1. (a) If $a \in \mathbb{R}$ is such that $0 \le a < \epsilon$ for any $\epsilon > 0$, then show that $a = 0$.

(b) Find all values of x that satisfy $|x - 1| > |x + 1|$. Sketch the graph of this inequality.

(c) Find the supremum and infimum, if they exist, of the following sets:

(i) $\left\{\cos\frac{n\pi}{2} : n \in \mathbb{N}\right\}$

(ii) $\left\{\frac{x+2}{3} : x > 3\right\}$

(d) Show that $\operatorname{Sup}\left\{1-\frac{1}{n} : n \in \mathbb{N}\right\} = 1$.

2. (a) Let S be a non-empty subset of $\mathbb{R}$ that is bounded. Prove that

$$\operatorname{Inf} S = -\operatorname{Sup}\{-s : s \in S\}$$

(b) State and prove the Archimedean Property of real numbers.

(c) If $S = \left\{\frac{1}{n} - \frac{1}{m} : n, m \in \mathbb{N}\right\}$, find *Inf* S and Sup S.

(d) Define a convergent sequence. Show that the limit of a convergent sequence is unique.

3. (a) Using the definition of limit, show that

$$\lim_{n\to\infty} \frac{2n+3}{3n-7} = \frac{2}{3}$$

(b) Show that $\lim_{n\to\infty}\left(n^{1/n}\right) = 1$.

(c) State and prove the Sandwich Theorem for sequences.

(d) Show that every increasing sequence which is bounded above is convergent.

4. (a) Let $x_1 = 1$ and $x_{n+1} = \sqrt{2 + x_n}$ for all $n \ge 1$. Prove that $\langle x_n \rangle$ converges and find its limit.

(b) Prove that every Cauchy sequence is bounded.

(c) Show that the sequence $\langle x_n \rangle$ where

$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, for all $n \in \mathbb{N}$,

does not converge.

(d) Find the limit superior and limit inferior of the following sequences:

(i) $x_n = (-2)^n\left(1 + \frac{1}{n}\right)$, for all $n \in \mathbb{N}$

(ii) $x_n = (-1)^n\left(\frac{1}{n}\right)$, for all $n \in \mathbb{N}$

5. (a) Show that the geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ converges if and only if $|r| < 1$.

(b) Find the sum of the following series, if it converges,

$$\sum \frac{1}{(n+a)(n+a+1)}, \ : a > 0$$

(c) Find the rational number which is the sum of the series represented by the repeating decimal $0.\overline{15}$.

(d) Check the convergence of the following series:

(i) $\sum \frac{1}{\log n}$, $n \ge 2$

(ii) $\sum \tan^{-1}\left(\frac{1}{n}\right)$

6. (a) State the Ratio Test (limit form) for positive series. Using this test or otherwise, check the convergence of the following series:

(i) $\sum\left(\frac{n!}{n^n}\right)$

(ii) $\sum\left(\frac{n!}{e^n}\right)$

(b) Check the convergence of the following series:

(i) $\sum_{n=2}^{\infty}\left(\frac{\log n}{n^2}\right)$

(ii) $\sum\left(\frac{n^{n^2}}{(n+1)^{n^2}}\right)$

(c) Define absolute convergence of a series. Show that every absolutely convergent series is convergent. Is the converse true? Justify your answer.

(d) Check the following series for absolute or conditional convergence :

(i) $\sum(-1)^{n+1}\left(\frac{n}{n^2+1}\right)$

(ii) $\sum_{n=2}^{\infty}(-1)^n\left(\frac{1}{n^2+(-1)^n}\right)$