

1. (a) (i) Find a cubic equation with rational coefficients having the roots

$$\frac{1}{2}, \frac{1}{2} + \sqrt{2}, \text{ stating the result used.}$$

- (ii) Find an upper limit to the roots of

$$x^5 + 4x^4 - 7x^2 - 40x + 1 = 0. \quad (4+3.5)$$

- (b) Find all the integral roots of

$$x^4 + 4x^3 + 8x + 32 = 0. \quad (7.5)$$

- (c) Find all the rational roots of

$$y^4 - \frac{40}{3}y^3 + \frac{130}{3}y^2 - 40y + 9 = 0. \quad (7.5)$$

2. (a) Express $\arg(\bar{z})$ and $\arg(-z)$ in terms of $\arg(z)$.

Find the geometric image for the complex number

$$z, \text{ such that } \arg(-z) \in \left(\frac{\pi}{6}, \frac{\pi}{3}\right). \quad (2+2+3.5)$$

- (b) Find $|z|$, $\arg z$, $\text{Arg } z$, $\arg \bar{z}$, $\arg(-z)$ for

$$z = (1-i)(6+6i) \quad (7.5)$$

- (c) Find the cube roots of $z = 1+i$ and represent them geometrically to show that they lie on a circle of radius $(2)^{1/6}$. (7.5)

3. (a) Solve $y^3 - 15y - 126 = 0$ using Cardan's method. (7.5)

- (b) Let n be a natural number. Given n consecutive integers, $a, a+1, a+2, \dots, a+(n-1)$, show that one of them is divisible by n . (7.5)

- (c) Let a and b be two integers such that $\gcd(a, b) = g$. Show that there exists integers m and n such that $g = ma + nb$. (7.5)

4. (a) Let a be an integer such that a is not divisible by 7. Show that $a \equiv 5^k \pmod{7}$ for some integer k . (7.5)

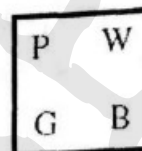
- (b) Let a and b be two integers such that 3 divides $(a^2 + b^2)$. Show that 3 divides a and b both. (7.5)

- (c) Solve the following pair of congruences, if possible. If no solution exists, explain why? (7.5)

$$x + 5y \equiv 3 \pmod{9}$$

$$4x + 5y \equiv 1 \pmod{9}$$

5. (a) Consider a square with four corners labelled as follows:



Describe the following motions graphically:

(i) R_0 = Rotation of 0 degree.

(ii) R_{90} = Rotation of 90 degrees counterclockwise.

(iii) R_{180} = Rotation of 180 degrees counterclockwise.

(iv) R_{270} = Rotation of 270 degrees counterclockwise.

(v) H = Flip about horizontal axis.

(vi) V = Flip about vertical axis.

(vii) D = Flip about the main diagonal.

(viii) $D1$ = Flip about the other diagonal.

Identify the motion that can act as identity under the composition of two motions. Further, find out the inverse of each motion. (3.5+1+3)

- (b) Show that the set $G = \{f_1, f_2, f_3, f_4\}$, is a group under the composition of functions defined as, $f \circ g(x) = f(g(x))$ for f, g in G , where $f_1(x) = x$, $f_2(x) = -x$, $f_3(x) = 1/x$, $f_4(x) = -1/x$ for all non-zero real number x . (7.5)

- (c) Define the inverse of an element in a group G . Show that $(a.b)^{-1} = b^{-1}.a^{-1}$ for all a, b in G . Further show that if $(a.b)^{-1} = a^{-1}.b^{-1}$ for all a, b in G , then G is Abelian. (4+3.5)

6. (a) Define $Z(G)$, the center of a group G . Show that $Z(G)$ is a subgroup of G . (2+5.5)

- (b) Define order of an element a in group G . Further show that if order of a is n , and $a^m = e$, where m is an integer, then n divides m . (2+5.5)

- (c) Find the generators of the cyclic group Z_{30} . Further describe all the subgroups of Z_{30} and find the generators of the subgroup of order 15 in Z_{30} . (2+3.5+2)