Week 9 Method of Characteristics

- ($\approx$ #2 HW8) ρ : function of x and t

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = f(x)\rho & \leftarrow \text{ODE w.r.t. } t. \\ \rho(x,0) = \rho_0(x) \end{cases}$$

To see this easier, set $\rho_x(t) := \rho(x,t)$ ← function of t, x is a parameter

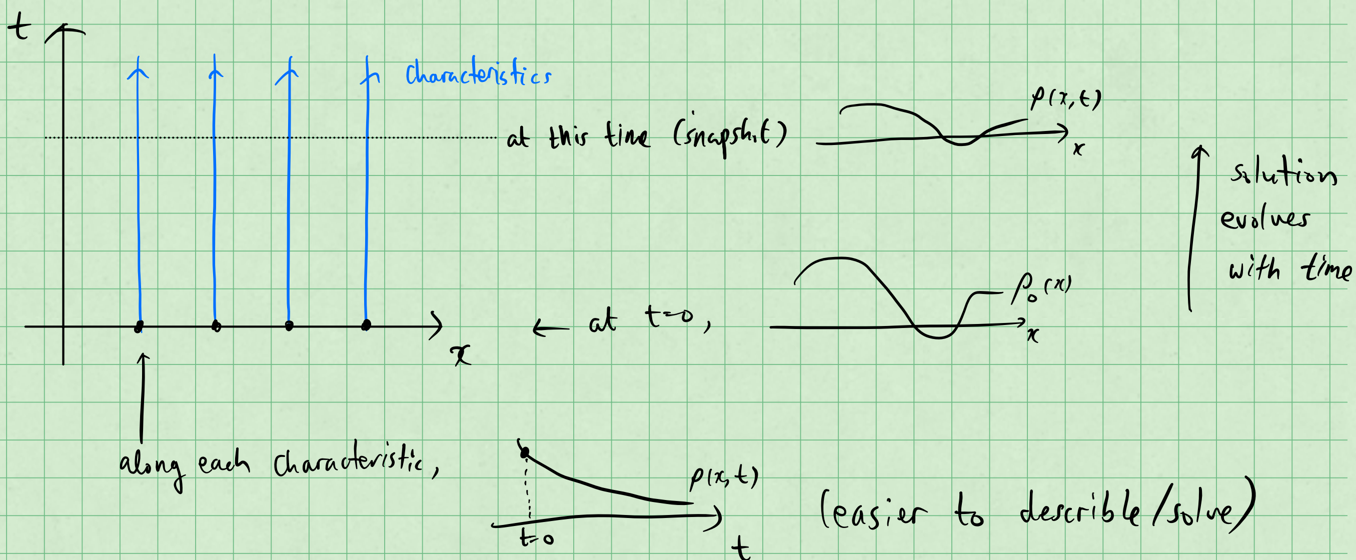
Then $\dot{\rho}_x = f(x)\rho$

$\Rightarrow \rho_x(t) = \rho_x(0)e^{f(x)t}$

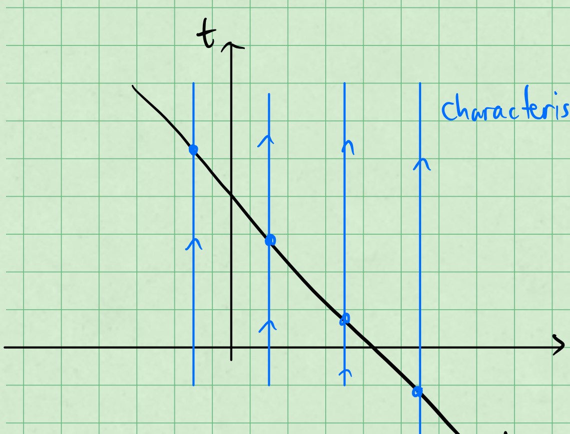
$\Rightarrow \rho(x,t) = \rho(x,0)e^{f(x)t} = \rho_0(x)e^{f(x)t}$

This is one of the simplest case for the method of characteristics;

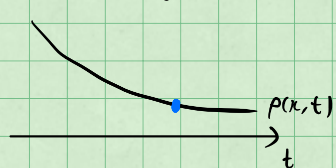
the characteristic lines are vertical lines $x \equiv$ const.



- ($\approx$ #3 HW8) What if we are given the a part of the solution, not along $t=0$ (I.C.) but along a curve that crosses the characteristics?



Characteristics → Still, along the characteristics, we know how the sol'n evolves.



$$\boxed{\text{HINT} \quad \dot{y} = Ay \Rightarrow y(t) = y(0)e^{At} = y(\tau)e^{A(t-\tau)} \quad \forall \tau \in \mathbb{R}}$$

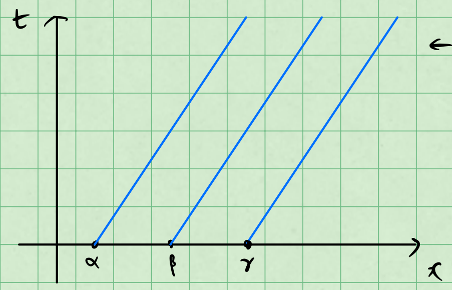
Sol'n is given on this curve (e.g. the observer moves over time)

- (≈ #4 HW8) Another simple case is

$$\begin{cases} u_t + c\,u_x = 0 \\ u(0,x) = u_0(x) \end{cases} \qquad u : (t,x) \longmapsto u(t,x)$$

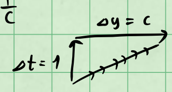
Consider a family of parametrized curves (lines) $y(t; y_0) = y_0 + ct$

(When $c > 0$)



← $y(t)$ with different starting pts $y_0 = \alpha, \beta, \gamma$

. slope of the curve $= \frac{1}{c}$



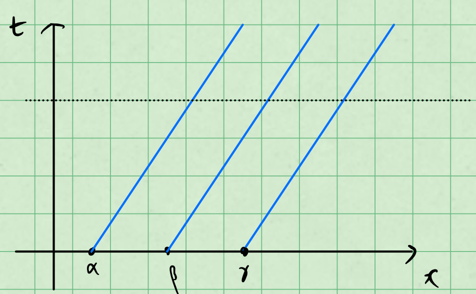
Along each curve, we'll see how the solution u evolves.

u along y : $u(t, y(t))$

$$\frac{d}{dt}\big(u(t, y(t))\big) = u_t(t, y(t)) + \dot{y}\, u_x(t, y(t))$$

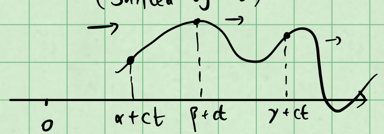
$$= u_t + c\,u_x = 0 \qquad \text{(By the PDE !!)}$$

Thus u is constant along each $y(t)$

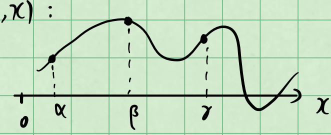


at this time,

(Shifted by ct)



← at $t=0$, $u(0,x)$:



Conservation Laws the Divergence Theorem

<u>Recall</u> (Divergence Theorem)

$V \subseteq \mathbb{R}^n$ compact, ∂V (boundary of V) : smooth

$F : \mathbb{R}^n \to \mathbb{R}^n$: continuously diff. vector field.

$$\Rightarrow \int_{\partial V} F \cdot \hat{n} \, dS = \int_V \nabla \cdot F \, dV$$

Now, let $\rho : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ denote the density,
$(t, x) \mapsto \rho(t, x)$

$v : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ denote the velocity field (Eulerian)
$(t, x) \mapsto v(t, x)$

and $F : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ denote the "flux" density
$(t, x) \longmapsto F(t, x)$

In Eulerian framework (i.e., fixed) when we consider an <u>arbitrary</u> volume element $V \subset \mathbb{R}^n$, the mass of the material in V at time t_0 is :

* Mass in V at t_0 : $\int_V \rho(t_0, x) \, dV$

If time passes and at time t_1, we get

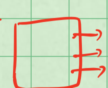
* Mass in V at t_1 : $\int_V \rho(t_1, x) \, dV$.

The change in mass must be caused by the incoming / outgoing flux.

Net flux going from V / into V at time t :

* Net flux at t : $-\int_{\partial V} F(t, x) \cdot \hat{n} \, dS$

Q Why negative?

A 

(because $\hat{n}$ is OUTWARD normal!)

The net amount of material moved from V / into V from time t_0 to t_1 is :

* Net change in mass through $[t_0, t_1]$: $-\int_{t_0}^{t_1} \int_{\partial V} F(t, x) \cdot \hat{n} \, dS$.

Thus, we get

$$\int_V \rho(t_1,x) - \rho(t_0,x) \, dV = -\int_{t_0}^{t_1} \int_{\partial V} F(t,x) \cdot \hat{n} \, dS \overset{\text{Div.Thm}}{=} -\int_{t_0}^{t_1} \int_V \nabla_x \cdot F \, dV$$

for arbitrary $t_0, t_1 \in \mathbb{R}$ and $V \subseteq \mathbb{R}^n$.

$\underline{\text{LHS}} = \int_V \int_{t_0}^{t_1} \frac{\partial}{\partial t} \rho(t,x) \, dt \, dV$, thus,

$$\int_V \int_{t_0}^{t_1} \underbrace{\frac{\partial}{\partial t} \rho(t,x) + \nabla_x \cdot F(t,x)}_{\Rightarrow \text{ The integrand must be identically zero}} \, dt \, dV = 0 \qquad \forall t_0, t_1 \in \mathbb{R}, \; V \subseteq \mathbb{R}^n$$

$$\Rightarrow \quad \frac{\partial}{\partial t} \rho(t,x) + \nabla_x \cdot F(t,x) = 0$$

"Continuity equation"

(A stronger version of the conservation law)

("local" conservation)

When the flux F is given by $F(t,x) = \rho(t,x) \, v(t,x)$,

we get $\rho_t + \nabla \cdot (\rho v) = 0$

(Mass conservation.)

※ If the fluid is incompressible, (i.e., ρ is constant, or volumetric strain rate $= 0$)

we have the volume continuity eq. $\nabla \cdot v = 0$

※ Conservation of linear momentum gives the famous Navier-Stokes eq.

($\approx$ #5HW8) • $\rho_t + (\rho v)_x = 0$ is non-linear w.r.t. ρ, unless v is not a function of ρ

(e.g. $v \equiv 1$, $\rho_t + \rho_x = 0$ is linear!)

Q. What is $v = x^2 e^{-t}$?

- What are the characteristics?

$$\rho_t + v\rho_x = -\rho v_t$$

$\Rightarrow$ Find the curves $t \mapsto y(t)$ s.t. $\frac{d}{dt}(\rho(t, y(t))) = \rho_t + v\rho_x$

$\Leftrightarrow \quad \dot{y} = v(t, y(t))$

If $v(t,x) = P(x)\, Q(t)$ (separable)

then $\dot{y} = \frac{dy}{dt} = P(y(t))\, Q(t)$

$$\Rightarrow \quad \frac{1}{P(y)}\, dy = Q(t)\, dt$$

$$\Rightarrow \quad \int \frac{1}{P(y)}\, dy = \int Q(t)\, dt$$

Solve for y

Then $y(t) = (\dots)$ function of $y(0)$ and t

- How does the solution evolve over time?

$\Rightarrow$ Reveal the behavior of the solution along the characteristics!

Recall : $y(t)$ is defined s.t. $\frac{d}{dt}(\rho(t, y(t))) = \rho_t + v\rho_x = -v_x \rho$

Set $p(t) := \rho(t, y(t))$ then

$$\dot{p} = -v_x(t, y(t))\, p$$

$$\frac{d}{dt}(\log p) = -v_x(t, y(t))$$

Note: $y(t)$ is a function of t and $y(0)$

$$\Rightarrow \quad \log p(t)/p(0) = -\int v_x(t, y(t))\, dt$$

$$p(t) = p(0)\, e^{-\int v_x(t, y(t))\, dt}$$

Then set $x = y(t)$, solve for $y(0)$ (in terms of x and t),

and you get $\rho(t,x)$ in terms of x and t!

$$\left(※ \quad \frac{d}{dt} \int_{x_1(t)}^{x_2(t)} \rho(t,x)\, dx = \int_{x_1(t)}^{x_2(t)} -(\rho v)_x\, dx + \rho(x_2, t)\, \underset{\substack{\uparrow \\ v(t,x_2)}}{\dot{x}_2} - \rho(x_1, t)\, \dot{x}_1 \right.$$

$$\left. = 0 \quad \text{"Transport thm"} \right)$$