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Chapter 1

Introduction

The objects of a geometric algebra are called *multivectors*. Multivectors generalize objects like directed lines, planes, and volumes. An important property of multivectors is they have *orientation*, i.e., a sense of up/down, inside/outside, etc. The following sections introduce examples of geometric algebras and the operations on their multivectors.

1.1 2D Euclidean Space and $U(1)$

1.1.1 Geometric Algebra of Euclidean 2-Space

Multivectors are composed of *basis one-vectors*. In Euclidean 2-space, these are unit vectors along the positive x and y axes. They are denoted by e_1 and e_2, respectively (the use of e here is shorthand for Euclidean. Different symbols may be used for bases in other algebras, as will be seen later).

The most important operation between multivectors is the *geometric product*. If a and b are multivectors, the geometric product of a and b is written ab. The geometric product is the sum of two simpler products, the *dot* (inner) and *wedge* (outer) products. In Euclidean 2-space, these are equivalent to the familiar dot and cross products of vector algebra. The dot product of a and b is written $a \cdot b$ and the wedge product is written $a \wedge b$. The geometric product, then, is written

$$ab = a \cdot b + a \wedge b.$$

The inner products of basis one-vectors amongst themselves defines the *signature* of an algebra,

$$
\begin{aligned}
e_1 \cdot e_1 &= 1 \\
e_2 \cdot e_2 &= 1 \\
e_1 \cdot e_2 &= 0
\end{aligned}
$$

Basis one-vectors represent directed unit lines. An oriented plane can be made by wedging the one-vectors together to form a basis *bivector*.

$$
\begin{aligned}
e_1 \wedge e_2 &= x\text{-}y \text{ plane, counterclockwise orientation} \\
e_i \wedge e_j &= 0 \quad (i = j)\,.
\end{aligned}
$$

The plane can be flipped by reversing the wedge product.

$$e_2 \wedge e_1 = -e_1 \wedge e_2 = x\text{-}y \text{ plane, clockwise orientation}$$

More generally, dot products are *symmetric* and wedge products are *antisymmetric*

$$
\begin{aligned}
\frac{1}{2}(ab + ba) &= \frac{1}{2}(a \cdot b + a \wedge b + b \cdot a + b \wedge a) = a \cdot b, \\
\frac{1}{2}(ab - ba) &= \frac{1}{2}(a \cdot b + a \wedge b - b \cdot a - b \wedge a) = a \wedge b.
\end{aligned}
$$

Since unit vectors are *orthogonal*,

$$
\begin{aligned}
e_i e_j &= e_i \cdot e_j + e_i \wedge e_j = 1 && (i = j) \\
e_i e_j &= e_i \cdot e_j + e_i \wedge e_j = e_i \wedge e_j && (i \neq j)
\end{aligned}
$$

explicit dots and wedges are unnecessary when writing basis vectors. We simply write $e_1 e_2$ instead of $e_1 \wedge e_2$.

A basis *zero-vector* is a scalar.

A general multivector m in Euclicean 2-space is a linear combination of basis vectors,

$$m = s + a_1 e_1 + a_2 e_2 + b e_1 e_2.$$

The following formula is seldom used[1], but for completeness, the product of two general multivectors,

$$
\begin{aligned}
m &= s + a_1 e_1 + a_2 e_2 + b e_1 e_2, \\
n &= r + c_1 e_1 + c_2 e_2 + d e_1 e_2
\end{aligned}
$$

is

$$
\begin{aligned}
mn = {} & (rs + a_1 b_1 + a_2 b_2 - bd) \\
& + (ra_1 + sc_1 - a_2 d + c_2 b)\, e_1 \\
& + (ra_2 + sc_2 + a_1 d - c_1 b)\, e_2 \\
& + (a_1 c_2 - a_2 c_1)\, e_1 e_2.
\end{aligned}
$$

[1] One application is coding computer algebra systems.

The basis vector formed by multiplying all basis one-vectors is called the *unit pseudoscalar* and is denoted by I. In 2-space, $I = e_1 e_2$. This is in direct analogy to $i = \sqrt{-1}$ from complex numbers, as shown below.

Often, we'll be interested in *even* multivectors, i.e., linear combinations of zero-vectors, bivectors, four-vectors, etc. The product of even multivectors

$$m = a + b e_1 e_2 = a + bI,$$
$$n = c + d e_1 e_2 = c + dI$$

is

$$mn = (ac - bd) + (ad + bc)\, I,$$

which is the formula for multiplying two complex numbers. From this perspective, the i from complex algebra can be thought of as a counterclockwise-oriented plane.

Unit psuedoscalars satisfy $I^2 = -1$. For this, we define the *reverse* operator on multivectors, which reverses the order of basis vectors. The reverse of $I = e_1 e_2$ in 2-space is

$$\tilde{I} = e_2 e_1 = -I.$$

Note that an odd number of swaps (just 1 in euclidean 2-space) is required to reverse I, so the signature of the algebra doesn't require modification to satisfy $I^2 = -1$. In other algebras, like Minkowski spacetime, the signature will need modification to satisfy this requirement.

The square of a multivector is defined by multiplying a multivector by its reverse. For example,

$$I^2 = \tilde{I} I = e_3 e_2 e_1 \left(e_1 e_2 e_3 \right) = -1.$$

We can use I to compute the *dual* of a multivector simply by multiplying. The dual M of a multivector m is

$$M = Im.$$

For example,

$$I e_2 = \left(e_1 e_2 \right) e_2 = e_1.$$

If m spans a subspace of Euclidean 2-space, its dual spans the remaining subspace needed to fill out 2-space. This is the same as the orthogonal complement in linear algebra.

1.1.2 $U(1)$ as a Geometric Algebra

In later chapters, we'll discuss symmetries in field theories. Many of these symmetries traditionally involve unitary groups, so we show how unitary groups can be represented with geometric algebras here.

Unitary groups $U(n)$ are groups of $n \times n$ unitary matrices, i.e., matrices U where $U^\dagger U = I$. For $n = 1$, this is the group of unit complex numbers. As shown above, this group equivalent to the group of even multivectors in Euclidean 2-space of unit magnitude.

Later, this will be important when discussing symmetries for quantum electrodynamics.

1.2 3D Euclidean Space and $SU(2)$

1.2.1 Geometric Algebra of Euclidean 2-Space

In Euclidean 3-space, the three basis vectors are e_1, e_2, and e_3. Their inner products satisfy,

$$\begin{aligned} e_1 \cdot e_1 &= 1 \\ e_2 \cdot e_2 &= 1 \\ e_3 \cdot e_3 &= 1 \\ e_i \cdot e_j &= 0 \quad (i \neq j) \end{aligned}$$

The wedge products are,

$$\begin{aligned} e_1 \wedge e_2 &= x\text{-}y \text{ plane, normal along } +z \\ e_2 \wedge e_3 &= y\text{-}z \text{ plane, normal along } +x \\ e_3 \wedge e_1 &= z\text{-}x \text{ plane, normal along } +y \\ e_i \wedge e_j &= 0 \quad (i = j) \end{aligned}$$

Flipping the planes,

$$\begin{aligned} e_2 \wedge e_1 = -e_1 \wedge e_2 &= x\text{-}y \text{ plane, normal along } -z \\ e_3 \wedge e_2 = -e_2 \wedge e_3 &= y\text{-}z \text{ plane, normal along } -x \\ e_1 \wedge e_3 = -e_3 \wedge e_1 &= z\text{-}x \text{ plane, normal along } -y. \end{aligned}$$

The unit volume/pseudoscalar is,

$$I = e_1 e_2 e_3 = e_1 \wedge e_2 \wedge e_3.$$

It's reverse is

$$\tilde{I} = e_3 e_2 e_1 = -I.$$

Again, an odd number of swaps (3) is required to reverse I, so no signature modification is required to satisfy $I^2 = -1$.

A general multivector m in Euclicean 3-space is a scalar plus a linear combination of basis vectors,

$$m = s + a_1 e_1 + a_2 e_2 + a_3 e_3 + b_1 e_2 e_3 + b_2 e_3 e_1 + b_3 e_1 e_2 + c e_1 e_2 e_3.$$

1.2.2 $SU(2)$ as a Geometric Algebra

1.3 Minkowski Spacetime

Minkowski spacetime has four dimensions, one timelike, three spacelike. These are denoted by γ_i instead of e_i. The algebra's signature needs modification to satisfy $I^2 = -1$. Given $I = \gamma_0\gamma_1\gamma_2\gamma_3$,

$$\tilde{I} = \gamma_3\gamma_2\gamma_1\gamma_0 = I.$$

An even number of swaps (6) is required to reverse I. So, to satisfy $I^2 = -1$, we need to modify the algebra's signature to achieve an overall negative sign. This can be done by setting the square 1 or 3 basis vectors to be negative. We choose the signature,

$$\begin{aligned}
\gamma_0 \cdot \gamma_0 &= 1 \\
\gamma_1 \cdot \gamma_1 &= -1 \\
\gamma_2 \cdot \gamma_2 &= -1 \\
\gamma_3 \cdot \gamma_3 &= -1 \\
\gamma_i \cdot \gamma_j &= 0 \quad (i \neq j)\,.
\end{aligned}$$

1.4 Conformal 3-Space

Conformal algebras are created by taking an underlying space and adding a spacelike and timelike dimension to it. If we append e_0 (timelike) and e_4 (spacelike) to Euclidean 3-space, we have the basis e_0, e_1, e_2, e_3, e_4. To determine the signature, reverse I,

$$\tilde{I} = e_4e_3e_2e_1e_0 = I.$$

This requires 10 swaps, so one, three, or five basis vectors should carry a negative sign. We choose e_0,

$$\begin{aligned}
e_0 \cdot e_0 &= -1 \\
e_1 \cdot e_1 &= 1 \\
e_2 \cdot e_2 &= 1 \\
e_3 \cdot e_3 &= 1 \\
e_4 \cdot e_4 &= 1 \\
e_i \cdot e_j &= 0 \quad (i \neq j)
\end{aligned}$$

The wedge products are,

$$
\begin{aligned}
e_1 \wedge e_2 &= x\text{-}y \text{ plane, normal along } +z \\
e_2 \wedge e_3 &= y\text{-}z \text{ plane, normal along } +x \\
e_3 \wedge e_1 &= z\text{-}x \text{ plane, normal along } +y \\
e_i \wedge e_j &= 0 \quad (i = j)
\end{aligned}
$$

Flipping the planes,

$$
\begin{aligned}
e_2 \wedge e_1 = -e_1 \wedge e_2 &= x\text{-}y \text{ plane, normal along } -z \\
e_3 \wedge e_2 = -e_2 \wedge e_3 &= y\text{-}z \text{ plane, normal along } -x \\
e_1 \wedge e_3 = -e_3 \wedge e_1 &= z\text{-}x \text{ plane, normal along } -y.
\end{aligned}
$$

Chapter 2

Derivatives, Integrals, and Geometric Calculus

2.1 The Geometric Derivative

Geometric Algebra is extended to Geometric Calculus by adding geometric derivatives and integrals. The geometric derivative is denoted by ∇ and acts on multivector-valued functions over a given domain. In summation notation, the geometric derivative of a function F is:

$$\nabla F(x) = e_i \partial_i F(x).$$

The following sections demonstrate the geometric derivative in example spaces.

2.1.1 Euclidean 2-Space

Real-valued Functions

Consider a real-valued function f over x and y in Euclidean 2-space. Its geometric derivative is

$$\nabla f = (\partial_x f)\, e_1 + (\partial_y f)\, e_2,$$

i.e., the geometric derivative of a real-valued function is its gradient.

Vector-valued Functions

Consider a vector-valued function $g(x, y) = u(x, y)\, e_1 + v(x, y)\, e_2$. It's derivative is

$$\begin{aligned}\nabla g &= (e_1 \partial_x + e_2 \partial_y)\,(u e_1 + v e_2) \\ &= (\partial_x u + \partial_y v) + (\partial_x v - \partial_y u)\, e_1 e_2,\end{aligned}$$

i.e., the derivative of a vector-valued function in Euclidean 2-space is the complex derivative. Grouping terms above, we see that, because of the geometric product, geometric derivatives separate into inner (gradient) and outer (curl) products,

$$\nabla g = \nabla \cdot g + \nabla \wedge g.$$

The geometric derivative contains both divergence and curl from traditional vector calculus.

Bivector-valued Functions

Given $f(x, y) e_1 e_2$,

$$\nabla f = (\partial_x f)\, e_2 - (\partial_y f)\, e_1.$$

2.1.2 Euclidean 3-Space

Real-valued Functions

The geometric derivative acting on a real-valued function f in 3-space is

$$\nabla f = (\partial_x f)\, e_1 + (\partial_y f)\, e_2 + (\partial_z f)\, e_3.$$

Again, this is simply the gradient of f.

Vector-valued Functions

Given a vector-valued function $g(x, y) = u(x, y)\, e_1 + v(x, y)\, e_2 + w(x, y)\, e_3$. It's derivative is

$$\begin{aligned}
\nabla g &= (e_1\partial_x + e_2\partial_y + e_3\partial_z)(ue_1 + ve_2 + we_3) \\
&= (\partial_x u + \partial_y v + \partial_z w) + \\
&= (\partial_x v - \partial_y u)\, e_1 e_2 + \\
&= (\partial_y w - \partial_z v)\, e_2 e_3 + \\
&= (\partial_z u - \partial_x w)\, e_3 e_1 .
\end{aligned}$$

2.1.3 Minkowski Spacetime

2.2 Gauge Covariant Derivative

In field theories, symmetries of the Lagrangian for the system determine the kinematic equations for that system. Consider a field theory in $\psi(x)$ with the following Lagrangian,

$$\mathcal{L} = \overline{\psi} D \psi .$$

Here, D is a derivative operator. We'd like this Lagrangian to be invariant under the following field transformations,

$$\psi' = e^{-\lambda}\psi ,$$

so that

$$\mathcal{L} = \overline{\psi} D \psi = \overline{\psi}' D' \psi' .$$

Transformations can be *global*, where λ is constant, or *local*, where λ is a function of x. We only consider local transformations here. Expanding the right side,

$$\overline{\psi}' D' \psi' = \overline{\psi} e^{\lambda} D' e^{-\lambda} \psi .$$

So D transforms, $D' = e^{-\lambda} D e^{\lambda}$.

If we let $D = \partial$, the derivative product rule breaks invariance,

$$\begin{aligned}
\mathcal{L} &= \overline{\psi}' D \psi' \\
&= \overline{\psi}' \partial \psi' \\
&= \overline{\psi} e^{\lambda} \partial e^{-\lambda} \psi \\
&= \left(\overline{\psi} e^{\lambda}\right) e^{-\lambda} \left((\partial\psi) - (\partial\lambda)\,\psi\right) \\
&= \overline{\psi}\partial\psi - \overline{\psi}\,(\partial\lambda)\,\psi .
\end{aligned}$$

There's an extra $\overline{\psi}\,(\partial\lambda)\,\psi$ term, the *gauge* term. The usual way to offset this is to add a *gauge field* A from D, then determine its transformation properties. Letting $D = \partial + A$, invariance requires

$$
\begin{aligned}
D' &= \partial + A' \\
&= e^{-\lambda} D e^{\lambda} \\
&= e^{-\lambda} \left(\partial + A\right) e^{\lambda} \\
&= e^{-\lambda} \left(e^{\lambda}\partial\lambda + e^{\lambda}\partial + A e^{\lambda}\right) \\
&= \partial + \left(e^{-\lambda} A e^{\lambda} + \partial\lambda\right).
\end{aligned}
$$

This shows the gauge field transforms $A' = e^{-\lambda} A e^{\lambda} + \partial\lambda$.

2.3 Cauchy's Integral Theorem and Formula

Chapter 3

Yang-Mills Theories

Chapter 4

Feynman Path Integrals