Contents

Chapter 1

Introduction

The objects of a geometric algebra are called *multivectors*. Multivectors generalize objects like directed lines, planes, and volumes. An important property of multivectors is they have *orientation*, i.e., a sense of up/down, inside/outside, etc. The following sections introduce examples of geometric algebras and the operations on their multivectors.

1.1 2D Euclidean Space and $U(1)$

1.1.1 Geometric Algebra of Euclidean 2-Space

Multivectors are composed of *basis one-vectors*. In Euclidean 2-space, these are unit vectors along the positive x and y axes. They are denoted by e_1 and e_2, respectively (the use of e here is shorthand for Euclidean. Different symbols may be used for bases in other algebras, as will be seen later).

The most important operation between multivectors is the *geometric product*. If a and b are multivectors, the geometric product of a and b is written ab. The geometric product is the sum of two simpler products, the *dot* (inner) and *wedge* (outer) products. In Euclidean 2-space, these are equivalent to the familiar dot and cross products of vector algebra. The dot product of a and b is written $a \cdot b$ and the wedge product is written $a \wedge b$. The geometric product, then, is written

$$ab = a \cdot b + a \wedge b.$$

The inner products of basis one-vectors amongst themselves defines the *signature* of an algebra,

$$\begin{aligned} e_1 \cdot e_1 &= 1 \\ e_2 \cdot e_2 &= 1 \\ e_1 \cdot e_2 &= 0 \end{aligned}$$

Basis one-vectors represent directed unit lines. An oriented plane can be made by wedging the one-vectors together to form a basis *bivector*.

$$\begin{aligned} e_1 \wedge e_2 &= x\text{-}y \text{ plane, counterclockwise orientation} \\ e_i \wedge e_j &= 0 \quad (i = j)\,. \end{aligned}$$

The plane can be flipped by reversing the wedge product.

$$e_2 \wedge e_1 = -e_1 \wedge e_2 = x\text{-}y \text{ plane, clockwise orientation}$$

More generally, dot products are *symmetric* and wedge products are *antisymmetric*

$$\begin{aligned} \frac{1}{2}(ab + ba) &= \frac{1}{2}(a \cdot b + a \wedge b + b \cdot a + b \wedge a) = a \cdot b, \\ \frac{1}{2}(ab - ba) &= \frac{1}{2}(a \cdot b + a \wedge b - b \cdot a - b \wedge a) = a \wedge b. \end{aligned}$$

Since unit vectors are *orthogonal*,

$$\begin{aligned} e_i e_j &= e_i \cdot e_j + e_i \wedge e_j = 1 && (i = j) \\ e_i e_j &= e_i \cdot e_j + e_i \wedge e_j = e_i \wedge e_j && (i \neq j) \end{aligned}$$

explicit dots and wedges are unnecessary when writing basis vectors. We simply write $e_1 e_2$ instead of $e_1 \wedge e_2$.

A basis *zero-vector* is a scalar.

A general multivector m in Euclicean 2-space is a linear combination of basis vectors,

$$m = s + a_1 e_1 + a_2 e_2 + b e_1 e_2.$$

The following formula is seldom used[1], but for completeness, the product of two general multivectors,

$$\begin{aligned} m &= s + a_1 e_1 + a_2 e_2 + b e_1 e_2, \\ n &= r + c_1 e_1 + c_2 e_2 + d e_1 e_2 \end{aligned}$$

is

$$\begin{aligned} mn = {} & (rs + a_1 b_1 + a_2 b_2 - bd) \\ & + (ra_1 + sc_1 - a_2 d + c_2 b)\, e_1 \\ & + (ra_2 + sc_2 + a_1 d - c_1 b)\, e_2 \\ & + (a_1 c_2 - a_2 c_1)\, e_1 e_2. \end{aligned}$$

[1]One application is coding computer algebra systems.

The basis vector formed by multiplying all basis one-vectors is called the *unit pseudoscalar* and is denoted by I. In 2-space, $I = e_1 e_2$. This is in direct analogy to $i = \sqrt{-1}$ from complex numbers, as shown below.

Often, we'll be interested in *even* multivectors, i.e., linear combinations of zero-vectors, bivectors, four-vectors, etc. The product of even multivectors

$$m = a + be_1 e_2 = a + bI,$$
$$n = c + de_1 e_2 = c + dI$$

is

$$mn = (ac - bd) + (ad + bc)\, I,$$

which is the formula for multiplying two complex numbers. From this perspective, the i from complex algebra can be thought of as a counterclockwise-oriented plane.

Unit psuedoscalars satisfy $I^2 = -1$. For this, we define the *reverse* operator on multivectors, which reverses the order of basis vectors. The reverse of $I = e_1 e_2$ in 2-space is

$$\tilde{I} = e_2 e_1 = -I.$$

Note that an odd number of swaps (just 1 in euclidean 2-space) is required to reverse I, so the signature of the algebra doesn't require modification to satisfy $I^2 = -1$. In other algebras, like Minkowski spacetime, the signature will need modification to satisfy this requirement.

The square of a multivector is defined by multiplying a multivector by its reverse. For example,

$$I^2 = \tilde{I} I = e_3 e_2 e_1 \left(e_1 e_2 e_3\right) = -1.$$

We can use I to compute the *dual* of a multivector simply by multiplying. The dual M of a multivector m is

$$M = Im.$$

For example,

$$Ie_2 = \left(e_1 e_2\right) e_2 = e_1.$$

If m spans a subspace of Euclidean 2-space, its dual spans the remaining subspace needed to fill out 2-space. This is the same as the orthogonal complement in linear algebra.

1.1.2 $U(1)$ as a Geometric Algebra

In later chapters, we'll discuss symmetries in field theories. Many of these symmetries involve *unitary* groups. We show here and in later sections how unitary groups can be represented with geometric algebras.

Unitary groups $U(n)$ are groups of $n \times n$ unitary matrices, i.e., matrices U where $U^\dagger U = I$. For $n = 1$, this is the group of unit complex numbers. As shown above, this group equivalent to the group of even multivectors in Euclidean 2-space of unit magnitude.

1.2 3D Euclidean Space and $SU(2)$

1.2.1 Geometric Algebra of Euclidean 3-Space

In Euclidean 3-space, the three basis vectors are e_1, e_2, and e_3. Their inner products satisfy,

$$
\begin{aligned}
e_1 \cdot e_1 &= 1 \\
e_2 \cdot e_2 &= 1 \\
e_3 \cdot e_3 &= 1 \\
e_i \cdot e_j &= 0 \quad (i \neq j)
\end{aligned}
$$

The wedge products are,

$$
\begin{aligned}
e_1 \wedge e_2 &= x\text{-}y \text{ plane, normal along } +z \\
e_2 \wedge e_3 &= y\text{-}z \text{ plane, normal along } +x \\
e_3 \wedge e_1 &= z\text{-}x \text{ plane, normal along } +y \\
e_i \wedge e_j &= 0 \quad (i = j)
\end{aligned}
$$

Flipping the planes,

$$
\begin{aligned}
e_2 \wedge e_1 = -e_1 \wedge e_2 &= x\text{-}y \text{ plane, normal along } -z \\
e_3 \wedge e_2 = -e_2 \wedge e_3 &= y\text{-}z \text{ plane, normal along } -x \\
e_1 \wedge e_3 = -e_3 \wedge e_1 &= z\text{-}x \text{ plane, normal along } -y.
\end{aligned}
$$

The unit volume/pseudoscalar is,

$$I = e_1 e_2 e_3 = e_1 \wedge e_2 \wedge e_3.$$

It's reverse is

$$\tilde{I} = e_3 e_2 e_1 = -I.$$

Again, an odd number of swaps (3) is required to reverse I, so no signature modification is required to satisfy $I^2 = -1$.

A general multivector m in Euclicean 3-space is a scalar plus a linear combination of basis vectors,

$$m = s + a_1 e_1 + a_2 e_2 + a_3 e_3 + b_1 e_2 e_3 + b_2 e_3 e_1 + b_3 e_1 e_2 + c e_1 e_2 e_3.$$

1.2.2 $SU(2)$ as a Geometric Algebra

1.3 Minkowski Spacetime

Minkowski spacetime has four dimensions, one timelike, three spacelike. These are denoted by γ_i instead of e_i. The algebra's signature needs modification to satisfy $I^2 = -1$. Given $I = \gamma_0\gamma_1\gamma_2\gamma_3$,

$$\tilde{I} = \gamma_3\gamma_2\gamma_1\gamma_0 = I.$$

An even number of swaps (6) is required to reverse I. So, to satisfy $I^2 = -1$, we need to modify the algebra's signature to achieve an overall negative sign. This can be done by setting the squares of 1 or 3 basis vectors to be negative. We choose the signature,

$$\begin{aligned}
\gamma_0 \cdot \gamma_0 &= 1 \\
\gamma_1 \cdot \gamma_1 &= -1 \\
\gamma_2 \cdot \gamma_2 &= -1 \\
\gamma_3 \cdot \gamma_3 &= -1 \\
\gamma_i \cdot \gamma_j &= 0 \quad (i \neq j)\,.
\end{aligned}$$

1.4 Conformal 3-Space

Conformal algebras are created by taking an underlying space and adding a spacelike and timelike dimension to it. If we append e_0 (timelike) and e_4 (spacelike) to Euclidean 3-space, we have the basis e_0, e_1, e_2, e_3, e_4. To determine the signature, reverse I,

$$\tilde{I} = e_4e_3e_2e_1e_0 = I.$$

This requires 10 swaps, so one, three, or five basis vectors should carry a negative sign. We choose e_0,

$$\begin{aligned}
e_0 \cdot e_0 &= -1 \\
e_1 \cdot e_1 &= 1 \\
e_2 \cdot e_2 &= 1 \\
e_3 \cdot e_3 &= 1 \\
e_4 \cdot e_4 &= 1 \\
e_i \cdot e_j &= 0 \quad (i \neq j)
\end{aligned}$$

The wedge products are,

$$\begin{aligned}
e_1 \wedge e_2 &= x\text{-}y \text{ plane, normal along } +z \\
e_2 \wedge e_3 &= y\text{-}z \text{ plane, normal along } +x \\
e_3 \wedge e_1 &= z\text{-}x \text{ plane, normal along } +y \\
e_i \wedge e_j &= 0 \quad (i = j)
\end{aligned}$$

Flipping the planes,

$$
\begin{aligned}
e_2 \wedge e_1 &= -e_1 \wedge e_2 = x\text{-}y \text{ plane, normal along } -z \\
e_3 \wedge e_2 &= -e_2 \wedge e_3 = y\text{-}z \text{ plane, normal along } -x \\
e_1 \wedge e_3 &= -e_3 \wedge e_1 = z\text{-}x \text{ plane, normal along } -y.
\end{aligned}
$$

Chapter 2

Derivatives, Integrals, and Geometric Calculus

2.1 The Geometric Derivative

Geometric Algebra is extended to Geometric Calculus by adding geometric derivatives and integrals. The geometric derivative is denoted by ∇ and acts on multivector-valued functions over a given domain. In summation notation, the geometric derivative of a function F is:

$$\nabla F\left(x\right) = e_i \partial_i F\left(x\right).$$

The following sections demonstrate the geometric derivative in example spaces.

2.1.1 Euclidean 2-Space

Real Functions

Consider a real-valued function f over x and y in Euclidean 2-space. Its geometric derivative is

$$\nabla f = \left(\partial_x f\right) e_1 + \left(\partial_y f\right) e_2,$$

i.e., the geometric derivative of a real-valued function is its gradient.

Vector Functions

Consider a vector-valued function $g = ue_1 + ve_2$. It's derivative is

$$\begin{aligned} \nabla g &= (e_1\partial_x + e_2\partial_y)\,(ue_1 + ve_2) \\ &= (\partial_x u + \partial_y v) + (\partial_x v - \partial_y u)\,e_1e_2. \end{aligned}$$

In other words, the derivative of a vector-valued function in Euclidean 2-space is the complex derivative. In particular, we see that the geometric derivatives separates into inner (gradient) and outer (curl) products,

$$\nabla g = \nabla \cdot g + \nabla \wedge g.$$

The geometric derivative contains both divergence and curl from traditional vector calculus.

Bivector (Pseudoscalar) Functions

For fe_1e_2,

$$\nabla f = (\partial_x f)\,e_2 - (\partial_y f)\,e_1.$$

2.1.2 Euclidean 3-Space

Real Functions

The geometric derivative acting on a real-valued function f in 3-space is

$$\nabla f = (\partial_x f)\,e_1 + (\partial_y f)\,e_2 + (\partial_z f)\,e_3.$$

Again, this is simply the gradient of f.

Vector Functions

Given a vector-valued function $g = ue_1 + ve_2 + we_3$. Its derivative is

$$\begin{aligned} \nabla g &= (e_1\partial_x + e_2\partial_y + e_3\partial_z)\,(ue_1 + ve_2 + we_3) \\ &= (\partial_x u + \partial_y v + \partial_z w) + \\ &= (\partial_x v - \partial_y u)\,e_1e_2 + \\ &= (\partial_y w - \partial_z v)\,e_2e_3 + \\ &= (\partial_z u - \partial_x w)\,e_3e_1. \end{aligned}$$

Bivector Functions

A bivector function in 3D Euclidean space has the form

$$f = ue_1e_2 + ve_2e_3 + we_3e_1.$$

Its derivative is

$$\begin{aligned}\nabla f &= (e_1\partial_x + e_2\partial_y + e_3\partial_z)(ue_1e_2 + ve_2e_3 + we_3e_1)\\ &= (\partial_z w - \partial_y u)\, e_1\\ &+ (\partial_x u - \partial_z v)\, e_2\\ &+ (\partial_y v - \partial_x w)\, e_3\\ &+ (\partial_x v + \partial_y w + \partial_z u)\, e_1e_2e_3\end{aligned}$$

Pseudoscalar Functions

For $fe_1e_2e_3$,

$$\nabla f = (e_1\partial_x + e_2\partial_y + e_3\partial_z) f = \partial_x f e_2e_3 - \partial_y f e_1e_2 + \partial_z f e_1e_2.$$

2.1.3 Minkowski Spacetime

Real Functions

Vector Functions

Bivector Functions

Pseudovector Functions

Pseudoscalar Functions

2.1.4 Conformal 3-Space

2.2 Gauge Covariant Derivative

In field theories, symmetries of the Lagrangian for the system determine the kinematic equations for that system. Consider a field theory in $\psi(x)$ with the following Lagrangian,

$$\mathcal{L} = \overline{\psi} D\psi.$$

Here, D is a derivative operator. We'd like this Lagrangian to be invariant under the following field transformations,

$$\psi' = e^{-\lambda}\psi,$$

so that

$$\mathcal{L} = \overline{\psi} D\psi = \overline{\psi}' D'\psi'.$$

Transformations can be *global*, where λ is constant, or *local*, where λ is a function of x. We consider local transformations here. Expanding the right side,

$$\overline{\psi}' D' \psi' = \overline{\psi} e^{\lambda} D' e^{-\lambda} \psi.$$

So D transforms $D' = e^{-\lambda} D e^{\lambda}$.

If we let $D = \partial$, the derivative product rule breaks invariance,

$$\begin{aligned}
\mathcal{L} &= \overline{\psi}' D \psi' \\
&= \overline{\psi}' \partial \psi' \\
&= \overline{\psi} e^{\lambda} \partial e^{-\lambda} \psi \\
&= \left(\overline{\psi} e^{\lambda}\right) e^{-\lambda} \left(\left(\partial \psi\right) - \left(\partial \lambda\right) \psi\right) \\
&= \overline{\psi} \partial \psi - \overline{\psi} \left(\partial \lambda\right) \psi.
\end{aligned}$$

There's an extra $\overline{\psi} \left(\partial \lambda\right) \psi$ term, the *gauge* term. The usual way to offset this is to add a *gauge field* A to D, then determine its transformation properties. Letting $D = \partial + A$, invariance requires

$$\begin{aligned}
D' &= \partial + A' \\
&= e^{-\lambda} D e^{\lambda} \\
&= e^{-\lambda} \left(\partial + A\right) e^{\lambda} \\
&= e^{-\lambda} \left(e^{\lambda} \partial \lambda + e^{\lambda} \partial + A e^{\lambda}\right) \\
&= \partial + \left(e^{-\lambda} A e^{\lambda} + \partial \lambda\right).
\end{aligned}$$

This shows the gauge field transforms $A' = e^{-\lambda} A e^{\lambda} + \partial \lambda$.

2.3 Cauchy's Integral Theorem and Formula

Chapter 3

Yang-Mills Theories

Chapter 4

Feynman Path Integrals