Contents

Chapter 1

Introduction

The objects of a geometric algebra are called *multivectors*. Multivectors generalize objects like directed lines, planes, and volumes. An important property of multivectors is they have *orientation*, i.e., a sense of up/down, inside/outside, etc. The following sections introduce examples of geometric algebras and the operations on their multivectors.

1.1 2D Euclidean Space and $U(1)$

1.1.1 Geometric Algebra of Euclidean 2-Space

Multivectors are composed of *basis one-vectors*. In Euclidean 2-space, these are unit vectors along the positive x and y axes. They are denoted by e_1 and e_2, respectively (the use of e here is shorthand for Euclidean. Different symbols may be used for bases in other algebras, as will be seen later).

The most important operation between multivectors is the *geometric product*. If a and b are multivectors, the geometric product of a and b is written ab. The geometric product is the sum of two simpler products, the *dot* (inner) and *wedge* (outer) products. In Euclidean 2-space, these are equivalent to the familiar dot and cross products of vector algebra. The dot product of a and b is written $a \cdot b$ and the wedge product is written $a \wedge b$. The geometric product, then, is written

$$ab = a \cdot b + a \wedge b.$$

The inner products of basis one-vectors amongst themselves defines the *signature* of an algebra,

$$\begin{aligned} e_1 \cdot e_1 &= 1 \\ e_2 \cdot e_2 &= 1 \\ e_1 \cdot e_2 &= 0 \end{aligned}$$

Basis one-vectors represent directed unit lines. An oriented plane can be made by wedging the one-vectors together to form a basis *bivector*.

$$\begin{aligned} e_1 \wedge e_2 &= x\text{-}y \text{ plane, counterclockwise orientation} \\ e_i \wedge e_j &= 0 \quad (i = j)\,. \end{aligned}$$

The plane can be flipped by reversing the wedge product.

$$e_2 \wedge e_1 = -e_1 \wedge e_2 = x\text{-}y \text{ plane, clockwise orientation}$$

More generally, dot products are *symmetric* and wedge products are *antisymmetric*

$$\begin{aligned} \frac{1}{2}(ab + ba) &= \frac{1}{2}(a \cdot b + a \wedge b + b \cdot a + b \wedge a) = a \cdot b, \\ \frac{1}{2}(ab - ba) &= \frac{1}{2}(a \cdot b + a \wedge b - b \cdot a - b \wedge a) = a \wedge b. \end{aligned}$$

Since unit vectors are *orthogonal*,

$$\begin{aligned} e_i e_j &= e_i \cdot e_j + e_i \wedge e_j = 1 & (i = j) \\ e_i e_j &= e_i \cdot e_j + e_i \wedge e_j = e_i \wedge e_j & (i \neq j) \end{aligned}$$

explicit dots and wedges are unnecessary when writing basis vectors. We simply write $e_1 e_2$ instead of $e_1 \wedge e_2$.

A basis *zero-vector* is a scalar.

A general multivector m in Euclicean 2-space is a linear combination of basis vectors,

$$m = s + a_1 e_1 + a_2 e_2 + b e_1 e_2.$$

The following formula is seldom used[1], but for completeness, the product of two general multivectors,

$$\begin{aligned} m &= s + a_1 e_1 + a_2 e_2 + b e_1 e_2, \\ n &= r + c_1 e_1 + c_2 e_2 + d e_1 e_2 \end{aligned}$$

is

$$\begin{aligned} mn = {} & (rs + a_1 b_1 + a_2 b_2 - bd) \\ & + (r a_1 + s c_1 - a_2 d + c_2 b)\, e_1 \\ & + (r a_2 + s c_2 + a_1 d - c_1 b)\, e_2 \\ & + (a_1 c_2 - a_2 c_1)\, e_1 e_2. \end{aligned}$$

[1] One application is coding computer algebra systems.

The basis vector formed by multiplying all basis one-vectors is called the *unit pseudoscalar* and is denoted by I. In 2-space, $I = e_1 e_2$. This is in direct analogy to $i = \sqrt{-1}$ from complex numbers, as shown below.

Often, we'll be interested in *even* multivectors, i.e., linear combinations of zero-vectors, bivectors, four-vectors, etc. The product of even multivectors

$$
\begin{aligned}
m &= a + b e_1 e_2 = a + bI, \\
n &= c + d e_1 e_2 = c + dI
\end{aligned}
$$

is

$$mn = (ac - bd) + (ad + bc)\, I,$$

which is the formula for multiplying two complex numbers. From this perspective, the i from complex algebra can be thought of as a counterclockwise-oriented plane.

Unit psuedoscalars satisfy $I^2 = -1$. For this, we define the *reverse* operator on multivectors, which reverses the order of basis vectors. The reverse of $I = e_1 e_2$ in 2-space is

$$\tilde{I} = e_2 e_1 = -I.$$

Note that an odd number of swaps (just 1 in euclidean 2-space) is required to reverse I, so the signature of the algebra doesn't require modification to satisfy $I^2 = -1$. In other algebras, like Minkowski spacetime, the signature will need modification to satisfy this requirement.

The square of a multivector is defined by multiplying a multivector by its reverse. For example,

$$I^2 = \tilde{I} I = e_3 e_2 e_1 \,(e_1 e_2 e_3) = -1.$$

We can use I to compute the *dual* of a multivector simply by multiplying. The dual M of a multivector m is

$$M = Im.$$

For example,

$$I e_2 = (e_1 e_2)\, e_2 = e_1.$$

If m spans a subspace of Euclidean 2-space, its dual spans the remaining subspace needed to fill out 2-space. This is the same as the orthogonal complement in linear algebra.

1.1.2 $U(1)$ as a Geometric Algebra

In later chapters, we'll discuss symmetries in field theories. Many of these symmetries involve *unitary* groups. We show here and in later sections how unitary groups can be represented with geometric algebras.

Unitary groups $U(n)$ are groups of $n \times n$ unitary matrices, i.e., matrices U where $U^\dagger U = I$.

For $n = 1$, this is simply the group of unit complex numbers. As shown above, this group equivalent to the group of even multivectors in Euclidean 2-space of unit magnitude.

Unitary groups have *Lie algebras* and *generators*.

1.2 3D Euclidean Space and $SU(2)$

1.2.1 Geometric Algebra of Euclidean 3-Space

In Euclidean 3-space, the three basis vectors are e_1, e_2, and e_3. Their inner products satisfy,

$$\begin{aligned} e_1 \cdot e_1 &= 1 \\ e_2 \cdot e_2 &= 1 \\ e_3 \cdot e_3 &= 1 \\ e_i \cdot e_j &= 0 \quad (i \neq j) \end{aligned}$$

The wedge products are,

$$\begin{aligned} e_1 \wedge e_2 &= x\text{-}y \text{ plane, normal along } +z \\ e_2 \wedge e_3 &= y\text{-}z \text{ plane, normal along } +x \\ e_3 \wedge e_1 &= z\text{-}x \text{ plane, normal along } +y \\ e_i \wedge e_j &= 0 \quad (i = j) \end{aligned}$$

Flipping the planes,

$$\begin{aligned} e_2 \wedge e_1 = -e_1 \wedge e_2 &= x\text{-}y \text{ plane, normal along } -z \\ e_3 \wedge e_2 = -e_2 \wedge e_3 &= y\text{-}z \text{ plane, normal along } -x \\ e_1 \wedge e_3 = -e_3 \wedge e_1 &= z\text{-}x \text{ plane, normal along } -y. \end{aligned}$$

The unit volume/pseudoscalar is,

$$I = e_1 e_2 e_3 = e_1 \wedge e_2 \wedge e_3.$$

It's reverse is

$$\tilde{I} = e_3 e_2 e_1 = -I.$$

Again, an odd number of swaps (3) is required to reverse I, so no signature modification is required to satisfy $I^2 = -1$.

A general multivector m in Euclicean 3-space is a scalar plus a linear combination of basis vectors,

$$m = s + a_1 e_1 + a_2 e_2 + a_3 e_3 + b_1 e_2 e_3 + b_2 e_3 e_1 + b_3 e_1 e_2 + c e_1 e_2 e_3.$$

1.2.2 $SU(2)$ as a Geometric Algebra

1.3 Minkowski Spacetime

Minkowski spacetime has four dimensions, one timelike, three spacelike. These are denoted by γ_i instead of e_i. The algebra's signature needs modification to satisfy $I^2 = -1$. Given $I = \gamma_0\gamma_1\gamma_2\gamma_3$,

$$\tilde{I} = \gamma_3\gamma_2\gamma_1\gamma_0 = I.$$

An even number of swaps (6) is required to reverse I. So, to satisfy $I^2 = -1$, we need to modify the algebra's signature to achieve an overall negative sign. This can be done by setting the squares of 1 or 3 basis vectors to be negative. We choose the signature,

$$\begin{aligned}
\gamma_0 \cdot \gamma_0 &= 1 \\
\gamma_1 \cdot \gamma_1 &= -1 \\
\gamma_2 \cdot \gamma_2 &= -1 \\
\gamma_3 \cdot \gamma_3 &= -1 \\
\gamma_i \cdot \gamma_j &= 0 \quad (i \neq j)\,.
\end{aligned}$$

1.4 Conformal 3-Space

Conformal algebras are created by taking an underlying space and adding a spacelike and timelike dimension to it. If we append e_0 (timelike) and e_4 (spacelike) to Euclidean 3-space, we have the basis e_0, e_1, e_2, e_3, e_4. To determine the signature, reverse I,

$$\tilde{I} = e_4e_3e_2e_1e_0 = I.$$

This requires 10 swaps, so one, three, or five basis vectors should carry a negative sign. We choose e_0,

$$\begin{aligned}
e_0 \cdot e_0 &= -1 \\
e_1 \cdot e_1 &= 1 \\
e_2 \cdot e_2 &= 1 \\
e_3 \cdot e_3 &= 1 \\
e_4 \cdot e_4 &= 1 \\
e_i \cdot e_j &= 0 \quad (i \neq j)
\end{aligned}$$

The wedge products are,

$$\begin{aligned}
e_1 \wedge e_2 &= x\text{-}y \text{ plane, normal along } +z \\
e_2 \wedge e_3 &= y\text{-}z \text{ plane, normal along } +x \\
e_3 \wedge e_1 &= z\text{-}x \text{ plane, normal along } +y \\
e_i \wedge e_j &= 0 \quad (i = j)
\end{aligned}$$

Flipping the planes,

$$\begin{aligned}
e_2 \wedge e_1 = -e_1 \wedge e_2 &= x\text{-}y \text{ plane, normal along } -z \\
e_3 \wedge e_2 = -e_2 \wedge e_3 &= y\text{-}z \text{ plane, normal along } -x \\
e_1 \wedge e_3 = -e_3 \wedge e_1 &= z\text{-}x \text{ plane, normal along } -y.
\end{aligned}$$

Chapter 2

Derivatives, Integrals, and Geometric Calculus

2.1 The Geometric Derivative

Geometric Algebra is extended to Geometric Calculus by adding geometric derivatives and integrals. The geometric derivative is denoted by ∇.

$$\nabla F\left(x\right) = e_i \partial_i F\left(x\right).$$

The following sections demonstrate the geometric derivative in various spaces to give a practical sense of its usage. Then, we examine discrete *simplicial* calculus and show how its limits produce the geometric derivative and fundamental calculus theorems like Stokes' and Cauchy's Theorems. Finally, we examine the theory of Green's functions.

2.1.1 Euclidean 2-Space

Real Functions

Consider a real-valued function f over x and y in Euclidean 2-space. Its geometric derivative is

$$\nabla f = \left(\partial_x f\right) e_1 + \left(\partial_y f\right) e_2,$$

i.e., the geometric derivative of a real-valued function is its gradient.

Vector Functions

Consider a vector-valued function $g = ue_1 + ve_2$. It's derivative is

$$\begin{aligned}\nabla g &= (e_1\partial_x + e_2\partial_y)\,(ue_1 + ve_2) \\ &= (\partial_x u + \partial_y v) + (\partial_x v - \partial_y u)\,e_1e_2.\end{aligned}$$

In other words, the derivative of a vector-valued function in Euclidean 2-space is the complex derivative. In particular, we see that the geometric derivatives separates into inner (gradient) and outer (curl) products,

$$\nabla g = \nabla \cdot g + \nabla \wedge g.$$

The geometric derivative contains both divergence and curl from traditional vector calculus.

Bivector (Pseudoscalar) Functions

For fe_1e_2,

$$\nabla f = (\partial_x f)\,e_2 - (\partial_y f)\,e_1.$$

2.1.2 Euclidean 3-Space

Real Functions

The geometric derivative acting on a real-valued function f in 3-space is

$$\nabla f = (\partial_x f)\,e_1 + (\partial_y f)\,e_2 + (\partial_z f)\,e_3.$$

Again, this is simply the gradient of f.

Vector Functions

Given a vector-valued function $g = ue_1 + ve_2 + we_3$. Its derivative is

$$\begin{aligned}\nabla g &= (e_1\partial_x + e_2\partial_y + e_3\partial_z)\,(ue_1 + ve_2 + we_3) \\ &= (\partial_x u + \partial_y v + \partial_z w) + \\ &= (\partial_x v - \partial_y u)\,e_1e_2 + \\ &= (\partial_y w - \partial_z v)\,e_2e_3 + \\ &= (\partial_z u - \partial_x w)\,e_3e_1.\end{aligned}$$

Bivector Functions

A bivector function in 3D Euclidean space has the form

$$f = ue_1e_2 + ve_2e_3 + we_3e_1.$$

Its derivative is

$$\begin{aligned}\nabla f &= (e_1\partial_x + e_2\partial_y + e_3\partial_z)(ue_1e_2 + ve_2e_3 + we_3e_1)\\ &= (\partial_z w - \partial_y u)\, e_1\\ &+ (\partial_x u - \partial_z v)\, e_2\\ &+ (\partial_y v - \partial_x w)\, e_3\\ &+ (\partial_x v + \partial_y w + \partial_z u)\, e_1e_2e_3\end{aligned}$$

Pseudoscalar Functions

For $fe_1e_2e_3$,

$$\nabla f = (e_1\partial_x + e_2\partial_y + e_3\partial_z)\, f = \partial_x f e_2 e_3 - \partial_y f e_1 e_2 + \partial_z f e_1 e_2.$$

2.1.3 Minkowski Spacetime

Real Functions

Vector Functions

Bivector Functions

Pseudovector Functions

Pseudoscalar Functions

2.1.4 Conformal 3-Space

2.2 Gauge Covariant Derivative

In field theories, symmetries of the Lagrangian for the system determine the kinematic equations for that system. Consider a field theory in $\psi(x)$ with the following Lagrangian,

$$\mathcal{L} = \overline{\psi} D\psi.$$

Here, D is a derivative operator. We'd like this Lagrangian to be invariant under the following field transformations,

$$\psi' = e^{-\lambda}\psi,$$

so that

$$\mathcal{L} = \overline{\psi} D\psi = \overline{\psi}' D'\psi'.$$

Transformations can be *global*, where λ is constant, or *local*, where λ is a function of x. We consider local transformations here. Expanding the right side,

$$\overline{\psi}' D' \psi' = \overline{\psi} e^{\lambda} D' e^{-\lambda} \psi.$$

So D transforms $D' = e^{-\lambda} D e^{\lambda}$.

If we let $D = \partial$, the derivative product rule breaks invariance,

$$\begin{aligned}
\mathcal{L} &= \overline{\psi}' D \psi' \\
&= \overline{\psi}' \partial \psi' \\
&= \overline{\psi} e^{\lambda} \partial e^{-\lambda} \psi \\
&= \left(\overline{\psi} e^{\lambda}\right) e^{-\lambda} \left((\partial \psi) - (\partial \lambda) \psi\right) \\
&= \overline{\psi} \partial \psi - \overline{\psi} (\partial \lambda) \psi.
\end{aligned}$$

There's an extra $\overline{\psi} (\partial \lambda) \psi$ term, the *gauge* term. The usual way to offset this is to add a *gauge field* A to D, then determine its transformation properties. Letting $D = \partial + A$, invariance requires

$$\begin{aligned}
D' &= \partial + A' \\
&= e^{-\lambda} D e^{\lambda} \\
&= e^{-\lambda} (\partial + A) e^{\lambda} \\
&= e^{-\lambda} \left(e^{\lambda} \partial \lambda + e^{\lambda} \partial + A e^{\lambda}\right) \\
&= \partial + \left(e^{-\lambda} A e^{\lambda} + \partial \lambda\right).
\end{aligned}$$

This shows the gauge field transforms $A' = e^{-\lambda} A e^{\lambda} + \partial \lambda$.

2.3 Simplices

To derive integral formulas for geometric calculus, we need to cast the regions over which functions are integrated into the formalism of geometric algebra. For example, the 1D integral of f along a path γ is written

$$\int_{\gamma} f \, dx.$$

It's important to note that the differential dx encodes a *direction* along the positive x axis. Reversing the direction of dx reverses the sign of entire integral.

Similarly, if a function g is defined over two-dimensional region Ω, its integral is

$$\int_{\Omega} g \, dx dy.$$

Again, the differential $dxdy$ has a direction. Reversing it, $dydx = -dxdy$, changes the overall sign of the integral. This suggests that regions over which functions are integrated carry geometric content.

A *simplex* $a(k)$ is an oriented k-dimensional volume defined by the set of $k+1$ points $(a_0, ..., a_k)$. In one dimension, a simplex $a(1)$ is a directed line segment between two points, $a_1 - a_0$, with volume $|a_1 - a_0|$.

In two dimensions,

$$a(2) = (a_1 - a_0) \wedge (a_2 - a_0)$$

Its volume,

$$|a(2)| = \frac{1}{2} |(a_1 - a_0) \wedge (a_2 - a_0)|.$$

Generally,

$$a(k) = (a_1 - a_0) \wedge ... \wedge (a_k - a_0)$$

and volume,

$$|a(k)| = \frac{1}{k!} |(a_1 - a_0) \wedge ... \wedge (a_k - a_0)|.$$

For tidiness of notation, let $\overline{a_k} = a_k - a_0$, or an edge of a simplex. A simplex in two dimensions can be written

$$a(2) = \frac{1}{2} \overline{a_1} \wedge \overline{a_2}.$$

A *chain* of simplices $\mathcal{C}$ is a collection simplices $a^j(k)$, typically with edges or faces shared amongst simplices to form connected regions.

We can approximate an integral of a function g over a region Ω by breaking Ω up into a chain of simplices and taking the product of g with each simplex a^j, evaluated at one point on that simplex,

$$\sum g\left(a_0^j\right) a^j.$$

An integral definition for the derivative of a function f can be derived using simplex sums. It's motivated by the usual definition of a derivative in one dimension,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

We evaluate f at the boundary of some region of length h, then shrink the boundary.

Consider the following sum of $f = u\ e_1 + v\ e_2$ over a square region Ω with boundary $\partial\Omega$ formed the by the chain $\mathcal{C}$ formed by simplices in two dimensions

$$\begin{aligned}
a^0 &= \Delta x\ e_1 = \overline{a_1}^0 \\
a^1 &= \Delta y\ e_2 = \overline{a_1}^1 \\
a^2 &= -\Delta x\ e_1 = \overline{a_1}^2 \\
a^3 &= -\Delta y\ e_2 = \overline{a_1}^3
\end{aligned}$$

Summing f over this region,

$$\sum_{\mathcal{C}} f = \sum f\left(a_0^i\right) \overline{a_1}^i.$$

Expanding the sum,

$$\begin{aligned}\sum_{\mathcal{C}} f = &\left(u\left(a_0^0\right)\ e_1 + v\left(a_0^0\right)\ e_2\right) \Delta x\ e_1 \\ &+ \left(u\left(a_0^1\right)\ e_1 + v\left(a_0^1\right)\ e_2\right) \Delta y\ e_2 \\ &- \left(u\left(a_0^2\right)\ e_1 + v\left(a_0^2\right)\ e_2\right) \Delta x\ e_1 \\ &- \left(u\left(a_0^3\right)\ e_1 + v\left(a_0^3\right)\ e_2\right) \Delta y\ e_2.\end{aligned}$$

Taking geometric products and rearranging,

$$\begin{aligned}\sum_{\mathcal{C}} f = &\left(u\left(a_0^0\right) - u\left(a_0^2\right)\right) \Delta x + \left(v\left(a_0^1\right) - v\left(a_0^3\right)\right) \Delta y \\ &+ \left(u\left(a_0^1\right) - u\left(a_0^3\right)\right) \Delta y\ e_1 e_2 + \left(v\left(a_0^2\right) - v\left(a_0^0\right)\right) \Delta x\ e_1 e_2.\end{aligned}$$

Dividing through by the volume $|\Omega| = \Delta x \Delta y$,

$$\begin{aligned}\frac{\sum_{\mathcal{C}} f}{|\Omega|} = &\frac{\left(u\left(a_0^0\right) - u\left(a_0^2\right)\right)}{\Delta y} + \frac{\left(v\left(a_0^1\right) - v\left(a_0^3\right)\right)}{\Delta x} \\ &+ \frac{\left(u\left(a_0^1\right) - u\left(a_0^3\right)\right)}{\Delta x}\ e_1 e_2 + \frac{\left(v\left(a_0^2\right) - v\left(a_0^0\right)\right)}{\Delta y}\ e_1 e_2.\end{aligned}$$

In the limit $|\Omega| \to 0$,

$$\lim_{|\Omega| \to 0} \frac{\sum_{\mathcal{C}} f}{|\Omega|} = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)\ e_1 e_2.$$

This is the complex derivative. In general, for any dimension,

$$\partial f = \lim_{|\Omega| \to 0} \frac{\sum_{\partial\Omega} f}{|\Omega|}.$$

2.4 Cauchy's Integral Theorem and Formula

Chapter 3

Yang-Mills Theories

Chapter 4

Feynman Path Integrals

Chapter 5

Appendix 1: Calculations and Derivations

5.1 STA Lagrangian Invariance under $U(1)$

5.2 Conserved Currents

An important concept in physics is *conservation*, particularly as stated by *Noether's Theorem*. The theorem gives a relationship between symmetries in a system's Lagrangian and conserved quantities like momentum in rigid body mechanics, or probability and spin in quantum mechanics.

Consider a field theory involving a field ψ. The Lagrangian for the system is $\mathcal{L}(\psi, \partial\psi, x)$. First, use the variational principle to determine the equations of motion for this system. The action S for this system is

$$S = \int \mathcal{L}(\psi, \partial\psi, x).$$

The equations of motion can be derived by setting the variation to zero,

$$\delta S = \delta \int \mathcal{L}(\psi, \partial\psi, x) = 0.$$

The first-order variation $\delta\mathcal{L}$ is

$$\begin{aligned} \delta\mathcal{L} &= \mathcal{L}(\psi + \delta\psi, \partial\psi + \delta(\partial\psi), x + \delta x) - \mathcal{L}(\psi, \partial\psi, x) \\ &= \frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial\psi)}\delta(\partial\psi) + \frac{\partial\mathcal{L}}{\partial x}\delta x. \end{aligned}$$

For this discussion, assume $\partial\mathcal{L}/\partial x = 0$. The variation becomes

$$\begin{aligned}\delta S &= \int \left(\frac{\partial \mathcal{L}}{\partial \psi} \delta\psi + \frac{\partial \mathcal{L}}{\partial \left(\partial\psi\right)} \delta \left(\partial\psi\right) \right) \\ &= \int \left(\frac{\partial \mathcal{L}}{\partial \psi} \delta\psi + \frac{\partial \mathcal{L}}{\partial \left(\partial\psi\right)} \partial \left(\delta\psi\right) \right).\end{aligned}$$

In the second line, we've used equality of mixed partials, $\delta\left(\partial\psi\right) = \partial\left(\delta\psi\right).$

The two integrand terms are variations in $\delta\psi$ and $\delta\left(\partial\psi\right)$. They can be combined as terms in $\partial\psi$ using the product rule,

$$\partial \left(\frac{\partial \mathcal{L}}{\partial \left(\partial\psi\right)} \delta\psi \right) = \partial \left(\frac{\partial \mathcal{L}}{\partial \left(\partial\psi\right)} \right) \delta\psi + \frac{\partial \mathcal{L}}{\partial \left(\partial\psi\right)} \partial \left(\delta\psi\right),$$

so

$$\frac{\partial \mathcal{L}}{\partial \left(\partial\psi\right)} \partial \left(\delta\psi\right) = \partial \left(\frac{\partial \mathcal{L}}{\partial \left(\partial\psi\right)} \delta\psi \right) - \partial \left(\frac{\partial \mathcal{L}}{\partial \left(\partial\psi\right)} \right) \delta\psi.$$

. Substituting,

$$\delta S = \int \left(\frac{\partial \mathcal{L}}{\partial \psi} \delta\psi + \partial \left(\frac{\partial \mathcal{L}}{\partial \left(\partial\psi\right)} \delta\psi \right) - \partial \left(\frac{\partial \mathcal{L}}{\partial \left(\partial\psi\right)} \right) \delta\psi \right).$$

The middle term is vanishes since variations $\delta\psi$ are assumed to be zero on the integration region boundary,

$$\int \partial \left(\frac{\partial \mathcal{L}}{\partial \left(\partial\psi\right)} \delta\psi \right) = \left. \frac{\partial \mathcal{L}}{\partial \left(\partial\psi\right)} \delta\psi \right|_{\partial\Omega} = 0.$$

Finally, we're left with

$$\delta S = \int \left(\frac{\partial \mathcal{L}}{\partial \psi} - \partial \left(\frac{\partial \mathcal{L}}{\partial \left(\partial\psi\right)} \right) \right) \delta\psi.$$

A zero variation gives the equations of motion,

$$\frac{\partial \mathcal{L}}{\partial \psi} = \partial \left(\frac{\partial \mathcal{L}}{\partial \left(\partial\psi\right)} \right).$$

These are called the *Euler-Lagrange* equations. This equation of motion implies a relationship between symmetries and conserved quantities. If the Lagrangian is invariant under some field symmetry, i.e.,

$$\frac{\partial \mathcal{L}}{\partial \psi} = 0,$$

Then the following quantity is conserved,

$$\partial \left(\frac{\partial \mathcal{L}}{\partial \left(\partial\psi\right)} \right) = 0.$$

5.3 Classical Conservation of Momentum

The Lagrangian for a particle of mass m in classical mechanics is

$$\mathcal{L} = \frac{p^2}{2m} + V(x) = \frac{mv^2}{2} + V(x).$$

Applying the Euler-Lagrange equations,

$$\frac{d\mathcal{L}}{dx} - \frac{d}{dt}\left(\frac{d\mathcal{L}}{dv}\right) = \frac{dV}{dx} - \frac{d}{dt}(mv) = \frac{dV}{dx} - \frac{d(mv)}{dt}.$$

Requiring this to be zero,

$$\frac{d(mv)}{dt} = \frac{dp}{dt} = \frac{dV}{dx},$$

which is Newton's second law.

For a free particle, $V = 0$, and momentum is conserved,

$$\frac{d}{dt}\left(\frac{d\mathcal{L}}{dv}\right) = \frac{dp}{dt} = 0.$$

5.4 Relativistic Conservation of Energy-Momentum

A relativistic spacetime interval is

$$ds^2 = (dx^\mu)^2.$$

The action for a free particle is

$$S = \int \left((dx^\mu)^2\right)^{1/2}$$

with Lagrangian

$$\mathcal{L} = \left((dx^\mu)^2\right)^{1/2}.$$

. The Euler-Lagrange equations for this system are

$$\frac{\partial\mathcal{L}}{\partial x^\mu} - \frac{d}{d\tau}\left(\frac{\partial\mathcal{L}}{\partial(dx^\mu)}\right).$$

The first term is zero, and evaluating the second term,

$$\begin{aligned}\frac{d}{d\tau}\left(\frac{\partial\mathcal{L}}{\partial(dx^\mu)}\right) &= \frac{d}{d\tau}\left(\frac{dx^\mu}{\left((dx^\mu)^2\right)^{1/2}}\right)\\ &= \frac{d}{d\tau}(\gamma dx^\mu)\\ &= \frac{dp}{d\tau}\\ &= 0.\end{aligned}$$

5.5 Conservation of Probability in Schrodinger Theory

The Schrodinger equation for a free particle is

$$\frac{\partial \psi}{\partial t} = i\left(\frac{1}{2m}\nabla^2\right)\psi.$$

Complex conjugation gives the adjoint Schrodinger equation,

$$\frac{\partial \psi^*}{\partial t} = -i\left(\frac{1}{2m}\nabla^2\right)\psi^*.$$

Probability normalization requires

$$\int_\Omega \psi^* \psi \, dx = 1,$$

which is conserved through time,

$$\frac{\partial}{\partial t}\int_\Omega \psi^* \psi \, dx = \int_\Omega \left(\frac{\partial \psi^*}{\partial t}\psi + \psi^* \frac{\partial \psi}{\partial t}\right) dx = 0.$$

Using the Schrodinger equation and its adjoint to substitute for $\partial\psi/\partial t$ and $\partial\psi^*/\partial t$,

$$\begin{aligned} &\int_\Omega \left(\frac{\partial \psi^*}{\partial t}\psi + \psi^* \frac{\partial \psi}{\partial t}\right) dx \\ &= \frac{i}{2m}\int_\Omega \left(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right) dx. \end{aligned}$$

This can be simplified noting

$$\begin{aligned} &\nabla\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) \\ &= \left(\nabla\psi^*\right)\left(\nabla\psi\right) + \psi^*\left(\nabla^2\psi\right) - \left(\nabla\psi\right)\left(\nabla\psi^*\right) - \psi\left(\nabla^2\psi^*\right) \\ &= \psi^*\nabla^2\psi - \psi\nabla^2\psi^*. \end{aligned}$$

Substituting,

$$\begin{aligned} &\frac{i}{2m}\int_\Omega \left(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right) dx \\ &= \frac{i}{2m}\int_\Omega \nabla\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) dx \\ &= \frac{i}{2m}\int_{\partial\Omega} \left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) dx. \end{aligned}$$

The probability continuity equation can be obtained by letting,

$$\rho = \psi^*\psi$$

and

$$J = -\frac{i}{2m}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right),$$

where *rho* is probability density and J is probability current. Combining the integrals above,

$$\begin{aligned} &\int_\Omega \frac{\partial}{\partial t}\left(\psi^*\psi\right)\, dx - \frac{i}{2m}\int_\Omega \nabla\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)\, dx \\ = &\int_\Omega \left(\frac{\partial\rho}{\partial t} + \nabla J\right)\, dx. \end{aligned}$$

5.6 Conservation of Probability in Dirac Theory

Probability conservation in Schrodinger theory can be made relativistically invariant by modifying the probability density,

$$\rho = \psi^*\psi \to \psi^*\partial_t\psi - \psi\partial_t\psi^*.$$

We can now combine this to form probability 4-current,

$$J^\mu = \psi^*\partial_\mu\psi - \psi\partial_\mu\psi^*.$$

5.7 Derivative of General STA Versor

An STA versor ψ has 8 components,

$$\psi = \alpha + \beta + \gamma,$$

where α is a scalar, β is a bivector, and γ is a pseudoscalar.

5.8 Cauchy's, Green's, and Stokes' Integral Theorems

5.8.1 Euclidean 2-Space

To map complex variable theory to geometric algebra, use the correspondence $i \leftrightarrow e_1e_2$,

$$\begin{aligned} f &= u + iv \leftrightarrow u + v\ e_1e_2 \\ \partial_z &= \partial_x + i\partial_y \leftrightarrow \partial_x + \partial_y\ e_1e_2 \\ dz &= dx + idy \leftrightarrow dx + dy\ e_1e_2. \end{aligned}$$

Consider the integral of f around a region Ω bounded by γ,

$$\begin{aligned}\int_\gamma f\ dz &= \int_\gamma (u + v\ e_1e_2)\,(dx + dy\ e_1e_2)\\ &= \int_\gamma (u\ dx - v\ dy) + (v\ dx + u\ dy)\ \ e_1e_2.\end{aligned}$$

Comparing with df/dz integrated over Ω,

$$\begin{aligned}\int_\Omega \frac{df}{dz}\ dA &= \int_\Omega (\partial_x + \partial_y\ e_1e_2)\,(u + v\ e_1e_2)\ \ dxdy\ e_1e_2\\ &= \int_\Omega ((\partial_x u - \partial_y v) + (\partial_x v + \partial_y u)\ \ e_1e_2)\ \ dxdy\ e_1e_2\\ &= \int_\Omega ((\partial_x u - \partial_y v)\ \ e_1e_2 - (\partial_x v + \partial_y u))\ \ dxdy.\end{aligned}$$

By Green's Theorem,

$$\begin{aligned}\int_\gamma (u\ dx - v\ dy) &= -\int_\Omega (\partial_x v + \partial_y u)\ \ dxdy\\ \int_\gamma (v\ dx + u\ dy) &= \int_\Omega (\partial_x u - \partial_y v)\ \ dxdy,\end{aligned}$$

and so

$$\int_\gamma f\ dz = \int_\Omega \frac{df}{dz}\ dA.$$

Using $i \leftrightarrow e_1e_2$ is cumbersome. Instead, let

$$\begin{aligned}f &= u\ e_1 + v\ e_2\\ \nabla &= \partial_x\ e_1 + \partial_y\ e_2\\ dz &= dx\ e_1 + dy\ e_2\end{aligned}$$

Now we have

$$\begin{aligned}\int_\gamma f\ dz &= \int_\gamma (u\ e_1 + v\ e_2)\,(dx\ e_1 + dy\ e_2)\\ &= \int_\gamma (u\ dx + v\ dy) - (v\ dx - u\ dy)\ \ e_1e_2\end{aligned}$$

and

$$\begin{aligned}\int_\Omega \nabla f\ dA &= \int_\Omega (\partial_x\ e_1 + \partial_y\ e_2)\,(u\ e_1 + v\ e_2)\ \ dxdy\ e_1e_2\\ &= \int_\Omega ((\partial_x u + \partial_y v) + (\partial_x v - \partial_y u)\ \ e_1e_2)\ \ dxdy\ e_1e_2\\ &= \int_\Omega ((\partial_x u + \partial_y v)\ \ e_1e_2 - (\partial_x v - \partial_y u))\ \ dxdy.\end{aligned}$$

Chapter 6

Appendix 2: Wisdom Comes Later

6.1 Merit

Who deserves what? This person deserves love, deserves happiness. But there are no premises. You love, are happy. You aren't.

6.2 Brakes

How can learning and doing be different, what requirements? Thinking is a side-effect of confusion and frustration. No need to pause and divert until there's resistance.