

# Priority Queues: Binary Heaps

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Data Structures  
Data Structures and Algorithms

# Outline

- 1 Binary Trees
- 2 Basic Operations
- 3 Complete Binary Trees
- 4 Pseudocode
- 5 Heap Sort
- 6 Final Remarks

## Definition

Binary max-heap is a binary tree (each node has zero, one, or two children) where the value of each node is at least the values of its children.

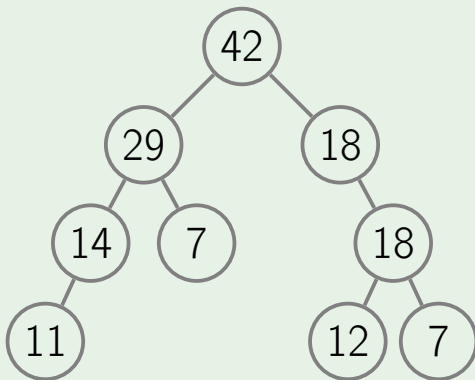
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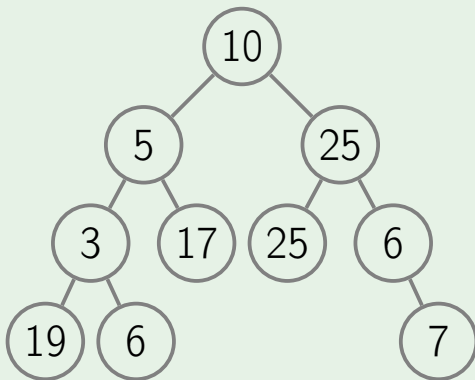
## In other words

For each edge of the tree, the value of the parent is at least the value of the child.

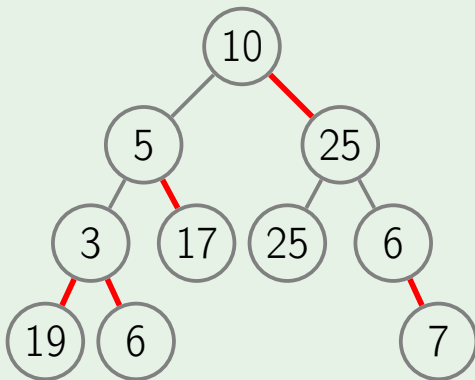
## Example: heap



Example: not a heap



Example: not a heap

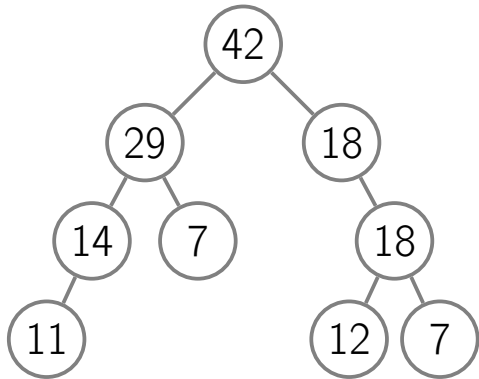


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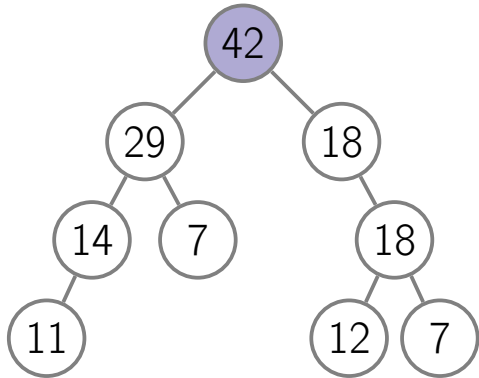


# GetMax



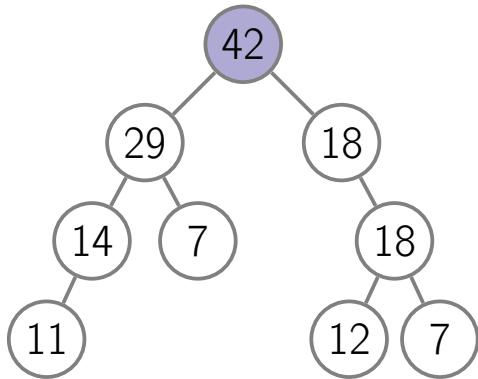
# GetMax

return the root  
value



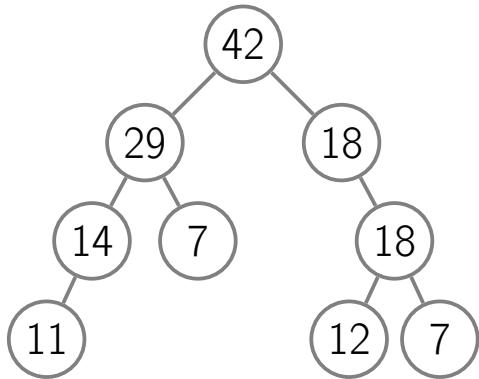
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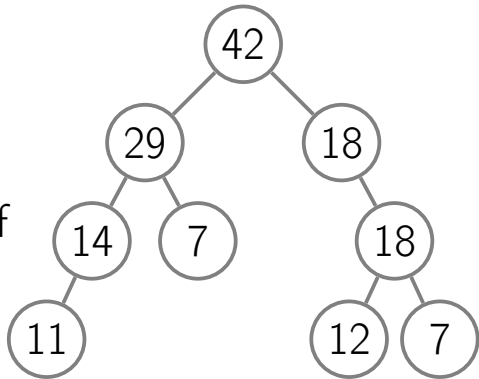
running time:  $O(1)$

# Insert



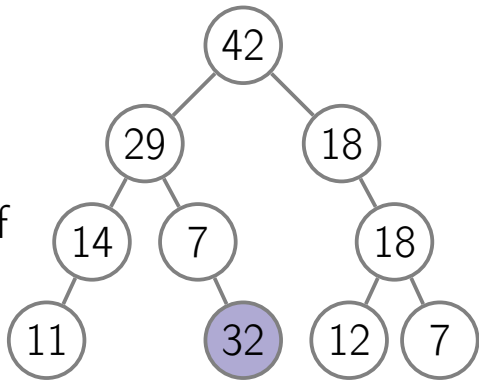
# Insert

attach a new  
node to any leaf



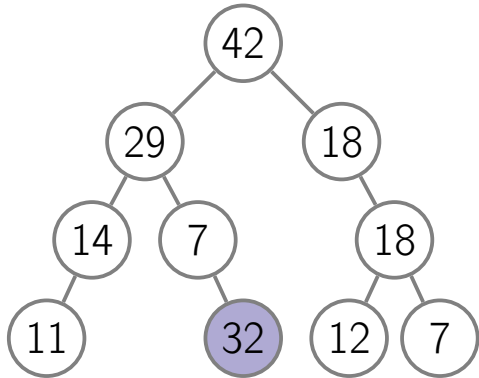
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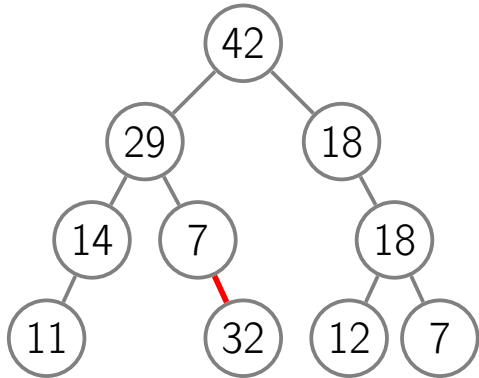
# Insert

this may violate  
the heap prop-  
erty



# Insert

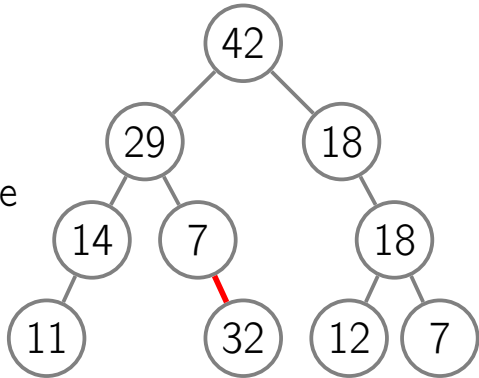
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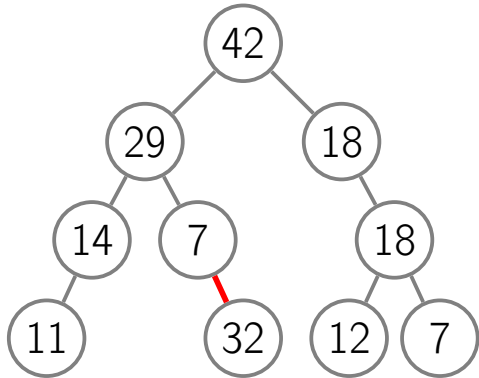
# Insert

to fix this, we  
let the new node  
sift up

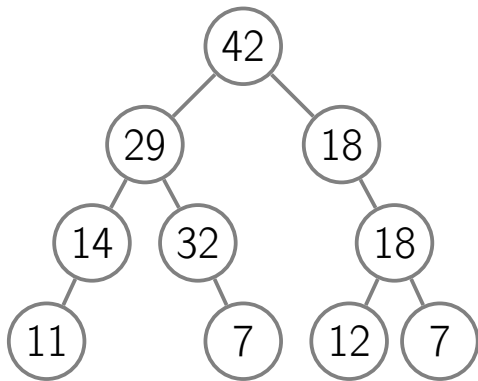


# SiftUp

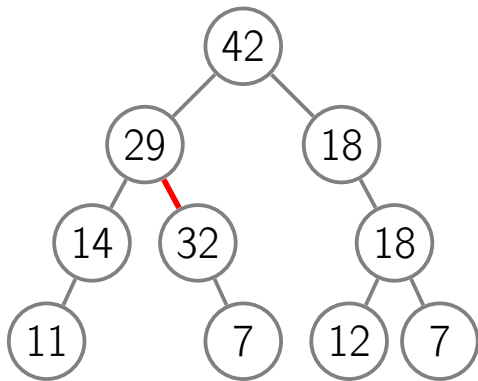
for this, we  
swap the prob-  
lematic node  
with its parent  
until the prop-  
erty is satisfied



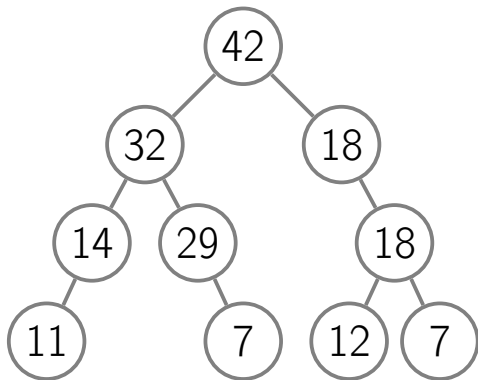
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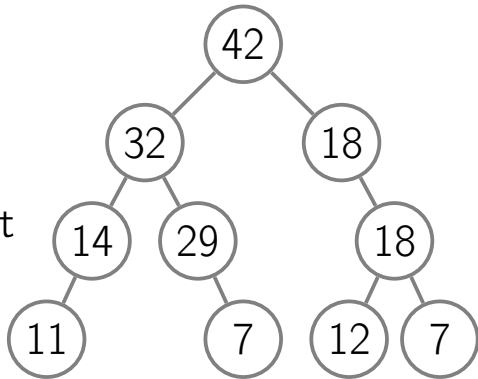


# SiftUp



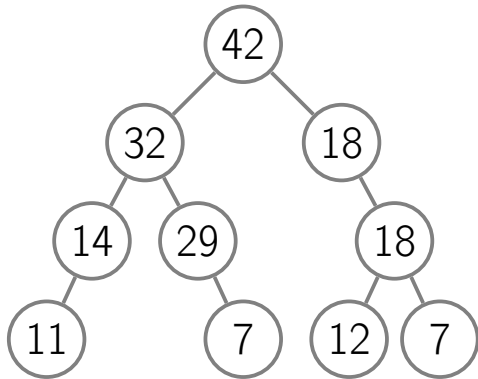
# SiftUp

**invariant:** heap  
property is vio-  
lated on at most  
one edge

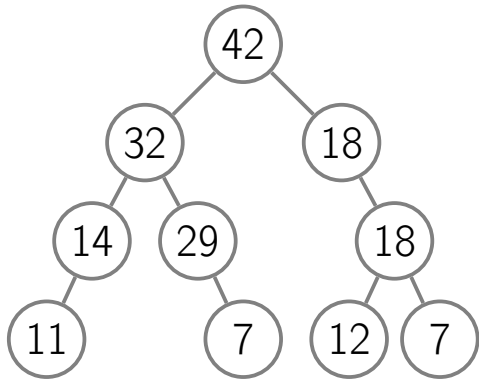


# SiftUp

this edge gets  
closer to the  
root while sift-  
ing up



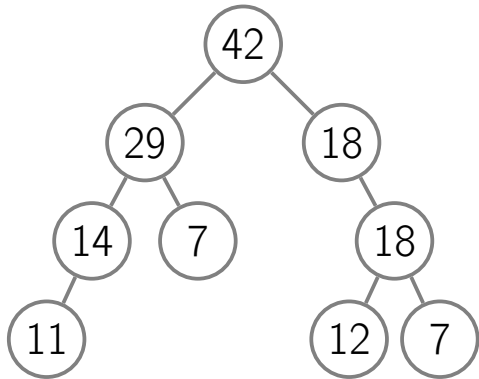
# SiftUp



running time:  $O(\text{tree height})$

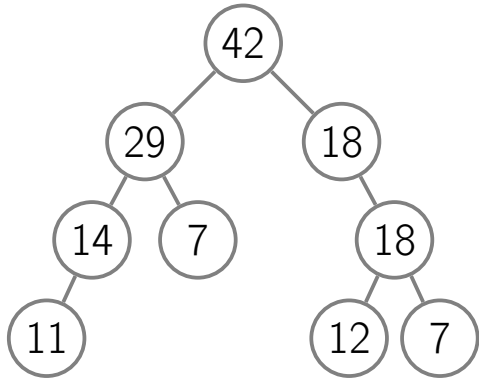


# ExtractMax



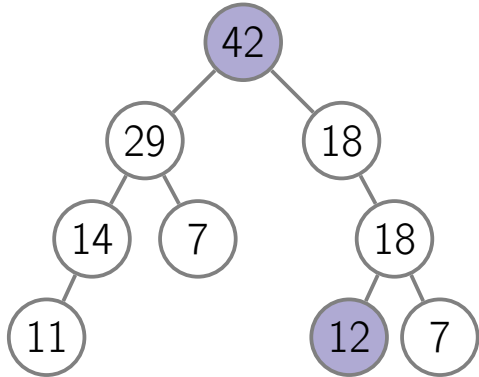
# ExtractMax

replace the root  
with any leaf



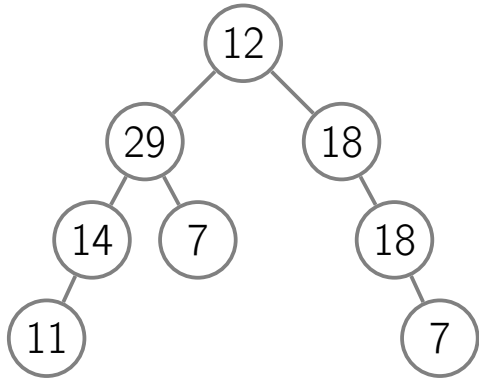
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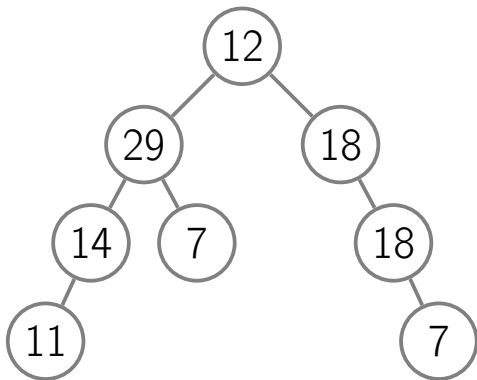
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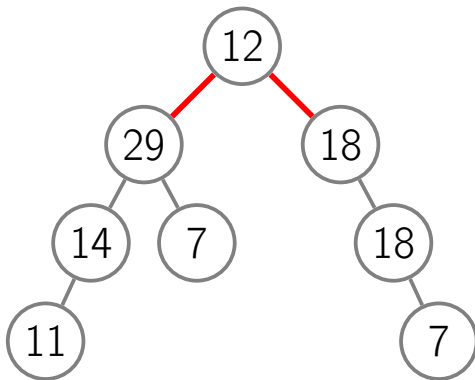
# ExtractMax

again, this may  
violate the heap  
property



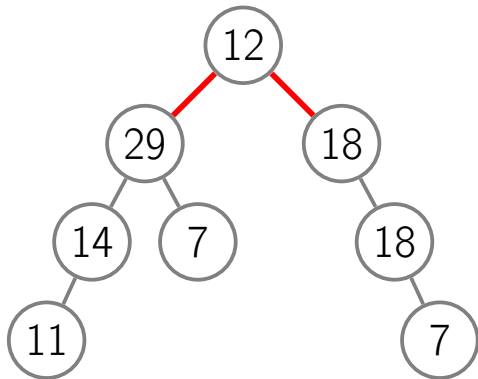
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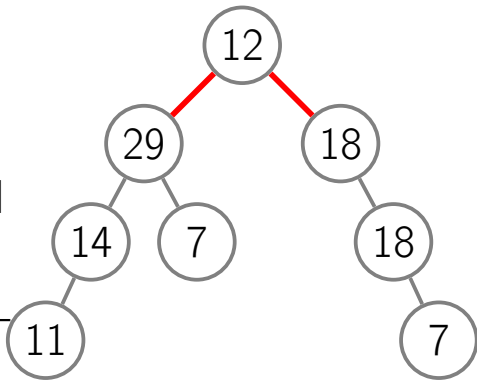
# ExtractMax

to fix it, we let  
the problematic  
node sift down



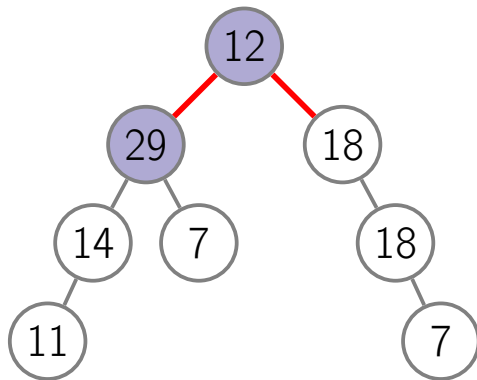
# SiftDown

for this, we  
swap the prob-  
lematic node  
with larger child  
until the heap  
property is satis-  
fied

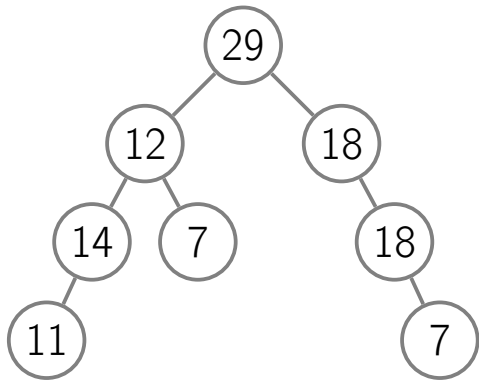




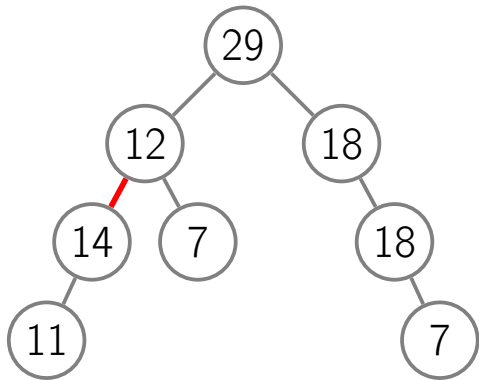
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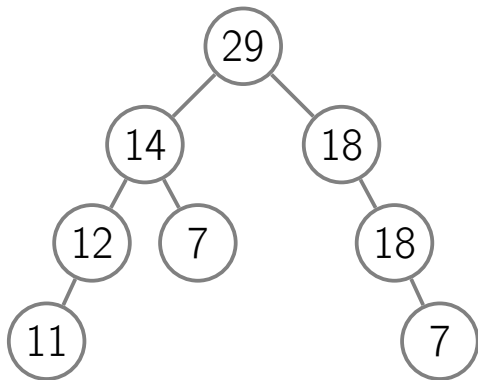
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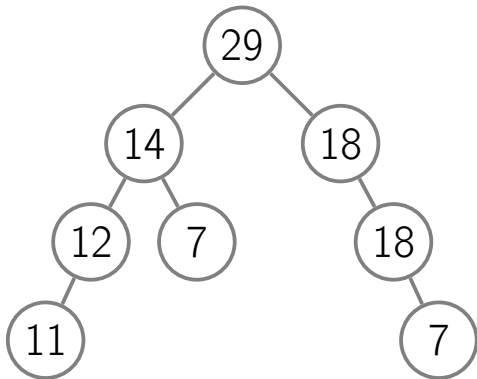


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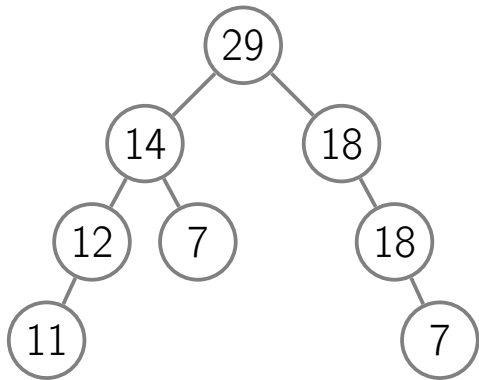


# SiftDown

we swap with  
the larger child  
which automatically fixes one  
of the two bad  
edges

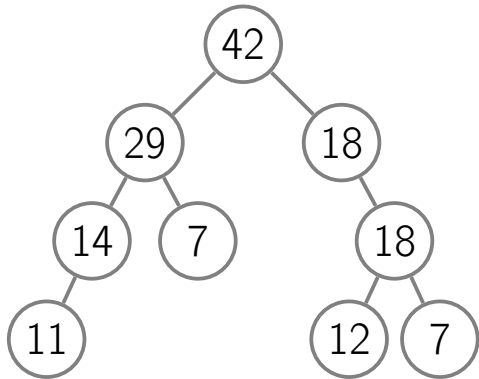


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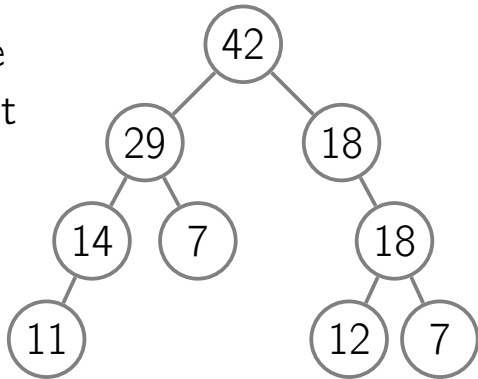
running time:  $O(\text{tree height})$

# ChangePriority



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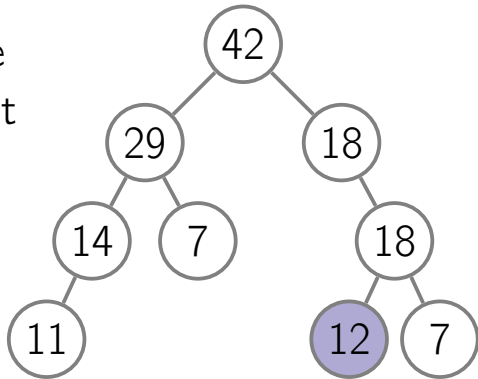
change the priority and let the changed element sift up or down depending on whether its priority decreased or increased





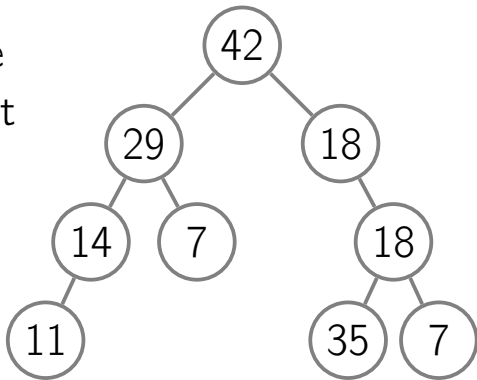
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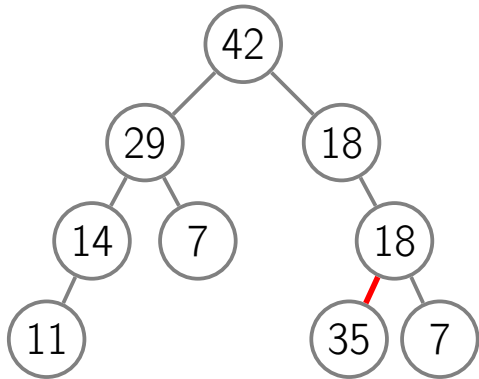


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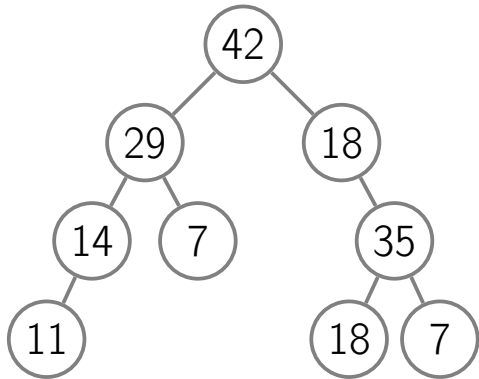
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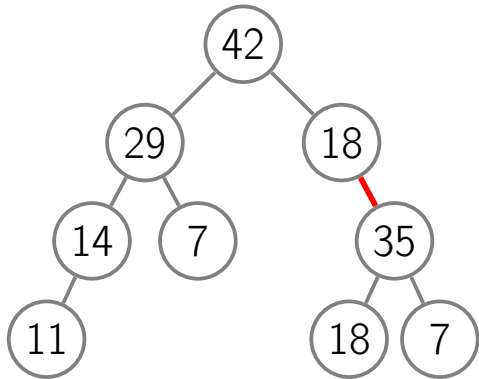
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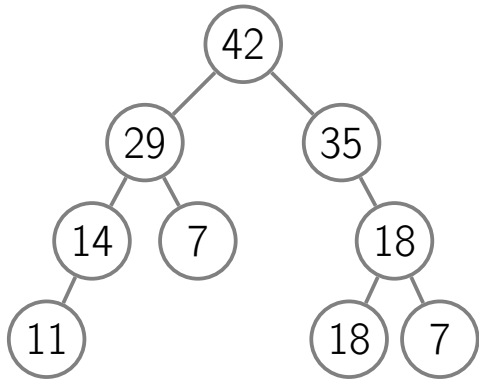
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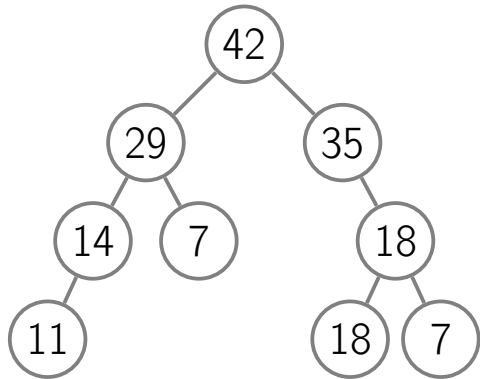
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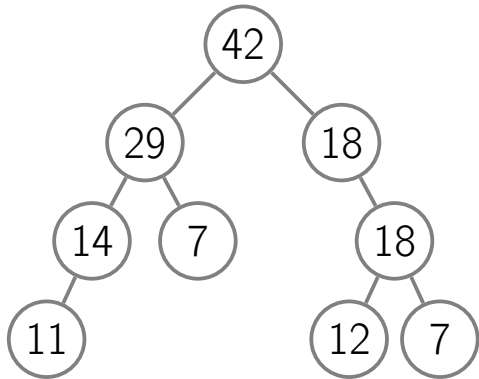


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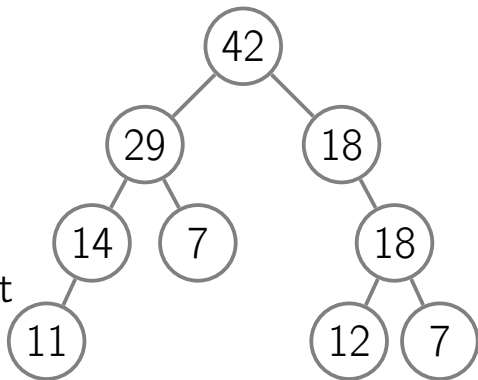
# Remove



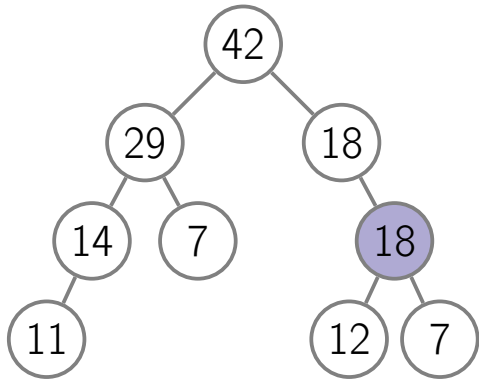


# Remove

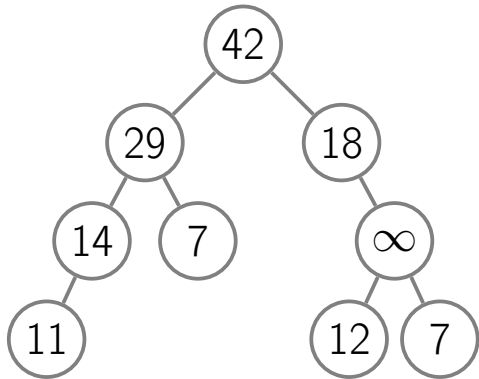
change the priority of the element to  $\infty$ ,  
let it sift up,  
and then extract maximum



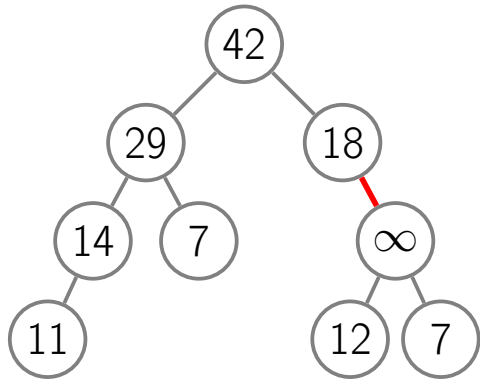
# Remove



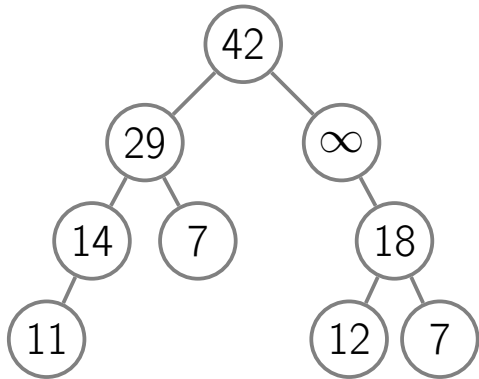
# Remove



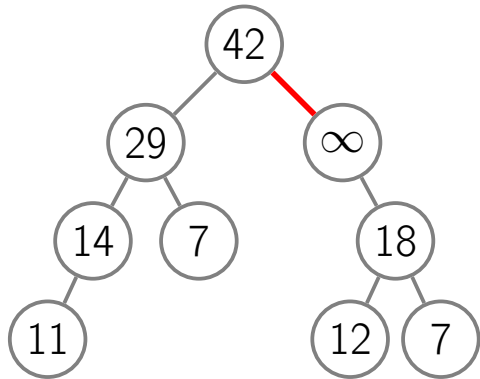
# Remove



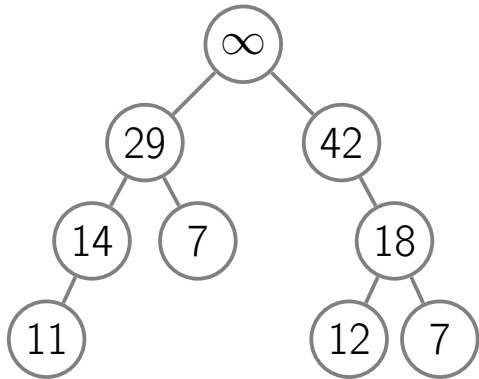
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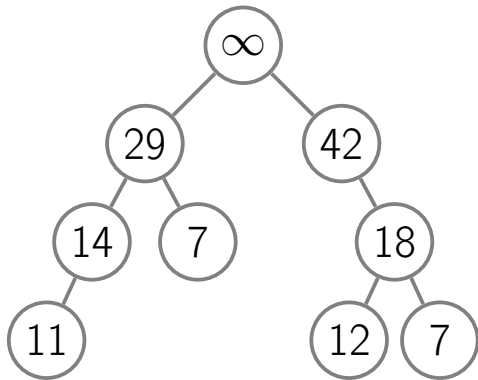


# Remove



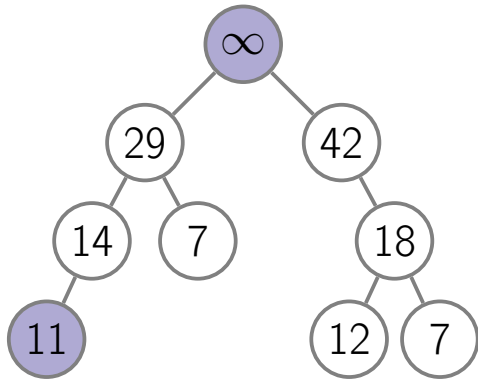
# Remove

now, call  
ExtractMax()

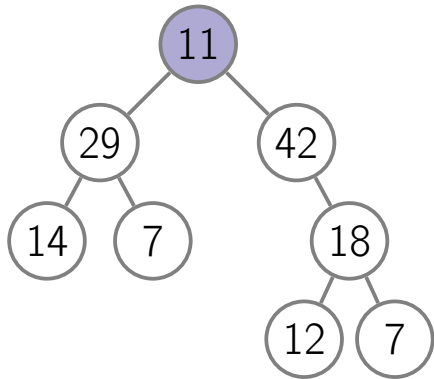




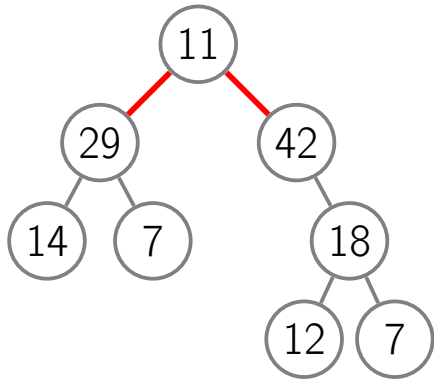
# Remove



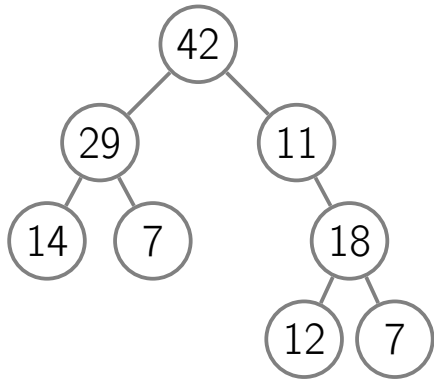
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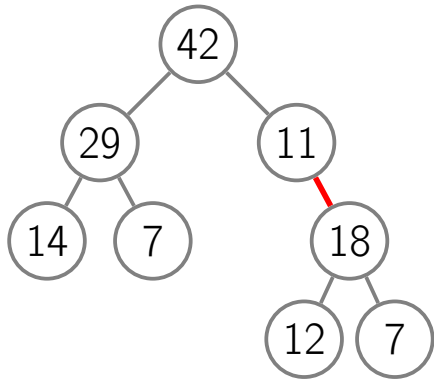
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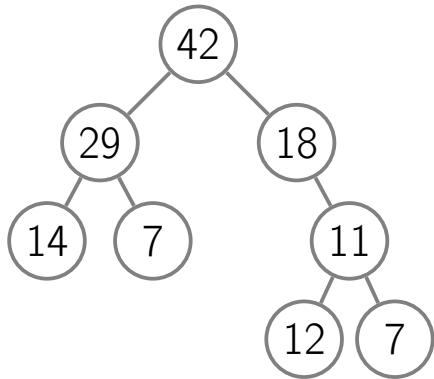
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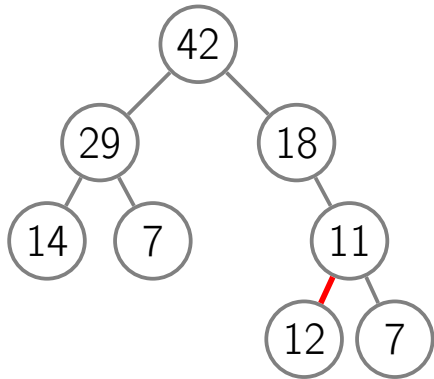
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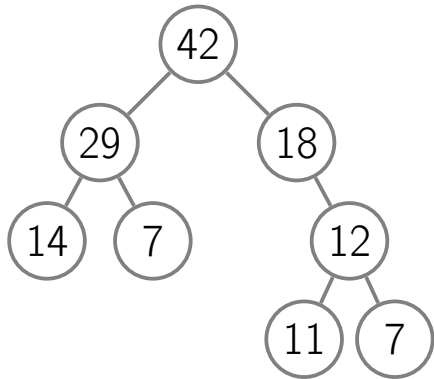
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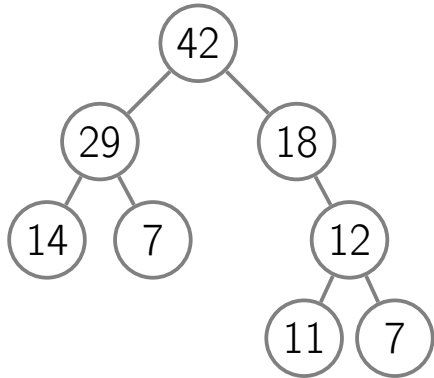


# Remove





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running time:  $O(\text{tree height})$

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- we definitely want a tree to be shallow

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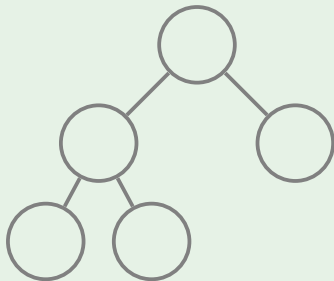
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# How to Keep a Tree Shallow?

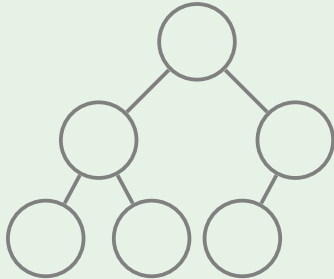
## Definition

A binary tree is **complete** if all its levels are filled except possibly the last one which is filled from left to right.

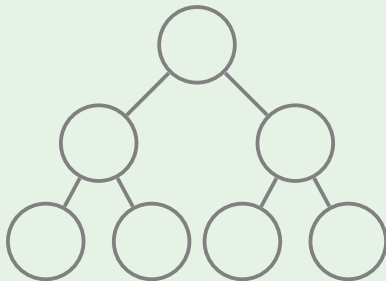
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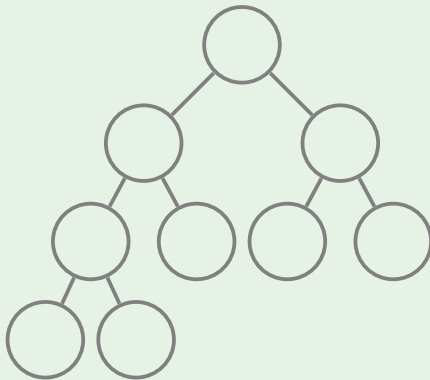


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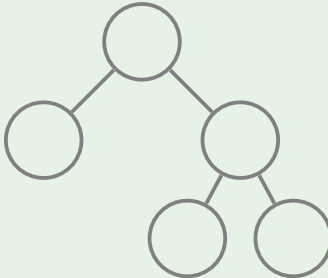




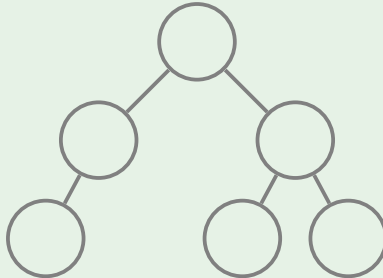
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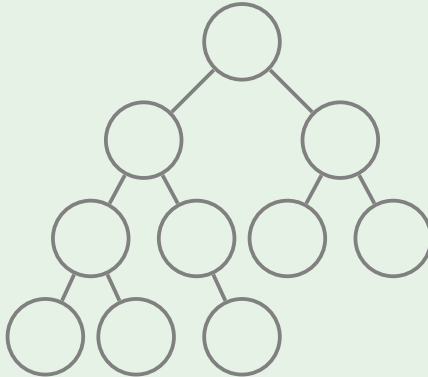
Example: **not** complete binary tree



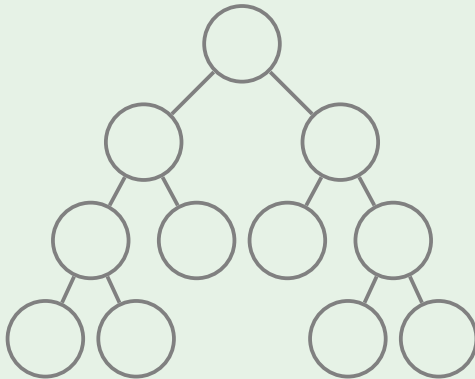
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Example: **not** complete binary tree



# First Advantage: Low Height

## Lemma

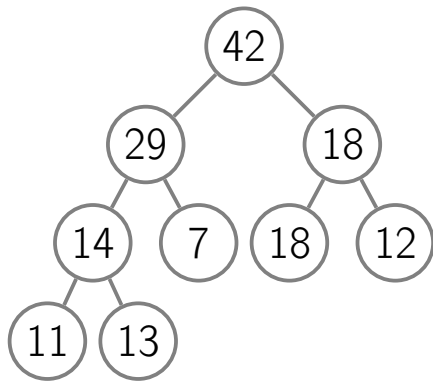
A complete binary tree with  $n$  nodes has height at most  $O(\log n)$ .

## Proof

- Complete the last level to get a full binary tree on  $n' \geq n$  nodes and the same number of levels  $\ell$ .
- Note that  $n' \leq 2n$ .
- Then  $n' = 2^\ell - 1$  and hence
$$\ell = \log_2(n' + 1) \leq \log_2(2n + 1) = O(\log n).$$

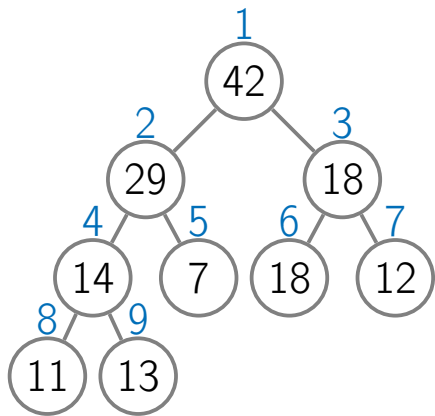


## Second Advantage: Store as Array

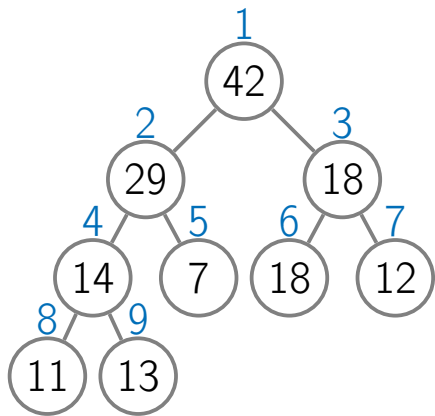




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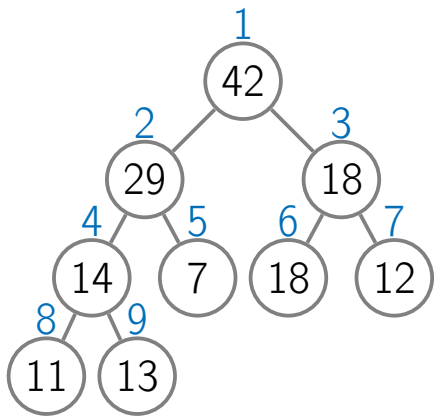


$$\text{parent}(i) = \lfloor \frac{i}{2} \rfloor$$

$$\text{leftchild}(i) = 2i$$

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$$\text{leftchild}(i) = 2i$$

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1	2	3	4	5	6	7	8	9
42	29	18	14	7	18	12	11	13

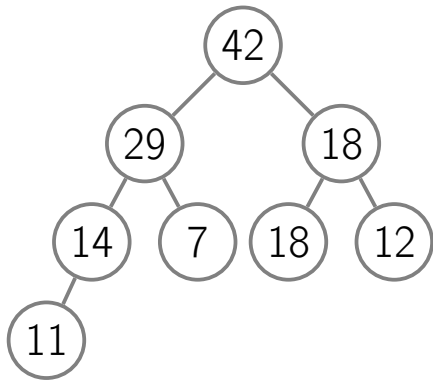
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- We need to keep the tree complete.
- Which binary heap operations modify the shape of the tree?
- Only Insert and ExtractMax (Remove changes the shape by calling ExtractMax).

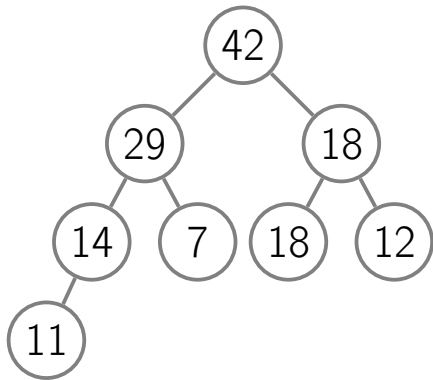
# Keeping the Tree Complete





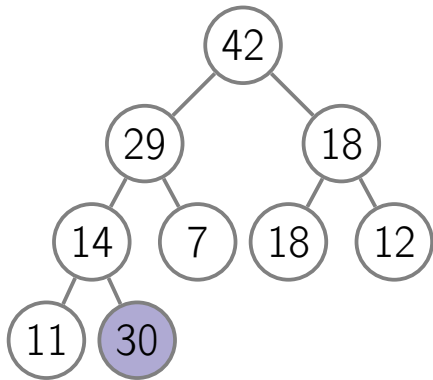
# Keeping the Tree Complete

to insert an element, insert it as a leaf in the **leftmost vacant position in the last level** and let it sift up



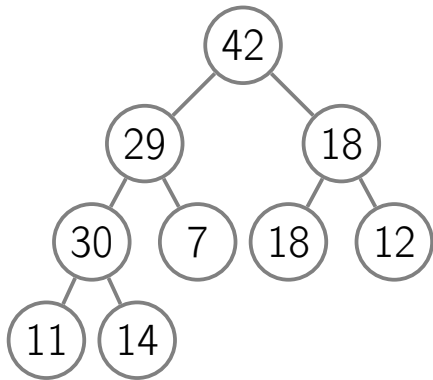
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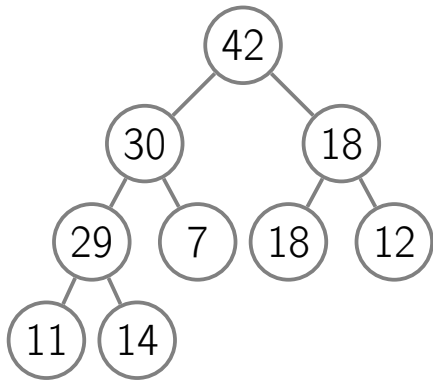
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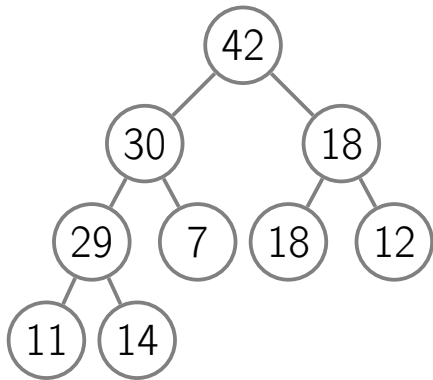
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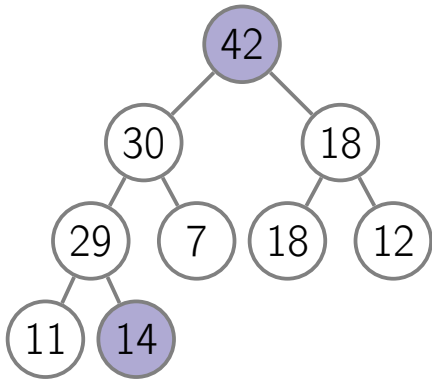
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replace the root  
by **the last leaf**  
and let it sift  
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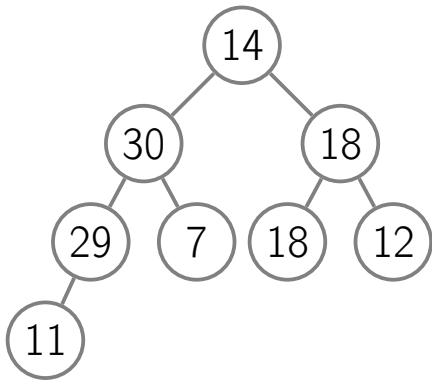
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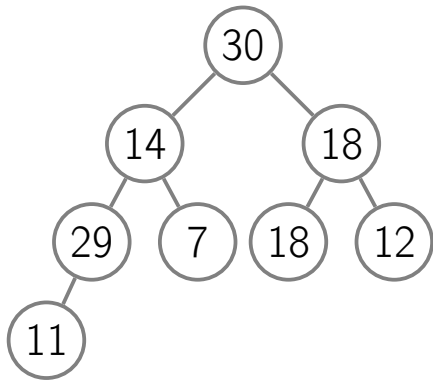
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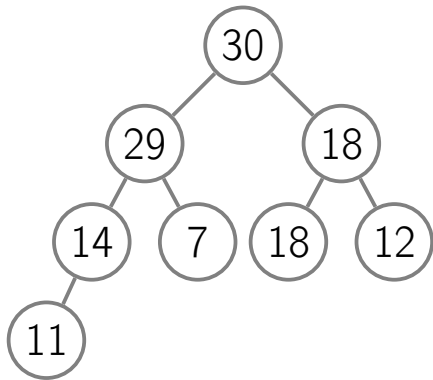
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# Outline

- 1 Binary Trees
- 2 Basic Operations
- 3 Complete Binary Trees
- 4 Pseudocode
- 5 Heap Sort
- 6 Final Remarks

# General Setting

- *maxSize* is the maximum number of elements in the heap

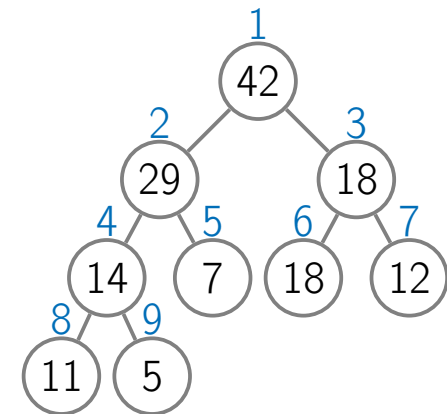
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- $H[1 \dots \textit{maxSize}]$  is an array of length *maxSize* where the heap occupies the first *size* elements

# Example



*size* = 9

*maxSize* = 13

	1	2	3	4	5	6	7	8	9	10	11	12	13
<i>H</i>	42	29	18	14	7	18	12	11	5	30	29	2	8

Parent( $i$ )

return  $\lfloor \frac{i}{2} \rfloor$

LeftChild( $i$ )

return  $2i$

RightChild( $i$ )

return  $2i + 1$

## SiftUp( $i$ )

```
while  $i > 1$  and  $H[\text{Parent}(i)] < H[i]$ :  
    swap  $H[\text{Parent}(i)]$  and  $H[i]$   
     $i \leftarrow \text{Parent}(i)$ 
```



## SiftDown(*i*)

$maxIndex \leftarrow i$

$\ell \leftarrow \text{LeftChild}(i)$

if  $\ell \leq size$  and  $H[\ell] > H[maxIndex]$ :

$maxIndex \leftarrow \ell$

$r \leftarrow \text{RightChild}(i)$

if  $r \leq size$  and  $H[r] > H[maxIndex]$ :

$maxIndex \leftarrow r$

if  $i \neq maxIndex$ :

swap  $H[i]$  and  $H[maxIndex]$

SiftDown( $maxIndex$ )

## Insert( $p$ )

```
if  $size = maxSize$ :  
    return ERROR  
 $size \leftarrow size + 1$   
 $H[size] \leftarrow p$   
SiftUp( $size$ )
```

## ExtractMax()

```
result  $\leftarrow H[1]$   
H[1]  $\leftarrow H[size]$   
size  $\leftarrow size - 1$   
SiftDown(1)  
return result
```

Remove( $i$ )

$H[i] \leftarrow \infty$

SiftUp( $i$ )

ExtractMax()

## ChangePriority( $i, p$ )

$oldp \leftarrow H[i]$

$H[i] \leftarrow p$

if  $p > oldp$ :

    SiftUp( $i$ )

else:

    SiftDown( $i$ )

# Summary

The resulting implementation is

- **fast**: all operations work in time  $O(\log n)$  (GetMax even works in  $O(1)$ )

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# Summary

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- **fast**: all operations work in time  $O(\log n)$  (GetMax even works in  $O(1)$ )
- **space efficient**: we store an array of priorities; parent-child connections are not stored, but are computed on the fly
- **easy to implement**: all operations are implemented in just a few lines of code



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# Sort Using Priority Queues

HeapSort( $A[1 \dots n]$ )

create an empty priority queue

for  $i$  from 1 to  $n$ :

    Insert( $A[i]$ )

for  $i$  from  $n$  downto 1:

$A[i] \leftarrow \text{ExtractMax}()$

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- Natural generalization of selection sort: instead of simply scanning the rest of the array to find the maximum value, use a smart data structure.
- Not in-place: uses additional space to store the priority queue.

# This lesson

**In-place heap sort algorithm.** For this, we will first turn a given array into a heap by permuting its elements.

# Turn Array into a Heap

BuildHeap( $A[1 \dots n]$ )

$size \leftarrow n$

for  $i$  from  $\lfloor n/2 \rfloor$  downto 1:  
    SiftDown( $i$ )

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- [Online visualization](#)
- Running time:  $O(n \log n)$

# In-place Heap Sort

HeapSort( $A[1 \dots n]$ )

BuildHeap( $A$ )

$\{size = n\}$

repeat  $(n - 1)$  times:

    swap  $A[1]$  and  $A[size]$

$size \leftarrow size - 1$

    SiftDown(1)

# Building Running Time

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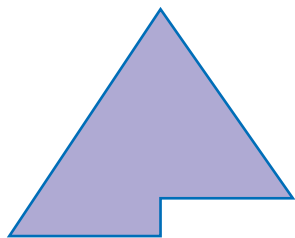
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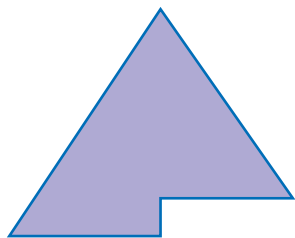
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- If a node is already close to the leaves, then sifting it down is fast.
- We have many such nodes!
- Was our estimate of the running time of BuildHeap too pessimistic?

# Building Running Time



# nodes	$T(\text{SiftDown})$
1	$\log_2 n$
2	
$\vdots$	$\vdots$
$\leq n/4$	2
$\leq n/2$	1

# Building Running Time



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$$\begin{aligned} T(\text{BuildHeap}) &\leq \frac{n}{2} \cdot 1 + \frac{n}{4} \cdot 2 + \frac{n}{8} \cdot 3 + \dots \\ &\leq n \cdot \sum_{i=1}^{\infty} \frac{i}{2^i} = 2n \end{aligned}$$

# Estimating the Sum

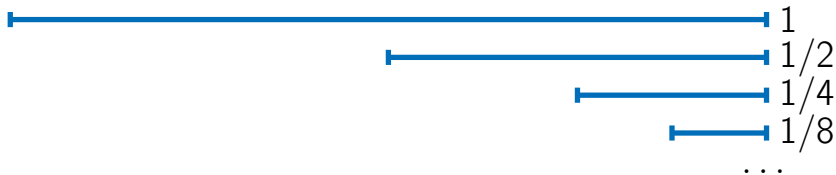


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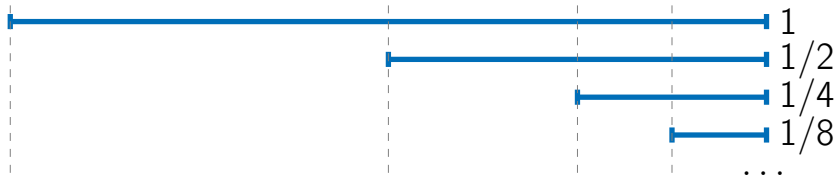
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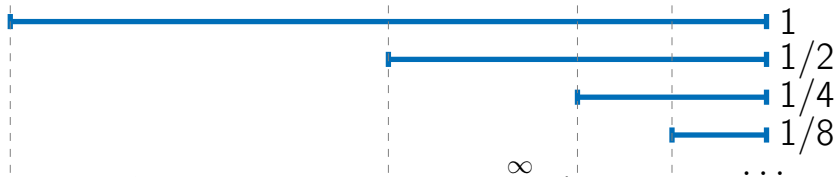
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## Partial sorting

**Input:** An array  $A[1 \dots n]$ , an integer  $1 \leq k \leq n$ .

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Can be solved in  $O(n)$  if  $k = O(\frac{n}{\log n})!$

PartialSorting( $A[1 \dots n], k$ )

BuildHeap( $A$ )

for  $i$  from 1 to  $k$ :

    ExtractMax()

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Running time:  $O(n + k \log n)$

# Summary

Heap sort is a time and space efficient comparison-based algorithm: has running time  $O(n \log n)$ , uses no additional space.

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# 0-based Arrays

Parent( $i$ )

return  $\lfloor \frac{i-1}{2} \rfloor$

LeftChild( $i$ )

return  $2i + 1$

RightChild( $i$ )

return  $2i + 2$

# Binary Min-Heap

## Definition

Binary **min**-heap is a binary tree (each node has zero, one, or two children) where the value of each node is **at most** the values of its children.

Can be implemented similarly.



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- The running time of SiftDown is  $O(d \log_d n)$ : on each level, we find the largest value among  $d$  children.

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- In an array/list implementation one operation is very fast ( $O(1)$ ) but the other one is very slow ( $O(n)$ ).
- Binary heap gives an implementation where both operations take  $O(\log n)$  time.
- Can be made also space efficient.