Questions

Problem 1: Polynomial Roots & Vieta's Formulas

(a) Given that one root of $x^3 + px^2 + qx + r = 0$ is the negative of another, prove that $r = pq$.

(b) Find all roots of $x^4 - 2x^3 - 21x^2 + 22x + 40 = 0$ given that they are in arithmetic progression.

(c) How would the roots change if the constant term were 80 instead of 40? Explain your reasoning.

(5+5+2 marks)

Problem 2: Complex Numbers

(a) Express $z = \sin a + i(1 + \cos a)$ in polar form.

(b) For the complex number:

$$z = \frac{(2\sqrt{3} + 2i)^8}{(1-i)^6} + \frac{(1+i)^6}{(2\sqrt{3} - 2i)^8}$$

compute $|z|$ and $\arg z$.

(c) Sketch the region in the complex plane defined by:

$$\{z \in \mathbb{C} \mid |z + 1 + i| < 3 \text{ and } 0 < \arg z < \frac{\pi}{6}\}$$

(4+6+4 marks)

Problem 3: Modular Arithmetic

(a) Solve $7x \equiv 8 \pmod{11}$.

(b) Prove: If $ac \equiv bc \pmod{n}$ and $\gcd(c, n) = 1$, then $a \equiv b \pmod{n}$.

(c) Solve the system of congruences:

$$\begin{cases} 2x + 3y \equiv 1 \pmod{6} \\ x + 3y \equiv 5 \pmod{6} \end{cases}$$

(3+4+5 marks)

Problem 4: Group Theory

(a) Show that $G = \{1, 5, 7, 11\}$ forms a group under multiplication modulo 12.

(b) Prove that for any group G and elements $a, b \in G$, $|aba^{-1}| = |b|$.

(c) Let $H_n = \{nx \mid x \in \mathbb{Z}\}$. Determine whether $H_2 \cup H_3$ is a subgroup of $\mathbb{Z}$.

(5+5+4 marks)

Problem 5: Roots of Unity & Cyclic Groups

(a) Let ε_j and ε_k be n-th roots of unity. Prove that:
   - $\varepsilon_j \cdot \varepsilon_k$ is an n-th root of unity
   - ε_j^{-1} is an n-th root of unity

(b) Prove that $\mathbb{Z}_n$ is cyclic under addition modulo n.

(c) Find all subgroups of $\mathbb{Z}_{48}$.

(4+4+6 marks)

Bonus Problem (Optional)

Problem 6: Using Bézout's Identity, prove that if $\gcd(a, b) = 1$ and $b \mid c$, then $\gcd(a, c) = 1$. **(8 marks)**