Week 2 2nd Order, Homogeneous, Autonomous Linear ODEs

$$\ddot{x} + p\dot{x} + qx = 0 \quad , \quad x(0) = x_0, \quad \dot{x}(0) = y_0$$

Let $y(t) := \dot{x}(t)$, then

$$\begin{cases} \dot{x} = y \\ \dot{y} = -py - qx \end{cases}$$

Therefore, by denoting $v := \begin{bmatrix} x \\ y \end{bmatrix}$ (i.e., $v : \mathbb{R} \to \mathbb{R}^2$),

$$\dot{v} = \underbrace{\begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix}}_{=:A} v = Av$$

E. values: $|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -q & -p-\lambda \end{vmatrix} = \lambda(p+\lambda) + q = \lambda^2 + p\lambda + q = 0$

$$\Leftrightarrow \quad \lambda_\pm = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

E. vectors: $\begin{bmatrix} -\lambda_\pm & 1 \\ -q & -p-\lambda_\pm \end{bmatrix} \begin{bmatrix} x_\pm \\ y_\pm \end{bmatrix} = 0 \Leftrightarrow -\lambda_\pm x_\pm + y_\pm = 0 \Leftrightarrow y_\pm = \lambda_\pm x_\pm$

$\Leftrightarrow \begin{bmatrix} x_\pm \\ y_\pm \end{bmatrix} = c \begin{bmatrix} 1 \\ \lambda_\pm \end{bmatrix}$ where $c \neq 0$ constant.

☆ Recall what we did in week 1. If $\lambda_+ \neq \lambda_-$ (i.e., $p^2 - 4q \neq 0$), then the eigenvectors $V_\pm = \begin{bmatrix} 1 \\ \lambda_\pm \end{bmatrix}$ span the whole space $\mathbb{R}^2$

and $A = \underbrace{\begin{bmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{bmatrix}}_{P} \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{bmatrix}^{-1} = PDP^{-1}$

$\dot{v} = Av \Leftrightarrow \frac{d}{dt}(P^{-1}v) = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}(P^{-1}v)$. Thus if $w := \begin{pmatrix} w_+ \\ w_- \end{pmatrix} = P^{-1}v$, then

$\dot{w}_\pm = \lambda_\pm w_\pm \Leftrightarrow w_\pm = A_\pm e^{\lambda_\pm t}$, where $A_\pm$ are constants ($= w_\pm(0)$).)

Then $v = Pw = \begin{bmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{bmatrix} \begin{bmatrix} A_+ e^{\lambda_+ t} \\ A_- e^{\lambda_- t} \end{bmatrix} = \begin{bmatrix} A_+ e^{\lambda_+ t} + A_- e^{\lambda_- t} \\ \cdots \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \leftarrow$ solution.

A Distinct Roots Thus, the general solution is, if $\lambda_+ \neq \lambda_-$ (i.e., $p^2-4q \neq 0$),

$$x = A_+ e^{\lambda_+ t} + A_- e^{\lambda_- t}.$$

$A_\pm$ can be determined by the initial condition:

A1 Two Real Roots $\begin{bmatrix} x(0) = x_0 \\ \dot{x}(0) = y_0 \end{bmatrix}$ gives $\begin{cases} x_0 = A_+ + A_- \\ y_0 = \lambda_+ A_+ + \lambda_- A_- \end{cases} \Leftrightarrow \begin{bmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{bmatrix}\begin{bmatrix} A_+ \\ A_- \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$

$$\Leftrightarrow \begin{bmatrix} A_+ \\ A_- \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{bmatrix}^{-1} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \frac{-1}{\lambda_+ - \lambda_-}\begin{pmatrix} \lambda_- & -1 \\ -\lambda_+ & 1 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

e.g. $\ddot{x} - 3\dot{x} + 2x = 0$. $p=-3, q=2 \rightarrow \lambda_+ = 2, \lambda_- = 1$, and

$x(t) = A_+ e^{2t} + A_- e^{t}$.

- If $x(0) = 3$, $\dot{x}(0) = 4$, then $\begin{bmatrix} A_+ \\ A_- \end{bmatrix} = -\frac{1}{2-1}\begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

and thus $x(t) = e^{2t} + 2e^{t}$.

A2 Complex Roots Note that when $p^2 - 4q < 0$, $\lambda_\pm = r \pm si$

where $r = -p/2$, $s = \sqrt{-(p^2-4q)}/2$

Then $A_+ e^{\lambda_+ t} + A_- e^{\lambda_- t}$

$$= (A_+ + A_-)e^{rt}\cos(st) + i(A_+ - A_-)e^{rt}\sin(st)$$

It is convenient to define $\begin{cases} B_+ = A_+ + A_- \\ B_- = i(A_+ - A_-) \end{cases}$

and denote the general solution by

$$x(t) = e^{rt}\left(B_+ \cos(st) + B_- \sin(st)\right)$$

$B_\pm$ can also be determined by the initial condition:

$\begin{cases} x_0 = x(0) = B_+ \\ y_0 = \dot{x}(0) = rB_+ + sB_- \end{cases} \Rightarrow \begin{bmatrix} 1 & 0 \\ r & s \end{bmatrix}\begin{bmatrix} B_+ \\ B_- \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} B_+ \\ B_- \end{bmatrix} = \frac{1}{s}\begin{bmatrix} s & 0 \\ -r & 1 \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ (Recall that $s > 0$)

e.g. $\ddot{x} + x = 0$

$p=0,\ q=1 \Rightarrow \lambda_\pm = \pm i,\ \begin{cases} r=0 \\ s=1 \end{cases} \Rightarrow x(t) = B_+ \cos t + B_- \sin t$

B_+ and B_- are determined by the initial condition $x(0)$ and $\dot{x}(0)$.

If $\begin{cases} x(0) = 1 \\ \dot{x}(0) = 2 \end{cases}$ then $\begin{bmatrix} B_+ \\ B_- \end{bmatrix} = \frac{1}{1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and

$x(t) = \cos t + 2\sin t$.

B Repeated Roots

Now, what if $\lambda_+ = \lambda_- = \lambda$? i.e., $p^2 - 4q = 0$ and A is not diagonalizable?

Recall the Jordan normal form which always exists:

$$A = PJP^{-1}, \text{ where } J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

Again, $P^{-1}\dot{v} = J(P^{-1}v)$ thus letting $w := \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = P^{-1}v$, we have

$$\begin{cases} \dot{w}_1 = \lambda w_1 + w_2 \\ \dot{w}_2 = \lambda w_2 \end{cases}$$

Solve using back substitution: $w_2 = e^{\lambda t}$, (up to a const. factor)

$$\dot{w}_1 = \lambda w_1 + e^{\lambda t} \Rightarrow w_1 = te^{\lambda t}$$

No need to find the matrix P of generalized eigenvectors.

Because $x = (v)_1 = (Pw)_1 = P_{11}w_1 + P_{12}w_2 = $ (linear comb. of w_1 and w_2) anyway.

Thus $x(t) = Ae^{\lambda t} + Bte^{\lambda t}$ and determine A and B with the initial condition.

$$\begin{cases} x_0 = x(0) = A \\ y_0 = \dot{x}(0) = \lambda A + B \end{cases} \Rightarrow \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 - \lambda x_0 \end{bmatrix}.$$

Application to a Spring-mass System

A spring - mass system w/ no forces other than a spring force $(-kx)$ and friction $(-cv)$

is governed by

$$ma = -cv - kx \qquad (\text{Newton's law} : F = ma)$$

whee $\begin{cases} a = \ddot{x} & \text{accerleration} \\ v = \dot{x} & \text{velocity} \\ \quad\; x & \text{displacement} \end{cases}$

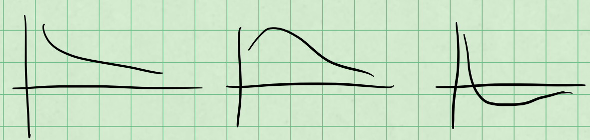
In other words, $\quad m\ddot{x} + c\dot{x} + kx = 0 \qquad$ (11.1 of the Textbook)

<u>Q</u> Classify, by using the theory covered today, 3 (possibly) qualitatively different classes of solutions ("oscillations") using the parameters $m, c,$ and k.

$D = c^2 - 4mk$

<u>A</u> $\quad c^2 - 4mk > 0 \quad$ (overdamped)

<u>B</u> $\quad c^2 = 4mk \quad$ (Critically damped)



<u>C</u> $\quad c^2 < 4mk \quad$ (underdamped)