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Bipolar aggregation using the Uninorms

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Abstract In bipolar aggregation the total score depends not just on previous score and the value of additional argument but on distribution of all other arguments as well. In addition the process of bipolar aggregation is not Markovian, i.e. aggregation is not associative. To model bipolar aggregation was introduced general R_G^* aggregation based on uninorms. By discarding associativity we built a variation of the uninorm using generating functions that can be applied as an intuitively appealing bipolar aggregation operator. This modified uninorm operator will allow us to control the aggregation depending on distribution of the arguments above and below the neutral element: the closer proportion of arguments below the neutral value to 1 or to 0 the closer bipolar aggregation is to some t-norm or t-conorm with desirable properties.

Keywords Bipolar · Uninorm · Fuzzy sets · Decision making

1 Introduction

In recent years applications of bipolar scales for decision-making have become widespread (Dubois and Prade 2008b). There is psychological evidence (Osgood et al. 1957) that scores which humans use to make judgments, lie on a bipolar scale, i.e. a scale that distinguishes between good (satisfactory) scores and bad (unsatisfactory) scores. Our objective here is to provide a class of very flexible aggregation operators that have neutral value serving as frontier between these two types of scores. These operators are based on uninorms (Yager and Rybalov 1996) that can be used to implement bipolar aggregation of values drawn from the unit interval.

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The key feature of bipolar aggregation is the differing effects of arguments above and below neutral element on the aggregated value. In bipolar aggregation the addition of an argument value above the neutral value can only serve to increase the current aggregated value while the addition of an argument value below the neutral value can only serve to decrease the current aggregated value. Suppose we are trying to locate a criminal who is described as tall man having long blond hair. If we see a person who is $6'3$ this will tend to support that we have a right person. However if this person is bald this information will tend to support that this is not a right person.

As bipolar aggregation is monotonic (the larger an argument the larger the aggregation score) and symmetric (the result of aggregation should not depend on order of arguments), operators describing it should possess these properties (Dubois and Prade 2008a; Dubois et al. 2008; Grabisch 2006; Saminger et al. 2006; Sicilia and Garcia 2004). One class of operators having all these features are uninorms (Yager and Rybalov 1996; DeBaets and Fodor 1999; De Baets 1998; Fodor et al. 1997; De Baets 1999).

Definition A **uninorm** $\boldsymbol{U}$ is a mapping $U : [0, 1] \times [0, 1] \to [0.1]$ having the following properties (Yager and Rybalov 1996):

1. $U(a, b) = U(b, a)$ (symmetry)
2. $U(a, b) \geq U(c, d)$ if $a \geq c$ and $b \geq d$ (monotonicity)
3. $U(a, U(b, c)) = U(U(a, b), c)$ (associativity)
4. There exists element $\boldsymbol{e}$ in [0,1] called the **identity** element such that for any *a in* [0, 1]: $U(a,\boldsymbol{e}) = a$

In uninorms identity is a proxy for the neutral element. Generally uninorms work by separately aggregating the values above the neutral value using a t-conorm and those below the neutral using a t-norm and then combining these two values. Uninorms work pretty well as bipolar aggregators if all arguments are either larger than neutral element or smaller than neutral element. If all arguments are larger than the neutral element, then aggregation score will be a t-conorm of them and hence not smaller than any of them; if all arguments are less than neutral element, then aggregation score will be a t-norm of them and hence not larger then any of them. If some arguments are larger than the neutral element and some are smaller, then to combine them we have to chose either a disjunctive or conjunctive uninorm i.e. uninorms where this combination is either maximum or minimum operator. This leads to a very brittle type of aggregation and can result in aggregations which, while formally correct, challenge our intuition as to what should be the results of a bipolar aggregation.

Our object here is to modify the pure uninorm to provide a more suitable bipolar aggregation operator. Let's look at the main properties of the uninorm operators in relation to the requirements of bipolar aggregation. It is obvious that monotonicity has to be preserved in bipolar aggregation. This aggregation should be commutative so that the order of arguments doesn't play a role. Also by definition bipolar aggregation should have neutral value. The other property of uninorms is associativity and as we see it is this property that affects the quality of the uninorm as a bipolar aggregation operator. Associativity is used to extend in a consistent way the aggregation process from n to $(n + 1)$ arguments. But precisely because of this it doesn't take into account either number or distribution of the arguments. It works like a Markovian process: to proceed from aggregation of n arguments to aggregation of $(n+1)$ arguments we need only know the result of aggregation of the first n elements and the value of $(n+1)$th argument. In the following we show that by discarding associativity we are able to provide a variation of the uninorm that can be used as an intuitively appealing bipolar aggregation operator. This modified uninorm operator will allow us to introduce sophisticated control over the aggregation in which we can fine-tune the aggregation depending on how many of the arguments are above and below the neutral element. That is while we still separately aggregate the values above and below the neutral element the combination of these two values will be effected by how many arguments are in each of the categories. In addition we introduce an additional level of tuning which enables a choice of the aggregation operator used in each of the two categories. Thus, for example, in aggregating the values above the neutral element while we still use a t-conorm, the choice of t-conorm can be controlled by the distribution of these arguments satisfying the condition.

In the second section of this paper R^* aggregation, introduced in Yager and Rybalov (1996) is used to model bipolar aggregation as weighted sum of t-norm and t-conorm. To achieve greater flexibility to model bipolar aggregation R^* aggregation was extended to R_e^* aggregation operator with identity where t-norms and t-conorms were modeled as uninorms. To get higher level control of bipolar aggregation we define the extension of R_e^* aggregation operator with identity by modeling aggregation of arguments below and above the neutral element by two separate uninorm *with their own identities*. In the third section we give examples how these uninorms can be constructed using generating functions. Properties of uninorms generated by using this approach were established. This model provides very sophisticated control over bipolar aggregation.

2 Bipolar aggregation and aggregation operators

In bipolar aggregation each additional argument reinforces the current score. T-norm and t-conorm aggregation also have this property. T-norms and t-conorms can be considered as uninorms whose identity elements are equal to 1 and 0 respectively:

Definition A **t-norm** $\boldsymbol{T}$ is a mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ having the following properties (Sicilia and Garcia 2004):

1. $T(a, b) = T(b, a)$ (symmetry)
2. $T(a, b) \geq T(c, d)$ if $a \geq c$ and $b \geq d$ (monotonicity)
3. $T(a, T(b, c)) = T(T(a, b), c)$ (associativity)
4. $For\ any\ a\ in\ [0, 1] : T(a, 1) = a$

Definition A **t-conorm** $\boldsymbol{S}$ is a mapping $S : [0, 1] \times [0, 1] \to [0, 1]$ having the following properties (Sicilia and Garcia 2004):

1. $S(a, b) = S(b, a)$ (symmetry)
2. $S(a, b) \geq S(c, d)$ if $a \geq c$ and $b \geq d$ (monotonicity)

3. $S(a, S(b, c)) = S(S(a, b), c)$ (associativity)
4. $For\ any\ a\ in\ [0, 1] : S(a, 0) = a$

In t-norm aggregation each additional argument makes the result of aggregation smaller, whereas in t-conorm aggregation each additional argument makes the result of aggregation larger. These properties allow them to be used for bipolar aggregation. The first step in this direction is R^* aggregation.

In Yager and Rybalov (1996) a class of commutative, monotone operators, that admit an identity but do not require the property of associativity, was introduced. They were called MICA operators. In (1994) Yager showed that MICA operators constitute the basic operators needed for aggregation in fuzzy system modeling. Some of MICA operators can be applied to bipolar aggregation. They are R^* aggregation operators:

Definition The $\boldsymbol{R}^*$ **aggregation** is defined as $R^*(e) = e$; $e \in [0, 1]$ is the neutral element (or identity of R^* aggregation);
Arguments $x_1, \ldots, x_n, y_1, \ldots, y_m$ are ordered:
$x_1, \ldots, x_n < \boldsymbol{e}$;
$y_1, \ldots, y_m > \boldsymbol{e}$;

$$R^*(x_1, \ldots, x_n, y_1, \ldots, y_m, e) = R^*(x_1, \ldots, x_n, y_1, \ldots, y_m)$$

$$R^*(x_1, \ldots, x_n, y_1, \ldots, y_m) = Q\left(\frac{n}{n+m}\right) \cdot T(x_1, \ldots, x_n) + \left[1 - Q\left(\frac{n}{n+m}\right)\right] \cdot S(y_1, \ldots, y_m)$$

$$Q\left(\frac{n}{n+m}\right) = \begin{cases} 1 & \text{if} \quad \frac{n}{n+m} \geq \theta_1 \quad or \quad \frac{n}{n+m} = \theta_1 = \theta_2 \\ \frac{n}{n+m} & \text{if} \quad \theta_2 < \frac{n}{n+m} < \theta_1 \\ 0 & \text{if} \quad \frac{n}{n+m} \leq \theta_2 \quad \text{and} \quad \theta_1 \neq \theta_2 \end{cases}$$

n is number of arguments below neutral element;
m is number of arguments above neutral element;
$\theta_1, \theta_2 \in [0, 1]$ are thresholds, $\theta_1 \geq \theta_2$;
$T(x_1, \ldots, x_n$ is t-norm for $n > 1$, $T(x) = x$;
$S(y_1, \ldots, y_m)$ is t-conorm for $m > 1$, $S(y) = y$.

R^* aggregation allows the neural element (or identity) to be any number in the interval $[0, 1]$. In R^* aggregation only those arguments, that are not equal to the neutral element, affect the result of aggregation. If $\boldsymbol{\theta}_1 = \boldsymbol{\theta}_2 = 0 R^*$ aggregation is equivalent to t-norm aggregation and if $\boldsymbol{\theta}_2 = \boldsymbol{\theta}_1 = 1$ then R^* aggregation is equivalent to t-conorm aggregation. Otherwise R^* aggregation is weighted $\left(\text{by ratio } r = \frac{n}{n+m}\right)$ average of t-norm and t-conorm aggregation, where arguments of t-norm are below the neutral element and arguments of t-conorm are above the neutral element. Thus we are able to control aggregation by assigning required thresholds. If $\boldsymbol{\theta}_1 = \boldsymbol{\theta}_2$ then R^* aggregation will be either t-norm or t-conorm aggregation $\left(\text{depending on ratio } r = \frac{n}{n+m}\right)$ and, therefore, will model bipolar aggregation.

The similar approach was taken by Grabisch et al. (2009) in chapter 9 where separable aggregation functions were employed but this approach has can cause some problems. In this case the aggregation value is calculated as distance (pseudo-difference) between two aggregating functions that operate separately on positive and negative arguments (on $[-1, 1]$ scale) that results in non-monotonicity of aggregation (as pseudo-difference is non-monotonic). This approach produces paradoxical outcomes, for example, in the case where almost all of arguments are close to -1 and just one (or several) argument is close to 1, results in aggregation value that is close to 1.

If the proportion of arguments, which have low rating, is above threshold then we want to perform t-norm aggregation. If, on the other hand, the proportion of arguments with low rating is below threshold, then we want to use t-conorm aggregation. In R^* aggregation t-norm and t-conorm have their identities equal to 1 and 0 respectively. To achieve more flexibility we will use the fact that all arguments in t-norm T are below identity $\boldsymbol{e}$, and all arguments t-conorm S are above $\boldsymbol{e}$. It was proven Fodor et al. (1997) that uninorms with all arguments below identity behave like t-norms and uninorms with all arguments above identity behave like t-conorms. Thus we can construct **aggregation operator with identity** by replacing a t-norm and a t-conorm with a uninorm.

Definition The R_e^* **aggregation operator with identity** is defined as $R_e^*(e) = e$
$e \in [0, 1]$ is the neutral element (or identity of R_e^* aggregation operator with identity);
Arguments $x_1, \ldots, x_n, y_1, \ldots, y_m$ are ordered:
$x_1, \ldots, x_n < \boldsymbol{e}$;
$y_1, \ldots, y_m > \boldsymbol{e}$;
n is number of arguments below neutral element;
m is number of arguments above neutral element.

$$
\begin{aligned}
&R_e^*(x_1, \ldots, x_n, y_1, \ldots, y_m, e) = R_e^*(x_1, \ldots, x_n, y_1, \ldots, y_m) \\
&R_e^*(x_1, \ldots, x_n, y_1, \ldots, y_m) = \frac{n}{n+m} U_e(x_1, \ldots, x_n) + \left(1 - \frac{n}{n+m}\right) \\
&\quad \times U_e(y_1, \ldots, y_m) = \frac{n}{n+m} U_e(x_1, \ldots, x_n) + \frac{m}{n+m} \cdot U_e(y_1, \ldots, y_m)
\end{aligned}
$$

$U_e(x_1, \ldots, x_n)$ is a uninorm with identity $\boldsymbol{e}$ for $n > 1$, $U_e(x) = x$.

This aggregation depends on ratio $r = \frac{n}{n+m}$. To achieve higher level of control of bipolar aggregation we have to do one more step.

We know that uninorms where all arguments are below identity $\boldsymbol{d}$ behave like t-norms on the interval $[0, d]$, and uninorms where all arguments are above identity $\boldsymbol{d}$ behave like t-conorms on the interval $[d, 1]$ (Fodor et al. 1997). These properties suggest the following definition

Definition The **general** R_G^* **aggregation** operator is defined as $R_e^*(e) = e$
$e \in [0, 1]$ is the neutral element (or identity of R_e^* aggregation operator with identity);
Arguments $x_1, \ldots, x_n, y_1, \ldots, y_m$ are ordered:
$x_1, \ldots, x_n < \boldsymbol{e}$;
$y_1, \ldots, y_m > \boldsymbol{e}$;

n is number of arguments below neutral element;
m is number of arguments above neutral element.

$$\begin{aligned} & R_G^*(x_1, \ldots, x_n, y_1, \ldots, y_m) \\ & = \frac{n}{n+m} U_d(x_1, \ldots, x_n) + \left(1 - \frac{n}{n+m}\right) \cdot U_f(y_1, \ldots, y_m) \\ & = \frac{n}{n+m} U_d(x_1, \ldots, x_n) + \frac{m}{n+m} \cdot U_f(y_1, \ldots, y) \end{aligned}$$

$x_1, \ldots, x_n < \boldsymbol{e}$;
$y_1, \ldots, y_m > \boldsymbol{e}$;
$e \in [0, 1]$ is the neutral element (or identity of the general R_G^* aggregation);
n is number of arguments below neutral element;
m is number of arguments above neutral element;
$U_d(x_1, \ldots, x_n)$ is uninorm with identity $d \geq e$ for $n > 1$, $U_d(x) = x$;
$U_f(y_1, \ldots, y_m)$ is uninorm with identity $f \leq e$ for $m > 1$, $U_f(y) = y$.

Theorem 1 *R_G^* aggregation is monotonic, symmetric, and has the identity* $\boldsymbol{e}$*, but it is not associative.*

Proof (Non-associativity is demonstrated in the example 1 in Sect. 3).

(1) Monotonicity

We have to prove that $R_G^*(x_1, \ldots, x_n, y_1, \ldots, y_m) < R_G^*(x_1', \ldots, x_n', y_1', \ldots, y_m')$ for

$$x_1 \leq x_1\prime, \ldots, x_n \leq x_n\prime, y_1 \leq y_1\prime, \ldots, y_m \leq y_m\prime$$

We have either $\frac{n\prime}{n\prime+m\prime} = \frac{n}{n+m}$ or $\frac{n\prime}{n\prime+m\prime} < \frac{n}{n+m}$ with $n' < n$ and $m' > m$

In the first case $n' = n$ and $m' = m$
$U_d(x_1, \ldots, x_n) \leq U_d(x_1', \ldots, x_n')$ (monotonicity of uninorms) and
$U_f(y_1, \ldots, y_m) \leq U_f(y_1', \ldots, y_m')$ Fodor et al. (1997)
Thus, $R_G^*(x_1, \ldots, x_n, y_1, \ldots, y_{m\prime}) \leq R_G^*(x_1', \ldots, x_n', y_1', \ldots, y_{m\prime}')$

In the second case

$U_d(x_1, \ldots, x_n) \leq U_d(x_1', \ldots, x_{n\prime}')$ (associativity of t-norms) and
$U_f(y_1, \ldots, y_m) \leq U_f(y_1', \ldots, y_{m\prime}')$ (associativity of t-conorms) Fodor et al. (1997)

Therefore

$$\frac{n}{n+m} \cdot U_d(x_1, \ldots, x_n) \leq \frac{n\prime}{n\prime + m\prime} \cdot U_d(x_1', \ldots, x_{n\prime}')$$

and

$$\frac{m}{n+m} \cdot U_f(y_1, \ldots, y_m) \leq \frac{m\prime}{n\prime + m\prime} \cdot U_f(y_1', \ldots, y_{m\prime}')$$

Thus

$$R_G^*(x_1, \ldots, x_n, y_1, \ldots, y_m) \le R_G^*(x_1', \ldots, x_{n'}', y_1', \ldots, y_{m'}')$$

(2) Symmetry

Symmetry of general R_G^* aggregation follows from symmetry of uninorms.

(3) Identity is equal to ***e*** follows from definition of $R_G^*(x_1, \ldots, x_n, y_1, \ldots, y_m)$

□

3 Generating functions and general R_G^* aggregation

The properties that general R_G^* aggregation should possess–dependence of strength of t-norms (on the interval $[0, d]$) and t-conorms (on the interval $[d, 1]$) on the closeness of the ratio $r = \frac{n}{n+m}$ to 0 or 1 correspondingly – can be achieved by the careful design of generation function.

To model the general R_G^* aggregation that has desirable properties we have to build corresponding uninorms used in its definition. Uninorms with identity ***d*** can be built using so-called generating functions $h(x)$ [cf. Aczel representation theorem (Aczel 1949; Dombi 1982; Klement et al. 1996)]:

$$U_d(x, y) = h^{-1}(h(x) + h(y))$$

where $h(x)$ is a monotonic continuous function with $h(0) = -\infty;_h (d) = 0;_h (1) = \infty$.

One such generating function that satisfies our requirements is the function $g_d(x) = \frac{x-d}{x\cdot(1-x)}$.

Theorem 2 *The uninorm* $G_d(x, y) = g_d^{-1}(g_d(x) + g_d(y))$, *where* $g_d(x) = \frac{x-d}{x\cdot(1-x)}$ *for* $x, y \in [0, d),_d < 1$ *is bounded from below:* $\inf\limits_d G_d(x, y) = \frac{1}{2}\min(x, y)$.

Proof As uninorm $G_d(x, y)$ in monotone in x, y then lower bound of $G_d(x, y)$ exists for $x, y < d$. Let $x \le y < d$, i.e. $\min(x, y) = x$. Then from monotonicity of the uninorm $G_d(x, y)$ follows that $\min\limits_y G_d(x, y) = G_d(x, x)$.

The formula for g_d^{-1} is $g_d^{-1}(z) = \frac{z-1+\sqrt{1-2\cdot z+z^2+4\cdot z\cdot d}}{2\cdot z}$ and

$$z = 2\frac{x-d}{x\cdot(1-x)} < 0$$

$$G_d(x, x) - \frac{x}{2} = \frac{2\frac{x-d}{x\cdot(1-x)} - 1 + \sqrt{1 - 2\cdot 2\frac{x-d}{x\cdot(1-x)} + \left(2\frac{x-d}{x\cdot(1-x)}\right)^2 + 4\cdot 2\frac{x-d}{x\cdot(1-x)}\cdot d}}{2\cdot 2\frac{x-d}{x\cdot(1-x)}}$$

$$-\frac{x}{2} > \text{ as } \quad \frac{x-d}{x\cdot(1-x)}\cdot d < 0$$

$$> \frac{1}{2} - \frac{x \cdot (1-x)}{4(x-d)} - \frac{x}{2} + \frac{\sqrt{1 + 2 \cdot 2\frac{x-d}{x\cdot(1-x)} + \left(2\frac{x-d}{x\cdot(1-x)}\right)^2}}{4\frac{x-d}{x\cdot(1-x)}}$$

$$\geq \frac{1}{2} - \frac{x \cdot (1-x)}{4(x-d)} - \frac{x}{2} + \frac{-1 - 2\frac{x-d}{x\cdot(1-x)}}{4\frac{x-d}{x\cdot(1-x)}} = \frac{x \cdot (1-x)}{2(d-x)} - \frac{x}{2} = \frac{x \cdot (1-d)}{2(d-x)} > 0$$

Thus $G_d(x,x) - \frac{x}{2} > 0$ and $G_d(x,x) > \frac{x}{2}$.
On other side

$$\lim_{d\to 1} G_d(x,x) = \lim_{d \to 1} \frac{2\frac{x-d}{x\cdot(1-x)} - 1 + \sqrt{1 - 2 \cdot 2\frac{x-d}{x\cdot(1-x)} + \left(2\frac{x-d}{x\cdot(1-x)}\right)^2 + 4 \cdot 2\frac{x-d}{x\cdot(1-x)} \cdot d}}{2 \cdot 2\frac{x-d}{x\cdot(1-x)}}$$

$$= \frac{1}{2} + \frac{x}{4} + \frac{\frac{2}{x} - 1}{-\frac{4}{x}} = \frac{1}{2} + \frac{x}{4} + \frac{x}{4} - \frac{1}{2} = \frac{x}{2}$$

As $G_d(x,y) \geq G_d(x,x)$ for $x \leq y$ then $\inf_d G_d(x,x) = \frac{x}{2}$ □

The structure of uninorm G_d provides opportunity to control the aggregation process using the parameter d. If the proportion r of arguments that are less that neutral element $\boldsymbol{e}$ approach 1 the value of the uninorm G_d should approach $\frac{\min_x(x_1,\ldots,x_n)}{2}$. This condition is satisfied for $d = \max(e, r)$. For the uninorm G_f the procedure is similar: if the proportion r of arguments that are less that neutral element $\boldsymbol{e}$ approach 0 the value the uninorm G_f should approach $\max_y(y_1, \ldots, y_m) + \frac{1}{2}[1 - \max_y(y_1, \ldots, y_m)]$. This condition is satisfied for $f = \min(e, r)$. (Figs. 1, 2).

Thus we build the mechanism of control of general R_G^* aggregation. This control depends on distribution of arguments relative to the neutral element.

Examples (1) Aggregation of scores $\{0.3, 0.4, 0.2, 0.8, 0.7\}$ with neutral element 0.5. Identities of uninorms are $d = 0.6$ and $f = 0.5$ and $R_G^*(\{0.3, 0.4, 0.2, 0.7, 0.8\}, 0.5) = 0.4310$.

This example of general R_G^* aggregation shows that R_G^* is non-associative as

$$R_G^*(\{0.4, R_G^*\{0.7, 0.8\}, 0.5), 0.5) = 0.6775 \quad \text{and}$$
$$R_G^*(\{R_G^*\{0.4, 0.7\}, 0.5), 0.8\}, 0.5) = 0.9001$$

(2) Aggregation of same scores $\{0.3, 0.4, 0.2, 0.8, 0.7\}$ but with neutral element equal to 0.75. Identities of uninorms are $d = 0.8$ and $f = 0.75$ and $R_G^*(\{0.3, 0.4, 0.2, 0.7, 0.8\}, 0.6, 0.4) = 0.2425$.

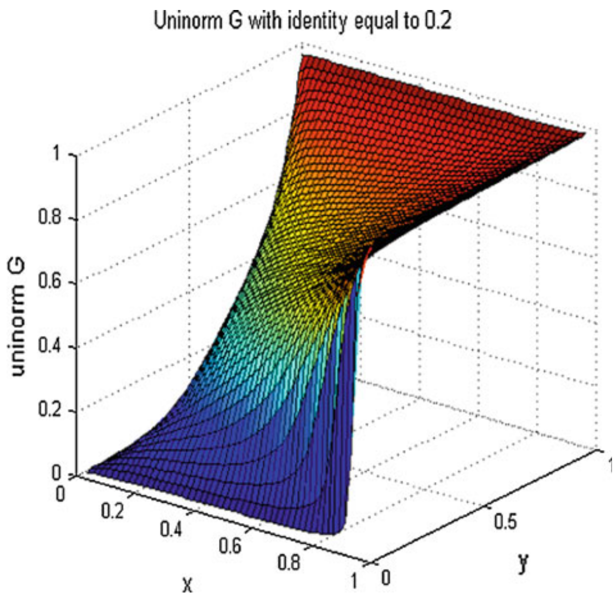

Fig. 1 Uninorm G with identity equal to 0.2

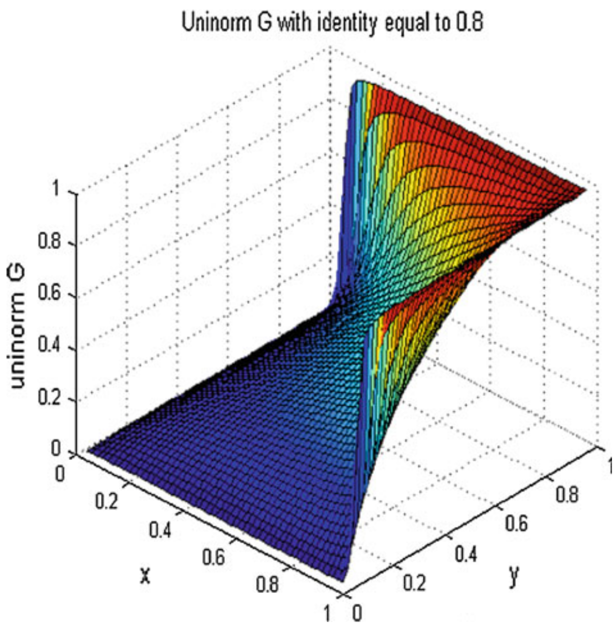

Fig. 2 Uninorm G with identity equal to 0.8

Another example of generating function that can be used to build uninorms used in general R_G^* aggregation satisfying our requirements is the function

$$h_d(x) = \log \frac{x^\alpha}{1 - x^\alpha}, \alpha > 0$$

with resulting uninorm

$$H_d(x, y) = h_d^{-1}(h_d(x) + h_d(y))$$

This uninorm has the identity

$$d_h = \frac{1}{2^{\frac{1}{\alpha}}}$$

Theorem 3 *The uninorm* $H_d(x, y) = h_d^{-1}(h_d(x) + h_d(y))$, *where* $h_d(x) = \log \frac{x^\alpha}{1-x^\alpha}$ *is bounded from below:* $\inf_d H_d(x, y) = x \cdot y$ *for* $x, y \in [0, d)$, $d < 1$.

Proof $h_d^{-1}(z) = \left[\frac{e^z}{1+e^z}\right]^{\frac{1}{\alpha}}$ and $H_d(x, y) = \left[\frac{x^\alpha \cdot y^\alpha}{(1-x^\alpha)\cdot(1-y^\alpha)+x^\alpha \cdot y^\alpha}\right]^{\frac{1}{\alpha}}$

$$H_d(x, y) = \left[\frac{x^\alpha \cdot y^\alpha}{1 - x^\alpha \cdot (1 - y^\alpha) - y^\alpha \cdot (1 - x^\alpha)}\right]^{\frac{1}{\alpha}}$$

As $x < 1$, $y < 1$, $\alpha > 0$ denominator in this expression is less than 1 and

$$H_d(x, y) = \left[\frac{x^\alpha \cdot y^\alpha}{1 - x^\alpha \cdot (1 - y^\alpha) - y^\alpha \cdot (1 - x^\alpha)}\right]^{\frac{1}{\alpha}} > \left[x^\alpha \cdot y^\alpha\right]^{\frac{1}{\alpha}} = x \cdot y$$

Thus $H_d(x, y) > x \cdot y$
For $d_h \to 1,_\alpha \to \infty$, $x^\alpha \to 0$, $y^\alpha \to 0$ and

$$\begin{aligned}\lim_{\alpha\to\infty} H_d(x, y) &= \lim_{\alpha\to\infty} \left[\frac{x^\alpha \cdot y^\alpha}{1 - x^\alpha \cdot (1 - y^\alpha) - y^\alpha \cdot (1 - x^\alpha)}\right]^{\frac{1}{\alpha}} = \lim_{\alpha\to\infty} \left[x^\alpha \cdot y^\alpha\right]^{\frac{1}{\alpha}} \\ &= x \cdot y\end{aligned}$$

Therefore $\inf_d H_d(x, y) = x \cdot y$ □

The structure of uninorm H_d allows to control the aggregation process using the parameter d. If the proportion r of arguments that are less that neutral element $\boldsymbol{e}$ approach 1 the value of the uninorm H_d should approach product aggregation $x_1 \cdot x_2 \cdot \dots \cdot x_n$. This condition is satisfied for $d = \max(e, r)$. (Figs. 3, 4).

For the uninorm H_f the procedure is similar: if the ratio $r = \frac{n}{n+m}$ approaches 0, the value of uninorm will approach the value of Łukasiewicz t-conorm:

$$S(a, b) = a + b - a \cdot b$$

The value of this uninorm is equal to

$$H_f(y_1, \dots, y_m) = 1 - H_{d'}(1 - y_1, \dots, 1 - y_m)$$

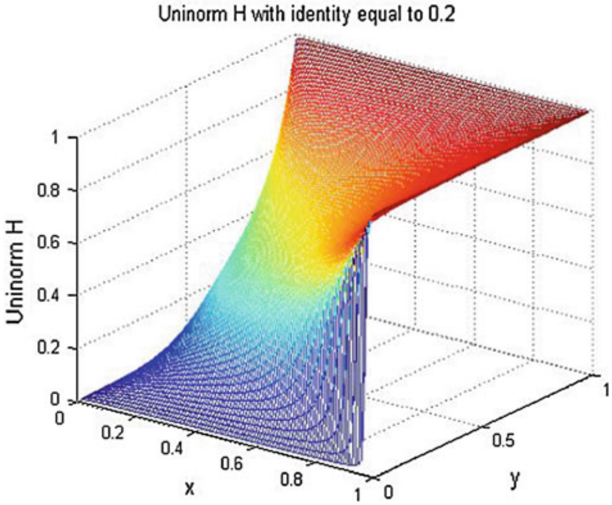

Fig. 3 Uninorm H with identity equal to 0.2

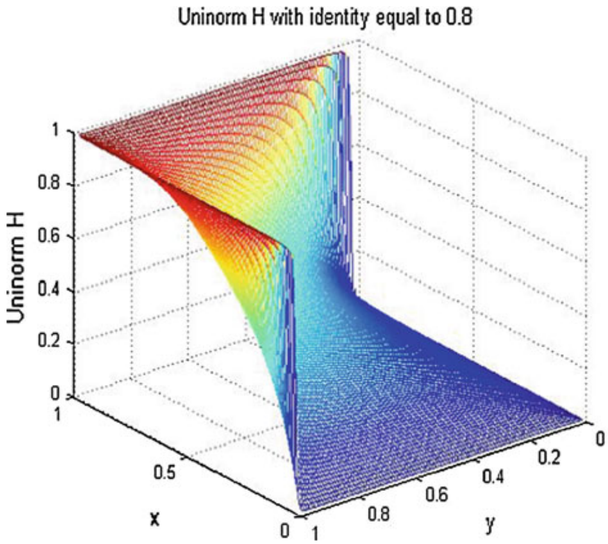

Fig. 4 Uninorm H with identity equal to 0.8

with identity d' equal to $d\prime = 1 - \min(e, r)$.

Examples (1) Aggregation of scores $\{0.3, 0.4, 0.2, 0.8, 0.7\}$ with neutral element 0.5. Identities of uninorms are $d = 0.6$ and $f = 0.5$ and $R^*_G(\{0.3, 0.4, 0.2, 0.7, 0.8\}, 0.5) = 0.3848$.

(2) Aggregation of same scores $\{0.3, 0.4, 0.2, 0.8, 0.7\}$ but with neutral element equal to 0.75. Identities of uninorms are $d = 0.8$ and $f = 0.25$ and $R^*_G(\{0.3, 0.4, 0.2, 0.7, 0.8\}, 0.6, 0.4) = 0.1757$

4 Conclusion

In bipolar aggregation the total score depends not just on previous score and the value of additional argument but on distribution of all other arguments as well. The process of bipolar aggregation is not Markovian, i.e. aggregation is not associative. In this paper general R^*_G aggregation operators were introduced to model bipolar aggregation. This type of aggregation allows to control the process of aggregation depending on distribution of arguments: the closer proportion of arguments below the neutral value to 1 or to 0 the closer bipolar aggregation is to chosen t-norm or t-conorm that have predefined lower or upper bounds.

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