Mathematical Physics I - BSc Physics Hons, DU

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Exam-Focused Notes

1. Differential Calculus & Taylor Series

Differential calculus is fundamental to physics, enabling us to describe rates of change and optimize functions. The Taylor series provides a powerful tool for approximating functions.

1.1 Taylor Series

The Taylor series allows for the expansion of a function $f(x)$ around a point $x = a$ as an infinite sum of terms, provided $f(x)$ is infinitely differentiable at $x = a$. The general form is given by:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f''(a)}{3!}(x-a)^3 + \cdots$$

Important Points:

- Maclaurin Series: When the expansion is done around $x = 0$, it is specifically called a Maclaurin series.
- Common Expansions to Remember: It is crucial to memorize the Taylor (or Maclaurin) series expansions for common functions as they frequently appear in physics problems:
  - $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$
  - $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$
  - $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

2. Ordinary Differential Equations (ODEs)

Ordinary Differential Equations are equations that involve an unknown function of one independent variable and its derivatives. They are ubiquitous in physics for modeling dynamic systems.

2.1 First-Order ODEs

These involve the first derivative of the unknown function.

2.1.1 Separable Equations   A first-order ODE is separable if it can be written in the form $\frac{dy}{dx} = f(x)g(y)$. The solution is found by separating the variables and integrating:

$$\frac{dy}{dx} = f(x)g(y) \Rightarrow \int \frac{1}{g(y)}dy = \int f(x)dx$$

2.1.2 Exact Equations   An ODE of the form $M(x, y)dx + N(x, y)dy = 0$ is exact if the condition $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ is met. If it's exact, a solution $\phi(x, y) = C$ exists such that $\frac{\partial \phi}{\partial x} = M$ and $\frac{\partial \phi}{\partial y} = N$.

2.2 Second-Order Linear ODEs

These involve the second derivative of the unknown function and are linear in the dependent variable and its derivatives.

2.2.1 Homogeneous   A second-order linear homogeneous ODE has the form $ay'' + by' + cy = 0$. The solution typically involves finding the roots of the characteristic equation $ar^2 + br + c = 0$.

2.2.2 Non-Homogeneous   For non-homogeneous equations of the form $ay'' + by' + cy = f(x)$, the general solution is the sum of the complementary function (CF, solution to the homogeneous part) and a particular integral (PI).

- Method of Undetermined Coefficients: Used when $f(x)$ is a polynomial, exponential, sine, cosine, or a combination thereof.
- Variation of Parameters: A more general method applicable when the method of undetermined coefficients is not suitable or for more complex $f(x)$.

2.3 Euler-Cauchy Equation

The Euler-Cauchy equation is a special type of second-order linear homogeneous ODE given by $x^2y'' + axy' + by = 0$.

- Solution Method: The standard approach is to substitute $y = x^r$ into the equation and solve for r. This leads to a characteristic equation for r.

3. Operator (D) Method

The D-operator method provides a convenient algebraic approach to solving linear ODEs, especially those with constant coefficients.

- Let $D = \frac{d}{dx}$.
- For solving equations like $(D^2 + 3D + 2)y = f(x)$:

1. Find Complementary Function (CF): Solve the homogeneous equation $(D^2 + 3D + 2)y = 0$ by finding the roots of the auxiliary equation $D^2 + 3D + 2 = 0$.
2. Find Particular Integral (PI): Determine the particular solution based on the form of the RHS $f(x)$ using standard rules for the D-operator.

4. Partial Differentiation & Extrema

Partial differentiation is essential when dealing with functions of multiple independent variables, common in thermodynamics, fluid dynamics, and electromagnetism.

4.1 Partial Derivatives

For a function $f(x, y, z, \ldots)$, the partial derivative with respect to one variable (e.g., x) is found by treating all other variables as constants.

- $\frac{\partial f}{\partial x}$
- $\frac{\partial f}{\partial y}$

4.2 Total Derivative

The total derivative extends the concept of a derivative to functions where the independent variables themselves depend on another variable. For $z = f(x, y)$ where $x = x(t)$ and $y = y(t)$, the total derivative of z with respect to t is:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

4.3 Maxima/Minima

To find local extrema (maxima or minima) of a multivariable function $f(x, y)$:

- First, find the critical points by setting the first partial derivatives to zero: $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$.
- Use Hessian Matrix: To determine the nature of these critical points (maxima, minima, or saddle points), construct the Hessian matrix:

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

- Check Determinant: Calculate the determinant $D = f_{xx}f_{yy} - (f_{xy})^2$:
  - If $D > 0$ and $f_{xx} > 0$, it's a local minimum.
  - If $D > 0$ and $f_{xx} < 0$, it's a local maximum.
  - If $D < 0$, it's a saddle point.
  - If $D = 0$, the test is inconclusive.

5. Vector Algebra & Calculus

Vectors are indispensable in physics for representing quantities that have both magnitude and direction, like force, velocity, and electric fields.

5.1 Scalar and Vector Products

5.1.1 Dot Product (Scalar Product)

The dot product of two vectors $\mathbf{A}$ and $\mathbf{B}$ yields a scalar quantity. It is defined as:

$$\mathbf{A} \cdot \mathbf{B} = AB \cos\theta$$

where A and B are the magnitudes of the vectors and θ is the angle between them. The dot product is used to find the work done by a force or the component of a vector along another.

5.1.2 Cross Product (Vector Product)

The cross product of two vectors $\mathbf{A}$ and $\mathbf{B}$ yields another vector that is perpendicular to both $\mathbf{A}$ and $\mathbf{B}$. Its magnitude is:

$$|\mathbf{A} \times \mathbf{B}| = AB \sin\theta$$

The direction is given by the right-hand rule, represented by the unit vector $\hat{n}$.

$$\mathbf{A} \times \mathbf{B} = AB \sin\theta \hat{n}$$

The cross product is used to calculate torque, angular momentum, and magnetic force.

5.2 Gradient, Divergence, Curl

These are fundamental vector differential operators that describe how scalar and vector fields change in space.

5.2.1 Gradient

The gradient of a scalar field $f(x, y, z)$ is a vector field that points in the direction of the greatest rate of increase of f, and its magnitude is that maximum rate of increase.

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

In Cartesian coordinates, it is often written as $\nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}$. The gradient is used to find conservative forces from potential energy functions.

5.2.2 Divergence The divergence of a vector field $\mathbf{A}(x, y, z)$ is a scalar field that measures the "outward flux" per unit volume at a given point. It indicates the source or sink strength of the field.

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Divergence is crucial in Gauss's law for electricity and magnetism.

5.2.3 Curl The curl of a vector field $\mathbf{A}(x, y, z)$ is a vector field that measures the "rotation" or "circulation" of the field at a given point.

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Curl is central to Ampere's law and Faraday's law in electromagnetism.

6. Dirac Delta Function

The Dirac delta function, $\delta(x)$, is an important generalized function (or distribution) used extensively in quantum mechanics, signal processing, and electromagnetism. It is often described as a function that is zero everywhere except at $x = 0$, where it is infinitely large, such that its integral over all space is one.

- Properties:
  - $\delta(x) = 0$ for $x \neq 0$
  - $\int_{-\infty}^{\infty} \delta(x)dx = 1$
  - Sifting Property: $\int_{-\infty}^{\infty} f(x)\delta(x - a)dx = f(a)$. This property is extremely useful for extracting the value of a function at a specific point.
  - Scaling Property: $\delta(ax) = \frac{1}{|a|}\delta(x)$.

7. Wronskian

The Wronskian is a determinant used to determine the linear independence of a set of functions, particularly solutions to linear ordinary differential equations.

- Given functions $y_1(x)$ and $y_2(x)$, the Wronskian $W(y_1, y_2)$ is defined as:

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$$

- Key Point: If $W \neq 0$ for all x in an interval, then the functions y_1 and y_2 are linearly independent on that interval. If $W = 0$ for all x in an interval, they are linearly dependent.
- Common PYQ: Check if e^x and $\ln x$ are independent using the Wronskian.

8. Polar & Axial Vectors

Vectors can be classified based on their transformation properties under spatial inversion (reversal of coordinate axes, i.e., $x \to -x$, $y \to -y$, $z \to -z$).

- Polar Vectors: These vectors change their sign under inversion. They transform like displacements (e.g., position vector $\mathbf{r}$, velocity $\mathbf{v}$, force $\mathbf{F}$).
- Axial Vectors (Pseudovectors): These vectors do not change their sign under inversion. They are typically defined as the cross product of two polar vectors (e.g., angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, magnetic field $\mathbf{B}$).

9. PYQ Practice Topics (Frequent)

Regular practice with these topics will be beneficial for exams.

Topic	Type of Question
Wronskian	Independence of solutions
Delta Function	Integration based on identity
Vector Algebra	Definitions of vector types
Taylor Series	Expansion of functions
ODEs	Solve using D-operator or exact method
Gradient/Curl/Divergence	Compute for given vector field

Good luck with your exams!