Week 8 Predator-Prey Model

50.2. Consider the predator-prey model with $b \neq 0$, equation 50.1. Calculate all possible equilibrium solutions. Compare these populations to the ones which occur if $b = 0$. Briefly explain the qualitative and quantitative differences between the two cases, $b = 0$ and $b \neq 0$.

Not going to "Solve" this for you (b/c it's a part of your HW7!)

But let's "Discuss" it, especially focusing on the diff. btw. Ch.50 of the book (or, equivalently, Lecture 17)

The system of equations is:

$$(\star) \text{ — } \begin{cases} \dot{F} = F(a - bF - cS) \\ \dot{S} = S(-k + \lambda F) \end{cases}$$

where
F : fish population
S : shark population

In Ch.50 [Textbook] or Lecture 17, $b = 0$ is assumed. All other params are positive.

Here, we can explain each param in biological / ecological terms.

- a : growth rate of fish, assuming no competition btw fish ($b=0$) and no sharks ($S=0$)

  (With $b=0$, $S=0$, we have $\dot{F} = aF$, thus $F(t)$ has an exponential growth: $F(t) = F(0)e^{at}$)

- b : Negative effect on the growth rate of F, caused by F.

  May include (but not limited to) the effects from, e.g.

  * Competition between fish, due to limited food source, etc.

- c : Negative (inhibiting) effect on the growth rate of F, caused by S.

  May depend on, e.g.,

  * how much fish each shark eats per time
  * how effective each shark is as a hunter

- k : (negative) growth rate of the shark population, in the absence of F.

  May depend on, e.g.

  * How long each shark can survive w/o fish

  Note: without fish ($F=0$), $\dot{S} = -kS$ so S diminishes exponentially ($S(t) = S(0)e^{-kt}$).

- λ : effect of fish population on the growth rate of shark.

  May depend on, e.g.

  * How much fish each shark needs to eat per day

Note that those parameters can be measured, or estimated from observations of the ecosystem.
On the other hand, there are some quantities resulting from (a purely mathematical) analysis of ($\star$):

For F,

- $\frac{a}{b}$: The carrying capacity of F, in the absence of S.
  (The population of F that is neither too much, nor too small for sustaining F.)

- $\frac{k}{\lambda}$: The population of F that is neither too much (so that $\dot{S} > 0$)
  nor too small (so that $\dot{S} < 0$)
  for sustaining S.

For S,

- $\frac{a}{c}$: The pop. of S that makes $\dot{F} = 0$ (controls the fish pop. just right)
  under no competition btw F (i.e., $b = 0$)

Among these quantities, two fish pop. $\frac{a}{b}$ and $\frac{k}{\lambda}$ will play an important role to the system.

Now, let's find the eq. pts :

$$\dot{F}=0 \Rightarrow F=0 \quad \text{or} \quad a-bF-cS=0$$

$$\dot{S}=0 \Rightarrow S=0 \quad \text{or} \quad \lambda F-k=0$$

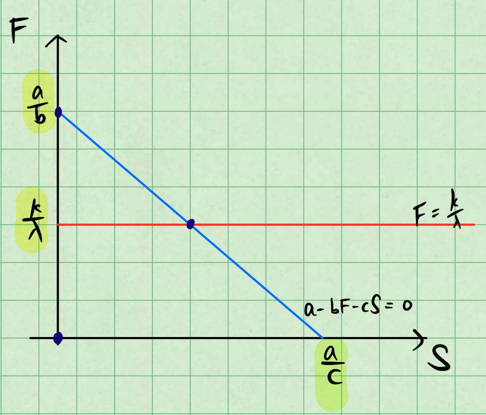
$$\Rightarrow (S^*,F^*) = (0,0),\ \left(0,\frac{a}{b}\right),\ \text{and}\ \left(\frac{a}{c}-\frac{bk}{c\lambda},\frac{k}{\lambda}\right),$$

But notice that $\left(\frac{a}{c}-\frac{bk}{c\lambda},\frac{k}{\lambda}\right)$ is meaningful only when it's on the 1st quadrant,

i.e, $\frac{a}{c}-\frac{bk}{c\lambda}=\frac{a\lambda-bk}{c\lambda}\geqslant 0 \iff a\lambda \geqslant bk \iff \frac{a}{b}\geqslant\frac{k}{\lambda}$ (!!!)

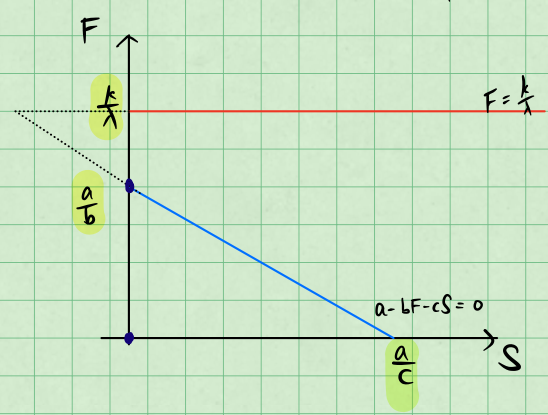
Thus we have the following two cases (except $\frac{a}{b}=\frac{k}{\lambda}$, which is left as an exercise)

Case (A) : $\frac{a}{b} > \frac{k}{\lambda}$



(Three fixed pts)

Case (B) : $\frac{a}{b} < \frac{k}{\lambda}$



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The red line $\lambda F-k$ and the blue $a-bF-cS$ are the nullclines, and we determine the directions of the arrows ($\leftarrow \rightarrow$ or $\uparrow \downarrow$) by investigating the sign of one when the other is zero.

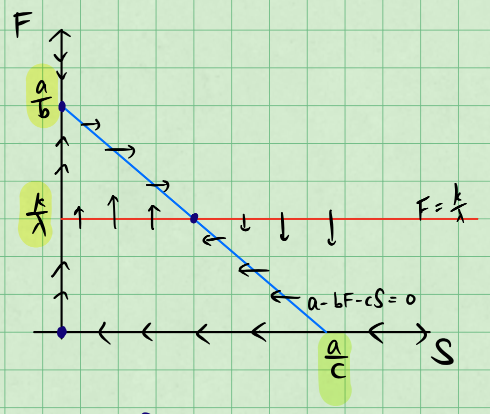
- On $\lambda F-k=0$, $\quad a-bF-cS \;\begin{cases} >0 & \text{if } S<\frac{a}{c}-\frac{bk}{c\lambda} \\ <0 & \text{if } S>\frac{a}{c}-\frac{bk}{c\lambda}\end{cases}$

- On $a-bF-cS=0$, $\quad \lambda F-k \;\begin{cases} >0 & \text{if } F<\frac{k}{\lambda} \\ <0 & \text{if } F>\frac{k}{\lambda}\end{cases}$ (doesn't depend on S, in fact)

Two other nullclines, $\begin{cases} F=0 & (\leftarrow\leftarrow \text{ always}) \\ S=0 & (\updownarrow \%) \end{cases}$ are easy to see.

Therefore, we have :

Case (A) : $\frac{a}{b} > \frac{k}{\lambda}$



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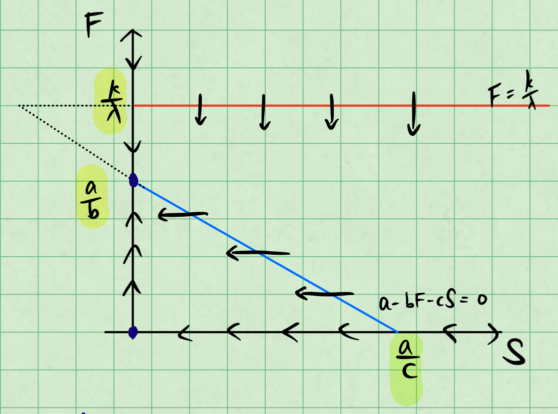
Case (B) : $\frac{a}{b} < \frac{k}{\lambda}$



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It seems like that, from the above figures,

- $(0,0)$ is always a saddle
- there is something rotating near $\left(\frac{a}{c}-\frac{bk}{c\lambda}, \frac{k}{\lambda}\right)$ in Case (A)
- $\left(0, \frac{a}{b}\right)$ is a saddle in Case (A)
- $\left(0, \frac{a}{b}\right)$ is stable in Case (B).

We will indeed they are correct, and discuss the stability of $\left(\frac{a}{c}-\frac{bk}{c\lambda}, \frac{k}{\lambda}\right)$ in Case (A), using linearization arguments.

Let $G(S,F) = (S(-k+\lambda F), F(a-bF-cS))$.

Then the Jacobian is

$$DG(S,F) = \begin{pmatrix} -k+\lambda F & \lambda S \\ -cF & a-2bF-cS \end{pmatrix}.$$

Thus,

- at $(0,0)$, $DG(0,0) = \begin{pmatrix} -k & 0 \\ 0 & a \end{pmatrix}$ and the eig. vals are $a>0$ and $-k<0$.

  Hence $(0,0)$ is a saddle.

- at $\left(0, \frac{a}{b}\right)$, $DG\left(0, \frac{a}{b}\right) = \begin{pmatrix} -k+\frac{\lambda a}{b} & 0 \\ -\frac{ac}{b} & -a \end{pmatrix}$ and the eig. vals are $-a<0$, and $-k+\frac{a\lambda}{b}$.

Here, again, it depends on $\frac{a}{b}$ vs. $\frac{k}{\lambda}$:

* Case (A) : $\frac{a}{b} > \frac{k}{\lambda}$: $-k + \frac{a\lambda}{b} = \lambda\left(\frac{a}{b} - \frac{k}{\lambda}\right) > 0$. Thus $(0, \frac{a}{b})$ is a saddle.
* Case (B) : $\frac{a}{b} < \frac{k}{\lambda}$: $-k + \frac{a\lambda}{b} < 0$. Thus $(0, \frac{a}{b})$ is a stable node.

· In Case (A), at $\left(\frac{a}{c} - \frac{bk}{c\lambda}, \frac{k}{\lambda}\right)$,

$$DG\left(\frac{a}{c} - \frac{bk}{c\lambda}, \frac{k}{\lambda}\right) = \begin{pmatrix} 0 & \frac{a\lambda - bk}{c} \\ -\frac{ck}{\lambda} & \frac{-bk}{\lambda} \end{pmatrix}.$$

$$\tau = \frac{-bk}{\lambda} < 0 \quad \text{and} \quad \Delta = bk\left(\frac{a}{b} - \frac{k}{\lambda}\right) > 0.$$

Thus $\left(\frac{a}{c} - \frac{bk}{c\lambda}, \frac{k}{\lambda}\right)$ is either a stable spiral or a stable node

(or a degenerate one, but still stable).

Note: $\tau^2 - 4\Delta = \frac{k}{\lambda}\left(\frac{b^2 k}{\lambda} - 4a\lambda + 4bk\right)$ can be $+$, 0, or $-$ in fact.

e.g. $k = \lambda = a = 1$, $\tau^2 - 4\Delta = b^2 + 4b - 4 = (b+2)^2 - 8$

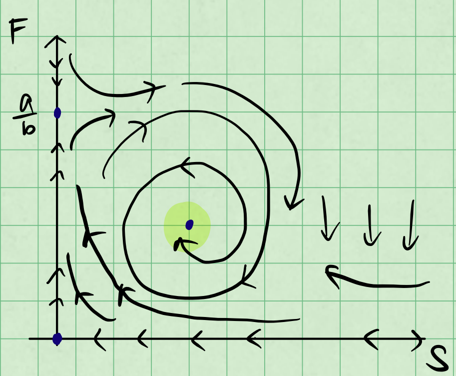
$= 0$ if $b = 2(\sqrt{2} - 1) \approx 0.83 < 1 = \frac{a\lambda}{k}$

> 0 if $b \in (2(\sqrt{2}-1), 1)$

< 0 if $b \in [0, 2(\sqrt{2}-1))$

To sum up, we have

Case (A) : $\frac{a}{b} > \frac{k}{\lambda}$

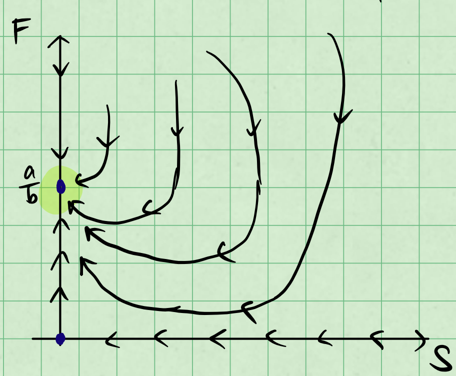


(Three fixed pts)

Converges to a dynamic equilibrium

$\left(\frac{a}{c} - \frac{bk}{c\lambda}, \frac{k}{\lambda}\right)$

Case (B) : $\frac{a}{b} < \frac{k}{\lambda}$



(Two fixed pts)

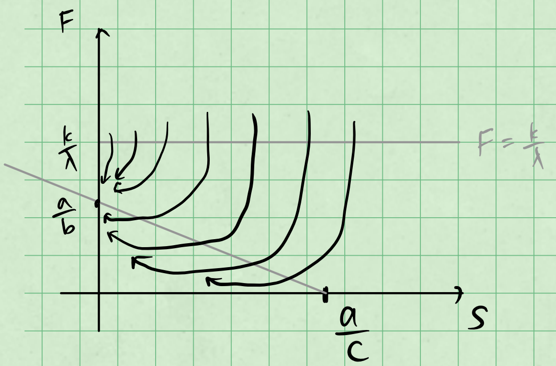
Sharks tend towards extinction

and fish goes to its carrying capacity $(0, \frac{a}{b})$

Note also that when $b \neq 0$, the system cannot be conservative (although it is when $b = 0$.)

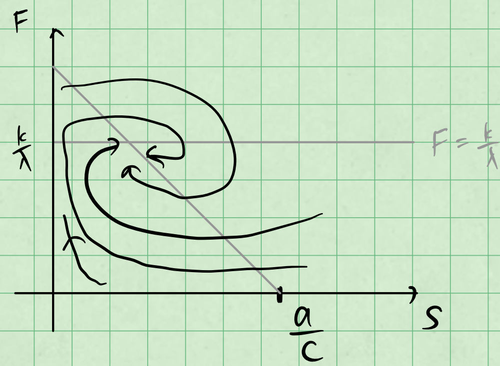
[Changing the parameter b: from a large # to 0, while fixing all other params]

$b > \frac{a\lambda}{k}$



$$\begin{cases} S \to 0 \\ F \to \frac{a}{b} \end{cases} \text{ as } t \to \infty$$

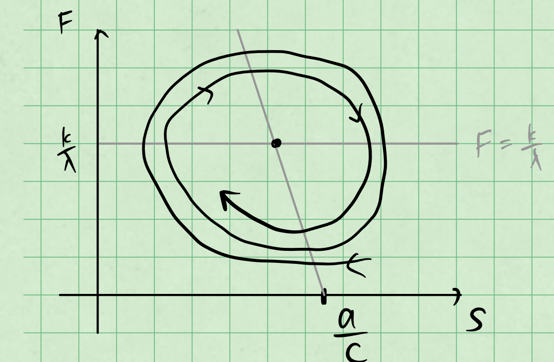
$b \lesssim \frac{a\lambda}{k}$



$$\begin{cases} S \to \frac{a}{c} - \frac{bk}{c\lambda} \\ F \to \frac{k}{\lambda} \end{cases} \text{ as } t \to \infty$$

As b gets smaller, more rotations:

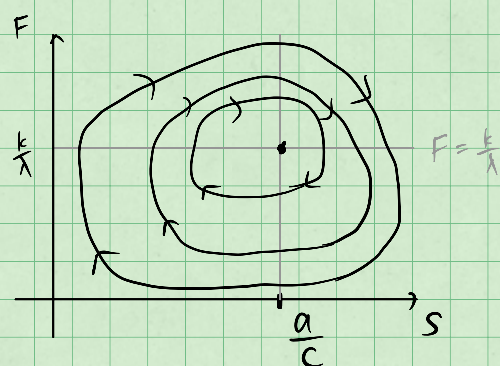
$b \ll \frac{a\lambda}{k}$



$$\begin{cases} S \to \frac{a}{c} - \frac{bk}{c\lambda} \\ F \to \frac{k}{\lambda} \end{cases} \text{ as } t \to \infty$$

And eventually, closed curves:

$b = 0$



S and F are periodic, oscillating around $\frac{a}{c}$ and $\frac{k}{\lambda}$, respectively.