Physics Examination Solutions with Detailed Explanations

Question 1

(a) Superposition of Harmonic Motions (10 marks)

i. **Given Expression:**

$$y = 4\cos^2\left(\frac{1}{2}t\right)\sin(1000t)$$

ii. **Trigonometric Identity Application:** We first simplify the $\cos^2$ term using the identity:

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

Applying this:

$$y = 4\left(\frac{1+\cos t}{2}\right)\sin(1000t) = 2(1+\cos t)\sin(1000t)$$

iii. **Expansion:**

$$y = 2\sin(1000t) + 2\cos t\sin(1000t)$$

iv. **Product-to-Sum Identity:** Using the identity $2\cos A\sin B = \sin(A+B)+\sin(A-B)$:

$$y = 2\sin(1000t) + \sin(1001t) + \sin(999t)$$

v. **Interpretation:** This represents the superposition of three independent harmonic motions:

- A wave with frequency 1000 rad/s and amplitude 2
- A wave with frequency 1001 rad/s and amplitude 1
- A wave with frequency 999 rad/s and amplitude 1

vi. **Physical Significance:** The original expression combines a high frequency carrier wave (1000 rad/s) with amplitude modulation from the lower frequency (1 rad/s) component.

(b) Lissajous Figure Conditions (5 marks)

i. **Basic Principle:** Lissajous figures result from the combination of two perpendicular harmonic motions:

$$x = A\cos(\omega_1 t + \phi_1)$$
$$y = B\cos(\omega_2 t + \phi_2)$$

ii. **Straight Line Conditions:** For a straight line to appear:

- The frequencies must be equal ($\omega_1 = \omega_2$)
- The phase difference must be an integer multiple of π ($\Delta\phi = n\pi$)

iii. **Mathematical Justification:** When $\omega_1 = \omega_2 = \omega$ and $\phi_2 - \phi_1 = n\pi$:

$$y = B\cos(\omega t + n\pi + \phi_1) = \pm B\cos(\omega t + \phi_1)$$

Thus:

$$y = \pm\frac{B}{A}x$$

which is the equation of a straight line.

iv. **Experimental Observation:** On an oscilloscope, this appears as a diagonal line whose slope depends on the amplitude ratio and phase relationship.

(c) Damped Harmonic Motion Solution (10 marks)

i. **Problem Setup:** Given:

- Mass $m = 3$ kg
- Restoring force $F = -12x$
- Damping force $F_d = -12v$
- Initial conditions: $x(0) = 10$, $v(0) = 0$

ii. **Equation of Motion:**

$$m\ddot{x} + b\dot{x} + kx = 0$$

Substituting values:

$$3\ddot{x} + 12\dot{x} + 12x = 0$$

iii. **Characteristic Equation:**

$$3r^2 + 12r + 12 = 0$$
$$r = \frac{-12 \pm \sqrt{144 - 144}}{6} = -2 \text{ (double root)}$$

iv. **General Solution:** For critical damping (double root):

$$x(t) = (A + Bt)e^{-2t}$$

v. **Applying Initial Conditions:**

- At $t = 0$: $x(0) = A = 10$
- Velocity: $\dot{x}(t) = (B - 2A - 2Bt)e^{-2t}$
- At $t = 0$: $\dot{x}(0) = B - 2A = 0 \Rightarrow B = 20$

vi. **Final Solutions:**

$$x(t) = (10 + 20t)e^{-2t}$$
$$v(t) = (-40t)e^{-2t}$$

vii. **Physical Interpretation:** The system exhibits critical damping, returning to equilibrium without oscillation. The position decays exponentially while the velocity shows a linear term multiplied by exponential decay.

(d) Normal Modes and Coordinates (5 marks)

i. **Normal Coordinates Definition:** Special coordinates that diagonalize the system's potential and kinetic energy matrices, decoupling the equations of motion.

ii. **Normal Modes Characteristics:**

- Each mode oscillates at a characteristic frequency (normal frequency)
- All parts of the system move with the same frequency and fixed phase relation
- Number of normal modes equals the system's degrees of freedom

iii. **Mathematical Representation:** For a coupled system with coordinates q_1, q_2, we find linear combinations:

$$Q_1 = aq_1 + bq_2$$
$$Q_2 = cq_1 + dq_2$$

such that the Lagrangian becomes:

$$L = \frac{1}{2}(\dot{Q}_1^2 + \dot{Q}_2^2) - \frac{1}{2}(\omega_1^2 Q_1^2 + \omega_2^2 Q_2^2)$$

iv. **Physical Significance:** Normal modes represent the fundamental vibration patterns where energy remains in individual modes without transferring between them.

(e) Stationary vs Progressive Waves (5 marks)

Feature	Stationary Waves	Progressive Waves
Formation	Result from interference of two identical waves traveling in opposite directions	Single wave propagating through medium
Energy	Confined between nodes, no net energy transfer	Propagates energy in direction of wave motion
Amplitude	Varies from zero at nodes to maximum at antinodes	Constant throughout medium
Phase	All points between nodes vibrate in phase	Phase changes continuously with position
Equation	$y = 2A\sin(kx)\cos(\omega t)$	$y = A\sin(kx - \omega t)$

(f) Quality Factor Calculation (5 marks)

i. **Given Parameters:**

- Frequency $f = 1$ Hz ($\omega = 2\pi$ rad/s)
- Amplitude halves in 5 seconds

ii. **Damping Analysis:** Amplitude decays as $A(t) = A_0 e^{-\beta t}$
When $A(t) = A_0/2$ at $t = 5$s:

$$\frac{1}{2} = e^{-5\beta} \Rightarrow \beta = \frac{\ln 2}{5} \approx 0.1386 \text{ s}^{-1}$$

iii. **Quality Factor:**

$$Q = \frac{\omega}{2\beta} = \frac{2\pi}{2 \times 0.1386} \approx 22.66$$

iv. **Physical Meaning:** The Q-factor of 22.66 indicates relatively weak damping, as the system completes about 22 oscillations before its energy decreases significantly.

Question 2

(a) Spring with Non-Negligible Mass (10 marks)

i. **System Description:** Spring with mass m_s, load mass M, and spring constant k

ii. **Effective Mass Concept:** The spring's mass contributes to the system's kinetic energy. Considering the velocity distribution along the spring:

$$KE_{spring} = \frac{1}{2}\int_0^L \left(\frac{x}{L}v\right)^2 \frac{m_s}{L}dx = \frac{1}{6}m_s v^2$$

iii. **Total Kinetic Energy:**

$$KE_{total} = \frac{1}{2}Mv^2 + \frac{1}{6}m_s v^2 = \frac{1}{2}\left(M + \frac{m_s}{3}\right)v^2$$

iv. **Period Calculation:**

$$T = 2\pi\sqrt{\frac{m_{eff}}{k}} = 2\pi\sqrt{\frac{M + m_s/3}{k}}$$

v. **Verification:** When $m_s \to 0$, reduces to standard formula $T = 2\pi\sqrt{M/k}$

(b) Lissajous Figure Analysis (5 marks)

i. **Given Equations:**

$$x = 10\cos(5\pi t)$$

$$y = 10\cos(10\pi t - \pi/4)$$

ii. **Frequency Ratio:** $\omega_y/\omega_x = 2/1$ (2:1 ratio)

iii. **Phase Difference:** $\phi = \pi/4$ (45 degrees)

iv. **Figure Characteristics:**

- Will show two vertical oscillations for each horizontal oscillation
- The phase difference causes the figure to be an open curve (not closed)
- Maximum x-value: 10 units
- Maximum y-value: 10 units
- Shape resembles a sideways "8" or infinity symbol with loops of unequal size

v. **Mathematical Description:** The parametric equations trace a curve satisfying:

$$y = 10\cos\left(2\cos^{-1}\left(\frac{x}{10}\right) - \frac{\pi}{4}\right)$$

Question 3

(a) Beats Formation (10 marks)

i. **Superposition Principle:** For two collinear SHMs with similar frequencies:

$$y_1 = A_1\cos(\omega_1 t)$$

$$y_2 = A_2\cos(\omega_2 t)$$

where $|\omega_1 - \omega_2| \ll \omega_1, \omega_2$

ii. **Resultant Wave:**

$$y = y_1 + y_2 = 2A\cos\left(\frac{\omega_1 - \omega_2}{2}t\right)\cos\left(\frac{\omega_1 + \omega_2}{2}t\right)$$

iii. **Beat Characteristics:**

- Amplitude envelope: $2A\cos\left(\frac{\Delta\omega}{2}t\right)$
- Carrier wave frequency: $\frac{\omega_1+\omega_2}{2}$
- Beat frequency: $f_{beat} = |f_1 - f_2| = \frac{|\omega_1-\omega_2|}{2\pi}$

iv. **Energy Considerations:** Energy transfers between the two waves, creating periodic amplitude maxima when they constructively interfere and minima when they destructively interfere.

v. **Practical Example:** When tuning two musical instruments, beats are heard as pulsations whose rate decreases as the frequencies get closer.

(b) RLC Circuit Analysis (5 marks)

i. **Given Parameters:**

- Peak-to-peak voltage: 40 V (amplitude $V_0 = 20$ V)
- $L = 100$ mH, $C = 1$ pF, $R = 100$

ii. **Resonance Frequency:**

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.1 \times 10^{-12}}} = 10^7 \text{ rad/s}$$

$$f_0 = \frac{\omega_0}{2\pi} \approx 1.59 \text{ MHz}$$

iii. **Quality Factor:**

$$Q = \frac{\omega_0 L}{R} = \frac{10^7 \times 0.1}{100} = 10,000$$

iv. **Bandwidth:**

$$\Delta\omega = \frac{\omega_0}{Q} = 1000 \text{ rad/s}$$

$$\Delta f = \frac{\Delta\omega}{2\pi} \approx 159 \text{ Hz}$$

v. **Significance:** The extremely high Q-factor indicates very sharp resonance, with the circuit being highly selective to frequencies near 1.59 MHz.

Question 4

(a) Power in Forced Damped Oscillator (10 marks)

i. **System Description:**

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos(\omega t)$$

ii. **Steady State Solution:**

$$x(t) = A\cos(\omega t - \phi)$$

where:

$$A = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (b\omega/m)^2}}$$

$$\tan\phi = \frac{b\omega}{m(\omega_0^2 - \omega^2)}$$

iii. **Instantaneous Power:**

$$P(t) = F(t)v(t) = F_0\cos(\omega t) \cdot [-\omega A \sin(\omega t - \phi)]$$

iv. **Average Power:**

$$P_{avg} = \frac{1}{T}\int_0^T P(t)dt = \frac{F_0 \omega A}{2}\sin\phi$$

v. **Power Dissipation:**

$$P_{diss} = bv^2 = b\omega^2 A^2 \sin^2(\omega t - \phi)$$

$$P_{diss,avg} = \frac{1}{2}b\omega^2 A^2$$

vi. **Equality Proof:** Using $\sin\phi = \frac{b\omega A}{F_0}$ from the amplitude expression:

$$P_{avg} = \frac{F_0 \omega A}{2} \cdot \frac{b\omega A}{F_0} = \frac{1}{2}b\omega^2 A^2 = P_{diss,avg}$$

(b) Coupled Oscillators Normal Modes (5 marks)

i. **System Setup:** Two masses M connected by three springs k:

k
M
k
M
k

ii. **Equations of Motion:**

$$M\ddot{x}_1 = -kx_1 + k(x_2 - x_1)$$
$$M\ddot{x}_2 = -k(x_2 - x_1) - kx_2$$

Simplifying:

$$\ddot{x}_1 + \frac{2k}{M}x_1 - \frac{k}{M}x_2 = 0$$
$$\ddot{x}_2 - \frac{k}{M}x_1 + \frac{2k}{M}x_2 = 0$$

iii. **Normal Mode Frequencies:** Assume solutions $x_i = A_i \cos(\omega t)$:

$$\begin{pmatrix} 2k/M - \omega^2 & -k/M \\ -k/M & 2k/M - \omega^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

Setting determinant to zero:

$$(2k/M - \omega^2)^2 - (k/M)^2 = 0$$

Solutions:

$$\omega_1 = \sqrt{k/M} \quad \text{(Symmetric mode)}$$
$$\omega_2 = \sqrt{3k/M} \quad \text{(Antisymmetric mode)}$$

iv. **Mode Shapes:**

- Symmetric: $A_1 = A_2$ (masses move in phase)
- Antisymmetric: $A_1 = -A_2$ (masses move out of phase)

Question 5

(a) String Vibration Modes (10 marks)

i. **Wave Equation:**

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

where $v = \sqrt{T/\mu}$

ii. **Boundary Conditions:** Fixed ends at $x = 0$ and $x = L$ require $y(0, t) = y(L, t) = 0$

iii. **Separation of Variables:** Assume $y(x, t) = X(x)T(t)$:

$$\frac{1}{v^2 T}\frac{d^2T}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2} = -k^2$$

iv. **Spatial Solution:**

$$X(x) = A\sin(kx) + B\cos(kx)$$

Boundary conditions give $B = 0$ and $kL = n\pi$:

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

v. **Temporal Solution:**

$$T_n(t) = C\cos(\omega_n t) + D\sin(\omega_n t)$$

where $\omega_n = vk = n\pi\sqrt{T/\mu L^2}$

vi. **Normal Modes:**

$$y_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)[A_n\cos(\omega_n t) + B_n\sin(\omega_n t)]$$

Frequencies:

$$f_n = \frac{n}{2L}\sqrt{\frac{T}{\mu}}$$

(b) String Tension Calculation (5 marks)

i. **Given Parameters:**

- Length $L = 0.5$ m
- Mass per unit length $\mu = 0.01$ kg/m
- Fundamental frequency $f_1 = 250$ Hz

ii. **Fundamental Frequency Relation:**

$$f_1 = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$$

iii. **Solve for Tension:**

$$250 = \frac{1}{1.0}\sqrt{\frac{T}{0.01}}$$

$$250 = \sqrt{100T}$$

$$62500 = 100T$$

$$T = 625 \text{ N}$$

iv. **Verification:** The calculated tension of 625 N is reasonable for a string of these dimensions to produce the given frequency.