1. (a) Find the equation of the tangent line to the graph of $(x^2 + y^2)^3 = 8x^2y^2$ at the point $(-1, 1)$.

   (b) Estimate the maximum error when approximating $f(x) = \sqrt{x}$ with a second order Taylor polynomial about $x = 4$ in the interval $[4, 4.2]$. (5+5)

2. (a) Consider the function $f(x) = 3^x$, prove that for all real numbers x_1, x_2

$$\frac{3^{x_1} + 3^{x_2}}{2} \geq 3^{\frac{x_1 + x_2}{2}}.$$

   (b) Is the statement "$3x \to 7$ as $x \to 1$" true? Using the $\varepsilon\delta$ definition of limits, justify your answer. (5+5)

3. (a) Suppose that a function $f(x)$ is convex. Show the restrictions on parameters "a" and "b" that will guarantee that $g(x) = af(x) + b$ is also convex even if f is not differentiable. (5+5)

   (b) Solve the following equation for x

$$1 + 2\log_4(x + 1) = 2\log_2 x$$

4. Let a and b be numbers, with $0 < a < b < 1$. Define the function f by

$$f(x) = b^x - a^x \text{ for } x \in [0, \infty).$$

   (i) Does the extreme value theorem guarantee the existence of maximum and minimum for this function?

   (ii) Find the value of x that minimizes $f(x)$ on this interval and find the minimum value.

   (iii) Show that this function has a single stationary point x^* at which

$$\left(\frac{b}{a}\right)^x = \frac{\ln(a)}{\ln(b)}$$

   (iv) Without calculating the value of x^*, show that the function attains a maximum at x^* by signing the first derivative around it. (10)

5. (a) Let $f(x)$ be a continuous function on $[0,1]$. Show that if $-1 \leq f(x) \leq 1$ for all $x \in [0,1]$, then there exists a $c \in [0,1]$ such that $[f(c)]^2 = c$.

   (b) A student chooses the number of hours $x \geq 0$ to attend tuition. The cost of x hours of tuition is $c(x)$, where c is a differentiable convex function. She wishes to maximize $(x) = g[\pi(x)] - c(x)$, where $\pi(x)$ is her probability of passing when she attends x hours of tuition. Assume that π is a differentiable concave function, and g is an increasing, differentiable and concave function.

   (i) Write down the first order condition for an interior solution $x^* > 0$ of this problem.

   (ii) Is the interior solution necessarily solving the maximization problem? Why or why not.

6. (a) A quadratic profit function $\pi(x) = ax^2 + bx + c$, where x = output, is used to reflect the following assumptions :

   (i) When $x = 0$, profits are negative.

   (ii) The profit function is strictly concave.

   (iii) The maximum profits occur at output level $= x^* > 0$.

   What restrictions need to be placed on values of the constants a, b and c to fulfil the above-mentioned assumptions?

   (b) For what values of p is the following inequality satisfied?

$$\frac{3p - 4}{p + 2} > 4 - p$$

(5+5)

7. (a) Determine all values of the constants A and B so that the following function is continuous for all values of x :

$$f(x) = \begin{cases} Ax - B, & \text{if } x \leq -1 \\ 2x^2 + 3Ax + B, & \text{if } -1 < x \leq 1 \\ 4, & \text{if } x > 1 \end{cases}$$

   (b) What value of the constant c allows the function $e^{-x}\sqrt{1 + cx}$ to be approximated around $x = 0$ by a horizontal straight line? (5+5)

8. (a) If population A is growing exponentially over the time and A_1, A_2 are the values at Time 't_1' and 't_2' respectively, then find the growth rate of population in terms of A_1, A_2, t_1, t_2.

   (b) A line P passes through the point $(2, -1)$ and has a slope of -5. A second line Q passes through the points $(-2, 3)$ and $(4, -3)$. Find the equations for the lines P and Q. Find their point of intersection R. Determine the equation of the line S that passes through the point $(3, 4)$ and is parallel to line Q. (5+5)

9. (a) Let a, b, m and n be fixed numbers, where $a < b$ and m and n are positive. Define the function g for all x by $g(x) = (b - x)^m (x - a)^n$. For the equation $g'(x) = 0$, find a solution x_0 that lies between a and b.

   (b) Given the function $f(x) = x^3 + cx + 1$, identify the local extrema of this function for different values of c. (5+5)

10. (a) Determine values of x in domain where $H(x) = (x^2 - 1)e^{(2-x^2)}$ is strictly increasing and strictly decreasing.

    (b) The Government revenue $R(t)$ and government spending $G(t)$, both as functions of time 't' in years, are given by :

$$R(t) = 100t - t^2 \quad \text{and} \quad G(t) = 50t + 10$$

    (i) Find the rate of change of the budget surplus $B(t) = R(t) - G(t)$ with respect to time 't'.

    (ii) Determine the positive value of 't' for which the proportional rate of change of the budget surplus equals the proportional rate of change of Government spending. (5+5)