The theory of games Lecture 2

More on game representations

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Announcements

- Reading for next week: 30–39; followed by 39–44.

Representation of Games in Normal Form

- Instead of using a tree, a table of strategies is used.
- Called “Normal Form” representation.
- Tree-based representation called “Extensive Form”.

		Player 2		
		L	M	R
Player 1	T	$2, 2$	$2, 0$	$0, 3$
	B	$3, 0$	$0, 9$	$1, 1$

Figure: Player 1 has 2 actions; Player 2 has 3 actions. General sum game.

With three or more players, multiple tables are needed.

Normal Form is particularly suitable for simultaneous play games

For example, Rock-Paper-Scissors. A zero-sum game.

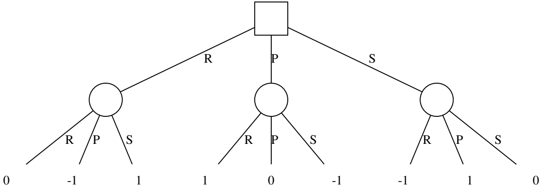
		Player 2		
		Rock	Paper	Scissors
Player 1	Rock	0	-1	1
	Paper	1	0	-1
	Scissors	-1	1	0

Figure: A Simultaneous-play game: Rock-Paper-Scissors.

Shown is payoff to Player 1.

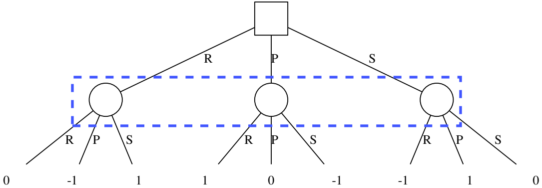
Simultaneous-play games are games of imperfect information!

Rock-Paper-Scissors in extensive form



What is missing? The information sets.

Rock-Paper-Scissors in extensive form



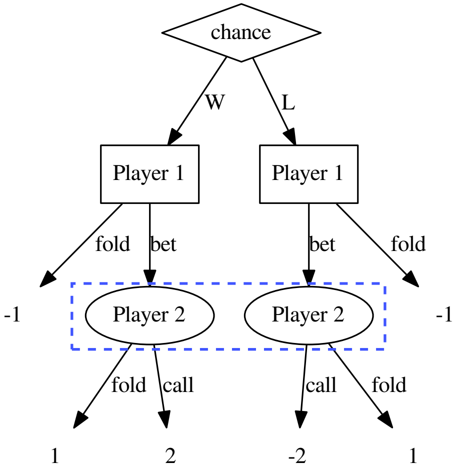
What is missing?

Games in normal form

Representation of games as a table of strategies is called *Normal Form*.

1. Each player simultaneously picks a strategy from their list of possible strategies.
2. They play the game.
3. The outcome can be read off in advance.

A boring way to play a game



What are the information sets?

2-player,2-card, tiny poker in normal form

Payoff to Player 1

		Player 2 Bet	Player 2 Fold
Player 1	Bet with W; Bet with L	0	1
	Bet with W; Fold with L	$1/2$	0
	Fold with W; Bet L	$-3/2$	0
	Fold with W; Fold with L	-1	-1

Figure: A turn-based game in normal form

Extensive form vs. Normal form

Extensive form: game represented as a game tree.

Normal form: game represented as a table of strategies.

Game trees are Huge

Strategy spaces are Huge

Measures of game size

1. Number of board positions which can occur in a game.
2. Number of decision nodes in the game tree. $\geq$ item 1.
3. Number of possible games $=$ number of terminal nodes
4. Number strategies. Sum of points 2 and 3. (Since every strategic decision leads to a terminal or non-terminal node.)

Very roughly, branching factor raised of depth of tree $\approx b^d$.

Here is the point

- With realistic-sized games, we work with game trees.

- Normal form is useful for theory, and for very small games.

Note: A tree can be searched much more efficiently than a list.

New topic

Solving games

Some definitions

Winning strategy: ensures a positive payoff for the player whatever the other players do.

Draw-ensuring strategy: ensures a payoff of at least zero for the player whatever the other players do.

An important theorem

Theorem

*For any **two-player, zero-sum** game with **perfect information** and **no chance**, that ends after a finite number of moves, one of the following is true.*

1. *Player 1 has a winning strategy;*
2. *Player 2 has a winning strategy;*
3. *Player 1 and Player 2 both have strategies which ensure at least a draw.*

Proof.

See notes on games, page 28. □

When such a strategy is found, the game is *solved*.

Questions to think about

Which of the assumptions are essential for the theorem to be true?

1. Two-player?
2. Zero-sum?
3. Perfect information?
4. No chance?

Levels of game solution

Ultra-weak: Proving which player can force a win, or draw for either without providing the strategy (non-constructive proof).

Weak: Provides the strategy whereby one player can win or either can draw, *starting at the beginning of the game*.

Strong: Providing the strategy which produces perfect play from *any* point in the game, even if mistakes have been made earlier.

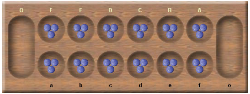
Source: Wikipedia article on “Solved game”

Solved games

- Connect 4 (1988) First player can force a win.

- Checkers/Draughts (2007) Either player can force a draw `http://www.sciencemag.org/content/317/5844/1518.`

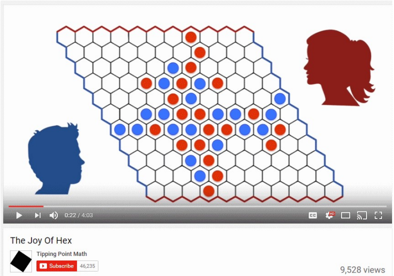
- Kalah $6, 6$ (2011) First player can force a win.

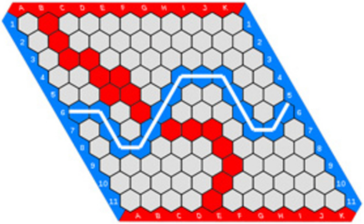


Solved games

- Tic Tac Toe - Both sides can ensure a draw.
- Hex - An ultra-weakly solved game. Player 1 can force a win.

Ultra-weakly solved game — hex





A game which always ends with a winner.

Nash's argument about hex (1952)

1. Hex cannot end in a draw; every game ends in a win.
2. Player 2 cannot have a winning strategy.
   - If Player 2 *had* a winning strategy, Player 1 could employ it, and win the game first,
   - pretending to be Player 2.
   - called a "strategy-stealing" argument.
3. Therefore, from the theorem above, Player 1 must have a winning strategy.

But the argument does not tell us what the winning strategy is.

Finding a winning strategy for Hex is an intrinsically hard computational problem.

Conclusions

1. Extensive form represents the game as a game tree.
2. Normal form represents the game as a list or table of strategies for each player.
   - Mathematically simple.
   - Computationally unfeasible for real games.
   - Strategy spaces tend to be exponential in the number of decision nodes associated with a player.
   - Game trees are linear in the number of decision nodes associated with all players.
3. An important theorem. Two-player, zero-sum games with perfect information and no chance can be solved.

Next topic

Techniques to solve games