Mathematics Exam Solutions

Question 1

(a) For the function $f(x) = \ln(4 - x^2)$

(i) Domain

The argument of the natural logarithm must be positive:

$$4 - x^2 > 0$$

$$x^2 < 4$$

$$-2 < x < 2$$

Domain: $(-2, 2)$

(ii) Asymptotes

- **Vertical asymptotes:** Occur where the function approaches infinity, which happens when $4 - x^2$ approaches 0:
$$\lim_{x \to -2^+} \ln(4 - x^2) = -\infty$$
$$\lim_{x \to 2^-} \ln(4 - x^2) = -\infty$$
So vertical asymptotes at $x = -2$ and $x = 2$.
- **Horizontal asymptotes:** None, as the domain is bounded.

(b) Linear approximation for $g(2.05)$

Given $g(2) = -4$ and $g'(x) = \sqrt{x^2 + 5}$.

First, find $g'(2)$:

$$g'(2) = \sqrt{2^2 + 5} = \sqrt{4 + 5} = 3$$

Linear approximation formula:

$$g(2.05) \approx g(2) + g'(2)(2.05 - 2)$$
$$g(2.05) \approx -4 + 3(0.05)$$
$$g(2.05) \approx -4 + 0.15 = -3.85$$

Estimate: $g(2.05) \approx -3.85$

Question 2

(a) Find the limit

$$\lim_{x\to\infty} \left[x \ln \left(1 - \frac{2}{3x} \right) \right]$$

Let $t = \frac{1}{x}$, so as $x \to \infty$, $t \to 0^+$:

$$\lim_{t\to 0^+} \frac{\ln(1 - \frac{2t}{3})}{t}$$

This is a 0/0 indeterminate form, so apply L'Hôpital's Rule:

$$\lim_{t\to 0^+} \frac{-\frac{2}{3}/(1 - \frac{2t}{3})}{1} = -\frac{2}{3}$$

Limit: $-\frac{2}{3}$

(b) Continuity of piecewise function

$$f(x) = \begin{cases} x^3 - 1 & \text{for } x < 2 \\ x^2 + 3 & \text{for } x \geq 2 \end{cases}$$

Check continuity at $x = 2$:

- Left limit: $\lim_{x \to 2^-} f(x) = 2^3 - 1 = 7$
- Right limit: $\lim_{x \to 2^+} f(x) = 2^2 + 3 = 7$
- Function value: $f(2) = 7$

Since all three are equal, the function is continuous at $x = 2$. The function is continuous everywhere because both pieces are polynomials.

Continuity: The function is continuous everywhere.

Question 3

(a) Check convergence

(i)

$$\sum_{k=0}^{\infty} b \left(1 + \frac{P}{100}\right)^{-k} \qquad P > 0$$

This is a geometric series with ratio $r = \left(1 + \frac{P}{100}\right)^{-1}$. Since $P > 0$, $0 < r < 1$, so the series converges.

Convergence: Converges (geometric series with $|r| < 1$)

(ii)

$$\sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{1}{n}\right)$$

This is an alternating series where:

1. $\frac{1}{n}$ decreases monotonically
2. $\lim_{n \to \infty} \frac{1}{n} = 0$

By the Alternating Series Test, it converges.

Convergence: Converges (alternating series test)

(b) Solve inequality

$$\frac{1}{y} - \frac{1}{y+1} \geq 1$$

Combine terms:

$$\frac{(y+1)-y}{y(y+1)} \geq 1$$
$$\frac{1}{y(y+1)} \geq 1$$
$$1 \geq y(y+1)$$
$$y^2 + y - 1 \leq 0$$

Find roots:

$$y = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$

The parabola opens upward, so the inequality holds between the roots:

$$\frac{-1-\sqrt{5}}{2} \leq y \leq \frac{-1+\sqrt{5}}{2}$$

But we must also consider where the denominator $y(y+1)$ is positive (for the original inequality to hold):

- For $y \in (-1, 0)$, denominator is negative, which would reverse the inequality
- For $y < -1$ or $y > 0$, denominator is positive

Thus the solution is:

$$y \in \left(\frac{-1-\sqrt{5}}{2}, -1\right) \cup \left(0, \frac{-1+\sqrt{5}}{2}\right]$$

Question 4

(a) Find set D

Given:

- $A = \{x : |x| < 1\} = (-1, 1)$

- $B = \{x : |x-1| \geq 1\} = (-\infty, 0] \cup [2, \infty)$

$A \cup B = (-\infty, 1) \cup [2, \infty)$

Given $A \cup B = \mathbb{R} - D$, then:

$$D = \mathbb{R} - (A \cup B) = [1, 2)$$

Set D: $[1, 2)$

(b) Graph transformation

The graph of $f(x) = \ln|x-2|$ is obtained from $f(x) = \ln|x|$ by:

1. Shifting the graph right by 2 units (horizontal shift)
2. The vertical asymptote moves from $x = 0$ to $x = 2$
3. The general shape remains the same

Question 5

(a)(i) Range of y

Given $y = (x-1)^2$ for $0 < x < 2$:

- At $x \to 0^+$, $y \to 1$
- At $x = 1$, $y = 0$
- At $x \to 2^-$, $y \to 1$

The minimum value is 0 at $x = 1$, and the maximum approaches 1 at the endpoints.

Range: $[0, 1)$

(a)(ii) Real root of $f(x) = 20x - e^{-4x}$

- $f(0) = -1$
- $f(1) = 20 - e^{-4} \approx 20 > 0$

By IVT, there's at least one root in $(0, 1)$.

Uniqueness:

$$f'(x) = 20 + 4e^{-4x} > 0 \text{ for all } x$$

Since f is strictly increasing, it can have only one root.

Conclusion: Exactly one real root.

(b)(i) Uniqueness of inverse

Assume F has two inverses G and H. Then:

$$F(G(x)) = x$$

$$F(H(x)) = x$$

But since F is one-to-one (required for inverse to exist), $G(x) = H(x)$.

Conclusion: Inverse is unique.

(b)(ii) Continuity of $|x| + |x - 1|$ on $[-1, 2]$

The function is composed of absolute value functions which are continuous everywhere. The sum of continuous functions is continuous.

Continuity: Continuous everywhere in $[-1, 2]$

Question 6

(a) Solve for x

Let $z = \ln(x + e)$. The equation becomes:

$$z^3 - (2z)^2 = z - 4$$

$$z^3 - 4z^2 - z + 4 = 0$$

Try $z = 1$:

$$1 - 4 - 1 + 4 = 0$$

Factor:

$$(z-1)(z^2 - 3z - 4) = 0$$

Solutions:

$$z = 1 \text{ or } z = \frac{3 \pm \sqrt{9+16}}{2} = \frac{3 \pm 5}{2}$$
$$z = 1, 4, -1$$

Now solve for x:

1. $\ln(x+e) = 1 \Rightarrow x + e = e \Rightarrow x = 0$
2. $\ln(x+e) = 4 \Rightarrow x + e = e^4 \Rightarrow x = e^4 - e$
3. $\ln(x+e) = -1 \Rightarrow x + e = e^{-1} \Rightarrow x = \frac{1}{e} - e$

Check domain:

- For $x = \frac{1}{e} - e$, $x + e = \frac{1}{e} > 0$ (valid)
- All solutions are valid.

Solutions: $x = 0, e^4 - e, \frac{1}{e} - e$

(b) Largest possible $f(4)$

Given:

- f continuous and differentiable on $[-3, 4]$
- $f(-3) = \gamma$
- $f'(x) \leq -17$

By Mean Value Theorem:

$$\frac{f(4) - f(-3)}{4 - (-3)} = f'(c) \leq -17$$
$$f(4) \leq \gamma + 7(-17) = \gamma - 119$$

Largest possible $f(4)$: $\gamma - 119$

Question 7

(a) Analyze $f(x) = \frac{2}{x^3}(2x+5)$

Simplify:

$$f(x) = \frac{4x+10}{x^3} = 4x^{-2} + 10x^{-3}$$

Find derivative:

$$f'(x) = -8x^{-3} - 30x^{-4} = \frac{-8x-30}{x^4}$$

Critical points:

$$-8x - 30 = 0 \Rightarrow x = -\frac{15}{4}$$

Second derivative:

$$f''(x) = 24x^{-4} + 120x^{-5}$$

At $x = -\frac{15}{4}$, $f'' < 0$ (local maximum)

Differentiability at 0: The function is undefined at $x = 0$ (vertical asymptote).

Conclusions:

- Local maximum at $x = -\frac{15}{4}$ (not global)
- Not differentiable at $x = 0$
- Cusp: None, but vertical asymptote at $x = 0$

(b) Solutions of $Ax = e^x$

Consider $f(x) = e^x - Ax$.

Find critical points:

$$f'(x) = e^x - A = 0 \Rightarrow x = \ln A$$

For two solutions:

- $f(\ln A) < 0$

$$e^{\ln A} - A\ln A < 0$$

$$A(1 - \ln A) < 0$$

$$\ln A > 1$$

$$A > e$$

As $x \to -\infty$, $f(x) \to 0^+$ (from above)
As $x \to \infty$, $f(x) \to \infty$

Thus for $e < A < \infty$, there are exactly two solutions.

Question 8

(a) Elasticity of y wrt x

Given $x^3y^3 + 3x^3 = 2$.

Differentiate implicitly:

$$3x^2y^3 + 3x^3y^2y' + 9x^2 = 0$$

$$y' = \frac{-3x^2y^3 - 9x^2}{3x^3y^2} = \frac{-y^3 - 3}{xy^2}$$

Elasticity:

$$E = \frac{dy}{dx} \cdot \frac{x}{y} = \frac{-y^3 - 3}{xy^2} \cdot \frac{x}{y} = \frac{-y^3 - 3}{y^3} = -1 - \frac{3}{y^3}$$

Elasticity: $-1 - \frac{3}{y^3}$

(b) Find α and β

Given $f(x) = \alpha x e^{-\beta x}$ with maximum $f(2) = 1$.

Conditions:

1. $f(2) = 1$:

$$2\alpha e^{-2\beta} = 1$$

2. $f'(2) = 0$:

$$f'(x) = \alpha e^{-\beta x} - \alpha\beta x e^{-\beta x}$$

$$f'(2) = \alpha e^{-2\beta}(1 - 2\beta) = 0$$

   Since $\alpha \neq 0$ and $e^{-2\beta} \neq 0$, $1 - 2\beta = 0 \Rightarrow \beta = \frac{1}{2}$

Substitute back:

$$2\alpha e^{-1} = 1 \Rightarrow \alpha = \frac{e}{2}$$

Values: $\alpha = \frac{e}{2}$, $\beta = \frac{1}{2}$

Question 9

(a) Find the limit

$$\lim_{x \to 0^+} \frac{1 - (1 + x^\alpha)^{-\beta}}{x}$$

This is a 0/0 form, so apply L'Hôpital's Rule:

$$\lim_{x \to 0^+} \frac{\beta\alpha x^{\alpha-1}(1 + x^\alpha)^{-\beta-1}}{1}$$
$$= \beta\alpha \cdot 0^{\alpha-1} \cdot 1$$

- If $\alpha > 1$, limit is 0
- If $\alpha = 1$, limit is β
- If $0 < \alpha < 1$, limit is $+\infty$

Limit: Depends on α:

- $\alpha > 1$: 0
- $\alpha = 1$: β
- $0 < \alpha < 1$: $+\infty$

(b) Population estimate

Given $F(t) = 40 - \frac{8}{t+2}$.

Estimate change over 6 months (0.5 years):

$$dF \approx F'(t) \cdot dt$$

$$F'(t) = \frac{8}{(t+2)^2}$$

At $t = 0$:

$$F'(0) = \frac{8}{4} = 2$$

$$dF \approx 2 \times 0.5 = 1 \text{ million}$$

Estimated increase: 1 million

Question 10

(a) Tangent to $y = x^3$

At point (a, a^3), the slope is $3a^2$.

Tangent line:

$$y - a^3 = 3a^2(x - a)$$

$$y = 3a^2x - 2a^3$$

Find intersection with $y = x^3$:

$$x^3 = 3a^2x - 2a^3$$

$$x^3 - 3a^2x + 2a^3 = 0$$

We know $x = a$ is a root, so factor:

$$(x - a)(x^2 + ax - 2a^2) = 0$$

Other root:

$$x = \frac{-a \pm \sqrt{a^2 + 8a^2}}{2} = \frac{-a \pm 3a}{2}$$

$$x = a \text{ (double root) or } x = -2a$$

At $x = -2a$: Slope is $3(-2a)^2 = 12a^2$, which is $4 \times 3a^2$.

Conclusion: The slope at the second intersection point is four times the original slope.

(b) Existence of c

Let $h(x) = f(x) - g(x)$.

Given:

- $h(a) = f(a) - g(a) > 0$
- $h(b) = f(b) - g(b) < 0$

Since f and g are continuous, h is continuous. By IVT, there exists $c \in (a, b)$ such that $h(c) = 0$, i.e., $f(c) = g(c)$.

Question 11

(a) Taylor approximation for $\ln(1.1)$

Third degree Taylor polynomial for $\ln(1 + x)$ at 0:

$$f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3$$

$$f(0) = 0$$

$$f'(x) = \frac{1}{1+x} \Rightarrow f'(0) = 1$$

$$f''(x) = -\frac{1}{(1+x)^2} \Rightarrow f''(0) = -1$$

$$f'''(x) = \frac{2}{(1+x)^3} \Rightarrow f'''(0) = 2$$

Thus:

$$\ln(1 + x) \approx x - \frac{x^2}{2} + \frac{x^3}{3}$$

For $\ln(1.1)$, set $x = 0.1$:

$$\ln(1.1) \approx 0.1 - \frac{0.01}{2} + \frac{0.001}{3} \approx 0.095333$$

Error bound:

$$R_3(x) = \frac{f^{(4)}(c)}{4!}x^4$$

$$f^{(4)}(x) = -\frac{6}{(1+x)^4}$$

$$|R_3(0.1)| \leq \frac{6}{24}(0.1)^4 = 0.000025$$

(b) Inflection point of $f(x) = x|x|$

For $x \geq 0$, $f(x) = x^2$, $f''(x) = 2$
For $x < 0$, $f(x) = -x^2$, $f''(x) = -2$

At $x = 0$:

- f'' changes from negative to positive
- Thus $(0, 0)$ is an inflection point
- But $f''(0)$ does not exist (left and right limits don't match)

Graph: V-shaped curve with vertex at origin, smooth transition but sharp second derivative change.

Question 12

For $f(x) = \frac{3}{x^4 - x^2 + 1}$

(a) Increasing/decreasing intervals

Find derivative:

$$f'(x) = -3(4x^3 - 2x)/(x^4 - x^2 + 1)^2$$

Critical points:

$$4x^3 - 2x = 0$$

$$2x(2x^2 - 1) = 0$$

$$x = 0, \pm\frac{1}{\sqrt{2}}$$

Intervals:

1. $x < -\frac{1}{\sqrt{2}}$: $f' > 0$ (increasing)
2. $-\frac{1}{\sqrt{2}} < x < 0$: $f' < 0$ (decreasing)
3. $0 < x < \frac{1}{\sqrt{2}}$: $f' > 0$ (increasing)
4. $x > \frac{1}{\sqrt{2}}$: $f' < 0$ (decreasing)

(b) Local extrema

- Local maxima at $x = \pm\frac{1}{\sqrt{2}}$
- Local minimum at $x = 0$

(c) Global extrema

As $x \to \pm\infty$, $f(x) \to 0$. The maximum value occurs at $x = \pm\frac{1}{\sqrt{2}}$:

$$f\left(\frac{1}{\sqrt{2}}\right) = \frac{3}{\frac{1}{4} - \frac{1}{2} + 1} = \frac{3}{3/4} = 4$$

Minimum value at $x = 0$:

$$f(0) = 3$$

Global maximum: 4 at $x = \pm\frac{1}{\sqrt{2}}$
Global minimum: 3 at $x = 0$

Question 13

(a) Population growth $P(t) = \frac{a}{b+e^{-at}}$

(i) dP/dt **at** $t = 0$

$$P'(t) = \frac{a^2 e^{-at}}{(b + e^{-at})^2}$$

At $t = 0$:

$$P'(0) = \frac{a^2}{(b+1)^2}$$

(ii) Proportional growth rate

$$\frac{P'}{P} = \frac{ae^{-at}}{b + e^{-at}}$$

(iii) Limiting value

As $t \to \infty$, $e^{-at} \to 0$, so:

$$P(t) \to \frac{a}{b}$$

(iv) Most rapid growth

Find maximum of $P'(t)$. Set $P''(t) = 0$:

$$P''(t) = \frac{-a^3 e^{-at}(b + e^{-at})^2 + 2a^3 e^{-2at}(b + e^{-at})}{(b + e^{-at})^4} = 0$$

$$-a^3 e^{-at}(b + e^{-at}) + 2a^3 e^{-2at} = 0$$

$$-b - e^{-at} + 2e^{-at} = 0$$

$$e^{-at} = b$$

$$t = \frac{-\ln b}{a}$$

(b) Diamond value

Given $F(t) = 25000(1.75)^{4\sqrt{t}}$ and continuous interest at 7%.

We need to find when the growth rate equals 7%:

$$\frac{F'(t)}{F(t)} = 0.07$$

$$\ln(1.75) \cdot 4 \cdot \frac{1}{2\sqrt{t}} = 0.07$$

$$\frac{2\ln(1.75)}{\sqrt{t}} = 0.07$$

$$\sqrt{t} = \frac{2\ln(1.75)}{0.07} \approx 8.04$$

$$t \approx 64.6 \text{ years}$$

Holding time: Approximately 64.6 years