Week 7 The SIR Model

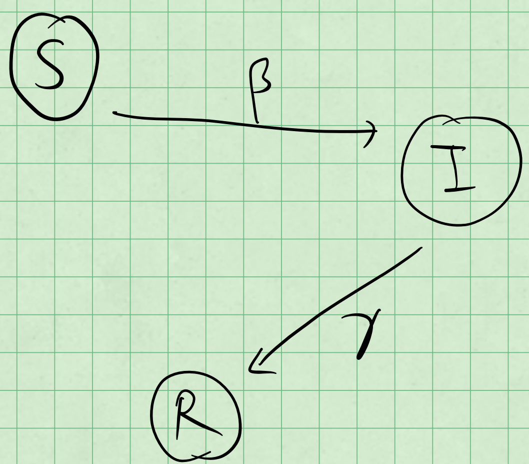
Simulate the spread of infectious diseases

$$\left\{\begin{array}{lll} S: & \text{Susceptible} & \text{fraction of the population} \\ I: & \text{Infected} & \text{fraction of the population} \\ R: & \text{Recovered} & \text{fraction of the population} \end{array}\right) \Rightarrow$$

$S + I + R = 1$

(They are fractions)

and $S, I, R \in [0, 1]$



Parameters

- β : measures, e.g.
  - Infectiousness
  - avg. # of contacts /person /time
- γ : measures, e.g.
  - the recovery rate

    (an individual is infectious for an avg. time period $1/\gamma$)

$$\left.\begin{aligned} \dot{S} &= -\beta S I \\ \dot{R} &= \gamma I \end{aligned}\right\} \Rightarrow \quad \dot{I} = \beta S I - \gamma I$$

Don't need all three variables, because $S + I + R = 1$.

Let's remove R, then

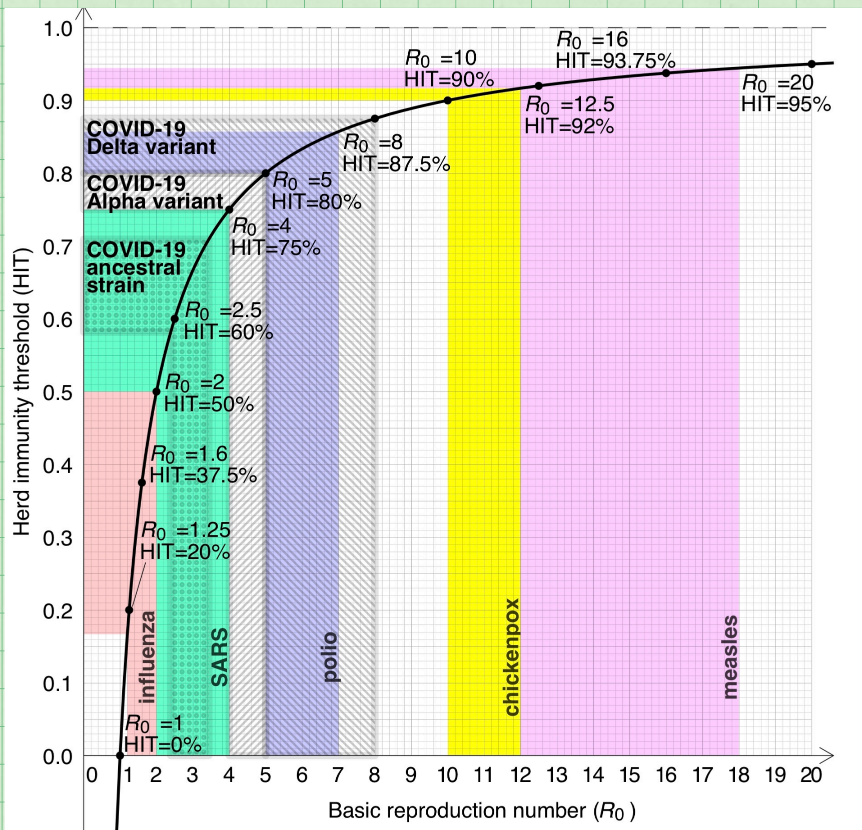
$$\left\{\begin{aligned} \dot{S} &= -\beta S I \\ \dot{I} &= \beta S I - \gamma I \end{aligned}\right.$$

⑥ The Sign of $\dot{I}$ and the Basic Reproduction Number

Note that $\dot{I} = I(\beta S - \gamma)$, and

$$\dot{I} \begin{cases} > 0 & \text{if } S > \gamma/\beta \\ = 0 & \text{if } S = \gamma/\beta \\ < 0 & \text{if } S < \gamma/\beta \end{cases}$$

Thus, the quantity γ/β plays an important role here, and is called the "basic reproduction number" often denoted by R_0.

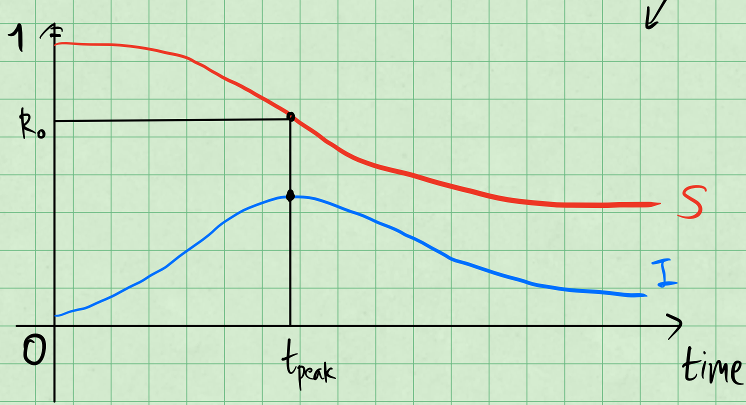


The figure on the left shows R_0 for various diseases.

Because $\dot{S} < 0$ as long as $I, S > 0$,
If $S(0) > R_0$, I increases and S decreases, until S becomes R_0.
At $S = R_0$, $\dot{I} = 0$ and S still decreases, and as soon as $S < R_0$, $\dot{I} < 0$.

Thus, I has its peak (maximum) when $S = R_0$.

See the figure below.



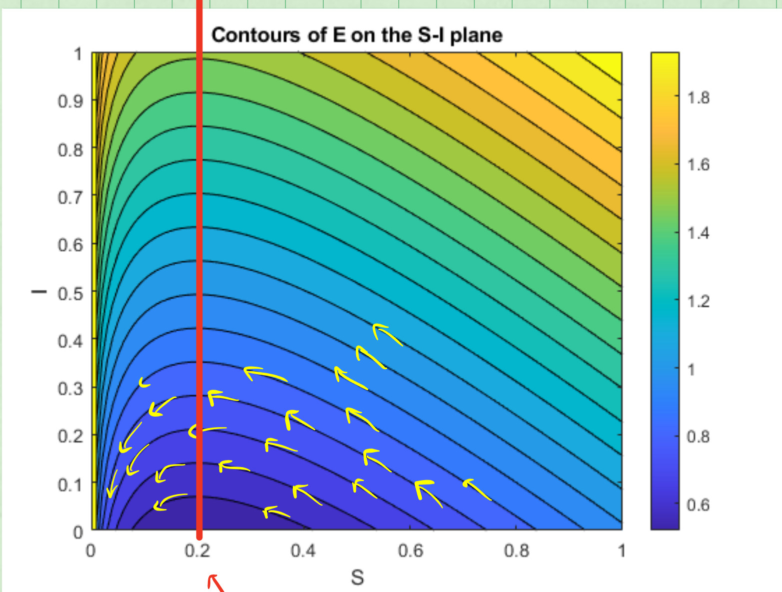
◎ Phase Portrait and an Energy Function

Since $\dot{I} + \dot{S} = -\gamma I = \frac{\gamma}{\beta} \cdot (-\beta S I)/S = \frac{\gamma}{\beta} \frac{\dot{S}}{S} = \frac{\gamma}{\beta} \frac{d}{dt}(\log(S))$

We have $\frac{d}{dt}\left(I + S - \frac{\gamma}{\beta} \log(S)\right) = 0$.

Set $E(S, I) := I + S - \frac{\gamma}{\beta} \log(S)$ then it is a conserved quantity.

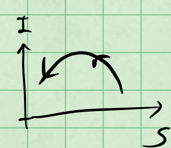
i.e., the system is conservative.



The figure on the left shows the contours of E.

They also coincide with the phase portrait, i.e., the solution curves on the S-I plane.

From the analysis on the sign of $\dot{I}$, we notice the arrows' direction:



I has its peak at $S = R_0 = 0.2$ in this figure.
(maximized)

◎ Simulation on C++

We model the spread of a disease as the following, WITHOUT any knowledge on the ODE (SIR model)

1. An individual i meets M_i people per unit time

   * Average of M_i = Average contacts per person per time, λ, is given

   (Assume $M_i \sim \text{Poisson}(\lambda)$.)

2. Per each contact, if one of them is Infected and the other is Susceptible, the chance to transmit the disease is p.

* Then β in the SIR model can be computed as $\beta = \lambda p$

(avg. contacts / person / time × chance to transmit / contact)

= avg. infection-producing contacts per person per time

3. Each infected individual i will be recovered after r_i units of time (e.g. days)

* The average of r_i = avg. period of time for recovery, ρ, is given.

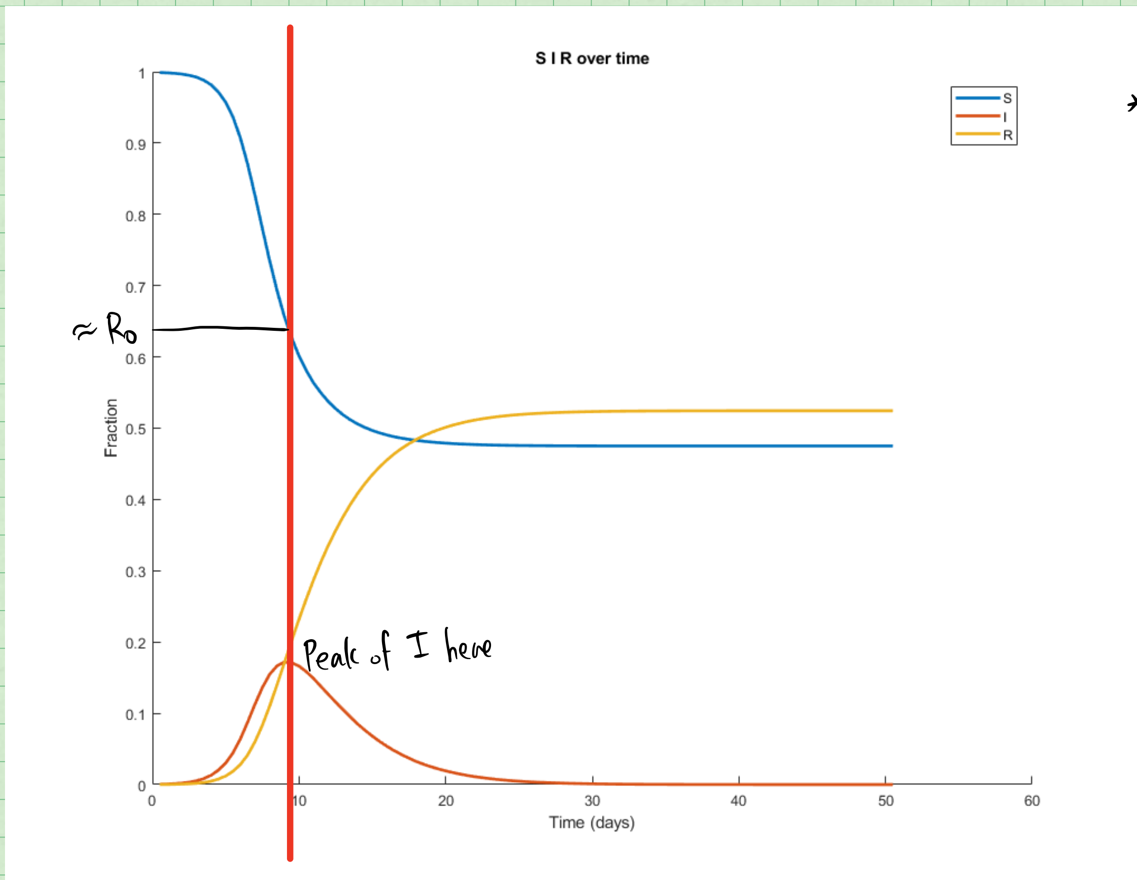
* Assume $r_i \sim \text{exponential}(1/\rho)$

(Look up "exponential distribution" for more detail)

Thus the simulation will have the following parameters:

- N : total population (e.g. 1,000,000 people)
- I_0 : Initial fraction of the infected (e.g. 0.01 % = 0.0001)
- Unit of time : e.g. 1 day, ½ days, or 7 days, etc.
- ρ : Avg. period for recovery (e.g. 3 days)
- λ : Avg. contacts / person / day (e.g. meeting 4 people per day)
- p : Chance to transmit (e.g. 12.5 % = 0.125)
- T : Observation period (e.g. 150 days total)

- Using the example parameters highlighted above, we get the following



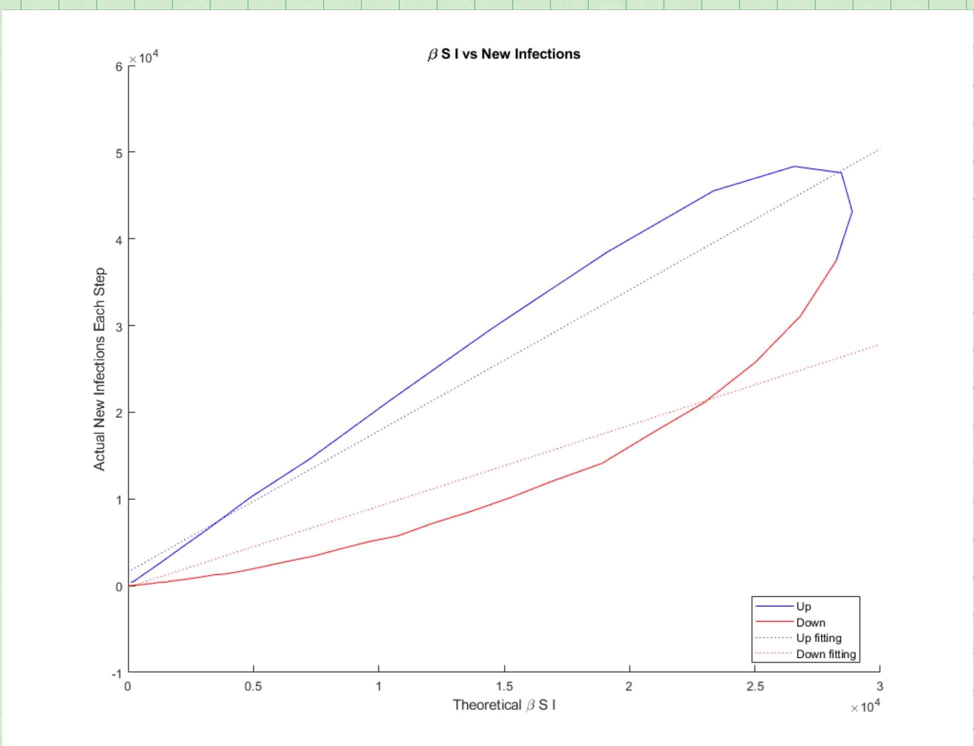
* $R_0 = 2/3 \approx 0.67$

And S at its peak is 0.6568.

Hence, the behavior is well-predicted by the SIR model.

Furthermore, we can also check that

$$\#(\text{new infections}) \approx \beta S I \quad (= -\dot{S} \text{ in the SIR model})$$



They don't match perfectly, but certainly there's a positive correlation, and #(new infections) can be approximated by $\beta S I$.

* To improve the SIR model, we can find a more proper approximation than $\beta S I$. But it's beyond the scope of this course.