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Chapter 1

Introduction

The objects of a geometric algebra are called *multivectors*. Multivectors generalize objects like directed lines, planes, and volumes. An important property of multivectors is they have *orientation*, i.e., a sense of up/down, inside/outside, etc. The following sections introduce examples of geometric algebras and the operations on their multivectors.

1.1 2D Euclidean Space and $U(1)$

1.1.1 Geometric Algebra of Euclidean 2-Space

Multivectors are composed of *basis one-vectors*. In Euclidean 2-space, these are unit vectors along the positive x and y axes. They are denoted by e_1 and e_2 , respectively (the use of e here is shorthand for Euclidean. Different symbols may be used for bases in other algebras, as will be seen later).

The most important operation between multivectors is the *geometric product*. If a and b are multivectors, the geometric product of a and b is written ab . The geometric product is the sum of two simpler products, the *dot* (inner) and *wedge* (outer) products. In Euclidean 2-space, these are equivalent to the familiar dot and cross products of vector algebra. The dot product of a and b is written $a \cdot b$ and the wedge product is written $a \wedge b$. The geometric product, then, is written

$$ab = a \cdot b + a \wedge b.$$

The inner products of basis one-vectors amongst themselves defines the *signature*

of an algebra,

$$\begin{aligned} e_1 \cdot e_1 &= 1 \\ e_2 \cdot e_2 &= 1 \\ e_1 \cdot e_2 &= 0 \end{aligned}$$

Basis one-vectors represent directed unit lines. An oriented plane can be made by wedging the one-vectors together to form a basis *bivector*.

$$\begin{aligned} e_1 \wedge e_2 &= x\text{-}y \text{ plane, counterclockwise orientation} \\ e_i \wedge e_j &= 0 \quad (i = j). \end{aligned}$$

The plane can be flipped by reversing the wedge product.

$$e_2 \wedge e_1 = -e_1 \wedge e_2 = x\text{-}y \text{ plane, clockwise orientation}$$

More generally, dot products are *symmetric* and wedge products are *antisymmetric*

$$\begin{aligned} \frac{1}{2}(ab + ba) &= \frac{1}{2}(a \cdot b + a \wedge b + b \cdot a + b \wedge a) = a \cdot b, \\ \frac{1}{2}(ab - ba) &= \frac{1}{2}(a \cdot b + a \wedge b - b \cdot a - b \wedge a) = a \wedge b. \end{aligned}$$

Since unit vectors are *orthogonal*,

$$\begin{aligned} e_i e_j &= e_i \cdot e_j + e_i \wedge e_j = 1 & (i = j) \\ e_i e_j &= e_i \cdot e_j + e_i \wedge e_j = e_i \wedge e_j & (i \neq j) \end{aligned}$$

explicit dots and wedges are unnecessary when writing basis vectors. We simply write $e_1 e_2$ instead of $e_1 \wedge e_2$.

A basis *zero-vector* is a scalar.

A general multivector m in Euclidean 2-space is a linear combination of basis vectors,

$$m = s + a_1 e_1 + a_2 e_2 + b e_1 e_2.$$

The following formula is seldom used¹, but for completeness, the product of two general multivectors,

$$\begin{aligned} m &= s + a_1 e_1 + a_2 e_2 + b e_1 e_2, \\ n &= r + c_1 e_1 + c_2 e_2 + d e_1 e_2 \end{aligned}$$

is

$$\begin{aligned} mn &= (rs + a_1 b_1 + a_2 b_2 - bd) \\ &\quad + (ra_1 + sc_1 - a_2 d + c_2 b) e_1 \\ &\quad + (ra_2 + sc_2 + a_1 d - c_1 b) e_2 \\ &\quad + (a_1 c_2 - a_2 c_1) e_1 e_2. \end{aligned}$$

¹One application is coding computer algebra systems.

The basis vector formed by multiplying all basis one-vectors is called the *unit pseudoscalar* and is denoted by I . In 2-space, $I = e_1 e_2$. This is in direct analogy to $i = \sqrt{-1}$ from complex numbers, as shown below.

Often, we'll be interested in *even* multivectors, i.e., linear combinations of zero-vectors, bivectors, four-vectors, etc. The product of even multivectors

$$\begin{aligned} m &= a + b e_1 e_2 = a + b I, \\ n &= c + d e_1 e_2 = c + d I \end{aligned}$$

is

$$mn = (ac - bd) + (ad + bc) I,$$

which is the formula for multiplying two complex numbers. From this perspective, the i from complex algebra can be thought of as a counterclockwise-oriented plane.

Unit pseudoscalars satisfy $I^2 = -1$. For this, we define the *reverse* operator on multivectors, which reverses the order of basis vectors. The reverse of $I = e_1 e_2$ in 2-space is

$$\tilde{I} = e_2 e_1 = -I.$$

Note that an odd number of swaps (just 1 in euclidean 2-space) is required to reverse I , so the signature of the algebra doesn't require modification to satisfy $I^2 = -1$. In other algebras, like Minkowski spacetime, the signature will need modification to satisfy this requirement.

The square of a multivector is defined by multiplying a multivector by its reverse. For example,

$$I^2 = \tilde{I} I = e_3 e_2 e_1 (e_1 e_2 e_3) = -1.$$

We can use I to compute the *dual* of a multivector simply by multiplying. The dual M of a multivector m is

$$M = I m.$$

For example,

$$I e_2 = (e_1 e_2) e_2 = e_1.$$

If m spans a subspace of Euclidean 2-space, its dual spans the remaining subspace needed to fill out 2-space. This is the same as the orthogonal complement in linear algebra.

1.1.2 $U(1)$ as a Geometric Algebra

In later chapters, we'll discuss symmetries in field theories. Many of these symmetries involve *unitary* groups. We show here and in later sections how unitary groups can be represented with geometric algebras.

Unitary groups $U(n)$ are groups of $n \times n$ unitary matrices, i.e., matrices U where $U^\dagger U = I$.

For $n = 1$, this is simply the group of unit complex numbers. As shown above, this group equivalent to the group of even multivectors in Euclidean 2-space of unit magnitude.

Unitary groups have *Lie algebras* and *generators*.

1.2 3D Euclidean Space and $SU(2)$

1.2.1 Geometric Algebra of Euclidean 3-Space

In Euclidean 3-space, the three basis vectors are e_1 , e_2 , and e_3 . Their inner products satisfy,

$$\begin{aligned} e_1 \cdot e_1 &= 1 \\ e_2 \cdot e_2 &= 1 \\ e_3 \cdot e_3 &= 1 \\ e_i \cdot e_j &= 0 \quad (i \neq j) \end{aligned}$$

The wedge products are,

$$\begin{aligned} e_1 \wedge e_2 &= x\text{-}y \text{ plane, normal along } +z \\ e_2 \wedge e_3 &= y\text{-}z \text{ plane, normal along } +x \\ e_3 \wedge e_1 &= z\text{-}x \text{ plane, normal along } +y \\ e_i \wedge e_j &= 0 \quad (i = j) \end{aligned}$$

Flipping the planes,

$$\begin{aligned} e_2 \wedge e_1 &= -e_1 \wedge e_2 = x\text{-}y \text{ plane, normal along } -z \\ e_3 \wedge e_2 &= -e_2 \wedge e_3 = y\text{-}z \text{ plane, normal along } -x \\ e_1 \wedge e_3 &= -e_3 \wedge e_1 = z\text{-}x \text{ plane, normal along } -y. \end{aligned}$$

The unit volume/pseudoscalar is,

$$I = e_1 e_2 e_3 = e_1 \wedge e_2 \wedge e_3.$$

It's reverse is

$$\tilde{I} = e_3 e_2 e_1 = -I.$$

Again, an odd number of swaps (3) is required to reverse I , so no signature modification is required to satisfy $I^2 = -1$.

A general multivector m in Euclidean 3-space is a scalar plus a linear combination of basis vectors,

$$m = s + a_1 e_1 + a_2 e_2 + a_3 e_3 + b_1 e_2 e_3 + b_2 e_3 e_1 + b_3 e_1 e_2 + c e_1 e_2 e_3.$$

1.2.2 $SU(2)$ as a Geometric Algebra

1.3 Minkowski Spacetime

Minkowski spacetime has four dimensions, one timelike, three spacelike. These are denoted by γ_i instead of e_i . The algebra's signature needs modification to satisfy $I^2 = -1$. Given $I = \gamma_0\gamma_1\gamma_2\gamma_3$,

$$\tilde{I} = \gamma_3\gamma_2\gamma_1\gamma_0 = I.$$

An even number of swaps (6) is required to reverse I . So, to satisfy $I^2 = -1$, we need to modify the algebra's signature to achieve an overall negative sign. This can be done by setting the squares of 1 or 3 basis vectors to be negative. We choose the signature,

$$\begin{aligned}\gamma_0 \cdot \gamma_0 &= 1 \\ \gamma_1 \cdot \gamma_1 &= -1 \\ \gamma_2 \cdot \gamma_2 &= -1 \\ \gamma_3 \cdot \gamma_3 &= -1 \\ \gamma_i \cdot \gamma_j &= 0 \quad (i \neq j).\end{aligned}$$

1.4 Conformal 3-Space

Conformal algebras are created by taking an underlying space and adding a spacelike and timelike dimension to it. If we append e_0 (timelike) and e_4 (spacelike) to Euclidean 3-space, we have the basis e_0, e_1, e_2, e_3, e_4 . To determine the signature, reverse I ,

$$\tilde{I} = e_4e_3e_2e_1e_0 = I.$$

This requires 10 swaps, so one, three, or five basis vectors should carry a negative sign. We choose e_0 ,

$$\begin{aligned}e_0 \cdot e_0 &= -1 \\ e_1 \cdot e_1 &= 1 \\ e_2 \cdot e_2 &= 1 \\ e_3 \cdot e_3 &= 1 \\ e_4 \cdot e_4 &= 1 \\ e_i \cdot e_j &= 0 \quad (i \neq j)\end{aligned}$$

The wedge products are,

$$\begin{aligned} e_1 \wedge e_2 &= x\text{-}y \text{ plane, normal along } +z \\ e_2 \wedge e_3 &= y\text{-}z \text{ plane, normal along } +x \\ e_3 \wedge e_1 &= z\text{-}x \text{ plane, normal along } +y \\ e_i \wedge e_j &= 0 \quad (i = j) \end{aligned}$$

Flipping the planes,

$$\begin{aligned} e_2 \wedge e_1 &= -e_1 \wedge e_2 = x\text{-}y \text{ plane, normal along } -z \\ e_3 \wedge e_2 &= -e_2 \wedge e_3 = y\text{-}z \text{ plane, normal along } -x \\ e_1 \wedge e_3 &= -e_3 \wedge e_1 = z\text{-}x \text{ plane, normal along } -y. \end{aligned}$$

Chapter 2

Derivatives, Integrals, and Geometric Calculus

2.1 The Geometric Derivative

Geometric Algebra is extended to Geometric Calculus by adding geometric derivatives and integrals. The geometric derivative is denoted by ∇ and acts on multivector-valued functions over a given domain. In summation notation, the geometric derivative of a function F is:

$$\nabla F(x) = e_i \partial_i F(x).$$

The following sections demonstrate the geometric derivative in example spaces.

2.1.1 Euclidean 2-Space

Real Functions

Consider a real-valued function f over x and y in Euclidean 2-space. Its geometric derivative is

$$\nabla f = (\partial_x f) e_1 + (\partial_y f) e_2,$$

i.e., the geometric derivative of a real-valued function is its gradient.

Vector Functions

Consider a vector-valued function $g = ue_1 + ve_2$. It's derivative is

$$\begin{aligned}\nabla g &= (e_1\partial_x + e_2\partial_y)(ue_1 + ve_2) \\ &= (\partial_x u + \partial_y v) + (\partial_x v - \partial_y u)e_1e_2.\end{aligned}$$

In other words, the derivative of a vector-valued function in Euclidean 2-space is the complex derivative. In particular, we see that the geometric derivatives separates into inner (gradient) and outer (curl) products,

$$\nabla g = \nabla \cdot g + \nabla \wedge g.$$

The geometric derivative contains both divergence and curl from traditional vector calculus.

Bivector (Pseudoscalar) Functions

For fe_1e_2 ,

$$\nabla f = (\partial_x f)e_2 - (\partial_y f)e_1.$$

2.1.2 Euclidean 3-Space**Real Functions**

The geometric derivative acting on a real-valued function f in 3-space is

$$\nabla f = (\partial_x f)e_1 + (\partial_y f)e_2 + (\partial_z f)e_3.$$

Again, this is simply the gradient of f .

Vector Functions

Given a vector-valued function $g = ue_1 + ve_2 + we_3$. Its derivative is

$$\begin{aligned}\nabla g &= (e_1\partial_x + e_2\partial_y + e_3\partial_z)(ue_1 + ve_2 + we_3) \\ &= (\partial_x u + \partial_y v + \partial_z w) + \\ &= (\partial_x v - \partial_y u)e_1e_2 + \\ &= (\partial_y w - \partial_z v)e_2e_3 + \\ &= (\partial_z u - \partial_x w)e_3e_1.\end{aligned}$$

Bivector Functions

A bivector function in 3D Euclidean space has the form

$$f = ue_1e_2 + ve_2e_3 + we_3e_1.$$

Its derivative is

$$\begin{aligned}
 \nabla f &= (e_1 \partial_x + e_2 \partial_y + e_3 \partial_z) (ue_1e_2 + ve_2e_3 + we_3e_1) \\
 &= (\partial_z w - \partial_y u) e_1 \\
 &\quad + (\partial_x u - \partial_z v) e_2 \\
 &\quad + (\partial_y v - \partial_x w) e_3 \\
 &\quad + (\partial_x v + \partial_y w + \partial_z u) e_1 e_2 e_3
 \end{aligned}$$

Pseudoscalar Functions

For $f e_1 e_2 e_3$,

$$\nabla f = (e_1 \partial_x + e_2 \partial_y + e_3 \partial_z) f = \partial_x f e_2 e_3 - \partial_y f e_1 e_2 + \partial_z f e_1 e_2.$$

2.1.3 Minkowski Spacetime

Real Functions

Vector Functions

Bivector Functions

Pseudovector Functions

Pseudoscalar Functions

2.1.4 Conformal 3-Space

2.2 Gauge Covariant Derivative

In field theories, symmetries of the Lagrangian for the system determine the kinematic equations for that system. Consider a field theory in $\psi(x)$ with the following Lagrangian,

$$\mathcal{L} = \bar{\psi} D \psi.$$

Here, D is a derivative operator. We'd like this Lagrangian to be invariant under the following field transformations,

$$\psi' = e^{-\lambda} \psi,$$

so that

$$\mathcal{L} = \bar{\psi} D \psi = \bar{\psi}' D' \psi'.$$

Transformations can be *global*, where λ is constant, or *local*, where λ is a function of x . We consider local transformations here. Expanding the right side,

$$\bar{\psi}' D' \psi' = \bar{\psi} e^\lambda D' e^{-\lambda} \psi.$$

So D transforms $D' = e^{-\lambda} D e^\lambda$.

If we let $D = \partial$, the derivative product rule breaks invariance,

$$\begin{aligned} \mathcal{L} &= \bar{\psi}' D \psi' \\ &= \bar{\psi}' \partial \psi' \\ &= \bar{\psi} e^\lambda \partial e^{-\lambda} \psi \\ &= (\bar{\psi} e^\lambda) e^{-\lambda} ((\partial \psi) - (\partial \lambda) \psi) \\ &= \bar{\psi} \partial \psi - \bar{\psi} (\partial \lambda) \psi. \end{aligned}$$

There's an extra $\bar{\psi} (\partial \lambda) \psi$ term, the *gauge* term. The usual way to offset this is to add a *gauge field* A to D , then determine its transformation properties. Letting $D = \partial + A$, invariance requires

$$\begin{aligned} D' &= \partial + A' \\ &= e^{-\lambda} D e^\lambda \\ &= e^{-\lambda} (\partial + A) e^\lambda \\ &= e^{-\lambda} (e^\lambda \partial \lambda + e^\lambda \partial + A e^\lambda) \\ &= \partial + (e^{-\lambda} A e^\lambda + \partial \lambda). \end{aligned}$$

This shows the gauge field transforms $A' = e^{-\lambda} A e^\lambda + \partial \lambda$.

2.3 Cauchy's Integral Theorem and Formula

Chapter 3

Yang-Mills Theories

Chapter 4

Feynman Path Integrals

Chapter 5

Appendix 1: Calculations and Derivations

5.1 STA Lagrangian Invariance under $U(1)$

5.2 Conserved Currents

An important concept in physics is *conservation*, particularly as stated by *Noether's Theorem*. The theorem gives a relationship between symmetries in a system's Lagrangian and conserved quantities like momentum in rigid body mechanics, or probability and spin in quantum mechanics.

Consider a field theory involving a field ψ . The Lagrangian for the system is $\mathcal{L}(\psi, \partial\psi, x)$. First, use the variational principle to determine the equations of motion for this system. The action S for this system is

$$S = \int \mathcal{L}(\psi, \partial\psi, x).$$

The equations of motion can be derived by setting the variation to zero,

$$\delta S = \delta \int \mathcal{L}(\psi, \partial\psi, x) = 0.$$

The first-order variation $\delta\mathcal{L}$ is

$$\begin{aligned} \delta\mathcal{L} &= \mathcal{L}(\psi + \delta\psi, \partial\psi + \delta(\partial\psi), x + \delta x) - \mathcal{L}(\psi, \partial\psi, x) \\ &= \frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial\psi)}\delta(\partial\psi) + \frac{\partial\mathcal{L}}{\partial x}\delta x. \end{aligned}$$

For this discussion, assume $\partial\mathcal{L}/\partial x = 0$. The variation becomes

$$\begin{aligned}\delta S &= \int \left(\frac{\partial\mathcal{L}}{\partial\psi} \delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial\psi)} \delta(\partial\psi) \right) \\ &= \int \left(\frac{\partial\mathcal{L}}{\partial\psi} \delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial\psi)} \partial(\delta\psi) \right).\end{aligned}$$

In the second line, we've used equality of mixed partials, $\delta(\partial\psi) = \partial(\delta\psi)$.

The two integrand terms are variations in $\delta\psi$ and $\delta(\partial\psi)$. They can be combined as terms in $\partial\psi$ using the product rule,

$$\partial \left(\frac{\partial\mathcal{L}}{\partial(\partial\psi)} \delta\psi \right) = \partial \left(\frac{\partial\mathcal{L}}{\partial(\partial\psi)} \right) \delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial\psi)} \partial(\delta\psi),$$

so

$$\frac{\partial\mathcal{L}}{\partial(\partial\psi)} \partial(\delta\psi) = \partial \left(\frac{\partial\mathcal{L}}{\partial(\partial\psi)} \delta\psi \right) - \partial \left(\frac{\partial\mathcal{L}}{\partial(\partial\psi)} \right) \delta\psi.$$

. Substituting,

$$\delta S = \int \left(\frac{\partial\mathcal{L}}{\partial\psi} \delta\psi + \partial \left(\frac{\partial\mathcal{L}}{\partial(\partial\psi)} \delta\psi \right) - \partial \left(\frac{\partial\mathcal{L}}{\partial(\partial\psi)} \right) \delta\psi \right).$$

The middle term is vanishes since variations $\delta\psi$ are assumed to be zero on the integration region boundary,

$$\int \partial \left(\frac{\partial\mathcal{L}}{\partial(\partial\psi)} \delta\psi \right) = \frac{\partial\mathcal{L}}{\partial(\partial\psi)} \delta\psi \Big|_{\partial\Omega} = 0.$$

Finally, we're left with

$$\delta S = \int \left(\frac{\partial\mathcal{L}}{\partial\psi} - \partial \left(\frac{\partial\mathcal{L}}{\partial(\partial\psi)} \right) \right) \delta\psi.$$

A zero variation gives the equations of motion,

$$\frac{\partial\mathcal{L}}{\partial\psi} = \partial \left(\frac{\partial\mathcal{L}}{\partial(\partial\psi)} \right).$$

This equation of motion implies a relationship between symmetries and conserved quantities. If the Lagrangian is invariant under some field symmetry, i.e.,

$$\frac{\partial\mathcal{L}}{\partial\psi} = 0,$$

Then the following quantity is conserved,

$$\partial \left(\frac{\partial\mathcal{L}}{\partial(\partial\psi)} \right) = 0.$$