Mathematical Physics I

Questions

Q1. Vector Calculus and Field Theory

(a) Prove the vector identity:

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

using index notation. (5 marks)

(b) For the vector field $\mathbf{F} = 2x(y^2 + z^3)\hat{\imath} - 2x^2 y\hat{\jmath} + 3x^2 z^2 \hat{k}$:

- Show that $\mathbf{F}$ is conservative.
- Find its scalar potential ϕ.
- Calculate the work done in moving a particle from $(-1, 2, 1)$ to $(2, 3, 4)$.

(10 marks)

(c) Compute the directional derivative of $\Phi = 2xz - y^2$ at the point $(1, 3, 2)$ in the direction of maximum increase. (5 marks)

Q2. Differential Equations

(a) Solve the Cauchy-Euler equation:

$$x^2 \frac{d^2 y}{dx^2} + 4x \frac{dy}{dx} + 2y = e^x$$

using the substitution $x = e^t$. (8 marks)

(b) Use the method of **variation of parameters** to solve:

$$\frac{d^2 y}{dx^2} + y = x - \cot x$$

(7 marks)

(c) Solve the radioactive decay problem:

$$\frac{dN}{dt} = kN, \quad N(0) = 50\,\text{mg}, \quad N(2) = 45\,\text{mg}$$

and find the mass after 4 hours. (5 marks)

Q3. Theorems and Applications

(a) State and prove **Green's theorem** in the plane. Use it to derive the formula for the area enclosed by a curve C:

$$\text{Area} = \frac{1}{2} \oint_C (x\,dy - y\,dx)$$

(10 marks)

(b) Verify **Stokes' theorem** for the vector field $\mathbf{F} = y^2\hat{\imath} - (x+z)\hat{\jmath} + yz\hat{k}$ over the unit square in the xy-plane. (10 marks)

Q4. Probability and Distributions

(a) Derive the **mean** and **variance** of a Poisson distribution:

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

(7 marks)

(b) A student population has 70% government school attendees. If 15 representatives are chosen randomly:

- Find the mean and standard deviation of government school students in the sample.
- What is the probability that exactly 10 are from government schools?

(8 marks)

(c) For the probability distribution:

X	-3	6	9
$P(X = x)$	$1/6$	$1/2$	$1/3$

compute $E(X)$ and $E((2X+1)^2)$. (5 marks)

Q5. Special Topics

(a) Show that the scalar product $\mathbf{A} \cdot \mathbf{B}$ is **invariant under rotation**. Use matrix notation in your proof. (6 marks)

(b) Find the **Taylor series expansion** of $f(x) = \ln(1+x)$ about $x = 0$ and use it to approximate $\ln(0.9)$. (4 marks)

(c) Solve the **inexact differential equation**:

$$(2x \log x - xy)dy + 2y dx = 0$$

by finding an integrating factor. (5 marks)

(d) Evaluate the surface integral:

$$\iint_S \mathbf{r} \cdot \hat{n}\, dS$$

where S is a closed surface, using the divergence theorem. (5 marks)

Useful Formulas

- $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$
- Green's Theorem: $\oint_C (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$
- Poisson PMF: $P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$
- Taylor series: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$