1. Attempt any six questions : (6×3=18)

(a) Obtain all the roots of the equation :

$$z^3 - i = 0; \; i = \sqrt{-1}$$

(b) Show that $tanh^{-1}(z) = \frac{1}{2} ln \frac{1+z}{1-z}$

(c) In the finite z-plane, determine and classify the singularities of the function:

$$f(z) = tan^{-1}(z^2 + 4z + 5)$$

(d) Solve $\frac{1}{2\pi i} \oint_C \frac{e^z \, dz}{z-1}$; C is $|z-1| = 2$.

(e) Find the residue at $z = 0$ for $f(z) = \frac{\cosh(z)}{z^3}$.

(f) If $\mathcal{F}^{-1}[F(k)] = f(x)$, show that

$$\mathcal{F}^{-1}[F(k-a)] = e^{iax} f(x); a > 0.$$

(g) General solution of 1-d wave equation is given as:

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi t}{L}\right)$$

If $0 \le x \le L$ and $y(x, 0) = x$, determine c_n.

2. (a) If $z = 4\, e^{i\pi/3}$, evaluate $|e^{iz}|$. (4)

(b) Given $\tan(x + iy) = u + iv$, show that

$$u = \frac{\sin(2x)}{\cos(2x) + \cosh(2y)}$$

and $$v = \frac{\sinh(2y)}{\cos(2x) + \cosh(2y)}$$ (6)

(c) Prove that the function $u(x, y) = 2x(1 - y)$ is harmonic and hence find $v(x, y)$ such that $f(z) = u + iv$ is analytic. Also, express $f(z)$ in terms of z, where $z = x + iy$. (8)

3. (a) State and verify Cauchy's theorem for the function:

$$f(z) = 3z + 2i$$

and C is a triangle with vertices $1 + i$, $-1 \pm i$.

(8)

(b) Solve the integral

$$\oint_C \frac{z^2\, dz}{(z^2 + 9)(z^2 + 4)^2}; \; C \text{ is } |z| = 1 \qquad (5)$$

(c) Expand

$$f(z) = \frac{z}{(z + 1)(z - 2)}$$

in a Laurent series valid for the annular domain $0 < |z - 2| < 3$. (5)

4. Using residue theorem and suitable contour, solve any two real integrals : (2×9=18)

(a) $\int_0^{\infty} \frac{x^2}{x^4+1} dx$

(b) $\int_0^{2\pi} \frac{d\theta}{\cos\theta + 2\sin\theta + 3}$

(c) $\int_0^{\infty} \frac{x \sin 2x}{x^2+9} dx$

5. (a) Find $\mathcal{F}^{-1}\left(\frac{1}{k^2 - 4k + 29}\right)$. (8)

(b) Show that $\mathcal{F}_c\left(\frac{1}{\sqrt{x}}\right) = \frac{1}{\sqrt{k}}$. (5)

(c) Obtain the function $q(x)$, if $\mathcal{F}_s[q(x)] = e^{-2k}$. (5)

6. (a) Using the method of separation of variables, find the solution of the following partial differential equation :

$$4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$$

such that $u(0, y) = 3e^{-y}$. (4)

(b) Solve one-dimensional heat equation:

$$\frac{\partial u}{\partial t} = 9\frac{\partial^2 u}{\partial x^2} \qquad (0 \leq x \leq L).$$

such that $u(0, t) = 0, \ u(L, t) = 0$

and $u(x, 0) = x(L - x)$ (14)

Some useful formulae:

1. $\int_0^\infty x^n \, exp(-ax^m) \, dx = \frac{1}{m \, a^{(n+1)/m}} \Gamma\left(\frac{n+1}{m}\right);$

$n > -1; \ a, m > 0$

2. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

3. $\mathcal{F}^{-1}\left(\frac{1}{k^2 + a^2}\right) = \frac{\sqrt{2\pi}}{2a} e^{-a|x|}; \; a > 0$

4. $\mathcal{F}^{-1}[a\, g(k) + b\, h(k)] = a\, \mathcal{F}^{-1}[g(k)] + b\, \mathcal{F}^{-1}[h(k)]$
(*a* and *b* are constants)

5. Use the following definition for the Fourier transform of f(x) :

$$\mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

6. Use the following definition for the Fourier sine transform of f(x) :

$$\mathcal{F}_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin(kx)\, dx$$

7. Use the following definition for the Fourier cosine transform of f(x) :

$$\mathcal{F}_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos(kx)\, dx$$

8. Use the following definition for the convolution of two functions f(x) and g(x) :

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy$$

9. Some useful formulae are given at the end.