1. (a) (i) Find a cubic equation with rational coefficients having the roots

$$\frac{1}{2}, \frac{1}{2}+\sqrt{2}, \text{ stating the result used.}$$

(ii) Find an upper limit to the roots of

$x^5 + 4x^4 - 7x^2 - 40x + 1 = 0.$ (4+3.5)

(b) Find all the integral roots of

$x^4 + 4x^3 + 8x + 32 = 0.$ (7.5)

(c) Find all the rational roots of

$y^4 - \frac{40}{3}y^3 + \frac{130}{3}y^2 - 40y + 9 = 0.$ (7.5)

2. (a) Express arg $(\overline{z})$ and arg $(-z)$ in terms of arg(z). Find the geometric image for the complex number z, such that $\arg(-z) \in \left(\frac{\pi}{6}, \frac{\pi}{3}\right)$. (2+2+3.5)

(b) Find $|z|$, arg z, Arg z, arg $\overline{z}$, arg$(-z)$ for $z = (1-i)(6+6i)$ (7.5)

(c) Find the cube roots of $z = 1+i$ and represent them geometrically to show that they lie on a circle of radius $(2)^{1/6}$. (7.5)

3. (a) Solve $y^3 - 15y - 126 = 0$ using Cardan's method. (7.5)

(b) Let n be a natural number. Given n consecutive integers, a, a + 1, a + 2, ..., a + (n−1), show that one of them is divisible by n. (7.5)

(c) Let a and b be two integers such that gcd(a, b) = g. Show that there exists integers m and n such that g = ma + nb. (7.5)

4. (a) Let a be an integer such that a is not divisible by 7. Show that $a \equiv 5^k \pmod 7$ for some integer k. (7.5)

(b) Let a and b be two integers such that 3 divides $(a^2 + b^2)$. Show that 3 divides a and b both. (7.5)

(c) Solve the following pair of congruences, if possible. If no solution exists, explain why? (7.5)

$x + 5y \equiv 3 \pmod 9$

$4x + 5y \equiv 1 \pmod 9$

5. (a) Consider a square with four corners labelled as follows :

P	W
G	B

Describe the following motions graphically:

(i) R_0 = Rotation of 0 degree.

(ii) R_{90} = Rotation of 90 degrees counterclockwise.

(iii) R_{180} = Rotation of 180 degrees counterclockwise.

(iv) R_{270} = Rotation of 270 degrees counterclockwise.

(v) H = Flip about horizontal axis.

(vi) V = Flip about vertical axis.

(vii) D = Flip about the main diagonal.

(viii) D1 = Flip about the other diagonal.

Identify the motion that can act as identity under the composition of two motions. Further, find out the inverse of each motion. (3.5+1+3)

(b) Show that the set G = $(f_1, f_2, f_3, f_4\}$, is a group under the composition of functions defined as, $fog(x) = f(g(x))$ for f, g in G, where $f_1(x) = x$, $f_2(x) = -x$, $f_3(x) = 1/x$, $f_4(x) = -1/x$ for all non-zero real number x. (7.5)

(c) Define the inverse of an element in a group G. Show that $(a.b)^{-1} = b^{-1}.a^{-1}$ for all a, b in G. Further show that if $(a.b)^{-1} = a^{-1}.b^{-1}$ for all a, b in G, then G is Abelian. (4+3.5)

6. (a) Define Z(G), the center of a group G. Show that Z(G) is a subgroup of G. (2+5.5)

(b) Define order of an element a in group G. Further show that if order of a is n, and $a^m = e$, where m is an integer, then n divides m. (2+5.5)

(c) Find the generators of the cyclic group Z_{30}. Further describe all the subgroups of Z_{30} and find the generators of the subgroup of order 15 in Z_{30}. (2+3.5+2)