Week 3

Recall that the solution of a 2^{nd} order linear ODE

$$\ddot{x} + p\dot{x} + qx = 0$$

is given by

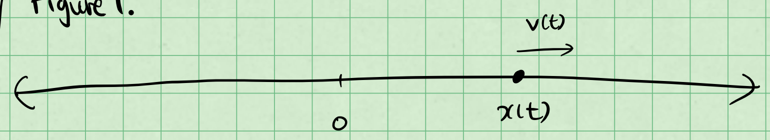
$$x(t) = \begin{cases} A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} & \text{if } p^2 > 4q \\ A_1 e^{\lambda t} + A_2 t e^{\lambda t} & \text{if } p^2 = 4q \\ e^{rt}\left(B_1 \cos(st) + B_2 \sin(st)\right) & \text{if } p^2 < 4q \end{cases}$$

where $\lambda = \dfrac{-p \pm \sqrt{p^2 - 4q}}{2}$, $r = -\dfrac{p}{2}$, $s = \dfrac{\sqrt{4q - p^2}}{2}$.

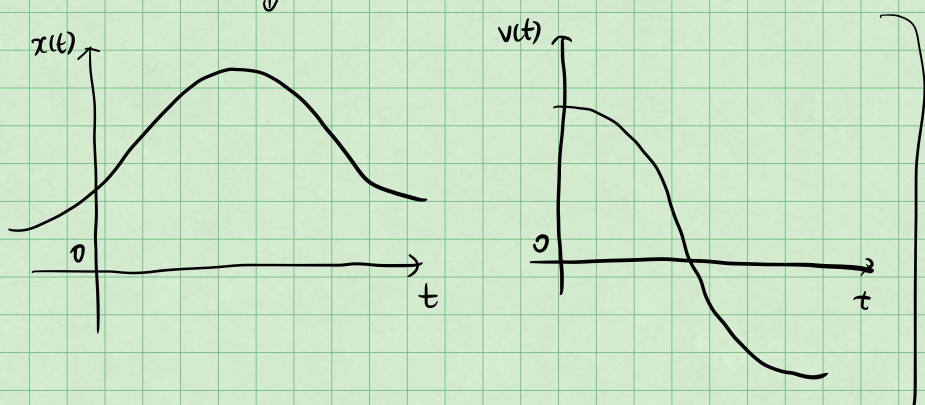
Today, we will see more qualitative behavior through the phase plane.

For easier (maybe more intuitive) understanding, call $x(t)$ the displacement and $v(t) := \dot{x}(t)$ the velocity of a particle, in 1-D.

Figure 1.



← Description of the particle's state (i.e., the displacement and the velocity) at a certain time t



← Description of each state variable $x(t)$ and $v(t)$ over time

Figure 2.

Figure 1 shows both of the state variables at a certain time, while Figure 2 shows a more complete information over time, for a certain solution (associated with some initial condition.)

A phase portrait is drawn on the phase plane, where each axis represents each state variable (x and v in this case.)

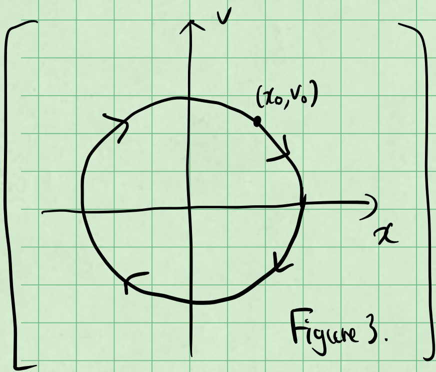


Figure 3.

A solution $\vec{x}(t) = (x(t), \underset{\dot{x}(t)}{\overset{}{v(t)}})$ with a certain initial condition (x_0, v_0) is, then, a parametrized curve on the plane. (See Figure 3.)

Although it is (a bit) harder to track how fast the particle is moving along the curve, (e.g. $v(t) = x(t)^2 + 1$ from HW1.) the phase portrait conveys the information (qualitative behavior) of the family of solutions for various initial conditions

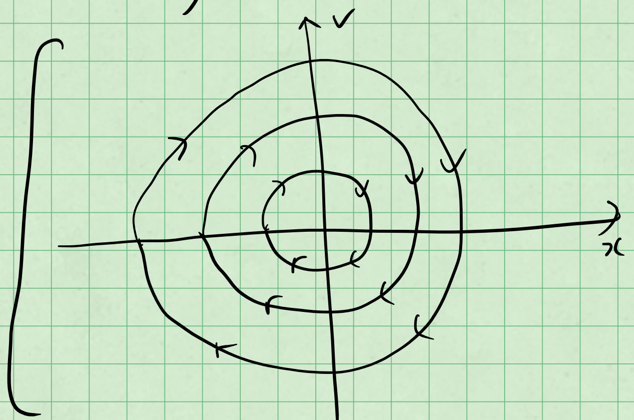


Figure 4.
sol. of $\ddot{x}(t) = -x(t)$
with various initial conditions

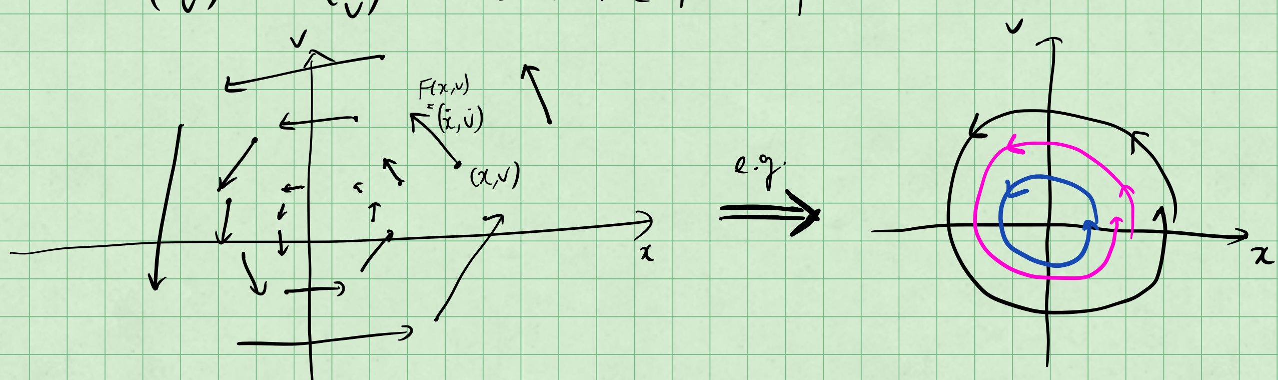
Therefore, it is a quite useful (powerful) tool to understand ODEs.

Now, let's get back to the 2^{nd} order linear case :

$$\begin{cases} \ddot{x} + p\dot{x} + q x = 0 \\ x(0) = x_0 \\ \dot{x}(0) = v_0 \end{cases} \iff \begin{cases} \begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} \\ x(0) = x_0, \quad v(0) = v_0 \end{cases}$$

The portrait of the trajectories of the vector field (flow)

$F : \begin{pmatrix} x \\ v \end{pmatrix} \mapsto \begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix}$ is called the phase portrait.



Thus, for linear ODEs of the form $\dot{x} = A\dot{x}$,

it is (again) very important to understand the map A,

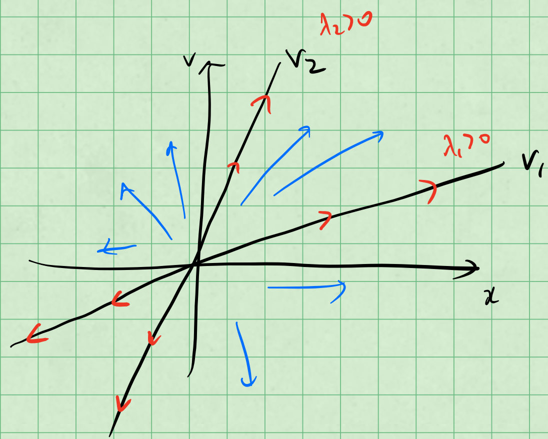
and we can do some quality analysis through the eigenvalues / eigenvectors of A.

<u>Notation</u> $\begin{cases} Av_1 = \lambda_1 v_1 \\ Av_2 = \lambda_2 v_2 \end{cases}$ (If $\lambda_1 = \lambda_2$, we denote it by just λ.)

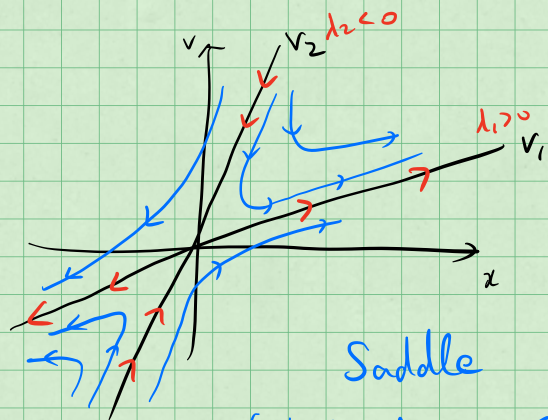
$v_1 \neq 0 \neq v_2$.

<u>Case 1</u> λ_1, λ_2 both real $\Rightarrow$ Depends on the signs of λ_1 and λ_2

If both $-$, compare their magnitudes.



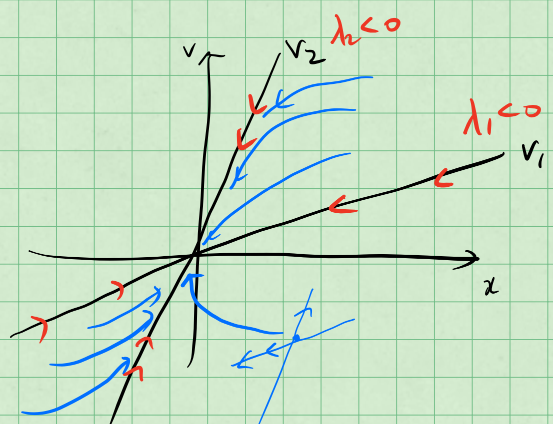
Unstable



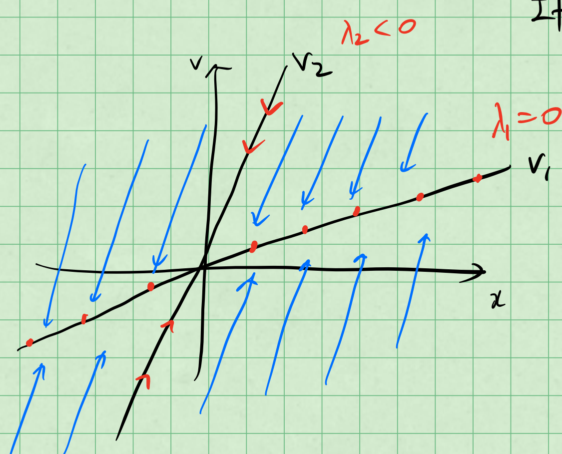
Saddle

(stable along $\langle v_2 \rangle$,

unstable otherwise)



$|\lambda_1| > |\lambda_2| > 0$



If $\lambda_1 = 0$,

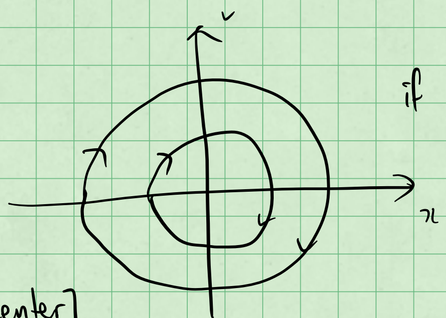
the sol. doesn't move along the $\langle v_1 \rangle$ direction.

Case 2 Complex eigenvalues.

In this case both eigenvectors are in $\mathbb{C}^2 \setminus \mathbb{R}^2$ (i.e., imaginary part $\neq 0$) and it's hard to analyze the sol. in $\mathbb{R}^2$ with only e.values / vectors.

Thus we use the solution $x(t) = e^{rt}\left(B_1 \cos(st) + B_2 \sin(st)\right)$ here.

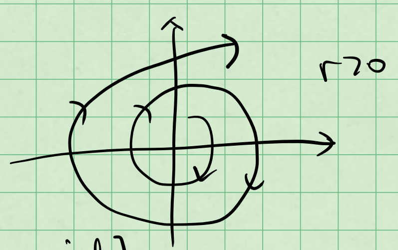
$$\left(\begin{cases} r = \frac{-p}{2} \\ s = \frac{\sqrt{-(p^2-4q)}}{2} \end{cases}, \text{ or, equivalently, } \begin{cases} r = \mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2) \\ s = \mathrm{Im}(\lambda_1) = -\mathrm{Im}(\lambda_2) \text{ (positive)} \end{cases} \right)$$



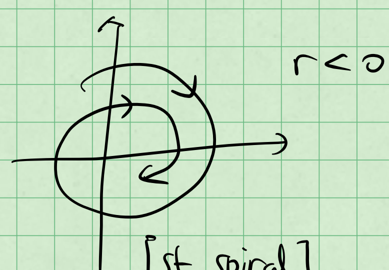
[center]

if $r=0$, no change in the "amplitude" term and only periodic terms alive.

If $r>0$, or $r<0$, it either explodes (to ∞) or decays (to 0).



[uns. spiral]

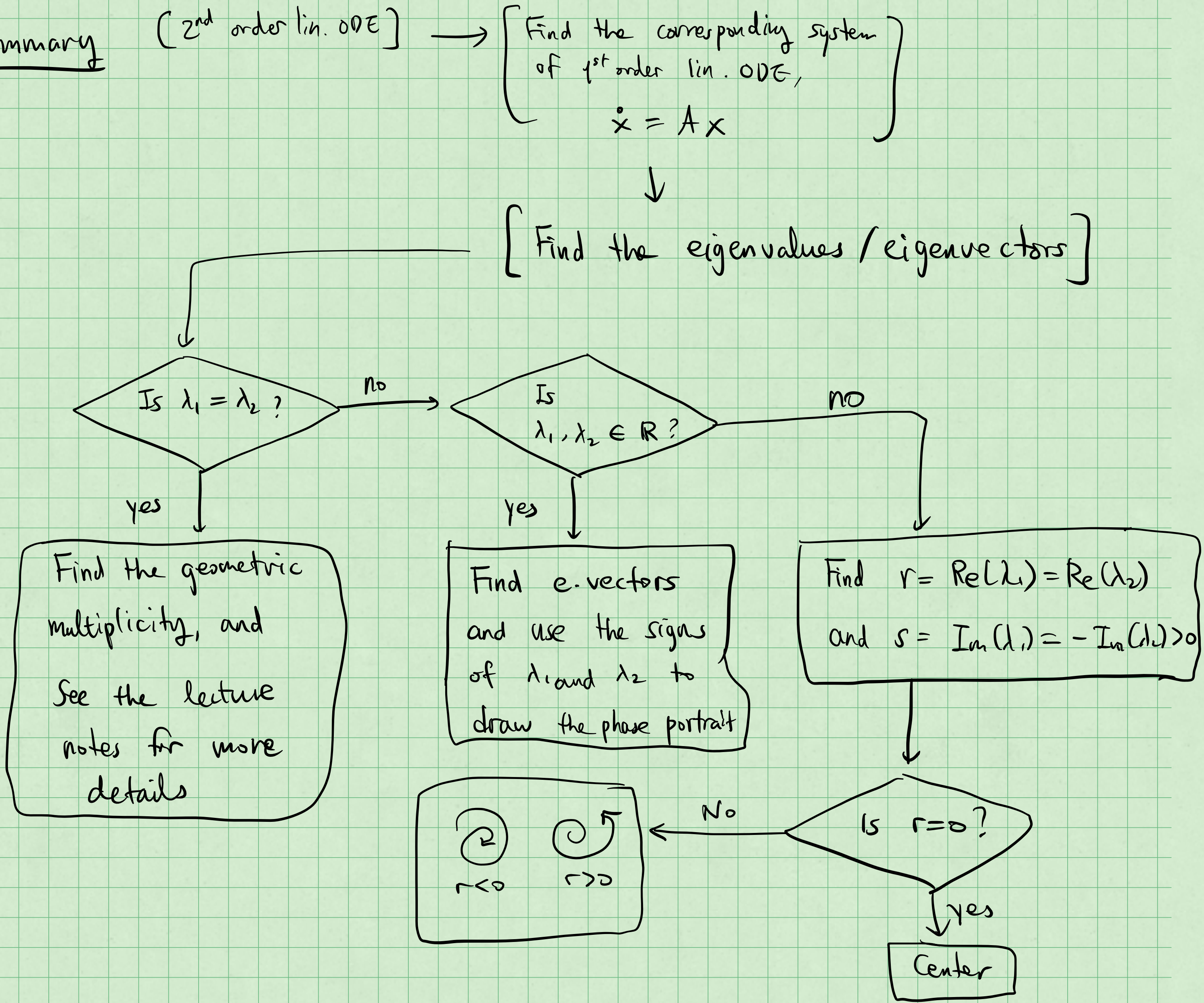


[st. spiral]

We've only seen the case of 2^{nd} order lin. ODEs, i.e., A has the form $\begin{bmatrix} 0 & 1 \\ -p & -q \end{bmatrix}$, but the theory above applies to any general matrix A, i.e., any "system of 1^{st} order linear ODEs".

(that is homogeneous and autonomous)

Summary


[2nd order lin. ODE]
[Find the corresponding system of 1st order lin. ODE, $\dot{x} = Ax$]
[Find the eigenvalues / eigenvectors]
Is $\lambda_1 = \lambda_2$?
no
yes
Find the geometric multiplicity, and See the lecture notes for more details
Is $\lambda_1, \lambda_2 \in \mathbb{R}$?
no
yes
Find e. vectors and use the signs of λ_1 and λ_2 to draw the phase portrait
Find $r = \mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2)$ and $s = \mathrm{Im}(\lambda_1) = -\mathrm{Im}(\lambda_2) > 0$
Is $r = 0$?
No
yes
$r < 0$
$r > 0$
Center

Exercise Problems

Draw the phase portraits for the following ODE's :

(A) $\dot{x} = x+y\,, \quad \dot{y} = 4x-2y$

(B) $\dot{x} = 2x+y\,, \quad \dot{y} = 3x+4y$

(C) $\dot{x} = ay\,, \quad \dot{y} = -bx\,, \qquad a, b > 0$

(d) $\dot{x} = x-y \quad \dot{y} = x+y$

(e) $\dot{x} = \lambda x + b\,, \quad \dot{y} = \lambda y\,, \qquad b \neq 0$