Advanced Mathematics Formula Sheet

1. Sequences & Limits

- **Limit Definition**:

$$\lim_{n\to\infty} a_n = L \iff \forall\epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } |a_n - L| < \epsilon \ \forall n \geq N.$$

- **Monotone Convergence Theorem**:
  - A bounded, monotone (increasing/decreasing) sequence converges.
- **Bolzano-Weierstrass Theorem**:
  - Every bounded sequence has a convergent subsequence.
- **Limit Theorems**:
  - If $\lim a_n = A$ and $\lim b_n = B$, then:

$$\lim(a_n \pm b_n) = A \pm B, \quad \lim(a_n b_n) = AB, \quad \lim\left(\frac{a_n}{b_n}\right) = \frac{A}{B} \ (B \neq 0).$$

2. Infinite Series

- **Convergence Tests**:
  - **Divergence Test**: If $\lim a_n \neq 0$, $\sum a_n$ diverges.
  - **Comparison Test**: If $0 \leq a_n \leq b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges.
  - **Ratio Test**:

$$\lim\left|\frac{a_{n+1}}{a_n}\right| = L \implies \begin{cases} L < 1 & \text{Converges,} \\ L > 1 & \text{Diverges,} \\ L = 1 & \text{Inconclusive.} \end{cases}$$

  - **Root Test**:

$$\lim |a_n|^{1/n} = L \implies \text{Same as Ratio Test.}$$

  - **Integral Test**: If f is positive, continuous, and decreasing on $[1,\infty)$, then $\sum f(n)$ converges iff $\int_1^\infty f(x)\,dx$ converges.
- **Geometric Series**:

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \quad (|r| < 1).$$

3. Continuity & Limits of Functions

- **Limit Definition**:

$$\lim_{x\to c} f(x) = L \iff \forall\epsilon > 0, \exists\delta > 0 \text{ s.t. } 0 < |x - c| < \delta \implies |f(x) - L| < \epsilon.$$

- **Continuity**:
  - f is continuous at c if $\lim_{x\to c} f(x) = f(c)$.
  - **Intermediate Value Theorem (IVT)**: If f is continuous on $[a,b]$ and k lies between $f(a)$ and $f(b)$, then $\exists c \in (a,b)$ s.t. $f(c) = k$.
  - **Extreme Value Theorem (EVT)**: A continuous function on a closed interval $[a,b]$ attains its maximum and minimum.

4. Differentiation

- **Derivative Definition**:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

- **Rules**:
  - **Sum**: $(f \pm g)' = f' \pm g'$

  - **Product**: $(fg)' = f'g + fg'$
  - **Quotient**: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$
  - **Chain Rule**: $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$

- **Mean Value Theorem (MVT)**:
  - If f is continuous on $[a, b]$ and differentiable on (a, b), then $\exists c \in (a, b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

- **Rolle's Theorem**:
  - Special case of MVT where $f(a) = f(b) \implies \exists c \in (a, b)$ s.t. $f'(c) = 0$.

5. Important Inequalities

- **Triangle Inequality**: $|a + b| \leq |a| + |b|$
- **Bernoulli's Inequality**: $(1 + x)^n \geq 1 + nx$ for $x > -1, n \in \mathbb{N}$.
- **Cauchy-Schwarz Inequality**:

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

6. Special Limits

- **Exponential**:

$$\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e, \quad \lim_{x\to 0}\frac{e^x - 1}{x} = 1.$$

- **Trigonometric**:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1, \quad \lim_{x\to 0}\frac{1 - \cos x}{x} = 0.$$

- **Logarithmic**:

$$\lim_{x\to 0}\frac{\ln(1 + x)}{x} = 1.$$

7. Taylor's Theorem

- **Taylor Expansion**:

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + R_n(x),$$

where $R_n(x)$ is the remainder term.