Elementary Real Analysis Exam

Mathematics (Honours)

Questions

Question 1. Supremum and Infimum

(a) Prove that for a non-empty set $S \subseteq \mathbb{R}$ bounded above, $\sup(aS) = a \cdot \sup(S)$ for $a > 0$.

(b) Find the supremum and infimum of the set $\left\{\cos\left(\frac{n\pi}{2}\right) : n \in \mathbb{N}\right\}$. Justify your answer.

(c) Show that the infimum of $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ is 0.

[Total: 15 marks]

Question 2. Sequences and Limits

(a) Using the ϵ-N definition, prove that $\lim_{n\to\infty} \frac{2n+3}{3n-7} = \frac{2}{3}$.

(b) Show that $\lim_{n\to\infty} n^{1/n} = 1$ using the binomial theorem or squeeze theorem.

(c) Prove that every convergent sequence is bounded. Provide an example of a bounded sequence that does not converge.

[Total: 20 marks]

Question 3. Series Convergence

(a) State and prove the Ratio Test for series convergence. Apply it to determine whether $\sum \frac{n!}{n^n}$ converges.

(b) Show that the alternating harmonic series $\sum(-1)^{n+1}\frac{1}{n}$ converges conditionally but not absolutely.

(c) Find the sum of the telescoping series $\sum_{n=1}^{\infty} \left(\frac{1}{n+a} - \frac{1}{n+a+1}\right)$.

[Total: 25 marks]

Question 4. Cauchy Sequences and Completeness

(a) Prove that every Cauchy sequence in $\mathbb{R}$ is convergent.

(b) Show that the sequence $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is not Cauchy (and hence diverges).

(c) State the Monotone Convergence Theorem and use it to prove that the recursive sequence $x_1 = 1$, $x_{n+1} = \sqrt{2 + x_n}$ converges to 2.

[Total: 25 marks]

Question 5. Advanced Limit Theorems

(a) Prove that if $x_n > 0$ and $\lim_{n\to\infty} x_n = L > 0$, then $\lim_{n\to\infty} \sqrt[n]{x_n} = 1$.

(b) Investigate the convergence of:

- $\sum \frac{\log n}{n^2}$ (use comparison test)
- $\sum (-1)^n \frac{1}{n^2 + (-1)^n}$ (check absolute/conditional convergence)

(c) Define lim sup and lim inf of a sequence. Compute these for $x_n = (-1)^n \left(1 - \frac{1}{n}\right)$.

[Total: 15 marks]