1. (a) Prove that the order of a permutation of a finite set written in disjoint cycle form is the least common multiple of the lengths of the cycles. (7.5)

(b) (i) Let S_n denote the symmetric group of degree n. In S_3, find elements α and β such that $|\alpha| = 2$, $|\beta| = 2$ and $|\alpha\beta| = 3$.

(ii) Let $\beta \in S_7$ and $\beta^4 = (2\ 1\ 4\ 3\ 5\ 6\ 7)$. Then find β. (3,4.5)

(c) (i) Give two reasons to show that the set of odd permutations in S_n is not a subgroup of S_n.

(ii) Define even and odd permutations and show that the set of even permutations in S_n is a subgroup of S_n. (3, 4.5)

2. (a) (i) Let a be an element in a group G such that $|a| = 15$. Find all left cosets of $\langle a^5 \rangle$ in $\langle a \rangle$.

(ii) State and prove Lagrange's theorem.

(3,4.5)

(b) Suppose that G is a group with more than one element and G has no proper, non-trivial subgroups. Prove that $|G|$ is prime. (7.5)

(c) Let $\mathbb{C}^*$ be the group of non-zero complex numbers under multiplication and let $H = \{a + bi \in \mathbb{C}^* \mid a^2 + b^2 = 1\}$. Give a geometrical description of the coset $(3 + 4i)H$. Give a geometrical description of the coset $(c + di)H$. (7.5)

3. (a) (i) Let G be a group and H be its subgroup. Prove that if H has index 2 in G, then H is normal in G.

(ii) If a group G has a unique subgroup H of some finite order, then show that H is normal in G. (3,4.5)

(b) (i) Prove that a factor group of a cyclic group is cyclic. Is converse true? Justify your answer.

(ii) Let G be a group and let Z(G) be the center of G. If G/Z(G) is cyclic, then show that G is Abelian. (3,4.5)

(c) (i) Let ϕ be a group homomorphism from group G_1 to group G_2 and H be a subgroup of G_1. Show that if H is cyclic, then $\phi(H)$ is cyclic.

(ii) How many homomorphisms are there from $\mathbb{Z}_{20}$ to $\mathbb{Z}_8$? How many are there onto $\mathbb{Z}_8$?

(3,4.5)

4. (a) (i) Suppose that ϕ is a homomorphism from U(30) to U(30) and Ker $\phi = \{1,11\}$. If $\phi(7) = 7$, find all the elements of U(30) that map to 7.

(ii) Let ϕ be a homomorphism from a group G_1 to group G_2. Show that $\phi(a) = \phi(b)$ iff aKer ϕ = bKer ϕ. (3,4.5)

(b) (i) Is U(8) isomorphic to U(10)? Justify your answer.

(ii) Show that any infinite cyclic group is isomorphic to the group of integers under addition. (3,4.5)

(c) If ϕ is an onto homomorphism from group G_1 to group G_2, then prove that $G_1/\text{Ker}\,\phi$ is isomorphic to G_2. Hence show that if G_1 is finite, then order of G_2 divides the order of G_1. (7.5)

5. (a) Let G be a group and let $a \in G$. Define the inner automorphism of G induced by a. Show that the set of all inner automorphisms of a group G, denoted by Inn(G), forms a subgroup of Aut(G), the group of all automorphisms of G. Find $\text{Inn}(D_4)$. (7.5)

(b) Prove that the order of an element in a direct product of a finite number of finite groups is the lcm of the orders of the components of the element, i.e., $|(g_1, g_2, \ldots, g_n)| = \text{lcm}(|g_1|, |g_2|, \ldots, |g_n|)$. Also, find the number of elements of order 7 in $\mathbb{Z}_{49} \oplus \mathbb{Z}_7$. (7.5)

(c) Without doing any calculations in $\text{Aut}(\mathbb{Z}_{105})$, determine how many elements of $\text{Aut}(\mathbb{Z}_{105})$ have order 6. (7.5)

6. (a) For any group G, prove that $G/Z(G) \cong \text{Inn}(G)$.

(7.5)

(b) Define the internal direct product of a collection of subgroups of a group G. Let $\mathbb{R}^*$ denote the group of all nonzero real numbers under multiplication. Let $\mathbb{R}^+$ denote the group of all positive real numbers under multiplication. Prove that $\mathbb{R}^*$ is the internal direct product of $\mathbb{R}^+$ and the subgroup $\{1, -1\}$. (7.5)

(c) The set $G = \{1,4,11,14,16,19,26,29,31,34,41,44\}$ is a group under multiplication modulo 45. Write G as an external and an internal direct product of cyclic groups of prime-power order.

(7.5)