Advanced Calculus Exam Solutions

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June 25, 2025

Question 1

(a)

Proof. Assume for contradiction that $a \neq 0$. Then $a > 0$. Take $\epsilon = \frac{a}{2} > 0$. By the given condition, $a < \frac{a}{2}$, which implies $2a < a$ or $a < 0$. This contradicts $a \geq 0$. Therefore, $a = 0$. $\square$

(b)

We solve $|x - 1| > |x + 1|$:

Square both sides:

$$\begin{aligned}(x-1)^2 &> (x+1)^2\\ x^2 - 2x + 1 &> x^2 + 2x + 1\\ -4x &> 0\\ x &< 0\end{aligned}$$

The solution is all real numbers x such that $x < 0$.

Graph description: The inequality holds for all points to the left of the origin on the number line.

(c)

Find the supremum and infimum:

(i)

$\left\{\cos \frac{n\pi}{2} : n \in \mathbb{N}\right\}$

The sequence of values is periodic:

- $n = 1$: $\cos \frac{\pi}{2} = 0$
- $n = 2$: $\cos \pi = -1$

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- $n = 3$: $\cos\frac{3\pi}{2} = 0$
- $n = 4$: $\cos 2\pi = 1$, and repeats.

Thus, the set is $\{-1, 0, 1\}$.

- Supremum = 1
- Infimum = -1

(ii)

$\left\{\frac{x+2}{3} : x > 3\right\}$

For $x > 3$, $\frac{x+2}{3} > \frac{5}{3}$. As x approaches 3 from above, the expression approaches $\frac{5}{3}$, and as x approaches infinity, it approaches infinity.

- Infimum = $\frac{5}{3}$
- Supremum does not exist (set is unbounded above)

(d)

Proof. To show Sup $\left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\} = 1$:
1. For all $n \in \mathbb{N}$, $1 - \frac{1}{n} < 1$, so 1 is an upper bound.
2. For any $\epsilon > 0$, choose $n > \frac{1}{\epsilon}$ (by Archimedean property). Then $1 - \frac{1}{n} > 1 - \epsilon$.

Thus, 1 is the least upper bound. □

Question 2

(a)

Proof. Let $\alpha = \text{Inf } S$ and $\beta = \text{Sup } \{-s : s \in S\}$.
1. For all $s \in S$, $s \geq \alpha \Rightarrow -s \leq -\alpha$. Thus $-\alpha$ is an upper bound for $\{-s\}$.
2. Since β is the least upper bound, $\beta \leq -\alpha$.
3. Conversely, $-s \leq \beta \Rightarrow s \geq -\beta$, so $-\beta$ is a lower bound for S.
4. Since α is the greatest lower bound, $\alpha \geq -\beta$.

Thus $\alpha = -\beta$. □

(b)

Archimedean Property: For any $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > x$.

Proof. Assume for contradiction that $\mathbb{N}$ is bounded above. Then by completeness, $\mathbb{N}$ has a supremum s. But then $s - 1$ is not an upper bound, so there exists $n \in \mathbb{N}$ with $n > s - 1$. Then $n + 1 > s$, contradicting s being the supremum. □

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(c)

For $S = \left\{\frac{1}{n} - \frac{1}{m} : n, m \in \mathbb{N}\right\}$:

- The maximum occurs when n is minimized and m is maximized: as $n = 1$ and $m \to \infty$, expression approaches 1.
- The minimum occurs when $m = 1$ and $n \to \infty$, expression approaches -1.
- All intermediate values are achieved.

Thus:

- Sup S = 1
- Inf S = -1

(d)

Definition: A sequence (x_n) converges to L if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - L| < \epsilon$.

Uniqueness of limit. Suppose L and L' are both limits. For any $\epsilon > 0$, there exist N_1, N_2 such that:

- $n \geq N_1 \Rightarrow |x_n - L| < \epsilon/2$
- $n \geq N_2 \Rightarrow |x_n - L'| < \epsilon/2$

For $n \geq \max(N_1, N_2)$:

$$|L - L'| \leq |L - x_n| + |x_n - L'| < \epsilon$$

Since ϵ is arbitrary, $L = L'$. □

Question 3

(a)

Proof. For any $\epsilon > 0$, choose $N > \frac{23}{9\epsilon}$. Then for $n \geq N$:

$$\left|\frac{2n+3}{3n-7} - \frac{2}{3}\right| = \left|\frac{23}{3(3n-7)}\right| < \frac{23}{9n} < \epsilon$$

□

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(b)

Proof. Let $x_n = n^{1/n} - 1 \geq 0$. By binomial theorem for $n \geq 2$:

$$n = (1 + x_n)^n \geq \frac{n(n-1)}{2} x_n^2$$

Thus $x_n \leq \sqrt{\frac{2}{n-1}} \to 0$ as $n \to \infty$. □

(c)

Sandwich Theorem: If $a_n \leq b_n \leq c_n$ for all $n \geq N$ and $\lim a_n = \lim c_n = L$, then $\lim b_n = L$.

Proof. For any $\epsilon > 0$, there exists N' such that for $n \geq N'$:

$$L - \epsilon < a_n \leq b_n \leq c_n < L + \epsilon$$

Thus $|b_n - L| < \epsilon$. □

(d)

Proof. Let (x_n) be increasing and bounded above. By completeness, $S = \{x_n\}$ has a supremum L. For any $\epsilon > 0$, $L - \epsilon$ is not an upper bound, so there exists N with $x_N > L - \epsilon$. By monotonicity, for all $n \geq N$:

$$L - \epsilon < x_N \leq x_n \leq L$$

Thus $|x_n - L| < \epsilon$. □

Question 4

(a)

Proof. 1. **Monotonicity:** By induction, $x_{n+1} > x_n$ and bounded above by 2.
2. **Bounded:** $x_n < 2$ for all n (induction).
3. By monotone convergence, limit L exists.
4. Taking limit: $L = \sqrt{2 + L} \Rightarrow L^2 - L - 2 = 0 \Rightarrow L = 2$. □

(b)

Proof. Take $\epsilon = 1$. There exists N such that for all $m, n \geq N$, $|x_m - x_n| < 1$. Then for $n \geq N$:

$$|x_n| \leq |x_N| + 1$$

Thus $\{x_n\}$ is bounded by $\max(|x_1|, \ldots, |x_{N-1}|, |x_N| + 1)$. □

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(c)

Proof. Consider $x_{2n} - x_n = \frac{1}{n+1} + \cdots + \frac{1}{2n} \geq \frac{n}{2n} = \frac{1}{2}$. If (x_n) converged, this difference would tend to 0. Contradiction. $\square$

(d)

Find limit superior and inferior:

(i)

$x_n = (-2)^n \left(1 + \frac{1}{n}\right)$
Subsequence $x_{2k} \to +\infty$, $x_{2k-1} \to -\infty$.

- $\limsup x_n = +\infty$
- $\liminf x_n = -\infty$

(ii)

$x_n = (-1)^n \left(\frac{1}{n}\right)$
Subsequence $x_{2k} \to 0$, $x_{2k-1} \to 0$.

- $\limsup x_n = 0$
- $\liminf x_n = 0$

Question 5

(a)

Proof. - If $|r| \geq 1$, terms don't tend to $0 \Rightarrow$ diverges.
- If $|r| < 1$, partial sums $S_n = a\frac{1-r^n}{1-r} \to \frac{a}{1-r}$. $\square$

(b)

Telescoping series:

$$\frac{1}{(n+a)(n+a+1)} = \frac{1}{n+a} - \frac{1}{n+a+1}$$

Thus sum $= \frac{1}{a+1}$.

(c)

$$0.\overline{15} = \frac{15}{99} = \frac{5}{33}$$

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(d)

Check convergence:

(i)

$\sum \frac{1}{\log n}$

By comparison with $\sum \frac{1}{n}$ (divergent) and $\frac{1}{\log n} > \frac{1}{n}$ for $n \geq 3$, series diverges.

(ii)

$\sum \tan^{-1}\left(\frac{1}{n}\right)$

Since $\tan^{-1}(1/n) \sim 1/n$ as $n \to \infty$, and $\sum 1/n$ diverges, this series diverges by limit comparison.

Question 6

(a)

Ratio Test: For $\sum a_n$, if $\lim \left|\frac{a_{n+1}}{a_n}\right| = L$:

- $L < 1$: converges
- $L > 1$: diverges
- $L = 1$: inconclusive

(i)

$\sum \frac{n!}{n^p}$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^p} \cdot \frac{n^p}{n!} = (n+1)\left(\frac{n}{n+1}\right)^p \to \infty$$

Thus diverges for all p.

(ii)

$\sum \frac{n!}{e^n}$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{e^{n+1}} \cdot \frac{e^n}{n!} = \frac{n+1}{e} \to \infty$$

Thus diverges.

(b)

Check convergence:

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(i)

$\sum \frac{\log n}{n^2}$

Compare with $\sum \frac{1}{n^{3/2}}$. Since $\frac{\log n}{n^{1/2}} \to 0$, $\frac{\log n}{n^2} < \frac{1}{n^{3/2}}$ for large n. Thus converges by comparison.

(ii)

$\sum \frac{n^{n^2}}{(n+1)^{n^2}}$

$$a_n = \left(\frac{n}{n+1}\right)^{n^2} = \left(1 - \frac{1}{n+1}\right)^{n^2} \approx e^{-n}$$

Thus series behaves like $\sum e^{-n}$ (convergent geometric series), so converges.

(c)

Absolute convergence: $\sum |a_n|$ converges.

Proof. For any $\epsilon > 0$, there exists N such that $\sum_{k=m}^{n} |a_k| < \epsilon$ for $n > m \geq N$. Then $|\sum_{k=m}^{n} a_k| \leq \sum_{k=m}^{n} |a_k| < \epsilon$, so $\sum a_n$ converges by Cauchy criterion. $\square$

Converse false: $\sum (-1)^n/n$ converges conditionally but not absolutely.

(d)

Check convergence:

(i)

$\sum (-1)^{n+1} \frac{n}{n^2+1}$

Alternating series with $\frac{n}{n^2+1}$ decreasing to 0 $\Rightarrow$ converges. $\sum |a_n|$ diverges by comparison with $\sum \frac{1}{n}$. Thus conditionally convergent.

(ii)

$\sum (-1)^n \frac{1}{n^2+(-1)^n}$

For even n, $a_n \approx \frac{1}{n^2}$; for odd n, $a_n \approx -\frac{1}{n^2}$. Thus $\sum |a_n|$ converges by comparison with $\sum \frac{1}{n^2}$. Absolutely convergent.

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