Mathematics Exam Solutions

Question 1

(a)(i) Find a cubic equation with rational coefficients having the roots $\frac{1}{2}, \frac{1}{2} + \sqrt{2}$

Solution:

To construct a cubic equation with the given roots and rational coefficients, we must include the conjugate root since $\sqrt{2}$ is irrational. Therefore, the three roots are:

$$r_1 = \frac{1}{2}, \quad r_2 = \frac{1}{2} + \sqrt{2}, \quad r_3 = \frac{1}{2} - \sqrt{2}$$

Using Vieta's formulas for a general cubic equation $x^3 + bx^2 + cx + d = 0$:

1. Sum of roots:

$$r_1 + r_2 + r_3 = \frac{1}{2} + \left(\frac{1}{2} + \sqrt{2}\right) + \left(\frac{1}{2} - \sqrt{2}\right) = \frac{3}{2} = -b$$

Thus, $b = -\frac{3}{2}$.

2. Sum of products of roots two at a time:

$$\begin{aligned} r_1r_2 + r_1r_3 + r_2r_3 &= \frac{1}{2}\left(\frac{1}{2} + \sqrt{2}\right) + \frac{1}{2}\left(\frac{1}{2} - \sqrt{2}\right) + \left(\frac{1}{2} + \sqrt{2}\right)\left(\frac{1}{2} - \sqrt{2}\right) \\ &= \left(\frac{1}{4} + \frac{\sqrt{2}}{2}\right) + \left(\frac{1}{4} - \frac{\sqrt{2}}{2}\right) + \left(\frac{1}{4} - 2\right) \\ &= \frac{3}{4} - 2 = -\frac{5}{4} = c \end{aligned}$$

3. Product of roots:

$$r_1r_2r_3 = \frac{1}{2}\left(\frac{1}{4} - 2\right) = \frac{1}{2}\left(-\frac{7}{4}\right) = -\frac{7}{8} = -d$$

Thus, $d = \frac{7}{8}$.

Therefore, the cubic equation is:

$$x^3 - \frac{3}{2}x^2 - \frac{5}{4}x + \frac{7}{8} = 0$$

To eliminate fractions, multiply through by 8:

$$8x^3 - 12x^2 - 10x + 7 = 0$$

(a)(ii) Find an upper limit to the roots of $x^5+4x^4-7x^2-40x+1=0$

Solution:

We can use Cauchy's bound to find an upper limit for the roots. For a polynomial:

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

an upper bound is:

$$1 + \max\{|a_{n-1}|, \ldots, |a_0|\}$$

Applying this to our polynomial:

$$1 + \max\{4, 0, 7, 40, 1\} = 1 + 40 = 41$$

Alternatively, using the more precise bound:

$$\max\left\{1, \sum_{k=0}^{n-1} |a_k|\right\} = \max\{1, 4+0+7+40+1\} = \max\{1, 52\} = 52$$

Thus, all real roots lie in the interval $[-52, 52]$, and more precisely in $[-41, 41]$.

(b) Find all integral roots of $x^4 + 4x^3 + 8x + 32 = 0$

Solution:

By the Rational Root Theorem, possible integer roots are factors of 32:

$$\pm1, \pm2, \pm4, \pm8, \pm16, \pm32$$

We test these systematically:

1. $f(1) = 1 + 4 + 8 + 32 = 45 \neq 0$ 2. $f(-1) = 1 - 4 - 8 + 32 = 21 \neq 0$ 3. $f(2) = 16 + 32 + 16 + 32 = 96 \neq 0$ 4. $f(-2) = 16 - 32 - 16 + 32 = 0 \rightarrow x = -2$ is a root

Now perform polynomial division or synthetic division to factor out $(x + 2)$:

Using synthetic division:

$$\begin{array}{r|rrrrr} -2 & 1 & 4 & 0 & 8 & 32 \\ & & -2 & -4 & 8 & -32 \\ \hline & 1 & 2 & -4 & 16 & 0 \end{array}$$

Now we have:

$$(x + 2)(x^3 + 2x^2 - 4x + 16) = 0$$

Continue testing possible roots on the cubic:

1. $f(-4) = -64 + 32 + 16 + 16 = 0 \rightarrow x = -4$ is a root

Perform synthetic division again:

$$\begin{array}{r|rrrr} -4 & 1 & 2 & -4 & 16 \\ & & -4 & 8 & -16 \\ \hline & 1 & -2 & 4 & 0 \end{array}$$

Now the factorization is:

$$(x + 2)(x + 4)(x^2 - 2x + 4) = 0$$

The quadratic $x^2 - 2x + 4$ has discriminant:

$$D = (-2)^2 - 4(1)(4) = 4 - 16 = -12 < 0$$

Thus, it has no real roots.

Therefore, the only integral roots are:

$$x = -2 \text{ and } x = -4$$

(c) Find all rational roots of $y^4 - \frac{40}{3}y^3 + \frac{130}{3}y^2 - 40y + 9 = 0$

Solution:

First, we eliminate fractions by multiplying through by 3:

$$3y^4 - 40y^3 + 130y^2 - 120y + 27 = 0$$

By the Rational Root Theorem, possible rational roots are of the form $\pm\frac{p}{q}$ where p divides 27 and q divides 3:

$$\pm 1, \pm 3, \pm 9, \pm 27, \pm\frac{1}{3}, \pm\frac{9}{3}$$

We test these systematically:

1. $f(1) = 3 - 40 + 130 - 120 + 27 = 0 \rightarrow y = 1$ is a root 2. $f(3) = 243 - 1080 + 1170 - 360 + 27 = 0 \rightarrow y = 3$ is a root

Now perform polynomial division or synthetic division to factor out $(y - 1)$:

Using synthetic division:

$$\begin{array}{r|rrrrr} 1 & 3 & -40 & 130 & -120 & 27 \\ & & 3 & -37 & 93 & -27 \\ \hline & 3 & -37 & 93 & -27 & 0 \end{array}$$

Now we have:

$$(y - 1)(3y^3 - 37y^2 + 93y - 27) = 0$$

We know $y = 3$ is also a root, so factor it out from the cubic:

Using synthetic division:

$$\begin{array}{r|rrrr} 3 & 3 & -37 & 93 & -27 \\ & & 9 & -84 & 27 \\ \hline & 3 & -28 & 9 & 0 \end{array}$$

Now the factorization is:

$$(y - 1)(y - 3)(3y^2 - 28y + 9) = 0$$

Solve the quadratic equation:

$$y = \frac{28 \pm \sqrt{(-28)^2 - 4(3)(9)}}{6} = \frac{28 \pm \sqrt{784 - 108}}{6} = \frac{28 \pm \sqrt{676}}{6} = \frac{28 \pm 26}{6}$$

Thus:

$$y = \frac{54}{6} = 9 \quad \text{and} \quad y = \frac{2}{6} = \frac{1}{3}$$

Therefore, all rational roots are:

$$y = 1, \quad y = \frac{1}{3}, \quad y = 3, \quad y = 9$$

Question 2

(a) Express $\arg(\overline{z})$ and $\arg(-z)$ in terms of $\arg(z)$. Find the geometric image for complex numbers z such that $\arg(-z) \in \left(\frac{\pi}{6}, \frac{\pi}{3}\right)$.

Solution:

For any non-zero complex number z:

1. The argument of the conjugate:

$$\arg(\overline{z}) = -\arg(z) \pmod{2\pi}$$

This is because conjugation reflects z across the real axis.

2. The argument of the negative:

$$\arg(-z) = \arg(z) + \pi \pmod{2\pi}$$

This is because negation rotates z by π radians.

For the geometric image when $\arg(-z) \in \left(\frac{\pi}{6}, \frac{\pi}{3}\right)$:

$$\arg(z) = \arg(-z) - \pi \in \left(-\frac{5\pi}{6}, -\frac{2\pi}{3}\right)$$

This represents all complex numbers z in the sector bounded by the angles $-\frac{5\pi}{6}$ and $-\frac{2\pi}{3}$ from the positive real axis, excluding the origin.

(b) Find $|z|$, $\arg z$, $\operatorname{Arg} z$, $\arg \overline{z}$, $\arg(-z)$ for $z = (1-i)(6+6i)$

Solution:

First compute the product:

$$(1-i)(6+6i) = 6(1-i)(1+i) = 6(1-i^2) = 6(1+1) = 12$$

Thus:

$$\begin{aligned}
|z| &= 12 \\
\arg z &= 0 \quad \text{(since } z \text{ lies on the positive real axis)} \\
\operatorname{Arg} z &= 0 \quad \text{(principal value)} \\
\arg \overline{z} &= 0 \quad \text{(same as } z \text{ since } z \text{ is real)} \\
\arg(-z) &= \pi \quad \text{(rotation by } \pi \text{ radians)}
\end{aligned}$$

(c) Find the cube roots of $z = 1 + i$ and represent them geometrically

Solution:

First express z in polar form:

$$|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\arg z = \frac{\pi}{4}$$

The cube roots are given by:

$$z_k = 2^{1/6}\left[\cos\left(\frac{\pi/4 + 2k\pi}{3}\right) + i\sin\left(\frac{\pi/4 + 2k\pi}{3}\right)\right], \quad k = 0, 1, 2$$

Calculating each root:

$$z_0 = 2^{1/6}\left[\cos\left(\frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{12}\right)\right]$$
$$z_1 = 2^{1/6}\left[\cos\left(\frac{\pi}{4} + \frac{2\pi}{3}\right) + i\sin\left(\frac{\pi}{4} + \frac{2\pi}{3}\right)\right] = 2^{1/6}\left[\cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right)\right]$$
$$z_2 = 2^{1/6}\left[\cos\left(\frac{\pi}{4} + \frac{4\pi}{3}\right) + i\sin\left(\frac{\pi}{4} + \frac{4\pi}{3}\right)\right] = 2^{1/6}\left[\cos\left(\frac{19\pi}{12}\right) + i\sin\left(\frac{19\pi}{12}\right)\right]$$

Geometrically, these roots lie on a circle centered at the origin with radius $2^{1/6} \approx 1.122$, spaced at angles of $\frac{2\pi}{3}$ radians (120 degrees) apart.

Question 3

(a) Solve $y^3 - 15y - 126 = 0$ using Cardan's method

Solution:

The equation is already in depressed cubic form $y^3 + py + q = 0$ with:

$$p = -15, \quad q = -126$$

Cardano's formula gives:

$$y = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$$

First compute discriminant:

$$\Delta = \left(\frac{-126}{2}\right)^2 + \left(\frac{-15}{3}\right)^3 = (-63)^2 + (-5)^3 = 3969 - 125 = 3844 > 0$$

Since $\Delta > 0$, there is one real root and two complex conjugate roots.

Compute the real root:

$$y = \sqrt[3]{63 + \sqrt{3844}} + \sqrt[3]{63 - \sqrt{3844}} = \sqrt[3]{63 + 62} + \sqrt[3]{63 - 62} = \sqrt[3]{125} + \sqrt[3]{1} = 5 + 1 = 6$$

Verification:

$$6^3 - 15(6) - 126 = 216 - 90 - 126 = 0$$

To find the other roots, perform polynomial division:

$$(y - 6)(y^2 + 6y + 21) = 0$$

The quadratic gives:

$$y = \frac{-6 \pm \sqrt{36 - 84}}{2} = \frac{-6 \pm \sqrt{-48}}{2} = -3 \pm 2\sqrt{3}i$$

Thus, the solutions are:

$$y = 6, \quad y = -3 + 2\sqrt{3}i, \quad y = -3 - 2\sqrt{3}i$$

(b) Show that among any n consecutive integers, one is divisible by n

Solution:

Consider the sequence of n consecutive integers:

$$a, a+1, a+2, \ldots, a+(n-1)$$

When any integer is divided by n, the possible remainders are:

$$0, 1, 2, \ldots, n-1$$

By the pigeonhole principle: 1. There are n consecutive integers 2. There are exactly n possible remainder classes modulo n 3. Therefore, each remainder must appear exactly once in the sequence

In particular, there must be exactly one integer in the sequence with remainder 0 when divided by n, meaning it is divisible by n.

(c) Bézout's Identity: For integers a, b with $\gcd(a, b) = g$, show $\exists m, n \in \mathbb{Z}$ such that $g = ma + nb$

Solution:

Consider the set of all linear combinations:

$$S = \{ma + nb \mid m, n \in \mathbb{Z}\}$$

Key steps in the proof: 1. S contains positive integers (e.g., when $m = 1, n = 0$) 2. By the well-ordering principle, S has a smallest positive element d 3. Show d divides every element of S: - For any $x \in S$, write $x = qd + r$ with $0 \leq r < d$ - Then $r = x - qd \in S$, so minimality of d forces $r = 0$ 4. Since $a, b \in S$, d is a common divisor 5. Any common divisor c of a, b divides d (as d is a linear combination) 6. Therefore d is the greatest common divisor

Thus, there exist integers m, n such that $\gcd(a, b) = ma + nb$.

Question 4

(a) For integer a not divisible by 7, show $a \equiv 5^k \pmod 7$ for some k

Solution:

The multiplicative group modulo 7 is cyclic of order 6. We verify that 5 is a primitive root:

Compute powers of 5 modulo 7:

$$\begin{aligned}
5^1 &\equiv 5 \pmod 7 \\
5^2 &\equiv 25 \equiv 4 \pmod 7 \\
5^3 &\equiv 5 \cdot 4 \equiv 20 \equiv 6 \pmod 7 \\
5^4 &\equiv 5 \cdot 6 \equiv 30 \equiv 2 \pmod 7 \\
5^5 &\equiv 5 \cdot 2 \equiv 10 \equiv 3 \pmod 7 \\
5^6 &\equiv 5 \cdot 3 \equiv 15 \equiv 1 \pmod 7
\end{aligned}$$

Since 5 generates all non-zero residues, every $a \not\equiv 0 \pmod 7$ can be expressed as 5^k for some k.

(b) If $3 \mid (a^2 + b^2)$, show $3 \mid a$ and $3 \mid b$

Solution:

Examine squares modulo 3:

$$\begin{aligned} 0^2 &\equiv 0 \pmod 3 \\ 1^2 &\equiv 1 \pmod 3 \\ 2^2 &\equiv 4 \equiv 1 \pmod 3 \end{aligned}$$

Possible sums modulo 3:

$$\begin{aligned} 0 + 0 &\equiv 0 \\ 0 + 1 &\equiv 1 \\ 1 + 1 &\equiv 2 \end{aligned}$$

For $a^2 + b^2 \equiv 0 \pmod 3$, the only possibility is $a^2 \equiv b^2 \equiv 0$, which implies $a \equiv b \equiv 0 \pmod 3$.

(c) Solve the system: $\begin{cases} x + 5y \equiv 3 \pmod 9 \\ 4x + 5y \equiv 1 \pmod 9 \end{cases}$

Solution:

Subtract the first equation from the second:

$$3x \equiv -2 \equiv 7 \pmod 9$$

We need to solve $3x \equiv 7 \pmod 9$. However:

$$\gcd(3, 9) = 3 \nmid 7$$

Thus, there is no solution.

Alternatively, multiply first equation by 4:

$$4x + 20y \equiv 12 \pmod 9 \Rightarrow 4x + 2y \equiv 3 \pmod 9$$

Subtract second equation:

$$-3y \equiv 2 \pmod 9 \Rightarrow 3y \equiv 7 \pmod 9$$

Again, no solution since $\gcd(3, 9) \nmid 7$.

Question 5

(a) Square Symmetry Group

Solution:

For square with vertices labeled:

$$\begin{matrix} P & W \\ G & B \end{matrix}$$

The 8 symmetries are:

- R_0: Identity (no change)
- R_{90}: Rotate 90 degrees counterclockwise ($P \to G \to B \to W \to P$)
- R_{180}: Rotate 180 degrees ($P \leftrightarrow B$, $W \leftrightarrow G$)
- R_{270}: Rotate 270 degrees counterclockwise ($P \to W \to B \to G \to P$)
- H: Flip horizontally ($P \leftrightarrow G$, $W \leftrightarrow B$)
- V: Flip vertically ($P \leftrightarrow W$, $G \leftrightarrow B$)
- D: Flip across main diagonal ($P \leftrightarrow B$)
- D': Flip across other diagonal ($W \leftrightarrow G$)

The identity is R_0. Inverses:

- $R_{90}^{-1} = R_{270}$
- $R_{180}^{-1} = R_{180}$
- $H^{-1} = H$, $V^{-1} = V$
- $D^{-1} = D$, $D'^{-1} = D'$

(b) Show $G = \{f, f_y, f_z, f_x\}$ forms a group under composition

Solution:

Assuming:

$$\begin{aligned} f(x) &= x \quad \text{(identity)} \\ f_y(x) &= -x \\ f_z(x) &= \frac{1}{x} \\ f_x(x) &= -\frac{1}{x} \end{aligned}$$

Verify group axioms:

- Closure: All compositions remain in G
- Associativity: Function composition is always associative
- Identity: f serves as identity
- Inverses: Each element is its own inverse

(c) Group Inverse Properties

Solution:

1. For any group G, $(ab)^{-1} = b^{-1}a^{-1}$ because:

$$(ab)(b^{-1}a^{-1}) = a(bb^{-1})a^{-1} = aea^{-1} = aa^{-1} = e$$

2. If $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$, then:

$$b^{-1}a^{-1} = a^{-1}b^{-1} \implies ab = ba$$

Thus G is Abelian.

Question 6

(a) Center of a Group $Z(G)$

Solution:

Definition:

$$Z(G) = \{z \in G \mid zg = gz \ \forall g \in G\}$$

Subgroup proof:

- Closure: $z_1, z_2 \in Z(G) \implies z_1z_2g = z_1gz_2 = gz_1z_2$
- Identity: $eg = ge$ for all g
- Inverses: $z \in Z(G) \implies z^{-1}g = (g^{-1}z)^{-1} = (zg^{-1})^{-1} = gz^{-1}$

(b) Order of an Element

Solution:

The order of $a \in G$ is the smallest positive n such that $a^n = e$.
If $a^m = e$ and order is n, write $m = qn + r$ with $0 \leq r < n$. Then:

$$e = a^m = a^{qn+r} = (a^n)^q a^r = e^q a^r = a^r$$

By minimality of n, $r = 0$, so $n \mid m$.

(c) Cyclic Group $\mathbb{Z}_{xy}$

Solution:

Assuming $\gcd(x, y) = 1$, the generators of $\mathbb{Z}_{xy}$ are integers k with $\gcd(k, xy) = 1$.
Subgroups correspond to divisors of xy. For subgroup of order 15 (assuming 15 divides xy), its generators are elements of order 15 in $\mathbb{Z}_{xy}$.