- HW2 / Midterm Review
- HW3 Q & A, Review on Lecture

HW2 / Midterm Review

* Don't copy other students' answers

Some of you had very similar answers (which are wrong) in the midterm.

ex:

4(b)

(b) (5 points) With what order will Newton's method converge to $x_\star = 3$ when applied using this $f(x)$?

Let $P_n = n^3-5n^2+3n+9$, we want to find α such that $\lim\limits_{n\to\infty} \frac{|P_{n+1}-3|}{|P_n-3|^\alpha} = \lambda$.

$$\lim_{n\to\infty} \frac{|(n+1)^3-5(n+1)^2+3(n+1)+9-3|}{|n^3-5n^2+3n+9-3|^\alpha} = \lim_{n\to\infty} \left|\frac{n^3-2n^2-4n+5}{(n^3-5n^2+3n+6)^\alpha}\right| = \begin{cases} 1 & \text{if } \alpha = 1 \\ 0 \text{ or } \infty & \text{otherwise} \end{cases}$$

Thus, f converges to 3 $\boxed{\text{linearly (order 1)}}$.

They are not p_n

(b) (5 points) With what order will Newton's method converge to $x_\star = 3$ when applied using this $f(x)$?

Let $P_n = n^3-5n^2+3n+9$, we want to find α s.t. $\lim\limits_{n\to\infty} \frac{|P_{n+1}-3|}{|P_n-3|^\alpha} = \lambda$

$$\lim_{n\to\infty} \frac{|(n+1)^3-5(n+1)^2+3(n+1)+9-3|}{|n^3-5n^2+3n+9-3|^\alpha} = \lim_{n\to\infty} \left|\frac{n^3-2n^2-4n+5}{(n^3-5n^2+3n+6)^\alpha}\right| = \begin{cases} 1 & \text{if } \alpha = 1 \\ 0 & \text{otherwise} \end{cases}$$

Thus, f converges to 3 linearly.

HW2 (See discussion Recording)

Midterm

Q4

$f(x) = x^3 - 5x^2 + 3x + 9$

(a) multiplicity of $x_* = 3$?

(Sol. 1) $f(3) = f'(3) = 0 \neq f''(3)$

$f(3) = 3^3 - 5\cdot 3^2 + 3\cdot 3 + 9 = 0$

$f'(3) = 3x^2 - 10x + 3 \Big|_{x=3} = 0$

$f''(3) = 6x - 10 \Big|_{x=3} \neq 0$

(Sol. 2) $f(x) = (x-3)^2 (x+1)$

multip. $= 2$

(b) Newton's method to $x_* = 3$.

Order of conv. ?

Ans. multip. > 1 at $x_* = 3$

Order of Conv. of Newton's method $= 1$ (linear)

$$\lim_{k\to\infty} \frac{|x_{k+1} - 3|}{|x_k - 3|^\alpha} \leftarrow \forall \alpha > 1, \nexists$$

$\alpha = 1, \rightarrow \square < 1$

(c) modified Newton's method

$$x_{k+1} = x_k - \frac{f(x_k)f'(x_k)}{f'(x_k)^2 - f(x_k)f''(x_k)}$$

$$\frac{1}{\frac{f'}{f} - \frac{f''}{f'}}$$

Newton's $\frac{f}{f'} = \frac{1}{f'/f}$

☆ Other questions for midterm ?

HW3 Hints, and Review

- Approx. of f by P
  $\|f-P\| < \varepsilon$
- Interpolation on $\{x_0, x_1, \cdots, x_n\}$
  $f(x_i) = P(x_i) \ \forall i = 0, \cdots, n$

- Interpolations

(1) Vandermonde Matrix (theoretically simple / useful)

Interpolate f by $P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

$$= [a_0, a_1, \cdots, a_n] \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^n \end{bmatrix}$$

$$= [1 \ x \cdots x^n] \begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix}$$

$$\begin{cases} f(x_0) = P(x_0) \\ f(x_1) = P(x_1) \\ \quad \vdots \\ f(x_n) = P(x_n) \end{cases}$$

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & & & & \\ 1 & x_n & & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} f(x_0) \\ \vdots \\ f(x_n) \end{bmatrix}$$

$(n+1)$ by $(n+1)$ matrix

$$Ax = b$$

$$x = A^{-1} b$$

(2) Lagrange interpolation (useful, but lacks extendability)

$P_n = \{$ poly. with deg $\le n \} \le$ all poly.

basis of P_n : $\{1, x, x^2, \cdots, x^n\}$

Interp. Points $\{x_0, x_1, \cdots, x_n\}$

$$B_k(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_n)}{x-x_k} = \prod_{j\neq k}(x-x_j)$$

$$\begin{cases} B_k(x_j) = 0 & j \neq k \\ B_k(x_k) = \prod_{j\neq k}(x_k - x_j) \end{cases}$$

$$L_k(x) = \frac{B_k(x)}{B_k(x_k)} = \frac{\prod_{j\neq k}(x-x_j)}{\prod_{j\neq k}(x_k-x_j)}$$

$$L_k(x_j) = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

$f(x_j) = P(x_j) \quad \forall j = 0, \cdots, n$

Then, $P(x) = f(x_0)L_0(x) + f(x_1)L_1(x) + \cdots + f(x_n)L_n(x)$

$\{x_0, \cdots, x_n, x_{n+1}\}$

(3) Newton's divided differences

$$P_n(x) = c_0 \cdot 1 + c_1 (x - x_0) + c_2 (x-x_0)(x-x_1)$$
$$+ c_3 (x-x_0)(x-x_1)(x-x_2)$$
$$+ \cdots$$
$$+ c_n (x-x_0) \cdots (x-x_{n-1})$$

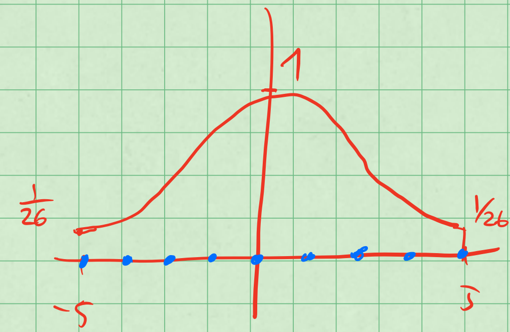
$x_0, \cdots, x_n, x_{n+1}$

$$P_{n+1} = P_n(x) + c_{n+1} (x-x_0) \cdots (x-x_n)$$

Error Analysis

e.g. Runge Phenomenon

$$f(x) = \frac{1}{1+x^2}, \qquad [-5, 5]$$



We can prove

$$\lim_{n\to\infty}\left(\sup_{x\in[-5,5]}\left|f(x) - P_{2n}(x)\right|\right) = \infty$$

Review the lecture note!!!

Poly. of deg. n

Thm . $f(x_i) = P(x_i) \quad \forall i = 0, \cdots, n. \qquad P \in P_n$

$$f(x) - P(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!}\prod_{i=0}^{n}(x - x_i) =: R(x)$$

$$\sup_{x\in[a,b]}|f(x) - P(x)| = \sup_{x\in[a,b]}|R(x)| = \sup_{x\in[a,b]}\left|\frac{f^{(n+1)}(\xi_x)}{(n+1)!}\prod_{i=0}^{n}(x-x_i)\right|$$

$$\leq \sup_{x\in[a,b]}\left|\frac{\prod_{i=0}^{n}(x-x_i)}{(n+1)!}\right| \times \sup_{x\in[a,b]}\left|f^{(n+1)}(x)\right|$$

$$\leq 10^{-4}$$

Newton's div. diff

$*\quad P_n(x) = C_0 + C_1(x-x_0) + \cdots + C_n(x-x_0)\cdots(x-x_n)$

Prove that $C_k = f[x_0, x_1, \cdots, x_k]$

for all $k = 0, \cdots, n$

3.3.7 (9th ed.)

Construct P of degree 3

$$f[x_{k-1}, x_k] = \frac{f[x_k] - f[x_{k-1}]}{x_k - x_{k-1}}$$

	x	$f[x_k]$	$f[x_{k-1}, x_k]$	$f[x_{k-2}, x_{k-1}, x_k]$
x_0:	-0.1	5.30		
			$\frac{-3.3}{0.1} = -33$	
x_1:	0.0	2.00		$\frac{38.95}{0.3} = 129.8\dot{3}$
			$\frac{+1.19}{0.2} = +5.95$	
x_2:	0.2	3.19		$\frac{-27.85}{0.3} = -92.8\dot{3}$
			$\frac{-2.19}{0.1} = -21.9$	
x_3:	0.3	1.00		

$$f[x_0, x_1, x_2, x_3] = \frac{-222.\dot{6}}{0.4} = -556.\dot{6}$$

$$P_n(x) = f[x_0] + f[x_0, x_1](x-x_0)$$

$$+ f[x_0, x_1, x_2](x-x_0)(x-x_1)$$

$$+ f[x_0, x_1, x_2, x_3](x-x_0)(x-x_1)(x-x_2)$$

$$= 5.30 - 33(x+0.1) + 129.8\dot{3}(x+0.1)x - 556.\dot{6}(x+0.1)x(x-0.2)$$