The theory of games Lecture 4

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Announcements

Reading for today: pp 40 – 60 (proofs optional).

Simplified poker: Section 2.7. Solution to a more complex version.

Reading notes: Finish Chapter 2.

Best response strategy

Definition: A *best response strategy* is the strategy for a given player which yields the highest payoff to that player with the strategies of all the other players held fixed.

Nash Equilibrium

(page 31 of the notes)

Definition: A strategy profile $(s_1^*, s_2^*, \ldots, s_k^*)$ for a game with k players, is a ***Nash equilibrium*** if each strategy is a best response to all of the others.

- It is not a strategy; it is a choice of strategy for *all* players in the game.
- If the players are playing the Nash, no player has any incentive to change its strategy *unilaterally*.

Mini-max approach

For 2-player, zero-sum games in normal form:

Player 1: Finds the minimum payoff for each strategy and plays the strategy which maximizes it.

Player 2: Finds the maximum payoff to Player 1 for each strategy and plays the strategy which minimizes it.

If such a pair of strategies exists - they constitute a Nash equilibrium.

Mini-max is a worst-case analysis

From the prospective of both players:

- for each of my strategies, find the most harm my opponent can do to me;
- choose the strategy which makes that least bad.

Dominance

We also saw how the concept of dominance could be used to remove strategies from consideration.

- Strategy A dominates strategy B if the payoffs of A are better than the payoffs of B independent of what the opponent does.
- Dominated strategies can be removed.

Next topic

What about games with no *apparent* mini-max solutions?

		Column	
		A	B
Row	1	$(-1, 1)$	$(1, -1)$
	2	$(1, -1)$	$(-1, 1)$

Does this have a mini-max solution? Does it have a Nash equilibria?
We will answer these questions.

Mixed Strategies — Normal form

A mixed strategy is a strategy for a player in which:

- plays probabilistic combination of pure strategies;
- receives a probabilistic combination of payoffs.

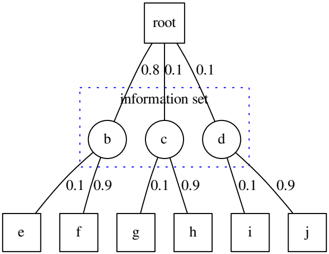
Normal Form:

- Assign probability q_i to the pure strategy i.
- Where $0 \leq q_i \leq 1$ and $\sum_i q_i = 1$.
- Choose strategy i with probability q_i.
- Get the appropriate payoff with probability q_i.

Mixed strategies in extensive form

Extensive form:

- At each node where the player has a decision, assign a probability function to each of the possible choices.



When mixed strategies are needed

Mixed strategies can be required when there is hidden information

- Extensive form games with hidden information (e.g. poker).
- Normal form games always have hidden information (the hidden strategy of the opponent(s).)

Mixed-strategy Nash equilibria

- Probabilistic combinations of pure strategies.
- **Nash:** All games with finite number of players and finite number of moves have at least one Nash equilibrium (mixed or pure).
- Mixed strategies may be needed games with hidden information.

Finding Nash equilbria — General case

General sum games: Best algorithm seems to be the Lemke-Howson algorithm, which can be exponential in time.

Zero-sum games in normal form: Generally can be solved using linear programming.

We will look at simple cases. The full tools of linear programming are not required.

A simple example: big brother — little brother game

- Two brothers Donald (older) and Eric (younger) go to the same high school.
- There are two cafes in the school: Red and Blue where they lunch every day.
- Donald wants to be in a different cafe than little brother,
- but Eric always wants to be near big brother.

		Eric: Red	Eric: Blue
Donald	Red	$(-1, 1)$	$(1, -1)$
	Blue	$(1, -1)$	$(-1, 1)$

The mixed strategies

Each choose the lunch spot probabilistically,

$$\text{Donald's choice } = \begin{cases} \text{Red cafe} & \text{with probability } x; \\ \text{Blue cafe} & \text{with probability } 1 - x. \end{cases}$$

$$\text{Eric's choice } = \begin{cases} \text{Red cafe} & \text{with probability } y; \\ \text{Blue cafe} & \text{with probability } 1 - y. \end{cases}$$

Expected payoff

- Since the strategies are probabilistic, so are the payoffs.
- So we need the expected payoffs.
- Let $U_1(S_1, S_2)$ is the payoff to Player 1 for a given set of pure strategies, (S_1, S_2),
- Expected (average) payoff — $\mathrm{Ex}\left[\mathrm{U}_1(\mathrm{S}_1, \mathrm{S}_2)\right]$ for pure or mixed strategies (S_1, S_2).

Optimal choice for Donald

To find the optimal value of x for Donald,

1. Find the expected payoffs against each of the strategies of the opponent.
2. Find the mixed strategy x which makes them equal.

Optimizing Donald's decision

If Eric always goes to the Red cafe:

$$\begin{aligned} \mathrm{Ex}\left[\mathrm{U}_1(\mathrm{S}_1(\mathrm{x}), \mathrm{Red})\right] &= (-1)x + (1)(1-x) \quad (1)\\ &= 1 - 2x. \quad (2) \end{aligned}$$

If Eric always goes to the Blue cafe:

$$\begin{aligned} \mathrm{Ex}\left[\mathrm{U}_1(\mathrm{S}_1(\mathrm{x}), \mathrm{Blue})\right] &= (1)x + (-1)(1-x) \quad (3)\\ &= 2x - 1. \quad (4) \end{aligned}$$

Task: Find the value of x such that

$$\mathrm{Ex}\left[\mathrm{U}_1(\mathrm{S}_1(\mathrm{x}), \mathrm{Red})\right] = \mathrm{Ex}\left[\mathrm{U}_1(\mathrm{S}_1(\mathrm{x}), \mathrm{Blue})\right]$$

$\boxed{x = 1/2}$

Why must the expected payoffs be equal?

If they are not equal then:

- Eric will have a pure strategy best response which will be higher.

If they are equal

- Eric will get the same payoff, whatever strategy he uses.
- Why?
- So Eric will have no incentive to unilaterally deviate from whatever he does.

Mathematical

- Let $S_1(x)$ be Donald's strategy. (A function of x.)
- Let Eric use a pure strategy: Red/Blue
- Suppose, expected payoff to Donald is bigger when Eric plays Red than Blue,

$$\mathrm{Ex}\left[\mathrm{U}_1(\mathrm{S}_1(\mathrm{x}), \mathrm{Red})\right] > \mathrm{Ex}\left[\mathrm{U}_1(\mathrm{S}_1(\mathrm{x}), \mathrm{Blue})\right]$$

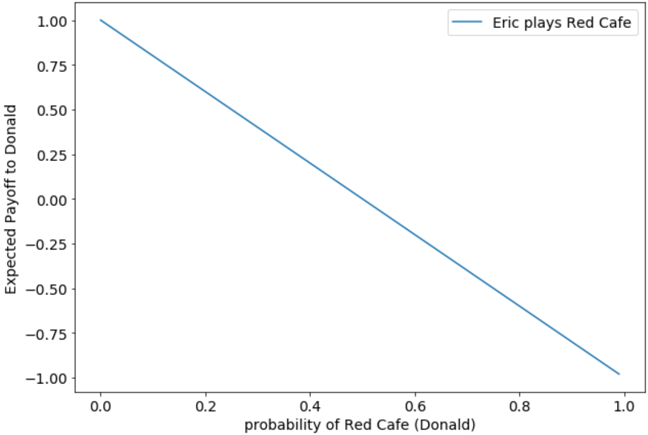
- Then Eric should play Blue against Donald's strategy $S_1(x)$.
- Then Donald should play Red against Eric's strategy Blue

$\vdots$

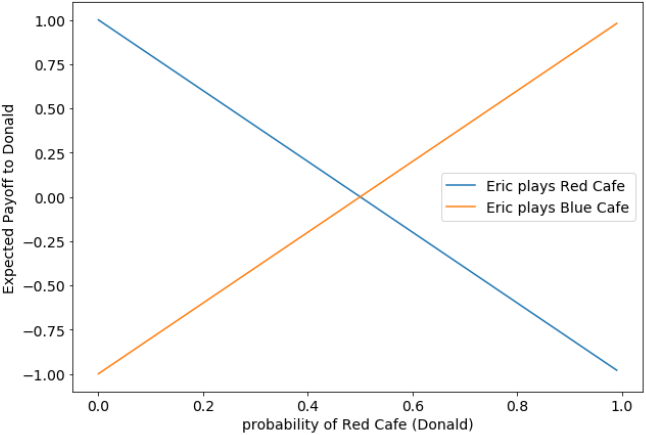
Relation to Mini-Max

This is the maximum of the minima in mixed strategy space.

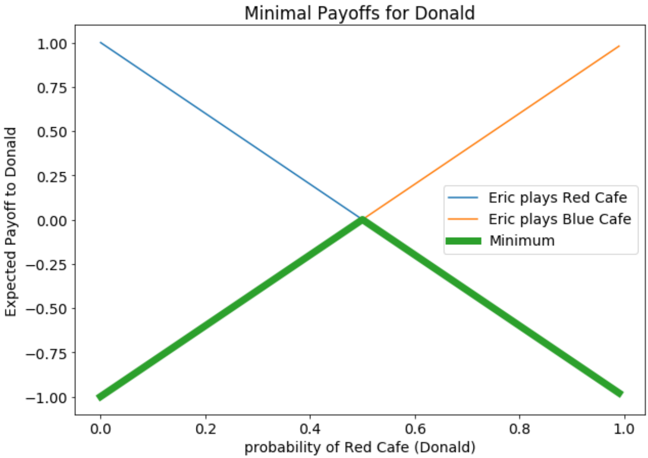
Graphical depiction



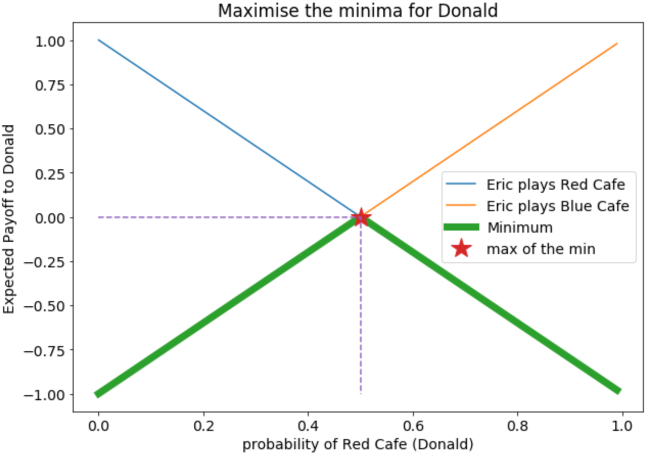
Graphical depiction



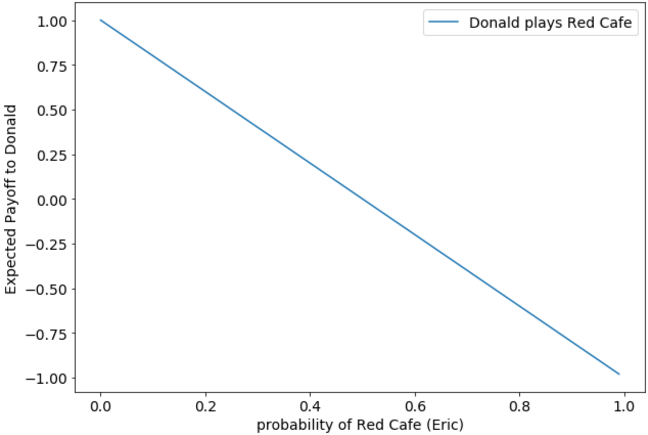
Graphical depiction



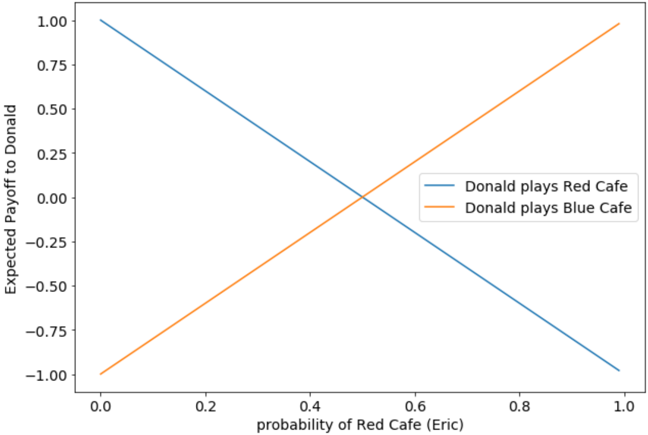
Donald is playing the maximum of the minima



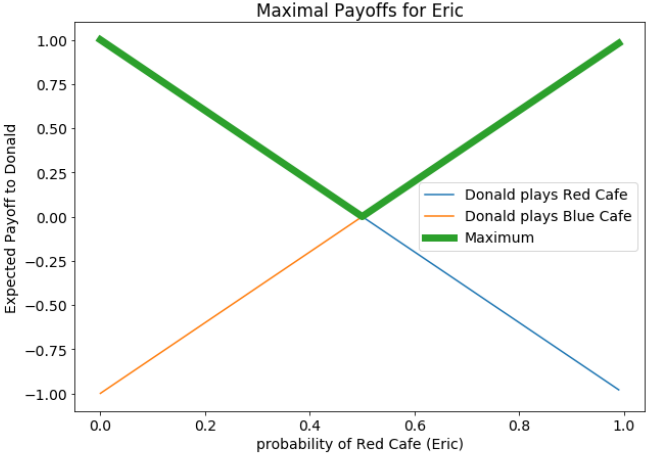
Graphical depiction from Eric's perspective



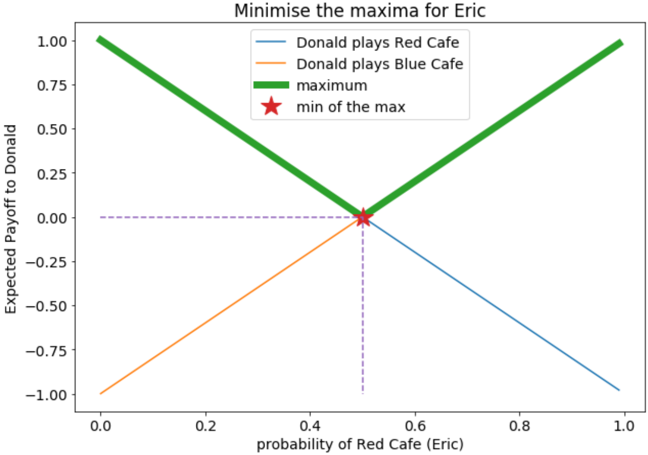
Graphical depiction from Eric's perspective



Graphical depiction from Eric's perspective



Eric plays the minimum of the maxima



The Nash equilibria

- $x = \frac{1}{2}$ and $y = \frac{1}{2}$.
- Donald and Eric both choose their lunch cafe independently with a random probability of 0.5.
- They are each happy half the time and unhappy half the time, on average.
- Neither has an incentive to unilaterally deviate, because *their payoff will not change*

Interesting property of the solution

A common feature of mixed strategy Nash equilibria

1. If one player plays its component of the of the *mixed* Nash equilibrium, the payoff is indifferent to what the other player does.

2. The other player must also play its component of the Nash equilibrium to force the first player to play its component and not take advantage.

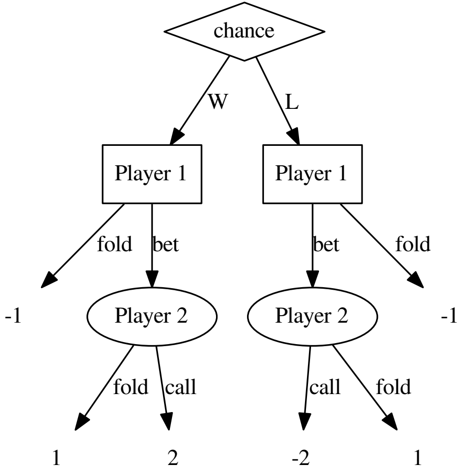
Key ideas

1. Introduce probabilities over pure strategies.
2. Set the expected payoffs against each opponent's pure strategy to be equal to find the value of the probabilities.
3. With 2-action, 2-player games, Nash equilibria can be found graphically.

2-Player, 2-card tiny poker

Equipment: A deck of cards consisting of two types of cards: W and L.

1. Each player puts £1 into the pot.
2. Player 1 draws a card from the deck, but does not show it to player 2.
3. Player 1 can then bet or fold. If he folds, player 2 gets the pot. If he bets, he puts another £1 into the pot.
4. If Player 1 bet, player 2 can call or fold. If he folds, player 1 gets the pot.
5. If player 2 calls, player 2 puts another £1 into the pot; there is a showdown, and player 1 must show his card.
6. W means player 1 wins; L means player 2 wins.


chance
W
L
Player 1
Player 1
fold
bet
bet
fold
-1
Player 2
Player 2
-1
fold
call
call
fold
1
2
-2
1


chance

Tiny Poker in normal form

		Student victim	
		Always bet	Always fold
Teacher	Bet with W; Bet with L	0	1
	Bet with W; Fold with L	$1/2$	0
	Fold with W; Bet with L	$-3/2$	0
	Fold with W; Fold with L	-1	-1

Finding the solution for tiny poker

Question: Is there a Nash equilibrium involving pure strategies only?

Answer: No!

1. If player 1 always bets, then player 2 should always bet.
2. But if player 2 always bets, then player 1 should fold with the losing card.
3. But if player 1 always folds with the losing card, then player 2 should always fold.
4. But if player 2 always folds, then player 1 should always bet.
5. And round and round it goes.

Now we play the game

(fun, fun, fun)

What did I do?

- With a winning card, I always bet. (Why?)
- With a losing card, I bet $1/3$ of the time.
- You should have bet $2/3$ of the time.

We seek a mixed strategy for tiny poker

Let

B^1_W: the probability of player 1 betting with the winning card.

B^1_L: the probability of player 1 betting with the losing card.

B^2: the probability of player 2 betting.

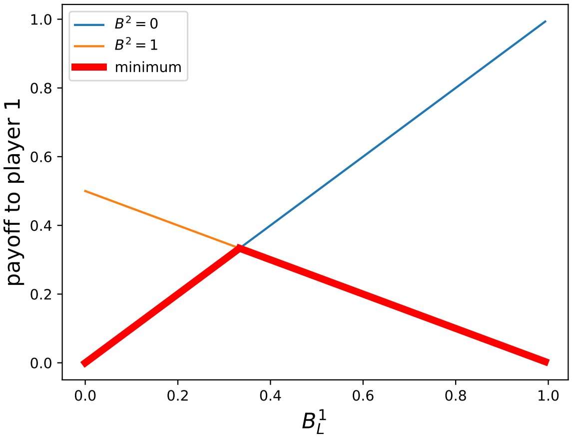
U: the payoff to player 1; $-U$ is the payoff to player 2

What should the value of B^1_W be?

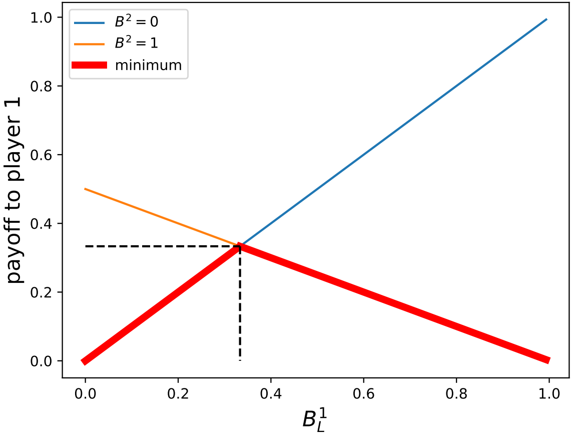
It is $B^1_W = 1$, by dominance.

▸ Back to Tiny Poker Trees

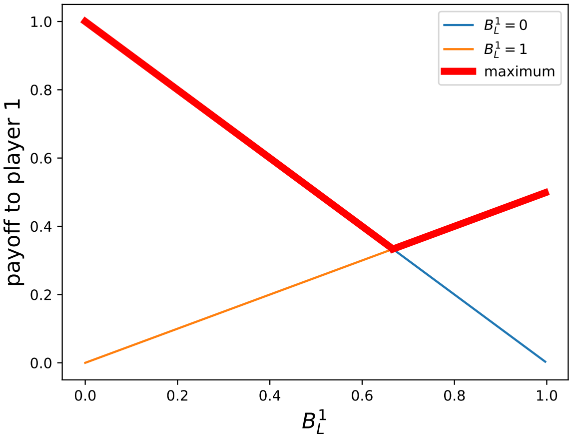
A graphical solution — for player 1



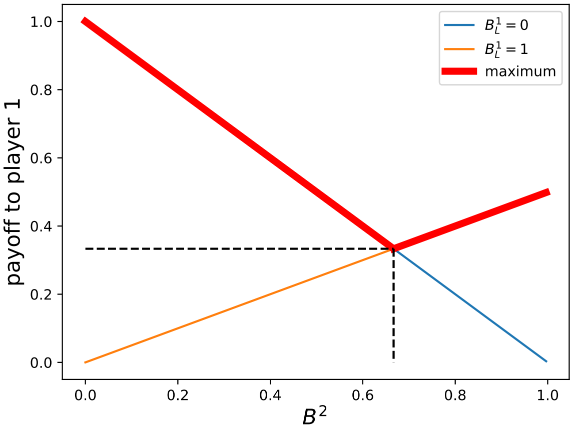
A graphical solution — for player 1



A graphical solution — for player 2



A graphical solution — for player 2



Player 1's mixed strategies against Player 2's pure strategies

Player 2 always bets: Draws the W card Draws the L card

$$U(B_L^1, B^2 = 1) = \frac{1}{2}\left[2 + (-2)B_L^1 + (-1)(1 - B_L^1)\right]$$
$$= \frac{1}{2}\left(1 - B_L^1\right).$$

Player 2 always folds:

$$U(B_L^1, B^2 = 0) = \frac{1}{2}\left[1 + (1)B_L^1 + (-1)(1 - B_L^1)\right]$$
$$= B_L^1.$$

Solution

- When $U(B_L^1, B^2 = 1) = U(B_L^1, B^2 = 0)$.
- Why?
- Otherwise, Player 2 would have a best pure strategy.
- Can be view as a Max-min solution.

We can solve for $B_L^1 = 1/3$. So, bluff one third of the time for an expected payoff of $1/3$ per game.

Player 2's mixed strategies against Player 1's pure strategies

Assume that Player 2 knows that B^1_W is 1.

Player 1 always bets with losing card: Draws the W card Draws the L card

$$U(B^1_L = 1, B^2) = \frac{1}{2}\left[(2)B^2 + (1)(1 - B^2) + (-2)B^2 + (1)(1 - B^2)\right],$$
$$= \left(1 - B^2\right).$$

Player 1 always folds with losing card:

$$U(B^1_L = 0, B^2) = \frac{1}{2}\left[(2)B^2 + (1)(1 - B^2) - 1\right]$$
$$= \frac{B^2}{2}.$$

Setting $U(B^1_L = 1, B^2) = U(B^1_L = 0, B^2)$ gives: $B^2 = 2/3$ with a payoff to Player 2 of $(-1/3)$.

Conclusions

1. There exists a mini-max equilibrium for all two-player games with perfect information, even with chance.
2. With imperfect information, we have to allow for mixed strategies.
3. All finite games have at least one Nash equilibrium.

Generalized Rock-Paper-Scissors

		Player 2		
		Rock	*Paper*	*Scissors*
Player 1	*Rock*	0	-1	1
	Paper	1	0	-1
	Scissors	-1	1	0

Solution: 1/3, 1/3, 1/3

		Player 2		
		Rock	*Paper*	*Scissors*
Player 1	*Rock*	0	-1	1000
	Paper	1	0	-1
	Scissors	-1000	1	0

Solution: ?, ?, ?

Question

- Could the Nash equilibrium consist of pure strategies?
- Is both players playing $1/3, 1/3, 1/3$ still a Nash equilibrium?
- Why or why not?

Strategies

Player 1:

- x_R - the probability of playing Rock;
- x_P - the probability of playing Paper;
- x_S - the probability of playing Scissors.

Obviously, $x_R + x_P + x_S = 1$

Player 2: Likewise with y_R, y_P, and y_S.

Payoff to player 1

$$U^{(1)}(R) = [-y_P + 1000y_S]$$
$$U^{(1)}(P) = [y_R - y_S]$$
$$U^{(1)}(S) = [-1000y_R + y_P]$$

To make a Nash equilibrium, make the terms in square brackets the same.
Why?

To make a Nash equilibrium, make the terms in square brackets the same.

Why?

- Because, if otherwise, the biggest one would be the best response, and thereby would constitute a *pure Nash equilibrium strategy*.
- Thus,

$$U^{(1)}(R) = [K]$$
$$U^{(1)}(P) = [K]$$
$$U^{(1)}(S) = [K]$$

$$U^{(1)} = K.$$

So Player 2 also wants to make K as small as possible.
Any thoughts on the value of K?

Solving

We need to solve the linear equation set of equations.

$$\begin{aligned} [-y_P + 1000y_S] &= K \\ [y_R - y_S] &= K \\ [-1000y_R + y_P] &= K, \end{aligned}$$

with

$$y_R + y_P + y_S = 1,$$

which can only be solved if $K = 0$.

Solution

$$\begin{aligned} y_R &= \frac{1}{1002}; \\ y_P &= \frac{1000}{1002}; \\ y_S &= \frac{1}{1002}; \end{aligned}$$

By symmetry, x_R, x_P, x_S are likewise.
So, almost always play paper.

General-sum games

In a two-action, two-player not zero-sum game, it works like this.

- Player 1 chooses its probabilities so that the payoff to *Player 2* is the same whatever strategy Player 2 plays.
- Player 2 chooses its probabilities so that the payoff to *Player 1* is the same whatever strategy Player 1 plays.

Then,

- neither player will have an incentive to deviate from its mixed strategy;
- So, it will constitue a Nash equilbrium.

General sum games

Player 1: strategies: $\mathbf{s}_1 \in \{s_1^1, s_1^2\}$, and x is the probability of playing strategy s_1^1.

Player 2: strategies: $\mathbf{s}_2 \in \{s_2^1, s_2^2\}$, and y is the probability of playing strategy s_1^1.

So, subscripts denote which player; superscripts denote which strategy.

Let $U_i(S_1, S_2)$ be the payoff to player i when Player 1 plays S_i.

Of course, $U_1(S_1, S_2) \neq -U_2(S_1, S_2)$; it is general sum.

Mixed-strategy Nash equilibrium

If:

Player 1: chooses x so that,

$$\mathrm{Ex}\left[\mathrm{U}_2(\mathrm{x}, \mathrm{s}_2^1)\right] = \mathrm{Ex}\left[\mathrm{U}_2(\mathrm{x}, \mathrm{s}_2^2)\right] \tag{5}$$

And

Player 2: choose y so that,

$$\mathrm{Ex}\left[\mathrm{U}_1(\mathrm{s}_1^1, \mathrm{y})\right] = \mathrm{Ex}\left[\mathrm{U}_1(\mathrm{s}_1^2), \mathrm{y}\right] \tag{6}$$

Then: Neither player will do better by deviating its strategy.

This will be a Nash equilibrium!

Conclusions on mixed strategies

- Mixed strategies are probabilistic mixtures of pure strategies.
- Once mixed strategies are allowed, equilibria exist in all finite games (Nash).
- For two-player, zero-sum games, finding equilibria is a linear problem.
- Mixed strategies are often required to prevent the opponents from predicting your actions.