1. (a) Let $a \geq 0$, $b \geq 0$ prove that $a^2 \leq b^2 \Leftrightarrow a \leq b$.

(b) Determine and sketch the set of pairs (x, y) on $\mathbb{R} \times \mathbb{R}$ satisfying the inequality $|x| \leq |y|$.

(c) Find the supremum and infimum, if they exist, of the following sets :

(i) $\left\{\sin\frac{n\pi}{2} : n \in \mathbb{N}\right\}$

(ii) $\left\{\left(\frac{1}{x} : x > 0\right)\right\}$

(d) Show that Sup $\left\{1+\frac{1}{n} : n \in \mathbb{N}\right\} = 2$.

2. (a) Let S be a non-empty bounded subset of $\mathbb{R}$. Let $a > 0$ and let $aS = \{as : s \in S\}$. Prove that

$$\text{Sup }(aS) = a(\text{ Sup } S)$$

(b) If x and y are positive rational numbers with $x <$ then show that there exists a rational number such that $x < r < y$.

(c) Show that $\inf\left\{\frac{1}{n} : n \in \mathbb{N}\right\} = 0$.

(d) Show that every convergent sequence is bounded. Is the converse true? Justify.

3. (a) Using definition of limit, show that

$$\lim_{n\to\infty} \frac{n^2+3n+5}{2n^2+5n+7} = \frac{1}{2}$$

(b) Show that if $c > 0$, $\lim\limits_{n\to\infty} (c)^{1/n} = 1$.

(c) Show that, if $x_n \geq 0$ for all n, and $\langle x_n \rangle$ is convergent then $\langle \sqrt{x_n} \rangle$ is also convergent and

$$\lim_{n\to\infty} \sqrt{x_n} = \sqrt{\lim_{n\to\infty} x_n}$$

(d) Show that every increasing sequence which is bounded above is convergent.

4. (a) Let $x_1 = 1$ and $x_{n+1} = \sqrt{2x_n}$ for all n. Prove that $\langle x_n \rangle$ is convergent and find its limit.

(b) Prove that every Cauchy sequence is convergent.

(c) Show that the sequence $\langle x_n \rangle$ defined by

$$x_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}, \text{ for all } n \in \mathbb{N}$$

is convergent.

(d) Find the limit superior and limit inferior of the following sequences :

(i) $x_n = (-1)^n \left(1+\frac{1}{n}\right)$, for all $n \in \mathbb{N}$

(ii) $x_n = \left(1+\frac{1}{n}\right)^{n+1}$, for all $n \in \mathbb{N}$

5. (a) Show that if a series Σa_n converges, then the sequence $\langle a_n \rangle$ converges to 0.

(b) Determine, if the following series converges, using the definition of convergence, $\sum \log\left(\frac{a_n}{a_{n+1}}\right)$ given that $a_n > 0$ for each n, $\lim\limits_{n\to\infty} a_n = a$, $a > 0$.

(c) Find the rational number which is the sum of the series represented by the repeating decimal $0.\dot{9}8\dot{7}$.

(d) Check the convergence of the following series :

(i) $\sum \frac{1}{2^n + n}$

(ii) $\sum \sin\left(\frac{1}{n^2}\right)$

6. (a) State the Root Test (limit form) for positive series. Using this test or otherwise, check the convergence of the following series

(i) $\sum \left(n^{1/n} - 1\right)^n$

(ii) $\sum \left(\frac{n^{n^2}}{(n+1)^{n^2}}\right)$

(b) Check the convergence of the following series :

(i) $\sum_{n=2}^{\infty} \left(\frac{1}{n \log n}\right)$

(ii) $\sum \left(\frac{n!}{n^n}\right)$

(c) Define absolute convergence of a series. Show that every absolutely convergent series is convergent. Is the converse true? Justify your answer.

(d) Check the following series for absolute or conditional convergence :

(i) $\sum (-1)^{n+1} \left(\frac{n}{n(n+3)}\right)$

(ii) $\sum (-1)^{n+1} \left(\frac{1}{n+1}\right)$