Statistics Exam Solutions

Question 1

(a) Test scores analysis

Given:

- Group I: 10, 10, 10, 15, 35, 75, 90, 95, 100, 175, 420, 490, 515, 515, 790
- Group II: 0, 5, 5, 15, 30, 45, 50, 50, 50, 60, 75, 110, 140, 240, 330

(i) Calculate quartiles

For Group I (n=15):

$$\text{Median position} = \frac{n+1}{2} = 8$$
$$Q_1 \text{ position} = \frac{n+1}{4} = 4$$
$$Q_3 \text{ position} = \frac{3(n+1)}{4} = 12$$
$$\Rightarrow \text{Median} = 95,\ Q_1 = 15,\ Q_3 = 490$$

For Group II (n=15):

$$\text{Median} = 50,\ Q_1 = 15,\ Q_3 = 110$$

(ii) Identify outliers

Group I:

$$\text{IQR} = Q_3 - Q_1 = 490 - 15 = 475$$
$$\text{Lower fence} = Q_1 - 1.5 \times \text{IQR} = 15 - 712.5 = -697.5$$
$$\text{Upper fence} = Q_3 + 1.5 \times \text{IQR} = 490 + 712.5 = 1202.5$$
$$\text{No outliers (all values within fences)}$$

Group II:

$$\text{IQR} = 110 - 15 = 95$$
$$\text{Upper fence} = 110 + 142.5 = 252.5$$
$$330 \text{ is an extreme outlier}$$

(b) Parking probability

(i) Total arrangements

$$20! \text{ possible arrangements}$$

(ii) Alternating probability

$$\text{Favorable arrangements} = 2 \times 10! \times 10!$$
$$\text{Probability} = \frac{2 \times 10! \times 10!}{20!}$$

Question 2

(a) Trimmed mean

Given data: 2, 4, 6, 7, 11, 21, 81, 90, 105, 121

27% trimmed mean

$$27\% \text{ of } 10 = 2.7 \approx 3 \text{ removed from each end}$$
$$\text{Trimmed data} = 7, 11, 21, 81, 90$$
$$\text{Trimmed mean} = \frac{7 + 11 + 21 + 81 + 90}{5} = 42$$

Trimming percentage

$$\frac{1}{10} = 10\% \text{ trimming per end} \Rightarrow 20\% \text{ total}$$

(b) Probability of tag loss

(i) At least one tag lost

$$\begin{aligned} P(A_1 \cup A_2) &= 1 - P(\text{none lost}) \\ &= 1 - (0.7 \times 0.7) = 0.51 \end{aligned}$$

(ii) Exactly one given at least one

$$\begin{aligned} P(\text{exactly one}) &= 2 \times 0.3 \times 0.7 = 0.42 \\ P(\text{exactly one}|\text{at least one}) &= \frac{0.42}{0.51} \approx 0.8235 \end{aligned}$$

Question 3

(a) Plant survival

(i) Probability plant is alive

$$\begin{aligned} P(\text{Alive}) &= P(\text{Water})P(\text{Alive}|\text{Water}) + P(\text{No water})P(\text{Alive}|\text{No water}) \\ &= 0.9 \times 0.85 + 0.1 \times 0.2 = 0.785 \end{aligned}$$

(ii) Neighbor forgot given plant died

$$\begin{aligned} P(\text{No water}|\text{Dead}) &= \frac{P(\text{Dead}|\text{No water})P(\text{No water})}{P(\text{Dead})} \\ &= \frac{0.8 \times 0.1}{0.215} \approx 0.3721 \end{aligned}$$

(b) Crash test results

Sample proportion

$$\hat{p} = \frac{7}{10} = 0.7$$

Additional successes needed

$$0.8 \times 25 = 20 \text{ successes needed}$$
$$20 - 7 = 13 \text{ more from 15 cars}$$

Question 4

(a) Airline overbooking

(i) Accommodate all

$$P(Y \leq 50) = \sum_{y=45}^{50} P(y) = 0.83$$

(ii) Cannot accommodate

$$P(Y > 50) = 1 - 0.83 = 0.17$$

(iii) Standby probabilities

- 1st standby: $P(Y \leq 49) = 0.66$
- 3rd standby: $P(Y \leq 47) = 0.27$

(b) Win-win game

Expected values

$$E(X) = 1 \times 0.4 + 2 \times 0.3 + 3 \times 0.1 + 4 \times 0.2 = 2.1$$
$$E(1000/X) = 1000 \left(\frac{0.4}{1} + \frac{0.3}{2} + \frac{0.1}{3} + \frac{0.2}{4} \right) \approx 633.33$$

Recommend random prize (higher expected value).

Question 6

(a) Defective items

PMF (Hypergeometric)

$$P(X = k) = \frac{C(3,k)C(7,3-k)}{C(10,3)}$$

$$P(0) = \frac{35}{120} \approx 0.2917$$
$$P(1) = \frac{63}{120} = 0.525$$
$$P(2) = \frac{21}{120} = 0.175$$
$$P(3) = \frac{1}{120} \approx 0.0083$$

CDF

$$F(0) = 0.2917$$
$$F(1) = 0.8167$$
$$F(2) = 0.9917$$
$$F(3) = 1$$

(b) Uniform distribution

Median

$$\text{Median} = \frac{A+B}{2}$$

90th percentile

$$P_{90} = A + 0.9(B - A)$$

Question 7

(a) Bus waiting time

Probability

$$\text{Total time} = 30 \text{ minutes}$$
$$\text{Favorable intervals} = (7:00 - 7:03) \text{ and } (7:15 - 7:18)$$
$$P(\text{wait} \geq 12) = \frac{6}{30} = 0.2$$

(b) Poisson claims

(i) Fewer than 3 claims

$$P(X < 3) = e^{-5}\left(1 + 5 + \frac{25}{2}\right) \approx 0.1247$$

(ii) 4 claims in 3 of 5 days

$$P(4) = \frac{e^{-5}5^4}{24} \approx 0.1755$$
$$\text{Binomial} = C(5,3)(0.1755)^3(0.8245)^2 \approx 0.0367$$

(c) System reliability

Probability functional

$$P = \sum_{k=4}^{6} \frac{C(15,k)C(5,6-k)}{C(20,6)} \approx 0.8687$$

Expected working components

$$E = 6 \times \frac{15}{20} = 4.5$$

Question 8

(a) Poisson colds

(i) No more than 2 colds

$$\text{With supplements}(\lambda = 3):P(X \leq 2) \approx 0.4232$$
$$\text{Without supplements}(\lambda = 5):P(X \leq 2) \approx 0.1247$$

(ii) Combined probability

$$0.7 \times 0.4232 + 0.3 \times 0.1247 \approx 0.3336$$

(b) Smoker probability

Normal approximation

$$\mu = 750 \times 0.3 = 225$$
$$\sigma = \sqrt{750 \times 0.3 \times 0.7} \approx 12.55$$

(i) Fewer than 200

$$P(X < 200) \approx P(Z < -2.03) \approx 0.0212$$

(ii) 240 or more

$$P(X \geq 240) \approx P(Z > 1.16) \approx 0.1230$$

(c) Battery lifespan

(i) Last more than 4 years

$$P(X > 4) = e^{-4/6} \approx 0.5134$$

(ii) Additional 5 years given 3

$$P(X > 8|X > 3) = P(X > 5) = e^{-5/6} \approx 0.4346$$

Question 9

(a) Credit card purchases

(i) Expectation and variance

$$E(X) = np = 10 \times 0.7 = 7$$
$$\text{Var}(X) = np(1-p) = 10 \times 0.7 \times 0.3 = 2.1$$

(ii) Probability range

$$P(5 \leq X \leq 8) \approx 0.8033$$

(b) Poisson death approximation

$$\lambda = 5, \;\; P(X \leq 6) \approx 0.7622$$

(c) SAT scores

(i) 30th percentile

$$z \approx -0.524 \Rightarrow x \approx 464.12$$

(ii) Score of 700

$$z = \frac{700 - 527}{120} \approx 1.4417 \Rightarrow P(Z > 1.4417) \approx 0.0746$$

Question 13

Central Limit Theorem

$$\begin{aligned} \text{Total weeks} &= 36 \\ \mu_T &= 36 \times 50 = 1800 \\ \sigma_T &= \sqrt{36 \times 16} = 24 \\ P(1728 < T < 1872) &= P(-3 < Z < 3) \approx 0.9974 \end{aligned}$$

Question 14

(i) Sample mean probability

$$\sigma_{\bar{X}} = \frac{3200}{\sqrt{36}} \approx 533.33$$
$$P(52000 < \bar{X} < 55000) \approx 0.9944$$

(ii) Small sample size

With n=12, CLT less reliable unless population is normal.

Problem 5: Continuous Random Variable (Weight of an Article)

(i) Calculate the mean of the random variable Z

Given the probability density function:

$$f(z) = \begin{cases} (z-8) & \text{for } 8 \le z \le 9 \\ (10-z) & \text{for } 9 < z \le 10 \\ 0 & \text{otherwise} \end{cases}$$

Solution:
First, we verify that $f(z)$ is a valid PDF by checking that it integrates to 1:

$$\int_8^{10} f(z)dz = \int_8^9 (z-8)dz + \int_9^{10} (10-z)dz = \frac{1}{2} + \frac{1}{2} = 1$$

The mean (expected value) is calculated as:

$$E[Z] = \int_8^{10} zf(z)dz = \int_8^9 z(z-8)dz + \int_9^{10} z(10-z)dz$$

$$= \left[\frac{z^3}{3} - 4z^2\right]_8^9 + \left[5z^2 - \frac{z^3}{3}\right]_9^{10} = \frac{17}{3} + \frac{17}{3} = \frac{34}{3}$$

$$\boxed{E[Z] = \frac{34}{3} \text{ grams}}$$

(ii) Expected profit per article

Given:

- Selling price: Rs. 2
- Refund if weight < 8.25g
- Cost function: $\frac{z}{15} + 0.35$

Solution:
1. Probability of refund:

$$P(Z < 8.25) = \int_8^{8.25} (z-8)dz = \frac{(0.25)^2}{2} = \frac{1}{32}$$

2. Expected revenue:

$$\text{Revenue} = 2 - 2 \times \frac{1}{32} = \frac{31}{16} \text{ Rs.}$$

3. Expected cost:

$$E[\text{Cost}] = \frac{E[Z]}{15} + 0.35 = \frac{34/3}{15} + 0.35 = \frac{34}{45} + \frac{7}{20}$$

4. Expected profit:

$$\text{Profit} = \frac{31}{16} - \left(\frac{34}{45} + \frac{7}{20}\right) \approx 0.484 \text{ Rs.}$$

$$\boxed{\text{Expected profit} = \frac{31}{16} - \frac{34}{45} - \frac{7}{20} \text{ Rs.}}$$

Problem 11: Joint Probability Density Function

(i) Compute the constant c

Given the joint PDF:

$$f(x,y) = \begin{cases} c(1-x)(2-y) & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Solution:
The constant c is found by normalizing the PDF:

$$\int_0^1 \int_0^1 c(1-x)(2-y)dydx = 1$$

$$c \times \frac{3}{2} \times \frac{1}{2} = 1 \Rightarrow c = \frac{4}{3}$$

$$\boxed{c = \frac{4}{3}}$$

(ii) Marginal PDFs of X and Y

Marginal PDF of X:

$$f_X(x) = \int_0^1 \frac{4}{3}(1-x)(2-y)dy = 2(1-x)$$

Marginal PDF of Y:

$$f_Y(y) = \int_0^1 \frac{4}{3}(1-x)(2-y)dx = \frac{2}{3}(2-y)$$

$$\boxed{f_X(x) = 2(1-x) \text{ for } 0 \le x \le 1} \quad \boxed{f_Y(y) = \frac{2}{3}(2-y) \text{ for } 0 \le y \le 1}$$

Problem 12: Discrete Joint Probability

(i) Joint PMF Table

Given marginal distributions:

$$P_X(1) = \frac{1}{2}, P_X(2) = \frac{1}{2}$$

$$P_Y(1) = \frac{5}{12}, P_Y(2) = \frac{1}{3}, P_Y(3) = \frac{1}{4}$$

Since X and Y are independent, joint probabilities multiply:

$$P(X = x, Y = y) = P_X(x)P_Y(y)$$

	Y=1	Y=2	Y=3
X=1	$\frac{5}{24}$	$\frac{1}{6}$	$\frac{1}{8}$
X=2	$\frac{5}{24}$	$\frac{1}{6}$	$\frac{1}{8}$

(ii) $P(X + Y \leq 3)$

We sum probabilities where $X + Y \leq 3$:

$$P(1, 1) + P(1, 2) + P(2, 1) = \frac{5}{24} + \frac{4}{24} + \frac{5}{24} = \frac{14}{24} = \frac{7}{12}$$

$$\boxed{P(X + Y \leq 3) = \frac{7}{12}}$$