Statistics Exam Solutions

Question 1

(a) Test scores analysis

Given:

- Group I: 10, 10, 10, 15, 35, 75, 90, 95, 100, 175, 420, 490, 515, 515, 790
- Group II: 0, 5, 5, 15, 30, 45, 50, 50, 50, 60, 75, 110, 140, 240, 330

(i) Calculate quartiles

For Group I (n=15):

$$\text{Median position} = \frac{n+1}{2} = 8$$

$$Q_1 \text{ position} = \frac{n+1}{4} = 4$$

$$Q_3 \text{ position} = \frac{3(n+1)}{4} = 12$$

$$\Rightarrow \text{Median} = 95,\ Q_1 = 15,\ Q_3 = 490$$

For Group II (n=15):

$$\text{Median} = 50,\ Q_1 = 15,\ Q_3 = 110$$

(ii) Identify outliers

Group I:

$$\text{IQR} = Q_3 - Q_1 = 490 - 15 = 475$$
$$\text{Lower fence} = Q_1 - 1.5 \times \text{IQR} = 15 - 712.5 = -697.5$$
$$\text{Upper fence} = Q_3 + 1.5 \times \text{IQR} = 490 + 712.5 = 1202.5$$
$$\text{No outliers (all values within fences)}$$

Group II:

$$\text{IQR} = 110 - 15 = 95$$
$$\text{Upper fence} = 110 + 142.5 = 252.5$$
$$330 \text{ is an extreme outlier}$$

(b) Parking probability

(i) Total arrangements

$$20! \text{ possible arrangements}$$

(ii) Alternating probability

$$\text{Favorable arrangements} = 2 \times 10! \times 10!$$
$$\text{Probability} = \frac{2 \times 10! \times 10!}{20!}$$

Question 2

(a) Trimmed mean

Given data: 2, 4, 6, 7, 11, 21, 81, 90, 105, 121

27% trimmed mean

$$27\% \text{ of } 10 = 2.7 \approx 3 \text{ removed from each end}$$
$$\text{Trimmed data} = 7, 11, 21, 81, 90$$
$$\text{Trimmed mean} = \frac{7 + 11 + 21 + 81 + 90}{5} = 42$$

Trimming percentage

$$\frac{1}{10} = 10\% \text{ trimming per end} \Rightarrow 20\% \text{ total}$$

(b) Probability of tag loss

(i) At least one tag lost

$$\begin{aligned} P(A_1 \cup A_2) &= 1 - P(\text{none lost}) \\ &= 1 - (0.7 \times 0.7) = 0.51 \end{aligned}$$

(ii) Exactly one given at least one

$$P(\text{exactly one}) = 2 \times 0.3 \times 0.7 = 0.42$$

$$P(\text{exactly one}|\text{at least one}) = \frac{0.42}{0.51} \approx 0.8235$$

Question 3

(a) Plant survival

(i) Probability plant is alive

$$\begin{aligned} P(\text{Alive}) &= P(\text{Water})P(\text{Alive}|\text{Water}) + P(\text{No water})P(\text{Alive}|\text{No water}) \\ &= 0.9 \times 0.85 + 0.1 \times 0.2 = 0.785 \end{aligned}$$

(ii) Neighbor forgot given plant died

$$\begin{aligned} P(\text{No water}|\text{Dead}) &= \frac{P(\text{Dead}|\text{No water})P(\text{No water})}{P(\text{Dead})} \\ &= \frac{0.8 \times 0.1}{0.215} \approx 0.3721 \end{aligned}$$

(b) Crash test results

Sample proportion

$$\hat{p} = \frac{7}{10} = 0.7$$

Additional successes needed

$$0.8 \times 25 = 20 \text{ successes needed}$$
$$20 - 7 = 13 \text{ more from 15 cars}$$

Question 4

(a) Airline overbooking

(i) Accommodate all

$$P(Y \leq 50) = \sum_{y=45}^{50} P(y) = 0.83$$

(ii) Cannot accommodate

$$P(Y > 50) = 1 - 0.83 = 0.17$$

(iii) Standby probabilities

- 1st standby: $P(Y \leq 49) = 0.66$
- 3rd standby: $P(Y \leq 47) = 0.27$

(b) Win-win game

Expected values

$$E(X) = 1 \times 0.4 + 2 \times 0.3 + 3 \times 0.1 + 4 \times 0.2 = 2.1$$
$$E(1000/X) = 1000 \left(\frac{0.4}{1} + \frac{0.3}{2} + \frac{0.1}{3} + \frac{0.2}{4} \right) \approx 633.33$$

Recommend random prize (higher expected value).

Question 6

(a) Defective items

PMF (Hypergeometric)

$$P(X = k) = \frac{C(3, k)C(7, 3 - k)}{C(10, 3)}$$

$$P(0) = \frac{35}{120} \approx 0.2917$$
$$P(1) = \frac{63}{120} = 0.525$$
$$P(2) = \frac{21}{120} = 0.175$$
$$P(3) = \frac{1}{120} \approx 0.0083$$

CDF

$$F(0) = 0.2917$$
$$F(1) = 0.8167$$
$$F(2) = 0.9917$$
$$F(3) = 1$$

(b) Uniform distribution

Median

$$\text{Median} = \frac{A + B}{2}$$

90th percentile

$$P_{90} = A + 0.9(B - A)$$

Question 7

(a) Bus waiting time

Probability

$$\begin{aligned}\text{Total time} &= 30 \text{ minutes} \\ \text{Favorable intervals} &= (7:00-7:03) \text{ and } (7:15-7:18) \\ P(\text{wait} \geq 12) &= \frac{6}{30} = 0.2\end{aligned}$$

(b) Poisson claims

(i) Fewer than 3 claims

$$P(X<3) = e^{-5}\left(1+5+\frac{25}{2}\right) \approx 0.1247$$

(ii) 4 claims in 3 of 5 days

$$\begin{aligned}P(4) &= \frac{e^{-5}5^4}{24} \approx 0.1755 \\ \text{Binomial} &= C(5,3)(0.1755)^3(0.8245)^2 \approx 0.0367\end{aligned}$$

(c) System reliability

Probability functional

$$P = \sum_{k=4}^{6} \frac{C(15,k)C(5,6-k)}{C(20,6)} \approx 0.8687$$

Expected working components

$$E = 6 \times \frac{15}{20} = 4.5$$

Question 8

(a) Poisson colds

(i) No more than 2 colds

$$\begin{aligned}\text{With supplements}(\lambda = 3) &: P(X \leq 2) \approx 0.4232 \\ \text{Without supplements}(\lambda = 5) &: P(X \leq 2) \approx 0.1247\end{aligned}$$

(ii) Combined probability

$$0.7 \times 0.4232 + 0.3 \times 0.1247 \approx 0.3336$$

(b) Smoker probability

Normal approximation

$$\begin{aligned}\mu &= 750 \times 0.3 = 225 \\ \sigma &= \sqrt{750 \times 0.3 \times 0.7} \approx 12.55\end{aligned}$$

(i) Fewer than 200

$$P(X < 200) \approx P(Z < -2.03) \approx 0.0212$$

(ii) 240 or more

$$P(X \geq 240) \approx P(Z > 1.16) \approx 0.1230$$

(c) Battery lifespan

(i) Last more than 4 years

$$P(X > 4) = e^{-4/6} \approx 0.5134$$

(ii) Additional 5 years given 3

$$P(X > 8 | X > 3) = P(X > 5) = e^{-5/6} \approx 0.4346$$

Question 9

(a) Credit card purchases

(i) Expectation and variance

$$E(X) = np = 10 \times 0.7 = 7$$
$$\text{Var}(X) = np(1-p) = 10 \times 0.7 \times 0.3 = 2.1$$

(ii) Probability range

$$P(5 \leq X \leq 8) \approx 0.8033$$

(b) Poisson death approximation

$$\lambda = 5,\ P(X \leq 6) \approx 0.7622$$

(c) SAT scores

(i) 30th percentile

$$z \approx -0.524 \Rightarrow x \approx 464.12$$

(ii) Score of 700

$$z = \frac{700 - 527}{120} \approx 1.4417 \Rightarrow P(Z > 1.4417) \approx 0.0746$$

Question 13

Central Limit Theorem

$$\text{Total weeks} = 36$$
$$\mu_T = 36 \times 50 = 1800$$
$$\sigma_T = \sqrt{36 \times 16} = 24$$
$$P(1728 < T < 1872) = P(-3 < Z < 3) \approx 0.9974$$

Question 14

(i) Sample mean probability

$$\sigma_{\bar{X}} = \frac{3200}{\sqrt{36}} \approx 533.33$$

$$P(52000 < \bar{X} < 55000) \approx 0.9944$$

(ii) Small sample size

With n=12, CLT less reliable unless population is normal.