Assignment 1

1. (Math)

Groups have the following four properties: the closure, the associativity, the existence of an identity element, and the existence of an inverse element. In order to prove that Euclidean transformations can form groups, the above four properties must be proved. The proof of each property is as follows:

$$M_i = \begin{bmatrix} R_i & \mathbf{t}_i \\ \mathbf{0}^T & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

where $R_i \in \mathbb{R}^{3\times 3}$ is an orthonormal matrix, $\det(R_i) = 1$, and $\mathbf{t}_i \in \mathbb{R}^{3\times 1}$ is a vector. In addition, the group multiplication of this group is matrix multiplication in the usual sense.

① The closure

Choose any two matrices M_a and M_b from $\{M_i\}$, we have

$$M_a \times M_b = \begin{bmatrix} R_a & \mathbf{t}_a \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} R_b & \mathbf{t}_b \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} R_a R_b & R_a \mathbf{t}_b + \mathbf{t}_a \\ \mathbf{0}^T & 1 \end{bmatrix}$$

Then, just prove: $R_a R_b \in \mathbb{R}^{3\times 3}$ is an orthonormal matrix, $\det(R_a R_b) = 1$, and $\mathbf{t}_i \in \mathbb{R}^{3\times 1}$ is a vector.

$$(R_a R_b)^T (R_a R_b) = R_b^T R_a^T R_a R_b$$

Due to satisfy: R_a, $R_b \in \mathbb{R}^{3\times 3}$ are orthonormal matrices, thus:

$$R_a^T R_a = I, R_b^T R_b = I$$

$$(R_a R_b)^T (R_a R_b) = R_b^T R_a^T R_a R_b = I$$

This proves that $R_a R_b \in \mathbb{R}^{3\times 3}$ is an orthogonal matrix.

$$\det(R_a R_b) = \det(R_a)\det(R_b) = 1$$

Obviously, $R_a \mathbf{t}_b + \mathbf{t}_a \in \mathbb{R}^{3\times 1}$ is a vector. Thus:

$$M_a \times M_b \in \{M_i\}$$

Satisfied the closure

② The associativity

Choose any three matrices M_a, M_b and M_c from $\{M_i\}$, we have

$$(M_a \times M_b) \times M_c = \left(\begin{bmatrix} R_a & \mathbf{t}_a \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} R_b & \mathbf{t}_b \\ \mathbf{0}^T & 1 \end{bmatrix} \right) \begin{bmatrix} R_c & \mathbf{t}_c \\ \mathbf{0}^T & 1 \end{bmatrix}$$

$$= \begin{bmatrix} R_a R_b & R_a \mathbf{t}_b + \mathbf{t}_a \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} R_c & \mathbf{t}_c \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} R_a R_b R_c & R_a R_b \mathbf{t}_c + R_a \mathbf{t}_b + \mathbf{t}_a \\ \mathbf{0}^T & 1 \end{bmatrix}$$

$$M_a \times (M_b \times M_c) = \begin{bmatrix} R_a & \mathbf{t}_a \\ \mathbf{0}^T & 1 \end{bmatrix} \left(\begin{bmatrix} R_b & \mathbf{t}_b \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} R_c & \mathbf{t}_c \\ \mathbf{0}^T & 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} R_a & \mathbf{t}_a \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} R_b R_c & R_b \mathbf{t}_c + \mathbf{t}_b \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} R_a R_b R_c & R_a R_b \mathbf{t}_c + R_a \mathbf{t}_b + \mathbf{t}_a \\ \mathbf{0}^T & 1 \end{bmatrix}$$

Thus, $(M_a \times M_b) \times M_c = M_a \times (M_b \times M_c)$.

Satisfied the associativity

③ The existence of an identity element

Let $E = \begin{bmatrix} I_{3\times3} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}$, where $I_{3\times3}$ is a third-order identity matrix. Choose any one matrix M_a from $\{M_i\}$, we have

$$M_a \times E = \begin{bmatrix} R_a & \mathbf{t}_a \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} I_{3\times3} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} R_a & \mathbf{t}_a \\ \mathbf{0}^T & 1 \end{bmatrix} = M_a$$

$$E \times M_a = \begin{bmatrix} I_{3\times3} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} R_a & \mathbf{t}_a \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} R_a & \mathbf{t}_a \\ \mathbf{0}^T & 1 \end{bmatrix} = M_a$$

Thus, $\exists E = \begin{bmatrix} I_{3\times3} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}$, $\forall M_a \in \{M_i\}$, $E \times M_a = M_a \times E = M_a$.

Satisfied the existence of an identity element

④ The existence of an inverse element

Choose any one matrix M_a from $\{M_i\}$, we have

$$\begin{bmatrix} R_a & \mathbf{t}_a \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} R_a^{-1} & -R_a^{-1}\mathbf{t}_a \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} I_{3\times3} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} = E$$

$$\begin{bmatrix} R_a^{-1} & -R_a^{-1}\mathbf{t}_a \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} R_a & \mathbf{t}_a \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} I_{3\times3} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} = E$$

Then, just prove: $\begin{bmatrix} R_a^{-1} & -R_a^{-1}\mathbf{t}_a \\ \mathbf{0}^T & 1 \end{bmatrix} \in \{M_i\}$.

$$(R_a^{-1})^T R_a^{-1} = (R_a R_a^T)^{-1} = I_{3\times3}$$

According to the properties of the inverse matrix, we have

$$\det(R_a^{-1}) = \frac{1}{\det(R_a)} = 1$$

$$-R_a^{-1}\mathbf{t}_a \in \mathbb{R}^{3\times1}$$

Thus, $\begin{bmatrix} R_a^{-1} & -R_a^{-1}\mathbf{t}_a \\ \mathbf{0}^T & 1 \end{bmatrix} \in \{M_i\}$. Define $\begin{bmatrix} R_a^{-1} & -R_a^{-1}\mathbf{t}_a \\ \mathbf{0}^T & 1 \end{bmatrix}$ as M_a^{-1}.

Thus, $\forall M_a \in \{M_i\}$, $\exists M_a^{-1} \in \{M_i\}$, $M_a \times M_a^{-1} = M_a^{-1} \times M_a = E$.

Satisfied the existence of an inverse element

2. (Math)

$$M = \begin{bmatrix} \sum_{(x_i, y_i) \in w} (I_x)^2 & \sum_{(x_i, y_i) \in w} (I_x I_y) \\ \sum_{(x_i, y_i) \in w} (I_x I_y) & \sum_{(x_i, y_i) \in w} (I_y)^2 \end{bmatrix}$$

In fact, the above formula is equivalent to:

$$M = \begin{bmatrix} \sum_{(x_i, y_i) \in w} \left(\frac{\partial f(x_i, y_i)}{\partial x}\right)^2 & \sum_{(x_i, y_i) \in w} \left(\frac{\partial f(x_i, y_i)}{\partial x} \cdot \frac{\partial f(x_i, y_i)}{\partial y}\right) \\ \sum_{(x_i, y_i) \in w} \left(\frac{\partial f(x_i, y_i)}{\partial x} \cdot \frac{\partial f(x_i, y_i)}{\partial y}\right) & \sum_{(x_i, y_i) \in w} \left(\frac{\partial f(x_i, y_i)}{\partial y}\right)^2 \end{bmatrix}$$

a)

$$\begin{bmatrix} \sum_{(x_i, y_i) \in w} \left(\frac{\partial f(x_i, y_i)}{\partial x}\right)^2 & \sum_{(x_i, y_i) \in w} \left(\frac{\partial f(x_i, y_i)}{\partial x} \cdot \frac{\partial f(x_i, y_i)}{\partial y}\right) \\ \sum_{(x_i, y_i) \in w} \left(\frac{\partial f(x_i, y_i)}{\partial x} \cdot \frac{\partial f(x_i, y_i)}{\partial y}\right) & \sum_{(x_i, y_i) \in w} \left(\frac{\partial f(x_i, y_i)}{\partial y}\right)^2 \end{bmatrix}$$

$$= \begin{pmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \\ \frac{\partial f}{\partial y_1} & \frac{\partial f}{\partial y_2} & \cdots & \frac{\partial f}{\partial y_n} \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial y_1} \\ \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial y_2} \\ \vdots & \vdots \\ \frac{\partial f}{\partial x_n} & \frac{\partial f}{\partial y_n} \end{pmatrix} = (I_x \quad I_y)^T (I_x \quad I_y)$$

It can be seen that the matrix M can be expressed as $A^T A$, where $A = (I_x \quad I_y)$.

Take any vector $\boldsymbol{x}$, $\boldsymbol{x}^T M \boldsymbol{x} = \boldsymbol{x}^T A^T A \boldsymbol{x} = (A\boldsymbol{x})^T (A\boldsymbol{x}) = \|A\boldsymbol{x}\|_2^2 \geq 0$.

So, it can be proved that M is positive semi-definite.

b)

$$[x \quad y] M \begin{bmatrix} x \\ y \end{bmatrix} = [x \quad y][I_x \quad I_y]^T [I_x \quad I_y] \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= x^2 I_x^T I_x + xy I_y^T I_x + xy I_x^T I_y + y^2 I_y^T I_y$$

$$= x^2 \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}\right)^2 + 2xy \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \cdot \frac{\partial f}{\partial y_i}\right) + y^2 \sum_{i=1}^{n} \left(\frac{\partial f}{\partial y_i}\right)^2$$

$$= 1$$

According to the general equation of a conic section:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

If the curve represents an ellipse, it must satisfy:

$$\Delta = B^2 - 4AC < 0$$

According to the Cauchy-Schwarz inequality, we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

And because M is positive definite, so $[x \quad y]M\begin{bmatrix} x \\ y \end{bmatrix} - 1 = 0$ satisfies:

$$\Delta = \left(2\sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\cdot\frac{\partial f}{\partial y_i}\right)\right)^2 - 4\sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\right)^2\sum_{i=1}^{n}\left(\frac{\partial f}{\partial y_i}\right)^2 < 0$$

So $[\boldsymbol{x} \quad \boldsymbol{y}]\boldsymbol{M}\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \mathbf{1}$ represents an ellipse.

c)

Since M is a symmetric matrix, there must exist an orthogonal matrix P that satisfies:

$$M = P^T\Lambda P = P^{-1}\Lambda P$$

where $P^T P = P^{-1}P = I$, $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$.

$$[\boldsymbol{x} \quad \boldsymbol{y}]M\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = [\boldsymbol{x} \quad \boldsymbol{y}]P^T\Lambda P\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \left[P\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix}\right]^T \Lambda P\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix}$$

Since P is an orthogonal matrix, so $P\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix}$ is equivalent to doing the same rotation transformation for all points on the ellipse.

Since the rotation transformation does not change the shape of the ellipse, so

$$[\boldsymbol{x} \quad \boldsymbol{y}]\boldsymbol{M}\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \left[P\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix}\right]^T \Lambda P\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = 1$$

has the same semi-major axis and semi-minor axis with

$$[\boldsymbol{x} \quad \boldsymbol{y}]\Lambda\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = [\boldsymbol{x} \quad \boldsymbol{y}]\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = 1$$

The above equation simplifies to:

$$\lambda_1 x^2 + \lambda_2 y^2 = \frac{x^2}{\left(\frac{1}{\sqrt{\lambda_1}}\right)^2} + \frac{y^2}{\left(\frac{1}{\sqrt{\lambda_2}}\right)^2} = 1$$

So the length of its semi-major axis is $\frac{1}{\sqrt{\lambda_2}}$ while the length of its semi-minor axis is $\frac{1}{\sqrt{\lambda_1}}$.

3. (Math)

Consider the formula: $\boldsymbol{x}^T(A^TA)\boldsymbol{x} = (A\boldsymbol{x})^T(A\boldsymbol{x}) = \|A\boldsymbol{x}\|_2^2$.

Since $A \in \mathbb{R}^{m\times n}$, $m > n$, $rank(A) = n$, so $A\boldsymbol{x} = \boldsymbol{0}$ only if $\boldsymbol{x} = \boldsymbol{0}$

So A^TA is positive definite.

Since A^TA is a positive definite matrix. According to the relevant conclusions of positive definite matrices, the eigenvalues of matrix A^TA are all positive numbers. So $det(A^TA) = \lambda_1\lambda_2 \ldots \lambda_n \neq 0$.

So, it can be proved that A^TA is non-singular (or in other words, it is invertible).

4. (Programming)

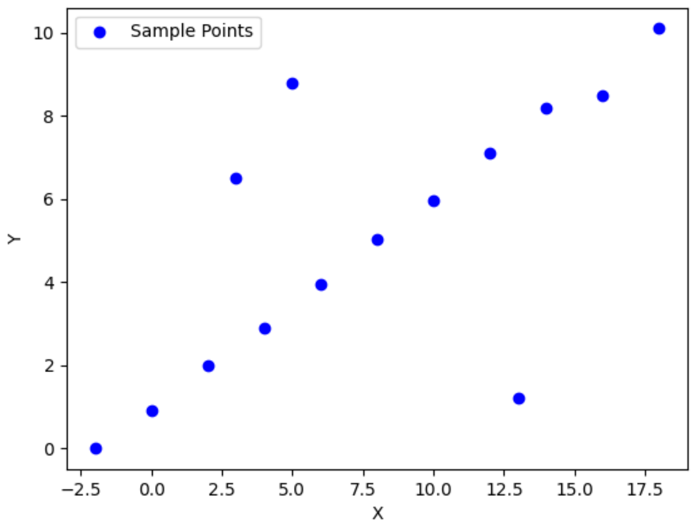
Platform: PyCharm Community Edition 2023.2

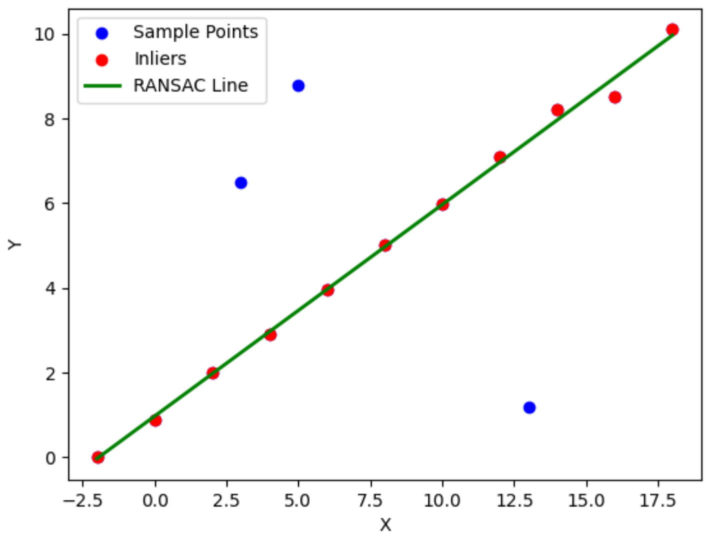
Python version: 3.11

Python libraries: numpy matplotlib

Code location: ./Code/Programming4/RANSAC.py

Results are as follows:





5. (Programming)

Platform: PyCharm Community Edition 2023.2

Python version: 3.11

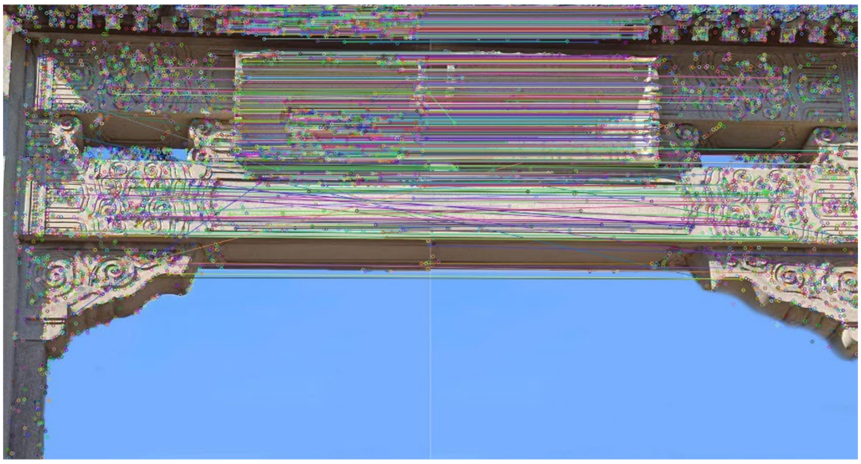
Python libraries: numpy opencv-python

Code location: ./Code/Programming5

Results are as follows:

Original Picture

Key-points Detection



Match Result

Stitch Result 1

Stitch Result 2

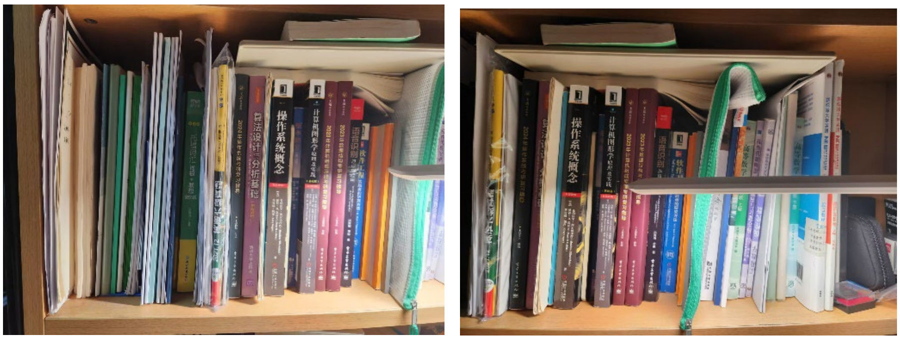
6. (Programming)

Platform: Visual Studio 2022

Libraries: opencv 4.8.0

Code location: ./Code/Programming6

Results are as follows:



Original Picture

Match Result