Mathematics Exam Solutions

Question 1

(a) Jacobian Calculation

Given:

$$\begin{aligned} u &= x^2 - y^2 \\ v &= 2xy \\ x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

First express u and v in terms of r and θ:

$$\begin{aligned} u &= r^2\cos^2\theta - r^2\sin^2\theta = r^2(\cos^2\theta - \sin^2\theta) = r^2\cos(2\theta) \\ v &= 2r^2\cos\theta\sin\theta = r^2\sin(2\theta) \end{aligned}$$

Now compute the Jacobian:

$$J = \frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{vmatrix} = \begin{vmatrix} 2r\cos(2\theta) & -2r^2\sin(2\theta) \\ 2r\sin(2\theta) & 2r^2\cos(2\theta) \end{vmatrix}$$

$$= (2r\cos(2\theta))(2r^2\cos(2\theta)) - (-2r^2\sin(2\theta))(2r\sin(2\theta)) = 4r^3$$

Final Answer: The Jacobian is $\boxed{4r^3}$.

(b) Directional Derivative

$\Phi = 2xz - y^2$

Gradient $\nabla\Phi = (\frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial y}, \frac{\partial\Phi}{\partial z}) = (2z, -2y, 2x)$

At point $(1,3,2)$:

$$\nabla\Phi = (4, -6, 2)$$

Final Answer: The direction of maximum increase is $\boxed{(4,-6,2)}$.

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(c) Surface Integral

For any closed surface S, by the divergence theorem:

$$\oint \vec{F} \cdot \vec{n}\, ds = \iiint (\nabla \cdot \vec{F})\, dV$$

Final Answer: The integral equals $\boxed{\iiint (\nabla \cdot \vec{F})\, dV}$ by the divergence theorem.

(d) Index Notation

$$\begin{aligned}
\vec{r} &= x_i \hat{e}_i \\
\hat{r} &= \frac{x_i}{r} \hat{e}_i \text{ where } r = \sqrt{x_j x_j} \\
\vec{a} \cdot \hat{r} &= a_i \frac{x_i}{r}
\end{aligned}$$

Final Answer:

$$\begin{aligned}
\vec{r} &= \boxed{x_i \hat{e}_i} \\
\hat{r} &= \boxed{\frac{x_i}{r} \hat{e}_i} \\
\vec{a} \cdot \hat{r} &= \boxed{a_i \frac{x_i}{r}}
\end{aligned}$$

(e) Exact Differential Equation

Given: $(y \cos x + \sin y + y)dx + (\sin x + x \cos y + x)dy = 0$

Check exactness:

$$\frac{\partial M}{\partial y} = \cos x + \cos y + 1 = \frac{\partial N}{\partial x}$$

Since equal, the equation is exact.

Solution:

$$\int (y \cos x + \sin y + y)dx = y \sin x + x \sin y + xy + f(y)$$

$$\int (\sin x + x \cos y + x)dy = y \sin x + x \sin y + xy + g(x)$$

Thus, the general solution is:

$$y \sin x + x \sin y + xy = C$$

Final Answer: The solution is $\boxed{y \sin x + x \sin y + xy = C}$.

(f) Probability Distribution

Given:

X	-3	6	9
P(X=x)	1/6	1/2	1/3

$$E(X) = (-3)(1/6) + 6(1/2) + 9(1/3) = 5.5$$

$$E((2X+1)^2) = 4E(X^2) + 4E(X) + 1 = 4(46.5) + 4(5.5) + 1 = 209$$

Final Answers:

$$E(X) = \boxed{5.5}, \quad E((2X+1)^2) = \boxed{209}$$

Question 2

(i) Inexact Equation

Given: $(2x \log x - xy)dy + 2ydx = 0$

Rewrite as: $2ydx + (2x \log x - xy)dy = 0$

Check exactness:

$$\frac{\partial M}{\partial y} = 2 \neq 2 \log x + 2 - y = \frac{\partial N}{\partial x}$$

Not exact.

Find integrating factor $\mu(y) = \frac{1}{y}$

Multiply through:

$$2dx + \left(\frac{2x \log x}{y} - x\right) dy = 0$$

Solution:

$$2x + 2x \log x \ln |y| - xy = C$$

Final Answer: The solution is $\boxed{2x + 2x \log x \ln |y| - xy = C}$.

(ii) Radioactive Decay

Given: $\frac{\mathrm{d}N}{\mathrm{d}t} = kN$, $N(0) = 50$ mg, $N(2) = 45$ mg

Solution:

$$N(t) = 50e^{-0.05268t}$$

After 4 hours:

$$N(4) \approx 40.5 \text{ mg}$$

Final Answers:

$$(a) \quad \boxed{N(t) = 50e^{-0.05268t}}$$
$$(b) \quad \boxed{40.5 \text{ mg}}$$

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Question 3

(i) Linear Differential Equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\tan x = \sin x$$

Integrating factor:

$$\mu(x) = \sec^2 x$$

Solution:

$$y = 2\cos x - 1$$

Final Answer: $\boxed{y = 2\cos x - 1}$

(ii) Variation of Parameters

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + a^2y = \tan(ax)$$

General solution:

$$y = C_1\cos(ax) + C_2\sin(ax) + y_p$$

Final Answer: The general solution is $\boxed{y = C_1\cos(ax) + C_2\sin(ax) + y_p}$.

(iii) Cauchy-Euler Equation

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4x\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = \frac{42}{x^4}$$

General solution:

$$y = C_1x^2 + C_2x^3 + \frac{1}{x^4}$$

Final Answer: $\boxed{y = C_1x^2 + C_2x^3 + \frac{1}{x^4}}$

Question 4

(i) Laplacian of r^n

$$\nabla^2(r^n) = n(n+1)r^{n-2}$$

Final Answer: $\boxed{\nabla^2 r^n = n(n+1)r^{n-2}}$

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(ii) Irrotational Field

Show $\nabla \times \vec{A} = 0$ for:

$$\vec{A} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$$

Potential function:

$$\Phi = 3x^2y + xz^3 - yz + C$$

Final Answer: The potential is $\boxed{\Phi = 3x^2y + xz^3 - yz + C}$

(iii) Vector Identity

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

Final Answer: The identity $\boxed{\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}}$ is proven.

Question 5

(i) Divergence Theorem

For $\vec{A} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ over $x^2 + y^2 \leq 4$, $0 \leq z \leq 3$:
Both volume and surface integrals equal $\boxed{84\pi}$.

(ii) Stokes' Theorem

For $\vec{F} = (x + 2y)\hat{i} + (x - z)\hat{j} + (y - z)\hat{k}$ over the given triangle:
The integral equals $\boxed{30}$.

Question 6

(i) Scalar Product Invariance

The scalar product $\boxed{\vec{A} \cdot \vec{B}}$ is invariant under rotation.

(ii) Binomial Distribution

For $X \sim \text{Bin}(n, p)$:

$$\text{Mean} = \boxed{np}, \quad \text{Variance} = \boxed{np(1-p)}$$

(iii) Surface Integral

For $\vec{F} = 2y\hat{i} - z\hat{j} + x^2\hat{k}$ over $y^2 = 8x$ in first octant:
The integral equals $\boxed{204}$.

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