Mathematical Methods for Economics (MME) - BA Economics Hons, DU

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BA Economics Hons, DU - Semester I: Exam-Focused Notes

This document provides a comprehensive and detailed overview of essential mathematical methods used in economics, tailored for students of BA Economics Honours.

1. Basics of Logic and Set Theory

Logic and Set Theory form the foundational language for formalizing economic concepts and arguments.

Logic

- Propositions: A proposition is a declarative statement that is either true or false, but not both. Examples include "The price of apples is rising" or "All consumers are rational."
- Truth Tables: These are tabular representations used to show the truth value of a compound proposition for all possible truth values of its constituent simple propositions. They are crucial for analyzing the validity of logical arguments.
- Logical Connectives: These are symbols or words used to combine propositions to form more complex propositions:
  - AND ($\wedge$): Conjunction. True only if all propositions are true.
  - OR ($\vee$): Disjunction. True if at least one proposition is true.
  - NOT ($\neg$): Negation. Reverses the truth value of a proposition.
  - IF-THEN ($\rightarrow$): Implication or Conditional. $P \rightarrow Q$ means "If P, then Q." It is false only when P is true and Q is false.
  - Example: If P = "It is raining" and Q = "I will stay home", then $P \rightarrow Q$ means: "If it rains, I will stay home."

Sets

A set is a well-defined collection of distinct objects, called elements.

- Types of Sets:
  - Finite Set: Contains a countable number of elements (e.g., set of all prime numbers less than 10).
  - Infinite Set: Contains an uncountable number of elements (e.g., set of all real numbers).
  - Null Set (or Empty Set, $\emptyset$ or $\{\}$): A set containing no elements.
  - Universal Set (U): The set of all elements under consideration in a particular context.
  - Subsets: A set A is a subset of set B (denoted $A \subseteq B$) if every element of A is also an element of B.
- Set Operations:
  - Union ($A \cup B$): The set containing all elements that are in A, or in B, or in both.
  - Intersection ($A \cap B$): The set containing all elements that are common to both A and B.
  - Difference ($A - B$): The set containing all elements that are in A but not in B.

PYQ: Prove De Morgan's Laws using truth tables and set diagrams.

2. Functions and Graphs

Functions are mathematical relationships that map elements from one set (domain) to another set (codomain). Graphs visually represent these relationships.

Functions

- Definition: A function $f : X \to Y$ assigns to each element in set X (the domain) exactly one element in set Y (the codomain). In economics, demand and supply curves are common functions.
- Types of Functions:
  - Linear Functions: Of the form $y = mx + c$, representing a straight line. Used in simple cost functions, supply curves.
  - Quadratic Functions: Of the form $y = ax^2 + bx + c$, representing a parabola. Used in cost functions (U-shaped average cost) or total revenue curves.
  - Polynomial Functions: Generalization of linear and quadratic functions.
  - Exponential Functions: Of the form $y = a^x$. Used to model growth processes (e.g., compound interest, population growth).
  - Logarithmic Functions: Inverse of exponential functions. Used for diminishing marginal utility or elasticity calculations.
- Inverse Functions: If a function f maps x to y, its inverse function f^{-1} maps y back to x. Not all functions have inverses.

- Composite Functions: A function formed by applying one function to the result of another function, e.g., $f(g(x))$.

Graphs

- Plotting Curves: Visual representation of functions on a coordinate plane.
- Identifying Intercepts: Points where the graph crosses the x-axis (x-intercept, where $y = 0$) or y-axis (y-intercept, where $x = 0$).
- Turning Points: Points where the slope of the curve changes sign (from positive to negative for a maximum, or negative to positive for a minimum). These correspond to local maxima or minima.

PYQ: Sketch and analyze the function $f(x) = x^2 - 4x + 3$. (This involves finding intercepts, vertex, and general shape).

3. Limits and Continuity

Limits and continuity are fundamental concepts for understanding the behavior of functions and are prerequisites for calculus.

Limit

- Definition of $\lim_{x \to a} f(x)$: The limit of a function $f(x)$ as x approaches a particular value a is the value that $f(x)$ gets arbitrarily close to as x gets arbitrarily close to a, without necessarily equaling a.
- Techniques for Evaluation:
  - Direct Substitution: If the function is well-behaved at $x = a$ (e.g., polynomial), simply substitute a into $f(x)$.
  - Factorization: For indeterminate forms (e.g., $\frac{0}{0}$), factorize the numerator and denominator to cancel common terms before substitution.
  - L'Hôpital's Rule: (Not explicitly mentioned but a common technique for indeterminate forms involving derivatives).

Continuity

- Definition: A function $f(x)$ is continuous at a point $x = a$ if three conditions are met:
  1. $f(a)$ is defined (the function exists at a).
  2. $\lim_{x \to a} f(x)$ exists (the limit exists as x approaches a).
  3. $\lim_{x \to a} f(x) = f(a)$ (the limit value is equal to the function's value at a).

  Informally, a continuous function can be drawn without lifting the pen from the paper.

PYQ: Evaluate limits and check continuity of given functions.

4. Differentiation

Differentiation is a powerful tool used to find the rate of change of a function. In economics, it's widely used for marginal analysis.

Rules

- Power Rule: If $f(x) = x^n$, then $f'(x) = nx^{n-1}$.
- Product Rule: If $f(x) = u(x)v(x)$, then $f'(x) = u'(x)v(x) + u(x)v'(x)$.
- Quotient Rule: If $f(x) = \frac{u(x)}{v(x)}$, then $f'(x) = \frac{u'(x)v(x)-u(x)v'(x)}{[v(x)]^2}$.
- Chain Rule: If $f(x) = g(h(x))$, then $f'(x) = g'(h(x)) \cdot h'(x)$. Used when one function is nested inside another.

Economic Applications

- Marginal Cost (MC): The change in total cost resulting from producing one additional unit of output. It is the derivative of the total cost function with respect to quantity ($MC = \frac{dTC}{dQ}$).
- Marginal Revenue (MR): The change in total revenue resulting from selling one additional unit of output. It is the derivative of the total revenue function with respect to quantity ($MR = \frac{dTR}{dQ}$).
- Elasticity of Demand (E_d): Measures the responsiveness of quantity demanded to a change in price.

  $$E_d = \frac{dQ}{dP} \cdot \frac{P}{Q}$$

  This formula can be adapted for other elasticities (e.g., income elasticity, cross-price elasticity).

PYQ: Find and interpret marginal revenue from a given demand function.

5. Partial Derivatives

Partial derivatives are used for functions of multiple independent variables, allowing us to examine the effect of changing one variable while holding others constant.

Multivariable Functions

- For a function $f(x, y)$, the partial derivative with respect to x, denoted $\frac{\partial f}{\partial x}$, is found by treating y as a constant and differentiating with respect to x. Similarly, for $\frac{\partial f}{\partial y}$, treat x as a constant.
- Example: For $f(x, y) = x^2 + y^2$
  - $\frac{\partial f}{\partial x} = 2x$
  - $\frac{\partial f}{\partial y} = 2y$

Economic Use

- Utility Maximization: In consumer theory, utility functions $U(x, y)$ (where x, y are quantities of goods) describe a consumer's satisfaction. Partial derivatives allow us to find marginal utility with respect to each good.
- Production Functions: Describe the relationship between inputs (e.g., labor L, capital K) and output Q.
  - Cobb-Douglas Production Function: A common form is $Q = AL^{\alpha}K^{\beta}$, where A, α, β are constants. Partial derivatives yield the marginal product of labor ($\frac{\partial Q}{\partial L}$) and marginal product of capital ($\frac{\partial Q}{\partial K}$). These measure the additional output from an extra unit of input, holding other inputs constant.

PYQ: Calculate partial derivatives and interpret them in economic models.

6. Optimization: Unconstrained

Unconstrained optimization involves finding the maximum or minimum of a function without any external restrictions on the variables.

First Order Conditions (FOC)

- To find potential maxima or minima, set the first derivative of the function equal to zero: $\frac{df}{dx} = 0$. These points are called critical points.

Second Order Conditions (SOC)

- After finding critical points, the second derivative test helps determine whether a critical point is a local maximum, local minimum, or an inflection point:
  - If $\frac{d^2f}{dx^2} > 0$ at the critical point, it is a local minimum.
  - If $\frac{d^2f}{dx^2} < 0$ at the critical point, it is a local maximum.
  - If $\frac{d^2f}{dx^2} = 0$, the test is inconclusive, and higher-order derivatives or graphical analysis may be needed.
- Example: Maximize $f(x) = -x^2 + 4x$
  - FOC: $\frac{df}{dx} = -2x + 4 = 0 \Rightarrow x = 2$
  - SOC: $\frac{d^2f}{dx^2} = -2$. Since $-2 < 0$, $x = 2$ corresponds to a local maximum.

PYQ: Solve optimization problems without constraints.

7. Optimization: Constrained (Lagrange Multiplier)

Constrained optimization involves finding the maximum or minimum of a function subject to one or more equality constraints. The Lagrange multiplier method is a powerful technique for this.

Method

- To maximize (or minimize) a function $f(x, y)$ subject to a constraint $g(x, y) = c$:
  - Formulate the Lagrangian function ($\mathcal{L}$):

$$\mathcal{L} = f(x, y) - \lambda(g(x, y) - c)$$

    where λ (lambda) is the Lagrange multiplier, representing the marginal change in the objective function per unit change in the constraint.
  - Take partial derivatives of $\mathcal{L}$ with respect to x, y, and λ, and set them equal to zero (FOC for constrained optimization):
    * $\frac{\partial \mathcal{L}}{\partial x} = 0$
    * $\frac{\partial \mathcal{L}}{\partial y} = 0$
    * $\frac{\partial \mathcal{L}}{\partial \lambda} = 0$ (This last condition simply recovers the original constraint $g(x, y) = c$).
  - Solve the system of equations to find the optimal values of x, y, and λ.

PYQ: Use the Lagrange method to find utility maximization subject to a budget constraint.

8. Matrices and Determinants

Matrices are rectangular arrays of numbers used to represent and manipulate data, especially in systems of linear equations. Determinants are scalar values associated with square matrices.

Operations

- Addition: Matrices can be added if they have the same dimensions, by adding corresponding elements.
- Multiplication: Matrix multiplication is defined for matrices where the number of columns in the first matrix equals the number of rows in the second. The result is a new matrix.
- Transpose: The transpose of a matrix (denoted A^T) is obtained by interchanging its rows and columns.

Determinants

- For 2x2 Matrices: For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $|A| = ad - bc$.
- For 3x3 Matrices (and higher): Calculated using cofactor expansion.
- Properties: Determinants have various properties (e.g., if two rows/columns are identical, determinant is zero; determinant of a triangular matrix is the product of its diagonal elements).
- Cramer's Rule: A method for solving systems of linear equations using determinants.

Inverse

- The inverse of a square matrix A (denoted A^{-1}) is a matrix such that $A \cdot A^{-1} = I$ (identity matrix). An inverse exists if and only if the determinant of A is non-zero.
- Formula: $A^{-1} = \frac{1}{|A|} \cdot adj(A)$, where $adj(A)$ is the adjugate (or adjoint) matrix of A.

PYQ: Solve a system of equations using matrix inverse or Cramer's Rule.

9. Linear Equations and Input-Output Analysis

Linear equations are foundational in economics, often representing relationships between variables with constant rates of change. Input-Output analysis is a specific application using linear systems to model interdependencies in an economy.

Linear Systems

- Representation: A system of linear equations can be represented in matrix form as $AX = B$, where A is the coefficient matrix, X is the column vector of variables, and B is the column vector of constants.
- Solving with Inverse Matrices: If A is invertible, the solution is $X = A^{-1}B$.

Leontief Input-Output Model

- This model analyzes the interdependencies among different sectors of an economy. It shows how the output of one industry is used as input by other industries and to satisfy final demand.
- Basic Equation: $X = AX + D$
  - X: Total output vector (vector of gross output for each sector).
  - A: Technology matrix (or input coefficient matrix), where a_{ij} represents the amount of input from sector i required to produce one unit of output in sector j.
  - D: Final demand vector (vector of net demand for each sector).
- Solution for Total Output: The equation can be rearranged to solve for the total output required to satisfy a given final demand:

  $$X = (I - A)^{-1}D$$

  where I is the identity matrix. The matrix $(I - A)^{-1}$ is known as the Leontief inverse.

PYQ: Solve a 2-sector input-output problem with a given consumption matrix.

10. PYQ Practice Topics (Frequent)

The following topics are frequently asked in examinations and require thorough preparation:

Topic	Type of Question
Set Theory & Logic	Prove laws (e.g., De Morgan's), solve logic problems
Functions	Identify, sketch, interpret functions and their graphs
Differentiation	Calculate derivatives and apply them in economic contexts (e.g., marginal analysis)
Partial Derivatives	Find and interpret from multivariable functions (e.g., production, utility functions)
Optimization	Solve unconstrained (FOC/SOC) and constrained (Lagrangian) problems
Matrices	Perform operations, solve systems of equations, find inverse/determinant
Input-Output	Calculate total output using the Leontief model