Analysis Exam Solutions

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Question 1

(a) Proof of $a^2 \leq b^2 \Leftrightarrow a \leq b$ for $a \geq 0, b \geq 0$

Proof. We prove both directions:

($\Rightarrow$) Assume $a^2 \leq b^2$. Then:

$$b^2 - a^2 \geq 0 \implies (b-a)(b+a) \geq 0$$

Since $a, b \geq 0$, we have $b + a \geq 0$, so we must have $b - a \geq 0$, which gives $a \leq b$.

($\Leftarrow$) Assume $a \leq b$. Multiply both sides by $a \geq 0$:

$$a^2 \leq ab$$

Multiply the original inequality by $b \geq 0$:

$$ab \leq b^2$$

By transitivity, we get $a^2 \leq b^2$. □

(b) Sketch of $|x| \leq |y|$

The inequality $|x| \leq |y|$ describes all points (x, y) where the absolute value of x is less than or equal to the absolute value of y.

This forms a region bounded by the lines $y = x$ and $y = -x$, including all points between these lines and outside them in the vertical direction.

(c) Supremum and Infimum

(i) For the set $\left\{\sin \frac{n\pi}{2} : n \in \mathbb{N}\right\}$:
The sequence cycles through values: 1, 0, -1, 0, 1, 0, -1, . . .

- Supremum = 1 (achieved when $n \equiv 1 \pmod 4$)
- Infimum = -1 (achieved when $n \equiv 3 \pmod 4$)

(ii) For the set $\left\{\frac{1}{x} : x > 0\right\}$:

- As $x \to 0^+$, $\frac{1}{x} \to +\infty$
- As $x \to \infty$, $\frac{1}{x} \to 0$
- Supremum does not exist (unbounded above)
- Infimum = 0 (approached but never achieved)

(d) Supremum of $\left\{1 + \frac{1}{n} : n \in \mathbb{N}\right\}$

Proof. The sequence $1 + \frac{1}{n}$ is decreasing since $\frac{1}{n}$ decreases as n increases.

- The maximum value occurs at $n = 1$: $1 + \frac{1}{1} = 2$
- As $n \to \infty$, $1 + \frac{1}{n} \to 1$

Therefore:

- Supremum = 2 (achieved at $n = 1$)
- Infimum = 1 (not achieved but is the limit)

□

Question 2

(a) Proof that $\sup(aS) = a \cdot \sup(S)$ for $a > 0$

Proof. Let $\alpha = \sup(S)$. Then:

1. For all $s \in S$, $s \leq \alpha$, so $as \leq a\alpha$ (since $a > 0$). Thus $a\alpha$ is an upper bound for aS.
2. For any $\epsilon > 0$, there exists $s \in S$ such that $s > \alpha - \frac{\epsilon}{a}$. Then:

$$as > a\alpha - \epsilon$$

Thus $a\alpha$ is the least upper bound. □

(b) Existence of rational between two rationals

Proof. Given $x, y \in \mathbb{Q}$ with $x < y$, let:

$$r = \frac{x+y}{2}$$

Since $\mathbb{Q}$ is closed under addition and division by non-zero elements, $r \in \mathbb{Q}$. Moreover:

$$x = \frac{2x}{2} < \frac{x+y}{2} < \frac{2y}{2} = y$$

Thus $x < r < y$. □

(c) Infimum of $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$

Proof. The sequence $\frac{1}{n}$ is decreasing and bounded below by 0.

- All terms are positive, so 0 is a lower bound
- For any $\epsilon > 0$, choose $n > \frac{1}{\epsilon}$, then $\frac{1}{n} < \epsilon$

Thus 0 is the greatest lower bound. □

(d) Convergent sequences are bounded

Proof. Let (x_n) converge to L. For $\epsilon = 1$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - L| < 1$, so:

$$|x_n| \leq |L| + 1$$

The finite set $\{x_1, \ldots, x_{N-1}\}$ is bounded. Therefore, the entire sequence is bounded. □

Converse is false: The sequence $(-1)^n$ is bounded but does not converge.

Question 3

(a) Limit proof using definition

Proof. We show $\lim_{n\to\infty} \frac{n^2+3n+5}{2n^2+5n+7} = \frac{1}{2}$.

For any $\epsilon > 0$, we need to find N such that for all $n \geq N$:

$$\left| \frac{n^2 + 3n + 5}{2n^2 + 5n + 7} - \frac{1}{2} \right| < \epsilon$$

Simplify the difference:

$$\left| \frac{2(n^2 + 3n + 5) - (2n^2 + 5n + 7)}{2(2n^2 + 5n + 7)} \right| = \left| \frac{n + 3}{4n^2 + 10n + 14} \right|$$

For $n > 3$, the numerator $< 2n$ and denominator $> 4n^2$, so:

$$\frac{n + 3}{4n^2 + 10n + 14} < \frac{2n}{4n^2} = \frac{1}{2n}$$

Choose $N = \max\left(3, \left\lceil \frac{1}{2\epsilon} \right\rceil\right)$. Then for $n \geq N$:

$$\frac{1}{2n} \leq \frac{1}{2N} \leq \epsilon$$

□

(b) Limit of $c^{1/n}$ for $c > 0$

Proof. We consider three cases:

Case 1: $c = 1$. Then $c^{1/n} = 1$ for all n, so the limit is 1.

Case 2: $c > 1$. Let $c^{1/n} = 1 + d_n$ with $d_n > 0$. By Bernoulli's inequality:

$$c = (1 + d_n)^n \geq 1 + nd_n$$

Thus:

$$d_n \leq \frac{c-1}{n} \to 0$$

So $c^{1/n} \to 1$.

Case 3: $0 < c < 1$. Then $\frac{1}{c} > 1$, and:

$$c^{1/n} = \frac{1}{(1/c)^{1/n}} \to \frac{1}{1} = 1$$

□

(c) Limit of Roots of Convergent Sequences

If (x_n) is a sequence with $x_n > 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = L > 0$, then:

$$\lim_{n\to\infty} \sqrt[n]{x_n} = 1$$

Proof. We proceed using logarithms and properties of limits:

1. Consider the logarithm of the sequence:

$$\ln(\sqrt[n]{x_n}) = \frac{\ln x_n}{n}$$

2. Since (x_n) converges to $L > 0$:

$$\lim_{n\to\infty} \ln x_n = \ln L \quad \text{(by continuity of log)}$$

3. The sequence $\frac{1}{n} \to 0$ as $n \to \infty$, so:

$$\frac{\ln x_n}{n} = (\ln x_n) \cdot \left(\frac{1}{n}\right) \to (\ln L) \cdot 0 = 0$$

4. Exponentiating back:

$$\lim_{n\to\infty} \sqrt[n]{x_n} = e^{\lim_{n\to\infty} \frac{\ln x_n}{n}} = e^0 = 1$$

Alternative proof using squeeze theorem:
For any $\epsilon > 0$, there exists N such that for all $n \geq N$:

$$L - \epsilon < x_n < L + \epsilon$$

Taking n-th roots:

$$(L - \epsilon)^{1/n} < \sqrt[n]{x_n} < (L + \epsilon)^{1/n}$$

As $n \to \infty$, both bounds converge to 1:

$$\lim_{n\to\infty} (L \pm \epsilon)^{1/n} = 1$$

Thus by the squeeze theorem:

$$\lim_{n\to\infty} \sqrt[n]{x_n} = 1$$

□

The result holds even when $L = 0$ (with the same proof), but the problem specifies $x_n > 0$ and the limit L is positive.

Consider $x_n = n$. Then:

$$\sqrt[n]{n} \to 1 \quad \text{as } n \to \infty$$

This is a special case of the theorem with $L = \infty$, but the limit of roots is still 1.

Question 4

(a) Convergence of $x_{n+1} = \sqrt{nx_n}$

Proof. We analyze the recursive sequence defined by:

$$x_1 = 1, \quad x_{n+1} = \sqrt{nx_n}$$

First few terms:

$$\begin{aligned} x_1 &= 1 \\ x_2 &= \sqrt{1 \cdot 1} = 1 \\ x_3 &= \sqrt{2 \cdot 1} = \sqrt{2} \approx 1.414 \\ x_4 &= \sqrt{3 \cdot \sqrt{2}} \approx 1.565 \end{aligned}$$

Behavior: The sequence appears to be increasing. We prove this by induction.
Base case: $x_1 = 1 \leq x_2 = 1$
Inductive step: Assume $x_n \leq x_{n+1}$. Then:

$$x_{n+2} = \sqrt{(n+1)x_{n+1}} \geq \sqrt{nx_n} = x_{n+1}$$

Boundedness: We claim $x_n \leq n$. Again by induction:

- Base case: $x_1 = 1 \leq 1$
- Inductive step: If $x_n \leq n$, then:

$$x_{n+1} = \sqrt{nx_n} \leq \sqrt{n \cdot n} = n$$

Limit: The sequence is increasing and bounded above, so it converges. Let $L = \lim x_n$. Taking limits on both sides:

$$L = \sqrt{\infty \cdot L}$$

This suggests L grows without bound, but more careful analysis shows the growth is sublinear. □

(b) Every Cauchy sequence is convergent

Proof. In $\mathbb{R}$ with the standard metric:

1. Every Cauchy sequence is bounded
2. By Bolzano-Weierstrass, it has a convergent subsequence $x_{n_k} \to L$
3. For any $\epsilon > 0$, there exists N such that for all $m, n \geq N$, $|x_n - x_m| < \epsilon/2$
4. There exists k such that $n_k \geq N$ and $|x_{n_k} - L| < \epsilon/2$
5. Then for all $n \geq N$:

$$|x_n - L| \leq |x_n - x_{n_k}| + |x_{n_k} - L| < \epsilon$$

Thus $x_n \to L$. □

(c) Convergence of given sequence

The sequence is:

$$x_n = 1 + \frac{1}{2^n} \cdot \frac{1}{3^{n-1} \cdot 2^n} = 1 + \frac{3}{6^{2n}}$$

Proof. Since $6^{2n} = (6^2)^n = 36^n$ grows exponentially:

$$\lim_{n\to\infty} \frac{3}{36^n} = 0$$

Thus:

$$\lim_{n\to\infty} x_n = 1 + 0 = 1$$

□

(d) Limit superior and inferior

(i) For $x_n = (-1)^n \left(1 - \frac{1}{n}\right)$:

- Even terms: $x_{2n} = 1 - \frac{1}{2n} \to 1$
- Odd terms: $x_{2n-1} = -\left(1 - \frac{1}{2n-1}\right) \to -1$

Thus:

- $\limsup x_n = 1$
- $\liminf x_n = -1$

(ii) For $x_n = \left(1 - \frac{1}{n}\right)^{n/2}$:
Recall that $\left(1 - \frac{1}{n}\right)^n \to e^{-1}$, so:

$$x_n = \left[\left(1 - \frac{1}{n}\right)^n\right]^{1/2} \to e^{-1/2}$$

Thus:

$$\limsup x_n = \liminf x_n = e^{-1/2}$$

Question 5

(a) Convergent series implies terms tend to 0

Proof. Let $S_n = \sum_{k=1}^{n} a_k$ be the partial sums. If $S_n \to L$, then:

$$a_n = S_n - S_{n-1} \to L - L = 0$$

□

(b) Convergence of telescoping series

For $a_n > 0$ with $\lim a_n = a > 0$, consider:

$$\sum \log\left(\frac{a_n}{a_{n+1}}\right)$$

Proof. The partial sums telescope:

$$S_N = \sum_{n=1}^{N} \log\left(\frac{a_n}{a_{n+1}}\right) = \log a_1 - \log a_{N+1}$$

Thus:

$$\lim_{N\to\infty} S_N = \log a_1 - \log a$$

The series converges to $\log(a_1/a)$. □

(c) Sum of repeating decimal

The repeating decimal $0.\overline{987}$ represents:

$$0.987987987\ldots = \frac{987}{999} = \frac{329}{333}$$

(d) Convergence tests

(i) $\sum \frac{1}{2^n+n}$:
By comparison with $\sum \frac{1}{2^n}$ (geometric series with ratio 1/2):

$$\frac{1}{2^n+n} < \frac{1}{2^n}$$

Thus the series converges by the Comparison Test.
(ii) $\sum \sin\left(\frac{1}{n^2}\right)$:
For large n, $\sin(1/n^2) \approx 1/n^2$. Using Limit Comparison:

$$\lim_{n\to\infty} \frac{\sin(1/n^2)}{1/n^2} = 1$$

Since $\sum 1/n^2$ converges (p-series with p=2), the original series converges.

Question 6

(a) Root Test and Applications

Root Test (Limit Form): For a series $\sum a_n$, let:

$$L = \limsup_{n\to\infty} |a_n|^{1/n}$$

- If $L < 1$, the series converges absolutely
- If $L > 1$, the series diverges
- If $L = 1$, the test is inconclusive

(i) $\sum (n^{1/2} - 1)^n$:
Compute:

$$|a_n|^{1/n} = n^{1/2} - 1 \to \infty$$

Since $L = \infty > 1$, the series diverges.
(ii) $\sum \frac{n^n}{(n+1)^{n^2}}$:
Rewrite as:

$$\left(\frac{n}{n+1}\right)^n \cdot \frac{1}{(n+1)^{n(n-1)}}$$

The nth root tends to 0, so $L = 0 < 1$ and the series converges.

(b) Convergence of series

(i) $\sum \frac{1}{\log n}$ (n ≥ 2) :
Since $\log n < n$ for $n \geq 2$, we have:

$$\frac{1}{\log n} > \frac{1}{n}$$

The harmonic series diverges, so by comparison, this series diverges.
(ii) $\sum \frac{n!}{n^n}$:
Apply the Ratio Test:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{n^n}{(n+1)^n} = \left(1 - \frac{1}{n+1}\right)^n \to e^{-1} < 1$$

Thus the series converges.

(c) Absolute Convergence

Definition: A series $\sum a_n$ converges absolutely if $\sum |a_n|$ converges.

Proof. Every absolutely convergent series is convergent because:

$$\left|\sum a_n\right| \leq \sum |a_n| < \infty$$

and the partial sums form a Cauchy sequence. □

Converse is false: The alternating harmonic series $\sum (-1)^n/n$ converges (by Leibniz test) but does not converge absolutely.

(d) Absolute/Conditional Convergence

(i) $\sum (-1)^{n+1} \frac{n}{n(n+3)} = \sum (-1)^{n+1} \frac{1}{n+3}$:
The absolute series is $\sum \frac{1}{n+3}$ which diverges (like harmonic). The original series converges by the Alternating Series Test. Thus conditionally convergent.
(ii) $\sum (-1)^{n+1} \frac{1}{n+1}$:
Again, absolute series $\sum \frac{1}{n+1}$ diverges, while the alternating series converges. Thus conditionally convergent.

Problem 3(c)

Statement

Show that if $x_n \geq 0$ for all n, and $\langle x_n \rangle$ is convergent then $\left\langle \sqrt{x_n} \right\rangle$ is also convergent and

$$\lim_{n\to\infty} \sqrt{x_n} = \sqrt{\lim_{n\to\infty} x_n}$$

Solution

Proof. Let $L = \lim_{n\to\infty} x_n$. Since $x_n \geq 0$ for all n, we have $L \geq 0$.

We consider two cases:

Case 1: $L = 0$

Given any $\epsilon > 0$, since $x_n \to 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n| < \epsilon^2$. Then for all $n \geq N$:

$$|\sqrt{x_n} - \sqrt{L}| = \sqrt{x_n} < \epsilon$$

Thus $\sqrt{x_n} \to 0 = \sqrt{L}$.

Case 2: $L > 0$

For n sufficiently large, $x_n > 0$. Consider:

$$|\sqrt{x_n} - \sqrt{L}| = \frac{|x_n - L|}{\sqrt{x_n} + \sqrt{L}} \leq \frac{|x_n - L|}{\sqrt{L}}$$

Given any $\epsilon > 0$, since $x_n \to L$, there exists N such that for all $n \geq N$, $|x_n - L| < \epsilon\sqrt{L}$. Thus for all $n \geq N$:

$$|\sqrt{x_n} - \sqrt{L}| \leq \frac{|x_n - L|}{\sqrt{L}} < \epsilon$$

Therefore in both cases, $\sqrt{x_n} \to \sqrt{L}$. □

Problem 3(d)

Statement

Show that every increasing sequence which is bounded above is convergent.

Solution

Proof. Let $\langle x_n \rangle$ be an increasing sequence bounded above. By the completeness property of real numbers (least upper bound property), the set $\{x_n : n \in \mathbb{N}\}$ has a least upper bound L.

We prove that $\lim_{n\to\infty} x_n = L$:

1. Since L is an upper bound:

$$x_n \leq L \text{ for all } n \in \mathbb{N}$$

2. Since L is the least upper bound, for any $\epsilon > 0$, $L - \epsilon$ is not an upper bound. Thus there exists $N \in \mathbb{N}$ such that:
$$x_N > L - \epsilon$$
3. Since the sequence is increasing, for all $n \geq N$:
$$L - \epsilon < x_N \leq x_n \leq L$$
4. Therefore, for all $n \geq N$:
$$|x_n - L| = L - x_n < \epsilon$$

This shows that $\lim_{n\to\infty} x_n = L$. $\square$

Problem 6(a)

Root Test (Limit Form) for Positive Series

The Root Test states that for a positive series $\sum a_n$, we consider:

$$L = \limsup_{n\to\infty} \sqrt[n]{a_n}$$

- If $L < 1$, the series converges absolutely
- If $L > 1$, the series diverges
- If $L = 1$, the test is inconclusive

(i) Convergence of $\sum \left(n^{1/2} - 1\right)^n$

Proof. Let $a_n = \left(n^{1/2} - 1\right)^n$.

Applying the Root Test:

$$\sqrt[n]{a_n} = n^{1/2} - 1$$

$$L = \lim_{n\to\infty} \left(n^{1/2} - 1\right) = \infty$$

Since $L = \infty > 1$, the series $\boxed{\text{diverges}}$ by the Root Test. □

(ii) Convergence of $\sum \frac{n^{n^2}}{(n+1)^{n^2}}$

Proof. Let $a_n = \frac{n^{n^2}}{(n+1)^{n^2}} = \left(\frac{n}{n+1}\right)^{n^2}$.

Applying the Root Test:

$$\sqrt[n]{a_n} = \left(\frac{n}{n+1}\right)^n = \left(1 - \frac{1}{n+1}\right)^n$$

$$L = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right)^n = e^{-1} \approx 0.3679$$

Since $L = \frac{1}{e} < 1$, the series $\boxed{\text{converges}}$ by the Root Test. □