P1

Neville's method

$$\left.\begin{array}{ll} x_i & P \\ x_{i+1} & P' \end{array}\right] \longrightarrow \frac{(x-x_i)P' - (x-x_{i+1})P}{x_{i+1}-x_i}$$

$$\begin{array}{llllll}
0 & x_0 & & & & \\
 & & P_{0,1} & & & \\
1/4 & x_1 & & P_{0,1,2} & & \\
 & & P_{1,2} & & P_{0,1,2,3} & \\
1/2 & x_2 & & P_{1,2,3} & & \\
 & & P_{2,3} & & & \\
3/4 & x_3 & & & &
\end{array}$$

at $x = 0.4$:

$$\begin{array}{c|lllll}
x & f & & & & \\
0 & 1 & & & & \\
 & & 2.6 & & & \\
1/4 & 2 & & y_2 & & \\
 & & y_1 & & 3.016 & \\
1/2 & y_0 & & 2.96 & & \\
 & & 2.4 & & & \\
3/4 & 8 & & & &
\end{array}$$

Solve Backward $\longleftarrow$

$$\begin{array}{l} y_0 \\ \quad > 2.4 \\ 8 \end{array} \Rightarrow \frac{(0.4-0.75)\,y_0 - (0.4-0.5)\cdot 8}{0.5-0.75} = 2.4$$

$$\Rightarrow y_0 = 4$$

P2

$f(x) = \ln(x) \qquad x_0 = 1, \quad x_1 = 1.1, \quad x_2 = 1.3, \quad x_3 = 1.4$

(a) Lagrange interpolating poly $L_{3,k}(x)$ for $k = 0, 1, 2, 3$

Recall : $L_{n,k}$ has the property

$$L_{n,k}(x_i) = 0 \qquad \forall i \in [n] \setminus \{k\} = \{0, 1, \dots, k-1, k+1, \dots, n\}$$

$$L_{n,k}(x_k) = 1$$

$$\Rightarrow \quad L_{3,0}(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)},$$

$$L_{3,1}(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}, \qquad \dots$$

(b) $$P_3(x) = \sum_{i=0}^{3} f(x_i) L_{3,i}(x)$$

(c) $$f(x) - P_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0) \cdots (x-x_n)$$

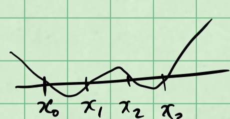
Thus an error bound :

$$\underbrace{\frac{1}{(n+1)!}\left(\max_{x_0 \le x \le x_n} \left|f^{(n+1)}(x)\right|\right)}_{(A)} \cdot \underbrace{\max_{x_0 \le x \le x_n} \prod_{i=0}^{3}(x-x_i)}_{(B)}$$

(A) : $f^{(n+1)} = -3!\, x^{-4}$, thus

$$(A) = \frac{1}{4} \max_{[x_0, x_n]} x^{-4} = \frac{1}{4}$$

(B) :



$q(x) = \prod (x - x_i)$

$$\begin{aligned} q'(x) = \; & (x-x_1)(x-x_2)(x-x_3) \\ & + (x-x_0)(x-x_2)(x-x_3) \\ & + (x-x_0)(x-x_1)(x-x_3) \\ & + (x-x_0)(x-x_1)(x-x_3), \end{aligned} \qquad \dots$$

* Use the node structure (equally spaced nodes) $\quad \nwarrow$ too complicated

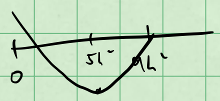
$\Rightarrow$ Set $\begin{cases} \frac{x_1+x_2}{2} = \bar{x} \\ h = \frac{x_1-x_0}{2} \end{cases}$ then $-3h \le x-\bar{x} \le 3h$.

$$g(x) = (x-(\bar{x}-3h))(x-(\bar{x}+3h))\,(x-(\bar{x}-h))\,(x-(\bar{x}+h))$$

$$= ((x-\bar{x})^2-9h^2)((x-\bar{x})^2-h^2)$$

$y = (x-\bar{x})^2, \quad 0 \le y \le 9h^2$

$(y-9h^2)(y-h^2) = y^2 - 10h^2y + 9h^4 = (y-5h^2)^2 - 16h^4$



- min at $y = 5h^2$, min val $= -16h^4$
- at $y=0$, max val $= 9h^4$

$\Rightarrow$ (B) $= 16h^4 = 16 \cdot (0.05)^4$

$\Rightarrow$ Err bd $= \frac{1}{4} \times 16 \times \left(\frac{1}{20}\right)^4 = 2.5 \times 10^{-5}$

P3

(a) P_5 using Lag. interpolation (skipped)

P_5 using Neville's method

$x = (\text{Year} - 1960)$

$$
\begin{array}{lllllll}
179{,}323 & 0 & & & & & \\
 & & > P_{0,1} & & & & \\
\vdots & 10 & & > P_{0,1,2} & & & \\
 & & > P_{1,2} & & > P_{0,1,2,3} & & \\
 & 20 & & > P_{1,2,3} & & > P_{0,1,2,3,4} & \\
 & & > P_{2,3} & & > P_{1,2,3,4} & & > \boxed{P_5} \\
 & 30 & & > P_{2,3,4} & & > P_{1,2,3,4,5} & \\
 & & > P_{3,4} & & > P_{2,3,4,5} & & \\
 & 40 & & > P_{3,4,5} & & & \\
 & & > P_{4,5} & & & & \\
 & 50 & & & & &
\end{array}
$$

$$P_{0,1} = \frac{(x-10)\cdot 179323 - (x-0)\cdot 203302}{0-10} = \cdots$$

See the code

P4, 5 See the code.