Equations of Motion for 1111.1655v2

Cosmological Solutions in Bimetric Gravity and their Observational Tests

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Notebook by Mikica Kocic, created 2015-11-07, last modified 2016-05-12

Initialization

Get[NotebookDirectory[] <> $PathnameSeparator <> "xAct-B1.m"]

```
------------------------------------------------------------
Loading xAct adapted to bimetric theory...
Copyright (C) 2014-2015 by Mikica B. Kocic, under GPL.
------------------------------------------------------------
Package xAct`xPerm`  version 1.2.3, {2015, 8, 23}
CopyRight (C) 2003-2015, Jose M. Martin-Garcia, under the General Public License.
Connecting to external MinGW executable...
Connection established.
------------------------------------------------------------
Package xAct`xTensor`  version 1.1.2, {2015, 8, 23}
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------------------------------------------------------------
Package xAct`xCoba`  version 0.8.3, {2015, 8, 23}
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------------------------------------------------------------
------------------------------------------------------------
Package xAct`TexAct`  version 0.3.7, {2015, 8, 23}
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------------------------------------------------------------
Context path: {xAct`Bim`, xAct`xCoba`, xAct`TexAct`, System`}
Memory in use: 15 488 280
Time used: 1.513
------------------------------------------------------------
```

Manifold and the accompanying abstract indices

DefManifold[$\mathcal{M}$, **4**, {a, b, c, d, e, f, g, h} **]**

```
** DefManifold: Defining manifold 𝓜.
** DefVBundle: Defining vbundle Tangent𝓜.
```

Chart 𝓑 on 𝓜 with the coordinate fields Scalars𝓑

```
Scalars𝓑 = Sequence[ t[], r[], θ[], ϕ[] ];

assumeReal[ t[], t[] > 0 ]
assumeReal[ r[], r[] > 0 ]

DefChart[ 𝓑, 𝓜, Range[0, 3], {Scalars𝓑} ]

** DefChart: Defining chart 𝓑.
** DefTensor: Defining coordinate scalar t[].
** DefTensor: Defining coordinate scalar r[].
** DefTensor: Defining coordinate scalar θ[].
** DefTensor: Defining coordinate scalar ϕ[].
** DefMapping: Defining mapping 𝓑.
** DefMapping: Defining inverse mapping i𝓑.
** DefTensor: Defining mapping differential tensor di𝓑[-ạ, i𝓑a].
** DefTensor: Defining mapping differential tensor d𝓑[-a, 𝓑ạ].
** DefBasis: Defining basis 𝓑. Coordinated basis.
** DefCovD: Defining parallel derivative PD𝓑[-a].
** DefTensor: Defining vanishing torsion tensor TorsionPD𝓑[a, -b, -c].
** DefTensor: Defining symmetric Christoffel tensor ChristoffelPD𝓑[a, -b, -c].
** DefTensor: Defining vanishing Riemann tensor RiemannPD𝓑[-a, -b, -c, d].
** DefTensor: Defining vanishing Ricci tensor RicciPD𝓑[-a, -b].
** DefTensor: Defining antisymmetric +1 density etaUp𝓑[a, b, c, d].
** DefTensor: Defining antisymmetric -1 density etaDown𝓑[-a, -b, -c, -d].
```

Fields (metric components)

Define fields (functions of coordinates) to be used as metric components.
Underscore operator denotes functions of `Φield` variables.

```
Φield = Sequence[ t[] ];

defineField /@ { a, X, Y, Υ };

** DefScalarFunction: Defining scalar function a.
** DefScalarFunction: Defining scalar function X.
** DefScalarFunction: Defining scalar function Y.
** DefScalarFunction: Defining scalar function Υ.

assumeReal[ a, a > 0 ]
assumeReal[ X, X > 0 ]
assumeReal[ Y, Y > 0 ]
```

Constants

```
DefConstantSymbol[ κ ]
assumeReal[ κ ]

** DefConstantSymbol: Defining constant symbol κ.
```

```
DefConstantSymbol[ m ]
assumeReal[ m, m > 0 ]
```

** DefConstantSymbol: Defining constant symbol m.

```
DefConstantSymbol[ M★, PrintAs → "M★" ]
assumeReal[ M★, M★ > 0 ]
```

** DefConstantSymbol: Defining constant symbol M★.

```
DefConstantSymbol[ ρ★, PrintAs → "ρ★" ]
assumeReal[ ρ★, ρ★ > 0 ]
```

** DefConstantSymbol: Defining constant symbol ρ★.

Beta parameters

```
DefInertHead[β]
DefConstantSymbol[ β[0] = β0, PrintAs → "β0" ];
DefConstantSymbol[ β[1] = β1, PrintAs → "β1" ];
DefConstantSymbol[ β[2] = β2, PrintAs → "β2" ];
DefConstantSymbol[ β[3] = β3, PrintAs → "β3" ];
DefConstantSymbol[ β[4] = β4, PrintAs → "β4" ];
```

** DefInertHead: Defining inert head β.

** DefConstantSymbol: Defining constant symbol β0.

** DefConstantSymbol: Defining constant symbol β1.

** DefConstantSymbol: Defining constant symbol β2.

** DefConstantSymbol: Defining constant symbol β3.

** DefConstantSymbol: Defining constant symbol β4.

```
assumeReal@{ β0, β1, β2, β3, β4 }
```

Define metric $g = E^{*}\,\eta$ through a vielbein E

$$E\blacksquare = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\underline{a}}{\sqrt{1-\kappa\, r[]^2}} & 0 & 0 \\ 0 & 0 & \underline{a}\, r[] & 0 \\ 0 & 0 & 0 & \underline{a}\, r[]\, \mathrm{Sin}[\theta[]] \end{pmatrix};$$

Define metric $f = L^{*}\,\eta$ through a vielbein L

$$L\blacksquare = \begin{pmatrix} \underline{X} & 0 & 0 & 0 \\ 0 & \frac{\underline{Y}}{\sqrt{1-\kappa\, r[]^2}} & 0 & 0 \\ 0 & 0 & \underline{Y}\, r[] & 0 \\ 0 & 0 & 0 & \underline{Y}\, r[]\, \mathrm{Sin}[\theta[]] \end{pmatrix};$$

Complete the initialization

```
Get[NotebookDirectory[] <> $PathnameSeparator <> "xAct-B2.m"]
```

** DefTensor: Defining symmetric metric tensor g⌘[-a, -b].

** DefTensor: Defining antisymmetric tensor epsilong⌘[-a, -b, -c, -d].

** DefTensor: Defining tetrametric Tetrag⌘[-a, -b, -c, -d].

** DefTensor: Defining tetrametric Tetrag⌘†[-a, -b, -c, -d].

** DefCovD: Defining covariant derivative CD[-a].

```
** DefTensor: Defining vanishing torsion tensor TorsionCD[a, -b, -c].
** DefTensor: Defining symmetric Christoffel tensor ChristoffelCD[a, -b, -c].
** DefTensor: Defining Riemann tensor RiemannCD[-a, -b, -c, -d].
** DefTensor: Defining symmetric Ricci tensor RicciCD[-a, -b].
** DefCovD:  Contractions of Riemann automatically replaced by Ricci.
** DefTensor: Defining Ricci scalar RicciScalarCD[].
** DefCovD:  Contractions of Ricci automatically replaced by RicciScalar.
** DefTensor: Defining symmetric Einstein tensor EinsteinCD[-a, -b].
** DefTensor: Defining Weyl tensor WeylCD[-a, -b, -c, -d].
** DefTensor: Defining symmetric TFRicci tensor TFRicciCD[-a, -b].
** DefTensor: Defining Kretschmann scalar KretschmannCD[].
** DefCovD:  Computing RiemannToWeylRules for dim 4
** DefCovD:  Computing RicciToTFRicci for dim 4
** DefCovD:  Computing RicciToEinsteinRules for dim 4
** DefTensor: Defining weight +2 density Detg⌘[]. Determinant.
** DefTensor: Defining weight +2 density Detg⌘ℬ[]. Determinant.
------------------------------------------------------------
** DefTensor: Defining symmetric metric tensor f⌘[-a, -b].
** DefTensor: Defining inverse metric tensor Invf⌘[a, b]. Metric is frozen!
** DefTensor: Defining antisymmetric tensor epsilonf⌘[-a, -b, -c, -d].
** DefTensor: Defining tetrametric Tetraf⌘[-a, -b, -c, -d].
** DefTensor: Defining tetrametric Tetraf⌘†[-a, -b, -c, -d].
** DefCovD: Defining covariant derivative CDf[-a].
** DefTensor: Defining vanishing torsion tensor TorsionCDf[a, -b, -c].
** DefTensor: Defining symmetric Christoffel tensor ChristoffelCDf[a, -b, -c].
** DefTensor: Defining Riemann tensor RiemannDownCDf[-a, -b, -c, -d].
** DefTensor: Defining Riemann tensor
 RiemannCDf[-a, -b, -c, d]. Antisymmetric only in the first pair.
** DefTensor: Defining symmetric Ricci tensor RicciCDf[-a, -b].
** DefCovD:  Contractions of Riemann automatically replaced by Ricci.
** DefTensor: Defining Ricci scalar RicciScalarCDf[].
** DefCovD:  Contractions of Ricci automatically replaced by RicciScalar.
** DefTensor: Defining symmetric Einstein tensor EinsteinCDf[-a, -b].
** DefTensor: Defining Weyl tensor WeylCDf[-a, -b, -c, -d].
** DefTensor: Defining symmetric TFRicci tensor TFRicciCDf[-a, -b].
** DefTensor: Defining Kretschmann scalar KretschmannCDf[].
** DefCovD:  Computing RiemannToWeylRules for dim 4
** DefCovD:  Computing RicciToEinsteinRules for dim 4
** DefTensor: Defining weight +2 density Detf⌘[]. Determinant.
------------------------------------------------------------
** DefTensor: Defining tensor A[a, -b].
** DefTensor: Defining tensor S[a, -b].
** DefTensor: Defining tensor iS[a, -b].
------------------------------------------------------------
```

```
** MetricCompute g: Metric[-1, -1]

** MetricCompute g: Metric[1, 1]

** MetricCompute g: DetMetric[]

** MetricCompute g: Christoffel[-1, -1, -1]

** DefTensor: Defining tensor ChristoffelCDPD𝓑[a, -b, -c].

** MetricCompute g: Christoffel[1, -1, -1]

** MetricCompute g: Riemann[-1, -1, -1, -1]

** MetricCompute g: Riemann[-1, -1, -1, 1]

** MetricCompute g: Riemann[-1, -1, 1, 1]

** MetricCompute g: Ricci[-1, -1]

** MetricCompute g: RicciScalar[]

** MetricCompute g: Einstein[-1, -1]

** MetricCompute g: DMetric[-1, -1, -1]

** MetricCompute g: DDMetric[-1, -1, -1, -1]

** MetricCompute g: Weyl[-1, -1, -1, -1]

** MetricCompute g: Kretschmann[]

** MetricCompute g: CDRiemann[-1, -1, -1, -1, -1]

------------------------------------------------------------

** MetricCompute f: Metric[-1, -1]

** MetricCompute f: Metric[1, 1]

** MetricCompute f: DetMetric[]

** DefTensor: Defining weight +2 density Detf⌘𝓑[]. Determinant.

** MetricCompute f: Christoffel[-1, -1, -1]

** DefTensor: Defining tensor ChristoffelCDfPD𝓑[a, -b, -c].

** MetricCompute f: Christoffel[1, -1, -1]

** MetricCompute f: Riemann[-1, -1, -1, -1]

** MetricCompute f: Riemann[-1, -1, -1, 1]

** MetricCompute f: Riemann[-1, -1, 1, 1]

** MetricCompute f: Ricci[-1, -1]

** MetricCompute f: RicciScalar[]

** MetricCompute f: Einstein[-1, -1]

** MetricCompute f: DMetric[-1, -1, -1]

** MetricCompute f: DDMetric[-1, -1, -1, -1]

** MetricCompute f: Weyl[-1, -1, -1, -1]

** MetricCompute f: Kretschmann[]

** MetricCompute f: CDRiemann[-1, -1, -1, -1, -1]

------------------------------------------------------------

** DefTensor: Defining tensor Vg[-a, -b].

** DefTensor: Defining tensor Vf[-a, -b].

------------------------------------------------------------

16.9261
```

Consistency check for elementary symmetric polynomials

```
ℰ[S[a, -b], 0] == 1;
ℰ[S[a, -b], 1] == S[a, -a] // ToCanonical;
ℰ[S[a, -b], 2] == 1/2 (S[a, -a] S[b, -b] - S[a, -b] S[b, -a]) // ToCanonical;
ℰ[S[a, -b], 3] ==
   1/6 (S[a, -a] S[b, -b] S[c, -c] - 3 S[a, -a] S[b, -c] S[c, -b] + 2 S[a, -b] S[b, -c] S[c, -a]) //
  ToCanonical;
% ∧
 %% ∧
 %%% ∧
 %%%%
```

```
True
```

```
𝒴[S[a, -b], 0] == g⌘[a, -b] // ToCanonical;
𝒴[S[a, -b], 1] == S[a, -b] - S[c, -c] g⌘[a, -b] // ToCanonical;
𝒴[S[a, -b], 2] == S[a, -c] S[c, -b] - S[a, -b] S[c, -c] +
    1/2 g⌘[a, -b] (S[c, -c] S[d, -d] - S[c, -d] S[d, -c]) // ToCanonical;
𝒴[S[a, -b], 3] == S[a, -c] S[c, -d] S[d, -b] - S[a, -c] S[c, -b] S[d, -d] +
    1/2 S[a, -b] (S[c, -c] S[d, -d] - S[c, -d] S[d, -c]) - 1/6 g⌘[a, -b] (S[d, -d] S[e, -e] S[c, -c] -
        3 S[d, -d] S[e, -c] S[c, -e] + 2 S[d, -e] S[e, -c] S[c, -d]) // ToCanonical;
% ∧
 %% ∧
 %%% ∧
 %%%%
```

```
True
```

Consistency check for the square root

```
A▮ == S▮.S▮ // Simplify
```

```
True
```

```
S▮ == Inverse@E▮.L▮ // Simplify
```

```
True
```

Display calculated variables

Paramater ranges

```
$Assumptions /. And → List // Column
```

```
t ∈ Reals
t > 0
r ∈ Reals
r > 0
a[t] ∈ Reals
a[t] > 0
X[t] ∈ Reals
X[t] > 0
Y[t] ∈ Reals
Y[t] > 0
κ ∈ Reals
m ∈ Reals
m > 0
M★ ∈ Reals
M★ > 0
ρ★ ∈ Reals
ρ★ > 0
(β0 | β1 | β2 | β3 | β4) ∈ Reals
```

Metric g

```
g⌘[-a, -b] // printMatrixComponents[ℬ]
```

» $g_{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{a^2}{1-\kappa r^2} & 0 & 0 \\ 0 & 0 & a^2 r^2 & 0 \\ 0 & 0 & 0 & a^2 r^2 \mathrm{Sin}[\theta]^2 \end{pmatrix}$

$g_{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{a^2}{1-\kappa r^2} & 0 & 0 \\ 0 & 0 & a^2 r^2 & 0 \\ 0 & 0 & 0 & a^2 r^2 \mathrm{Sin}[\theta]^2 \end{pmatrix}$

```
{"g = ", ({d# & /@ {Scalarsℬ}}.g▮.Transpose@{d# & /@ {Scalarsℬ}})[[1, 1]]} // printNice // Row
```

$$g = -(dt)^2 + a^2 (d\theta)^2 r^2 + \frac{a^2 (dr)^2}{1-\kappa r^2} + a^2 (d\phi)^2 r^2 \mathrm{Sin}[\theta]^2$$

Metric f

```
f⌘[-a, -b] // printMatrixComponents[ℬ]
```

» $f_{ab} = \begin{pmatrix} -X^2 & 0 & 0 & 0 \\ 0 & \frac{Y^2}{1-\kappa r^2} & 0 & 0 \\ 0 & 0 & Y^2 r^2 & 0 \\ 0 & 0 & 0 & Y^2 r^2 \mathrm{Sin}[\theta]^2 \end{pmatrix}$

$f_{ab} = \begin{pmatrix} -X^2 & 0 & 0 & 0 \\ 0 & \frac{Y^2}{1-\kappa r^2} & 0 & 0 \\ 0 & 0 & Y^2 r^2 & 0 \\ 0 & 0 & 0 & Y^2 r^2 \mathrm{Sin}[\theta]^2 \end{pmatrix}$

```
{"f = ", ({d# & /@ {Scalarsℬ}}.f▮.Transpose@{d# & /@ {Scalarsℬ}})[[1, 1]]} // printNice // Row
```

$$f = -X^2 (dt)^2 + Y^2 (d\theta)^2 r^2 + \frac{Y^2 (dr)^2}{1-\kappa r^2} + Y^2 (d\phi)^2 r^2 \mathrm{Sin}[\theta]^2$$

Square root

```
A[a, -b] // printMatrixComponents[ℬ]
```

» $$A^{a}{}_{b} = \begin{pmatrix} X^2 & 0 & 0 & 0 \\ 0 & \frac{Y^2}{a^2} & 0 & 0 \\ 0 & 0 & \frac{Y^2}{a^2} & 0 \\ 0 & 0 & 0 & \frac{Y^2}{a^2} \end{pmatrix}$$

$$A^{a}{}_{b} = \begin{pmatrix} X^2 & 0 & 0 & 0 \\ 0 & \frac{Y^2}{a^2} & 0 & 0 \\ 0 & 0 & \frac{Y^2}{a^2} & 0 \\ 0 & 0 & 0 & \frac{Y^2}{a^2} \end{pmatrix}$$

```
S[a, -b] // printMatrixComponents[ℬ]
```

» $$S^{a}{}_{b} = \begin{pmatrix} X & 0 & 0 & 0 \\ 0 & \frac{Y}{a} & 0 & 0 \\ 0 & 0 & \frac{Y}{a} & 0 \\ 0 & 0 & 0 & \frac{Y}{a} \end{pmatrix}$$

$$S^{a}{}_{b} = \begin{pmatrix} X & 0 & 0 & 0 \\ 0 & \frac{Y}{a} & 0 & 0 \\ 0 & 0 & \frac{Y}{a} & 0 \\ 0 & 0 & 0 & \frac{Y}{a} \end{pmatrix}$$

```
iS[a, -b] // printMatrixComponents[ℬ]
```

» $$[\![S^{-1}]\!]^{a}{}_{b} = \begin{pmatrix} \frac{1}{X} & 0 & 0 & 0 \\ 0 & \frac{a}{Y} & 0 & 0 \\ 0 & 0 & \frac{a}{Y} & 0 \\ 0 & 0 & 0 & \frac{a}{Y} \end{pmatrix}$$

$$[\![S^{-1}]\!]^{a}{}_{b} = \begin{pmatrix} \frac{1}{X} & 0 & 0 & 0 \\ 0 & \frac{a}{Y} & 0 & 0 \\ 0 & 0 & \frac{a}{Y} & 0 \\ 0 & 0 & 0 & \frac{a}{Y} \end{pmatrix}$$

```
g⌘[-a, -c] S[c, -b] // printComponents[ℬ, MatrixForm]
```

» $$g_{ac}\; S^{c}{}_{b} \doteq \begin{pmatrix} -X & 0 & 0 & 0 \\ 0 & \frac{a\,Y}{1-\kappa\,r^2} & 0 & 0 \\ 0 & 0 & a\,Y\,r^2 & 0 \\ 0 & 0 & 0 & a\,Y\,r^2\,\mathrm{Sin}[\theta]^2 \end{pmatrix}$$

$$g_{ac}\; S^{c}{}_{b} \doteq \begin{pmatrix} -X & 0 & 0 & 0 \\ 0 & \frac{a\,Y}{1-\kappa\,r^2} & 0 & 0 \\ 0 & 0 & a\,Y\,r^2 & 0 \\ 0 & 0 & 0 & a\,Y\,r^2\,\mathrm{Sin}[\theta]^2 \end{pmatrix}$$

Christoffel Symbols for *g*

```
ChristoffelCDPDℬ[a, -b, -c] // printNonZeroComponents[ℬ]
```

» $\Gamma[\nabla g,\mathcal{D}]^{0}{}_{11} \doteq \frac{a\,\dot{a}}{1-\kappa\,r^2}$

» $\Gamma[\nabla g,\mathcal{D}]^{0}{}_{22} \doteq a\,r^2\,\dot{a}$

» $\Gamma[\nabla g,\mathcal{D}]^{0}{}_{33} \doteq a\,r^2\,\mathrm{Sin}[\theta]^2\,\dot{a}$

» $\Gamma[\nabla g,\mathcal{D}]^{1}{}_{01} \doteq \frac{\dot{a}}{a}$

» $\Gamma[\nabla g,\mathcal{D}]^{1}{}_{10} \doteq \frac{\dot{a}}{a}$

» $\Gamma[\nabla g,\mathcal{D}]^{1}{}_{11} \doteq \frac{\kappa\,r}{1-\kappa\,r^2}$

» $\Gamma[\nabla g,\mathcal{D}]^{1}{}_{22} \doteq r\,(-1+\kappa\,r^2)$

» $\Gamma[\nabla g,\mathcal{D}]^{1}{}_{33} \doteq r\,(-1+\kappa\,r^2)\,\mathrm{Sin}[\theta]^2$

» $\Gamma[\nabla g,\mathcal{D}]^{2}{}_{02} \doteq \frac{\dot{a}}{a}$

» $\Gamma[\nabla g,\mathcal{D}]^{2}{}_{12} \doteq \frac{1}{r}$

» $\Gamma[\nabla g,\mathcal{D}]^{2}{}_{20} \doteq \frac{\dot{a}}{a}$

» $\Gamma[\nabla g,\mathcal{D}]^{2}{}_{21} \doteq \frac{1}{r}$

» $\Gamma[\nabla g,\mathcal{D}]^{2}{}_{33} \doteq -\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]$

» $\Gamma[\nabla g,\mathcal{D}]^{3}{}_{03} \doteq \frac{\dot{a}}{a}$

» $\Gamma[\nabla g,\mathcal{D}]^{3}{}_{13} \doteq \frac{1}{r}$

» $\Gamma[\nabla g,\mathcal{D}]^{3}{}_{23} \doteq \mathrm{Cot}[\theta]$

» $\Gamma[\nabla g,\mathcal{D}]^{3}{}_{30} \doteq \frac{\dot{a}}{a}$

» $\Gamma[\nabla g,\mathcal{D}]^{3}{}_{31} \doteq \frac{1}{r}$

» $\Gamma[\nabla g,\mathcal{D}]^{3}{}_{32} \doteq \mathrm{Cot}[\theta]$

Ricci Scalar for *g*

```
RicciScalarCD[] // printComponents[ℬ, Expand]
```

» $R[\nabla g] \doteq \frac{6\,\kappa}{a^2}+\frac{6\,\dot{a}^2}{a^2}+\frac{6\,\ddot{a}}{a}$

$R[\nabla g] \doteq \frac{6\,\kappa}{a^2}+\frac{6\,\dot{a}^2}{a^2}+\frac{6\,\ddot{a}}{a}$

Kretschmann for *g*

```
KretschmannCD[] // printComponents[ℬ, Expand]
```

» $K[\nabla g] \doteq \frac{12\,\kappa^2}{a^4}+\frac{24\,\kappa\,\dot{a}^2}{a^4}+\frac{12\,\dot{a}^4}{a^4}+\frac{12\,\ddot{a}^2}{a^2}$

$K[\nabla g] \doteq \frac{12\,\kappa^2}{a^4}+\frac{24\,\kappa\,\dot{a}^2}{a^4}+\frac{12\,\dot{a}^4}{a^4}+\frac{12\,\ddot{a}^2}{a^2}$

Einsten Tensor for *g*

```
EinsteinCD[-a, -b] // printNonZeroComponents[ℬ]
```

» $G[\nabla g]_{00} \doteq \frac{3\left(\kappa+\dot{a}^2\right)}{a^2}$

» $G[\nabla g]_{11} \doteq \frac{\kappa+\dot{a}^2+2\,a\,\ddot{a}}{-1+\kappa\,r^2}$

» $G[\nabla g]_{22} \doteq -r^2\left(\kappa+\dot{a}^2+2\,a\,\ddot{a}\right)$

» $G[\nabla g]_{33} \doteq -r^2\,\mathrm{Sin}[\theta]^2\left(\kappa+\dot{a}^2+2\,a\,\ddot{a}\right)$

Christoffel Symbols for *f*

```
ChristoffelCDfPD𝓑[a, -b, -c] // printNonZeroComponents[𝓑]
```

» $\Gamma[\nabla f,\mathcal{D}]^{0}{}_{00} \doteq \frac{\dot{X}}{X}$

» $\Gamma[\nabla f,\mathcal{D}]^{0}{}_{11} \doteq -\frac{Y\,\dot{Y}}{X^2\,(-1+\kappa\,r^2)}$

» $\Gamma[\nabla f,\mathcal{D}]^{0}{}_{22} \doteq \frac{Y\,r^2\,\dot{Y}}{X^2}$

» $\Gamma[\nabla f,\mathcal{D}]^{0}{}_{33} \doteq \frac{Y\,r^2\,\mathrm{Sin}[\theta]^2\,\dot{Y}}{X^2}$

» $\Gamma[\nabla f,\mathcal{D}]^{1}{}_{01} \doteq \frac{\dot{Y}}{Y}$

» $\Gamma[\nabla f,\mathcal{D}]^{1}{}_{10} \doteq \frac{\dot{Y}}{Y}$

» $\Gamma[\nabla f,\mathcal{D}]^{1}{}_{11} \doteq \frac{\kappa\,r}{1-\kappa\,r^2}$

» $\Gamma[\nabla f,\mathcal{D}]^{1}{}_{22} \doteq r\,(-1+\kappa\,r^2)$

» $\Gamma[\nabla f,\mathcal{D}]^{1}{}_{33} \doteq r\,(-1+\kappa\,r^2)\,\mathrm{Sin}[\theta]^2$

» $\Gamma[\nabla f,\mathcal{D}]^{2}{}_{02} \doteq \frac{\dot{Y}}{Y}$

» $\Gamma[\nabla f,\mathcal{D}]^{2}{}_{12} \doteq \frac{1}{r}$

» $\Gamma[\nabla f,\mathcal{D}]^{2}{}_{20} \doteq \frac{\dot{Y}}{Y}$

» $\Gamma[\nabla f,\mathcal{D}]^{2}{}_{21} \doteq \frac{1}{r}$

» $\Gamma[\nabla f,\mathcal{D}]^{2}{}_{33} \doteq -\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]$

» $\Gamma[\nabla f,\mathcal{D}]^{3}{}_{03} \doteq \frac{\dot{Y}}{Y}$

» $\Gamma[\nabla f,\mathcal{D}]^{3}{}_{13} \doteq \frac{1}{r}$

» $\Gamma[\nabla f,\mathcal{D}]^{3}{}_{23} \doteq \mathrm{Cot}[\theta]$

» $\Gamma[\nabla f,\mathcal{D}]^{3}{}_{30} \doteq \frac{\dot{Y}}{Y}$

» $\Gamma[\nabla f,\mathcal{D}]^{3}{}_{31} \doteq \frac{1}{r}$

» $\Gamma[\nabla f,\mathcal{D}]^{3}{}_{32} \doteq \mathrm{Cot}[\theta]$

Ricci Scalar for *f*

```
RicciScalarCDf[] // printComponents[ℬ, Expand]
```

» $$R[\nabla f] \doteq \frac{6\,\kappa}{Y^2} - \frac{6\,\dot{X}\,\dot{Y}}{X^3\,Y} + \frac{6\,\dot{Y}^2}{X^2\,Y^2} + \frac{6\,\ddot{Y}}{X^2\,Y}$$

$$R[\nabla f] \doteq \frac{6\,\kappa}{Y^2} - \frac{6\,\dot{X}\,\dot{Y}}{X^3\,Y} + \frac{6\,\dot{Y}^2}{X^2\,Y^2} + \frac{6\,\ddot{Y}}{X^2\,Y}$$

Kretschmann for *f*

```
KretschmannCDf[] // printComponents[ℬ, Expand]
```

» $$K[\nabla f] \doteq \frac{12\,\kappa^2}{Y^4} + \frac{24\,\kappa\,\dot{Y}^2}{X^2\,Y^4} + \frac{12\,\dot{X}^2\,\dot{Y}^2}{X^6\,Y^2} + \frac{12\,\dot{Y}^4}{X^4\,Y^4} - \frac{24\,\dot{X}\,\dot{Y}\,\ddot{Y}}{X^5\,Y^2} + \frac{12\,\ddot{Y}^2}{X^4\,Y^2}$$

$$K[\nabla f] \doteq \frac{12\,\kappa^2}{Y^4} + \frac{24\,\kappa\,\dot{Y}^2}{X^2\,Y^4} + \frac{12\,\dot{X}^2\,\dot{Y}^2}{X^6\,Y^2} + \frac{12\,\dot{Y}^4}{X^4\,Y^4} - \frac{24\,\dot{X}\,\dot{Y}\,\ddot{Y}}{X^5\,Y^2} + \frac{12\,\ddot{Y}^2}{X^4\,Y^2}$$

```
ToValues@KretschmannCDf[] /. Y′ → X a′;
%/12 // Simplify // printNice
```

$$\frac{X^4\,\left(\kappa + \dot{a}^2\right)^2 + Y^2\,\left(-\dot{a}\,\dot{X} + \ddot{Y}\right)^2}{X^4\,Y^4}$$

Einsten Tensor for *f*

```
EinsteinCDf[-a, -b] // printNonZeroComponents[ℬ, Expand]
```

» $$G[\nabla f]_{00} \doteq \frac{3\,X^2\,\kappa}{Y^2} + \frac{3\,\dot{Y}^2}{Y^2}$$

» $$G[\nabla f]_{11} \doteq \frac{\kappa}{-1 + \kappa\,r^2} - \frac{2\,Y\,\dot{X}\,\dot{Y}}{X^3\,(-1 + \kappa\,r^2)} + \frac{\dot{Y}^2}{X^2\,(-1 + \kappa\,r^2)} + \frac{2\,Y\,\ddot{Y}}{X^2\,(-1 + \kappa\,r^2)}$$

» $$G[\nabla f]_{22} \doteq -\kappa\,r^2 + \frac{2\,Y\,r^2\,\dot{X}\,\dot{Y}}{X^3} - \frac{r^2\,\dot{Y}^2}{X^2} - \frac{2\,Y\,r^2\,\ddot{Y}}{X^2}$$

» $$G[\nabla f]_{33} \doteq -\kappa\,r^2\,\mathrm{Sin}[\theta]^2 + \frac{2\,Y\,r^2\,\mathrm{Sin}[\theta]^2\,\dot{X}\,\dot{Y}}{X^3} - \frac{r^2\,\mathrm{Sin}[\theta]^2\,\dot{Y}^2}{X^2} - \frac{2\,Y\,r^2\,\mathrm{Sin}[\theta]^2\,\ddot{Y}}{X^2}$$

Equations of Motion

Equations of Motion, 3+1 decomposition

Equations

$$G^{(g)}{}_{ab} + \frac{m^d}{{m_g}^{d-2}}\,V^{(g)}{}_{ab} = \frac{1}{{m_g}^{d-2}}\,T^{(g)}{}_{a\,b}$$

$$V^{(g)}{}_{ab} := \sum_{n=0}^{d-1} (-1)^n\,\beta_n\,g_{ac}[Y^n(S)]^c{}_b$$

$$[Y^n(S)]^a{}_b = \sum_{k=0}^{n} (-1)^k\,\mathcal{E}_n(S)\left[S^{n-k}\right]^c{}_b$$

$$[Y^n(S)]^a{}_b = \frac{2}{\sqrt{-g}}\,g^{ac}\,\frac{\delta}{\delta g^{cb}}\left(\sqrt{-g}\;\mathcal{E}_n(S)\right)$$

where $S = \sqrt{g^{-1}\,f}$

After decomposition, we get the 3+1 Einstein system consiting of:

- The evolution equations

$$\partial_t\,\gamma_{ij} = -2\,N\,K_{ij} + \mathcal{L}_\beta\,\gamma_{ij} = -2\,N\,K_{ij} + D_i\,\beta_j + D_j\,\beta_i$$

$$\partial_t\,K_{ij} = -D_iD_j\,N + N\!\left(R_{ij} + K\,K_{ij} - 2\,K_{ik}\,K^k{}_j\right) + \mathcal{L}_\beta\,K_{ij} + \kappa\!\left(\tfrac{1}{2}\,\gamma_{ij}(\mathcal{S} - \rho) - \mathcal{S}_{ij}\right)$$

- The constraint equations

$R + K^2 - K_{ij} K^{ij} = 2\,\kappa\,\rho$ (Hamiltonian constraint)

$D_i(K^{ij} - \gamma^{ij} K) = \kappa\, p^j$ (momentum constraint)

where

$\rho := n^a n^b T_{ab}$ (energy density; aka E)

$p^i := -\gamma^{ic} n^d T_{cd}$ (momentum density; aka j^i or S^i)

$S_{ij} := \gamma_i{}^c \gamma_j{}^d T_{cd}$ (stress tensor)

Here we have $\mathcal{S} := \gamma^{ij} S_{ij}$, and $\mathcal{S} - \rho = T = g^{ab} T_{ab}$.

Compare to (2.4), (2.5):

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \frac{m^2}{2} \sum_{n=0}^{3} (-1)^n \beta_n \left[g_{\mu\lambda} Y^{\lambda}_{(n)\nu}(\sqrt{g^{-1}f}) + g_{\nu\lambda} Y^{\lambda}_{(n)\mu}(\sqrt{g^{-1}f}) \right] = \tfrac{1}{M_g^2} T_{\mu\nu}\,, \quad (2.4)$$

$$\bar{R}_{\mu\nu} - \tfrac{1}{2} f_{\mu\nu} \bar{R} + \frac{m^2}{2M_\star^2} \sum_{n=0}^{3} (-1)^n \beta_{4-n} \left[f_{\mu\lambda} Y^{\lambda}_{(n)\nu}(\sqrt{f^{-1}g}) + f_{\nu\lambda} Y^{\lambda}_{(n)\mu}(\sqrt{f^{-1}g}) \right] = 0\,, \quad (2.5)$$

EoM for *g*

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \frac{m^2}{2} \sum_{n=0}^{3} (-1)^n \beta_n \left[g_{\mu\lambda} Y^{\lambda}_{(n)\nu}(\sqrt{g^{-1}f}) + g_{\nu\lambda} Y^{\lambda}_{(n)\mu}(\sqrt{g^{-1}f}) \right] = \tfrac{1}{M_g^2} T_{\mu\nu}\,, \quad (2.4)$$

```
EinsteinCD[-a, -b] // printNonZeroComponents[ℬ]
```

» $G[\nabla g]_{00} \doteq \dfrac{3\,(\kappa + \dot{a}^2)}{a^2}$

» $G[\nabla g]_{11} \doteq \dfrac{\kappa + \dot{a}^2 + 2\,a\,\ddot{a}}{-1 + \kappa\, r^2}$

» $G[\nabla g]_{22} \doteq -r^2\,(\kappa + \dot{a}^2 + 2\,a\,\ddot{a})$

» $G[\nabla g]_{33} \doteq -r^2\,\mathrm{Sin}[\theta]^2\,(\kappa + \dot{a}^2 + 2\,a\,\ddot{a})$

```
Vg[-a, -b] // printNonZeroComponents[ℬ, Expand]
```

» $[\![\underset{(g)}{V}]\!]_{00} \doteq -\beta_0 - \dfrac{3\,Y\,\beta_1}{a} - \dfrac{3\,Y^2\,\beta_2}{a^2} - \dfrac{Y^3\,\beta_3}{a^3}$

» $[\![\underset{(g)}{V}]\!]_{11} \doteq -\dfrac{a^2\,\beta_0}{-1 + \kappa\, r^2} - \dfrac{a^2\,X\,\beta_1}{-1 + \kappa\, r^2} - \dfrac{2\,a\,Y\,\beta_1}{-1 + \kappa\, r^2} - \dfrac{2\,a\,X\,Y\,\beta_2}{-1 + \kappa\, r^2} - \dfrac{Y^2\,\beta_2}{-1 + \kappa\, r^2} - \dfrac{X\,Y^2\,\beta_3}{-1 + \kappa\, r^2}$

» $[\![\underset{(g)}{V}]\!]_{22} \doteq a^2\,\beta_0\,r^2 + a^2\,X\,\beta_1\,r^2 + 2\,a\,Y\,\beta_1\,r^2 + 2\,a\,X\,Y\,\beta_2\,r^2 + Y^2\,\beta_2\,r^2 + X\,Y^2\,\beta_3\,r^2$

» $[\![\underset{(g)}{V}]\!]_{33} \doteq a^2\,\beta_0\,r^2\,\mathrm{Sin}[\theta]^2 + a^2\,X\,\beta_1\,r^2\,\mathrm{Sin}[\theta]^2 +$
$2\,a\,Y\,\beta_1\,r^2\,\mathrm{Sin}[\theta]^2 + 2\,a\,X\,Y\,\beta_2\,r^2\,\mathrm{Sin}[\theta]^2 + Y^2\,\beta_2\,r^2\,\mathrm{Sin}[\theta]^2 + X\,Y^2\,\beta_3\,r^2\,\mathrm{Sin}[\theta]^2$

```
EinsteinCD[-a, -b] + m² Vg[-a, -b] // printNonZeroComponents[ℬ, Expand]
```

» $G[\nabla g]_{00} + m^2\,[\![\underset{(g)}{V}]\!]_{00} \doteq -m^2\,\beta_0 - \dfrac{3\,m^2\,Y\,\beta_1}{a} - \dfrac{3\,m^2\,Y^2\,\beta_2}{a^2} - \dfrac{m^2\,Y^3\,\beta_3}{a^3} + \dfrac{3\,\kappa}{a^2} + \dfrac{3\,\dot{a}^2}{a^2}$

» $G[\nabla g]_{11} + m^2\,[\![\underset{(g)}{V}]\!]_{11} \doteq$
$-\dfrac{a^2\,m^2\,\beta_0}{-1 + \kappa\, r^2} - \dfrac{a^2\,m^2\,X\,\beta_1}{-1 + \kappa\, r^2} - \dfrac{2\,a\,m^2\,Y\,\beta_1}{-1 + \kappa\, r^2} - \dfrac{2\,a\,m^2\,X\,Y\,\beta_2}{-1 + \kappa\, r^2} - \dfrac{m^2\,Y^2\,\beta_2}{-1 + \kappa\, r^2} - \dfrac{m^2\,X\,Y^2\,\beta_3}{-1 + \kappa\, r^2} + \dfrac{\kappa}{-1 + \kappa\, r^2} + \dfrac{\dot{a}^2}{-1 + \kappa\, r^2} + \dfrac{2\,a\,\ddot{a}}{-1 + \kappa\, r^2}$

» $G[\nabla g]_{22} + m^2\,[\![\underset{(g)}{V}]\!]_{22} \doteq$
$a^2\,m^2\,\beta_0\,r^2 + a^2\,m^2\,X\,\beta_1\,r^2 + 2\,a\,m^2\,Y\,\beta_1\,r^2 + 2\,a\,m^2\,X\,Y\,\beta_2\,r^2 + m^2\,Y^2\,\beta_2\,r^2 + m^2\,X\,Y^2\,\beta_3\,r^2 - \kappa\,r^2 - r^2\,\dot{a}^2 - 2\,a\,r^2\,\ddot{a}$

» $G[\nabla g]_{33} + m^2\,[\![\underset{(g)}{V}]\!]_{33} \doteq$
$a^2\,m^2\,\beta_0\,r^2\,\mathrm{Sin}[\theta]^2 + a^2\,m^2\,X\,\beta_1\,r^2\,\mathrm{Sin}[\theta]^2 + 2\,a\,m^2\,Y\,\beta_1\,r^2\,\mathrm{Sin}[\theta]^2 + 2\,a\,m^2\,X\,Y\,\beta_2\,r^2\,\mathrm{Sin}[\theta]^2 +$
$m^2\,Y^2\,\beta_2\,r^2\,\mathrm{Sin}[\theta]^2 + m^2\,X\,Y^2\,\beta_3\,r^2\,\mathrm{Sin}[\theta]^2 - \kappa\,r^2\,\mathrm{Sin}[\theta]^2 - r^2\,\mathrm{Sin}[\theta]^2\,\dot{a}^2 - 2\,a\,r^2\,\mathrm{Sin}[\theta]^2\,\ddot{a}$

```
EinsteinCD[{0, -ℬ}, {0, -ℬ}] + m² Vg[{0, -ℬ}, {0, -ℬ}] // printComponents[ℬ, Expand]
```

» $G[\nabla g]_{00} + m^2 \left[\!\left[\underset{(g)}{V} \right]\!\right]_{00} \doteq -m^2 \beta_0 - \frac{3 m^2 Y \beta_1}{a} - \frac{3 m^2 Y^2 \beta_2}{a^2} - \frac{m^2 Y^3 \beta_3}{a^3} + \frac{3\kappa}{a^2} + \frac{3\dot{a}^2}{a^2}$

$G[\nabla g]_{00} + m^2 \left[\!\left[\underset{(g)}{V} \right]\!\right]_{00} \doteq -m^2 \beta_0 - \frac{3 m^2 Y \beta_1}{a} - \frac{3 m^2 Y^2 \beta_2}{a^2} - \frac{m^2 Y^3 \beta_3}{a^3} + \frac{3\kappa}{a^2} + \frac{3\dot{a}^2}{a^2}$

```
RicciScalarCD[] // printComponents[ℬ, Expand]
```

» $R[\nabla g] \doteq \frac{6\kappa}{a^2} + \frac{6\dot{a}^2}{a^2} + \frac{6\ddot{a}}{a}$

$R[\nabla g] \doteq \frac{6\kappa}{a^2} + \frac{6\dot{a}^2}{a^2} + \frac{6\ddot{a}}{a}$

Compare to (2.14), (2.15):

$$-3\left(\frac{\dot{a}}{a}\right)^2 - 3\frac{k}{a^2} + m^2\left[\beta_0 + 3\beta_1\frac{Y}{a} + 3\beta_2\frac{Y^2}{a^2} + \beta_3\frac{Y^3}{a^3}\right] = \frac{1}{M_g^2}T^0{}_0\,, \tag{2.14}$$

$$-2\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 - \frac{k}{a^2} + m^2\left[\beta_0 + 2\beta_1\left(\frac{Y}{a} + \frac{\dot{Y}}{\dot{a}}\right) + \beta_2\left(\frac{Y^2}{a^2} + 2\frac{Y\dot{Y}}{a\dot{a}}\right) + \beta_3\frac{Y^2\dot{Y}}{a^2\dot{a}}\right] = \frac{1}{M_g^2}T^1{}_1\,. \tag{2.15}$$

EoM for *f*

$$\bar{R}_{\mu\nu} - \tfrac{1}{2}f_{\mu\nu}\bar{R} + \frac{m^2}{2M_\star^2}\sum_{n=0}^{3}(-1)^n\beta_{4-n}\left[f_{\mu\lambda}Y^\lambda_{(n)\nu}(\sqrt{f^{-1}g}) + f_{\nu\lambda}Y^\lambda_{(n)\mu}(\sqrt{f^{-1}g})\right] = 0\,, \tag{2.5}$$

```
EinsteinCDf[-a, -b] // printNonZeroComponents[ℬ, Expand]
```

» $G[\nabla f]_{00} \doteq \frac{3 X^2 \kappa}{Y^2} + \frac{3\dot{Y}^2}{Y^2}$

» $G[\nabla f]_{11} \doteq \frac{\kappa}{-1 + \kappa r^2} - \frac{2 Y \dot{X} \dot{Y}}{X^3(-1 + \kappa r^2)} + \frac{\dot{Y}^2}{X^2(-1 + \kappa r^2)} + \frac{2 Y \ddot{Y}}{X^2(-1 + \kappa r^2)}$

» $G[\nabla f]_{22} \doteq -\kappa r^2 + \frac{2 Y r^2 \dot{X}\dot{Y}}{X^3} - \frac{r^2 \dot{Y}^2}{X^2} - \frac{2 Y r^2 \ddot{Y}}{X^2}$

» $G[\nabla f]_{33} \doteq -\kappa r^2 \mathrm{Sin}[\theta]^2 + \frac{2 Y r^2 \mathrm{Sin}[\theta]^2 \dot{X}\dot{Y}}{X^3} - \frac{r^2 \mathrm{Sin}[\theta]^2 \dot{Y}^2}{X^2} - \frac{2 Y r^2 \mathrm{Sin}[\theta]^2 \ddot{Y}}{X^2}$

Compare to (A.8):

$$\begin{aligned}
R_{00} - \frac{1}{2}f_{00}R &= 3\frac{\dot{Y}^2}{Y^2} + 3\frac{kX^2}{Y^2}\,,\\
R_{11} - \frac{1}{2}f_{11}R &= -\frac{1}{X^2(1-kr^2)}\left(2\ddot{Y} - 2\frac{Y\dot{Y}\dot{X}}{X} + \dot{Y}^2 + kX^2\right)\,,\\
R_{22} - \frac{1}{2}f_{22}R &= -\frac{r^2}{X^2}\left(2\ddot{Y} - 2\frac{Y\dot{Y}\dot{X}}{X} + \dot{Y}^2 + kX^2\right)\,,\\
R_{33} - \frac{1}{2}f_{33}R &= -\frac{r^2\sin^2\theta}{X^2}\left(2\ddot{Y} - 2\frac{Y\dot{Y}\dot{X}}{X} + \dot{Y}^2 + kX^2\right)\,.
\end{aligned} \tag{A.8}$$

```
Vf[-a, -b] // printNonZeroComponents[ℬ, Expand]
```

» $$[\![\underset{(f)}{V}]\!]_{00} \doteq -\frac{a^3 X^2 \beta_1}{Y^3} - \frac{3 a^2 X^2 \beta_2}{Y^2} - \frac{3 a X^2 \beta_3}{Y} - X^2 \beta_4$$

» $$[\![\underset{(f)}{V}]\!]_{11} \doteq -\frac{a^2 \beta_1}{X(-1+\kappa r^2)} - \frac{a^2 \beta_2}{-1+\kappa r^2} - \frac{2 a Y \beta_2}{X(-1+\kappa r^2)} - \frac{2 a Y \beta_3}{-1+\kappa r^2} - \frac{Y^2 \beta_3}{X(-1+\kappa r^2)} - \frac{Y^2 \beta_4}{-1+\kappa r^2}$$

» $$[\![\underset{(f)}{V}]\!]_{22} \doteq \frac{a^2 \beta_1 r^2}{X} + a^2 \beta_2 r^2 + \frac{2 a Y \beta_2 r^2}{X} + 2 a Y \beta_3 r^2 + \frac{Y^2 \beta_3 r^2}{X} + Y^2 \beta_4 r^2$$

» $$[\![\underset{(f)}{V}]\!]_{33} \doteq \frac{a^2 \beta_1 r^2 \mathrm{Sin}[\theta]^2}{X} + a^2 \beta_2 r^2 \mathrm{Sin}[\theta]^2 + \frac{2 a Y \beta_2 r^2 \mathrm{Sin}[\theta]^2}{X} + 2 a Y \beta_3 r^2 \mathrm{Sin}[\theta]^2 + \frac{Y^2 \beta_3 r^2 \mathrm{Sin}[\theta]^2}{X} + Y^2 \beta_4 r^2 \mathrm{Sin}[\theta]^2$$

```
(EinsteinCDf[-a, -b] + μ^2 Vf[-a, -b] // printNonZeroComponents[ℬ, Expand]) /.
  μ^2 → HoldForm@ m^2/M★^2
```

» $$\mathrm{G}[\nabla f]_{00} + \mu^2 [\![\underset{(f)}{V}]\!]_{00} \doteq \frac{3 X^2 \kappa}{Y^2} - \frac{a^3 X^2 \beta_1 \mu^2}{Y^3} - \frac{3 a^2 X^2 \beta_2 \mu^2}{Y^2} - \frac{3 a X^2 \beta_3 \mu^2}{Y} - X^2 \beta_4 \mu^2 + \frac{3 \dot{Y}^2}{Y^2}$$

» $$\mathrm{G}[\nabla f]_{11} + \mu^2 [\![\underset{(f)}{V}]\!]_{11} \doteq \frac{\kappa}{-1+\kappa r^2} - \frac{a^2 \beta_1 \mu^2}{X(-1+\kappa r^2)} - \frac{a^2 \beta_2 \mu^2}{-1+\kappa r^2} - \frac{2 a Y \beta_2 \mu^2}{X(-1+\kappa r^2)} - \frac{2 a Y \beta_3 \mu^2}{-1+\kappa r^2} - \frac{Y^2 \beta_3 \mu^2}{X(-1+\kappa r^2)} - \frac{Y^2 \beta_4 \mu^2}{-1+\kappa r^2} - \frac{2 Y \dot{X} \dot{Y}}{X^3(-1+\kappa r^2)} + \frac{\dot{Y}^2}{X^2(-1+\kappa r^2)} + \frac{2 Y \ddot{Y}}{X^2(-1+\kappa r^2)}$$

» $$\mathrm{G}[\nabla f]_{22} + \mu^2 [\![\underset{(f)}{V}]\!]_{22} \doteq -\kappa r^2 + \frac{a^2 \beta_1 \mu^2 r^2}{X} + a^2 \beta_2 \mu^2 r^2 + \frac{2 a Y \beta_2 \mu^2 r^2}{X} + 2 a Y \beta_3 \mu^2 r^2 + \frac{Y^2 \beta_3 \mu^2 r^2}{X} + Y^2 \beta_4 \mu^2 r^2 + \frac{2 Y r^2 \dot{X} \dot{Y}}{X^3} - \frac{r^2 \dot{Y}^2}{X^2} - \frac{2 Y r^2 \ddot{Y}}{X^2}$$

» $$\mathrm{G}[\nabla f]_{33} + \mu^2 [\![\underset{(f)}{V}]\!]_{33} \doteq -\kappa r^2 \mathrm{Sin}[\theta]^2 + \frac{a^2 \beta_1 \mu^2 r^2 \mathrm{Sin}[\theta]^2}{X} + a^2 \beta_2 \mu^2 r^2 \mathrm{Sin}[\theta]^2 + \frac{2 a Y \beta_2 \mu^2 r^2 \mathrm{Sin}[\theta]^2}{X} + 2 a Y \beta_3 \mu^2 r^2 \mathrm{Sin}[\theta]^2 + \frac{Y^2 \beta_3 \mu^2 r^2 \mathrm{Sin}[\theta]^2}{X} + Y^2 \beta_4 \mu^2 r^2 \mathrm{Sin}[\theta]^2 + \frac{2 Y r^2 \mathrm{Sin}[\theta]^2 \dot{X} \dot{Y}}{X^3} - \frac{r^2 \mathrm{Sin}[\theta]^2 \dot{Y}^2}{X^2} - \frac{2 Y r^2 \mathrm{Sin}[\theta]^2 \ddot{Y}}{X^2}$$

Note from the text p. 20:

"For the equations of motion, we raise one index with $f^{\mu\nu}$ and consider the 00- as well as the 11-component ."

```
f⌘[{0, ℬ}, {a, ℬ}] (EinsteinCDf[{-a, -ℬ}, {0, -ℬ}] + HoldForm@ m^2/M★^2 Vf[{-a, -ℬ}, {0, -ℬ}]) //
  printComponents[ℬ, Expand]
```

» $$f^{0a}\left(\mathrm{G}[\nabla f]_{a0} + \frac{m^2}{M_\star^2} [\![\underset{(f)}{V}]\!]_{a0}\right) \doteq -\frac{3\kappa}{Y^2} + \frac{a^3 \beta_1 \frac{m^2}{M_\star^2}}{Y^3} + \frac{3 a^2 \beta_2 \frac{m^2}{M_\star^2}}{Y^2} + \frac{3 a \beta_3 \frac{m^2}{M_\star^2}}{Y} + \beta_4 \frac{m^2}{M_\star^2} - \frac{3 \dot{Y}^2}{X^2 Y^2}$$

$$f^{0a}\left(\mathrm{G}[\nabla f]_{a0} + \frac{m^2}{M_\star^2} [\![\underset{(f)}{V}]\!]_{a0}\right) \doteq -\frac{3\kappa}{Y^2} + \frac{a^3 \beta_1 \frac{m^2}{M_\star^2}}{Y^3} + \frac{3 a^2 \beta_2 \frac{m^2}{M_\star^2}}{Y^2} + \frac{3 a \beta_3 \frac{m^2}{M_\star^2}}{Y} + \beta_4 \frac{m^2}{M_\star^2} - \frac{3 \dot{Y}^2}{X^2 Y^2}$$

```
f⌘[{1, ℬ}, {a, ℬ}] (EinsteinCDf[{-a, -ℬ}, {1, -ℬ}] + HoldForm@ m^2/M★^2 Vf[{-a, -ℬ}, {1, -ℬ}]) //
  printComponents[ℬ, Expand]
```

» $$f^{1a}\left(\mathrm{G}[\nabla f]_{a1} + \frac{m^2}{M_\star^2} [\![\underset{(f)}{V}]\!]_{a1}\right) \doteq -\frac{\kappa}{Y^2} + \frac{a^2 \beta_1 \frac{m^2}{M_\star^2}}{X Y^2} + \frac{a^2 \beta_2 \frac{m^2}{M_\star^2}}{Y^2} + \frac{2 a \beta_2 \frac{m^2}{M_\star^2}}{X Y} + \frac{\beta_3 \frac{m^2}{M_\star^2}}{X} + \frac{2 a \beta_3 \frac{m^2}{M_\star^2}}{Y} + \beta_4 \frac{m^2}{M_\star^2} + \frac{2 \dot{X} \dot{Y}}{X^3 Y} - \frac{\dot{Y}^2}{X^2 Y^2} - \frac{2 \ddot{Y}}{X^2 Y}$$

$$f^{1a}\left(\mathrm{G}[\nabla f]_{a1} + \frac{m^2}{M_\star^2} [\![\underset{(f)}{V}]\!]_{a1}\right) \doteq -\frac{\kappa}{Y^2} + \frac{a^2 \beta_1 \frac{m^2}{M_\star^2}}{X Y^2} + \frac{a^2 \beta_2 \frac{m^2}{M_\star^2}}{Y^2} + \frac{2 a \beta_2 \frac{m^2}{M_\star^2}}{X Y} + \frac{\beta_3 \frac{m^2}{M_\star^2}}{X} + \frac{2 a \beta_3 \frac{m^2}{M_\star^2}}{Y} + \beta_4 \frac{m^2}{M_\star^2} + \frac{2 \dot{X} \dot{Y}}{X^3 Y} - \frac{\dot{Y}^2}{X^2 Y^2} - \frac{2 \ddot{Y}}{X^2 Y}$$

Compare to (A.9), (A.10):

$$-3\frac{\dot{Y}^2}{X^2Y^2}-3\frac{k}{Y^2}+\frac{m^2}{M_*^2}\left[\frac{a^3}{Y^3}\beta_1+3\frac{a^2}{Y^2}\beta_2+3\frac{a}{Y}\beta_3+\beta_4\right]=0, \tag{A.9}$$

$$0=-\frac{1}{X^2}\left(2\frac{\ddot{Y}}{Y}-2\frac{\dot{Y}\dot{X}}{YX}+\frac{\dot{Y}^2}{Y^2}\right)-\frac{k}{Y^2}$$
$$+\frac{m^2}{M_*^2}\left[\frac{a^2}{XY^2}\beta_1+\left(\frac{a^2}{Y^2}+\frac{2a}{XY}\right)\beta_2+\left(2\frac{a}{Y}+\frac{1}{X}\right)\beta_3+\beta_4\right]. \tag{A.10}$$

Notes on X

Imposing that f EoM (A.9) and (A.10) are not independent (related by the operator $1+\frac{Y}{3\,Y'}\partial_t$) is equivalent to imposing the Bianchi constraint ($X = Y' / a'$)

```
eq$f$00 =
 f⌘[{0, ℬ}, {a, ℬ}] (EinsteinCDf[{-a, -ℬ}, {0, -ℬ}] + μ^2 Vf[{-a, -ℬ}, {0, -ℬ}]) // ToBasis[
     ℬ] // ToBasis[ℬ] // TraceBasisDummy // ComponentArray // ToValues // FullSimplify
```

$$\frac{1}{X[t]^2\,Y[t]^3}\left(X[t]^2\left(-3\,\kappa\,Y[t]+\mu^2\left(\beta_1\,a[t]^3+3\,\beta_2\,a[t]^2\,Y[t]+3\,\beta_3\,a[t]\,Y[t]^2+\beta_4\,Y[t]^3\right)\right)-3\,Y[t]\,Y'[t]^2\right)$$

```
eq$f$11 = (1/(-1 + κ r^2) (1 - κ r^2))^-1 f⌘[{1, ℬ}, {a, ℬ}]
    (EinsteinCDf[{-a, -ℬ}, {1, -ℬ}] + μ^2 Vf[{-a, -ℬ}, {1, -ℬ}]) // ToBasis[ℬ] //
   ToBasis[ℬ] // TraceBasisDummy // ComponentArray // ToValues // FullSimplify
```

$$\frac{1}{X[t]^3\,Y[t]^2}\Big(-\mu^2\,a[t]^2\,X[t]^2\,(\beta_1+\beta_2\,X[t])-2\,\mu^2\,a[t]\,X[t]^2\,(\beta_2+\beta_3\,X[t])\,Y[t]-$$
$$\beta_3\,\mu^2\,X[t]^2\,Y[t]^2+X[t]^3\left(\kappa-\beta_4\,\mu^2\,Y[t]^2\right)-2\,Y[t]\,X'[t]\,Y'[t]+X[t]\left(Y'[t]^2+2\,Y[t]\,Y''[t]\right)\Big)$$

Now, use X from the Bianchi `X[t] a'[t] - Y'[t] == 0`

```
Block[{X},
 X[t_] := D[Y[t], t] / D[a[t], t];
 eq$a11 = eq$f$00 /. X[t[]] → D[Y[t[]], t[]] / D[a[t[]], t[]] // FullSimplify // Expand //
    Collect[#, μ^2] &;
 eq$a12 = eq$f$11 /. X[t[]] → D[Y[t[]], t[]] / D[a[t[]], t[]] // FullSimplify // Expand //
    Collect[#, μ^2] &;
]
```

Compare (A.11) and (A.12) to what we obtained

$$0=-3\frac{\dot{a}^2}{Y^2}-3\frac{k}{Y^2}+\frac{m^2}{M_*^2}\left[\frac{a^3}{Y^3}\beta_1+3\frac{a^2}{Y^2}\beta_2+3\frac{a}{Y}\beta_3+\beta_4\right], \tag{A.11}$$
$$0=-2\frac{\dot{a}\ddot{a}}{Y\dot{Y}}-\frac{\dot{a}^2}{Y^2}-\frac{k}{Y^2}+\frac{m^2}{M_*^2}\left[\frac{a^2\dot{a}}{\dot{Y}Y^2}\beta_1+\left(\frac{a^2}{Y^2}+\frac{2a\dot{a}}{\dot{Y}Y}\right)\beta_2+\left(2\frac{a}{Y}+\frac{\dot{a}}{\dot{Y}}\right)\beta_3+\beta_4\right]. \tag{A.12}$$

```
eq$a11
eq$a12
```

$$\mu^2\left(\beta_4+\frac{\beta_1\,a[t]^3}{Y[t]^3}+\frac{3\,\beta_2\,a[t]^2}{Y[t]^2}+\frac{3\,\beta_3\,a[t]}{Y[t]}\right)-\frac{3\,\kappa}{Y[t]^2}-\frac{3\,a'[t]^2}{Y[t]^2}$$

$$\frac{\kappa}{Y[t]^2}+\frac{a'[t]^2}{Y[t]^2}+$$
$$\mu^2\left(-\beta_4-\frac{\beta_2\,a[t]^2}{Y[t]^2}-\frac{2\,\beta_3\,a[t]}{Y[t]}-\frac{\beta_3\,a'[t]}{Y'[t]}-\frac{\beta_1\,a[t]^2\,a'[t]}{Y[t]^2\,Y'[t]}-\frac{2\,\beta_2\,a[t]\,a'[t]}{Y[t]\,Y'[t]}\right)+\frac{2\,a'[t]\,a''[t]}{Y[t]\,Y'[t]}$$

Verify that these are the same applying $(3+(Y/\dot{Y})\partial_t)/3$ on EoM 00,

```
eq$a11$alt =
  (eq$a11 + Y[t[]]/(3 D[Y[t[]], t[]]) D[eq$a11, t[]]) // FullSimplify // Expand // Collect[#, μ^2] &
```

$$-\frac{\kappa}{Y[t]^2}-\frac{a'[t]^2}{Y[t]^2}+\mu^2\left(\beta_4+\frac{\beta_2\,a[t]^2}{Y[t]^2}+\frac{2\,\beta_3\,a[t]}{Y[t]}+\frac{\beta_3\,a'[t]}{Y'[t]}+\frac{\beta_1\,a[t]^2\,a'[t]}{Y[t]^2\,Y'[t]}+\frac{2\,\beta_2\,a[t]\,a'[t]}{Y[t]\,Y'[t]}\right)-\frac{2\,a'[t]\,a''[t]}{Y[t]\,Y'[t]}$$

```
eq$a11$alt + eq$a12 // FullSimplify
```

0

What happens without the Bianchi?

```
(eq$f$00 + Y[t[]]/(3 D[Y[t[]], t[]]) D[eq$f$00, t[]]) + eq$f$11 // FullSimplify
```

$$\left(\mu^2\left(\beta_1\,a[t]^2+2\,\beta_2\,a[t]\,Y[t]+\beta_3\,Y[t]^2\right)\left(X[t]\,a'[t]-Y'[t]\right)\right)\Big/\left(X[t]\,Y[t]^2\,Y'[t]\right)$$

The two equations are not the same. They imply `X[t] a'[t] - Y'[t] == 0` (the same condition that comes from the Bianchi).

In conclusion, imposing that $(3+(Y/\dot{Y})\partial_t)/3$ on EoM (A.9) to get (A.10) has the same effect as using the Bianchi.

Bianchi contraint for *g*

```
CD[-a][m^2 g⌘[a, c] Vg[-c, -b]] // HoldForm;
% // ReleaseHold // ToBasis[ℬ] // ToBasis[ℬ] // TraceBasisDummy // ComponentArray //
    ToValues // printNice // FullSimplify // Part[#, 1] &;
{%%, " = ", %, " = ", 0} // Row
```

$$\nabla g_a\left[m^2\;g^{ac}\;\left[\!\left[\underset{(g)}{V}\right]\!\right]_{cb}\right]\;=\;\frac{3\,m^2\left(a^2\,\beta_1+2\,a\,Y\,\beta_2+Y^2\,\beta_3\right)\left(-X\,\dot{a}+\dot{Y}\right)}{a^3}\;=\;0$$

Compare to (2.12):

$$\frac{3m^2}{a}\left[\beta_1+2\frac{Y}{a}\beta_2+\frac{Y^2}{a^2}\beta_3\right]\left(\dot{Y}-\dot{a}X\right)=0. \tag{2.12}$$

Bianchi contraint for *f*

```
CDf[-a][m^2/M★ f⌘[a, c] Vf[-c, -b]] // HoldForm;
% // ReleaseHold // ToBasis[ℬ] // ToBasis[ℬ] // TraceBasisDummy // ComponentArray //
    ToValues // printNice // FullSimplify // Part[#, 1] &;
{%%, " = ", %, " = ", 0} // Row
```

$$\nabla f_a\left[\frac{m^2\;f^{ac}\;\left[\!\left[\underset{(f)}{V}\right]\!\right]_{cb}}{M_\star}\right]\;=\;\frac{3\,m^2\left(a^2\,\beta_1+2\,a\,Y\,\beta_2+Y^2\,\beta_3\right)\left(X\,\dot{a}-\dot{Y}\right)}{M_\star\,X\,Y^3}\;=\;0$$

```
withBianchi[e_] := Block[{X}, X[t_] := D[Y[t], t] / D[a[t], t]; e]
```

```
RicciScalarCDf[] // ToBasis[ℬ] // ToValues // withBianchi // Simplify // Expand // printNice
```

$$\frac{6\,\kappa}{Y^2}+\frac{6\,\dot{a}^2}{Y^2}+\frac{6\,\dot{a}\,\ddot{a}}{Y\,\dot{Y}}$$

The Bianchi constraint (2.12), when calculated for `f`, has `X Y` in the denominator.
Hence, we get (2.13) (i.e. $X=\frac{dY}{da}$) as a solution provided that $X\neq 0$ and $Y\neq 0$.

3+1 Decomposition of g and f

Define manifold Σ_t and chart Σ_B on Σ_t with the coordinate fields $\{r, \theta, \phi\}$

```
DefManifold[Σt, 3, {i, j, k, l}]
** DefManifold: Defining manifold Σt.
** DefVBundle: Defining vbundle TangentΣt.
```

```
DefChart[ΣB, Σt, Range[1, 3], {Scalars𝓑}[[2;;]], ChartColor → RGBColor[0, 0.7, 0]]
** DefChart: Defining chart ΣB.
** DefMapping: Defining mapping ΣB.
** DefMapping: Defining inverse mapping iΣB.
** DefTensor: Defining mapping differential tensor diΣB[-ẹ, iΣBi].
** DefTensor: Defining mapping differential tensor dΣB[-i, ΣBẹ].
** DefBasis: Defining basis ΣB. Coordinated basis.
** DefCovD: Defining parallel derivative PDΣB[-i].
** DefTensor: Defining vanishing torsion tensor TorsionPDΣB[i, -j, -k].
** DefTensor: Defining symmetric Christoffel tensor ChristoffelPDΣB[i, -j, -k].
** DefTensor: Defining vanishing Riemann tensor RiemannPDΣB[-i, -j, -k, l].
** DefTensor: Defining vanishing Ricci tensor RicciPDΣB[-i, -j].
** DefTensor: Defining antisymmetric +1 density etaUpΣB[i, j, k].
** DefTensor: Defining antisymmetric -1 density etaDownΣB[-i, -j, -k].
```

Define the γ as the induced metric of g on Σ_t (i.e., the spatial part of g)

```
DefMetric[1, γ⌘[-i, -j], cdγ, SymbolOfCovD → {"|", "D(g)"}, PrintAs -> "γ"];
** DefTensor: Defining symmetric metric tensor γ⌘[-i, -j].
** DefTensor: Defining antisymmetric tensor epsilonγ⌘[-i, -j, -k].
** DefCovD: Defining covariant derivative cdγ[-i].
** DefTensor: Defining vanishing torsion tensor Torsioncdγ[i, -j, -k].
** DefTensor: Defining symmetric Christoffel tensor Christoffelcdγ[i, -j, -k].
** DefTensor: Defining Riemann tensor Riemanncdγ[-i, -j, -k, -l].
** DefTensor: Defining symmetric Ricci tensor Riccicdγ[-i, -j].
** DefCovD:  Contractions of Riemann automatically replaced by Ricci.
** DefTensor: Defining Ricci scalar RicciScalarcdγ[].
** DefCovD:  Contractions of Ricci automatically replaced by RicciScalar.
** DefTensor: Defining symmetric Einstein tensor Einsteincdγ[-i, -j].
** DefTensor: Defining vanishing Weyl tensor Weylcdγ[-i, -j, -k, -l].
** DefTensor: Defining symmetric TFRicci tensor TFRiccicdγ[-i, -j].
** DefTensor: Defining Kretschmann scalar Kretschmanncdγ[].
** DefCovD:  Computing RiemannToWeylRules for dim 3
** DefCovD:  Computing RicciToTFRicci for dim 3
** DefCovD:  Computing RicciToEinsteinRules for dim 3
** DefTensor: Defining weight +2 density Detγ⌘[]. Determinant.
```

```
MetricInBasis[γ⌘, -ΣB, g▮[[2;;,2;;]]];
```

```
γ⌘[-i, -j] // printMatrixComponents[ΣB]
```

```
** DefTensor: Defining weight +2 density Detγ⌘ΣB[]. Determinant.
```

» $\gamma_{ij} = \begin{pmatrix} \frac{a^2}{1-\kappa r^2} & 0 & 0 \\ 0 & a^2 r^2 & 0 \\ 0 & 0 & a^2 r^2 \mathrm{Sin}[\theta]^2 \end{pmatrix}$

$\gamma_{ij} = \begin{pmatrix} \frac{a^2}{1-\kappa r^2} & 0 & 0 \\ 0 & a^2 r^2 & 0 \\ 0 & 0 & a^2 r^2 \mathrm{Sin}[\theta]^2 \end{pmatrix}$

```
MetricCompute[γ⌘, ΣB, All, CVSimplify → Simplify]
```

```
** DefTensor: Defining tensor ChristoffelcdγPDΣB[i, -j, -k].
```

The Ricci scalar for γ

```
RicciScalarcdγ[] // printComponents[ΣB, Expand]
```

» $R\left[\underset{(g)}{D}\right] \doteq \frac{6\,\kappa}{a^2}$

$R\left[\underset{(g)}{D}\right] \doteq \frac{6\,\kappa}{a^2}$

The Ricci tensor for γ

```
Riccicdγ[-i, -j] // printMatrixComponents[ΣB]
```

» $R\left[\underset{(g)}{D}\right]_{ij} = \begin{pmatrix} \frac{2\kappa}{1-\kappa r^2} & 0 & 0 \\ 0 & 2\,\kappa\, r^2 & 0 \\ 0 & 0 & 2\,\kappa\, r^2 \mathrm{Sin}[\theta]^2 \end{pmatrix}$

$R\left[\underset{(g)}{D}\right]_{ij} = \begin{pmatrix} \frac{2\kappa}{1-\kappa r^2} & 0 & 0 \\ 0 & 2\,\kappa\, r^2 & 0 \\ 0 & 0 & 2\,\kappa\, r^2 \mathrm{Sin}[\theta]^2 \end{pmatrix}$

The extrinsic curvature $K^{(g)}{}_{ij} = -\frac{1}{2N}\left(\partial_t - \mathcal{L}_{\beta}\right)\gamma_{ij}$ for γ where $N = 1$, $\beta = 0$.

Here we have simply $K_{ij} = -\frac{1}{2}\,\partial_t\,\gamma_{ij}$ since the shift vector is 0.

```
Quiet@UndefTensor[Kγ⌘]
```

```
DefTensor[Kγ⌘[-i, -j], Σt, Symmetric[{-i, -j}]]
```

```
** DefTensor: Defining tensor Kγ⌘[-i, -j].
```

```
γ⌘[-i, -j] // ToBasis[ΣB] // ToBasis[ΣB] // TraceBasisDummy // ComponentArray // ToValues;
```

```
Kγ▮ = -1/2 D[%, t[]] // Simplify;
```

```
AllComponentValues[Kγ⌘[-{i, ΣB}, -{j, ΣB}], Kγ▮];
```

```
Kγ⌘[-i, -j] // printMatrixComponents[ΣB]
```

» $K\gamma⌘_{ij} = \begin{pmatrix} \frac{a\,\dot{a}}{-1+\kappa r^2} & 0 & 0 \\ 0 & -a\, r^2\, \dot{a} & 0 \\ 0 & 0 & -a\, r^2 \mathrm{Sin}[\theta]^2\, \dot{a} \end{pmatrix}$

$K\gamma⌘_{ij} = \begin{pmatrix} \frac{a\,\dot{a}}{-1+\kappa r^2} & 0 & 0 \\ 0 & -a\, r^2\, \dot{a} & 0 \\ 0 & 0 & -a\, r^2 \mathrm{Sin}[\theta]^2\, \dot{a} \end{pmatrix}$

Define the φ as the induced metric of f on Σ_t (i.e., the spatial part of f)

```
DefMetric[1, φ⌘[-i, -j], cdφ, SymbolOfCovD -> {"|", "D(f)"}, PrintAs -> "φ"];
```

DefMetric::old : There are already metrics {γ⌘} in vbundle TΣt.

```
** DefTensor: Defining symmetric metric tensor φ⌘[-i, -j].
** DefTensor: Defining inverse metric tensor Invφ⌘[i, j]. Metric is frozen!
** DefTensor: Defining antisymmetric tensor epsilonφ⌘[-i, -j, -k].
** DefCovD: Defining covariant derivative cdφ[-i].
** DefTensor: Defining vanishing torsion tensor Torsioncdφ[i, -j, -k].
** DefTensor: Defining symmetric Christoffel tensor Christoffelcdφ[i, -j, -k].
** DefTensor: Defining Riemann tensor RiemannDowncdφ[-i, -j, -k, -l].
** DefTensor: Defining Riemann tensor
 Riemanncdφ[-i, -j, -k, l]. Antisymmetric only in the first pair.
** DefTensor: Defining symmetric Ricci tensor Riccicdφ[-i, -j].
** DefCovD:  Contractions of Riemann automatically replaced by Ricci.
** DefTensor: Defining Ricci scalar RicciScalarcdφ[].
** DefCovD:  Contractions of Ricci automatically replaced by RicciScalar.
** DefTensor: Defining symmetric Einstein tensor Einsteincdφ[-i, -j].
** DefTensor: Defining vanishing Weyl tensor Weylcdφ[-i, -j, -k, -l].
** DefTensor: Defining symmetric TFRicci tensor TFRiccicdφ[-i, -j].
** DefTensor: Defining Kretschmann scalar Kretschmanncdφ[].
** DefCovD:  Computing RiemannToWeylRules for dim 3
** DefCovD:  Computing RicciToEinsteinRules for dim 3
** DefTensor: Defining weight +2 density Detφ⌘[]. Determinant.
```

```
MetricInBasis[φ⌘, -ΣB, f■[[2;;,2;;]]];
```

```
φ⌘[-i, -j] // printMatrixComponents[ΣB]
```

```
** DefTensor: Defining weight +2 density Detφ⌘ΣB[]. Determinant.
```

» $$\varphi_{ij} = \begin{pmatrix} \frac{Y^2}{1-\kappa r^2} & 0 & 0 \\ 0 & Y^2 r^2 & 0 \\ 0 & 0 & Y^2 r^2 \mathrm{Sin}[\theta]^2 \end{pmatrix}$$

$$\varphi_{ij} = \begin{pmatrix} \frac{Y^2}{1-\kappa r^2} & 0 & 0 \\ 0 & Y^2 r^2 & 0 \\ 0 & 0 & Y^2 r^2 \mathrm{Sin}[\theta]^2 \end{pmatrix}$$

```
MetricCompute[φ⌘, ΣB, All, CVSimplify → Simplify]
```

```
** DefTensor: Defining tensor ChristoffelcdφPDΣB[i, -j, -k].
```

The Ricci scalar for φ

```
RicciScalarcdφ[] // printComponents[ΣB, Expand]
```

» $$R\left[\underset{(f)}{D}\right] \doteq \frac{6\kappa}{Y^2}$$

$$R\left[\underset{(f)}{D}\right] \doteq \frac{6\kappa}{Y^2}$$

The Ricci tensor for φ

```
Riccicdφ[-i, -j] // printMatrixComponents[ΣB]
```

» $R\left[\underset{(f)}{D}\right]_{ij} = \begin{pmatrix} \frac{2\kappa}{1-\kappa r^2} & 0 & 0 \\ 0 & 2\kappa r^2 & 0 \\ 0 & 0 & 2\kappa r^2 \mathrm{Sin}[\theta]^2 \end{pmatrix}$

$R\left[\underset{(f)}{D}\right]_{ij} = \begin{pmatrix} \frac{2\kappa}{1-\kappa r^2} & 0 & 0 \\ 0 & 2\kappa r^2 & 0 \\ 0 & 0 & 2\kappa r^2 \mathrm{Sin}[\theta]^2 \end{pmatrix}$

The extrinsic curvature $K^{(f)}{}_{ij} = -\frac{1}{2N}(\partial_t - \mathcal{L}_{\boldsymbol{\beta}})\,\varphi_{ij}$ for φ where $N = X$, $\boldsymbol{\beta} = 0$.

Here we have simply $K_{ij} = -\frac{1}{2X}\partial_t\varphi_{ij}$ since the shift vector is 0.

```
Quiet@UndefTensor[Kφ⌘]

DefTensor[Kφ⌘[-i, -j], Σt, Symmetric[{-i, -j}]]
```

** DefTensor: Defining tensor Kφ⌘[-i, -j].

```
φ⌘[-i, -j] // ToBasis[ΣB] // ToBasis[ΣB] // TraceBasisDummy // ComponentArray // ToValues;

Kφ▮ = -1/(2 X) D[%, t[]] // Simplify;

AllComponentValues[Kφ⌘[-{i, ΣB}, -{j, ΣB}], Kφ▮];

Kφ⌘[-i, -j] // printMatrixComponents[ΣB]
```

» $K\varphi⌘_{ij} = \begin{pmatrix} \frac{Y\dot{Y}}{X(-1+\kappa r^2)} & 0 & 0 \\ 0 & -\frac{Y r^2 \dot{Y}}{X} & 0 \\ 0 & 0 & -\frac{Y r^2 \mathrm{Sin}[\theta]^2 \dot{Y}}{X} \end{pmatrix}$

$K\varphi⌘_{ij} = \begin{pmatrix} \frac{Y\dot{Y}}{X(-1+\kappa r^2)} & 0 & 0 \\ 0 & -\frac{Y r^2 \dot{Y}}{X} & 0 \\ 0 & 0 & -\frac{Y r^2 \mathrm{Sin}[\theta]^2 \dot{Y}}{X} \end{pmatrix}$

Hamiltonian Constraints

The Hamiltonian constraint for g

Hamiltonian contraint: $\left(R + K^2 - K_{ij}K^{ij} + 2\kappa V_\rho\right)_{(g)} = 0 = \frac{2}{M_g{}^2}\rho_g{}^{\mathrm{matter}}$.

```
RicciScalarcdγ[] + γ⌘[i, k] Kγ⌘[-i, -k] γ⌘[j, l] Kγ⌘[-j, -l] -
  γ⌘[i, k] γ⌘[j, l] Kγ⌘[-i, -j] Kγ⌘[-k, -l]
% // ToBasis[ΣB] // ToBasis[ΣB] // TraceBasisDummy // ComponentArray // ToValues;
% /. BasisValues[-ΣB, ℬ];
%/2 // Simplify // Expand // printNice
```

$R\left[\underset{(g)}{D}\right] + K\gamma⌘_{ik}\; K\gamma⌘_{jl}\; \gamma^{ik}\; \gamma^{jl} - K\gamma⌘_{ij}\; K\gamma⌘_{kl}\; \gamma^{ik}\; \gamma^{jl}$

$\frac{3\kappa}{a^2} + \frac{3\dot{a}^2}{a^2}$

```
g⌘[{0, ℬ}, a] g⌘[{0, ℬ}, b] m² Vg[-a, -b] // ToBasis[ℬ] // TraceBasisDummy // ToValues //
    Simplify // Expand // printNice
```

$-m^2\beta_0 - \frac{3 m^2 Y \beta_1}{a} - \frac{3 m^2 Y^2 \beta_2}{a^2} - \frac{m^2 Y^3 \beta_3}{a^3}$

```
-% - %%
```

$$m^2\,\beta_0 + \frac{3\,m^2\,Y\,\beta_1}{a} + \frac{3\,m^2\,Y^2\,\beta_2}{a^2} + \frac{m^2\,Y^3\,\beta_3}{a^3} - \frac{3\,\kappa}{a^2} - \frac{3\,\dot{a}^2}{a^2}$$

```
g⌘[{0, ℬ}, {a, ℬ}] (EinsteinCD[{-a, -ℬ}, {0, -ℬ}] + m^2 Vg[{-a, -ℬ}, {0, -ℬ}])
eq214 = % // ToBasis[ℬ] // TraceBasisDummy // ToValues // Simplify // Expand;
% // printNice
```

$$g^{0a}\left(G[\nabla g]_{a0} + m^2\,[\![\underset{(g)}{V}]\!]_{a0}\right)$$

$$m^2\,\beta_0 + \frac{3\,m^2\,Y\,\beta_1}{a} + \frac{3\,m^2\,Y^2\,\beta_2}{a^2} + \frac{m^2\,Y^3\,\beta_3}{a^3} - \frac{3\,\kappa}{a^2} - \frac{3\,\dot{a}^2}{a^2}$$

$$-3\left(\frac{\dot a}{a}\right)^2 - 3\frac{k}{a^2} + m^2\left[\beta_0 + 3\beta_1\frac{Y}{a} + 3\beta_2\frac{Y^2}{a^2} + \beta_3\frac{Y^3}{a^3}\right] = \frac{1}{M_g^2}T^0{}_0\,, \qquad (2.14)$$

The Hamiltonian constraint for *f*

Hamiltonian contraint: $(R + K^2 - K_{ij}\,K^{ij} + 2\,\kappa\,V_\rho)_{(f)} = 0.$

```
RicciScalarcdφ[] + φ⌘[i, k] Kφ⌘[-i, -k] φ⌘[j, l] Kφ⌘[-j, -l] -
  φ⌘[i, k] φ⌘[j, l] Kφ⌘[-i, -j] Kφ⌘[-k, -l]
% // ToBasis[ΣB] // ToBasis[ΣB] // TraceBasisDummy // ComponentArray // ToValues;
% /. BasisValues[-ΣB, ℬ];
%/2 // Simplify // Expand // printNice
```

$$R\left[\underset{(f)}{D}\right] + K\varphi⌘_{ik}\;K\varphi⌘_{jl}\;\varphi^{ik}\;\varphi^{jl} - K\varphi⌘_{ij}\;K\varphi⌘_{kl}\;\varphi^{ik}\;\varphi^{jl}$$

$$\frac{3\,\kappa}{Y^2} + \frac{3\,\dot{Y}^2}{X^2\,Y^2}$$

```
X^2 f⌘[{0, ℬ}, a] f⌘[{0, ℬ}, b] (m^2/M⋆^2) Vf[-a, -b] // ToBasis[ℬ] // TraceBasisDummy // ToValues //
    Simplify // Expand // printNice
```

$$-\frac{a^3\,m^2\,\beta_1}{M_\star{}^2\,Y^3} - \frac{3\,a^2\,m^2\,\beta_2}{M_\star{}^2\,Y^2} - \frac{3\,a\,m^2\,\beta_3}{M_\star{}^2\,Y} - \frac{m^2\,\beta_4}{M_\star{}^2}$$

```
-% - %%
```

$$\frac{a^3\,m^2\,\beta_1}{M_\star{}^2\,Y^3} + \frac{3\,a^2\,m^2\,\beta_2}{M_\star{}^2\,Y^2} + \frac{3\,a\,m^2\,\beta_3}{M_\star{}^2\,Y} + \frac{m^2\,\beta_4}{M_\star{}^2} - \frac{3\,\kappa}{Y^2} - \frac{3\,\dot{Y}^2}{X^2\,Y^2}$$

```
f⌘[{0, ℬ}, {a, ℬ}] (EinsteinCDf[{-a, -ℬ}, {0, -ℬ}] + HoldForm@ (m^2/M⋆^2) Vf[{-a, -ℬ}, {0, -ℬ}])
eq216 = % // ReleaseHold // ToBasis[ℬ] // TraceBasisDummy // ToValues // Simplify // Expand
% // withBianchi // printNice
```

$$f^{0a}\left(G[\nabla f]_{a0} + \frac{m^2}{M_\star{}^2}\,[\![\underset{(f)}{V}]\!]_{a0}\right)$$

$$\frac{m^2\,\beta_4}{M_\star{}^2} + \frac{m^2\,\beta_1\,a[t]^3}{M_\star{}^2\,Y[t]^3} - \frac{3\,\kappa}{Y[t]^2} + \frac{3\,m^2\,\beta_2\,a[t]^2}{M_\star{}^2\,Y[t]^2} + \frac{3\,m^2\,\beta_3\,a[t]}{M_\star{}^2\,Y[t]} - \frac{3\,Y'[t]^2}{X[t]^2\,Y[t]^2}$$

$$\frac{a^3\,m^2\,\beta_1}{M_\star{}^2\,Y^3} + \frac{3\,a^2\,m^2\,\beta_2}{M_\star{}^2\,Y^2} + \frac{3\,a\,m^2\,\beta_3}{M_\star{}^2\,Y} + \frac{m^2\,\beta_4}{M_\star{}^2} - \frac{3\,\kappa}{Y^2} - \frac{3\,\dot{a}^2}{Y^2}$$

$$-3\left(\frac{\dot a}{Y}\right)^2 - 3\frac{k}{Y^2} + \frac{m^2}{M_\star^2}\left(\beta_4 + 3\beta_3\frac{a}{Y} + 3\beta_2\frac{a^2}{Y^2} + \beta_1\frac{a^3}{Y^3}\right) = 0\,, \qquad (2.16)$$

Relation between f/g extrinsic curvatures

```
Kφ▮ == (Y/a) Kγ▮ // withBianchi
```

```
True
```

Quartic Equation

Introduce: $\Upsilon = \underline{Y} / \underline{a}$

$$\Upsilon \equiv \frac{Y}{a}, \quad \rho_\star \equiv \rho / 3M_g^2 m^2 . \tag{3.1}$$

```
eq32 = Block[{Y},
    1/(3 m^2) eq214 + ρ⋆ /. {∂t[] a → H a, Y[t_] :> ϒ[t] a[t] }
  ] // Expand;
% // printNice
```

$\frac{\beta_0}{3} - \frac{\kappa}{a^2 m^2} + \rho_\star + \beta_1 \Upsilon + \beta_2 \Upsilon^2 + \frac{\beta_3 \Upsilon^3}{3} - \frac{H[t]^2}{m^2}$

$$\frac{\beta_3}{3}\Upsilon^3 + \beta_2 \Upsilon^2 + \beta_1 \Upsilon + \frac{\beta_0}{3} + \rho_\star - \frac{H^2}{m^2} - \frac{k}{m^2 a^2} = 0 \tag{3.2}$$

We assumed $\underline{Y} \neq 0$, hence $\Upsilon = \underline{Y} / \underline{a} \neq 0$ so we can multiply Eq. (2.16) by Υ^2.

```
eq33 = Block[{Y},
    Y^2/(3 m^2 a^2) eq216 /. {∂t[] Y → X H a, Y[t_] :> ϒ[t] a[t] }
  ] // Expand;
% // printNice
```

$\frac{\beta_2}{M_\star^2} - \frac{\kappa}{a^2 m^2} + \frac{\beta_1}{3 M_\star^2 \Upsilon} + \frac{\beta_3 \Upsilon}{M_\star^2} + \frac{\beta_4 \Upsilon^2}{3 M_\star^2} - \frac{H[t]^2}{m^2}$

$$\frac{\beta_4}{3M_\star^2}\Upsilon^2 + \frac{\beta_3}{M_\star^2}\Upsilon + \frac{\beta_1}{3M_\star^2}\frac{1}{\Upsilon} + \frac{\beta_2}{M_\star^2} - \frac{H^2}{m^2} - \frac{k}{m^2 a^2} = 0 . \tag{3.3}$$

Eqs. (3.2) and (3.3) combined gives the quartic equation (3.4):

```
eq34 = ϒ (eq32 - eq33);
Replace[Apart[%, ϒ], Plus → List, {1}, Heads → True] // FullSimplify // Total;
% // printNice
```

$-\frac{\beta_1}{3 M_\star^2} + \frac{1}{3}\left(\beta_0 - \frac{3 \beta_2}{M_\star^2} + 3 \rho_\star\right) \Upsilon + \left(\beta_1 - \frac{\beta_3}{M_\star^2}\right) \Upsilon^2 + \frac{1}{3}\left(3 \beta_2 - \frac{\beta_4}{M_\star^2}\right) \Upsilon^3 + \frac{\beta_3 \Upsilon^4}{3}$

$$\frac{\beta_3}{3}\Upsilon^4 + \left(\beta_2 - \frac{\beta_4}{3M_\star^2}\right)\Upsilon^3 + \left(\beta_1 - \frac{\beta_3}{M_\star^2}\right)\Upsilon^2 + \left(\rho_\star + \frac{\beta_0}{3} - \frac{\beta_2}{M_\star^2}\right)\Upsilon - \frac{\beta_1}{3M_\star^2} = 0 . \tag{3.4}$$

From the paper: "This determines Υ as a function of $\rho_\star$. The analytic solutions are straightforward to derive but lengthy to display. For generic values of parameters there is the further complexity that in some cases one has to match different branches of solutions in order to keep Υ real for all ρ, although a real solution always exists. In order to keep the discussion transparent we simply state some generic properties here and then consider a few special cases in more detail."

Notes regarding singularities

We have two possible genuine singularities (where Ricci/Kretschmann scalars degenerate):

- *g*: at $a = 0$,
- *f*: at $X = 0$, or at $Y = 0$ (where $Y = 0$ is equivalent to $\Upsilon = Y / a = 0$).

These singularities correspond to a diverging expansion scalar (see the section below on the Raycahaudhury equation), that is, they are are a Big Bang / Gnab Gib for the respective metric.

The evolution of the coupled PDE for *g* and *f* **stop** at the singularity.

The eventual matching of two branches across $\Upsilon = 0$ would be a splicing of two different *f* spacetimes (two "*f* universes" at the crossover big bang / gnab gib). If the lapse *X* changes the sign, then the branch of the square root dictates the change of sign of *Y* too.

When matching: a^2, X^2 and Y^2 are part of the metric so they must be continuously differentiable.

Raychaudhuri equation

The Raychaudhuri equation:

$$\dot{\Theta} + \frac{1}{3}\Theta + \frac{\kappa}{2}[\rho + 3\,p] = 0$$

$$\frac{\ddot{a}}{a} + \frac{\kappa}{2}\,\frac{1}{3}\,(\rho + 3\,p) = H^2 + \dot{H} + \frac{\kappa}{2}\,\frac{1}{3}\,(\rho + 3\,p) = 0$$

obtained from R_{00} in the locally comoving frame $u^a\,u_a = -1$:

```
DefTensor[u[-a], M]
```

```
** DefTensor: Defining tensor u[-a].
```

The expansion scalar Θ for *g*

```
AllComponentValues[u[{-a, -B}], {-1, 0, 0, 0}];
u[-a] // printNonZeroComponents[B]
```

» $u_0 \doteq -1$

```
g⌘[a, b] u[-a] u[-b] // printComponents[B]
```

» $g^{ab}\;u_a\;u_b \doteq -1$

$g^{ab}\;u_a\;u_b \doteq -1$

```
g⌘[a, b] CD[-a]@u[-b] // printComponents[B]
```

» $g^{ab}\,(\nabla g_a\,u_b) \doteq \frac{3\,\dot{a}}{a}$

$g^{ab}\,(\nabla g_a\,u_b) \doteq \frac{3\,\dot{a}}{a}$

The expansion scalar Θ for *f*

```
AllComponentValues[u[{-a, -B}], {-X, 0, 0, 0}];
u[-a] // printNonZeroComponents[B]
```

» $u_0 \doteq -X$

```
f⌘[a, b] u[-a] u[-b] // printComponents[B]
```

» $f^{ab}\;u_a\;u_b \doteq -1$

$f^{ab}\;u_a\;u_b \doteq -1$

```
f⌘[a, b] CDf[-a]@u[-b] // printComponents[ℬ]
```

» $f^{ab}\ \left(\nabla f_a\, u_b\right)\ \doteq\ \frac{3\,\dot{Y}}{X\,Y}$

$f^{ab}\ \left(\nabla f_a\, u_b\right)\ \doteq\ \frac{3\,\dot{Y}}{X\,Y}$

TODO

Analyze the phase space $a, \dot{a}, Y, \dot{Y}, X, \dot{X}$

we know from the Bianchi contraint $X = \frac{\dot{Y}}{\dot{a}}$

and also $\Upsilon = \frac{Y}{a}$,

$\dot{\Upsilon} = \dot{Y} / a - Y\,\dot{a} \big/ a^2 = \left(\dot{Y} / \dot{a}\right)\ (\dot{a} / a) - Y\,\dot{a} \big/ a^2 = X\,H - \Upsilon\,H = (X - \Upsilon)\ \frac{\dot{a}}{a}$

```
eq34$pn = eq34 // Simplify // printNice
```

$$\frac{1}{3\,M_\star^2}\left(-\beta_1 + \left(-3\,\beta_2 + M_\star^2\ (\beta_0 + 3\,\rho_\star)\right)\,\Upsilon + 3\,\left(M_\star^2\,\beta_1 - \beta_3\right)\,\Upsilon^2 + \left(3\,M_\star^2\,\beta_2 - \beta_4\right)\,\Upsilon^3 + M_\star^2\,\beta_3\,\Upsilon^4\right)$$

```
∂t[] eq34 // Simplify // Expand
% // Simplify // Apart
% / ∂t[] ϒ
% /. ∂t[] ϒ → (X - ϒ) ∂t[] a / a
eq34d$pn = % // printNice
```

$$\frac{1}{3}\,\beta_0\,\Upsilon'[t] - \frac{\beta_2\,\Upsilon'[t]}{M_\star^2} + \rho_\star\,\Upsilon'[t] + 2\,\beta_1\,\Upsilon[t]\,\Upsilon'[t] - \frac{2\,\beta_3\,\Upsilon[t]\,\Upsilon'[t]}{M_\star^2} + 3\,\beta_2\,\Upsilon[t]^2\,\Upsilon'[t] - \frac{\beta_4\,\Upsilon[t]^2\,\Upsilon'[t]}{M_\star^2} + \frac{4}{3}\,\beta_3\,\Upsilon[t]^3\,\Upsilon'[t]$$

$$\frac{1}{3\,M_\star^2}\left(M_\star^2\,\beta_0 - 3\,\beta_2 + 3\,M_\star^2\,\rho_\star + 6\,M_\star^2\,\beta_1\,\Upsilon[t] - 6\,\beta_3\,\Upsilon[t] + 9\,M_\star^2\,\beta_2\,\Upsilon[t]^2 - 3\,\beta_4\,\Upsilon[t]^2 + 4\,M_\star^2\,\beta_3\,\Upsilon[t]^3\right)\,\Upsilon'[t]$$

$$\frac{1}{3\,M_\star^2}\left(M_\star^2\,\beta_0 - 3\,\beta_2 + 3\,M_\star^2\,\rho_\star + 6\,M_\star^2\,\beta_1\,\Upsilon[t] - 6\,\beta_3\,\Upsilon[t] + 9\,M_\star^2\,\beta_2\,\Upsilon[t]^2 - 3\,\beta_4\,\Upsilon[t]^2 + 4\,M_\star^2\,\beta_3\,\Upsilon[t]^3\right)$$

$$\frac{1}{3\,M_\star^2}\left(M_\star^2\,\beta_0 - 3\,\beta_2 + 3\,M_\star^2\,\rho_\star + 6\,M_\star^2\,\beta_1\,\Upsilon[t] - 6\,\beta_3\,\Upsilon[t] + 9\,M_\star^2\,\beta_2\,\Upsilon[t]^2 - 3\,\beta_4\,\Upsilon[t]^2 + 4\,M_\star^2\,\beta_3\,\Upsilon[t]^3\right)$$

$$\frac{1}{3\,M_\star^2}\left(M_\star^2\,\beta_0 - 3\,\beta_2 + 3\,M_\star^2\,\rho_\star + 6\,M_\star^2\,\beta_1\,\Upsilon - 6\,\beta_3\,\Upsilon + 9\,M_\star^2\,\beta_2\,\Upsilon^2 - 3\,\beta_4\,\Upsilon^2 + 4\,M_\star^2\,\beta_3\,\Upsilon^3\right)$$

```
eq34$pn // printNice
```

$$\frac{1}{3\,M_\star^2}\left(-\beta_1 + \left(-3\,\beta_2 + M_\star^2\ (\beta_0 + 3\,\rho_\star)\right)\,\Upsilon + 3\,\left(M_\star^2\,\beta_1 - \beta_3\right)\,\Upsilon^2 + \left(3\,M_\star^2\,\beta_2 - \beta_4\right)\,\Upsilon^3 + M_\star^2\,\beta_3\,\Upsilon^4\right)$$

```
With[
 {
  p1 = eq34$pn,
  p214 = eq214,
  p216 = eq216
 },
 Manipulate[
  Plot[
   {Sign@p1 Log10@Abs[10^2 p1] / 2},
   {ϒ, -10, 10}, AspectRatio → 0.5, ImageSize → 500,
   PlotRange → {-5, 5},
   Exclusions → {p1 == 0}, ExclusionsStyle → Red,
   AxesLabel → {ϒ, "sgn log10 Eq. (3.4)"},
   AxesStyle → Directive[FontFamily → "Times", Medium]
  ],
  {
    {
     {"ρ⋆", Control[{{ρ⋆, 0.1, ""}, -10, 10}]},
     {"M⋆", Control[{{M⋆, 0.1, ""}, 0.01, 3}]},
     {"m", Control[{{m, 0.1, ""}, 0.01, 3}]},
     {"κ", Control[{{κ, 0, ""}, {-1, 0, 1}}]},
     {},
     {Button[" Setup Eqs ",
       Print["Eq 2.14"]; Print[p214 == 0 /. {t[] → t}];
       Print["Eq 2.16"]; Print[p216 == 0 /. {t[] → t}];
       Print["Eq 3.4"]; Print[p1 == 0];
       Print[(Y[t] → ϒ a[t]) /. Solve[p1 == 0, ϒ]];
      ], SpanFromLeft}
    } // Grid[#, Alignment → Left] &,
   Spacer[20],
   {
     {"β0", Control[{{β0, 0, ""}, -3, 3}]},
     {"β1", Control[{{β1, -0.5, ""}, -3, 3}]},
     {"β2", Control[{{β2, 0, ""}, -3, 3}]},
     {"β3", Control[{{β3, 0.5, ""}, -3, 3}]},
     {"β4", Control[{{β4, 0.5, ""}, -3, 3}]}
    } // Grid
  } // Row
 ]]
```

Used Resourses

```
Print[ xAct`xCore`Private`bars ]
Print[ "Memory in use: ", Style[ MemoryInUse[] - mem$1, Blue ],
 " + ", Style[ mem$1 - mem$0, Blue ] ]
Print[ "Time used: ", Style[ TimeUsed[] - cpu$1, Blue ],
 " + ", Style[ cpu$1 - cpu$0, Blue ] ]
Print[ xAct`xCore`Private`bars ]
```

```
------------------------------------------------------------
Memory in use: 40 568 304 + 15 488 280
Time used: 36.44 + 1.513
------------------------------------------------------------
```