1. Attempt any **five** questions : $(5\times3=15)$

(a) Prove that even function can have no sine terms in its Fourier expansion.

(b) Determine whether the functions $\cos 2x$ and $\cos x$ are orthogonal or not in the interval $(0, 2\pi)$.

(c) Evaluate : $\int_0^{\pi/2} \cos^6\theta \, d\theta$.

(d) Find the value of $\Gamma\left(\frac{-5}{2}\right)$.

(e) Show that for integral values of n, $AJ_n(x) + BJ_{-n}(x)$ is not a general solution of Bessel equation of order n.

(f) Prove : $P_n'(1) = \frac{n(n+1)}{2}$.

(g) Find whether $x = 1$ is an ordinary, regular or irregular singular point of the given differential equation :

$$x^2\left(1-x^2\right)y'' + \frac{2}{x}y' + 4y = 0$$

(h) Determine whether or not $u(x, y) = 4e^{-3x} \cos 3y$ is a solution of given partial differential equation :

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

2. (a) Find the Fourier series expansion of a periodic function given by : (10)

$$f(t) = \begin{cases} E_o \sin t, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \end{cases}$$

(b) Evaluate : $\int_0^2 x(8 - x^3)^{1/3} dx$ (5)

3. Consider a periodic function f(x) of period 2π such that

$$f(x) = \pi - x, \ 0 < x < \pi$$

(a) Plot odd extension of f(x) in the range $(-3\pi, 3\pi)$. (3)

(b) Find its half-range Fourier Sine Series. (6)

(c) Show that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... = \frac{\pi}{4}$ (3)

(d) Also, prove that $1 + \frac{1}{4} + \frac{1}{9} + ... = \frac{\pi^2}{6}$ (3)

4. Consider the following differential equation :

$$4x^2 y'' + 4xy' + (x^2 - 1)y = 0$$

(a) Find whether x = 0 is an ordinary, regular or irregular singular point. (3)

(b) Using Frobenius method, determine the roots of indicial equation and hence find the first solution. (4,5)

(c) Also, find the second solution. (3)

5. (a) Prove that orthogonality relation for Legendre polynomials is given by

$$\int_{-1}^{1} P_n(x)\, P_m(x)\, dx = \begin{cases} \dfrac{2}{2n+1}, & m = n \\ 0, & m \neq n \end{cases} \quad (10)$$

(b) The generating function of Legendre polynomials is given by :

$$(1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x),$$

Using this generating function, prove that :

$$P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x) \quad (5)$$

6. Given, $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$

Verify that :

(i) $\cos(x\sin\theta) = J_0(x) + 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta + \ldots$

(ii) $\sin(x\sin\theta) = 2J_1(x)\sin\theta + 2J_3(x)\sin 3\theta + \ldots$

Hence prove that :

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\,d\theta$$ (3,3,9)

7. Using the method of separation of variables, find the general solution of 2-D wave .equation for the case of symmetrical circular membrane (radius = a):

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2};\ c>0$$

subject to the conditions :

$u(a, t) = 0$, $\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$ and $u(r, 0) = u_0(r)$ (15)

8. (a) Using the method of separation of variables, solve the following differential equation :

$$\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}.$$

when $u(0, y) = 8e^{-3y} + 4e^{-5y}$. (5)

(b) A homogeneous rod of conducting material of length 100 cm has its ends kept at zero temperature and the initial temperature is

$$u(x, 0) = \begin{cases} x, & 0 \le x \le 50 \\ 100 - x, & 50 \le x \le 100 \end{cases}$$

Using 1-D heat equation, find the temperature $u(x, t)$ at any time. (10)