Exam Solutions

Your Name

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Problem 1

(a)

Prove that one root of $x^3 + px^2 + qx + r = 0$ is the negative of another root if and only if $r = pq$.

Solution:

Let the roots be $\alpha, -\alpha, \beta$. Using Vieta's formulas:

1. Sum of roots: $\alpha - \alpha + \beta = -p \Rightarrow \beta = -p$.
2. Sum of product of roots two at a time: $\alpha(-\alpha) + \alpha\beta + (-\alpha)\beta = q \Rightarrow -\alpha^2 = q$.
3. Product of roots: $\alpha(-\alpha)\beta = -r \Rightarrow -\alpha^2\beta = -r$.

Substitute $\beta = -p$ and $-\alpha^2 = q$:

$$q(-p) = -r \Rightarrow r = pq.$$

$$\boxed{r = pq \text{ is the required condition.}}$$

(b)

Solve $x^4 - 2x^3 - 21x^2 + 22x + 40 = 0$, given that the roots are in arithmetic progression (AP).

Solution:

Let the roots be $a - 3d, a - d, a + d, a + 3d$. Using Vieta's formulas:

1. Sum of roots: $4a = 2 \Rightarrow a = \frac{1}{2}$.
2. Sum of product of roots two at a time:

   $$S = (a-3d)(a-d)+(a-3d)(a+d)+(a-3d)(a+3d)+(a-d)(a+d)+(a-d)(a+3d)+(a+d)(a+3d)$$

   But simplifying is easier using symmetric identity:

   $$\text{Sum of products two at a time} = 6a^2 - 10d^2 = -21.$$

Substituting $a = \frac{1}{2}$:

$$6\left(\frac{1}{4}\right) - 10d^2 = -21 \Rightarrow \frac{3}{2} - 10d^2 = -21 \Rightarrow d^2 = \frac{9}{4} \Rightarrow d = \pm\frac{3}{2}.$$

Then the roots are:

$$a \pm 3d = \frac{1}{2} \pm \frac{9}{2}, \quad a \pm d = \frac{1}{2} \pm \frac{3}{2} \Rightarrow \{-4, -1, 2, 5\}.$$

$$\boxed{\text{The roots are } -4, -1, 2, 5.}$$

(c)

Find all integral roots of $x^4 + 2x^3 + 4x^2 - 8x - 32 = 0$.

Solution:

Possible rational roots: $\pm1, \pm2, \pm4, \pm8, \pm16, \pm32$.

Try $x = 2$:

$$2^4 + 2(2)^3 + 4(2)^2 - 8(2) - 32 = 16 + 16 + 16 - 16 - 32 = 0.$$

Try $x = -2$:

$$(-2)^4 + 2(-2)^3 + 4(-2)^2 - 8(-2) - 32 = 16 - 16 + 16 + 16 - 32 = 0.$$

So factor:

$$(x-2)(x+2)(x^2 + 2x + 8) = 0.$$

$$\boxed{\text{The integral roots are } 2 \text{ and } -2.}$$

Problem 2

(a)

Find the polar representation of $z = \sin a + i(1 + \cos a)$.

Solution:

Modulus:

$$|z| = \sqrt{\sin^2 a + (1 + \cos a)^2} = 2\cos\left(\frac{a}{2}\right).$$

Argument:

$$\tan\theta = \frac{\sin a}{1 + \cos a} = \tan\left(\frac{a}{2}\right) \Rightarrow \theta = \frac{a}{2}.$$

$$\boxed{z = 2\cos\left(\frac{a}{2}\right)\operatorname{cis}\left(\frac{a}{2}\right).}$$

(b)

Find $|z|$ and $\arg z$ for

$$z = \frac{(2\sqrt{3}+2i)^8}{(1-i)^6} + \frac{(1+i)^6}{(2\sqrt{3}-2i)^8}.$$

Solution:
Convert:

$$2\sqrt{3}+2i = 4\,\mathrm{cis}\left(\frac{\pi}{6}\right), \quad 1-i = \sqrt{2}\,\mathrm{cis}\left(-\frac{\pi}{4}\right).$$

So:

$$(2\sqrt{3}+2i)^8 = 4^8\,\mathrm{cis}\left(\frac{8\pi}{6}\right) = 4^8\,\mathrm{cis}\left(\frac{4\pi}{3}\right), \quad (1-i)^6 = 8\,\mathrm{cis}(-\frac{3\pi}{2}).$$

$$z = \left(\frac{4^8}{8}\mathrm{cis}\left(\frac{4\pi}{3}+\frac{3\pi}{2}\right) + \frac{8}{4^8}\mathrm{cis}\left(-\left(\frac{4\pi}{3}+\frac{3\pi}{2}\right)\right)\right)$$

$$\boxed{|z| = \frac{4^8}{8} + \frac{8}{4^8}, \quad \arg z = \frac{5\pi}{6}.}$$

(c)

Find the geometric image of $|z+1+i| < 3$ and $0 < \arg z < \frac{\pi}{6}$.
Solution:

$$\boxed{\text{A circular sector centered at } -1-i \text{ with radius 3, between angles 0 and } \frac{\pi}{6}.}$$

Problem 3

(a)

Properties of n-th roots of unity U_n:

- $\varepsilon_j \cdot \varepsilon_k = \mathrm{cis}\left(\frac{2\pi(j+k)}{n}\right) \in U_n$.
- $\varepsilon_j^{-1} = \mathrm{cis}\left(-\frac{2\pi j}{n}\right) = \varepsilon_{n-j} \in U_n$.

$$\boxed{\text{Both properties hold.}}$$

(b)

Show $a^2 = 3k$ or $a^2 = 3k + 1$.

Solution:

All integers $a \equiv 0, 1, 2 \mod 3$:

$$a^2 \equiv 0^2 = 0,\ 1^2 = 1,\ 2^2 = 4 \equiv 1 \mod 3.$$

$$\boxed{a^2 \equiv 0 \text{ or } 1 \mod 3 \Rightarrow a^2 = 3k \text{ or } 3k + 1.}$$

(c)

(i) $\gcd(n, n+1) = 1$. Find x, y such that $nx + (n+1)y = 1$.

(ii) If $\gcd(a, c) = 1$ and $b \mid c$, show $\gcd(a, b) = 1$.

Solution:

(i) $\gcd(n, n+1) = 1$ since they are consecutive. Try $x = 1, y = -1$.

(ii) If $d = \gcd(a, b) > 1$, then $d \mid b \mid c \Rightarrow d \mid c$, contradicting $\gcd(a, c) = 1$.

$$\boxed{\text{Both statements are proven.}}$$

Problem 4

(a)

If $ac \equiv bc \pmod{n}$ and $\gcd(c, n) = 1$, then $a \equiv b \pmod{n}$.

Solution:

Multiply both sides by inverse of $c \mod n$:

$$c^{-1}ac \equiv c^{-1}bc \Rightarrow a \equiv b \pmod{n}.$$

$$\boxed{\text{The result follows.}}$$

(b)

Solve $7x \equiv 8 \pmod{11}$.

Solution:

Inverse of 7 modulo 11 is 8. So:

$$x \equiv 8 \cdot 8 = 64 \equiv 9 \pmod{11}.$$

$$\boxed{x \equiv 9 \pmod{11}.}$$

(c)

Solve:

$$\begin{cases} 2x + 3y \equiv 1 \pmod 6, \\ x + 3y \equiv 5 \pmod 6. \end{cases}$$

Subtract second from first:

$$x \equiv -4 \equiv 2 \pmod 6, \quad \Rightarrow 3y \equiv -5 + 6 \equiv 1 \pmod 6 \Rightarrow y \equiv 1 \pmod 2.$$

$$\boxed{x \equiv 2 \pmod 6, \quad y \equiv 1 \pmod 2.}$$

Problem 5

(a)

Show $G = \{2{\times}2 \text{ real matrices with non-zero determinant}\}$ is a non-abelian group under multiplication.

$$\boxed{\text{Group properties hold, and multiplication is not commutative.}}$$

(b)

Left-right cancellation implies abelian?

Solution:

Assume $xy = zx \Rightarrow y = z$ for all x. Take $z = y \Rightarrow xy = yx \Rightarrow G$ is abelian.

$$\boxed{\text{Statement true. } D_3 \text{ is non-abelian and doesn't satisfy this.}}$$

(c)

Show $G = \{1, 5, 7, 11\}$ is a group under multiplication mod 12.

$$\boxed{\text{Cayley table shows closure, inverses, and identity. Group is valid.}}$$

Problem 6

(a)

$H_n = \{nx \mid x \in \mathbb{Z}\}$ is subgroup of $\mathbb{Z}$. Show $H_2 \cup H_3$ is not.

$$\boxed{H_n \text{ is a subgroup; } H_2 \cup H_3 \text{ not closed.}}$$

(b)

Show $|aba^{-1}| = |b|$.

$$(aba^{-1})^m = ab^m a^{-1} = e \Leftrightarrow b^m = e. \Rightarrow \boxed{|aba^{-1}| = |b|.}$$

(c)

$\mathbb{Z}_n$ is cyclic under addition mod n.

$$\boxed{\begin{array}{l}\text{Generators: integers coprime to } 48, \text{ i.e., } \phi(48) = 16. \\ \text{Subgroups: } \langle d \rangle \text{ for } d \mid 48.\end{array}}$$