Advanced Physics Formula Sheet

1. Vector Calculus

Vector Algebra

- **Dot Product**:
$$\vec{A} \cdot \vec{B} = |A||B| \cos\theta = A_x B_x + A_y B_y + A_z B_z$$
- **Cross Product**:
$$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\hat{\imath} + (A_z B_x - A_x B_z)\hat{\jmath} + (A_x B_y - A_y B_x)\hat{k}$$
- **Scalar Triple Product**:
$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$$
- **Vector Triple Product**:
$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

Differential Operators

- **Gradient**:
$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{\imath} + \frac{\partial\phi}{\partial y}\hat{\jmath} + \frac{\partial\phi}{\partial z}\hat{k}$$
- **Divergence**:
$$\nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$
- **Curl**:
$$\nabla \times \vec{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\imath} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{k}$$
- **Laplacian**:
$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

Integral Theorems

- **Gauss's Divergence Theorem**:
$$\iiint_V (\nabla \cdot \vec{F}) dV =_S \vec{F} \cdot d\vec{S}$$
- **Stokes' Theorem**:
$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{l}$$

2. Coordinate Systems

Cylindrical (ρ, ϕ, z)

- **Del Operator**:
$$\nabla = \hat{\rho}\frac{\partial}{\partial\rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{z}\frac{\partial}{\partial z}$$
- **Laplacian**:
$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}$$

Spherical (r, θ, ϕ)

- **Del Operator**:
$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$$
- **Laplacian**:
$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

3. Special Functions

Gamma Function

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$$

$$\Gamma(n+1) = n\Gamma(n), \quad \Gamma(1/2) = \sqrt{\pi}$$

Bessel Functions

$$x^2 y'' + xy' + (x^2 - n^2)y = 0$$

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(n+k+1)} \left(\frac{x}{2}\right)^{n+2k}$$

Legendre Polynomials

$$(1-x^2)y'' - 2xy' + l(l+1)y = 0$$

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2-1)^l$$

4. Complex Analysis

Cauchy-Riemann Equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Cauchy Integral Formula

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz$$

Residue Theorem

$$\oint_C f(z)dz = 2\pi i \sum \operatorname{Res}(f, a_k)$$

5. Fourier Analysis

Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\frac{n\pi x}{L} dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\frac{n\pi x}{L} dx$$

Fourier Transform

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk$$

6. PDE Solutions

Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(x,t) = f(x-ct) + g(x+ct)$$

Heat Equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

$$u(x,t) = \frac{1}{\sqrt{4\pi\alpha t}} \int_{-\infty}^{\infty} f(\xi) e^{-(x-\xi)^2/4\alpha t} d\xi$$

Laplace's Equation

$$\nabla^2 u = 0$$

$$\text{In spherical: } u(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

7. Tensors (Basics)

Metric Tensor

$$ds^2 = g_{ij} dx^i dx^j$$

Christoffel Symbols

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right)$$

8. Green's Functions

$$LG(x,x') = \delta(x - x')$$

$$\text{E.g., 1D: } G(x,x') = \frac{1}{2k} e^{-k|x-x'|}$$

9. Integral Transforms

Laplace Transform

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t) dt$$

Convolution Theorem

$$\mathcal{L}\{f * g\} = F(s)G(s)$$