1. (a) Prove that one root of $x^3 + px^2 + qx + r = 0$ is negative of another root if and only if $r = pq$. (7.5)

(b) Solve $x^4 - 2x^3 - 21x^2 + 22x + 40 = 0$, whose roots are in arithmetical progression. (7.5)

P.T.O.

(c) Find all the integral roots of

$x^7 + 2x^5 + 4x^4 - 8x^2 - 32 = 0$. (7.5)

2. (a) Find the polar representation of the complex number

$z = \text{Sin } a + i(1 + \text{Cos } a),\ a \in [0, 2\pi)$ (7.5)

(b) Find $|z|$ and arg z for $z = \dfrac{(2\sqrt{3}+2i)^8}{(1-i)^6} + \dfrac{(1+i)^6}{(2\sqrt{3}-2i)^8}$

(7.5)

(c) Find the geometric image for the complex number z such that

$|z + 1 + i| < 3$ and $0 < \arg z < \dfrac{\pi}{6}$ (7.5)

3. (a) Let $U_n = \{\epsilon_0, \epsilon_1, \epsilon_2, \ldots \epsilon_{n-1}\}$ be the set of n^{th} roots of unity, where $\epsilon_k = \cos(2k\pi/n) + i\sin(2k\pi/n)$, $k \in \{0, 1, 2, \ldots, n-1\}$. Prove the following :

(i) Prove that $\epsilon_j . \epsilon_k \in U_n$ for all $j, k \in \{0, 1, 2, \ldots, n-1\}$

(ii) $\epsilon_j^{-1} \in U_n$ for all $j \in \{0, 1, 2, \ldots, n-1\}$ (4+3.5)

(b) Let a be an integer, show that there exists an integer k such that

$a^2 = 3k$ or $a^2 = 3k + 1$. (7.5)

(c) (i) Prove that gcd(n, n+1) = 1 for every natural number n. Find integers x and y such that $n.x + (n+1).y = 1$.

(ii) Let a, b and c be three natural numbers such that gcd(a, c) = 1 and b divides c. Prove that gcd(a, b) = 1. (4+3.5)

4. (a) Let $n > 1$ be a fixed natural number. Let a, b, c be three integers such that $ac \equiv bc \pmod{n}$ and gcd(c, n) = 1. Prove that $a \equiv b \pmod{n}$. (7.5)

(b) Solve the congruence, $7x \equiv 8 \pmod{11}$. (7.5)

(c) Solve the following pair of congruences:

$2x + 3y \equiv 1 \pmod{6}$

$x + 3y \equiv 5 \pmod{6}$ (7.5)

5. (a) Let G be the set of all 2×2 real matrices with non-zero determinant. Show that G is a group under the operation of matrix multiplication. Further show that it is not an Abelian Group. (7.5)

P.T.O.

(b) Let G be a group such that for any x, y, z in the group, xy = zx implies y = z (called left-fight cancellation property). Show that G is Abelian. Give an Example of a non-abelian group in which left-right cancellation property does not hold. (7.5)

(c) Show that the set G = {1, 5, 7, 11} is a group under multiplication modulo 12 with the help of the Cayley table. (7.5)

6. (a) Show that for any integer n, the set $H_n = \{n.x \mid x \in Z\}$ is a subgroup of the group Z of integers under the operation of addition. Further show that $H_2 \cup H_3$ is not a subgroup of Z. (5.5+2)

(b) Let G be a group. Show that $|aba^{-1}| = |b|$ for all a and b in G (|x| denotes the order of an element x in G). (7.5)

(c) Show that the group $Z_n = \{0, 1, 2, \ldots, n-1\}$ is cyclic under the operation of addition modulo n. How many generators Z_n have? Further, describe all the subgroups of Z_{40}. (2+1+4.5)