Exam Solutions

Question 1

(a)

Find the equation of the tangent line to the graph of $(x^2+y^2)^3 = 8x^2y^2$ at the point $(-1,1)$.

Solution:

We use implicit differentiation to find the slope of the tangent line at $(-1,1)$.

Given: $(x^2+y^2)^3 = 8x^2y^2$

Differentiating both sides with respect to x:

$$3(x^2+y^2)^2(2x+2y\frac{dy}{dx}) = 8(2xy^2 + x^2 \cdot 2y\frac{dy}{dx})$$

At $(-1,1)$:

$$\begin{aligned}
3((-1)^2+(1)^2)^2(2(-1)+2(1)\frac{dy}{dx}) &= 8(2(-1)(1)^2 + (-1)^2 \cdot 2(1)\frac{dy}{dx}) \\
3(1+1)^2(-2+2\frac{dy}{dx}) &= 8(-2+2\frac{dy}{dx}) \\
3(4)(-2+2\frac{dy}{dx}) &= -16+16\frac{dy}{dx} \\
-24+24\frac{dy}{dx} &= -16+16\frac{dy}{dx} \\
8\frac{dy}{dx} &= 8 \\
\frac{dy}{dx} &= 1
\end{aligned}$$

Using point-slope form:

$$y-1 = 1(x-(-1))$$

$$y = x + 2$$

Final Answer: $\boxed{y = x + 2}$

(b)

Estimate the maximum error when approximating $f(x) = \sqrt{x}$ with a second order Taylor polynomial about $x = 4$ in the interval $[4, 4.2]$.

Solution:

Taylor series about $x = 4$:

$$f(x) \approx f(4) + f'(4)(x-4) + \frac{f''(4)}{2}(x-4)^2$$

Compute derivatives:

$$f'(x) = \frac{1}{2}x^{-1/2} \Rightarrow f'(4) = \frac{1}{4}$$
$$f''(x) = -\frac{1}{4}x^{-3/2} \Rightarrow f''(4) = -\frac{1}{32}$$

The remainder term is:

$$R_2(x) = \frac{f'''(c)}{6}(x-4)^3$$

where $c \in [4, 4.2]$

Third derivative:

$$f'''(x) = \frac{3}{8}x^{-5/2}$$

Maximum error occurs at $x = 4.2$ and $c = 4$ (since $f'''(x)$ is decreasing):

$$|R_2(4.2)| \leq \left|\frac{3}{8 \cdot 4^{5/2}} \cdot \frac{(0.2)^3}{6}\right| = \frac{3}{8 \cdot 32} \cdot \frac{0.008}{6} \approx 1.5625 \times 10^{-6}$$

Final Answer: $\boxed{1.5625 \times 10^{-6}}$

Question 2

(a)

Prove that for all real numbers x_1, x_2, $\frac{3^{x_1}+3^{x_2}}{2} \geq 3^{\frac{x_1+x_2}{2}}$

Solution:

This is the AM-GM inequality applied to 3^{x_1} and 3^{x_2}:

$$\frac{3^{x_1} + 3^{x_2}}{2} \geq \sqrt{3^{x_1} \cdot 3^{x_2}} = 3^{\frac{x_1+x_2}{2}}$$

The inequality holds because exponential functions are convex.

Final Answer: $\boxed{\text{The inequality holds by the AM-GM inequality and convexity of exponential functions.}}$

(b)

Is the statement "$3x \to 7$ as $x \to 1$" true? Justify using ϵ-δ definition.

Solution:

The statement is false. The correct limit is $\lim_{x\to 1} 3x = 3$.

Using ϵ-δ definition: For any $\epsilon > 0$, choose $\delta = \epsilon/3$. Then:

$$0 < |x - 1| < \delta \Rightarrow |3x - 3| = 3|x - 1| < 3\delta = \epsilon$$

Thus, the limit is 3, not 7.

Final Answer: $\boxed{\text{False, the limit is 3, not 7.}}$

Question 3

(a)

Restrictions on a and b for $g(x) = af(x) + b$ to be convex when f is convex.

Solution:

For g to be convex:

$$\begin{aligned}
g(\lambda x + (1-\lambda)y) &\leq \lambda g(x) + (1-\lambda)g(y) \\
af(\lambda x + (1-\lambda)y) + b &\leq \lambda(af(x) + b) + (1-\lambda)(af(y) + b) \\
af(\lambda x + (1-\lambda)y) + b &\leq a\lambda f(x) + a(1-\lambda)f(y) + b
\end{aligned}$$

Since f is convex, the inequality holds if $a \geq 0$. No restriction on b is needed as it cancels out.

Final Answer: $\boxed{a \geq 0 \text{ and } b \text{ can be any real number}}$

(b)

Solve $1 + 2\log_x(x+1) = 2\log_x x$

Solution:

Simplify the equation:

$$\begin{aligned}
1 + 2\log_x(x+1) &= 2 \quad (\text{since } \log_x x = 1) \\
2\log_x(x+1) &= 1 \\
\log_x(x+1) &= \frac{1}{2} \\
x^{1/2} &= x+1 \\
\sqrt{x} &= x+1
\end{aligned}$$

Square both sides:

$$\begin{aligned}
x = x^2 + 2x + 1 \\
x^2 + x + 1 = 0
\end{aligned}$$

This has no real solutions (discriminant $D = 1 - 4 = -3 < 0$).

Final Answer: $\boxed{\text{No real solutions}}$

Question 4

(i)

Does the extreme value theorem guarantee the existence of maximum and minimum for $f(x) = b^x - a^x$ on $[0, \infty)$?

Solution:

No, the extreme value theorem applies to closed and bounded intervals. $[0, \infty)$ is not bounded, so the theorem doesn't guarantee extrema.

Final Answer: $\boxed{\text{No, because the interval is not bounded.}}$

(ii)

Find the value of x that minimizes $f(x)$ and the minimum value.

Solution:

At $x = 0$:

$$f(0) = b^0 - a^0 = 1 - 1 = 0$$

For $x > 0$, $f(x) > 0$ since $b > a$. Thus, the minimum is 0 at $x = 0$.

Final Answer: $\boxed{x = 0 \text{ with minimum value } 0}$

(iii)

Show the stationary point condition $\left(\frac{b}{a}\right)^x = \frac{\ln a}{\ln b}$.

Solution:

Find $f'(x) = b^x \ln b - a^x \ln a$.

Set $f'(x^*) = 0$:

$$b^{x^*} \ln b = a^{x^*} \ln a$$
$$\left(\frac{b}{a}\right)^{x^*} = \frac{\ln a}{\ln b}$$

Final Answer: $\boxed{\text{Shown by setting the derivative equal to zero.}}$

(iv)

Show that x^* is a maximum by analyzing the first derivative.

Solution:

For $x < x^*$, $f'(x) > 0$ (function increasing). For $x > x^*$, $f'(x) < 0$ (function decreasing). Thus, x^* is a maximum by the first derivative test.

Final Answer: $\boxed{x^* \text{ is a maximum as the derivative changes from positive to negative.}}$

Question 5

(a)

Show there exists $c \in [0, 1]$ such that $[f(c)]^2 = c$.

Solution:

Define $g(x) = [f(x)]^2 - x$. Then:

$$g(0) = [f(0)]^2 \geq 0$$
$$g(1) = [f(1)]^2 - 1 \leq 0$$

If either equals 0, we're done. Otherwise, by IVT, there exists $c \in (0, 1)$ such that $g(c) = 0$.

Final Answer: $\boxed{\text{Exists by Intermediate Value Theorem.}}$

(b)

(i)

First order condition for interior solution $x^* > 0$.

Solution:

The student maximizes $U(x) = g(\pi(x)) - c(x)$.

FOC:

$$g'(\pi(x^*)) \cdot \pi'(x^*) - c'(x^*) = 0$$

Final Answer: $\boxed{g'(\pi(x^*)) \cdot \pi'(x^*) = c'(x^*)}$

(ii)

Is the interior solution necessarily solving the maximization problem?

Solution:

Not necessarily. The problem requires checking:

1. Second order condition: $g''(\pi(x))(\pi'(x))^2 + g'(\pi(x))\pi''(x) - c''(x) \leq 0$

2. Boundary at $x = 0$: Compare $U(0)$ with $U(x^*)$

Final Answer: $\boxed{\text{Not necessarily, need to verify second order conditions and boundary behavior.}}$

Question 6

(a)

Restrictions on a, b, c for the profit function $\pi(x) = ax^2 + bx + c$.

Solution:

1. $\pi(0) = c < 0$
2. Strictly concave: $a < 0$
3. Maximum at $x^* > 0$: $\pi'(x^*) = 2ax^* + b = 0 \Rightarrow x^* = -b/(2a) > 0 \Rightarrow b > 0$ (since $a < 0$)

Final Answer: $\boxed{a < 0, b > 0, c < 0}$

(b)

Solve $\frac{3p-4}{p+2} > 4 - p$.

Solution:

First, $p \neq -2$.

Case 1: $p > -2$

$$\begin{aligned} 3p - 4 &> (4 - p)(p + 2) \\ 3p - 4 &> -p^2 + 2p + 8 \\ p^2 + p - 12 &> 0 \\ (p + 4)(p - 3) &> 0 \end{aligned}$$

Solution: $p < -4$ or $p > 3$, but $p > -2$, so $p > 3$

Case 2: $p < -2$

$$\begin{aligned} 3p - 4 &< (4-p)(p+2) \\ p^2 + p - 12 &< 0 \\ -4 &< p < 3 \end{aligned}$$

But $p < -2$, so $-4 < p < -2$

Final Answer: $\boxed{p \in (-4, -2) \cup (3, \infty)}$

Question 7

(a)

Find A and B for continuity of $f(x)$.

Solution:

Continuity at $x = -1$:

$$\begin{aligned} \lim_{x \to -1^-} f(x) &= A(-1) - B = -A - B \\ \lim_{x \to -1^+} f(x) &= 2(-1)^2 + 3A(-1) + B = 2 - 3A + B \\ \text{Set equal: } -A - B &= 2 - 3A + B \\ 2A - 2B &= 2 \\ A - B &= 1 \end{aligned}$$

Continuity at $x = 1$:

$$\begin{aligned} \lim_{x \to 1^-} f(x) &= 2(1)^2 + 3A(1) + B = 2 + 3A + B \\ \lim_{x \to 1^+} f(x) &= 4 \\ \text{Set equal: } 2 + 3A + B &= 4 \\ 3A + B &= 2 \end{aligned}$$

Solve system:

$$\begin{aligned} A - B &= 1 \\ 3A + B &= 2 \\ \text{Add: } 4A = 3 &\Rightarrow A = 3/4 \\ \text{Then } B &= A - 1 = -1/4 \end{aligned}$$

Final Answer: $\boxed{A = \frac{3}{4}, B = -\frac{1}{4}}$

(b)

Find c for horizontal approximation at $x = 0$.

Solution:

Let $f(x) = e^{-x}\sqrt{1 + cx}$

Compute $f'(x) = -e^{-x}\sqrt{1 + cx} + e^{-x} \cdot \frac{c}{2\sqrt{1+cx}}$

At $x = 0$:

$$f'(0) = -1 \cdot 1 + 1 \cdot \frac{c}{2} = 0$$
$$-1 + c/2 = 0 \Rightarrow c = 2$$

Final Answer: $\boxed{c = 2}$

Question 8

(a)

Growth rate of exponentially growing population A.

Solution:

Exponential growth: $A(t) = A_0 e^{rt}$

Given $A_1 = A_0 e^{rt_1}$, $A_2 = A_0 e^{rt_2}$

Divide: $\frac{A_2}{A_1} = e^{r(t_2 - t_1)}$

Take ln: $r = \frac{\ln(A_2/A_1)}{t_2 - t_1}$

Final Answer: $\boxed{r = \frac{\ln(A_2) - \ln(A_1)}{t_2 - t_1}}$

(b)

Find equations of lines P and Q, their intersection, and line S.

Solution:

Line P: slope -5 through $(2, -1)$

$$y + 1 = -5(x - 2)$$
$$y = -5x + 9$$

Line Q: through $(-2, 3)$ and $(4, -3)$ Slope: $\frac{-3-3}{4-(-2)} = -1$ Equation: $y - 3 = -1(x + 2)$

$$y = -x + 1$$

Intersection R: Set $y = -5x + 9 = -x + 1$

$$-4x = -8 \Rightarrow x = 2$$
$$y = -2 + 1 = -1$$

R is at $(2, -1)$

Line S: parallel to Q (slope -1) through $(3, 4)$

$$y - 4 = -1(x - 3)$$
$$y = -x + 7$$

Final Answers:

- Line P: $\boxed{y = -5x + 9}$
- Line Q: $\boxed{y = -x + 1}$
- Intersection R: $\boxed{(2, -1)}$
- Line S: $\boxed{y = -x + 7}$

Question 9

(a)

Find solution x_0 between a and b for $g'(x) = 0$.

Solution:

Given $g(x) = (b-x)^m (x-a)^n$

Logarithmic differentiation:

$$\begin{aligned}
\frac{g'}{g} &= \frac{-m}{b-x} + \frac{n}{x-a} = 0 \\
\frac{n}{x-a} &= \frac{m}{b-x} \\
n(b-x) &= m(x-a) \\
nb - nx &= mx - ma \\
x(m+n) &= nb + ma \\
x_0 &= \frac{nb+ma}{m+n}
\end{aligned}$$

Final Answer: $\boxed{x_0 = \frac{ma+nb}{m+n}}$

(b)

Identify local extrema of $f(x) = x^3 + cx + 1$ for different c.

Solution:

Find critical points:

$$f'(x) = 3x^2 + c = 0 \Rightarrow x^2 = -c/3$$

Case 1: $c > 0$ - no real solutions, no extrema
Case 2: $c = 0$ - double root at $x = 0$ (inflection point, no extremum)
Case 3: $c < 0$ - two critical points $x = \pm\sqrt{-c/3}$

Second derivative test:

$$f''(x) = 6x$$

At $x = \sqrt{-c/3}$: $f'' > 0$ (local minimum)
At $x = -\sqrt{-c/3}$: $f'' < 0$ (local maximum)

Final Answer: $\boxed{\text{For } c < 0\text{: local max at } x = -\sqrt{-c/3},\ \text{local min at } x = \sqrt{-c/3}\text{. For } c \geq 0\text{: no local ext}}$

Question 10

(a)

Determine where $H(x) = (x^2 - 1)e^{x^2+1}$ is strictly increasing/decreasing.

Solution:

Find derivative:

$$\begin{aligned} H'(x) &= 2xe^{x^2+1} + (x^2 - 1) \cdot 2xe^{x^2+1} \\ &= 2xe^{x^2+1}(1 + x^2 - 1) \\ &= 2x^3 e^{x^2+1} \end{aligned}$$

Since $e^{x^2+1} > 0$ always:

- $H'(x) > 0$ when $x > 0$
- $H'(x) < 0$ when $x < 0$

Final Answers:

- Increasing: $\boxed{(0, \infty)}$
- Decreasing: $\boxed{(-\infty, 0)}$

(b)

(i)

Rate of change of budget surplus $B(t) = R(t) - G(t)$.

Solution:

$$\begin{aligned} B(t) &= (100t - t^2) - (50t + 10) = 50t - t^2 - 10 \\ B'(t) &= 50 - 2t \end{aligned}$$

Final Answer: $\boxed{B'(t) = 50 - 2t}$

(ii)

Find t where proportional rate of change of B equals that of G.

Solution:

Proportional rate of change:

$$\frac{B'(t)}{B(t)} = \frac{G'(t)}{G(t)}$$
$$\frac{50-2t}{50t-t^2-10} = \frac{50}{50t+10}$$

Cross-multiply:

$$\begin{aligned}(50-2t)(50t+10) &= 50(50t-t^2-10)\\ 2500t+500-100t^2-20t &= 2500t-50t^2-500\\ -100t^2+2480t+500 &= 2500t-50t^2-500\\ -50t^2-20t+1000 &= 0\\ 5t^2+2t-100 &= 0\end{aligned}$$

Quadratic formula:

$$t = \frac{-2 \pm \sqrt{4+2000}}{10} = \frac{-2 \pm \sqrt{2004}}{10}$$

Positive solution:

$$t \approx \frac{-2+44.77}{10} \approx 4.277$$

Final Answer: $\boxed{t \approx 4.277}$