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Some spherical gravitational waves in general relativity†

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Einstein's equations for empty space are solved for the class of metrics which admit a family of hypersurface-orthogonal, non-shearing, diverging null curves. Some of these metrics may be considered as representing a simple kind of spherical, outgoing radiation. (Among them are solutions admitting no Killing field whatsoever.) Examples of solutions to the Maxwell–Einstein equations with a similar geometry are also given.

1. Introduction

The aim of this paper is to provide a class of metrics some of which represent a very simple kind of spherical radiation. At great distances from the source, and over not too large regions of space, these metrics seem to be good approximations to actual radiation fields. They have an asymptotic structure similar to that of retarded waves in the linear approximation to Einstein's theory of gravitation. Some of the exact solutions exhibit a feature typical of waves: they depend on an arbitrary function of time. The geometric conditions imposed on our solutions, however, are too stringent to allow for completely realistic fields. In the most interesting cases, besides the point singularity representing the bounded source, there appear other singularities which can be pictured in three dimensions as occurring along lines extending from the origin to infinity.

We can form an intuitive picture of some of the new metrics by considering a special solution of Maxwell's equations in flat space. We write the line-element of Minkowski space in the form

$$\mathrm{d}s^2 = -2\rho^2 p^{-2}\,\mathrm{d}\zeta\,\mathrm{d}\bar{\zeta} + 2\,\mathrm{d}\rho\,\mathrm{d}\sigma + \mathrm{d}\sigma^2,$$

where $p = 1 + \frac{1}{2}\zeta\bar{\zeta}$, and introduce a complex null bivector,§

$$N_{kl} = 2\rho p^{-1}\sigma_{,[k}\zeta_{,l]}.$$

Then, if $A(\zeta, \sigma)$ is any function of ζ and σ analytic in ζ, the real part of

$$F_{kl} = A(\zeta, \sigma)\,\rho^{-1} p N_{kl} \tag{1}$$

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§ Throughout this paper the following conventions are used: Latin indices range and sum over 1, 2, 3, 4; a comma followed by indices denotes ordinary differentiation, a semicolon covariant differentiation; square index-brackets denote antisymmetrization over the indices enclosed.

represents a null electromagnetic field. Its physical components fall off like $1/\rho$ along the rays $\zeta =$ const., $\sigma =$ const., and, unless $A = 0$, they are singular for at least one value of ζ. For example, if A does not depend on ζ, some of the field components tend to infinity for $\zeta \to \infty$. Introducing the polar co-ordinates ϑ, ϕ by $\zeta = \sqrt{2}\,e^{i\phi}\cot\frac{1}{2}\vartheta$, one sees that in this case the field is singular along the straight line $\vartheta = 0$.

The real part of

$$\rho F_{kl} F_{mn}$$

with F_{kl} given by (1) has all the algebraic and differential properties of a linearized Riemann tensor. This null expanding solution of the linearized gravitational equations exhibits also at least one line singularity. Similar singularities occur among those exact null solutions described in this paper which correspond to spherical waves.

The congruence of null rays associated with the field (1) is geodetic, shear-free, hypersurface-orthogonal and diverging. The last two properties of the congruence are special but the first are general: every null electromagnetic or gravitational field defines a shear-free family of null geodesics (Robinson 1959). Conversely, one can associate a null solution of Maxwell's equations with every null congruence that is geodetic and shear-free (Robinson 1961). It has long been known that a gravitational field subject to a reasonable analogue of Sommerfeld's radiation conditions is asymptotically null (Trautman 1958). Asymptotically, therefore, it tends to a metric which admits a null, geodetic, shear-free congruence; and it seems reasonable to demand, in a simplified model, that such a congruence should exist everywhere. We need not fear that this condition would restrict us to null fields, since Sachs (1961) has shown that the condition is satisfied in all solutions with algebraically degenerate curvature tensors.

In the next section is given the canonical form for a metric which admits a shear-free, diverging and hypersurface-orthogonal null vector field σ_k, and which satisfies $R_{ik}\sigma^i\sigma^k = 0$. The remaining field equations are solved and discussed in §3. It is shown that solutions of the type considered here define two families of V_2 and admit a number of local and integral invariants (§4). The algebraic properties of the curvature tensor and its rate of change along rays are discussed in §5. Finally, the paper presents several explicit solutions of the Einstein and of the Maxwell–Einstein equations. Among them are null solutions of Einstein's equations analogous to the electromagnetic waves (1). Our class of fields contains the Schwarzschild metric as well as some static degenerate solutions of Levi-Civita's. We present also more general, type III and II, fields which have no obvious electromagnetic analogues.

2. The line-element

Let V_4 be a four-dimensional Riemann space with signature -2. Consider a family of hypersurfaces, $\sigma(x) =$ constant, such that $\sigma_{,i}$ is null

$$g^{ik}\sigma_{,i}\sigma_{,k} = 0. \tag{2}$$

The curves $x^i = x^i(\rho)$ defined by

$$\partial x^i/\partial \rho = g^{ik}\sigma_{,k} \tag{3}$$

are null geodesics; and ρ is an affine parameter. Each of these rays lies on a hypersurface of constant σ. It is therefore possible to introduce co-ordinates in V_4 such that $x^4 \equiv \sigma$ and the rays are co-ordinate lines of $x^3 \equiv \rho$ (i.e. $x^1 \equiv \xi$, $x^2 \equiv \eta$ and x^4 are constant along each ray). With this choice of co-ordinates, equation (3) can be written as $\delta_3^i = g^{i4}$, so that the contravariant metric tensor becomes

$$g^{ik} = \left(\begin{array}{cc:cc} \multicolumn{2}{c:}{P^2\gamma^{\iota\kappa}} & a & 0 \\ & & b & 0 \\ \hdashline a & b & -c & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \quad (\iota, \kappa = 1, 2), \tag{4}$$

where $\gamma^{\iota\kappa}$ is a negative-definite unimodular matrix.

The family of null hypersurfaces is invariant under a replacement of σ by a function of σ. By virtue of (3), this is accompanied by a change in the affine parameter on each hypersurface. The origin of ρ can be chosen on each ray at will; finally, a transformation of ξ, η into functions of ξ, η and σ is a simple relabelling of the rays. In other words, the form (4) of the metric is invariant under the following transformations ($\dot{\gamma} = \mathrm{d}\gamma/\mathrm{d}\sigma'$):

$$\xi = \alpha(\xi', \eta', \sigma'), \quad \eta = \beta(\xi', \eta', \sigma'), \quad \rho = \rho'/\dot{\gamma}(\sigma') + \delta(\xi', \eta', \sigma'), \quad \sigma = \gamma(\sigma').$$

Let us assume that the congruence of null rays defined by (3) is *shear-free*, so that $\sigma_{,i}$ is subject to

$$\sigma_{;kl}\sigma^{;kl} = \tfrac{1}{2}(\sigma_{;k}{}^{k})^2. \tag{5}$$

Writing this condition with g^{ik} given by (4) one obtains

$$\partial\gamma^{\iota\kappa}/\partial\rho = 0 \quad (\iota, \kappa = 1, 2).$$

Then, by a transformation of ξ and η, one can impose the further restriction

$$\gamma^{\iota\kappa} = -\delta^{\iota\kappa}.$$

The dependence of P on ρ can be obtained by integration of the field equation

$$R^{ik}\sigma_{,i}\sigma_{,k} = 0. \tag{6}$$

Indeed, by virtue of (5) and (6), the rate of change of $\sigma_{;k}{}^{k}$ along the rays is given by

$$\partial(\sigma_{;k}{}^{k})/\partial\rho = -\tfrac{1}{2}(\sigma_{;k}{}^{k})^2,$$

so that either $\sigma_{;k}{}^{k} = 0$, or $\rho - 2/\sigma_{;k}{}^{k}$ is independent of ρ. In the second case, the only one which will be considered here, we get

$$\sigma_{;k}{}^{k} = 2/\rho, \tag{7}$$

by a co-ordinate transformation of the form $\rho + \delta(\xi, \eta, \sigma) \to \rho$. On the other hand, $\sigma_{;k}{}^{k} = -2P^{-1}\partial P/\partial\rho$; so that $P = p/\rho$, where p is a function independent of ρ.

Thus we obtain

$$\mathrm{d}s^2 = -\rho^2 p^{-2}[(\mathrm{d}\xi - a\,\mathrm{d}\sigma)^2 + (\mathrm{d}\eta - b\,\mathrm{d}\sigma)^2] + 2\mathrm{d}\rho\,\mathrm{d}\sigma + c\,\mathrm{d}\sigma^2, \quad \partial p/\partial\rho = 0. \tag{8}$$

It is convenient to introduce a complex variable ζ through $\sqrt{2}\,\zeta = \xi + \mathrm{i}\eta$. The group of co-ordinate transformations which leave invariant the form (8) of the metric is

$$\zeta = \psi(\zeta', \sigma'), \quad \rho = \rho'/\dot{\gamma}(\sigma'), \quad \sigma = \gamma(\sigma'), \tag{9}$$

where ψ is a function analytic in ζ'.

3. The field equations

The role of different groups of field equations is clearly visible when one considers the projections of the Ricci tensor along the vectors of a null complex tetrad associated with the metric (8). Let the tetrad vectors ζ_k, $\bar{\zeta}_k$, τ_k and σ_k be such that

$$g_{ik} = \sigma_i\tau_k + \tau_i\sigma_k - \zeta_i\bar{\zeta}_k - \bar{\zeta}_i\zeta_k,$$

and $\sigma_k = \sigma_{,k}$.

Equation (6) is already satisfied by the form (8); the equations

$$R_{ik}\sigma^i\zeta^k = 0 \quad \text{and} \quad R_{ik}\zeta^i\zeta^k = 0 \tag{10}$$

are equivalent to

$$\partial(a+ib)/\partial\rho = 0 \quad \text{and} \quad \partial(a+ib)/\partial\bar{\zeta} = 0. \tag{11}$$

Therefore $a+ib$ is an analytic function of ζ and can be reduced to zero by a transformation (9). In any case, equations (11) assert that there exists a function q of σ, ξ and η, such that

$$a = \partial q/\partial\eta, \quad b = \partial q/\partial\xi, \tag{12}$$

and

$$\Delta q \equiv p^2(\partial^2/\partial\xi^2 + \partial^2/\partial\eta^2)\, q = 0. \tag{13}$$

If one normalizes the co-ordinates by the condition $q = 0$, then the line-element becomes simply

$$\mathrm{d}s^2 = -\rho^2 p^{-2}(\mathrm{d}\xi^2 + \mathrm{d}\eta^2) + 2\mathrm{d}\rho\,\mathrm{d}\sigma + c\,\mathrm{d}\sigma^2, \quad \partial p/\partial\rho = 0. \tag{14}$$

The group of transformations which preserve this form is given by (9) with ψ independent of σ'.

Next, from

$$R_{ik}\zeta^i\bar{\zeta}^k = 0 \tag{15}$$

it follows that

$$c = -\frac{2m}{\rho} + K - 2H\rho, \tag{16}$$

where

$$H = p^{-1}\frac{\partial p}{\partial\sigma} + \frac{\partial^2 q}{\partial\xi\,\partial\eta} + p^{-1}\frac{\partial p}{\partial\xi}\frac{\partial q}{\partial\eta} + p^{-1}\frac{\partial p}{\partial\eta}\frac{\partial q}{\partial\xi}, \tag{17}$$

$$K = \Delta\ln p, \tag{18}$$

and m is independent of ρ. The equation

$$R_{ik}\sigma^i\tau^k = 0$$

is then satisfied identically; and the two equations

$$R_{ik}\tau^i\zeta^k = 0 \tag{19}$$

tell us that m is a function of σ alone.

Finally, if the previous field equations are satisfied,

$$\rho^2 R_{ik}\tau^i\tau^k = \tfrac{1}{2}\Delta K - 2(\partial/\partial\sigma - 3H)\, m. \tag{20}$$

If m is non-zero then it can be always reduced to unity by means of a co-ordinate transformation (cf. next section). In the frame with q zero, the equation $R_{ik}\tau^i\tau^k = 0$ becomes

$$\partial\ln p/\partial\sigma + \tfrac{1}{12}\Delta\Delta\ln p = 0.$$

In this case, the entire solution is uniquely determined if p is specified on a single hypersurface of constant σ.

If $m = 0$, the last field equation reduces to

$$\Delta K = 0.$$

Here the dependence of p on σ is unrestricted; and we have information carrying waves.

4. Invariants

As we shall see in the next section, the curvature tensor determines the direction of $\sigma_{,k}$ uniquely, except in the case described as D, where $\sigma_{,k}$ points in one of two well-defined directions. With this qualification, σ is geometrically determined up to the transformation $\sigma \to \gamma(\sigma)$, and the tensor $\sigma_{,i}\rho_{,k}+\rho_{,i}\sigma_{,k}+c\sigma_{,i}\sigma_{,k}$ is fully determined. We have thus picked out two families of two-dimensional Riemannian spaces: $V_2(\rho,\sigma)$ with a line-element proportional to $\mathrm{d}l^2 = p^{-2}(\mathrm{d}\xi^2+\mathrm{d}\eta^2)$; and $\tilde{V}_2\,(\xi,\eta)$ with the line-element $\mathrm{d}\tilde{l}^2 = 2\,\mathrm{d}\rho\,\mathrm{d}\sigma+c\,\mathrm{d}\sigma^2$. K/ρ^2 and $2m/\rho^3$ are the Gaussian curvatures of V_2 and $\tilde{V}_2$, respectively.

Under a transformation (9), the quantities appearing in the metric transform according to the equations

$$\left.\begin{aligned} m' &= \dot{\gamma}^3 m, \quad K' = \dot{\gamma}^2 K, \\ H' &= \dot{\gamma}H+\ddot{\gamma}/\dot{\gamma}, \quad p' = \dot{\gamma}p/|\partial\psi/\partial\zeta'|. \end{aligned}\right\} \tag{21}$$

It is easy to construct invariants under these transformations; such are, for instance,

$$m\rho^{-3}, \quad K\rho^{-2}, \quad (3H-\mathrm{d}\ln m/\mathrm{d}\sigma)\,\rho^{-1},$$
$$|l|\,\rho^{-4} \quad \text{and} \quad |n|\,\rho^{-1}, \tag{22}$$

where $$l = 3m\rho^{-3}\frac{\partial}{\partial\zeta}\left(p^2\frac{\partial c}{\partial\zeta}\right)+\rho^{-2}p^2\left(\frac{\partial K}{\partial\zeta}\right)^2, \quad n = \rho^{-2}\frac{\partial}{\partial\zeta}\left(p^2\frac{\partial H}{\partial\zeta}\right). \tag{23}$$

If, moreover, I is an invariant, so are

$$\rho^{-2}\left|\frac{\partial}{\partial\zeta}\left(p^2\frac{\partial I}{\partial\zeta}\right)\right| \quad \text{and} \quad \rho^{-2}\Delta I.$$

One might be tempted to interpret m as the mass of the source, and solutions with $\mathrm{d}m/\mathrm{d}\sigma$ non-zero as representing radiation. However, the first of equations (21) shows that m can be always reduced locally to a constant. Unless there exists an independent way of fixing the retarded time co-ordinate σ, it is impossible to attach any meaning to the dependence of m on σ. On the other hand, $\mathrm{d}m/\mathrm{d}\sigma-3Hm \neq 0$ is an invariant statement, and it may be taken, very tentatively, as a criterion for radiation in the case $m \neq 0$.

In some cases one can also form integral invariants, such as

$$\int_{V_2} Kp^{-2}\,\mathrm{d}\xi\,\mathrm{d}\eta, \quad \int_{V_2} m^{\frac{2}{3}}p^{-2}\,\mathrm{d}\xi\,\mathrm{d}\eta,$$

if these integrals exist. If $R_{ik}\tau^i\tau^k = 0$ and there are no singularities, the latter invariant is a constant of motion. It is a special case of an invariant found for type II spaces by Sachs.

5. The curvature tensor

Let us now further specialize the tetrad vectors by requiring them to be parallelly propagated along rays. With $q = 0$, a possible choice is

$$\zeta_k = \rho p^{-1}\zeta_{,k}, \quad \tau_k = \rho_{,k} + \tfrac{1}{2}c\sigma_{,k}, \quad \sigma_k = \sigma_{,k}.$$

It is convenient to introduce three complex bivectors which are also covariantly constant along rays,

$$N_{kl} = 2\sigma_{[k}\zeta_{l]}, \quad M_{kl} = 2\sigma_{[k}\tau_{l]} + 2\bar{\zeta}_{[k}\zeta_{l]}, \quad L_{kl} = 2\tau_{[k}\bar{\zeta}_{l]}.$$

These bivectors are self-dual, in the sense that

$$\mathrm{i}^*N_{kl} = N_{kl}, \text{ etc.},$$

where
$$^*N_{kl} = \tfrac{1}{2}\sqrt{(-g)}\,\epsilon_{klmn}N^{mn}, \text{ etc.},$$

and ϵ_{klmn} is the Levi–Civita symbol. From the curvature tensor, we can form two duals

$$^*R_{klmn} = \tfrac{1}{2}\sqrt{(-g)}\,\epsilon_{klpq}R^{pq}{}_{mn},$$

$$R^*_{klmn} = \tfrac{1}{2}\sqrt{(-g)}\,\epsilon_{pqmn}R_{kl}{}^{pq},$$

which are equal if and only if $R_{kl} = \frac{1}{4}Rg_{kl}$.

If all the field equations are satisfied, the curvature tensor for our metric can be obtained from the formula

$$S_{klmn} = \frac{2m}{\rho^3}(M_{kl}M_{mn} - N_{kl}L_{mn} - L_{kl}N_{mn}) \\ + \frac{p}{\rho^2}\frac{\partial K}{\partial\zeta}(M_{kl}N_{mn} + N_{kl}M_{mn}) - \frac{1}{\rho^2}\frac{\partial}{\partial\zeta}\left(p^2\frac{\partial c}{\partial\zeta}\right)N_{kl}N_{mn}. \quad (24)$$

Here S_{klmn} is the complex self-dual tensor

$$S_{klmn} = R_{klmn} + \mathrm{i}^*R_{klmn}.$$

Equation (24) shows explicitly that the field is algebraically degenerate and that σ_k defines its propagation direction,†

$$\sigma_{[i}S_{k]lmn}\sigma^l\sigma^m = 0.$$

Goldberg & Sachs (1961) have generalized this result by proving that any empty-space solution admitting a null, geodetic, shear-free congruence is algebraically degenerate.

The Riemann tensor can be split into three parts differing in their rate of change along rays
$$S_{klmn} = \rho^{-3}D_{klmn} + \rho^{-2}\mathrm{III}_{klmn} + \rho^{-1}N_{klmn}, \quad (25)$$

† There are four principal null directions associated with a Riemann tensor in empty space (Ruse 1944; Debever 1959; Penrose 1960). A set of coincident principal null directions is here described as a propagation direction. To be quite precise, we should say that there are two propagation directions in the case described below as D.

where D_{klmn}, III_{klmn} and N_{klmn} are tensors of Petrov's type I degenerate (D), type III, and type II null (N), respectively

$$D_{klmn} = 2m(M_{kl}M_{mn} - N_{kl}L_{mn} - L_{kl}N_{mn}),$$

$$\mathrm{III}_{klmn} = p\frac{\partial K}{\partial \zeta}(M_{kl}N_{mn} + N_{kl}M_{mn}) - \frac{\partial}{\partial \zeta}\left(p^2\frac{\partial K}{\partial \zeta}\right)N_{kl}N_{mn},$$

$$N_{klmn} = 2\frac{\partial}{\partial \zeta}\left(p^2\frac{\partial H}{\partial \zeta}\right)N_{kl}N_{mn}.$$

They are covariantly constant along propagation rays. The decomposition (25) is a stronger form of one given by Sachs (1961): starting from the Bianchi identities, Sachs found for algebraically degenerate fields with hypersurface-orthogonal, diverging rays the slightly weaker law,

$$S_{klmn} = \rho^{-3}\mathrm{II}_{klmn} + \rho^{-2}\mathrm{III}_{klmn} + \rho^{-1}N_{klmn},$$

where II_{klmn} is a tensor of type II or D.

The three terms appearing in (24) are of type D, type III, and type N, respectively; and σ_k is their common propagation vector. It is seen by inspection that a solution is of type II or D, if $m \neq 0$, and of type III, N or flat, if $m = 0$.

In the latter case, type III occurs if $\partial K/\partial \zeta \neq 0$ (i.e. $l \neq 0$), type N occurs if $\partial K/\partial \zeta = 0$ (i.e. $l = 0$) and $n \neq 0$. The space is flat if $m = l = n = 0$.

For $m \neq 0$, the Riemann tensor may be written as the real part of

$$\frac{2m}{\rho^3}(M'_{kl}M'_{mn} - N_{kl}L'_{mn} - L'_{kl}N_{mn}) - \frac{l}{3m\rho^2}N_{kl}N_{mn},$$

where
$$M'_{kl} = M_{kl} + \frac{\rho p}{3m}\frac{\partial K}{\partial \zeta}N_{kl},$$

$$L'_{kl} = L_{kl} - \frac{\rho p}{6m}\frac{\partial K}{\partial \zeta}M_{kl} - \left(\frac{\rho p}{6m}\frac{\partial K}{\partial \zeta}\right)^2 N_{kl}.$$

Therefore
$$m \neq 0 \quad \text{and} \quad l = 0$$

are necessary and sufficient conditions for the field to be of type D.

Spaces of type D with $n = 0$ have the Gaussian curvature K independent of ξ and η; let us call them DS-spaces. The classification of our empty-space metrics can then be summarized in the table

		$l = 0$	
	$l \neq 0$	$n \neq 0$	$n = 0$
$m \neq 0$	II	D	DS
$m = 0$	III	N	0

where 0 stands for the flat space.

6. Special cases and explicit solutions

Examples of explicit solutions to the field equations can be given in several special cases. In general, it is convenient to choose $q = 0$; and this will be done in the following, unless otherwise specified. As has already been mentioned, either m is equal to

0, or it can be reduced to 1 by a co-ordinate transformation. Similarly, if $\partial K/\partial \zeta = 0$, then one can reduce K to one of the values -1, 0, or 1. Finally, the function H can be transformed away if it does not depend on ξ and η. In general, however, these three specializations cannot be achieved at the same time.

(i) Special solutions of type II,

$$m \neq 0, \quad \partial K/\partial \zeta \neq 0, \quad \partial H/\partial \zeta = 0.$$

H can be reduced to zero, and then $\partial p/\partial \sigma = 0$. It follows from the field equations that $p(\xi, \eta)$ must satisfy

$$\Delta K = \text{const.}, \tag{26}$$

and m is given by

$$m(\sigma) = m(0) + \tfrac{1}{4}\sigma \Delta K. \tag{27}$$

The curvature tensor for this case is given by

$$S_{klmn} = \rho^{-3} D_{klmn} + \rho^{-2} \mathrm{III}_{klmn}.$$

As an example, one can mention

$$p = \xi^{\frac{3}{2}}, \quad m = \text{const.} \neq 0.$$

(ii) Special solutions of type III,

$$m = 0, \quad \partial K/\partial \zeta \neq 0, \quad \partial H/\partial \zeta = 0.$$

The field equations reduce to $\Delta K = 0$. The curvature tensor goes like ρ^{-2}. As before, we can make $\partial p/\partial \sigma$ zero. Example: $p = \xi^{\frac{3}{2}}$, $m = 0$.

(iii) DS-spaces,

$$m \neq 0, \quad \partial K/\partial \zeta = 0, \quad n = 0.$$

If one normalizes σ by requiring m to be constant, then the field equations give $\partial p/\partial \sigma = 0$. Therefore K is a constant. The co-ordinates ξ, η may be chosen so that

$$p = 1 + \tfrac{1}{2} K \zeta \bar{\zeta}. \tag{28}$$

The metric for positive K is the Schwarzschild solution with a mass $mK^{-\frac{3}{2}}$. The remaining solutions are due to Levi–Civita. The constant K may be reduced to one of the three values -1, 0, 1. Alternatively, we can put m equal to 1 and retain K as a parameter. The solution with zero K then appears as a limiting case of the Schwarzschild solution, corresponding to infinite mass. We can gain some further insight into this limiting process by constructing a flat background metric for the Schwarzschild solution. Knowing the transformation properties of ρ, σ, and m we can easily verify that the tensor $m\rho^{-1}\sigma_{,k}\sigma_{,l}$ is invariantly defined; and the *background metric*

$$g_{kl} + 2m\rho^{-1}\sigma_{,k}\sigma_{,l}$$

is evidently flat. In the full metric, the subspace $\rho = 0$ is distinguished by the property that one scalar of the Riemann tensor is infinite there. In the background metric, it is a time-like line. This is one of the points of view from which the Schwarzschild solution may be identified as a model of a particle with a time-like path. From the same point of view, the limiting case, $K = 0$, looks like a model of a particle with a null path.

In each of these three cases, the Riemann tensor is proportional to ρ^{-3}.

(iv) A solution of Levi-Civita's, rediscovered by Newman & Tamburino (1961), may be written as

$$q = \xi, \quad m = 1, \quad p^{-2} = \mathrm{d}x/\mathrm{d}\eta,$$

where x is a function of η such that

$$\mathrm{d}x/\mathrm{d}\eta = -2x^3 + \mu x^2 + \nu x,$$

μ and ν being constants. This is a metric of type D. The subspace curvature K is variable,

$$K = 6x - \mu;$$

and ΔK is non-zero.

The Riemann tensor contains the ρ^{-1} term which seems characteristic of radiation. The metric, however, admits a time-like hypersurface-orthogonal Killing field. The solution might therefore be described as both static and radiative.

(v) Null solutions,

$$m = 0, \quad \partial K/\partial \zeta = 0, \quad n \neq 0.$$

Again K is locally reducible to -1, 0, or 1. Allowing q to be different from 0, one can put p in the form (28); and the only field equation is

$$\Delta q = 0.$$

The Riemann tensor is given by the real part of

$$2n\rho^{-1}N_{kl}N_{mn},$$

and, therefore, falls off like ρ^{-1}. These solutions and the plane gravitational waves have the same local geometry, since their curvature tensors are algebraically indistinguishable. The plane-fronted waves, characterized by the existence of a covariantly constant null vector, can be obtained from the diverging null solutions by a limiting process (Robinson & Trautman 1960).

The dependence of q on σ is completely arbitrary, a property which is typical of waves. The wave fronts are subspaces of constant ρ and σ; in the most interesting case, $K = 1$, they are spheres of radius ρ. These solutions represent, therefore a kind of spherical radiation. However, there appears at least one singularity on any wave front where $n \neq 0$. These line singularities are analogous to those occurring in the corresponding solutions in Maxwell's theory and in the linear approximation to Einstein's theory.

It is interesting to note that in the co-ordinate system defined by (8), (12), and (28) all components of the curvature tensor—covariant, contravariant, and mixed—are homogeneously linear in q. The contravariant metric tensor is linear in q. There is thus a very close relation between the rigorous solution and its linear approximation. It is conceivable, nevertheless, that the line singularity has a special significance in the rigorous case: P. G. Bergmann has suggested that it represents a flow of matter which restores to the source the energy carried away by radiation. Something of this sort is to be expected, since there are solutions, such as

$$m = 0, \quad p = 1 + \tfrac{1}{4}(\xi^2 + \eta^2), \quad q = \mathrm{e}^{\xi}\cos(\eta - \sigma),$$

in which there is no secular change which could be identified as loss of energy by the source. This solution, incidentally, does not admit any continuous group of motions.

7. Some solutions of the Einstein–Maxwell equations

Let F_{kl} denote the electromagnetic field bivector plus i times its dual. The combined Einstein and Maxwell equations can be written as

$$2R_{kl} = F_{mk}\bar{F}_l^m + \bar{F}_{mk}F_l^m, \tag{29}$$

$$F_{[kl,m]} = 0. \tag{30}$$

Assume now that F_{kl} admits $\sigma_{,i}$ satisfying (2), (5), and (7) as a principal null vector; i.e.

$$\sigma_{,[i}F_{k]l}\sigma^{,l} = 0. \tag{31}$$

Let ζ_k, $\bar{\zeta}_k$, τ_k, and $\sigma_k = \sigma_{,k}$ be again a null tetrad, then equations (29) and (31) imply that the first 5 empty-space field equations, (6) and (10), must be satisfied. Therefore, one is again led to consider the line-element (14). The most general solution of equation (31) can be written as

$$F_{kl} = e\rho^{-2}M_{kl} + fp\rho^{-1}N_{kl},$$

where e and f are complex functions of all co-ordinates. Maxwell's equations tell us that e does not depend on ρ and is analytic in ζ, $e = e(\zeta, \sigma)$, and that f has the form

$$f = A - \rho^{-1}\partial e/\partial\zeta,$$

where A is a complex function of ξ, η, and σ, subject to

$$\partial A/\partial\bar{\zeta} = \partial(ep^{-2})/\partial\sigma. \tag{32}$$

Contracting equation (29) with $\zeta^k\bar{\zeta}^l$ one obtains

$$c = e\bar{e}\rho^{-2} - 2M\rho^{-1} + K - 2H\rho.$$

where $\quad \partial M/\partial\rho = 0, \quad K = \Delta \ln p \quad \text{and} \quad H = p^{-1}\partial p/\partial\sigma.$

We shall not attempt here to solve the remaining field equations in the general case, but rather give explicit solutions in two special cases.

(i) *Generalized Reissner solutions*

If $A = 0$ then it can easily be shown from equation (32) that e can be reduced to a complex constant. The remaining equations then imply that p is a function of ξ and η subject to the condition (26) and $M = m(\sigma)$ is given by (27). The case when $K = 1$ and e is real corresponds to the Reissner solution for an electron.

(ii) *Solutions with a null electromagnetic field*

If $e = 0$, then $A = A(\zeta, \sigma)$ must be analytic in ζ, and $M = m(\sigma)$. The only further condition to be satisfied is

$$\tfrac{1}{2}\Delta K - 2(\partial/\partial\sigma - 3H)m = p^2A\bar{A}.$$

The analyticity of A implies $\Delta \ln |A| = 0$, and thus imposes an integrability condition on the left-hand side of this equation. In the case $K = 0$, for example, it follows from the integrability condition that H is independent of ξ and η and, consequently, is reducible to zero. In this co-ordinate system p is time-independent, and, since $K = 0$, can be chosen equal to 1. Therefore A is a function of σ only and

$$m(\sigma) = m(0) - \frac{1}{2}\int_0^\sigma |A(\lambda)|^2 \,\mathrm{d}\lambda.$$

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