Elementary Real Analysis (BSc Maths Hons, DU - Semester I) Exam-Focused Notes

1 Real Number System

This foundational section establishes the properties of real numbers, which are the basis for calculus and analysis.

1.1 Properties:

- **Field Properties:** The real numbers $\mathbb{R}$ with addition $(+)$ and multiplication $(\cdot)$ form a field, meaning they satisfy:
  - **Closure:** For any $a, b \in \mathbb{R}$, $a + b \in \mathbb{R}$ and $a \cdot b \in \mathbb{R}$.
  - **Associativity:** For any $a, b, c \in \mathbb{R}$, $(a+b)+c = a+(b+c)$ and $(a\cdot b)\cdot c = a\cdot(b\cdot c)$.
  - **Commutativity:** For any $a, b \in \mathbb{R}$, $a + b = b + a$ and $a \cdot b = b \cdot a$.
  - **Distributivity:** For any $a, b, c \in \mathbb{R}$, $a \cdot (b + c) = a \cdot b + a \cdot c$.
  - **Identity Elements:** There exist $0 \in \mathbb{R}$ such that $a + 0 = a$ and $1 \in \mathbb{R}$ such that $a \cdot 1 = a$.
  - **Inverse Elements:** For every $a \in \mathbb{R}$, there exists $-a \in \mathbb{R}$ such that $a+(-a) = 0$. For every $a \in \mathbb{R}, a \neq 0$, there exists $a^{-1} \in \mathbb{R}$ such that $a \cdot a^{-1} = 1$ [1].
- **Order Properties:** The real numbers are ordered, meaning they satisfy:
  - **Trichotomy:** For any $a, b \in \mathbb{R}$, exactly one of the following is true: $a < b$, $a = b$, or $a > b$ [1].
  - **Transitivity:** If $a < b$ and $b < c$, then $a < c$ [1].
  - **Compatibility with Operations:** If $a < b$, then $a + c < b + c$ for any $c \in \mathbb{R}$. If $a < b$ and $c > 0$, then $ac < bc$. If $a < b$ and $c < 0$, then $ac > bc$.

1.2 Completeness Axiom (Least Upper Bound Property):

- **Statement:** Every non-empty subset of real numbers that is bounded above has a least upper bound (supremum) in $\mathbb{R}$ [1]. This axiom is what distinguishes the real numbers from the rational numbers and is crucial for many theorems in analysis.

1.3 Bounds:

- **Upper Bound:** A number M is an upper bound of a set A if $x \leq M$ for all $x \in A$.
- **Lower Bound:** A number m is a lower bound of a set A if $x \geq m$ for all $x \in A$.
- **Supremum (sup A / Least Upper Bound):** The smallest among all upper bounds of a set A [1].
- **Infimum (inf A / Greatest Lower Bound):** The largest among all lower bounds of a set A [1].

PYQ Focus: Proving properties like $\sqrt{2}$ being irrational, and finding the supremum and infimum of given sets [1].

2 Sequences

Sequences are ordered lists of numbers, fundamental for understanding convergence and limits.

2.1 Definition:

- A **sequence** is a function $a : \mathbb{N} \to \mathbb{R}$, where $\mathbb{N}$ is the set of natural numbers and $\mathbb{R}$ is the set of real numbers [2]. We usually denote a sequence as $(a_n)_{n\in\mathbb{N}}$ or simply (a_n), where a_n is the n^{th} term.

2.2 Convergence:

- A sequence (a_n) is said to **converge** to a limit L (denoted $\lim_{n\to\infty} a_n = L$) if for every $\epsilon > 0$, there exists a natural number N such that for all $n > N$, $|a_n - L| < \epsilon$ [2]. This is the rigorous $\epsilon - N$ definition of convergence.

2.3 Boundedness:

- A sequence (a_n) is **bounded** if there exists a positive real number M such that $|a_n| \leq M$ for all $n \in \mathbb{N}$ [3].
- **Theorem:** Every convergent sequence is bounded. (The converse is not true; e.g., $(-1)^n$ is bounded but not convergent).

PYQ Focus: Proving the convergence of a sequence directly using the $\epsilon - N$ definition [3].

3 Monotonic Sequences

Monotonic sequences have a consistent trend (always increasing or always decreasing), which simplifies their convergence analysis.

3.1 Increasing/Decreasing Sequences:

- A sequence (a_n) is **increasing** if $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$ [4].
- A sequence (a_n) is **strictly increasing** if $a_{n+1} > a_n$ for all $n \in \mathbb{N}$ [4].
- Similarly, a sequence is **decreasing** if $a_{n+1} \leq a_n$, and **strictly decreasing** if $a_{n+1} < a_n$.
- A sequence is **monotonic** if it is either increasing or decreasing.

3.2 Monotone Convergence Theorem:

- **Statement:** Every bounded monotonic sequence is convergent [4]. This is a very powerful theorem as it guarantees convergence without needing to find the exact limit.

- **Proof Idea:** If an increasing sequence is bounded above, its set of terms has a supremum L. One can then show that L is the limit of the sequence. A similar argument applies to a decreasing sequence bounded below.

PYQ Focus: Showing that a given monotonic sequence converges using the Monotone Convergence Theorem [4].

4 Subsequences and Limit Points

These concepts refine the idea of convergence by looking at parts of a sequence.

4.1 Subsequences:

- A **subsequence** of a sequence (a_n) is a sequence (a_{n_k}) where (n_k) is a strictly increasing sequence of natural numbers, i.e., $n_1 < n_2 < n_3 < \dots$ [5]. This means we pick terms from the original sequence, maintaining their original order.
- **Example:** For $(a_n) = (1, 1/2, 1/3, 1/4, \dots)$, $(a_{2k}) = (1/2, 1/4, 1/6, \dots)$ is a subsequence.

4.2 Limit Point of a Sequence:

- A real number L is a **limit point** (or cluster point) of a sequence (a_n) if there exists a subsequence (a_{n_k}) that converges to L [5].
- **Bolzano-Weierstrass Theorem for Sequences:** Every bounded sequence in $\mathbb{R}$ has a convergent subsequence. (This implies every bounded sequence has at least one limit point).

PYQ Focus: Finding all limit points of a given sequence [5].

5 Cauchy Sequences

Cauchy sequences are fundamental because their convergence can be determined without knowing the limit beforehand.

5.1 Definition:

- A sequence (a_n) is a **Cauchy sequence** if for every $\epsilon > 0$, there exists a natural number N such that for all $m, n > N$, $|a_n - a_m| < \epsilon$ [6]. Intuitively, the terms of a Cauchy sequence get arbitrarily close to each other as the sequence progresses.

5.2 Theorem (Cauchy's Convergence Criterion):

- **Statement:** Every Cauchy sequence in $\mathbb{R}$ is convergent [7]. Conversely, every convergent sequence in $\mathbb{R}$ is a Cauchy sequence. This theorem establishes that the real numbers are "complete," meaning all Cauchy sequences in $\mathbb{R}$ converge to a point within $\mathbb{R}$.

PYQ Focus: Proving that a given sequence is a Cauchy sequence and then concluding its convergence based on the completeness of $\mathbb{R}$ [7].

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6 Series

Series are sums of terms of a sequence, and their convergence is a major topic in analysis.

6.1 Convergence of a Series:

- A series $\sum_{n=1}^{\infty} a_n$ is said to **converge** if its sequence of partial sums (s_n) converges, where $s_n = \sum_{k=1}^{n} a_k$ [8]. If the sequence of partial sums does not converge, the series diverges.

6.2 Tests for Convergence:

- **Term Test (Divergence Test):** If $\lim_{n\to\infty} a_n \neq 0$ (or the limit does not exist), then the series $\sum a_n$ diverges. (If $\lim_{n\to\infty} a_n = 0$, the test is inconclusive).
- **Comparison Test:** If $0 \leq a_n \leq b_n$ for all n, and $\sum b_n$ converges, then $\sum a_n$ converges. If $a_n \geq b_n \geq 0$ and $\sum b_n$ diverges, then $\sum a_n$ diverges [8].
- **Ratio Test:** For a series $\sum a_n$, let $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ [8].
  - If $L < 1$, the series converges absolutely [8].
  - If $L > 1$ or $L = \infty$, the series diverges [8].
  - If $L = 1$, the test is inconclusive.
- **Root Test:** For a series $\sum a_n$, let $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ [8].
  - If $L < 1$, the series converges absolutely [8].
  - If $L > 1$ or $L = \infty$, the series diverges [8].
  - If $L = 1$, the test is inconclusive.
- **Integral Test:** If $f(x)$ is a positive, continuous, and decreasing function for $x \geq 1$, then $\sum_{n=1}^{\infty} f(n)$ converges if and only if $\int_1^{\infty} f(x)dx$ converges.
- **Alternating Series Test (Leibniz Test):** For an alternating series $\sum(-1)^{n-1}b_n$ (where $b_n > 0$), if b_n is decreasing and $\lim_{n\to\infty} b_n = 0$, then the series converges.

PYQ Focus: Testing the convergence of a given series using the Ratio, Root, or Comparison tests [8].

7 Absolute and Conditional Convergence

These concepts classify the behavior of convergent series more precisely.

7.1 Definitions:

- A series $\sum a_n$ **converges absolutely** if the series of absolute values $\sum |a_n|$ converges [9].
- A series $\sum a_n$ **converges conditionally** if the series $\sum a_n$ converges, but the series of absolute values $\sum |a_n|$ diverges [9].

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- **Theorem:** If a series converges absolutely, then it converges. (The converse is false; a series can converge conditionally).

PYQ Focus: Demonstrating whether a given series converges conditionally or absolutely [9].

8 Limits and Continuity of Functions

This section extends the concept of limits from sequences to functions, leading to the definition of continuity.

8.1 Limit of a Function:

- A function $f : D \to \mathbb{R}$ has a limit L at a point a (denoted $\lim_{x\to a} f(x) = L$) if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in D$ satisfying $0 < |x - a| < \delta$, we have $|f(x) - L| < \epsilon$ [10]. This is the rigorous $\epsilon - \delta$ definition of a function limit.

8.2 Continuity of a Function:

- A function f is **continuous** at a point a if $\lim_{x\to a} f(x) = f(a)$ [11].
- This implies three conditions:
  1. $f(a)$ must be defined.
  2. $\lim_{x\to a} f(x)$ must exist.
  3. The limit must be equal to the function value at a.
- **Continuity on an Interval:** A function is continuous on an open interval if it is continuous at every point in the interval. It's continuous on a closed interval $[a, b]$ if it's continuous on (a, b) and also continuous from the right at a and from the left at b.

PYQ Focus: Proving that a given function is continuous at a point using the $\epsilon - \delta$ definition [11].

9 PYQ Practice Topics (Frequent)

This table summarizes the types of questions frequently asked in exams, correlating with the above topics:

Topic	Type of Question
Supremum/Infimum	Prove bounds or completeness [12]
Sequence Convergence	Use $\epsilon - N$ definition [12]
Monotonicity	Prove bounded + monotonic = convergent [12]
Cauchy Sequences	Prove a sequence is Cauchy [12]
Series Convergence	Apply ratio/root/comparison test [12]
Absolute vs Conditional	Prove classification of series [12]
Function Continuity	Prove via $\epsilon - \delta$ [12]

Table 1: Frequently Asked Questions in Elementary Real Analysis

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References

[1] Reference for Real Number System.

[2] Reference for Sequences Definition and Convergence.

[3] Reference for Boundedness and Sequence Convergence.

[4] Reference for Monotonic Sequences.

[5] Reference for Subsequences and Limit Points.

[6] Reference for Cauchy Sequences Definition.

[7] Reference for Cauchy's Convergence Criterion.

[8] Reference for Series Convergence Tests.

[9] Reference for Absolute and Conditional Convergence.

[10] Reference for Limit of a Function.

[11] Reference for Continuity of a Function.

[12] Reference for PYQ Practice Topics.

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