1. (a) (i) Define covering relation in an ordered set and finite ordered set. Prove that if X is any set, then the ordered set $\wp(X)$ equipped with the set inclusion relation given by $A \leq B$ iff $A \subseteq B$ for all $A, B \in \wp(X)$, a subset B of X covers a subset A of X iff $B = A \cup \{b\}$ for some $b \in X - A$.

(ii) State Zorn's Lemma.

(b) (i) Give an example of an ordered set (with diagram) with more than one maximal element but no greatest element. Specify maximal elements also.

(ii) Define when two sets have the same cardinality. Show that

- $\mathbb{N}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
- $\mathbb{Z}$ and $2\mathbb{Z}$

have the same cardinality.

(c) Let N_0 be the set of whole numbers equipped with the partial order $\leq$ defined by $m \leq n$ if and only if m divides n. Draw Hasse diagram for the subset $S = \{1,2,4,5,6,12,20,30,60\}$ of $(N_0, \leq)$. Find elements $a, b, c, d \in S$ such that $a \vee b$ and $c \wedge d$ does not exist in S.

2. (a) Define an order preserving map. In which of the following cases is the map $\varphi : P \to Q$ order preserving?

(i) $P = Q = (N_0, \leq)$ and $\varphi(x) = nx$ ($n \in N_0$ is fixed).

(ii) $P = Q = (\wp(N), \subseteq)$ and φ defined by

$$\varphi(U) = \begin{cases} \{1\}, & 1 \in U \\ \{2\}, & 2 \in U \text{ and } 1 \notin U, \\ \emptyset, & \text{otherwise} \end{cases}$$

where $\mathbb{N}_0$ be the set of whole numbers equipped with the partial order $\leq$ defined by $m \leq n$ iff m divides n and $\wp(\mathbb{N})$ be the power set of $\mathbb{N}$ equipped with the partial order given by $A \leq B$ iff $A \subseteq B$ for all $A, B \in \wp(\mathbb{N})$.

(b) For disjoint ordered sets P and Q define order relation on $P \cup Q$. Draw the diagram of ordered sets (i) 2×2 (ii) $3 \cup \overline{3}$ (iii) $M_2 \oplus M_3$ where $M_n = 1 \oplus \overline{n} \oplus 1$.

(c) Let $X = \{1,2,\ldots, n\}$ and define $\varphi: \wp(X) \to 2^n$ by $\varphi(A) = (\varepsilon_1,\ldots, \varepsilon_n)$ where

$$\varepsilon_i = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{if } i \notin A \end{cases}$$

Show that φ is an order-isomorphism.

3. (a) Let L and K be lattices and $f: L \to K$ a lattice homomorphism.

(i) Show that if $M \in \text{Sub } L$, then $f(M) \in \text{Sub } K$.

(ii) Show that if $N \in \text{Sub } K$, then $f^{-1}(N) \in \text{Sub}_0\, L$, where $\text{Sub}_0 L = \text{Sub } L \cup \varnothing$.

(b) Let L be a lattice.

(i) Assume that $b \leq a \leq b \vee c$ for $a, b, c \in L$. Show that $a \vee c = b \vee c$.

(ii) Show that the operations $\vee$ and $\wedge$ are isotone in L, i.e. $b \leq c \Rightarrow a \wedge b \leq a \wedge c$ and $a \vee b \leq a \vee c$.

(c) Let L and M be lattices. Show that the product $L \times M$ is a lattice under the operations $\vee$ and $\wedge$ defined as

$$(x_1,y_1) \vee (x_2,y_2) := (x_1 \vee x_2,\ y_1 \vee y_2),$$

$$(x_1,y_1) \wedge (x_2,y_2) := (x_1 \wedge x_2,\ y_1 \wedge y_2)$$

4. (a) Let L be a distributive lattice. Show that $\forall x, y, z \in$ L, the following laws are equivalent :

(i) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

(ii) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

(b) Define modular lattices. Show that every distributive lattice is modular. Is the converse true? Give arguments in support of your answer.

(c) (i) Prove that for any two elements x, y in a lattice L, the interval

$[x,y] := \{a \in L | x \leq a \leq y\}$ is a sublattice of L.

(ii) Let f be a monomorphism from a lattice L into a lattice M. Show that L is isomorphic to a sublattice of M.

5. (a) (i) Prove that $(x \wedge y)' = x' \vee y'$ and $(x \vee y)' = x' \wedge y'$ for all x, y in a Boolean algebra.

Deduce that $x \leq y \Leftrightarrow x' \geq y'$ for all $x, y \in B$.

(ii) Show that the lattice $B = (\{1,2,3,6,9,18\}, \gcd, \text{lcm})$ of all positive divisors of 18 does not form a Boolean algebra.

(b) Find the conjunctive normal form of

$$(x_1 + x_2 + x_3)(x_1x_2 + x_1'x_3)'$$

(c) Use a Karnaugh Diagram to simplify

$$p = x_1x_2x_3 + x_2x_3x_4 + x_1'x_2x_4' + x_1'x_2x_3x_4' + x_1'x_2x_4'$$

6. (a) Use the Quine-McCluskey method to find the minimal form of

$$wxyz' + wxy'z' + wx'yz + wx'yz' + w'x'yz + w'x'yz' + w'x'y'z$$

(b) Draw the contact diagram and determine the symbolic representation of the circuit given by

$$p = x_1x_2(x_3+x_4) + x_1x_3(x_5+x_6)$$

(c) Give mathematical models for the following random experiments

(i) when in tossing a die, all outcomes and all combinations are of interest.

(ii) when tossing a die, we are only interested whether the points are less than 3 or greater than or equal to 3.