# The theory of games Lecture 3

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# Announcements

Reading for today: pp 30 — 39

Reading for week: pp 40 —44 (proofs optional). Start reading Section 2.5.

Reading notes:

- Between page 42 and 47, “equilibrium” means with *pure* strategies.
- In general, “Nash equilibrium” includes mixed strategies (as we will see).
- Throughout, equilibrium and Nash equilibrium are equivalent.

# Next topic - How to find winning strategies

New concepts:

1. Best response;
2. Nash equilibrium.

# Best response

- For player $i$,
- With all the opponent strategies fixed,
- The best response is a strategy which gives the highest payoff to Player i.

# Best response

(Page 30 of the notes.)

- Let $s_i$ be the strategy of player $i$, $i = 1, \ldots, k$.
- **Definition:** $s_i^*$ is a *best response* to the collection of opponent strategies if and only if
  - no other strategy which player $i$ can play gives a higher payoff.

(There can be more than one best response strategy.)

# Existence of best response strategies

**Question:** is it always possible to find a best response to a fixed set of opponent strategies?

Finite strategy space: Yes. Check all strategies. Play (one of) the best one(s).

Infinite strategy space: Perhaps not. Counter example on the next slide.

# A game with no best response

A zero-sum game in Normal form (Simultaneous play).

1. Player 1 chooses a *positive* real number $X_1$.
2. Player 2 chooses a *positive* real number $X_2$.
3. If $X_2 > X_1$ Player 1 wins.
4. If $X_2 < X_1$ Player 2 wins.

Optimal strategies: Each player needs to play the smallest real number greater than zero.

No such number exists!

# Nash Equilibrium

(page 31 of the notes)

**Definition:** A strategy profile $(s_1^*, s_2^*, \ldots, s_k^*)$ for a game with $k$ players, is a ***Nash equilibrium*** if each strategy is a best response to all of the others.

- It is not a strategy; it is a choice of strategy for *all* players in the game.
- If the players are playing the Nash, no player has any incentive to change its strategy *unilaterally*.

# Relation to solving the game

For a two-player, zero-sum game with perfect information and no chance:

- Solving the game means finding the Nash equilibrium.
- Each is playing best response to the other.
- Which has the winning strategy. (Or do they both draw?)

# Simple examples

(Just check all four strategy pairs)

| | | Column 1 | Column 2 |
|---|---|---|---|
| Row | 1 | $(1, -1)$ | $(2, -2)$ |
| | 2 | $(0, 0)$ | $(1, -1)$ |

| | | Column 1 | Column 2 |
|---|---|---|---|
| Row | 1 | $(1, 1)$ | $(1, 0)$ |
| | 2 | $(0, 1)$ | $(2, 2)$ |

| | | Column 1 | Column 2 |
|---|---|---|---|
| Row | 1 | $(-1, 1)$ | $(1, -1)$ |
| | 2 | $(1, -1)$ | $(-1, 1)$ |

# Chicken

Two teenagers drive their cars straight at each other at high speed to prove they are not chicken.

- If one swerves and the other doesn't, the swerver is chicken and loses face.
- If both swerve, neither loses face.
- If neither swerve, the cars crash, and they both get smashed in the face.

| | | Bob: swerve | Bob: drive straight |
|---|---|---|---|
| Alice | swerve | $(0, 0)$ | $(-10, 10)$ |
| Alice | drive straight | $(10, -10)$ | $(-1000, -1000)$ |

Are there any Nash equilibria?

# Two-player, zero-sum games — finding equilibria

In what follows, two-player, zero-sum games are considered.
"The payoff" will mean *the payoff to Player 1*.

- Player 1 wants to **maximize** this;
- Player 2 wants to **minimize** this.

# Two-player, zero-sum games — the mini-max approach

(Example 2.3; Section 2.3 of notes)

A worst-case analysis

Player 1 (the maximizing player): for each of his strategies identifies the opponent strategy which gives the lowest payoff. Plays the strategy which maximizes this.

Player 2 (the minimizing player): for each of her strategies identifies the opponent strategy which gives the highest payoff[1]. Plays the strategy which minimizes this.

If both find the same pair of strategies, this strategy pair is a Nash equilibrium.

[1] to Player 1

# In mathematics

Definition: Let $a_{ij}$ be payoff to player 1, when player 1 plays its $i$th strategy, and player 2 plays its $j$th strategy.

Player 1:
- For each strategy $i$, find $m_i^1 = \min_j a_{ij}$.
- Play the strategy $i^* = \text{argmax}_i \, m_i^1$

Player 2:
- For each strategy $j$, find $m_j^2 = \max_i a_{ij}$.
- Play the strategy $j^* = \arg\min_j m_j^2$

Result: if $\max_i \min_j a_{ij} = a_{i^* j^*} = \min_j \max_i a_{ij}$, that point is a Nash equilibrium.

# Side note on `argmax`

Argmax is the *argument which maximizes*.
So, if $f(x)$ is a function on the unit interval, $x \in [0, 1]$,

- $\max_{x' \in [0,1]} f(x')$ is the maximum value which $f(x)$ takes on the unit interval
- $\arg\max_{x' \in [0,1]} f(x')$ is the $x$ that achieves that value.

So,

$$\text{If } x^* = \arg\max_{x' \in [0,1]} f(x'),$$
$$\text{then } f(x^*) = \max_{x' \in [0,1]} f(x').$$

# Mini-max approach

Mini-max approach is a *worst-case* analysis.

- The goal is to *minimize* the harm your opponent does to you.
- (rather than maximizing your own benefit).

# A zero-sum example

From the notes, Example 2.3 (page 33)

| | | Scott 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Amelia | 1 | 7 | 2 | 5 | 1 |
| | 2 | 2 | 2 | 3 | 4 |
| | 3 | 5 | 3 | 4 | 4 |
| | 4 | 3 | 2 | 1 | 6 |

# Dominance

(From Section 2.6, p51)

- A strategy $s_i$ *dominates* a strategy $s_i'$ if the payoff of $s$ against any opponent strategy is not less than that of $s'$ against the same opponent strategy, for all opponent strategies.
- In math:
  - let $s_i$ be a strategy for player i; $s_{-i}$ be the strategies of all players *except* $i$ .
  - strategy $s_i$ *dominates* a strategy $s_i'$ means

$$\text{payoff}(s_i, s_{-i}) \geq \text{payoff}(s_i', s_{-i}),$$

    for all possible opponent strategies, $s_{-i}$.
- Strategies which are dominated can be removed.

# Example of dominance in a zero-sum game

Player 2

| Player 1 | | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| | 1 | 4 | 3 | 1 | 1 |
| | 2 | 3 | 2 | 2 | 1 |
| | 3 | 4 | 4 | 3 | 2 |
| | 4 | 3 | 3 | 2 | 1 |

Player 1: strategy 3 dominates all others.

Player 2: strategy 4 dominates all others.

So, player 1 plays strategy 3, player 2 plays strategy 4.

# Dominance can be used to eliminate some strategies

|  |  | Player 2 |  |  |  |
|---|---|---|---|---|---|
|  |  | 1 | 2 | 3 | 4 |
| Player 1 | 1 | 2 | -2 | 1 | -1 |
|  | 2 | 0 | 0 | 1 | 0 |
|  | 3 | 1 | 2 | 1 | 0 |

Player 1: strategy 3 dominates strategy 2 (so remove strategy 2).

Player 2: strategy 4 dominates strategies 1 and 3 (so remove them).

# After elimination of dominated strategies

Player 2

| | | 2 | 4 |
|---|---|---|---|
| Player 1 | 1 | -2 | -1 |
| | 3 | 2 | 0 |

Then

Player 2

| | | 4 |
|---|---|---|
| Player 1 | 3 | 0 |

Solving yields:

Player 1: Strategy 3,

Player 2: Strategy 4,

forms a Nash equilibrium.

# Next topic

**What about games with no *apparent* mini-max solutions?**

| | | Column A | Column B |
|---|---|---|---|
| Row | 1 | $(-1, 1)$ | $(1, -1)$ |
| | 2 | $(1, -1)$ | $(-1, 1)$ |

Does this have a mini-max solution? We will answer this

Note here $\min_j \max_i a_{ij} > \max_i \min_j a_{ij}$.

# Mixed Strategies

A mixed strategy is a strategy for a player in which:

- plays probabilistic combination of pure strategies;
- receives a probabilistic combination of payoffs.

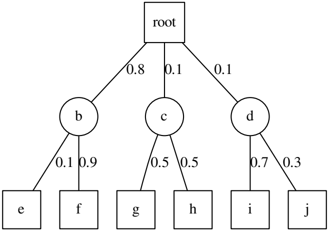
Normal Form:

- Assign probability $q_i$ to the pure strategy $i$.
- Where $0 \leq q_i \leq 1$ and $\sum_i q_i = 1$.
- Choose strategy $i$ with probability $q_i$.
- Get the appropriate payoff with probability $q_i$.

# Mixed strategies in extensive form

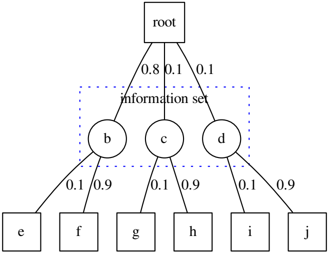
## Extensive form:

- At each node where the player has a decision, assign a probability function to each of the possible choices.



# Mixed strategies in extensive form

Extensive form: Needed when there is hidden information!



# Celebrated theorems

John von Neumann (1928): von Neumann showed that every zero-sum game with perfect information has solutions which minimize the maximum losses for the players, but mixed strategies may be required. von Neumann invented game theory as a mathematical discipline.

Nash (1950): Every game with a finite number of players in which each player has a finite number of pure strategies has at least one Nash equilibrium. Some will involve mixed strategies.

# When mixed strategies are needed

Mixed strategies *may* be required when there is hidden information

- Extensive form games with hidden information (e.g. poker).
- Normal form games always have hidden information (the hidden strategy of the opponent(s).)

Skip topic

# The prisoners dilemma

Alice and Bob commit a crime and are caught. They are taken to separate interrogation rooms, they can not communicate, and are told:

**If one of you gives evidence:** that person will go free; the other fully prosecuted.

**If both give evidence:** both will be prosecuted, with a bit of leniency.

**If neither give evidence:** both will be prosecuted on whatever evidence they have, for a lessor crime.

# The prisoners dilemma

| | | Bob: Talk | Bob: Be silent |
|---|---|---|---|
| Alice | Talk | $(-8, -8)$ | $(0, -10)$ |
| | Be silent | $(-10, 0)$ | $(-2, -2)$ |

- What is the Nash Equilibrium?
- What is the “best solution” for the pair of players?

stay silent: Usually referred to as *Cooperate*

Talk: Usually referred to as *Defect*

Why do we cooperate? What does game theory have to say about it?

# Conclusions

To find *pure-strategy* Nash equilibria in two-player, zero-sum games with no chance:

1. Exhaustive search over all strategy pairs
2. Use the mini-max method
3. Use dominance

To find *mixed-strategy* Nash equilibria in two-player, zero-sum games with no chance:
Come to the next lecture

# New concepts

1. **Best response:** The action(s) for a player which gives the highest payoff, against a set of fixed strategies for the other players.
2. A **Nash equilibrium** is a collection of strategies for all players such that each player is playing best response to all the others. Exists in general sum games.
3. **Mini-max approach:** For a zero-sum, two-player game in normal form, a strategy pair which is maximal in its columns and minimal in its rows is a Nash equilibrium.
4. **Dominance** can be used to reduce the number of strategies.