Analysis Problems and Solutions

Problem 1

(a)

This part proves that if a non-negative real number a is less than any positive ϵ, then a must be zero.

Proof. **Proof by Contradiction**: The proof assumes the opposite, i.e., $a > 0$.

Choosing ϵ: If $a > 0$, then ϵ can be chosen as $a/2$, which is also positive.

Contradiction: By the initial hypothesis ($a < \epsilon$), it would follow that $a < a/2$, which simplifies to $1 < 1/2$. This is a false statement, thus creating a contradiction.

Conclusion: Since the assumption ($a > 0$) leads to a contradiction, it must be false. Therefore, a must be 0. □

(b)

This section solves the inequality $|x - 1| > |x + 1|$ by considering different cases based on the values of x that make the expressions inside the absolute values change signs.

Case 1: $x \geq 1$

- In this case, both $(x - 1)$ and $(x + 1)$ are non-negative.
- The inequality becomes $x - 1 > x + 1$, which simplifies to $-1 > 1$. This is a false statement, meaning there are no solutions in this range.

Case 2: $-1 \leq x < 1$

- In this case, $(x - 1)$ is negative, so $|x - 1| = -(x - 1) = 1 - x$.
- $(x + 1)$ is non-negative, so $|x + 1| = x + 1$.
- The inequality becomes $1 - x > x + 1$, which simplifies to $-2x > 0$, and further to $x < 0$.
- Combining this with the case condition ($-1 \leq x < 1$), the solution for this case is $-1 \leq x < 0$.

Case 3: $x < -1$

- In this case, both $(x-1)$ and $(x+1)$ are negative.
- The inequality becomes $-(x-1) > -(x+1)$, which is $1-x > -x-1$. This simplifies to $1 > -1$. This statement is always true.
- Combining this with the case condition ($x < -1$), the solution for this case is $x < -1$.

Overall Solution: Combining the valid ranges from all cases ($x < 0$ from Case 2 and $x < -1$ from Case 3), the complete solution is $x < 0$.

(c)

This part finds the supremum (sup) and infimum (inf) for two sets.

(i) For $\{\cos(\frac{n\pi}{2}) : n \in \mathbb{N}\}$ **Values**: By substituting $n = 1, 2, 3, 4, 5, \ldots$, the sequence of values is $\cos(\pi/2) = 0$, $\cos(\pi) = -1$, $\cos(3\pi/2) = 0$, $\cos(2\pi) = 1$, $\cos(5\pi/2) = 0$, $\cos(3\pi) = -1$, and so on.

Supremum: The largest value that the set approaches or contains is 1.

Infimum: The smallest value that the set approaches or contains is -1.

(ii) For $\{\frac{x+2}{3} : x > 3\}$ **Lower Bound**: If $x > 3$, then $x + 2 > 5$, so $\frac{x+2}{3} > \frac{5}{3}$.

Infimum: The infimum is $\frac{5}{3}$.

Supremum: As x can be arbitrarily large (since $x > 3$), the values of $\frac{x+2}{3}$ can also be arbitrarily large. Therefore, the supremum does not exist.

(d)

This part proves that 1 is the least upper bound (supremum) of the set $S = \{1 - \frac{1}{n} : n \in \mathbb{N}\}$.

Proof. **1 as an Upper Bound**: As $n \in \mathbb{N}$, $\frac{1}{n}$ is always positive, so $1 - \frac{1}{n}$ will always be less than 1. Thus, 1 is clearly an upper bound.

1 as the Least Upper Bound: To show it's the least upper bound:

- Choose $n > 1/\epsilon$ (which is always possible due to the Archimedean property).
- Then, $\frac{1}{n} < \epsilon$.
- This implies $-\frac{1}{n} > -\epsilon$.
- Adding 1 to both sides gives $1 - \frac{1}{n} > 1 - \epsilon$.
- Since $1 - \frac{1}{n}$ is an element of S, this shows that for any $\epsilon > 0$, there's an element in S larger than $1 - \epsilon$, proving 1 is the least upper bound.

$\square$

Problem 2

(a)

This part proves the property that $\inf S = -\sup\{-s : s \in S\}$.

Proof. **Definition of Infimum**: Let $\alpha = \inf S$. This means $\alpha \leq s$ for all $s \in S$.

Relationship with $-s$: From $\alpha \leq s$, it follows that $-\alpha \geq -s$ for all $s \in S$. This implies that $-\alpha$ is an upper bound for the set $\{-s : s \in S\}$.

Definition of Supremum: Let $\beta = \sup\{-s : s \in S\}$. Since $-\alpha$ is an upper bound for $\{-s\}$, by the definition of supremum, $\beta \leq -\alpha$.

Proof by Contradiction for Equality: Suppose for contradiction that $\beta < -\alpha$.

- If $\beta < -\alpha$, then it means that $\alpha < -\beta$.
- Since β is the supremum of $\{-s\}$, for any $\epsilon' > 0$, there exists a $-s_0 \in \{-s\}$ such that $\beta - \epsilon' < -s_0$.
- This implies $s_0 < -\beta + \epsilon'$.
- If we choose ϵ' small enough such that $-\beta + \epsilon' < \alpha$, then $s_0 < \alpha$, which would contradict α being the lower bound for S.

Conclusion: Therefore, β must be equal to $-\alpha$, so $\inf S = -\sup\{-s : s \in S\}$. □

(b)

Archimedean Property

Theorem 1. *For any real number x, there exists a natural number n such that $n > x$.*

Proof. **Proof by Contradiction**:

- Assume the opposite: $\mathbb{N}$ is bounded above by some real number x.
- **Completeness Property**: By the completeness property of real numbers, if a non-empty set is bounded above, it has a supremum (least upper bound). Let this supremum be s.
- **Contradiction Derivation**: Since s is the least upper bound, $s - 1$ cannot be an upper bound (as it's smaller than the least upper bound).
- Therefore, there must exist some natural number $m \in \mathbb{N}$ such that $m > s - 1$.
- Rearranging this, we get $m + 1 > s$.
- However, since $m \in \mathbb{N}$, then $m + 1$ is also a natural number. This means we found a natural number $(m+1)$ that is greater than s, which contradicts our initial assumption that s is an upper bound for $\mathbb{N}$ (and specifically the least upper bound).

Conclusion: The initial assumption that $\mathbb{N}$ is bounded above must be false. Hence, the Archimedean property is true. □

(c)

This part finds the infimum and supremum for the set $S = \{\frac{1}{n} - \frac{1}{m} : n, m \in \mathbb{N}\}$.

Infimum: To make $\frac{1}{n} - \frac{1}{m}$ as small as possible:

- We want $\frac{1}{n}$ to be small, so n should be large (approaching infinity). In the limit, $\frac{1}{n} \to 0$.
- We want $\frac{1}{m}$ to be large, so m should be small. The smallest natural number for m is 1.
- Thus, the infimum is approximately $0 - \frac{1}{1} = -1$.

Supremum: To make $\frac{1}{n} - \frac{1}{m}$ as large as possible:

- We want $\frac{1}{n}$ to be large, so n should be small. The smallest natural number for n is 1.
- We want $\frac{1}{m}$ to be small, so m should be large (approaching infinity). In the limit, $\frac{1}{m} \to 0$.
- Thus, the supremum is approximately $\frac{1}{1} - 0 = 1$.

(d)

Convergence of a Sequence and Uniqueness of Limit.

Theorem 2. *The limit of a convergent sequence is unique.*

Proof. **Definition of Convergence**: A sequence (x_n) converges to a limit L if for every $\epsilon > 0$, there exists a natural number N such that for all $n \geq N$, the absolute difference $|x_n - L|$ is less than ϵ.

Uniqueness Proof:

- **Assumption**: Assume that a sequence (x_n) converges to two different limits, L and L', where $L \neq L'$.
- **Choosing** ϵ: Let $\epsilon = \frac{|L-L'|}{2} > 0$.
- **Applying Definition of Convergence**:
  - For $\epsilon/2$, exists N_1 such that for all $n \geq N_1$, $|x_n - L| < \epsilon/2$.
  - For $\epsilon/2$, exists N_2 such that for all $n \geq N_2$, $|x_n - L'| < \epsilon/2$.
- **Triangle Inequality**: For any $n \geq \max\{N_1, N_2\}$: $|L - L'| = |L - x_n + x_n - L'| \leq |L - x_n| + |x_n - L'| < \epsilon/2 + \epsilon/2 = \epsilon$.
- But this contradicts our choice of $\epsilon = |L - L'|/2$ since $|L - L'| < |L - L'|/2$ implies $1 < 1/2$.

Conclusion: Therefore, the limit of a convergent sequence must be unique. $\square$

Problem 3

(a)

This part involves proving the limit of a sequence using the $\epsilon - N$ definition.

The Sequence and Proposed Limit: $x_n = \frac{2n+3}{3n-7}$, limit is $\frac{2}{3}$.

Proof. We need to show that for any $\epsilon > 0$, there exists an N such that if $n \geq N$, then $\left|\frac{2n+3}{3n-7} - \frac{2}{3}\right| < \epsilon$.

- Simplify the expression: $\left|\frac{2n+3}{3n-7} - \frac{2}{3}\right| = \left|\frac{23}{3(3n-7)}\right| = \frac{23}{3(3n-7)}$ (for $n \geq 3$)
- We want $\frac{23}{3(3n-7)} < \epsilon$.
- Solve for n: $\frac{23}{3\epsilon} < 3n - 7 \Rightarrow n > \frac{23}{9\epsilon} + \frac{7}{3}$.

Choosing N: Let N be any integer greater than $\frac{23}{9\epsilon} + \frac{7}{3}$. Then for all $n \geq N$, the inequality holds. □

(b)

This part proves the limit of $n^{1/n}$ as $n \to \infty$.

Proof. **Rewriting the Expression**: $n^{1/n} = e^{\ln(n^{1/n})} = e^{\frac{\ln n}{n}}$.

Limit of the Exponent: $\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1/n}{1} = 0$ (by L'Hôpital's Rule).

Conclusion: Since e^x is continuous, $\lim_{n\to\infty} n^{1/n} = e^{\lim_{n\to\infty} \frac{\ln n}{n}} = e^0 = 1$. □

(c)

Sandwich Theorem (Squeeze Theorem).

Theorem 3. *If $a_n \leq b_n \leq c_n$ for all $n \geq N_0$, and $\lim a_n = \lim c_n = L$, then $\lim b_n = L$.*

Proof. For any $\epsilon > 0$, there exists:

- N_1 such that for $n \geq N_1$, $L - \epsilon < a_n < L + \epsilon$
- N_2 such that for $n \geq N_2$, $L - \epsilon < c_n < L + \epsilon$

Let $N = \max\{N_0, N_1, N_2\}$. Then for $n \geq N$: $L-\epsilon < a_n \leq b_n \leq c_n < L+\epsilon$, so $|b_n - L| < \epsilon$. □

(d)

Monotone Convergence Theorem for Increasing Sequences.

Theorem 4. *If (x_n) is increasing and bounded above, then it converges.*

Proof. Let $L = \sup\{x_n\}$. For any $\epsilon > 0$, exists $x_N > L - \epsilon$ (since L is least upper bound). By monotonicity, for all $n \geq N$: $L - \epsilon < x_N \leq x_n \leq L < L + \epsilon$, so $|x_n - L| < \epsilon$. □

Problem 4

(a)

Analyze the convergence of $x_n = \sqrt{2 + x_{n-1}}$ with $x_1 = \sqrt{2}$.

Monotonicity and Boundedness:

- By induction, $x_n < 2$ for all n:
  - Base case: $x_1 = \sqrt{2} < 2$
  - If $x_k < 2$, then $x_{k+1} = \sqrt{2 + x_k} < \sqrt{4} = 2$
- Sequence is increasing:
  - $x_1 = \sqrt{2} \approx 1.414$
  - $x_2 = \sqrt{2 + \sqrt{2}} \approx 1.847 > x_1$
  - If $x_k > x_{k-1}$, then $x_{k+1} = \sqrt{2 + x_k} > \sqrt{2 + x_{k-1}} = x_k$

Convergence: By Monotone Convergence Theorem, the sequence converges to L satisfying: $L = \sqrt{2 + L} \Rightarrow L^2 = 2 + L \Rightarrow (L - 2)(L + 1) = 0$.

Since $L \geq \sqrt{2} > 0$, we have $L = 2$.

(b)

Prove that Cauchy sequences are bounded.

Proof. Take $\epsilon = 1$ in the Cauchy definition. There exists N such that for all $n, m \geq N$, $|x_n - x_m| < 1$.

Fix $m = N$. Then for all $n \geq N$: $|x_n| \leq |x_n - x_N| + |x_N| < 1 + |x_N|$.

Let $M = \max\{|x_1|, |x_2|, ..., |x_{N-1}|, 1 + |x_N|\}$. Then $|x_n| \leq M$ for all n. □

(c)

Prove the divergence of the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$.

Proof. **Integral Test Proof**: Compare to $\int_1^\infty \frac{1}{x} dx = \lim_{b\to\infty} \ln b = \infty$.

Since the integral diverges and the terms are positive decreasing, the series diverges.

Grouping Terms Proof:

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \cdots$$
$$> 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots \to \infty$$

□

(d)

Find lim sup and lim inf for:

(i) $x_n = (-2)^n(1 + \frac{1}{n})$ For even $n = 2k$: $x_{2k} = 4^k(1 + \frac{1}{2k}) \to +\infty$
For odd $n = 2k - 1$: $x_{2k-1} = -2 \cdot 4^{k-1}(1 + \frac{1}{2k-1}) \to -\infty$
$\limsup x_n = +\infty$, $\liminf x_n = -\infty$

(ii) $x_n = (-1)^n \frac{1}{n}$ For even n: $x_n = \frac{1}{n} \to 0$
For odd n: $x_n = -\frac{1}{n} \to 0$
$\limsup x_n = \liminf x_n = 0$

Problem 5

(a)

Geometric Series $\sum_{n=0}^{\infty} ar^n$.
Partial Sums: $S_n = a\frac{1-r^n}{1-r}$ for $r \neq 1$.
Convergence:

- If $|r| < 1$, converges to $\frac{a}{1-r}$ (since $r^n \to 0$)
- If $|r| \geq 1$:
  - $r = 1$: $S_n = na \to \infty$ (unless $a = 0$)
  - $r = -1$: Oscillates, diverges (unless $a = 0$)
  - $|r| > 1$: $|r^n| \to \infty$, terms don't tend to zero

(b)

Telescoping Series $\sum_{n=1}^{\infty} \frac{1}{(n+a)(n+a+1)}$.
Partial Fractions: $\frac{1}{(n+a)(n+a+1)} = \frac{1}{n+a} - \frac{1}{n+a+1}$
Partial Sum: $S_N = \left(\frac{1}{1+a} - \frac{1}{2+a}\right) + \cdots + \left(\frac{1}{N+a} - \frac{1}{N+a+1}\right) = \frac{1}{1+a} - \frac{1}{N+a+1}$
Sum: As $N \to \infty$, $S_N \to \frac{1}{1+a}$.

(c)

Convert repeating decimal $0.\overline{15}$ to fraction.
Method 1 (Algebra):

$$x = 0.\overline{15}$$
$$100x = 15.\overline{15}$$
$$100x - x = 15 \Rightarrow x = \frac{15}{99} = \frac{5}{33}$$

Method 2 (Series): $0.\overline{15} = \frac{15}{100} + \frac{15}{100^2} + \cdots = \frac{15/100}{1-1/100} = \frac{15}{99} = \frac{5}{33}$

(d)

Test convergence:

(i) $\sum \frac{1}{\log n}$ For $n \geq 2$, $\log n < n \Rightarrow \frac{1}{\log n} > \frac{1}{n}$.
Since $\sum \frac{1}{n}$ diverges, by comparison $\sum \frac{1}{\log n}$ diverges.

(ii) $\sum \tan^{-1}(\frac{1}{n})$ Limit compare to $\sum \frac{1}{n}$:
$\lim_{n\to\infty} \frac{\tan^{-1}(1/n)}{1/n} = \lim_{x\to 0} \frac{\tan^{-1} x}{x} = 1$ (by L'Hôpital's).
Since $\sum \frac{1}{n}$ diverges, $\sum \tan^{-1}(\frac{1}{n})$ diverges.

Problem 6

(a)

Ratio Test: For $\sum a_n$, let $L = \lim |\frac{a_{n+1}}{a_n}|$.

- If $L < 1$: converges absolutely
- If $L > 1$: diverges
- If $L = 1$: inconclusive

(i) $\sum \frac{n!}{n^n}$ $\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{n^n}{(n+1)^n} = \frac{1}{(1+1/n)^n} \to \frac{1}{e} < 1$
Thus converges.

(ii) $\sum \frac{n!}{e^n}$ $\frac{a_{n+1}}{a_n} = \frac{n+1}{e} \to \infty > 1$
Thus diverges.

(b)

Test convergence:

(i) $\sum \frac{\log n}{n^2}$ For large n, $\log n < n^{0.5}$, so $\frac{\log n}{n^2} < \frac{1}{n^{1.5}}$.
$\sum \frac{1}{n^{1.5}}$ converges ($p = 1.5 > 1$), so by comparison, original converges.

(ii) $\sum \frac{-n^{n^2}}{(n+1)^{n^2}}$ Consider absolute convergence:
$|a_n|^{1/n} = \left(\frac{n}{n+1}\right)^n = \frac{1}{(1+1/n)^n} \to \frac{1}{e} < 1$
By Root Test, converges absolutely.

(c)

Absolute Convergence.
Definition: $\sum a_n$ converges absolutely if $\sum |a_n|$ converges.
Implications:

- Absolute convergence $\Rightarrow$ convergence
- Converse false: e.g., alternating harmonic series $\sum(-1)^n/n$ converges but not absolutely

(d)

Classify convergence:

(i) $\sum(-1)^{n+1}\frac{n}{n^2+1}$ **Alternating Series Test**:

- $b_n = \frac{n}{n^2+1} > 0$
- Decreasing for $n \geq 1$ (check derivative)
- $b_n \to 0$

Thus converges.
Absolute Convergence: Compare $\sum \frac{n}{n^2+1}$ to $\sum \frac{1}{n}$ (diverges), so converges conditionally.

(ii) $\sum(-1)^n\frac{1}{n^2+(-1)^n}$ **Absolute Convergence**: Compare $\sum \frac{1}{n^2+(-1)^n}$ to $\sum \frac{1}{n^2}$ (converges), so converges absolutely.