1. (a) Find the upper and lower Darboux integrals foı $f(x) = x^2$ on the interval $[0, b]$ and show that fı

$$\int_0^b x^2 = \frac{b^3}{3}.$$

(b) Let f be a bounded function on $[a, b]$. If P and Q are partitions of $[a, b]$ and $P \subseteq Q$, then prove that

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P)$$

(c) Let $f: [a, b] \rightarrow R$ be a bounded function on $[a, b]$. Prove that if f is integrable on $[a, b]$, then for each $\in > 0$, there exists a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \in$$

(d) Let $f(x) = 2x + 1$ over the interval $[0,2]$. Let

$P = \left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2\right\}$ be a partition of $[0,2]$. Compute

$U(f, P)$, $L(f, P)$ and $U(f, P) - L(f, P)$.

2. (a) Let f be an integrable function on [a, b]. Show that $-f$ is integrable on [a, b] and

$$\int_a^b (-f) = -\int_a^b f$$

(b) Let $f: [0,2] \to R$ be defined as

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ -1, & \text{if } x \text{ is irrational} \end{cases}$$

Calculate the upper and lower Darboux Integrals for f on the interval [0,2]. Is f integrable on [0,2]?

(c) Let $f: [a, b] \to R$ be a bounded function. Show that if f is integrable (Darboux) on [a, b], then it is Riemann integrable on [a, b].

(d) For a bounded function f on [a, b], define the Riemann Sum associated with a partition P. Hence, give Riemann's definition of integrability.

3. (a) Prove that every bounded piecewise monotonic function f on [a, b] is integrable.

(b) Show that if a function f is integrable on [a, b], then |f| is integrable on [a, b] and $\left|\int_a^b f\right| \leq \int_a^b |f|$.

(c) If f is a continuous, non-negative function on [a, b] and if $\int_a^b f = 0$, then prove that f is identically 0 on [a, b]. Give an example of a discontinuous non-zero function f on [0, 1] for which $\int_0^1 f = 0$.

(d) State and prove Fundamental Theorem of Calculus I.

4. (a) If u and v are continuous functions on [a, b] that are differentiable on (a, b), and u' and v' are integrable, prove that $\int_a^b uv' + \int_a^b u'v = u(b)v(b) - u(a)v(a)$. Hence evaluate $\int_0^{\pi} x \cos x$.

(b) Use the Fundamental Theorem of Calculus to calculate $\lim_{x \to 0} \frac{1}{x} \int_0^x e^{t^2} dt$.

(c) Let f be an integrable function on [a, b]. For x in [a, b], let $F(x) = \int_a^x f(t)dt$. Then show that F is uniformly continuous on [a, b]. For f(x) =

$$\begin{cases} 0, & t < 0 \\ t, & 0 \le t \le 1, \\ 4, & t > 1 \end{cases}$$

(i) Determine the function $F(x) = \int_0^x f(t)dt$.

(ii) Where is F continuous?

(d) For $t \in [0,1]$, define $F(t) = \begin{cases} 0, & t < \frac{1}{2} \\ 1, & t \geq \frac{1}{2} \end{cases}$ and let

$f(x) = x^2$, $x \in [0,1]$. Show that f is F-integrable

and that $\int_0^1 f\,dF = f\left(\frac{1}{2}\right)$.

5. (a) Find the volume of the solid generated when the region enclosed by the curves $x = \sqrt{y}$ and $x = y/4$ is revolved about the x – axis.

(b) Use cylindrical shells to find the volume of the solid generated when the region under $y = x^2$ is revolved about the line $y = -1$.

(c) Find the exact arc length of the curve $x = \frac{1}{3}(y^2 + 2)^{3/2}$ from $y = 0$ to $y = 1$.

(d) Find the area of the surface that is generated by revolving the portion of the curve $y = x^2$ between $x = 1$ and $x = 2$ about the y – axis.

6. (a) Discuss the convergence or divergence of the following improper integrals :

(i) $\int_0^1 \frac{1}{\sqrt{x}} dx$

(ii) $\int_{-\infty}^{+\infty} e^x \, dx$

(b) Find the value of r for which the integral $\int_0^1 x^{-r} \, dx$ exists or converges, and determine the value of the integral.

(c) Show that the improper integral $\int_1^\infty \frac{\sin x}{x^2} dx$ converges absolutely.

(d) Define the Gamma function $\Gamma(m)$. Prove that $\Gamma(m)$ converges if $m > 0$.