

Causality Constraints on Massive Gravity (1610.02033)

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Paper:

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Causality Constraints on Massive Gravity

<https://arxiv.org/abs/1610.02033>

Definitions:

The Minkowski null frame and its inverse:

$$\eta = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

`iη = Inverse@η; iη // MatrixForm`

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The vielbien of g :

$$\mathbf{E} = \begin{pmatrix} -F/2 & 1 & 0 \\ 1 & -G/2 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

The metric g and its null cone:

$$\mathbf{g} = \mathbf{E}^T \cdot \eta \cdot \mathbf{E};$$

`g // MatrixForm`

$$\begin{pmatrix} F & -1 - \frac{FG}{4} & 0 \\ -1 - \frac{FG}{4} & G & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{nc} \$ \mathbf{g} = (\{\{\mathbf{Tu}, \mathbf{Tv}, \mathbf{Tr}\}\} \cdot \mathbf{g} \cdot \{\{\mathbf{Tu}, \mathbf{Tv}, \mathbf{Tr}\}\}^T) \llbracket 1, 1 \rrbracket // \text{Simplify}$$

$$\mathbf{Tr}^2 + \frac{1}{2} (F \mathbf{Tu} - 2 \mathbf{Tv}) (2 \mathbf{Tu} - G \mathbf{Tv})$$

The inverse vielbien:

`itE = Inverse@Transpose@E // Simplify;`

`itE // MatrixForm`

$$\begin{pmatrix} \frac{2G}{4-FG} & \frac{4}{4-FG} & 0 \\ \frac{4}{4-FG} & \frac{2F}{4-FG} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{BesselK}[0, 10.^{-30}]$$

69.1935

The particular field F for the sandwich pp-wave solution:

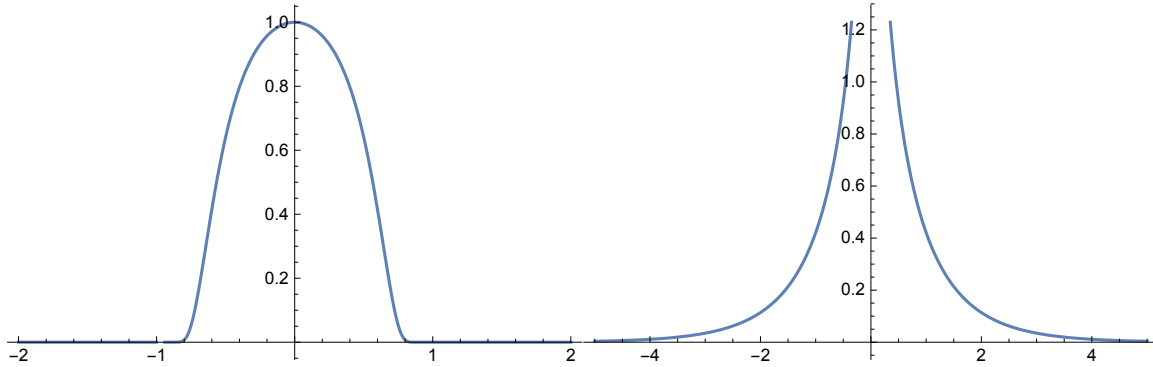
$$\mathbf{A}[\mathbf{u}_-, \mathbf{a}_-, \lambda_-] := \begin{cases} \mathbf{a} \mathbf{Exp}\left[-\frac{\lambda^2 u^2}{(u^2 - \lambda^2)^2}\right] & -\lambda < u < \lambda \\ 0 & \text{True} \end{cases}$$

$$A(u) = \begin{cases} a \exp\left[-\frac{\lambda^2 u^2}{(u^2 - \lambda^2)^2}\right] & \text{if } u \in [-\lambda, \lambda], \\ 0 & \text{otherwise,} \end{cases}$$

```
Fg[u_, r_, a_, λ_, m_] := A[u, a, λ] { BesselK[0, mAbs@r] mAbs@r ≥ 10-30
70 True
```

$$F = A(u)K_0(m|\vec{x}|)$$

```
Row@{Plot[A[u, 1, 1], {u, -2, 2}, ImageSize → 300],
Plot[BesselK[0, Abs@r], {r, -5, 5}, ImageSize → 300]}
```



The null cone of g :

Null cone field and null geodesics

```
ncEllipsed[loc_, lVec_, nVec_, bθ_, scale_] := With[
  {l = Normalize@lVec, n = Normalize@nVec},
  With[
    {
      a = Norm[l - n],
      c = Norm[l + n],
      b = Sin[bθ] scale
    },
    With[
      {p1 =  $\frac{a \sqrt{c^2 - b^2}}{c}$ , p2 =  $\frac{c^2 - b^2}{c}$ },
      Translate[Rotate[Scale[
        {
          Line[{{-p1, p2}, {0, 0}, {p1, p2}}],
          Line[{{-p1, -p2}, {0, 0}, {p1, -p2}}],
          Thick, Circle[{0, c}, {a, b}],
          Circle[{0, -c}, {a, b}, MinMax[{π - bθ/2, 2π + bθ/2}],
          Thin, Dashed, Circle[{0, -c}, {a, b}, MinMax[{bθ/2, π - bθ/2}]
        },
        scale], {{0, 1}, (1 + n)/2}], loc]
    ]]]
```

```
uiConeField[a_, λ_, m_, x_, r_, zt_,
  range_: 1.5, extraPanel_: False, in3D_: False] := Block[
  {F, G, lVec, nVec, nc, tvec$g, scale = range/1.5},
  F = Fg[N@u, N[x r], N@a, N@λ, N@m];
  G = Fg[N@v, N[(1 - x) r], N@a, N@λ, N@m];
  lVec[z_, t_] = {#[[2]] - #[[1]], #[[1]] + #[[2]]} / 21/2 &@
    (itE[[1, {1, 2}]] /. {u → (t - z) / 21/2, v → (t + z) / 21/2}});
  nVec[z_, t_] = {#[[2]] - #[[1]], #[[1]] + #[[2]]} / 21/2 &@
    (itE[[2, {1, 2}]] /. {u → (t - z) / 21/2, v → (t + z) / 21/2}});
```

```

nc = nc$g /. {u → (zt[[2]] - zt[[1]]) / 21/2, v → (zt[[2]] + zt[[1]]) / 21/2};
tvec$g = Normalize[ Normalize@itE[[1]] + Normalize@itE[[2]] ] /.
  {u → (zt[[2]] - zt[[1]]) / 21/2, v → (zt[[2]] + zt[[1]]) / 21/2};
Row@{
  Show[
    StreamPlot[lVec[z$v, t$v], {z$v, -2 range, 2 range}, {t$v, -2 range, 2 range},
      StreamStyle → {Opacity[0.5], Blue, "Line"},
      StreamPoints → ControlActive[{{0, 0}}, {
        Table[{-u/2, u/2}, {u, -2 range, 2 range, range/3}],
        0.1, 5 range}],
    ImageSize → {350, 350}
  ],
  StreamPlot[nVec[z$v, t$v], {z$v, -2 range, 2 range}, {t$v, -2 range, 2 range},
    StreamStyle → {Opacity[0.5], Red, "Line"},
    StreamPoints → ControlActive[{{0, 0}}, {
      Table[{v/2, v/2}, {v, -2 range, 2 range, range/3}],
      0.1, 5 range}]
  ],
  Graphics[{
    Arrow[{zt, zt + 0.6 {1, 1} scale}], Text[Style[v, 15], zt + 0.7 {0.9, 0.7} scale],
    Arrow[{zt, zt + 0.6 {-1, 1} scale}],
    Text[Style[u, 15], zt + 0.7 {-0.9, 0.7} scale],
    ncEllipsed[zt, lVec@@zt, nVec@@zt, 0.35/scale, 0.3 scale],
    Blue, Arrow[{zt, zt + 0.3 Normalize[lVec@@zt] scale}],
    Red, Arrow[{zt, zt + 0.3 Normalize[nVec@@zt] scale}],
    Black,
    Text[Style["z →", 15], Scaled@{0.87, 0.038}],
    Text[Style[Rotate["t →", 90 Degree], 15], Scaled@{0.04, 0.87}],
    If[extraPanel, Locator@zt, Point@zt]
  ]],
  PlotRange → range,
  Frame → True, BaseStyle → {FontFamily → "Cambria", 12}
],
Sequence@@ If[¬ extraPanel, {Nothing}, {
  Spacer[20],
  If[in3D, Graphics3D[
    GeometricTransformation[ContourPlot3D[
      nc,
      {Tv, -1, 1},
      {Tr, -1, 1},
      {Tu, -1, 1},
      PlotPoints → ControlActive[10, 50],
      Mesh → None, Contours → {0},
      ContourStyle → {White, Opacity[0.9], Specularity[1, 20]},
      Boxed → False, Axes → False,
      RegionFunction →
        Function[{vv, rr, uu}, Abs[{uu, vv, rr}.tvec$g] ≤ 0.7]
    ][[1]], RotationMatrix[-45 Degree, {0, 1, 0}]],
    Boxed → False, Axes → False,
    AxesLabel → {Row@{z, " →"}, None, Rotate[Row@{t, " →"}, 90 Degree]},

```

```

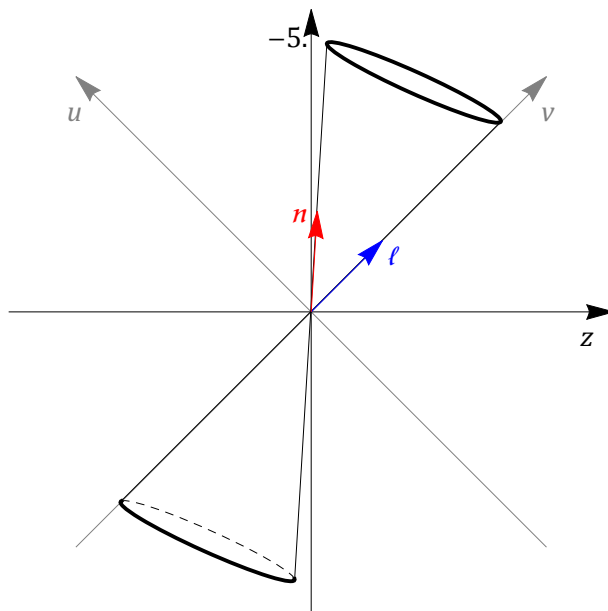
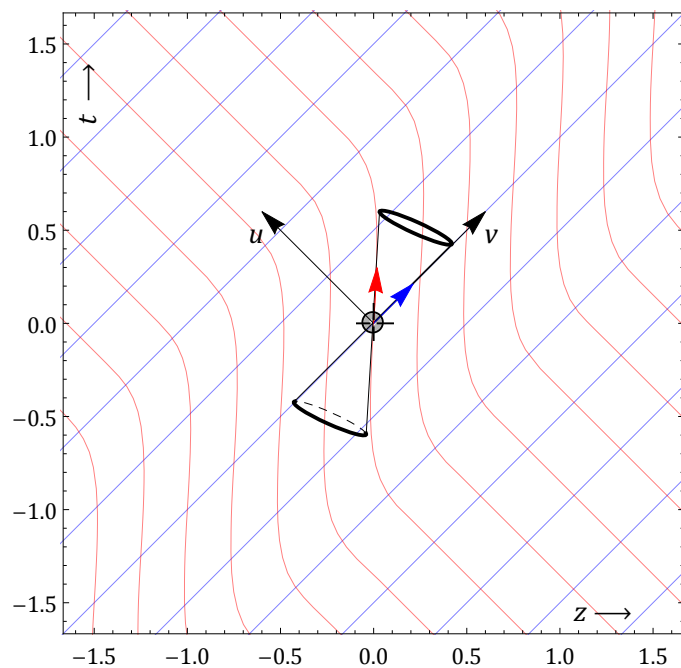
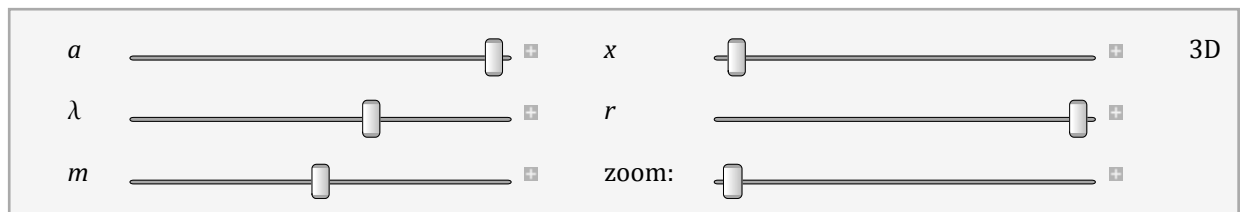
    Ticks → None, PlotRange → All,
    BaseStyle → {FontFamily → "Cambria", Bold, 15},
    ImageSize → {350, 300}, ViewPoint → Front
  ],
Graphics[{
  Gray,
  Arrow[0.7 {{-1, -1}, {1, 1}}], Text[Style[v, 15], 0.8 {0.88, 0.75}],
  Arrow[0.7 {{1, -1}, {-1, 1}}], Text[Style[u, 15], 0.8 {-0.88, 0.75}],
  Black,
  Arrow[0.9 {{0, -1}, {0, 1}}], Text[Style[t, 15], {-0.07, 0.82}],
  Arrow[0.9 {{-1, 0}, {1, 0}}], Text[Style[z, 15], {0.82, -0.07}],
  ncEllipsed[{0, 0}, lVec@@zt, nVec@@zt, 0.2, 0.4],
  Blue,
  Arrow[{{0, 0}, 0.3 Normalize[lVec@@zt]}],
  Text[Style[ℓ, 15], 0.3 RotationMatrix[-10 Degree].Normalize[lVec@@zt]],
  Red,
  Arrow[{{0, 0}, 0.3 Normalize[nVec@@zt]}],
  Text[Style[n, 15], 0.3 RotationMatrix[10 Degree].Normalize[nVec@@zt]]
},
PlotRange → 1,
BaseStyle → {FontFamily → "Cambria", 12},
ImageSize → 350
]
]]

```

```

Manipulate[
  uiConeField[a, λ, m, x, r, zt, zoom, True, in3D],
  Panel[Grid[{
    {
      Control[{{zt, {0, 0}}, Locator, Appearance → None}]; Spacer[10],
      "a",
      Control[{{a, 0.93, ""}, -0.93, 0.93}],
      Spacer[10],
      "x", Control[{{x, 0.01, ""}, 0.001, 0.999}],
      Spacer[10],
      "3D", Control[{{in3D, False, ""}, {True, False}}]
    },
    {
      "",
      "λ", Control[{{λ, 1, ""}, 0.1, 1.5}],
      "",
      "r", Control[{{r, 10, ""}, 0.5, 10}],
      "",
      "", Button["▶",
        Block[{F, G},
          F = Fg[N@u, N[x r], N@a, N@λ, N@m];
          G = Fg[N@v, N[(1 - x) r], N@a, N@λ, N@m];
          Do[
            zt = zt + 0.01 {#[[2]] - #[[1]], #[[1]] + #[[2]]} / 21/2 &@
              (itE[[2], {1, 2}]] /. {u → (zt[[2]] - zt[[1]]) / 21/2, v → (zt[[2]] + zt[[1]]) / 21/2},
            {30}
          ]
        ],
      ],
      Method → "Queued"
    },
    {
      "",
      "m", Control[{{m, 1, ""}, 0.5, 1.5}],
      "", "zoom:", Control[{{zoom, 1.5, ""}, 1.5, 10, 0.5}]
    }
  ], Alignment → Left], ImageSize → Scaled[1],
  BaseStyle → {FontFamily → "Cambria", FontSize → 14}],
  SaveDefinitions → True, Deployed → True, AppearanceElements → None, Paneled → False,
  FrameMargins → 0
]

```



Scattering on pp-waves

```

ticks[min_, max_] :=
  Join[Table[{i, Style[i, 12], {.01, 0}}, {i, Ceiling[min], Floor[max]}],
    Table[{j + .5, , {.005, 0}}, {j, Round[min], Round[max - 1], 1}]]

```

```

particlePosition[t_,  $\delta$ _, r$sep_, b$sep_] := If[t ≤ -2.5,
  {- $\delta$  - t, -r$sep/2 + b$sep, 0},
  If[t ≤ 0.5,
    {- $\delta$  + 2.5, -r$sep/2 + b$sep + (t + 2.5), 0},
    If[t ≤ 3,
      {- $\delta$  + 2.5 + (t - 0.5), r$sep/2 - b$sep, 0},
      {- $\delta$  + 5, r$sep/2 - b$sep - (t - 3), 0}
    ]
  ]
]

particleTime[t_,  $\delta$ _, r$sep_, b$sep_] := t

With[
  {r$sep = 4, b$sep = 1/2,  $\delta$  = 5},
  Manipulate[
    Row@{
      Show[
        Graphics3D[{
          If[$ControlActiveSetting, Nothing, {
            Arrowheads[{-0.024, 0.024}],
            Gray, Arrow@{{1, -r$sep/2, 0}, {1, r$sep/2, 0}},
            Black, Text[Style[r, 15], {1, 0, 0.4}],
            Gray, Arrow@{{0.5, -r$sep/2 + b$sep, 0}, {0.5, r$sep/2 - b$sep, 0}},
            Black, Text[Style[d, 15], {0.5, 0, 0.4}],
            Arrowheads[0.015],
            Gray, Arrow@{{-1, -r$sep/2 + b$sep + 0.5, 0}, {-1, -r$sep/2 + b$sep, 0}},
            Arrow@{{-1, -r$sep/2 - 0.5, 0}, {-1, -r$sep/2, 0}},
            Black, Text[Style[b, 15], {-1, -r$sep/2 + b$sep/2, 0.1}]
          ]],
          Arrowheads[0.024],
          If[t < -2.5, Gray, LightGray],
          Arrow@{{0, -r$sep/2 + b$sep, 0}, {- $\delta$  + 2.5, -r$sep/2 + b$sep, 0}},
          If[-2.5 ≤ t < 0.5, Gray, LightGray],
          Arrow@{{- $\delta$  + 2.5, -r$sep/2 + b$sep, 0}, {- $\delta$  + 2.5, r$sep/2 - b$sep, 0}},
          If[0.5 ≤ t < 3, Gray, LightGray],
          Arrow@{{- $\delta$  + 2.5, r$sep/2 - b$sep, 0}, {0, r$sep/2 - b$sep, 0}},
          If[3 < t, Gray, LightGray],
          Arrow@{{0, r$sep/2 - b$sep, 0}, {0, -r$sep/2 + b$sep, 0}},
          Arrowheads[0.024], Opacity[0.35], Specularity[1, 20],
          Blue, Arrow@Tube@{{- $\delta$ , -r$sep/2, 0}, { $\delta$ , -r$sep/2, 0}},
          Red, Arrow@Tube@{{ $\delta$ , r$sep/2, 0}, {- $\delta$ , r$sep/2, 0}},
          (* Particle *)
          Opacity[0.5], Specularity[1, 20], Gray,
          Sphere[particlePosition[t,  $\delta$ , r$sep, b$sep], 0.2],
          Sphere[{particlePosition[t,  $\delta$ , r$sep, b$sep][[1]],  $\delta$  - 0.1, 0.2}, 0.1]
        ],
        PlotRange → {{- $\delta$ ,  $\delta$ }, {- $\delta$ ,  $\delta$ }, {- $\delta$ ,  $\delta$ }/2},
        Boxed → True, Axes → True, AxesLabel → (Style[#, 15] & /@ {z, x, y}),
        FaceGrids → {{{-1, 0, 0}, {Range[- $\delta$ ,  $\delta$ , 0.5], Range[- $\delta$ ,  $\delta$ , 0.5]}}},
        FaceGridsStyle → Directive[Dashed, LightGray],

```

```

BaseStyle → {FontFamily → "Cambria", FontSize → 14},
Ticks → {ticks[-δ, δ], ticks[-δ, δ], ticks[-δ, δ]},
ImageSize → 600
],
SliceDensityPlot3D[ (* F[u], moving along v *)
Abs@Fg[(u*) (t - z) / 21/2 + 2, ((x + r$sep / 2)2 + y2)(1/2), a, λ, m],
ControlActive[{(t - z) / 21/2 + 2 == 0},
{Thread[(t - z) / 21/2 + 2 == Subdivide[-2 / 3 λ, 2 / 3 λ, 6]]}],
{z, -δ, δ},
{x, -δ, δ},
{y, -δ, δ},
PlotRange → {0.1, 5}, BoundaryStyle → None,
ColorFunction → (Directive[Blue, Opacity@Rescale[#, {0.1, 1}, {0.15, 0.9}]] &)
],
SliceDensityPlot3D[ (* F[v], moving along u *)
Abs@Fg[(v*) (t + z) / 21/2, ((x - r$sep / 2)2 + y2)(1/2), a, λ, m],
ControlActive[{(t + z) / 21/2 == 0},
{Thread[(t + z) / 21/2 == Subdivide[-2 / 3 λ, 2 / 3 λ, 6]]}],
{z, -δ, δ},
{x, -δ, δ},
{y, -δ, δ},
PlotRange → {0.1, 5}, BoundaryStyle → None,
ColorFunction → (Directive[Red, Opacity@Rescale[#, {0.1, 1}, {0.15, 0.9}]] &)
],
ParametricPlot3D[
{
{z, δ, A[(t - z) / 21/2 + 2, a, λ]}, (* u *)
{z, δ - 0.1, A[(t + z) / 21/2, a, λ]} (* v *)
},
{z, -δ, δ},
PlotStyle → {{Opacity[0.5], Blue}, {Opacity[0.5], Red}}
]
],
Spacer[20],
Show[
uiConeField[a, λ, m, 0.01, 10,
{particlePosition[t, δ, r$sep, b$sep][[1]], t}, 6][[1, 1]],
ParametricPlot[
{
{z, t + A[t - z, a, λ]}, (* u *)
{z, t + A[t + z, a, λ]} (* v *)
},
{z$z, -δ, δ},
Epilog → {PointSize[0.1], Point[{
particlePosition[t, δ, r$sep, b$sep][[1]], particleTime[t, δ, r$sep, b$sep]
}]}
],
PlotRange → {{-δ, δ}, {-δ, δ + 1}},
PlotStyle → {{Opacity[0.5], Blue}, {Opacity[0.5], Red}},
BaseStyle → {FontFamily → "Cambria", 12},

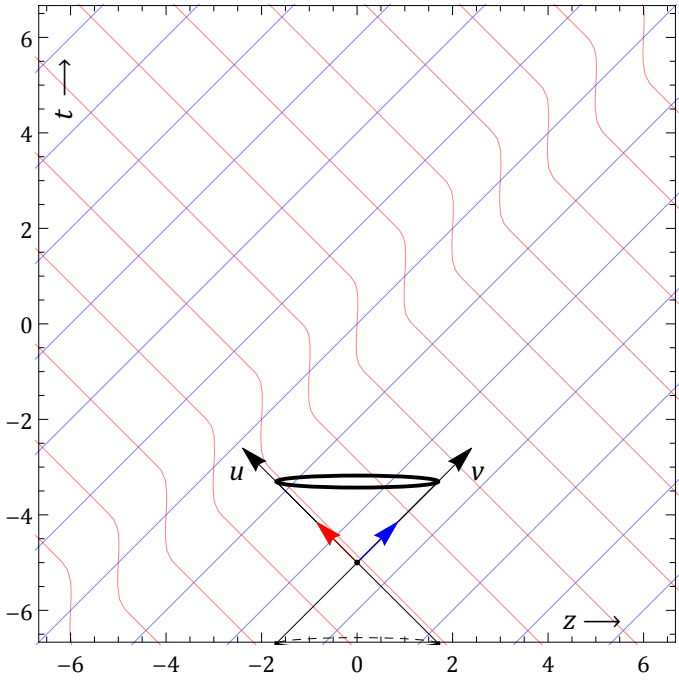
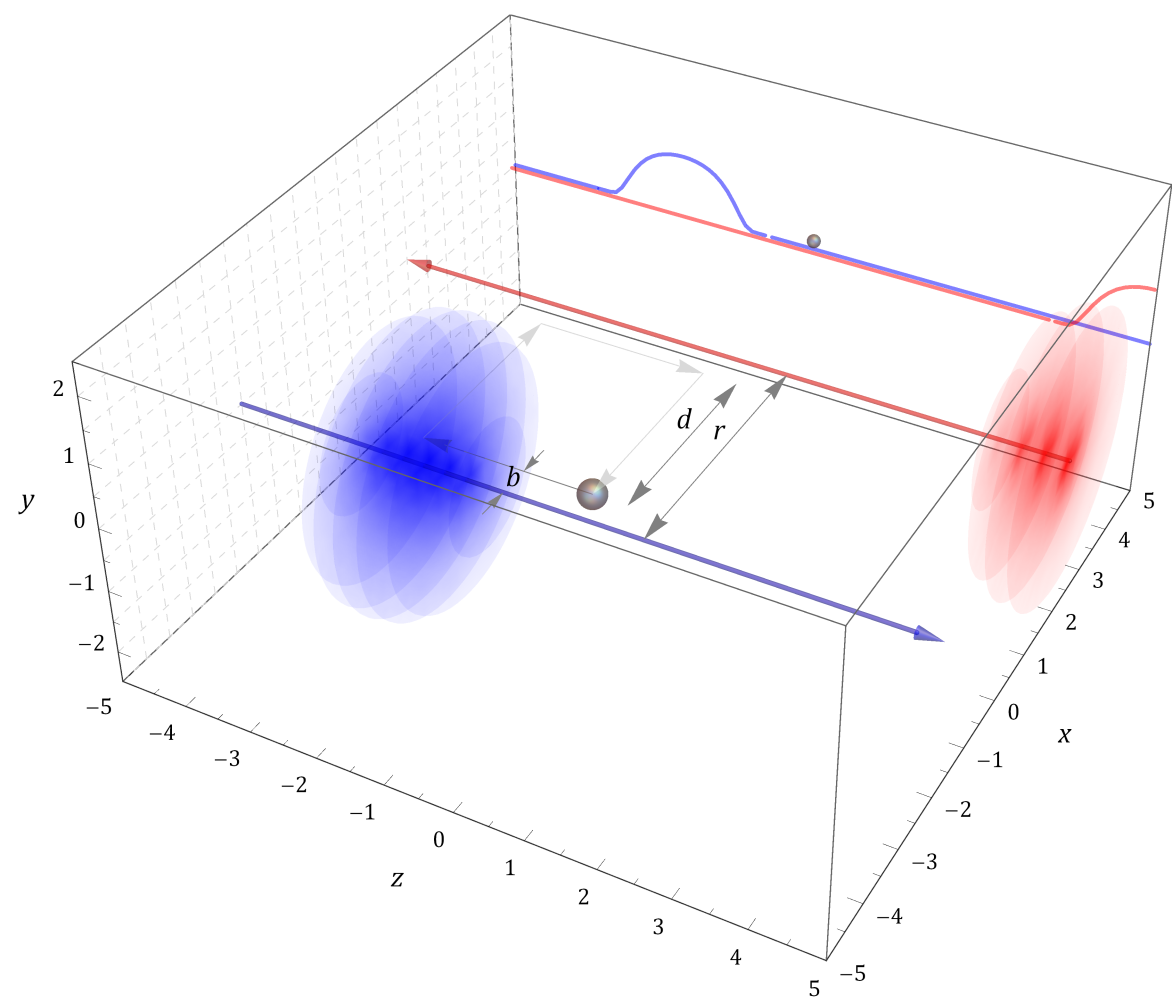
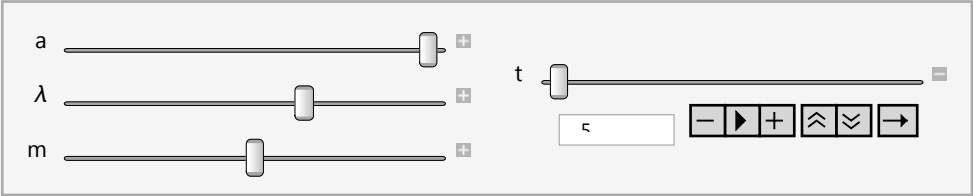
```



```

    FrameLabel → {Style[Row@{z, " →"}, 15, Darker@Green],
      Style[Row@{time, " →"}, 15, Darker@Green]},
    Frame → True, AspectRatio → 1,
    ImageSize → 400
  ]
]
},
Panel@Row[{
  Column@{
    Control[{{a, 0.93}, -0.93, 0.93}],
    Control[{{λ, 1}, 0.1, 1.5}],
    Control[{{m, 1}, 0.5, 1.5]}
  ],
  Spacer[20],
  Column@{
    Control[{{t, -δ}, -δ, δ + 1, 0.1, AnimationRate → 1/5, Appearance → "Open"}]
  }
}],
SaveDefinitions → True, AppearanceElements → None, Paneled → False
]
]

```



```

particleTime2[t_,  $\delta$ _, r$sep_, b$sep_] := If[t ≤ -3, t,
  If[t ≤ -2.5, -3 - (t + 3),
    If[t ≤ 1, -4 + (t + 2.5),
      If[t ≤ 2,
        0 - (t - 1),
        0 + (t - 2)
      ]
    ]
  ]
]

```