Week 6 Conservative Systems

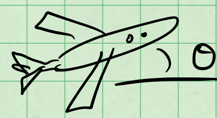
13.10 Exploration: Motion of a Glider

In this exploration we investigate the motion of a glider moving in an xy-plane, where x measures the horizontal direction and y the vertical direction. Let $v > 0$ be the velocity and θ the angle of the nose of the plane, with $\theta = 0$ indicating the horizontal direction. Besides gravity, there are two other forces that determine the motion of the glider: the drag (which is parallel to the velocity vector but in the opposite direction) and the lift (which is perpendicular to the velocity vector).

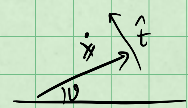
1. Assuming that both the drag and the lift are proportional to v^2, use Newton's second law to show that the system of equations governing the motion of the glider may be written as

$$\theta' = \frac{v^2 - \cos\theta}{v}$$
$$v' = -\sin\theta - Dv^2,$$

where $D \geq 0$ is a constant.
2. Find all equilibrium solutions for this system and use linearization to determine their types.
3. When $D = 0$ show that $v^3 - 3v\cos\theta$ is constant along solution curves. Sketch the phase plane in this case and describe the corresponding motion of the glider in the xy-plane.
4. Describe what happens when D becomes positive.



$$\begin{cases} \dot\theta = \dfrac{v^2 - \cos\theta}{v} \\ \dot v = -\sin\theta - Dv^2 \end{cases}$$



P1

$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\dot{\mathbf{x}} = \begin{pmatrix} \dot x \\ \dot y \end{pmatrix} = \begin{pmatrix} v\cos\theta \\ v\sin\theta \end{pmatrix}$, Let $\hat t := \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$ be the lift direction

$$\ddot{\mathbf{x}} = \underset{\text{(Gravitational)}}{\begin{pmatrix} 0 \\ -mg \end{pmatrix}} - \underset{\text{(Drag)}}{Dv^2\left(\frac{\dot{\mathbf{x}}}{v}\right)} + \underset{\text{(Lift)}}{Lv^2\hat t}$$

$$\begin{pmatrix} \dot v\cos\theta - v(\sin\theta)\dot\theta \\ \dot v\sin\theta + v(\cos\theta)\dot\theta \end{pmatrix} = \begin{pmatrix} 0 \\ -mg \end{pmatrix} - \begin{pmatrix} Dv^2\cos\theta \\ Dv^2\sin\theta \end{pmatrix} + \begin{pmatrix} -Lv^2\sin\theta \\ Lv^2\cos\theta \end{pmatrix} = \begin{pmatrix} -Dv^2\cos\theta - Lv^2\sin\theta \\ -mg - Dv^2\sin\theta + Lv^2\cos\theta \end{pmatrix}$$

$\parallel$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \dot v \\ v\dot\theta \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \dot v \\ v\dot\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} -Dv^2\cos\theta - Lv^2\sin\theta \\ -mg - Dv^2\sin\theta + Lv^2\cos\theta \end{pmatrix} =$$

$$= \begin{pmatrix} -Dv^2\cos^2\theta - Lv^2\cos\theta\sin\theta - mg\sin\theta - Dv\sin^2\theta + Lv^2\cos\theta\sin\theta \\ Dv^2\cos\theta\sin\theta + Lv^2\sin^2\theta - mg\cos\theta - Dv\sin\theta\cos\theta + Lv^2\cos^2\theta \end{pmatrix}$$

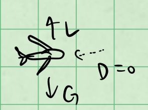
$$= \begin{pmatrix} -Dv^2 - mg\sin\theta \\ Lv^2 - mg\cos\theta \end{pmatrix}$$

$$\Rightarrow \begin{cases} \dot v = -Dv^2 - mg\sin\theta \\ \dot\theta = \dfrac{Lv^2 - mg\cos\theta}{v} \end{cases}.$$

Scale the units so that $mg = 1$ and $L = 1$ then we get the desired equations.

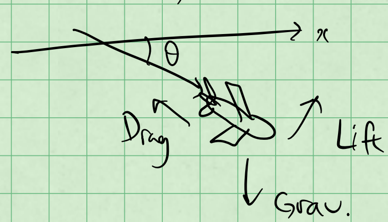
2. $\dot\theta = \dot v = 0 \Rightarrow \begin{cases} Dv^2 = -\sin\theta \\ v^2 = \cos\theta \end{cases} \Rightarrow v^2 = \cos\theta = -\dfrac{\sin\theta}{D}$

• if $D=0$, $\theta = 0$, $v^2 = 1$



$-D = \tan\theta$, $\theta = -\tan^{-1}(D)$, $v^2 = \cos\theta = \dfrac{1}{\sqrt{1+D^2}}$

$\Rightarrow v = (1+D^2)^{-1/4}$ (only take the + sign b/c $v \geq 0$)



$F(\theta, v) = \left(v - \dfrac{\cos\theta}{v}, -\sin\theta - Dv^2\right)$

$$DF(\theta, v) = \begin{pmatrix} \dfrac{\sin\theta}{v}, & -\cos\theta \\ 1 + \dfrac{\cos\theta}{v^2}, & -2Dv \end{pmatrix} \Rightarrow DF(-\tan^{-1}(D), (1+D^2)^{-1/4})$$

$$= \begin{pmatrix} \dfrac{-D}{(1+D^2)^{1/4}}, & -(1+D^2)^{-1/2} \\ 2, & \dfrac{-2D}{(1+D^2)^{1/4}} \end{pmatrix}$$

$\tau = \dfrac{-3D}{(1+D^2)^{1/4}} < 0 \quad (\tau = 0 \text{ for } D = 0)$

$\Delta = \dfrac{2D^2+2}{\sqrt{1+D^2}} = 2\sqrt{1+D^2}$

$\tau^2 - 4\Delta = \dfrac{1}{\sqrt{1+D^2}}(9D^2 - 8D^2 - 8)$

$= \dfrac{D^2-8}{\sqrt{1+D^2}} \quad \geq 0 \text{ if } D \geq \sqrt{8}, \quad < 0 \text{ if } D < \sqrt{8}$

3. $L(v,\theta) = v^3 - 3v\cos\theta$

$\dot{L} = \dot{v}L_v + \dot{\theta}L_\theta = \dot{v}(3v^2 - 3\cos\theta) + \dot{\theta}(3v\sin\theta)$

$= -3(\sin\theta + Dv^2)(v^2 - \cos\theta) + 3(v^2 - \cos\theta)\sin\theta$

$= -3(v^2 - \cos\theta)Dv^2 \quad = 0 \quad \text{if} \quad D = 0$

4. If $D > 0$, stable node (sink) $(D > \sqrt{8})$

spiral $(D < \sqrt{8})$