

1. (a) Prove that the order of a permutation of a finite set written in disjoint cycle form is the least common multiple of the lengths of the cycles. (7.5)

(b) (i) Let S_n denote the symmetric group of degree n . In S_3 , find elements α and β such that $|\alpha| = 2$, $|\beta| = 2$ and $|\alpha\beta| = 3$.

(ii) Let $\beta \in S_7$ and $\beta^4 = (2\ 1\ 4\ 3\ 5\ 6\ 7)$. Then find β . (3,4.5)

(c) (i) Give two reasons to show that the set of odd permutations in S_n is not a subgroup of S_n .

(ii) Define even and odd permutations and show that the set of even permutations in S_n is a subgroup of S_n . (3, 4.5)

2. (a) (i) Let a be an element in a group G such that

$|a| = 15$. Find all left cosets of $\langle a^5 \rangle$ in $\langle a \rangle$.

(ii) State and prove Lagrange's theorem.

(3,4.5)

(b) Suppose that G is a group with more than one element and G has no proper, non-trivial subgroups.

Prove that $|G|$ is prime. (7.5)

(c) Let \mathbb{C}^* be the group of non-zero complex numbers under multiplication and let $H = \{a + bi \in \mathbb{C}^* \mid a^2 + b^2 = 1\}$. Give a geometrical description of the coset $(3 + 4i)H$. Give a geometrical description of the coset $(c + di)H$. (7.5)

3. (a) (i) Let G be a group and H be its subgroup.
Prove that if H has index 2 in G , then H is normal in G .

(ii) If a group G has a unique subgroup H of some finite order, then show that H is normal in G . (3,4.5)

(b) (i) Prove that a factor group of a cyclic group is cyclic. Is converse true? Justify your answer.

(ii) Let G be a group and let $Z(G)$ be the center of G . If $G/Z(G)$ is cyclic, then show that G is Abelian. (3,4.5)

(c) (i) Let ϕ be a group homomorphism from group G_1 to group G_2 and H be a subgroup of G_1 . Show that if H is cyclic, then $\phi(H)$ is cyclic.

(ii) How many homomorphisms are there from \mathbb{Z}_{20} to \mathbb{Z}_8 ? How many are there onto \mathbb{Z}_8 ?

(3,4.5)

4. (a) (i) Suppose that ϕ is a homomorphism from $U(30)$ to $U(30)$ and $\text{Ker } \phi = \{1, 11\}$. If $\phi(7) = 7$, find all the elements of $U(30)$ that map to 7.

(ii) Let ϕ be a homomorphism from a group G_1 to group G_2 . Show that $\phi(a) = \phi(b)$ iff $a\text{Ker } \phi = b\text{Ker } \phi$. (3,4.5)

(b) (i) Is $U(8)$ isomorphic to $U(10)$? Justify your answer.

(ii) Show that any infinite cyclic group is isomorphic to the group of integers under addition. (3,4.5)

(c) If ϕ is an onto homomorphism from group G_1 to group G_2 , then prove that $G_1/\text{Ker } \phi$ is isomorphic to G_2 . Hence show that if G_1 is finite, then order of G_2 divides the order of G_1 . (7.5)

5. (a) Let G be a group and let $a \in G$. Define the inner automorphism of G induced by a . Show that the set of all inner automorphisms of a group G , denoted by $\text{Inn}(G)$, forms a subgroup of $\text{Aut}(G)$, the group of all automorphisms of G . Find $\text{Inn}(D_4)$. (7.5)

(b) Prove that the order of an element in a direct product of a finite number of finite groups is the lcm of the orders of the components of the element, i.e., $|(g_1, g_2, \dots, g_n)| = \text{lcm}(|g_1|, |g_2|, \dots, |g_n|)$. Also, find the number of elements of order 7 in $\mathbb{Z}_{49} \oplus \mathbb{Z}_7$. (7.5)

(c) Without doing any calculations in $\text{Aut}(\mathbb{Z}_{105})$, determine how many elements of $\text{Aut}(\mathbb{Z}_{105})$ have order 6. (7.5)

6. (a) For any group G , prove that $G/Z(G) \cong \text{Inn}(G)$. (7.5)
- (b) Define the internal direct product of a collection of subgroups of a group G . Let \mathbb{R}^* denote the group of all nonzero real numbers under multiplication. Let \mathbb{R}^+ denote the group of all positive real numbers under multiplication. Prove that \mathbb{R}^* is the internal direct product of \mathbb{R}^+ and the subgroup $\{1, -1\}$. (7.5)

(c) The set $G = \{1, 4, 11, 14, 16, 19, 26, 29, 31, 34, 41, 44\}$ is a group under multiplication modulo 45. Write G as an external and an internal direct product of cyclic groups of prime-power order.

(7.5)