Physics Examination Solutions

Question 1

(a) Simple Harmonic Motion and Superposition Principle (5 marks)

- **Definition:** Simple Harmonic Motion (SHM) is a periodic motion where the restoring force is directly proportional to the displacement from equilibrium and acts in the opposite direction. Mathematically:

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

where ω is the angular frequency of oscillation.

- **Superposition Principle Proof:** For a homogeneous linear differential equation $L(y) = 0$, if $y_1(t)$ and $y_2(t)$ are solutions, then any linear combination $y(t) = c_1 y_1(t) + c_2 y_2(t)$ is also a solution:

$$L(y) = L(c_1 y_1 + c_2 y_2) = c_1 L(y_1) + c_2 L(y_2) = c_1 \cdot 0 + c_2 \cdot 0 = 0$$

This proves the superposition principle holds for homogeneous linear equations governing SHM.

(b) Transverse Oscillation Frequency (5 marks)

For a mass m connected to two identical springs (spring constant k each) in the configuration shown:

- The restoring force for small transverse displacement y is $F = -2ky$
- Applying Newton's second law:

$$m\frac{d^2y}{dt^2} = -2ky$$

- The equation of motion becomes:

$$\frac{d^2y}{dt^2} + \frac{2k}{m}y = 0$$

- Comparing with standard SHM equation $\ddot{y} + \omega^2 y = 0$, we get:

$$\omega = \sqrt{\frac{2k}{m}}$$

(c) Damped Harmonic Oscillator Frequency (5 marks)

Given the differential equation:

$$8\frac{d^2y}{dt^2} + 24\frac{dy}{dt} + 48y = 0$$

- Divide by 8 to get standard form:

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 6y = 0$$

- Compare with general damped oscillator equation:

$$\ddot{y} + 2\beta\dot{y} + \omega_0^2 y = 0$$

where:

$$2\beta = 3 \Rightarrow \beta = 1.5\,\mathrm{s}^{-1}$$
$$\omega_0^2 = 6 \Rightarrow \omega_0 = \sqrt{6}\,\mathrm{rad/s}$$

- Frequency of damped oscillations:

$$\omega' = \sqrt{\omega_0^2 - \beta^2} = \sqrt{6 - (1.5)^2} = \sqrt{3.75} \approx 1.94\,\mathrm{rad/s}$$

(d) Logarithmic Damping Calculations (5 marks)

Given parameters:

- Initial amplitude $A_0 = 20\,\mathrm{cm}$
- Final amplitude $A = 2\,\mathrm{cm}$ after 100 oscillations
- Time period $T = 4.6\,\mathrm{s}$
- Amplitude ratio gives:

$$\frac{A_0}{A} = e^{n\beta T} \Rightarrow \ln(10) = 100 \times \beta \times 4.6$$

$$\beta = \frac{\ln(10)}{460} \approx 0.005\,\mathrm{s}^{-1}$$

- Logarithmic decrement:

$$\Lambda = \beta T \approx 0.023$$

- For 50% amplitude reduction:

$$\ln(2) = n \times 0.023 \Rightarrow n \approx 30\,\text{oscillations}$$

(e) Stationary vs Progressive Waves (5 marks)

Stationary Waves	**Progressive Waves**
Formed by interference of two identical waves	Single wave propagating through medium
Energy remains confined between nodes	Energy propagates with the wave
Nodes (zero amplitude) and antinodes (max amplitude)	No nodes, uniform amplitude
Particles between nodes vibrate in phase	Phase changes continuously

Question 2

(a) Spring Force Constants (5 marks)

For a spring cut into lengths l_1 and l_2 with $l_1/l_2 = n_1/n_2$:

- Force constant k is inversely proportional to length:

$$k \propto \frac{1}{l}$$

- New force constants:

$$k_1 = k\frac{l}{l_1} = k\frac{n_1 + n_2}{n_1}$$
$$k_2 = k\frac{l}{l_2} = k\frac{n_1 + n_2}{n_2}$$

(b) Harmonic Oscillator Energy (5 marks)

For an oscillator with mass m, amplitude a, and frequency ω:

- Total energy:

$$E = \frac{1}{2}m\omega^2 a^2$$

- Potential energy:

$$U = \frac{1}{2}m\omega^2 x^2$$

- Kinetic energy:

$$K = \frac{1}{2}m\omega^2(a^2 - x^2)$$

- When $K = U$:

$$\frac{1}{2}m\omega^2 x^2 = \frac{1}{2}m\omega^2(a^2 - x^2)$$
$$x = \pm\frac{a}{\sqrt{2}}$$

(c) RLC Circuit Analysis (5 marks)

Given parameters:

- $V_{pp} = 40\,\text{V} \Rightarrow V_0 = 20\,\text{V}$
- $L = 100\,\text{mH}$, $C = 1\,\mu\text{F}$, $R = 100\,\Omega$
- Resonant frequency:
$$\omega_0 = \frac{1}{\sqrt{LC}} = 3162\,\text{rad/s}$$
- Maximum current:
$$I_{max} = \frac{V_0}{R} = 0.2\,\text{A}$$
- Bandwidth:
$$\Delta\omega = \frac{R}{L} = 1000\,\text{rad/s}$$
- Quality factor:
$$Q = \frac{\omega_0}{\Delta\omega} \approx 3.16$$

Question 3

(a) Beats Formation (10 marks)

When two SHMs with nearly equal frequencies $\omega_1 \approx \omega_2$ and different amplitudes are superimposed:

- Resultant displacement:
$$x(t) = A_1 \cos\omega_1 t + A_2 \cos\omega_2 t$$
- Using trigonometric identity:
$$x(t) = 2A\cos\left(\frac{\omega_1 - \omega_2}{2}t\right)\cos\left(\frac{\omega_1 + \omega_2}{2}t\right)$$
where A is the average amplitude.
- The term $\cos\left(\frac{\omega_1-\omega_2}{2}t\right)$ causes amplitude modulation (beats)
- Beat frequency:
$$f_{beat} = |f_1 - f_2| = \frac{|\omega_1 - \omega_2|}{2\pi}$$
- Physical interpretation: The constructive and destructive interference of the two waves creates periodic variations in amplitude heard as beats.

(b) Lissajous Figure (5 marks)

Given perpendicular SHMs:

$$x = a\cos\pi t$$
$$y = a\cos\left(\frac{\pi t}{2}\right)$$

- Using trigonometric identity:

$$\cos\theta = 2\cos^2\left(\frac{\theta}{2}\right) - 1$$

$$y = a\left[2\cos^2\left(\frac{\pi t}{2}\right) - 1\right]$$

- Substitute x:

$$y = \frac{a}{2}(1 + \cos\pi t)$$
$$2y^2 = a^2(1 + \cos\pi t) = a^2 + a^2\cos\pi t = a^2 + ax$$

- The resulting trajectory is a parabola in the xy-plane.

Question 4

(a) Power in Forced Damped Oscillator (10 marks)

For a forced damped harmonic oscillator in steady state:

- Driving force: $F(t) = F_0\cos\omega t$
- Displacement: $x(t) = A\cos(\omega t - \phi)$
- Velocity: $v(t) = -\omega A\sin(\omega t - \phi)$
- Instantaneous power:

$$P(t) = F(t)v(t) = -F_0\omega A\cos\omega t\sin(\omega t - \phi)$$

- Average power over one period:

$$P_{avg} = \frac{1}{T}\int_0^T P(t)dt = \frac{1}{2}F_0\omega A\sin\phi$$

- Power dissipated by damping:

$$P_{diss} = bv^2 = b\omega^2A^2\sin^2(\omega t - \phi)$$

$$P_{diss,avg} = \frac{1}{2}b\omega^2A^2$$

- At steady state, the average input power equals average dissipated power:

$$\frac{1}{2}F_0\omega A\sin\phi = \frac{1}{2}b\omega^2 A^2$$

$$\Rightarrow F_0\sin\phi = b\omega A$$

which matches the known phase relationship for forced oscillations.

(b) Forced Oscillation Amplitude (5 marks)

Given parameters:

- $m = 0.1\,\text{kg}$, $k = 100\,\text{N/m}$
- Damping force $F_d = 5v \Rightarrow b = 5\,\text{Ns/m}$
- Driving force $F = 2\cos 20t \Rightarrow F_0 = 2\,\text{N}$, $\omega = 20\,\text{rad/s}$
- Natural frequency:

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{1000} \approx 31.62\,\text{rad/s}$$

- Amplitude formula:

$$A = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (b\omega/m)^2}}$$

$$A = \frac{20}{\sqrt{(1000-400)^2 + (5\times 20/0.1)^2}}$$

$$A = \frac{20}{\sqrt{360000+10000}} \approx 0.0033\,\text{m} = 3.3\,\text{mm}$$

Question 5

(a) Normal Modes of Vibrating String (10 marks)

For a string of length $3a$ with masses m at a and $2a$:

- **Transverse modes:**
  1. Symmetric mode (both masses move in same direction):

$$\omega_1 = \sqrt{\frac{T}{ma}}$$

  2. Antisymmetric mode (masses move in opposite directions):

$$\omega_2 = \sqrt{\frac{3T}{ma}}$$

- **Longitudinal modes:**
  1. Symmetric compression/extension:
  $$\omega_1' = \sqrt{\frac{2k}{m}}$$
  2. Antisymmetric mode:
  $$\omega_2' = \sqrt{\frac{6k}{m}}$$
- The normal modes are found by solving the coupled differential equations for small oscillations about equilibrium.

(b) Classical Wave Equation Derivation (5 marks)

Consider a general wave function $\psi(x,t)$:

- Second time derivative:
$$\frac{\partial^2\psi}{\partial t^2} = -\omega^2 A\cos(kx - \omega t)$$
- Second space derivative:
$$\frac{\partial^2\psi}{\partial x^2} = -k^2 A\cos(kx - \omega t)$$
- Using $v = \omega/k$:
$$\frac{\partial^2\psi}{\partial t^2} = v^2\frac{\partial^2\psi}{\partial x^2}$$
- This is the classical wave equation describing propagation of disturbances with speed v.