Homework 2

Tongji University 2022 Class Computer Science and Technology College Software Engineering Major Machine Intelligence Direction Computer Vision Course Assignment

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Semester: 2024-2025 Fall Semester

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Homogeneous Coordinates of the Point at Infinity

In the extended Euclidean plane (which can be regarded as a projective plane), points are represented by homogeneous coordinates (X, Y, Z), and lines are described by homogeneous equations of the form $aX + bY + cZ = 0$.

Consider the affine line $x - 3y + 4 = 0$. By introducing the homogeneous coordinate Z, it can be converted into its homogeneous form:

$$X - 3Y + 4Z = 0$$

Point at Infinity

Points at infinity lie on the line at infinity defined by $Z = 0$. To find the point at infinity corresponding to the given line, substitute $Z = 0$ into the homogeneous equation:

$$X - 3Y = 0$$

This simplifies to:

$$X = 3Y$$

Therefore, the point at infinity must satisfy $X = 3Y$ and $Z = 0$. Introducing a nonzero constant k, the homogeneous coordinates can be represented as:

$$(3k, k, 0)$$

where $k \neq 0$. Since homogeneous coordinates are equivalent up to a nonzero scalar multiple, any nonzero scalar k represents the same projective point at infinity.

Final Answer

The homogeneous coordinates of the point at infinity can be expressed as:

$$(3k, k, 0) \quad \text{with} \quad k \neq 0$$

Jacobian Matrix of Distortion Mapping

Below is the detailed derivation of the Jacobian matrix $\frac{\partial \mathbf{p}_d}{\partial \mathbf{p}_n^T}$ for the mapping from $\mathbf{p}_n = (x, y)^T$ to $\mathbf{p}_d = (x_d, y_d)^T$.

Given:

$$x_d = x(1 + k_1 r^2 + k_2 r^4 + k_3 r^6) + 2\rho_1 xy + \rho_2(r^2 + 2x^2)$$

$$y_d = y(1 + k_1 r^2 + k_2 r^4 + k_3 r^6) + 2\rho_2 xy + \rho_1(r^2 + 2y^2)$$

where $r^2 = x^2 + y^2$.

Define:

$$A = 1 + k_1 r^2 + k_2 r^4 + k_3 r^6$$

Thus:

$$x_d = xA + 2\rho_1 xy + \rho_2(r^2 + 2x^2)$$

$$y_d = yA + 2\rho_2 xy + \rho_1(r^2 + 2y^2)$$

We need to calculate:

$$\frac{\partial \mathbf{p}_d}{\partial \mathbf{p}_n^T} = \begin{bmatrix} \frac{\partial x_d}{\partial x} & \frac{\partial x_d}{\partial y} \\ \frac{\partial y_d}{\partial x} & \frac{\partial y_d}{\partial y} \end{bmatrix}$$

Calculate $\frac{\partial x_d}{\partial x}$

First, calculate $\frac{\partial A}{\partial x}$:

$$\frac{\partial A}{\partial x} = 2x(k_1 + 2k_2 r^2 + 3k_3 r^4)$$

Then:

$$\frac{\partial x_d}{\partial x} = A + x\frac{\partial A}{\partial x} + 2\rho_1 y + \frac{\partial}{\partial x}[\rho_2(r^2 + 2x^2)] = A + 2x^2(k_1 + 2k_2 r^2 + 3k_3 r^4) + 2\rho_1 y + 6\rho_2 x$$

Substituting A:

$$\frac{\partial x_d}{\partial x} = (1 + k_1 r^2 + k_2 r^4 + k_3 r^6) + 2x^2(k_1 + 2k_2 r^2 + 3k_3 r^4) + 2\rho_1 y + 6\rho_2 x$$

Calculate $\frac{\partial x_d}{\partial y}$

First, calculate $\frac{\partial A}{\partial y}$:

$$\frac{\partial A}{\partial y} = 2y(k_1 + 2k_2 r^2 + 3k_3 r^4)$$

Then:

$$\frac{\partial x_d}{\partial y} = x\frac{\partial A}{\partial y} + 2\rho_1 x + \frac{\partial}{\partial y}[\rho_2(r^2 + 2x^2)] = 2xy(k_1 + 2k_2r^2 + 3k_3r^4) + 2\rho_1 x + 2\rho_2 y$$

Calculate $\frac{\partial y_d}{\partial x}$

$$\frac{\partial y_d}{\partial x} = y\frac{\partial A}{\partial x} + 2\rho_2 y + \rho_1(2x) = 2xy(k_1 + 2k_2r^2 + 3k_3r^4) + 2\rho_2 y + 2\rho_1 x$$

Calculate $\frac{\partial y_d}{\partial y}$

$$\frac{\partial y_d}{\partial y} = A + y\frac{\partial A}{\partial y} + 2\rho_2 x + \frac{\partial}{\partial y}[\rho_1(r^2 + 2y^2)] = A + 2y^2(k_1 + 2k_2r^2 + 3k_3r^4) + 2\rho_2 x + 6\rho_1 y$$

Substituting A:

$$\frac{\partial y_d}{\partial y} = (1 + k_1r^2 + k_2r^4 + k_3r^6) + 2y^2(k_1 + 2k_2r^2 + 3k_3r^4) + 2\rho_2 x + 6\rho_1 y$$

Final Jacobian Matrix

Combining the above results, we have:

$$\frac{\partial \mathbf{p}_d}{\partial \mathbf{p}_n^T} = \begin{bmatrix} 1 + k_1r^2 + k_2r^4 + k_3r^6 + 2x^2(k_1 + 2k_2r^2 + 3k_3r^4) + 2\rho_1 y + 6\rho_2 x & 2xy(k_1 + 2k_2r^2 + 3k_3r^4) + 2(\rho_1 x + \rho_2 y) \\ 2xy(k_1 + 2k_2r^2 + 3k_3r^4) + 2(\rho_1 x + \rho_2 y) & 1 + k_1r^2 + k_2r^4 + k_3r^6 + 2y^2(k_1 + 2k_2r^2 + 3k_3r^4) + 2\rho_2 x + 6\rho_1 y \end{bmatrix}$$

Jacobian Matrix of Rotation Matrix

Rodrigues' Rotation Formula

Given the axis-angle representation $\mathbf{d} = \theta\mathbf{n}$, where θ is the rotation angle and $\mathbf{n}$ is the unit rotation axis, the rotation matrix R can be expressed using Rodrigues' rotation formula:

$$R = \cos\theta\, I + (1 - \cos\theta)\,\mathbf{n}\mathbf{n}^T + \sin\theta\,\mathbf{n}^\wedge$$

where:

- I is the 3×3 identity matrix.
- $\mathbf{n}^\wedge$ is the skew-symmetric matrix of $\mathbf{n}$:

$$\mathbf{n}^\wedge = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}$$

- $\alpha = \sin\theta$
- $\beta = \cos\theta$
- $\gamma = 1 - \cos\theta$

Substituting these notations, the rotation matrix simplifies to:

$$R = \beta I + \gamma\mathbf{n}\mathbf{n}^T + \alpha\mathbf{n}^\wedge$$

Vectorizing the Rotation Matrix

The rotation matrix R is a 3×3 matrix. To facilitate differentiation, we vectorize R into a 9×1 vector $\mathbf{r}$ in row-major order:

$$\mathbf{r} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{21} & r_{22} & r_{23} & r_{31} & r_{32} & r_{33} \end{bmatrix}^T$$

Detailed Calculation of $\mathbf{r}$

Expanding R using Rodrigues' formula:

$$R = \beta I + \gamma \mathbf{n}\mathbf{n}^T + \alpha \mathbf{n}^\wedge$$

Breaking it down element-wise, we obtain:

$$R = \begin{bmatrix} \beta + \gamma n_1^2 & \gamma n_1 n_2 - \alpha n_3 & \gamma n_1 n_3 + \alpha n_2 \\ \gamma n_2 n_1 + \alpha n_3 & \beta + \gamma n_2^2 & \gamma n_2 n_3 - \alpha n_1 \\ \gamma n_3 n_1 - \alpha n_2 & \gamma n_3 n_2 + \alpha n_1 & \beta + \gamma n_3^2 \end{bmatrix}$$

Vectorizing R:

$$\mathbf{r} = \begin{bmatrix} \beta + \gamma n_1^2 \\ \gamma n_1 n_2 - \alpha n_3 \\ \gamma n_1 n_3 + \alpha n_2 \\ \gamma n_2 n_1 + \alpha n_3 \\ \beta + \gamma n_2^2 \\ \gamma n_2 n_3 - \alpha n_1 \\ \gamma n_3 n_1 - \alpha n_2 \\ \gamma n_3 n_2 + \alpha n_1 \\ \beta + \gamma n_3^2 \end{bmatrix}$$

Calculating the Jacobian $\frac{d\mathbf{r}}{d\mathbf{n}^T}$

The Jacobian matrix $\frac{d\mathbf{r}}{d\mathbf{n}^T}$ is a 9×3 matrix where each element $\frac{\partial r_i}{\partial n_j}$ represents the partial derivative of the i-th component of $\mathbf{r}$ with respect to the j-th component of $\mathbf{n}$.

Partial Derivatives

1. **First Component** $r_1 = \beta + \gamma n_1^2$

$$\frac{\partial r_1}{\partial \mathbf{n}} = \begin{bmatrix} 2\gamma n_1 \\ 0 \\ 0 \end{bmatrix}$$

2. **Second Component** $r_2 = \gamma n_1 n_2 - \alpha n_3$

$$\frac{\partial r_2}{\partial \mathbf{n}} = \begin{bmatrix} \gamma n_2 \\ \gamma n_1 \\ -\alpha \end{bmatrix}$$

3. **Third Component** $r_3 = \gamma n_1 n_3 + \alpha n_2$

$$\frac{\partial r_3}{\partial \mathbf{n}} = \begin{bmatrix} \gamma n_3 \\ \alpha \\ \gamma n_1 \end{bmatrix}$$

4. **Fourth Component** $r_4 = \gamma n_2 n_1 + \alpha n_3$

$$\frac{\partial r_4}{\partial \mathbf{n}} = \begin{bmatrix} \gamma n_2 \\ \gamma n_1 \\ \alpha \end{bmatrix}$$

5. **Fifth Component** $r_5 = \beta + \gamma n_2^2$

$$\frac{\partial r_5}{\partial \mathbf{n}} = \begin{bmatrix} 0 \\ 2\gamma n_2 \\ 0 \end{bmatrix}$$

6. **Sixth Component** $r_6 = \gamma n_2 n_3 - \alpha n_1$

$$\frac{\partial r_6}{\partial \mathbf{n}} = \begin{bmatrix} -\alpha \\ \gamma n_3 \\ \gamma n_2 \end{bmatrix}$$

7. **Seventh Component** $r_7 = \gamma n_3 n_1 - \alpha n_2$

$$\frac{\partial r_7}{\partial \mathbf{n}} = \begin{bmatrix} \gamma n_3 \\ -\alpha \\ \gamma n_1 \end{bmatrix}$$

8. **Eighth Component** $r_8 = \gamma n_3 n_2 + \alpha n_1$

$$\frac{\partial r_8}{\partial \mathbf{n}} = \begin{bmatrix} \alpha \\ \gamma n_3 \\ \gamma n_2 \end{bmatrix}$$

9. **Ninth Component** $r_9 = \beta + \gamma n_3^2$

$$\frac{\partial r_9}{\partial \mathbf{n}} = \begin{bmatrix} 0 \\ 0 \\ 2\gamma n_3 \end{bmatrix}$$

Assembling the Jacobian Matrix

Combining all the partial derivatives, the Jacobian matrix $\frac{d\mathbf{r}}{d\mathbf{n}^T}$ is:

$$\frac{d\mathbf{r}}{d\mathbf{n}^T} = \begin{bmatrix} 2\gamma n_1 & 0 & 0 \\ \gamma n_2 & \gamma n_1 & -\alpha \\ \gamma n_3 & \alpha & \gamma n_1 \\ \gamma n_2 & \gamma n_1 & \alpha \\ 0 & 2\gamma n_2 & 0 \\ -\alpha & \gamma n_3 & \gamma n_2 \\ \gamma n_3 & -\alpha & \gamma n_1 \\ \alpha & \gamma n_3 & \gamma n_2 \\ 0 & 0 & 2\gamma n_3 \end{bmatrix}$$

Calculating the Jacobian Matrix of $\mathbf{r}$ with Respect to $\mathbf{d}$ ($\frac{\partial \mathbf{r}}{\partial \mathbf{d}^T}$)

Using the chain rule, the Jacobian matrix $\frac{\partial \mathbf{r}}{\partial \mathbf{d}^T}$ can be expressed as:

$$\frac{\partial \mathbf{r}}{\partial \mathbf{d}^T} = \frac{\partial \mathbf{r}}{\partial \theta}\frac{\partial \theta}{\partial \mathbf{d}^T} + \frac{\partial \mathbf{r}}{\partial \mathbf{n}^T}\frac{\partial \mathbf{n}}{\partial \mathbf{d}^T}$$

1. Calculate $\frac{\partial \theta}{\partial \mathbf{d}^T}$

Given $\theta = \|\mathbf{d}\| = \sqrt{d_1^2 + d_2^2 + d_3^2}$, the partial derivative of θ with respect to $\mathbf{d}$ is:

$$\frac{\partial \theta}{\partial \mathbf{d}^T} = \frac{\mathbf{d}^T}{\theta} = \mathbf{n}^T$$

2. Calculate $\frac{\partial \mathbf{n}}{\partial \mathbf{d}^T}$

Since $\mathbf{n} = \frac{\mathbf{d}}{\theta}$, differentiating with respect to $\mathbf{d}^T$ yields:

$$\frac{\partial \mathbf{n}}{\partial \mathbf{d}^T} = \frac{1}{\theta}\left(I - \mathbf{n}\mathbf{n}^T\right)$$

where I is the 3×3 identity matrix.

3. Calculate $\frac{\partial \mathbf{r}}{\partial \theta}$

From the expression of $\mathbf{r}$:

$$\mathbf{r} = \begin{bmatrix} \beta + \gamma n_1^2 \\ \gamma n_1 n_2 - \alpha n_3 \\ \gamma n_1 n_3 + \alpha n_2 \\ \gamma n_2 n_1 + \alpha n_3 \\ \beta + \gamma n_2^2 \\ \gamma n_2 n_3 - \alpha n_1 \\ \gamma n_3 n_1 - \alpha n_2 \\ \gamma n_3 n_2 + \alpha n_1 \\ \beta + \gamma n_3^2 \end{bmatrix}$$

Taking the derivative with respect to θ:

- $\frac{\partial \beta}{\partial \theta} = -\sin\theta = -\alpha$
- $\frac{\partial \gamma}{\partial \theta} = \sin\theta = \alpha$

Thus, the derivative $\frac{\partial \mathbf{r}}{\partial \theta}$ is:

$$\frac{\partial \mathbf{r}}{\partial \theta} = \begin{bmatrix} -\alpha + \alpha n_1^2 \\ \alpha n_1 n_2 - \cos\theta n_3 \\ \alpha n_1 n_3 + \cos\theta n_2 \\ \alpha n_2 n_1 + \cos\theta n_3 \\ -\alpha + \alpha n_2^2 \\ \alpha n_2 n_3 - \cos\theta n_1 \\ \alpha n_3 n_1 - \cos\theta n_2 \\ \alpha n_3 n_2 + \cos\theta n_1 \\ -\alpha + \alpha n_3^2 \end{bmatrix}$$

However, for simplification, if higher-order derivatives are negligible or specific assumptions are made, this term can be adjusted accordingly.

4. Substitute into the Chain Rule

Combining the above results, the Jacobian $\frac{\partial \mathbf{r}}{\partial \mathbf{d}^T}$ becomes:

$$\frac{\partial \mathbf{r}}{\partial \mathbf{d}^T} = \frac{\partial \mathbf{r}}{\partial \theta}\mathbf{n}^T + \frac{1}{\theta}\frac{\partial \mathbf{r}}{\partial \mathbf{n}^T}\left(I - \mathbf{n}\mathbf{n}^T\right)$$

Final Answer

The concrete form of the Jacobian matrix is:

$$\begin{bmatrix}
\frac{2\gamma n_1\left(1-n_1^2\right)}{\theta}+\alpha n_1\left(n_1^2-1\right) & -\frac{2\gamma n_1^2 n_2}{\theta}+\alpha n_2\left(n_1^2-1\right) & -\frac{2\gamma n_1^2 n_3}{\theta}+\alpha n_3\left(n_1^2-1\right) \\
n_1\left(\alpha n_1 n_2-\beta n_3\right)+\frac{\gamma n_2\left(1-2n_1^2\right)+\alpha n_1 n_3}{\theta} & n_2\left(\alpha n_1 n_2-\beta n_3\right)+\frac{\gamma n_1\left(1-2n_2^2\right)+\alpha n_2 n_3}{\theta} & n_3\left(\alpha n_1 n_2-\beta n_3\right)+\frac{\alpha\left(n_3^2-1\right)-2\gamma n_1 n_2 n_3}{\theta} \\
n_1\left(\alpha n_1 n_3+\beta n_2\right)+\frac{\gamma n_3\left(1-2n_1^2\right)-\alpha n_1 n_2}{\theta} & n_2\left(\alpha n_1 n_3+\beta n_2\right)+\frac{\alpha\left(1-n_2^2\right)-2\gamma n_1 n_2 n_3}{\theta} & n_3\left(\alpha n_1 n_3+\beta n_2\right)+\frac{\gamma n_1\left(1-2n_3^2\right)-\alpha n_2 n_3}{\theta} \\
n_1\left(\alpha n_1 n_2+\beta n_3\right)+\frac{\gamma n_2\left(1-2n_1^2\right)-\alpha n_1 n_3}{\theta} & n_2\left(\alpha n_1 n_2+\beta n_3\right)+\frac{\gamma n_1\left(1-2n_2^2\right)-\alpha n_2 n_3}{\theta} & n_3\left(\alpha n_1 n_2+\beta n_3\right)+\frac{\alpha\left(1-n_3^2\right)-2\gamma n_1 n_2 n_3}{\theta} \\
\frac{-2\gamma n_1 n_2^2}{\theta}+\alpha n_1\left(n_2^2-1\right) & \frac{2\gamma n_2\left(1-n_2^2\right)}{\theta}+\alpha n_2\left(n_2^2-1\right) & -\frac{2\gamma n_2^2 n_3}{\theta}+\alpha n_3\left(n_2^2-1\right) \\
n_1\left(\alpha n_2 n_3-\beta n_1\right)-\frac{\alpha\left(1-n_1^2\right)+2\gamma n_1 n_2 n_3}{\theta} & n_2\left(\alpha n_2 n_3-\beta n_1\right)+\frac{\gamma n_3\left(1-2n_2^2\right)+\alpha n_1 n_2}{\theta} & n_3\left(\alpha n_2 n_3-\beta n_1\right)+\frac{\alpha n_1 n_3+\gamma n_2\left(1-2n_3^2\right)}{\theta} \\
n_1\left(\alpha n_1 n_3-\beta n_2\right)+\frac{\alpha n_1 n_2+\gamma n_3\left(1-2n_1^2\right)}{\theta} & n_2\left(\alpha n_1 n_3-\beta n_2\right)-\frac{\alpha\left(1-n_2^2\right)+2\gamma n_1 n_2 n_3}{\theta} & n_3\left(\alpha n_1 n_3-\beta n_2\right)+\frac{\alpha n_2 n_3+\gamma n_1\left(1-2n_3^2\right)}{\theta} \\
n_1\left(\alpha n_2 n_3+\beta n_1\right)+\frac{\alpha\left(1-n_1^2\right)-2\gamma n_1 n_2 n_3}{\theta} & n_2\left(\alpha n_2 n_3+\beta n_1\right)+\frac{\gamma n_3\left(1-2n_2^2\right)-\alpha n_1 n_2}{\theta} & n_3\left(\alpha n_2 n_3+\beta n_1\right)+\frac{\gamma n_2\left(1-2n_3^2\right)-\alpha n_1 n_3}{\theta} \\
\frac{-2\gamma n_1 n_3^2}{\theta}+\alpha n_1\left(n_3^2-1\right) & -\frac{2\gamma n_2 n_3^2}{\theta}+\alpha n_2\left(n_3^2-1\right) & \frac{2\gamma n_3\left(1-n_3^2\right)}{\theta}+\alpha n_3\left(n_3^2-1\right)
\end{bmatrix}$$

Bird's Eye View Generation

Environment: Windows 11

Platform: PyCharm Professional 2024.1.4

Python version: 3.12.4

Python libraries: numpy opencv-Python

Results are as follows:

Camera calibration parameters

Reprojection Error

$$ret = 1.3526290383110415$$

Intrinsic Matrix

$$\text{mtx} = \begin{bmatrix} 1.06408820 \times 10^{3} & 0.00000000 \times 10^{0} & 6.97624043 \times 10^{2} \\ 0.00000000 \times 10^{0} & 1.05884544 \times 10^{3} & 3.67820618 \times 10^{2} \\ 0.00000000 \times 10^{0} & 0.00000000 \times 10^{0} & 1.00000000 \times 10^{0} \end{bmatrix}$$

Distortion Coefficients

$$\text{dist} = \begin{bmatrix} 2.19183009 \times 10^{-1} & -9.71999184 \times 10^{-1} & 8.92226849 \times 10^{-4} & -7.72790370 \times 10^{-3} & 9.61389806 \times 10^{-1} \end{bmatrix}$$

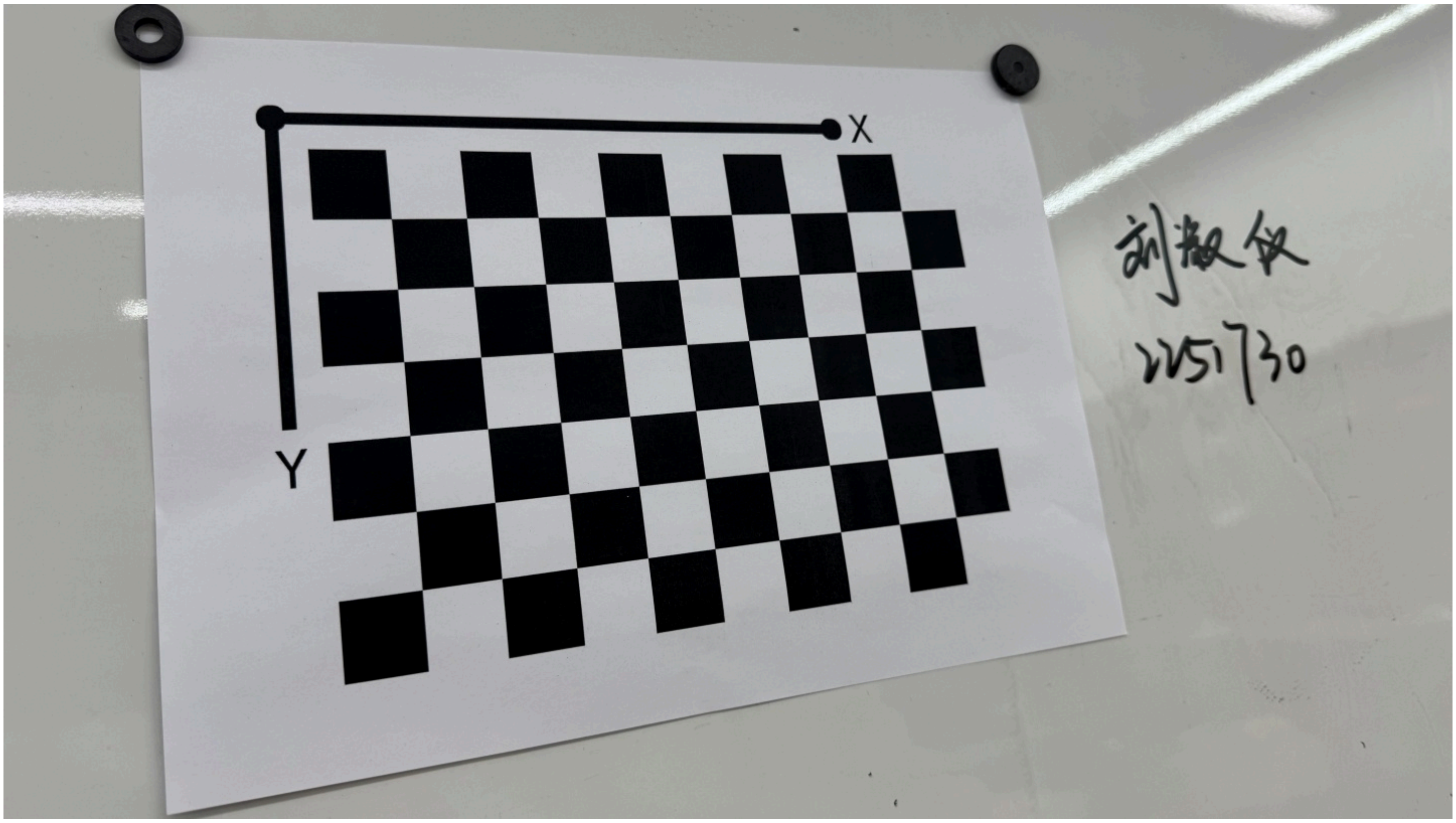
Rotation Vectors

$$\text{rvecs} = \left(\begin{bmatrix} -0.15060814 \\ 0.68259582 \\ -1.42548071 \end{bmatrix}, \begin{bmatrix} -0.62169292 \\ 0.02379807 \\ -1.54327886 \end{bmatrix}, \begin{bmatrix} -0.48797313 \\ 0.57757602 \\ -1.47210871 \end{bmatrix}, \begin{bmatrix} -0.00730448 \\ 0.19318482 \\ -1.58085337 \end{bmatrix}, \begin{bmatrix} 0.42666152 \\ 0.16677968 \\ -1.6343885 \end{bmatrix}, \begin{bmatrix} 0.42971655 \\ -0.20436555 \\ -1.63009947 \end{bmatrix}, \begin{bmatrix} 0.12150479 \\ -0.42919655 \\ -1.65266561 \end{bmatrix}, \begin{bmatrix} -0.29258934 \\ -0.22613081 \\ -1.57129171 \end{bmatrix}, \begin{bmatrix} -0.38838256 \\ 0.73023776 \\ -1.34886676 \end{bmatrix}, \begin{bmatrix} 0.47889099 \\ -0.40162315 \\ -1.70266642 \end{bmatrix} \right)$$

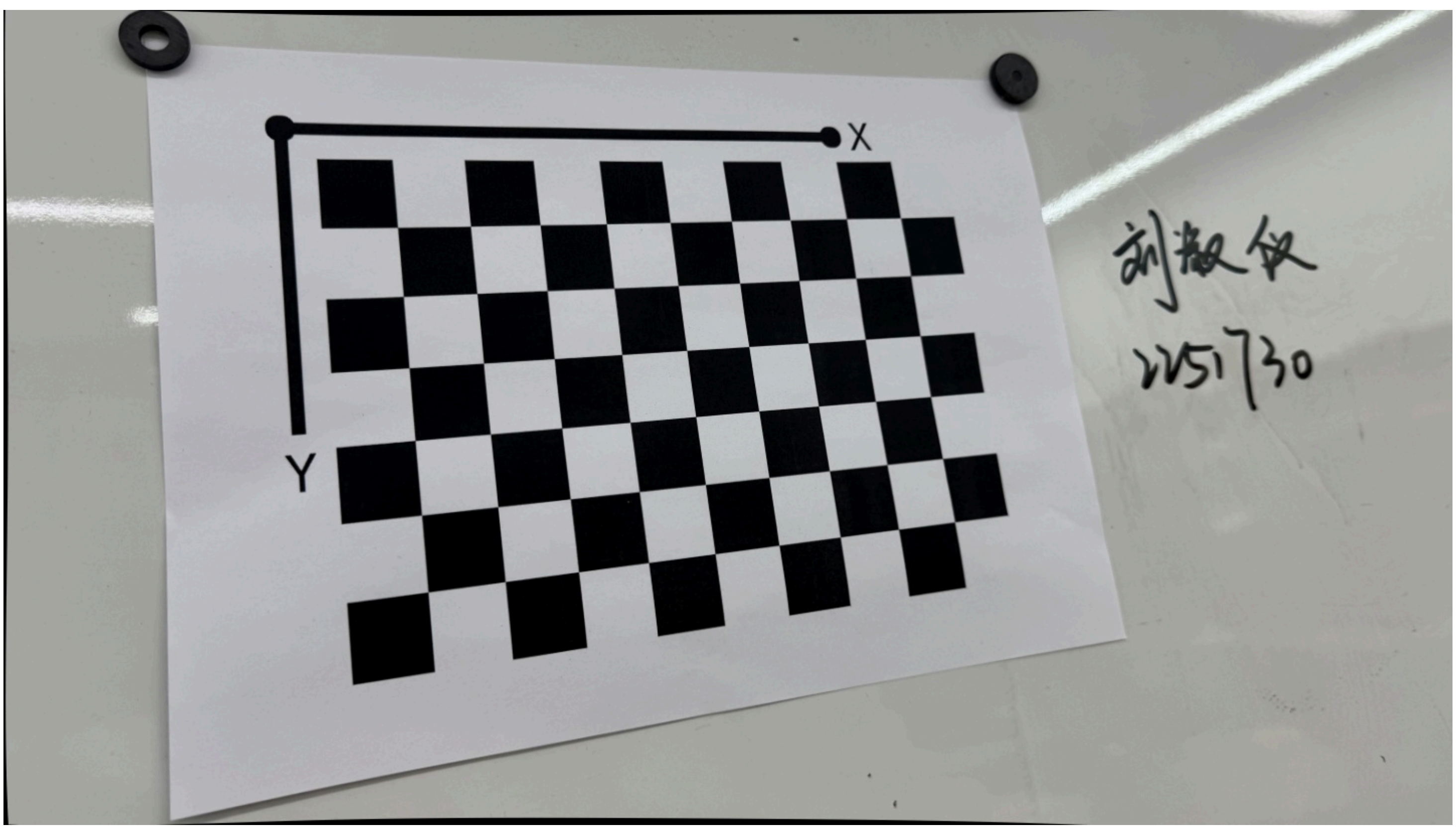
Translation Vectors

$$\text{tvecs} = \left(\begin{bmatrix} -0.16131816 \\ 0.04315679 \\ 0.4099573 \end{bmatrix}, \begin{bmatrix} -0.1625625 \\ 0.07084516 \\ 0.35582155 \end{bmatrix}, \begin{bmatrix} -0.17356461 \\ 0.05434441 \\ 0.38696302 \end{bmatrix}, \begin{bmatrix} -0.13625178 \\ 0.07758926 \\ 0.37540559 \end{bmatrix}, \begin{bmatrix} -0.08562375 \\ 0.08729008 \\ 0.34406331 \end{bmatrix}, \begin{bmatrix} -0.05401651 \\ 0.05972098 \\ 0.28404526 \end{bmatrix}, \begin{bmatrix} -0.07216266 \\ 0.05521112 \\ 0.27942144 \end{bmatrix}, \begin{bmatrix} -0.12059918 \\ 0.061236 \\ 0.32392025 \end{bmatrix}, \begin{bmatrix} -0.18402187 \\ 0.03017638 \\ 0.39936983 \end{bmatrix}, \begin{bmatrix} -0.04641674 \\ 0.05393896 \\ 0.27967062 \end{bmatrix} \right)$$

Original Image



Undistorted Image



Bird's Eye View