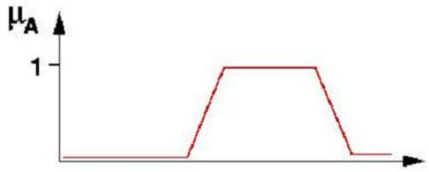
Fussy set operations

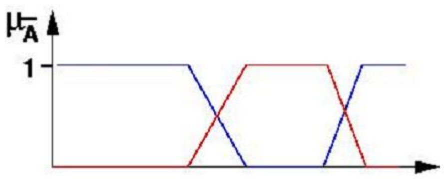
Compliment of A

COMPLEMENT

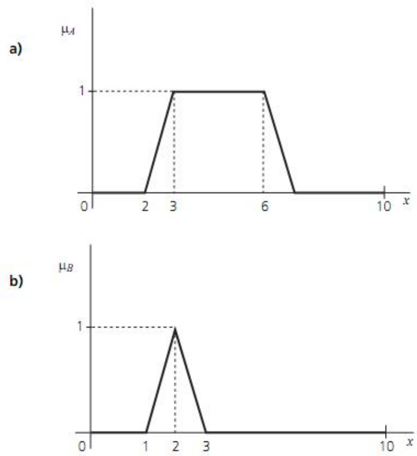
The absolute complement of a fuzzy set A is denoted by $\overline{A}$ and its membership function is defined by

$$\mu_{\overline{A}}(x) = 1 - \mu_A(x) \text{ for all } x \quad X$$





Fussy set operations

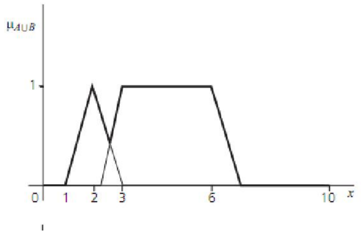


Fuzzy union

UNION

The union of two fuzzy sets A and B is a fuzzy set whose membership function is defined by

$$\mu_{A\cup B}(x) = \max\left[\mu_A(x), \mu_B(x)\right]$$

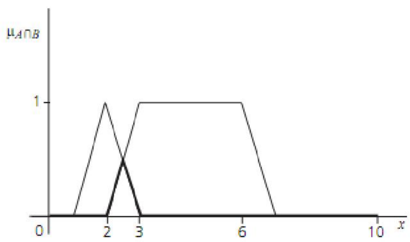


Fuzzy Intersection

INTERSECTION

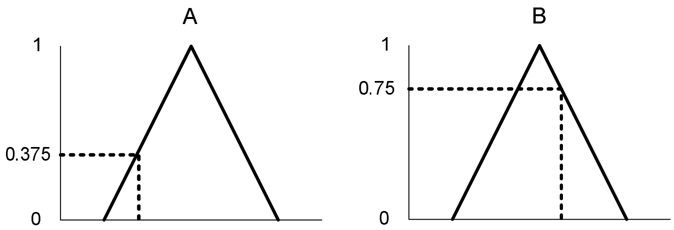
The intersection of two fuzzy sets A and B is a fuzzy set whose membership function is defined by

$$\mu_{A\cap B}(x) = \min[\mu_A(x), \mu_B(x)].$$



Fuzzy Union

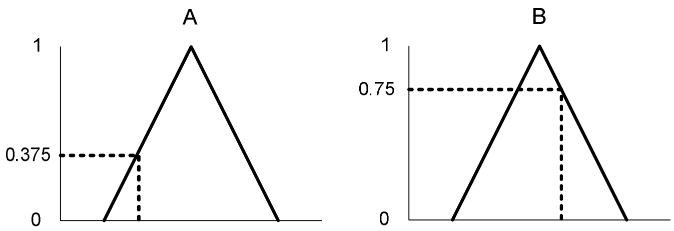
- $A \cup B \triangleq \max(A, B)$
- $A \cup B = C$ "Quality C is the union of Quality A and B"



$(A \vee B = C) \Rightarrow (C = 0.75)$

Fuzzy Conjunction

- $A \wedge B \triangleq \min(A, B)$
- $A \wedge B = C$ "Quality C is the conjunction of Quality A and B"



$(A \wedge B = C) \Rightarrow (C = 0.375)$

Properties of Fuzzy Sets

- Equality of two fuzzy sets
- Inclusion of one set into another fuzzy set
- Cardinality of a fuzzy set
- An empty fuzzy set
- α-cuts (alpha-cuts)

Properties of Fuzzy Sets

Commutativity

$$\underset{\sim}{A} \cup \underset{\sim}{B} = \underset{\sim}{B} \cup \underset{\sim}{A},$$
$$\underset{\sim}{A} \cap \underset{\sim}{B} = \underset{\sim}{B} \cap \underset{\sim}{A}.$$

Associativity

$$\underset{\sim}{A} \cup (\underset{\sim}{B} \cup \underset{\sim}{C}) = (\underset{\sim}{A} \cup \underset{\sim}{B}) \cup \underset{\sim}{C},$$
$$\underset{\sim}{A} \cap (\underset{\sim}{B} \cap \underset{\sim}{C}) = (\underset{\sim}{A} \cap \underset{\sim}{B}) \cap \underset{\sim}{C}.$$

Distributivity

$$\underset{\sim}{A} \cup (\underset{\sim}{B} \cap \underset{\sim}{C}) = (\underset{\sim}{A} \cup \underset{\sim}{B}) \cap (\underset{\sim}{A} \cup \underset{\sim}{C}),$$
$$\underset{\sim}{A} \cap (\underset{\sim}{B} \cup \underset{\sim}{C}) = (\underset{\sim}{A} \cap \underset{\sim}{B}) \cup (\underset{\sim}{A} \cap \underset{\sim}{C}).$$

Properties of Fuzzy Sets

Idempotency

$$\underset{\sim}{A} \cup \underset{\sim}{A} = \underset{\sim}{A},$$
$$\underset{\sim}{A} \cap \underset{\sim}{A} = \underset{\sim}{A}.$$

Identity

$$\underset{\sim}{A} \cup \phi = \underset{\sim}{A} \quad \text{and} \quad \underset{\sim}{A} \cap X = \underset{\sim}{A},$$
$$\underset{\sim}{A} \cap \phi = \phi \quad \text{and} \quad \underset{\sim}{A} \cup X = X.$$

Transtivity

$$\text{If } \underset{\sim}{A} \subset \underset{\sim}{B} \subset \underset{\sim}{C} \text{ then } \underset{\sim}{A} \subset \underset{\sim}{C}$$

Involution

$$\overline{\overline{\underset{\sim}{A}}} = \underset{\sim}{A}.$$

Equality

- Fuzzy set *A* is considered equal to a fuzzy set *B*, IF AND ONLY IF (iff):

  $\mu_A(x) = \mu_B(x), \forall x \in X$

  $A = 0.3/1 + 0.5/2 + 1/3$

  $B = 0.3/1 + 0.5/2 + 1/3$

  therefore $A = B$

Inclusion

- Inclusion of one fuzzy set into another fuzzy set. Fuzzy set $A \subseteq X$ is included in (is a subset of) another fuzzy set, $B \subseteq X$:

  $\mu_A(x) \leq \mu_B(x), \; \forall x \in X$

  Consider $X = \{1, 2, 3\}$ and sets A and B

  $A = 0.3/1 + 0.5/2 + 1/3;$

  $B = 0.5/1 + 0.55/2 + 1/3$

  then A is a subset of B, or $A \subseteq B$

Cardinality

- Cardinality of a non-fuzzy set, Z, is the number of elements in Z. BUT the cardinality of a fuzzy set A, the so-called SIGMA COUNT, is expressed as a SUM of the values of the membership function of A, $\mu_A(x)$:

  $card_A = \mu_A(x_1) + \mu_A(x_2) + \ldots \mu_A(x_n) = \Sigma \mu_A(x_i),$ for i=1..n

  Consider $X = \{1, 2, 3\}$ and sets A and B

  $A = 0.3/1 + 0.5/2 + 1/3;$
  $B = 0.5/1 + 0.55/2 + 1/3$

  $card_A = 1.8$
  $card_B = 2.05$

Empty Fuzzy Set

- A fuzzy set A is empty, IF AND ONLY IF:

  $\mu_A(x) = 0, \forall x \in X$

  Consider $X = \{1, 2, 3\}$ and set A

  $A = 0/1 + 0/2 + 0/3$

  then A is *empty*

Alpha-cut

- An α-cut or α-level set of a fuzzy set $A \subseteq X$ is an ORDINARY SET $A_\alpha \subseteq X$, such that:

  $A_\alpha = \{\mu_A(x) \geq \alpha, \forall x \in X\}$.

  Consider $X = \{1, 2, 3\}$ and set A

  $A = 0.3/1 + 0.5/2 + 1/3$

  then $A_{0.5} = \{2, 3\}$,

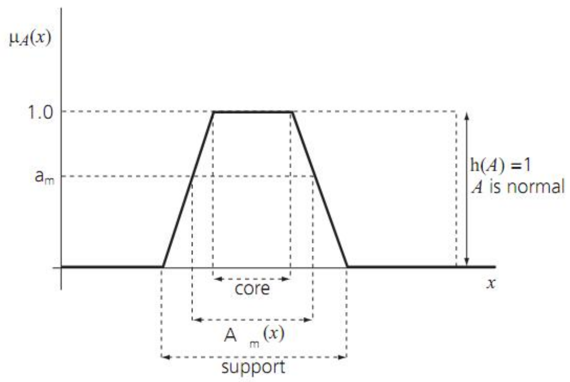
  $A_{0.1} = \{1, 2, 3\}$,

  $A_1 = \{3\}$

Fuzzy Sets Core and Support

- Assume *A* is a fuzzy subset of *X*:
- the **support** of *A* is the crisp subset of *X* consisting of all elements with membership grade:
  $supp(A) = \{x \mid \mu_A(x) > 0 \text{ and } x \in X\}$
- the **core** of *A* is the crisp subset of *X* consisting of all elements with membership grade:
  $core(A) = \{x \mid \mu_A(x) = 1 \text{ and } x \in X\}$

Features of Fuzzy Membership Function Core and Support



Fuzzy Set Math Operations

- $aA = \{a\mu_A(x), \forall x \in X\}$
  Let a =0.5, and
  A = {0.5/a, 0.3/b, 0.2/c, 1/d}
  then
  aA= {0.25/a, 0.15/b, 0.1/c, 0.5/d}

- $A^a = \{\mu_A(x)^a, \forall x \in X\}$
  Let a =2, and
  A = {0.5/a, 0.3/b, 0.2/c, 1/d}
  then
  A^a = {0.25/a, 0.09/b, 0.04/c, 1/d}

- …

Fuzzy Sets Examples

- Consider two fuzzy subsets of the set X,
  X = {a, b, c, d, e }

  referred to as A and B

  A = {1/a, 0.3/b, 0.2/c 0.8/d, 0/e}
  and
  B = {0.6/a, 0.9/b, 0.1/c, 0.3/d, 0.2/e}

Find support ,core , cardinality of A and B.
Also Find complement A , $A \cup B$ and $A \cap B$

Fuzzy Sets Examples

- Support:
  $supp(A) = \{a, b, c, d\}$
  $supp(B) = \{a, b, c, d, e\}$

- Core:
  $core(A) = \{a\}$
  $core(B) = \{\ \}$

- Cardinality:
  $card(A) = 1+0.3+0.2+0.8+0 = 2.3$
  $card(B) = 0.6+0.9+0.1+0.3+0.2 = 2.1$

Fuzzy Sets Examples

- Complement:
  $A = \{1/a, 0.3/b, 0.2/c\ 0.8/d, 0/e\}$
  $\neg A = \{0/a, 0.7/b, 0.8/c\ 0.2/d, 1/e\}$

- Union:
  $A \cup B = \{1/a, 0.9/b, 0.2/c, 0.8/d, 0.2/e\}$

- Intersection:
  $A \cap B = \{0.6/a, 0.3/b, 0.1/c, 0.3/d, 0/e\}$

Fuzzy Sets Examples

- *αA*:
  for *α*=0.5
  αA = {0.5/a, 0.15/b, 0.1/c, 0.4/d, 0/e}

- *A^α*:
  for *α*=2
  A^{α} = {1/a, 0.09/b, 0.04/c, 0.64/d, 0/e}

- *α*-cut:
  $A_{0.2}$ = {a, b, c, d}
  $A_{0.3}$ = {a, b, d}
  $A_{0.8}$ = {a, d}
  A_1 = {a}

1. Consider two fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$ find Complement, Union, Intersection, Difference,

$$\underset{\sim}{A} = \left\{ \frac{1}{2} + \frac{0.5}{3} + \frac{0.6}{4} + \frac{0.2}{5} + \frac{0.6}{6} \right\},$$
$$\underset{\sim}{B} = \left\{ \frac{0.5}{2} + \frac{0.8}{3} + \frac{0.4}{4} + \frac{0.7}{5} + \frac{0.3}{6} \right\}.$$

2. The membership functions for the two sensors in standard discrete form are

$$\underset{\sim}{S_1} = \left\{ \frac{0.5}{20} + \frac{0.65}{40} + \frac{0.85}{60} + \frac{1}{80} + \frac{1}{100} \right\}$$
$$\underset{\sim}{S_2} = \left\{ \frac{0.35}{20} + \frac{0.5}{40} + \frac{0.75}{60} + \frac{0.90}{80} + \frac{1}{100} \right\}$$

De Morgan's law.

Given

$$\underset{\sim}{P} = \left\{ \frac{0.1}{x_1} + \frac{0.2}{x_2} + \frac{0.7}{x_3} + \frac{0.5}{x_4} + \frac{0.4}{x_5} \right\},$$

$$\underset{\sim}{Q} = \left\{ \frac{0.9}{x_1} + \frac{0.6}{x_2} + \frac{0.3}{x_3} + \frac{0.2}{x_4} + \frac{0.8}{x_5} \right\}.$$

Find the following λ cut sets

(a) $\left(\overline{\underset{\sim}{P}}\right)_{0.2}$ (b) $(\underset{\sim}{Q})_{0.3}$ (c) $\left(\underset{\sim}{P} \cup \underset{\sim}{Q}\right)_{0.5}$ (d) $\left(\underset{\sim}{P} \cap \underset{\sim}{Q}\right)_{0.4}$

(e) $\left(\underset{\sim}{Q} \cup \overline{\underset{\sim}{P}}\right)_{0.8}$ (f) $\left(\underset{\sim}{P} \cup \overline{\underset{\sim}{P}}\right)_{0.2}$.

Fuzzy Relations

- **Cartesian Product of two Crisp Sets**

  Let **A** and **B** be two crisp sets in the universe of discourse **X** and **Y**..

  The Cartesian product of **A** and **B** is denoted by **A x B**

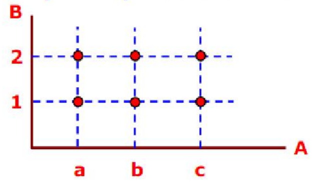
  Defined as $A \times B = \{ (a, b) \mid a \in A, \ b \in B \}$

Example :

Let $A = \{a, b, c\}$ and $B = \{1, 2\}$

then $A \times B = \{ (a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2) \}$

Graphic representation of A x B



- **Cartesian product of two Fuzzy Sets**

  Let **A** and **B** be two fuzzy sets in the universe of discourse **X** and **Y**.

  The Cartesian product of **A** and **B** is denoted by **A x B**

  Defined by their membership function $\mu_A(x)$ and $\mu_B(y)$ as

$$\mu_{A \times B}(x, y) = \min [\mu_A(x) , \mu_B(y)] = \mu_A(x) \wedge \mu_B(y)$$

or $\mu_{A \times B}(x, y) = \mu_A(x) \, \mu_B(y)$

for all $x \in X$ and $y \in Y$

Thus the Cartesian product **A x B** is a fuzzy set of ordered pair (x, y) for all $x \in X$ and $y \in Y$, with grade membership of (x, y) in $X \times Y$ given by the above equations .

In a sense Cartesian product of two Fuzzy sets is a Fuzzy Relation.

Fuzzy relation definition

Consider a Cartesian product

$A \times B = \{ (x, y) \mid x \in A, y \in B \}$

where **A** and **B** are subsets of universal sets U_1 and U_2.

Fuzzy relation on **A x B** is denoted by **R** or **R(x , y)** is defined as the set

$R = \{ ((x, y), \mu_R(x, y)) \mid (x, y) \in A \times B, \mu_R(x, y) \in [0,1] \}$

where $\mu_R(x, y)$ is a function in two variables called membership function.

- It gives the degree of membership of the ordered pair **(x , y)** in **R** associating with each pair **(x , y)** in **A x B** a real number in the interval **[0 , 1]**.
- The degree of membership indicates the degree to which **x** is in relation to **y**.

Example of Fuzzy Relation

$R = \{ ((x_1, y_1), 0)), ((x_1, y_2), 0.1)), ((x_1, y_3), 0.2)),$
$((x_2, y_1), 0.7)), ((x_2, y_2), 0.2)), ((x_2, y_3), 0.3)),$
$((x_3, y_1), 1)), ((x_3, y_2), 0.6)), ((x_3, y_3), 0.2)),$

The relation can be written in matrix form as

$R \triangleq$

y / x	y_1	Y_2	Y_3
x_1	0	0.1	0.2
X_2	0.7	0.2	0.3
X_3	1	0.6	0.2

Composition of Fussy sets

The operation composition combines the fuzzy relations in different variables, say **(x , y)** and **(y , z)** ; $x \in A$, $y \in B$, $z \in C$.

Consider the relations :

$R_1(x , y) = \{ ((x , y) , \mu_{R1}(x , y)) \mid (x , y) \in A \times B \}$

$R_2(y , z) = \{ ((y , y) , \mu_{R1}(y , z)) \mid (y , z) \in B \times C \}$

The domain of R_1 is $A \times B$ and the domain of R_2 is $B \times C$

Composition

Max-Min Composition

Definition : The Max-Min composition denoted by $R_1 \circ R_2$ with membership function $\mu_{R1 \circ R2}$ defined as

$$R_1 \circ R_2 = \{ ((x , z) , \max_{Y}(\min(\mu_{R1}(x , y) , \mu_{R2}(y , z))))\} , \quad (x , z) \in A \times C , \; y \in B$$

Thus $R_1 \circ R_2$ is relation in the domain $A \times C$

Composition

Example : Max-Min Composition

Consider the relations $R_1(x , y)$ and $R_2(y , z)$ as given below.

$R_1 \triangleq$

y / x	y_1	y_2	y_3
x_1	0.1	0.3	0
x_2	0.8	1	0.3

$R_2 \triangleq$

z / y	z_1	z_2	z_3
y_1	0.8	0.2	0
y_2	0.2	1	0.6
y_3	0.5	0	0.4

Note : Number of columns in the first table and second table are equal.

Compute max-min composition denoted by $R_1 \circ R_2$:

Step -1 Compute **min** operation

Consider row x_1 and column z_1 , means the pair (x_1 , z_1) for all y_j , $j = 1, 2, 3$, and perform **min** operation

$\min (\mu_{R1} (x_1 , y_1) , \mu_{R2} (y_1 , z_1)) = \min (0.1, 0.8) = 0.1,$

$\min (\mu_{R1} (x_1 , y_2) , \mu_{R2} (y_2 , z_1)) = \min (0.3, 0.2) = 0.2,$

$\min (\mu_{R1} (x_1 , y_3) , \mu_{R2} (y_3 , z_1)) = \min (0, 0.5) = 0,$

Step -2 Compute **max** operation (definition in previous slide).

For $x = x_1$, $z = z_1$, $y = y_j$, $j = 1, 2, 3,$

Calculate the grade membership of the pair (x_1 , z_1) as

$\{ (x_1 , z_1) , \max ((\min (0.1, 0.8), \min (0.3, 0.2), \min (0, 0.5))$

i.e. $\{ (x_1 , z_1) , \max(0.1, 0.2, 0) \}$

i.e. $\{ (x_1 , z_1) , 0.2 \}$

Hence the grade membership of the pair (x_1 , z_1) is 0.2 .

Similarly, find all the grade membership of the pairs

(x_1, z_2), (x_1, z_3), (x_2, z_1), (x_2, z_2), (x_2, z_3)

The final result is

$R_1 \circ R_2 =$

x \ z	z_1	z_2	z_3
x_1	0.1	0.3	0
x_2	0.8	1	0.3

Consider fuzzy relations:

$$\underset{\sim}{R} = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{cc} y_1 & y_2 \\ \multicolumn{2}{c}{\left[\begin{array}{cc} 0.7 & 0.6 \\ 0.8 & 0.3 \end{array}\right]} \end{array}, \quad \underset{\sim}{S} = \begin{array}{c} \\ y_1 \\ y_2 \end{array} \begin{array}{ccc} z_1 & z_2 & z_2 \\ \multicolumn{3}{c}{\left[\begin{array}{ccc} 0.8 & 0.5 & 0.4 \\ 0.1 & 0.6 & 0.7 \end{array}\right]} \end{array}.$$

Find the relation $T = \underset{\sim}{R} \circ \underset{\sim}{S}$ using max–min and max–product composition.

$$\begin{aligned} T &= \underset{\sim}{R} \circ \underset{\sim}{S} \\ \mu_T(x_1, z_1) &= \max[\min(0.7, 0.8), \min(0.6, 0.1)] \\ &= \max[0.7, 0.1] \\ &= 0.7, \\ \mu_T(x_1, z_2) &= \max[\min(0.7, 0.5), \min(0.6, 0.6)] \\ &= \max[0.5, 0.6] \\ &= 0.6, \end{aligned}$$

$$T = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{ccc} z_1 & z_2 & z_2 \\ \multicolumn{3}{c}{\left[\begin{array}{ccc} 0.7 & 0.6 & 0.7 \\ 0.8 & 0.5 & 0.4 \end{array}\right]} \end{array}.$$

Max–Product Composition

$$\begin{aligned}\mu_T(x_1, z_1) &= \max[\min(0.7 \times 0.8), \min(0.6 \times 0.1)]\\ &= \max[0.56, 0.06]\\ &= 0.56,\\ \mu_T(x_1, z_2) &= \max[\min(0.7 \times 0.5), \min(0.6 \times 0.6)]\\ &= \max[0.35, 0.36]\\ &= 0.36,\\ \mu_T(x_1, z_3) &= \max[\min(0.7 \times 0.4), \min(0.5 \times 0.7)]\\ &= \max[0.28, 0.35]\\ &= 0.35,\end{aligned}$$

$$\underset{\sim}{T} = \begin{bmatrix} 0.56 & 0.36 & 0.35 \\ 0.64 & 0.40 & 0.32 \end{bmatrix}.$$

In the field of computer networking there is an imprecise relationship between the level of use of a network communication bandwidth and the latency experienced in peer-to-peer communication. Let $\underset{\sim}{X}$ be a fuzzy set of use levels (in terms of the percentage of full bandwidth used) and $\underset{\sim}{Y}$ be a fuzzy set of latencies (in milliseconds) with the following membership function:

$$\underset{\sim}{X} = \left\{\frac{02}{10} + \frac{0.5}{20} + \frac{0.8}{40} + \frac{1.0}{60} + \frac{0.6}{80} + \frac{0.1}{100}\right\},$$

$$\underset{\sim}{Y} = \left\{\frac{0.3}{0.5} + \frac{0.6}{1} + \frac{0.9}{1.5} + \frac{1.0}{4} + \frac{0.6}{8} + \frac{0.3}{20}\right\}.$$

(a) Find the Cartesian product represented by the relation $\underset{\sim}{R} = \underset{\sim}{X} \times \underset{\sim}{Y}$.

Now, suppose we have second fuzzy set of bandwidth usage given by

$$\underset{\sim}{X} = \left\{\frac{0.3}{10} + \frac{0.6}{20} + \frac{0.7}{40} + \frac{0.9}{60} + \frac{1}{80} + \frac{0.5}{100}\right\}.$$

(b) Find $\underset{\sim}{S} = \underset{\sim 1\times 6}{Z} \circ \underset{\sim 6\times 6}{R}$ using (1) Max–min composition and (2) Using max–product composition.