

# Eigenvector Continuation for scattering with local chiral NN and optical potentials

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Christian Drischler

April 26, 2021 | INT 21-1b: Nuclear Forces for Precision Nuclear Physics

## This week's topic:

Improving nuclear forces with *novel fitting strategies* and higher orders in chiral EFT

1

using EC as an efficient emulator for scattering with local chiral interactions and optical potentials

2

statistical quantification and propagation of EFT truncation errors in nuclear matter calculations

## Keywords:

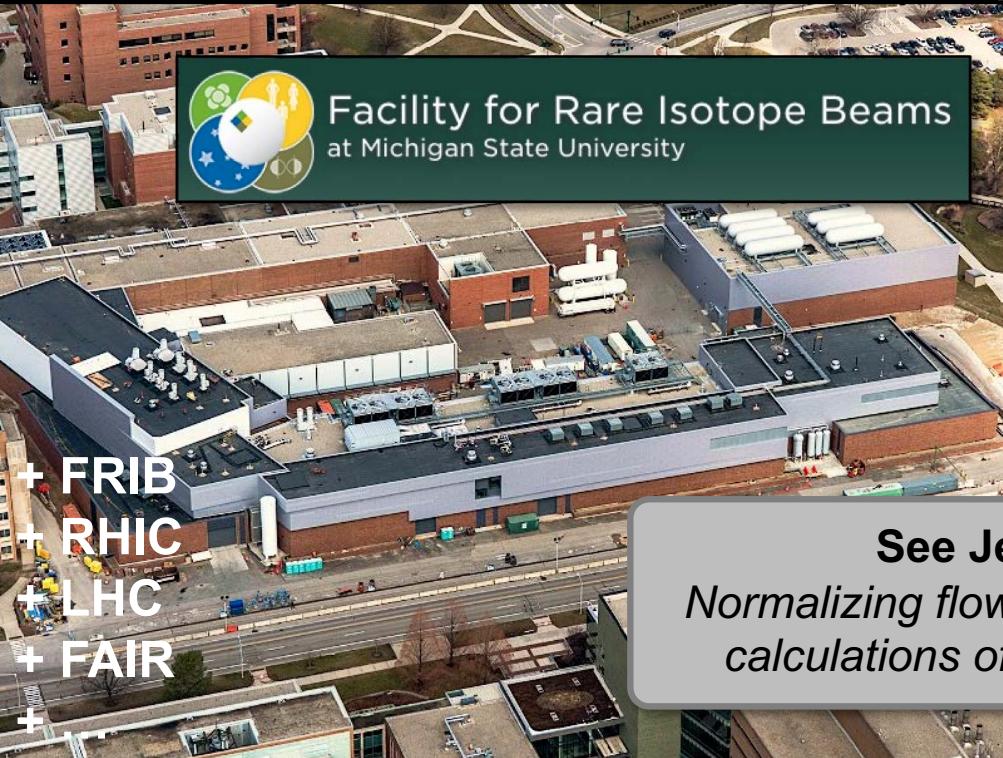
- + ChEFT + scattering
- + Variational principles
- + Eigenvector Continuation
- + Bayesian UQ
- + infinite nuclear matter
- + EFT truncation errors
- + ...

- + LVC
- + Virgo
- + GEO600
- + KAGRA
- + ...



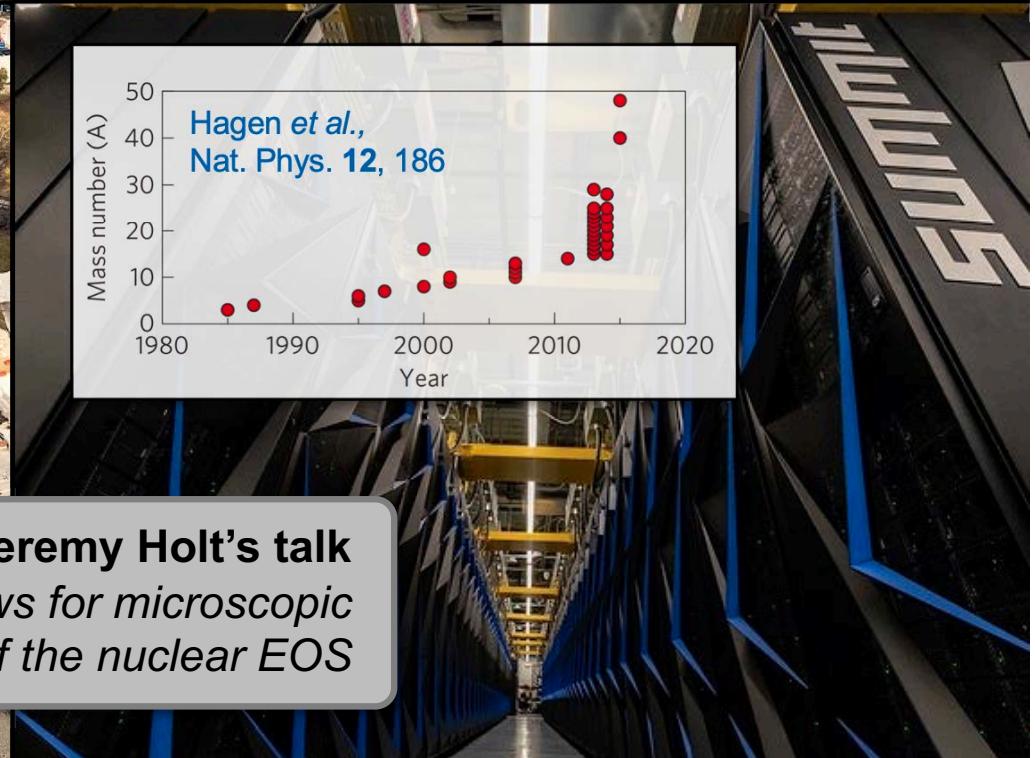
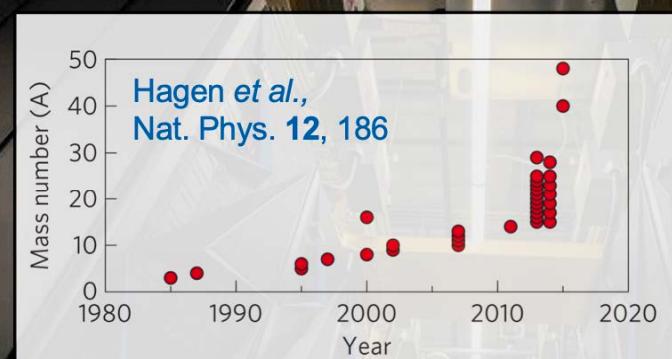
What is the secondary object  
in GW190425 and GW190814 ?

- + STROBE-X
- + eXTP
- + ...



- + FRIB
- + RHIC
- + LHC
- + FAIR
- + ...

See Jeremy Holt's talk  
*Normalizing flows for microscopic  
calculations of the nuclear EOS*



# Eigenvector Continuation for scattering with local chiral NN and optical potentials

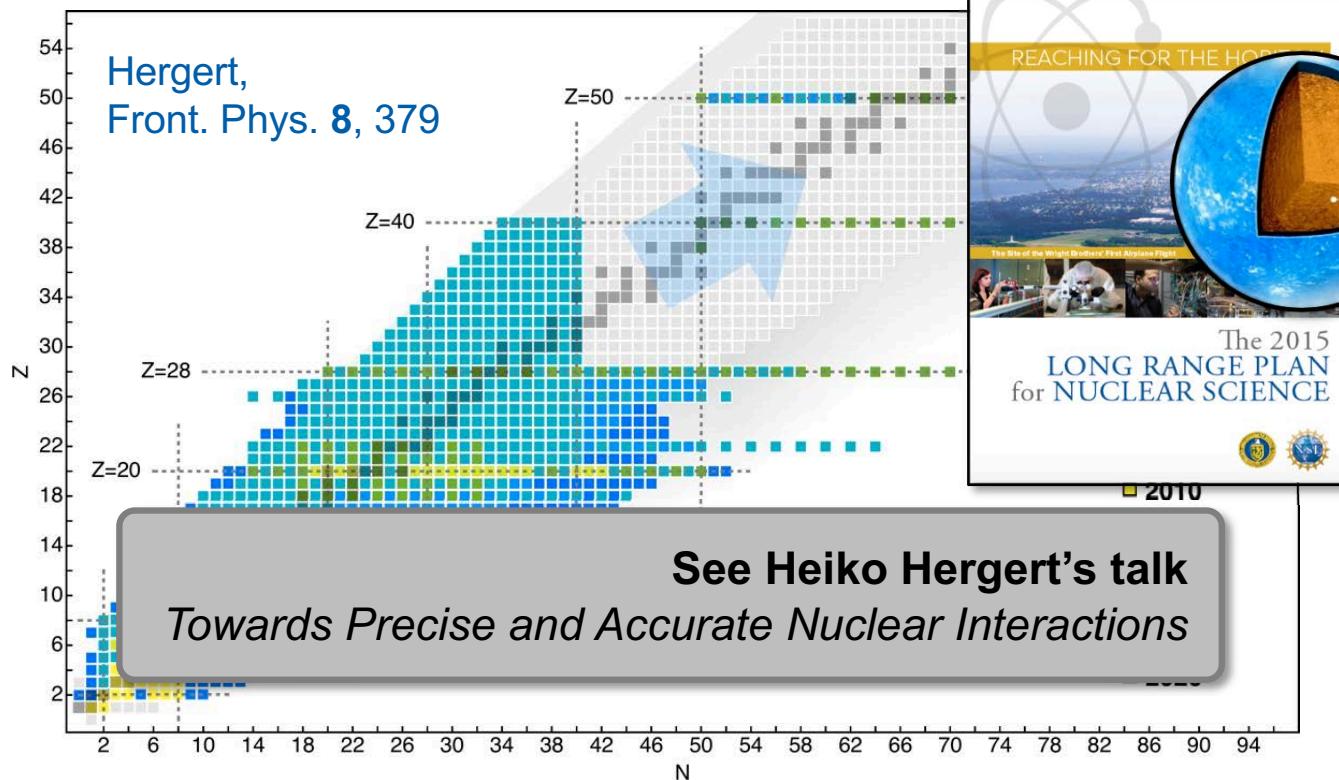
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CD, Haxton, McElvain *et al.*, arXiv:1910.07961 (PPNP in press)

**How** does the nuclear chart emerge from QCD?

**Where** do heavy elements come from?

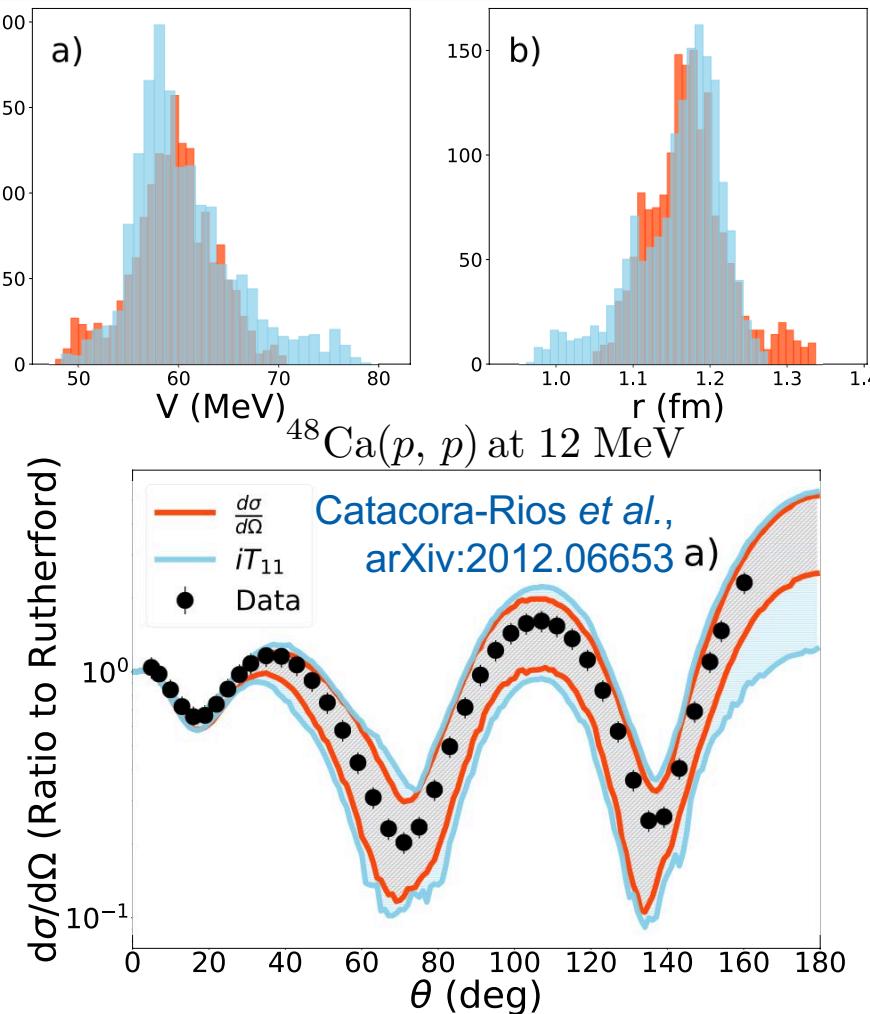
**How** does subatomic matter organize itself?



# Eigenvector Continuation for scattering with local chiral NN and optical potentials

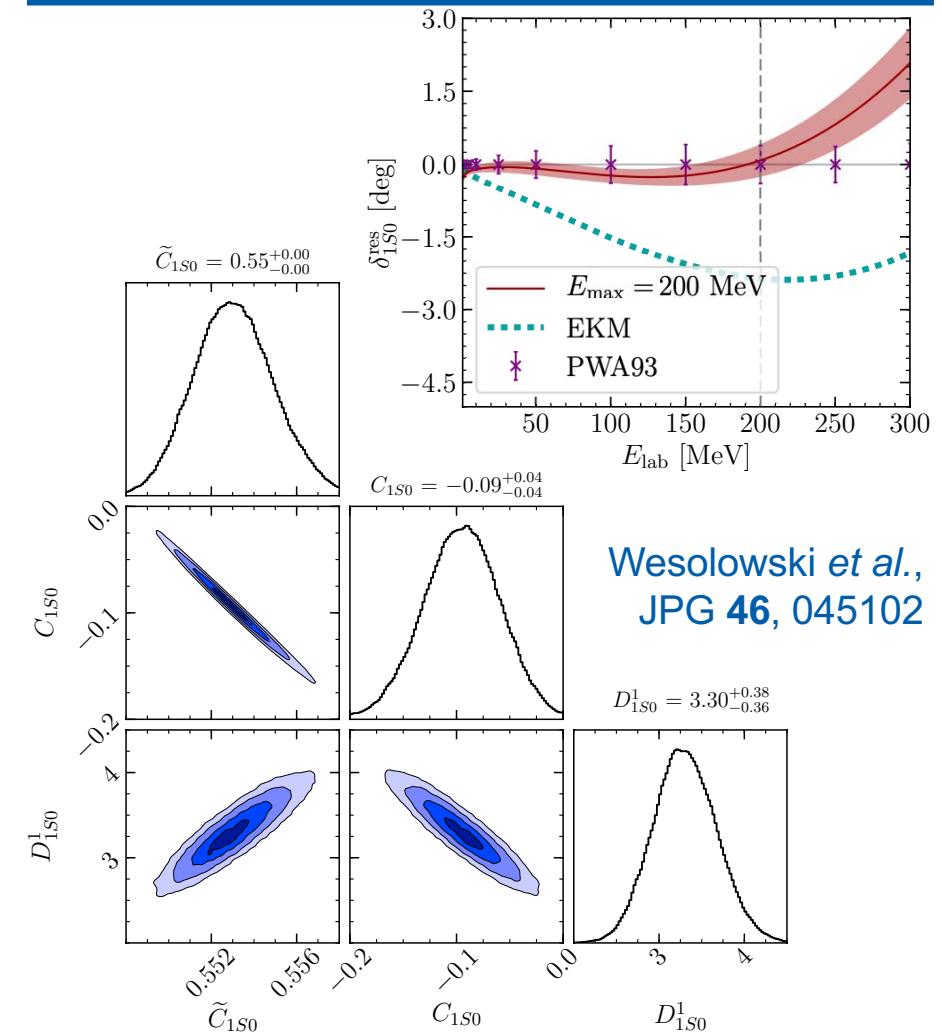
Bayesian uncertainty quantification

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## Optical potentials

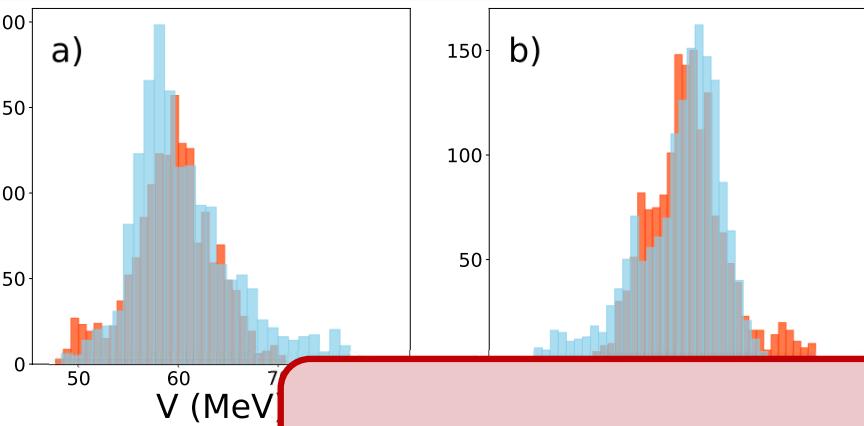
## Chiral potentials



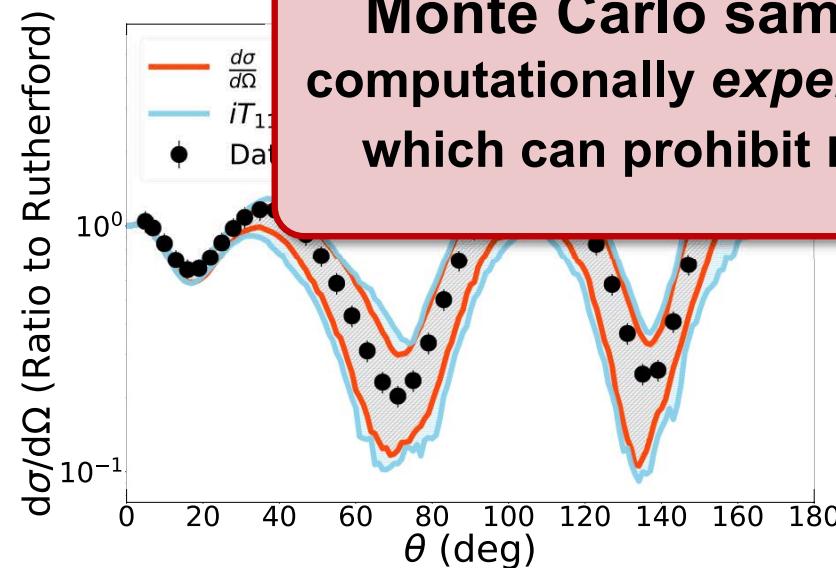
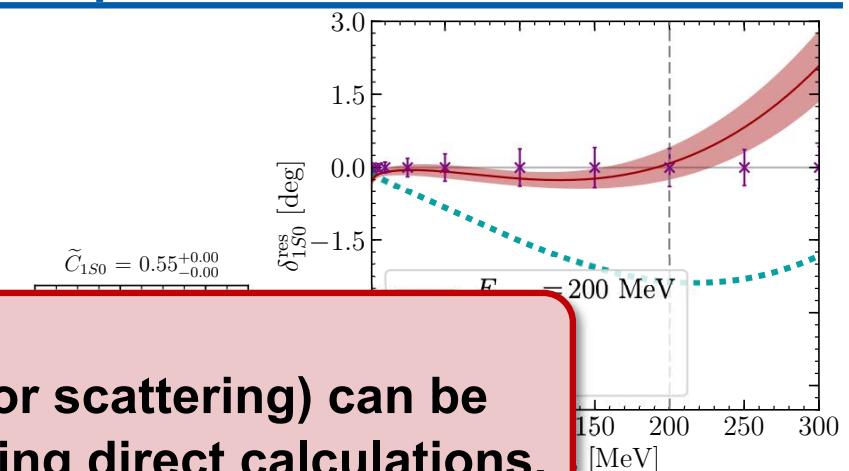
# Eigenvector Continuation for scattering with local chiral NN and optical potentials

Bayesian uncertainty quantification

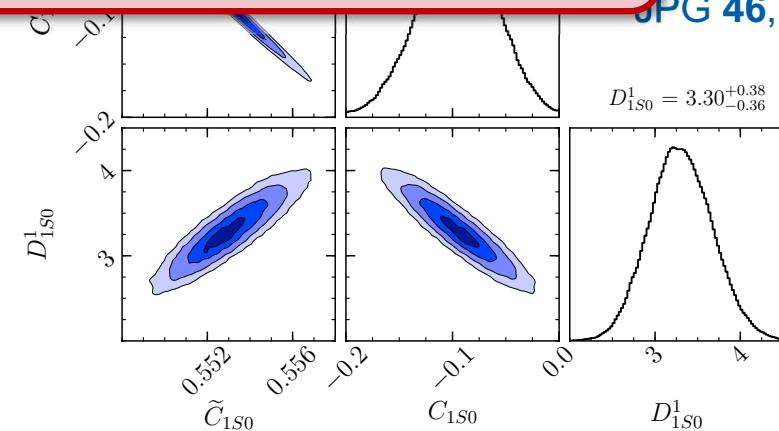
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## Chiral potentials



Monte Carlo sampling (for scattering) can be computationally expensive using direct calculations, which can prohibit meaningful Bayesian UQ.



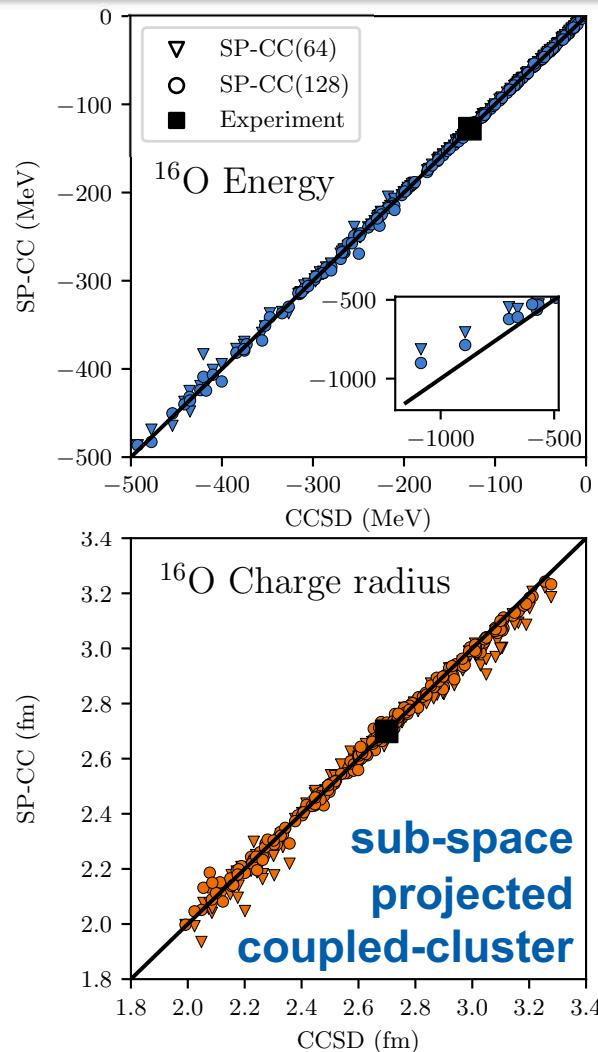
solowski et al.,  
JPG 46, 045102

## Optical potentials

# Eigenvector Continuation for scattering with local chiral NN and optical potentials

Rigorous UQ and global sensitivity analysis

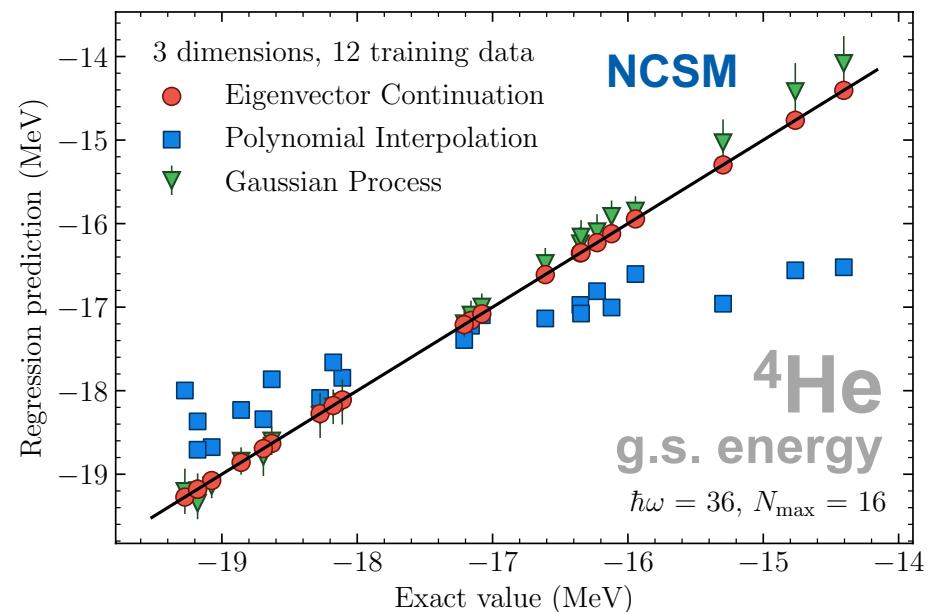
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Ekström & Hagen, PRL 123, 252501

Eigenvector Continuation  
Frame *et al.*, PRL 121, 032501

accurate & fast emulators  
efficient resummation tool

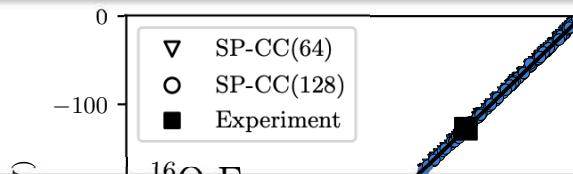


König, Ekström *et al.*, PLB 810, 135814  
Demol, Duguet *et al.*, PRC 101, 041302  
Wesolowski, Svensson *et al.*, arXiv:2104.04441

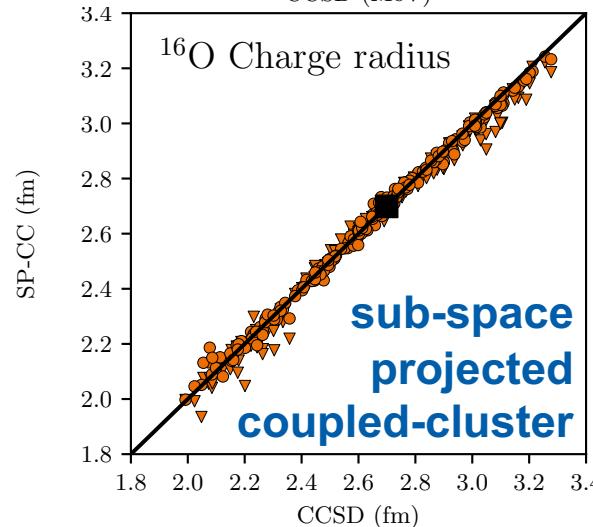
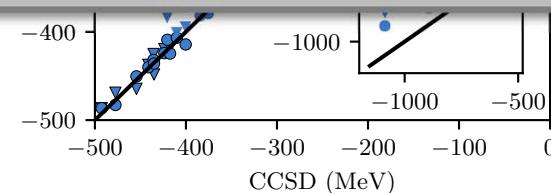
# Eigenvector Continuation for scattering with local chiral NN and optical potentials

Rigorous UQ and global sensitivity analysis

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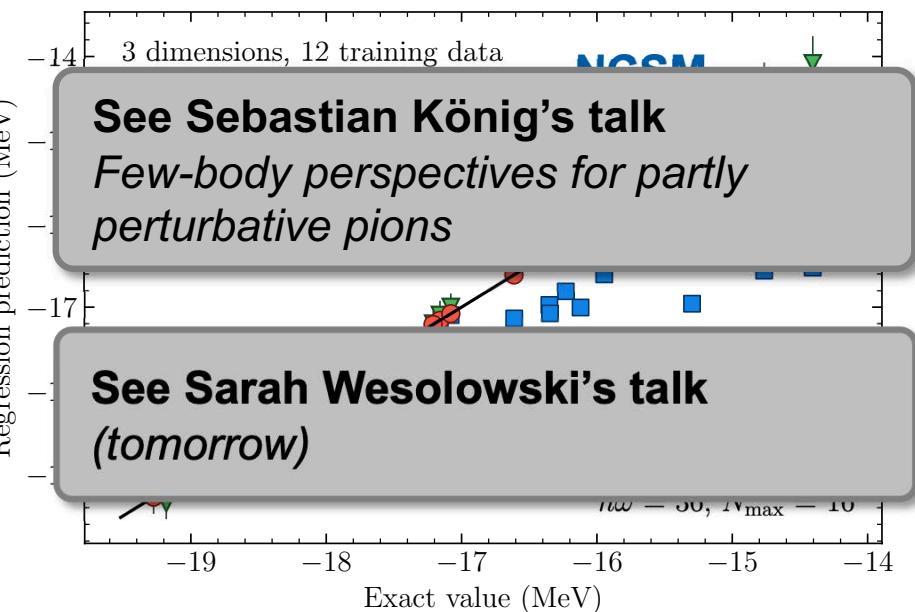
**See Gaute Hagen's talk**  
*Advances in CC computations of nuclei*



**Ekström & Hagen, PRL 123, 252501**

**Eigenvector Continuation**  
Frame *et al.*, PRL 121, 032501

accurate & fast emulators  
efficient resummation tool



**See Sebastian König's talk**  
*Few-body perspectives for partly perturbative pions*

König, Ekström *et al.*, PLB 810, 135814  
Demol, Duguet *et al.*, PRC 101, 041302  
Wesolowski, Svensson *et al.*, arXiv:2104.04441

# Eigenvector Continuation for scattering with local chiral NN and optical potentials

Kohn Variational Principle

Furnstahl, Garcia, Millican, and Zhang, PLB **809**, 10135719

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$$\beta [|\psi_{\text{trial}}\rangle] = \frac{K_\ell}{p} - 2\mu \langle \psi_{\text{trial}} | H(\boldsymbol{\theta}) - E | \psi_{\text{trial}} \rangle \quad p = \sqrt{2\mu E}$$

**stationary approximation to exact  $K_l$  matrix [accurate up to  $O(\delta u^2)$ ]**

$$u_{\ell,E}^{\text{trial}}(r) \sim \frac{1}{p} \sin(\eta_\ell) + \frac{K_\ell}{p} \cos(\eta_\ell) \quad \eta_\ell = kr - \frac{\pi}{2}\ell \quad H(\boldsymbol{\theta}) = T + V(\boldsymbol{\theta})$$



for EC in  $R$  matrix theory,  
see also Bai & Ren, PRC **103**, 014612

# Eigenvector Continuation for scattering with local chiral NN and optical potentials

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Kohn Variational Principle

1

Furnstahl, Garcia, Millican, and Zhang, PLB 809, 10135719

2

$$\beta [|\psi_{\text{trial}}\rangle] = \frac{K_\ell}{p} - 2\mu \langle \psi_{\text{trial}} | H(\theta) - E | \psi_{\text{trial}} \rangle \quad p = \sqrt{2\mu E}$$

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3

Training: solve RSE exactly for a set  $\{\theta_i\}_{i=1}^{N_b}$  and construct the trial wave function:

$$|\psi_{\text{trial}}\rangle = \sum_{i=1}^{N_b} c_i |\psi_E(\theta_i)\rangle$$

4

Given  $H(\theta)$ , the stationary point is obtained by simple linear algebra:

$$\delta\beta[\psi_{\text{trial}}] = 0 \quad \text{s.t.} \quad \sum_{i=1}^{N_b} c_i = 1$$

$$c_i = \sum_j (\Delta \tilde{U})_{ij}^{-1} \left( \frac{K_\ell^{(j)}(E)}{p} - \lambda \right) \quad \lambda = \frac{-1 + \sum_{ij} (\Delta \tilde{U})_{ij}^{-1} \frac{K_\ell^{(j)}(E)}{p}}{\sum_{ij} (\Delta \tilde{U})_{ij}^{-1}}$$

with matrix  $\Delta \tilde{U}_{ij} = 2\mu \langle \psi_E(\theta_i) | 2V(\theta) - V(\theta_i) - V(\theta_j) | \psi_E(\theta_j) \rangle$

5

Approximate  $K_l = \tan \delta_l$ :  $[K_\ell(E)]_{\text{exact}} \approx \sum_i c_i K_\ell^{(i)}(E) - \frac{p}{2} \sum_{ij} c_i \Delta \tilde{U}_{ij} c_j$

emulate

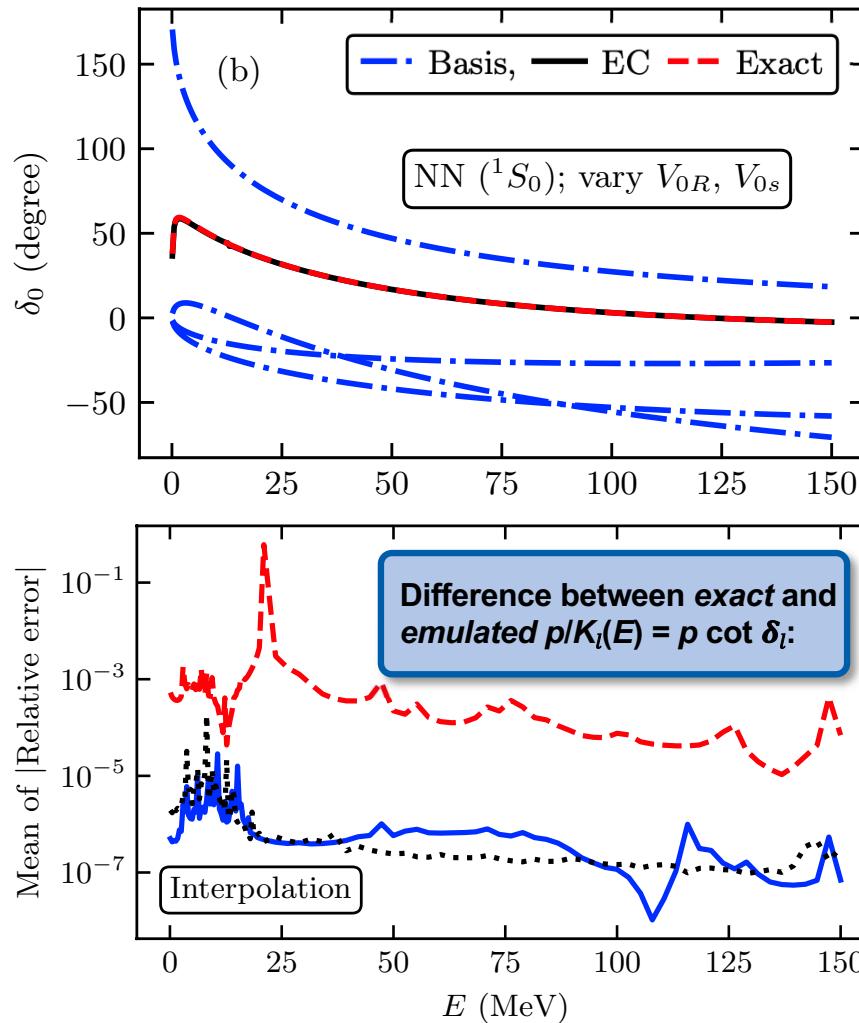
validate



# Eigenvector Continuation for scattering with local chiral NN and optical potentials

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Furnstahl, Garcia, Millican, and Zhang, PLB 809, 10135719



EC for scattering is effective for inter- and extrapolation

- local & nonlocal potentials
- incl. optical potentials
- high partial waves

Minnesota interaction:

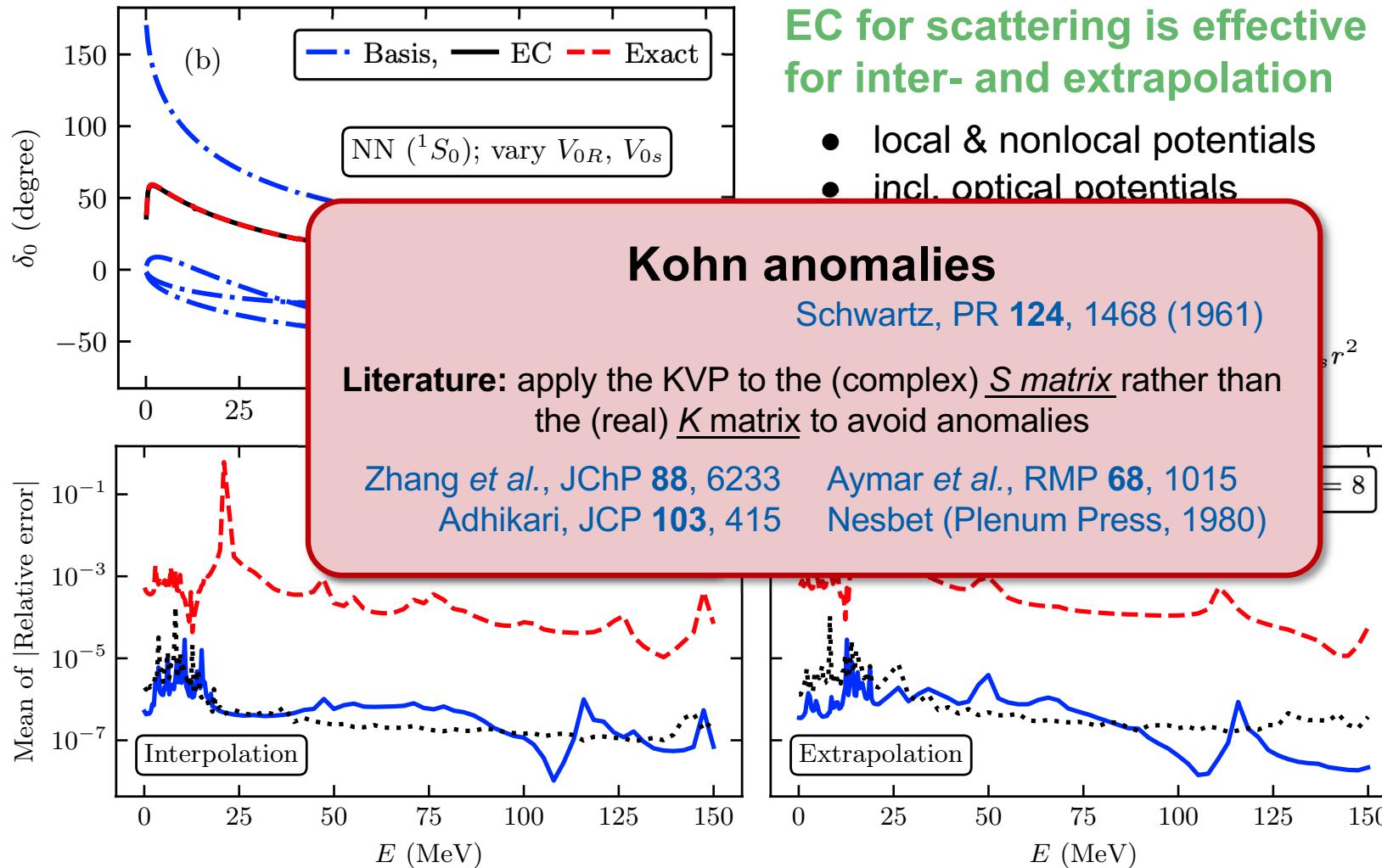
$$V_{^1S_0}(r) \equiv V_{0R} e^{-\kappa_R r^2} + V_{0s} e^{-\kappa_s r^2}$$



# Eigenvector Continuation for scattering with local chiral NN and optical potentials

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Furnstahl, Garcia, Millican, and Zhang, PLB 809, 10135719



# Eigenvector Continuation for scattering with local chiral NN and optical potentials

## Generalized Kohn Variational Principle

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Lucchese, PRA 40, 112

$$\beta_u [|\psi_{\text{trial}}\rangle] = L_\ell - \frac{2\mu}{\det u} \langle \psi_{\text{trial}} | H(\boldsymbol{\theta}) - E | \psi_{\text{trial}} \rangle$$

**stationary approximation to exact  $L_\ell$  matrix [accurate up to  $O(\delta u^2)$ ]**

$$u_{\ell,E}^{\text{trial}}(r) \sim \phi_0(r) + L \phi_1(r)$$

$$\phi_0(r) \sim u_{00} \sin(\eta_\ell) + u_{01} \cos(\eta_\ell)$$

$$\phi_1(r) \sim u_{10} \sin(\eta_\ell) + u_{11} \cos(\eta_\ell)$$

$$\eta_\ell = kr - \frac{\pi}{2}\ell$$



$$L = K \text{ for } u = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$L = S \text{ for } u = \begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix}$$

$$L = T \text{ for } u = \begin{bmatrix} 1 & 0 \\ i & 1 \end{bmatrix}$$

... any nonsingular matrix

$$K = \frac{u_{01} + u_{11}L}{u_{00} + u_{10}L}$$

for applications to N-d and p-d scattering, see  
Viviani, Kievsky, and Rosati, FBS 30, 3  
Kievsky, NPA 624, 125  
Kievsky, Viviani, and Rosati, NPA 577, 511

# Eigenvector Continuation for scattering with local chiral NN and optical potentials

## Generalized Kohn Variational Principle

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validate

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$$\lambda = \frac{-1 + \sum_{ij} (\Delta \tilde{U})_{ij}^{-1} L_\ell^{(j)}(E)}{\sum_{ij} (\Delta \tilde{U})_{ij}^{-1}}$$

with matrix  $\Delta \tilde{U}_{ij} = \frac{2\mu}{\det u} \langle \psi_E(\boldsymbol{\theta}_i) | 2V(\boldsymbol{\theta}) - V(\boldsymbol{\theta}_i) - V(\boldsymbol{\theta}_j) | \psi_E(\boldsymbol{\theta}_j) \rangle$

emulate

**Approximate  $L_\ell$ :**  $[L_\ell(E)]_{\text{exact}} \approx \sum_i c_i L_\ell^{(i)}(E) - \frac{1}{2} \sum_{ij} c_i \Delta \tilde{U}_{ij} c_j$

# Eigenvector Continuation for scattering with local chiral NN and optical potentials

Generalized KVP in practice

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CD, Giuliani, Quinonez *et al.*, in prep.

$$[L_\ell(E)]_{\text{exact}} \approx \sum_i c_i L_\ell^{(i)}(E) - \frac{1}{2} \sum_{ij} c_i \Delta \tilde{U}_{ij} c_j$$

## Considered a wide range of KVPs:

- $S, T, K$ , their inverses, and random complex parametrizations  $u$
- uncoupled channels
- local coordinate-space potentials

## Potentials $V(r; \theta)$ implemented:

- Minnesota (real, nonlinear in  $\theta$ )
- Woods-Saxon (real, nonlinear in  $\theta$ )
- Optical (complex, nonlinear in  $\theta$ )
- **Local chiral GT+** (real, linear in  $\theta$ )

We sample  $H(\theta)$  randomly for different channels, energies, and basis sizes, and investigate deviations of emulated from exact phase shifts.

To measure the efficacy of EC, we fit the  $c_i$ 's to the exact wave function.

$$|\psi_{\text{trial}}\rangle = \sum_{i=1}^{N_b} c_i |\psi_E(\theta_i)\rangle$$

The EC basis usually is still effective in energy regions where the KVP is affected by Kohn anomalies.

# Eigenvector Continuation for scattering with local chiral NN and optical potentials

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Complex vs real KVP

CD, Giuliani, Quinonez *et al.*, in prep.

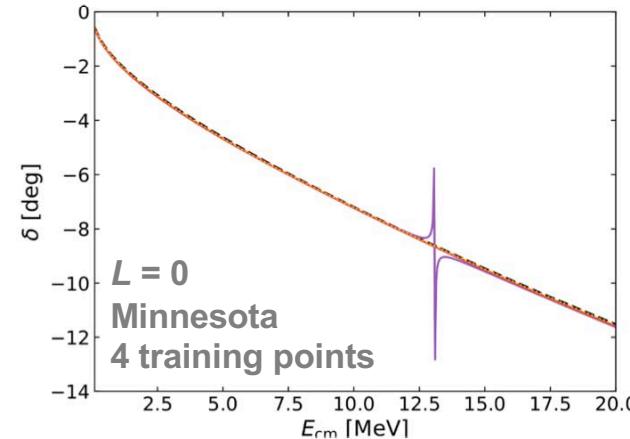
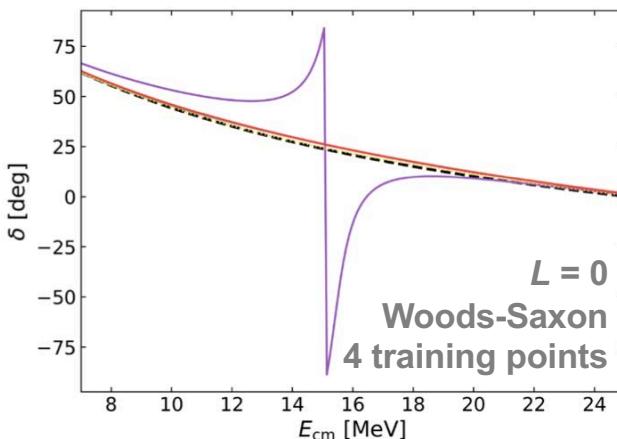
Why is the complex KVP less prone to Kohn anomalies



There are several sources of Kohn anomalies. For instance, no stationary point of the functional can be found if

$$\sum_{ij} (\Delta \tilde{U})_{ij}^{-1} = 0$$

(can be used to pinpoint anomalies)

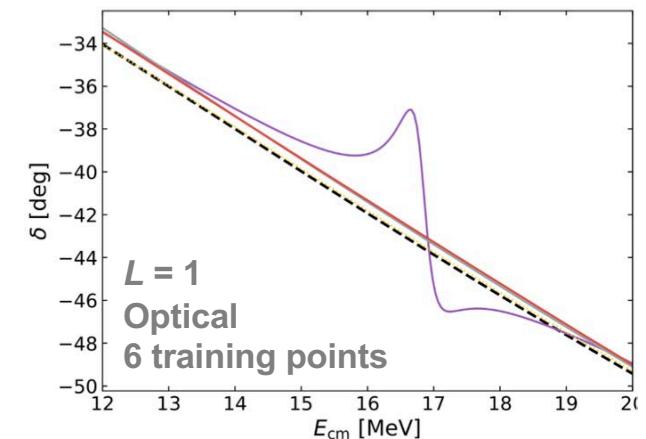


$$\lambda = \frac{-1 + \sum_{ij} (\Delta \tilde{U})_{ij}^{-1} L_\ell^{(j)}(E)}{\sum_{ij} (\Delta \tilde{U})_{ij}^{-1}}$$

**Advantage:** the denominator is complex for the complex KVP and/or optical potentials, hence improving issues with zero crossings

**No guarantee:** we found Kohn anomalies in the  $K$  matrix for all but chiral potentials.

**Other sources:** bound states, resonances...

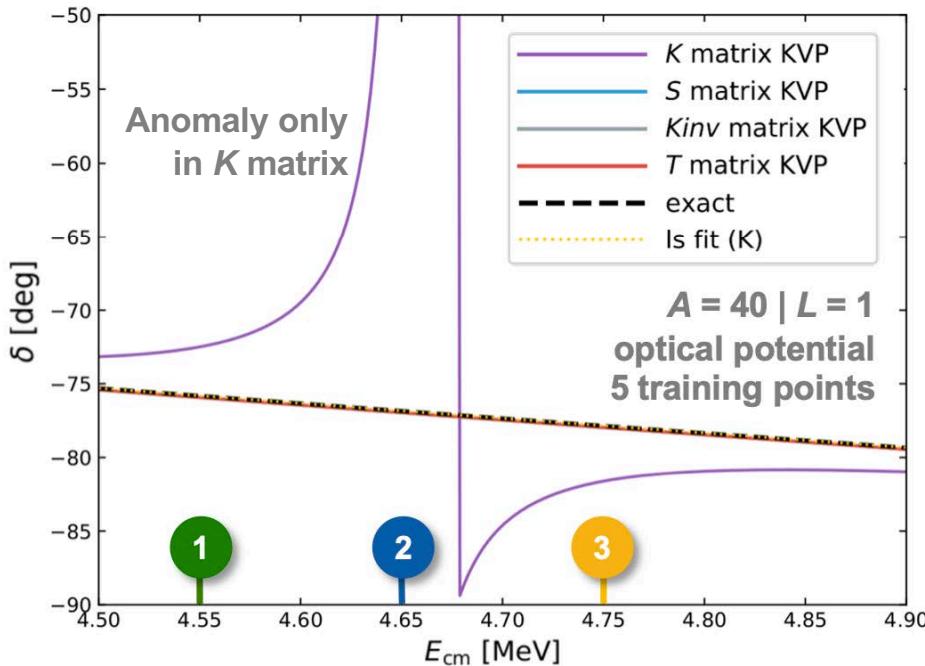


# Eigenvector Continuation for scattering with local chiral NN and optical potentials

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Phase shifts and (differential) cross sections

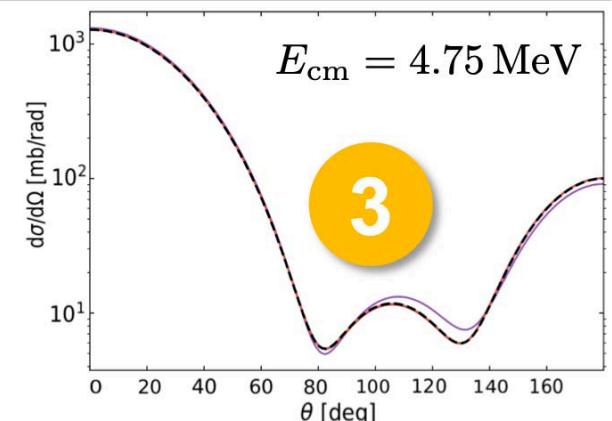
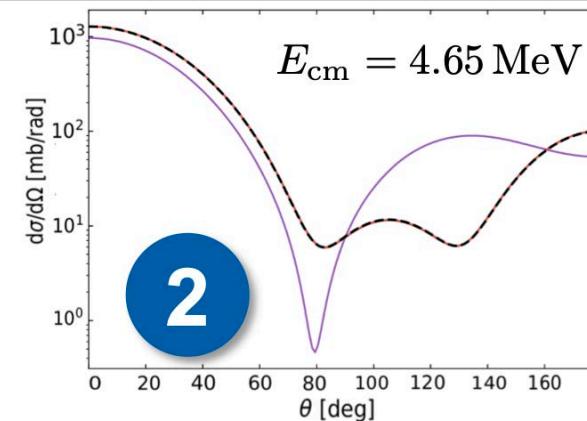
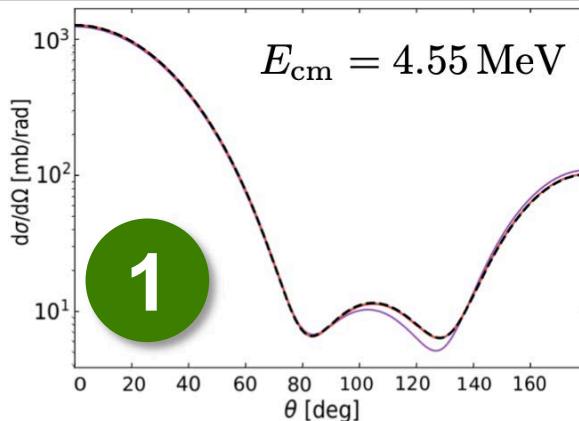
CD, Giuliani, Quinonez *et al.*, in prep.



$$\frac{d\sigma}{d\Omega} = \left| \frac{1}{2ik} \sum_{l=0}^{\infty} (2l+1) P_l(\cos \theta) (S_l - 1) \right|^2$$

Overall, the complex KVP reproduces well phase shifts and cross sections in contrast to the real KVP.  
In absence of anomalies, the real KVP may be slightly more accurate for some  $E$

differential cross sections



# Eigenvector Continuation for scattering with local chiral NN and optical potentials

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## Diagnostic tools

see also: Lucchese, PRA 40, 112

Does that imply that the complex KVP is free of *Kohn anomalies* ?

$$[L_\ell(E)]_{\text{exact}} \approx \sum_i c_i L_\ell^{(i)}(E) - \frac{1}{2} \sum_{ij} c_i \Delta \tilde{U}_{ij} c_j$$

## Diagnostic tools for anomalies:

- Use **multiple KVPs in parallel**
- Check KVPs for consistency

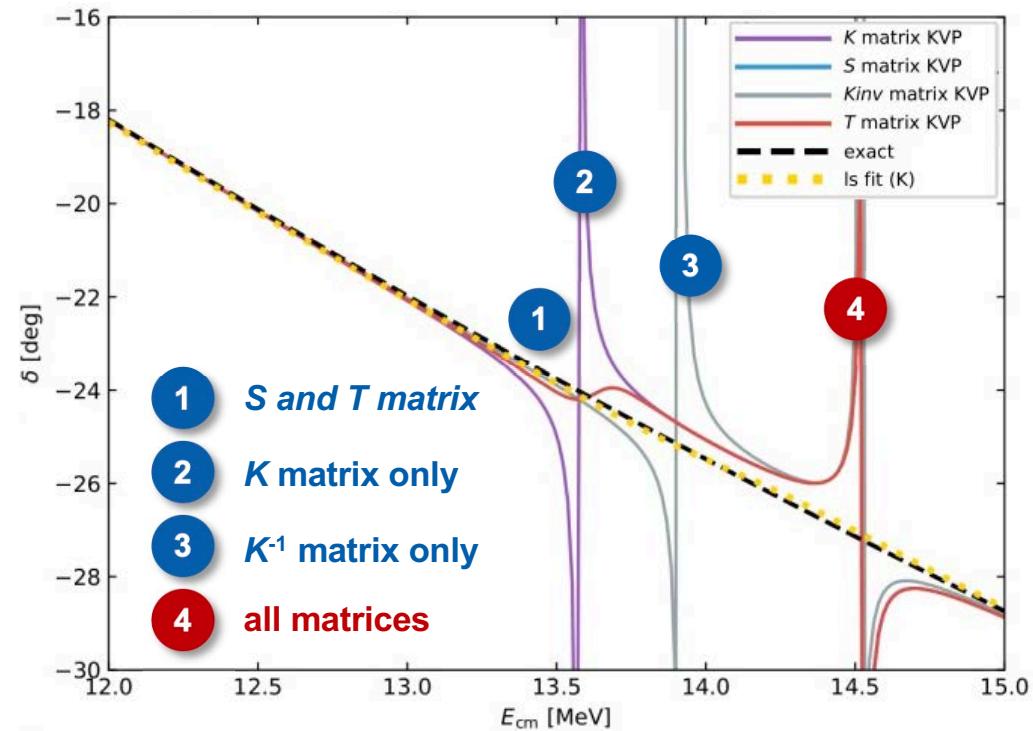
$$S = 1 + 2iT \quad L^{-1}L = 1$$

$$|S| = 1 \quad (\text{for real } V) \quad (\dots)$$

- Track and find zero crossings of

$$\Gamma(E) = \sum_{ij} (\Delta \tilde{U})_{ij}^{-1}$$

- **Change the size of the training basis** (double, add/remove, etc.)



see also: Lucchese, PRA 40, 112  
Viviani, Kievsky, and Rosati, FBS 30, 3

# Eigenvector Continuation for scattering with local chiral NN and optical potentials

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Numerical noise

CD, Giuliani, Quinonez *et al.*, in prep.

The condition number increases with increasing number of training points.

Methods to control the noise in matrix inversions are *not* efficient (fine-tuned)

- nugget regularization
- Moore-Penrose inverse

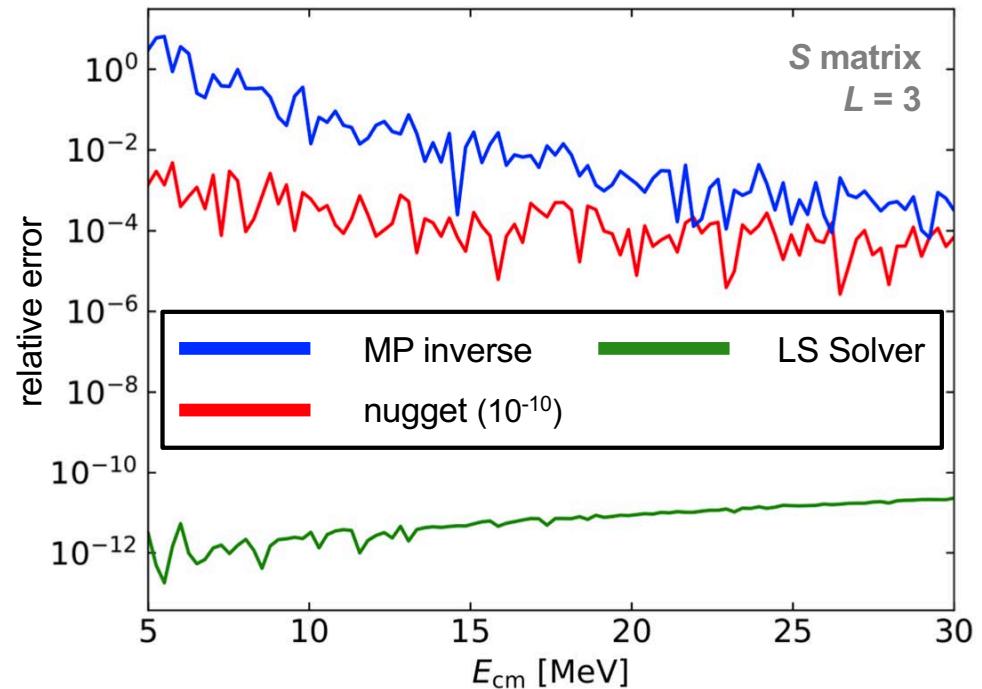
S matrix especially prone to noise, whereas the T matrix is more stable

Simple yet robust: find *stationary point* numerically (e.g., using an LS solver)

$$\begin{pmatrix} \Delta \tilde{U} & 1 \\ 1^T & 0 \end{pmatrix} \begin{pmatrix} c \\ \lambda \end{pmatrix} = \begin{pmatrix} L \\ 1 \end{pmatrix}$$

inverse-free,  
if possible

$$c_i = \sum_j (\Delta \tilde{U})_{ij}^{-1} \left( L_\ell^{(j)}(E) - \lambda \right)$$
$$\lambda = \frac{-1 + \sum_{ij} (\Delta \tilde{U})_{ij}^{-1} L_\ell^{(j)}(E)}{\sum_{ij} (\Delta \tilde{U})_{ij}^{-1}}$$



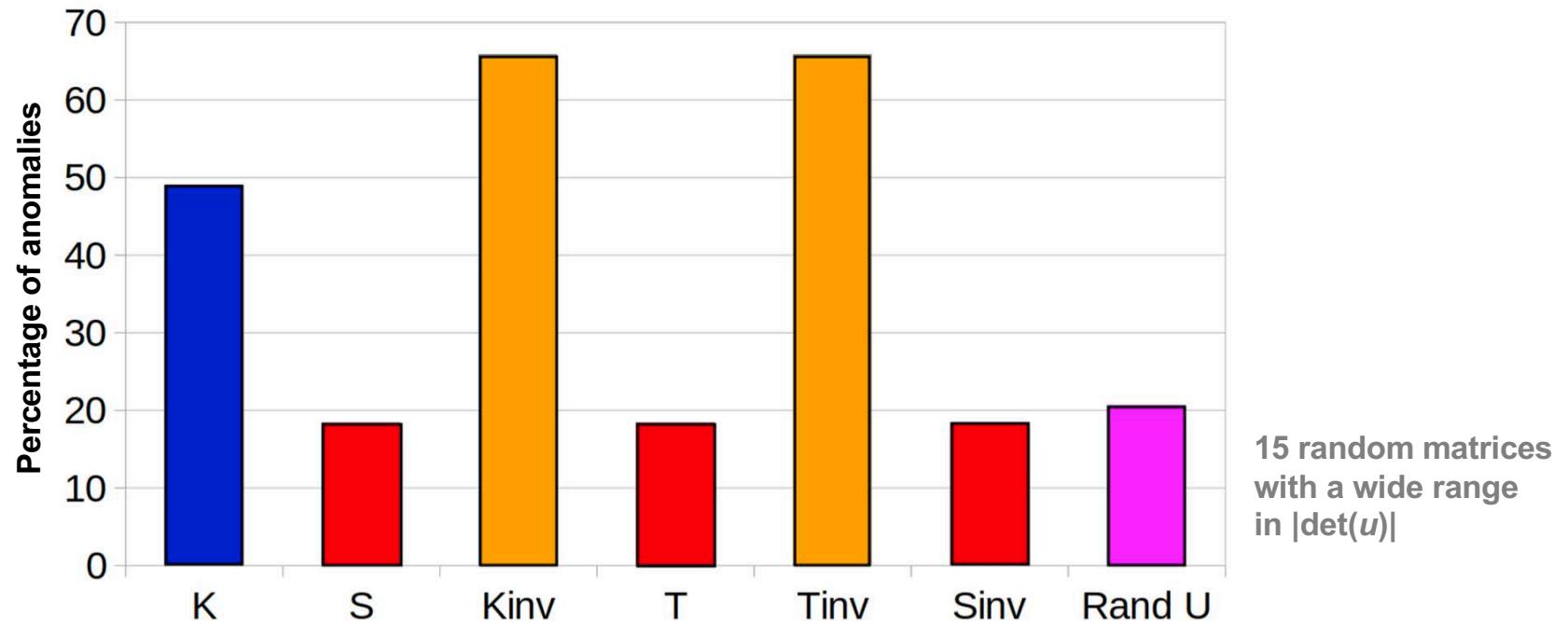
# Eigenvector Continuation for scattering with local chiral NN and optical potentials

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Abundance of Kohn anomalies

Which KVP performs best in terms of reducing *Kohn anomalies* ?

But be careful: this depends on the interaction, channel, ...



$L = 0$  |  $A = 40$  |  $E = 5\text{-}20$  MeV

Woods-Saxon potential

5 training points

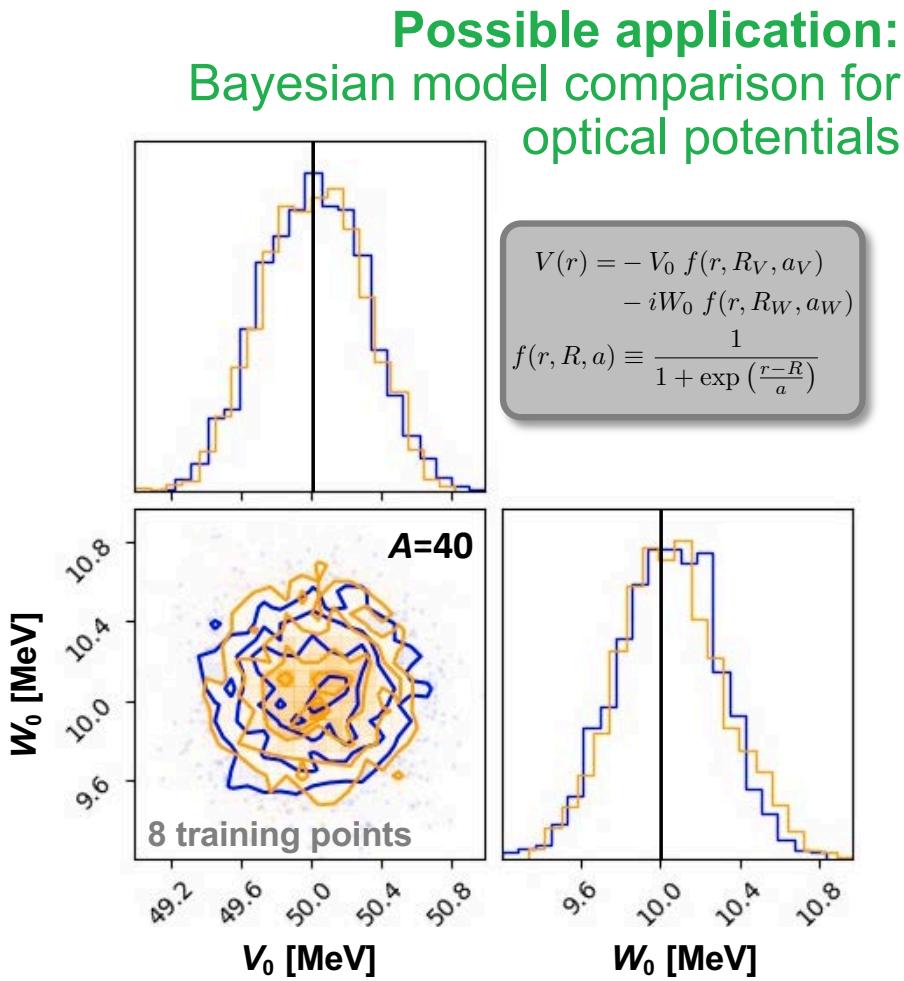
sample size: 500

# Eigenvector Continuation for scattering with local chiral NN and optical potentials

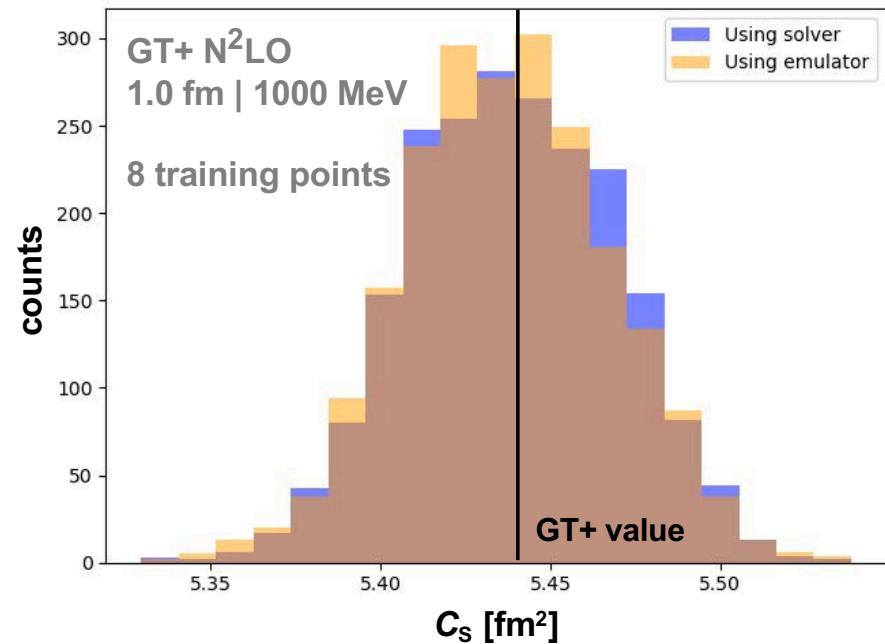
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Proof of principle: MCMC sampling

CD, Giuliani, Quinonez *et al.*, in prep.



## Local chiral potential



**Possible application:**  
parameter estimation for constructing next-generation (local) chiral interactions

Buchner (“UltraNest”), arXiv:2101.09604  
needs coupled channels; Kamimura, PTPS **62**, 236

# Eigenvector Continuation for scattering with local chiral NN and optical potentials

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Christian Drischler

April 26, 2021 | INT 21-1b: Nuclear Forces for Precision Nuclear Physics

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Improving nuclear forces with *novel fitting strategies* and higher orders in chiral EFT

1

using EC as an efficient emulator for scattering with local chiral interactions and optical potentials

2

statistical quantification and propagation of EFT truncation errors in nuclear matter calculations

## Keywords:

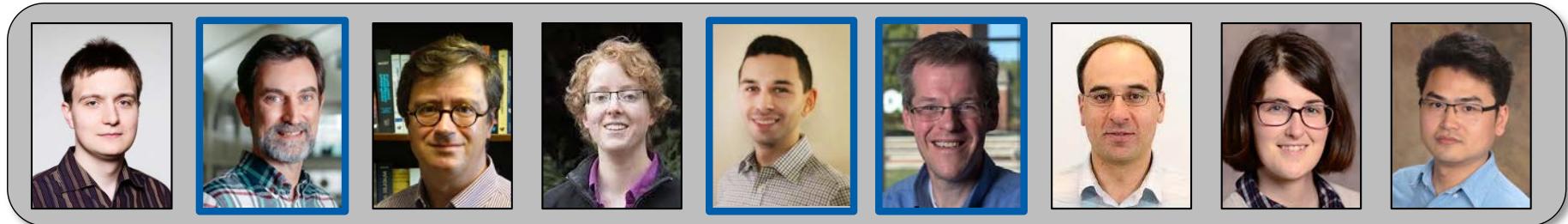
- + ChEFT + scattering
- + Variational principles
- + Eigenvector Continuation
- + Bayesian UQ
- + infinite nuclear matter
- + N<sup>3</sup>LO NN + 3N forces
- + ...

# Eigenvector Continuation for scattering with local chiral NN and optical potentials

New framework for UQ of the infinite-matter EOS

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[buqeye.github.io](https://buqeye.github.io)

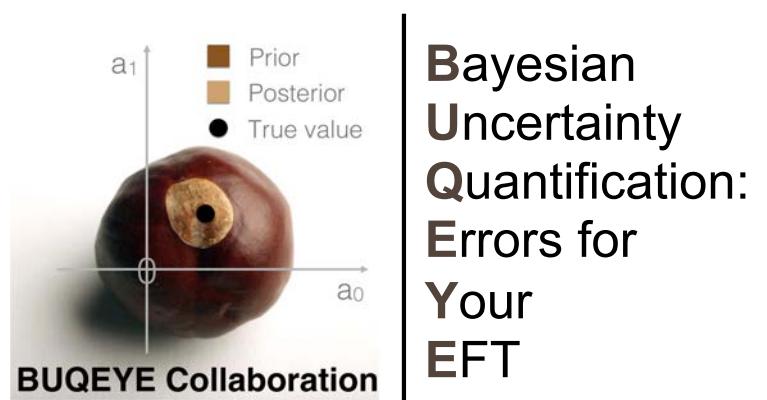


## CD, Furnstahl, Melendez, and Phillips

*How well do we know the neutron-matter equation of state at the densities inside neutron stars? A Bayesian approach with correlated uncertainties, PRL 125, 202702*

## CD, Melendez, Furnstahl, and Phillips

*Effective Field Theory Convergence Pattern of Infinite Nuclear Matter, PRC 102, 054315*



Bayesian  
Uncertainty  
Quantification:  
Errors for  
Your  
EFT

UQ framework available at  
<https://buqeye.github.io>

# Eigenvector Continuation for scattering with local chiral NN and optical potentials

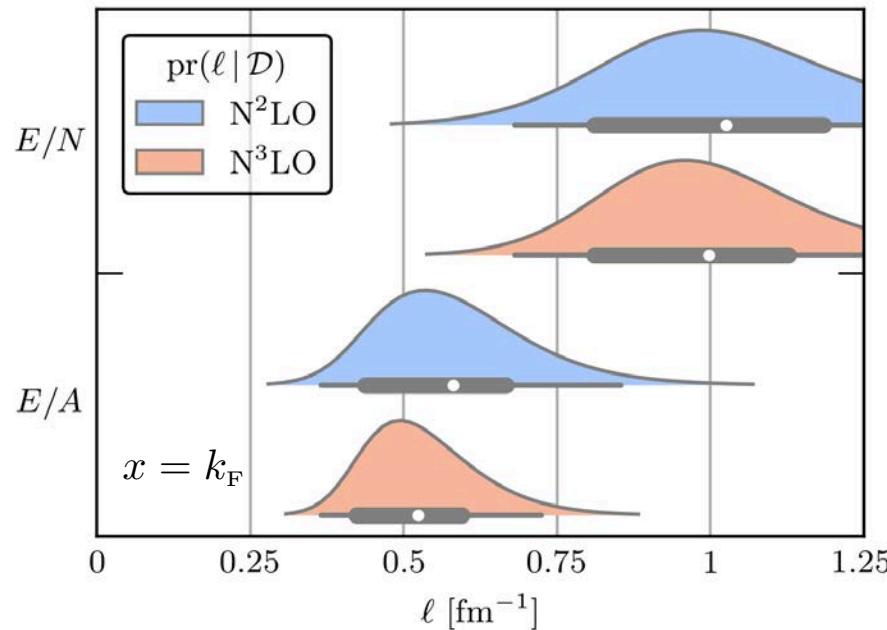
Propagating type-x uncertainties

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CD, Melendez *et al.*, PRC 102, 054315

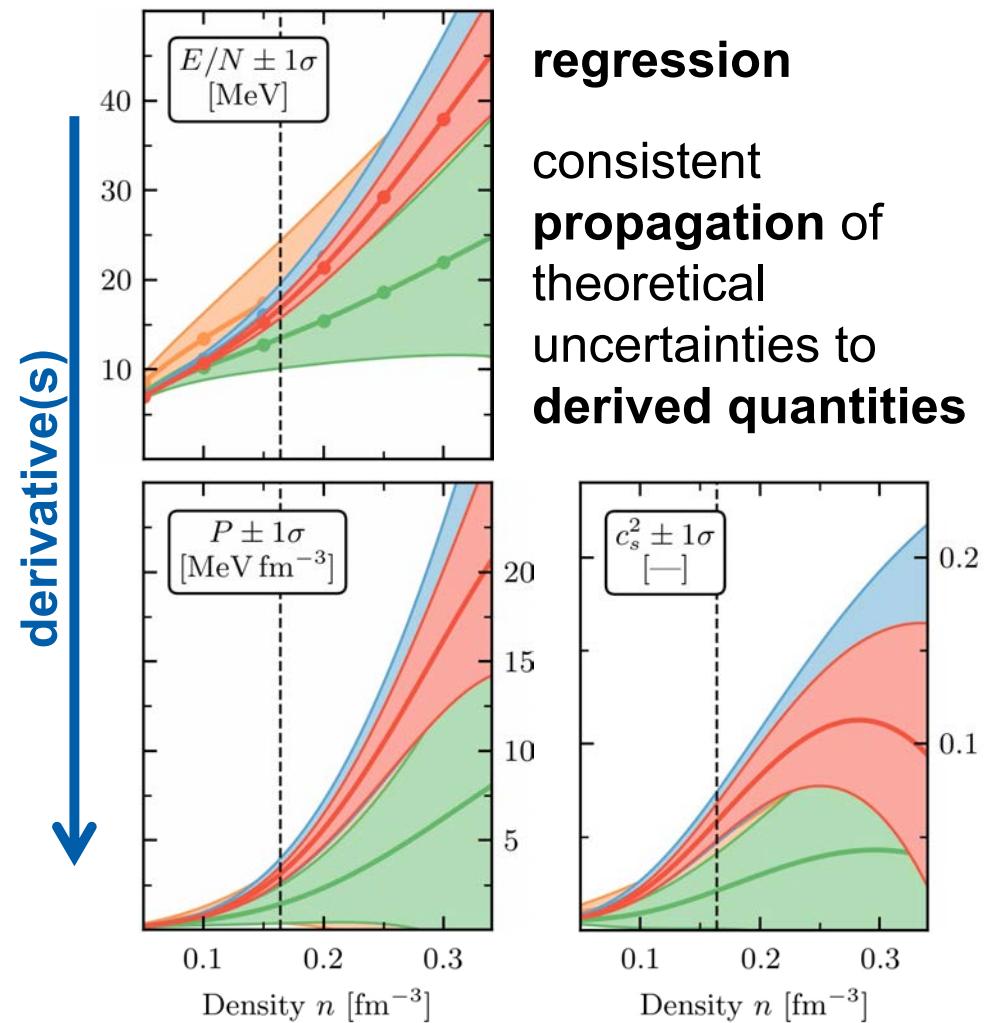
How correlated  
is nuclear matter ?

$\text{pr}(\ell | \mathcal{D})$   
correlation length



to be  
compared with

$$k_F^{\max} = \begin{cases} 2.2 \text{ fm}^{-1} & \text{PNM} \\ 1.7 \text{ fm}^{-1} & \text{SNM} \end{cases}$$



# Eigenvector Continuation for scattering with local chiral NN and optical potentials

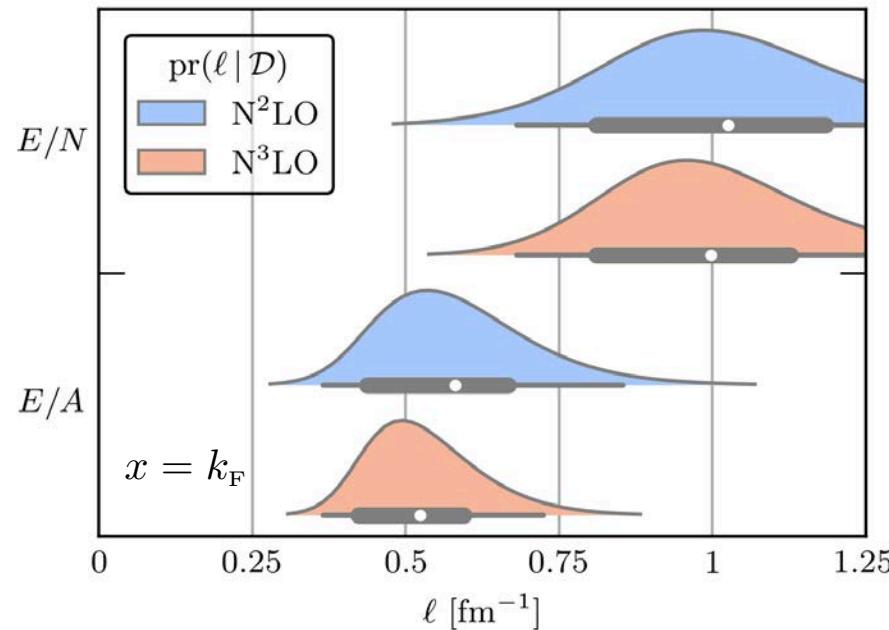
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Bayesian inference

CD, Melendez *et al.*, PRC 102, 054315

How correlated  
is nuclear matter ?

$\text{pr}(\ell | \mathcal{D})$   
correlation length

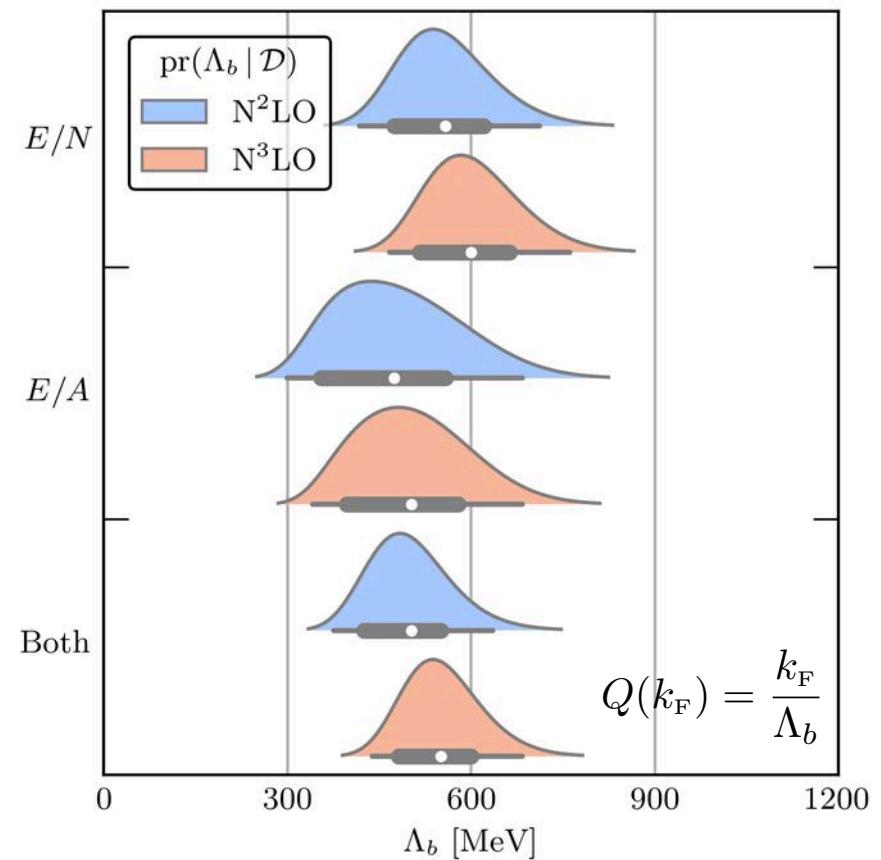


to be  
compared with

$$k_F^{\max} = \begin{cases} 2.2 \text{ fm}^{-1} & \text{PNM} \\ 1.7 \text{ fm}^{-1} & \text{SNM} \end{cases}$$

Where does the  
EFT break down ?

$\text{pr}(\Lambda_b | \mathcal{D})$   
breakdown scale

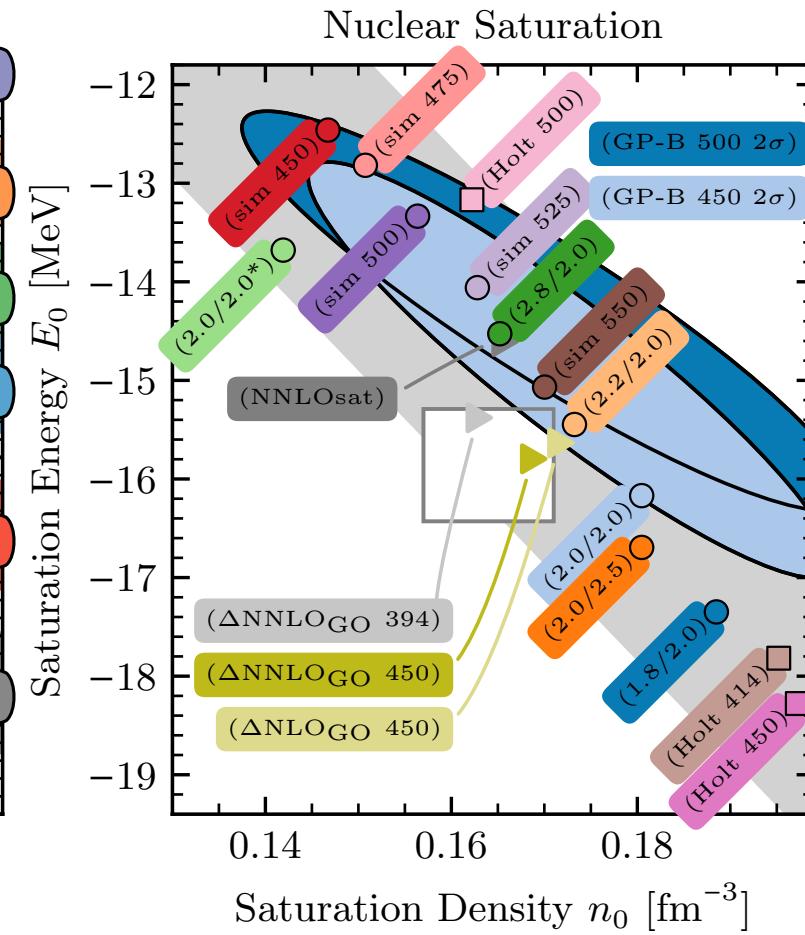
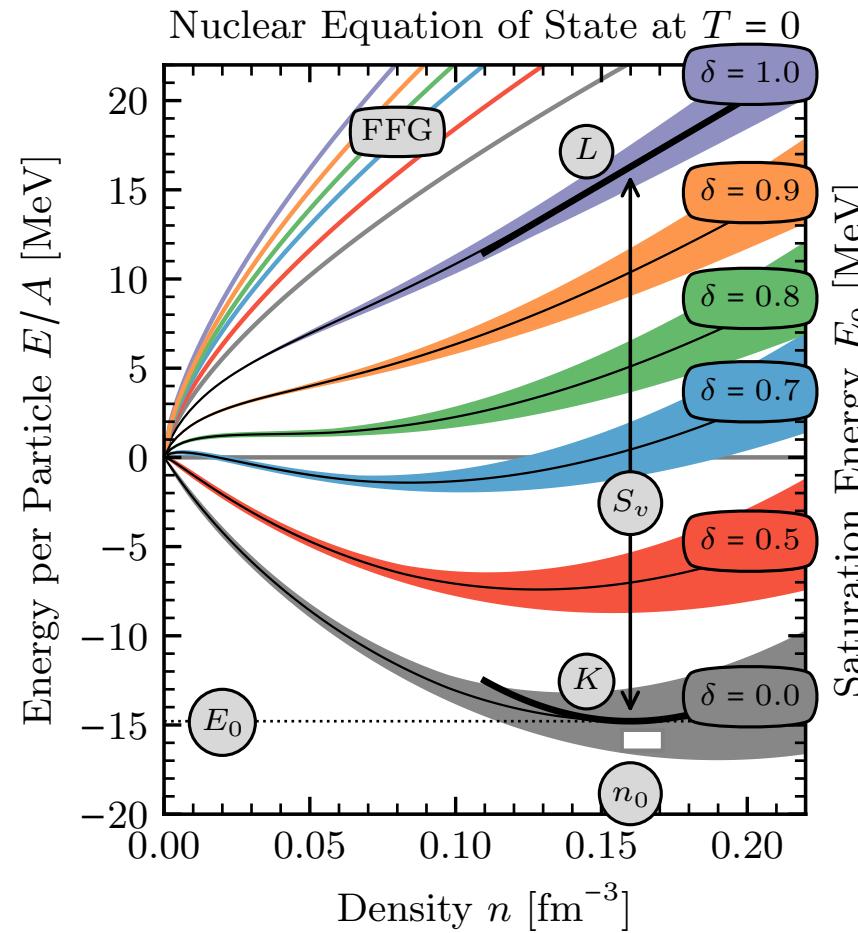


# Eigenvector Continuation for scattering with local chiral NN and optical potentials

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Parameters of the low-density EOS

CD, Holt, and Wellenhofer, arXiv:2101.01709



FFG: free Fermi gas;  $\delta = (n_n - n_p)/n$ : isospin asymmetry

for nuclear saturation, see also Atkinson *et al.*, PRC **102**, 044333; Dewulf *et al*, PRL **90**, 152501

Annotations:  $(\Lambda / \Lambda_{3N})$  in  $\text{fm}^{-1}$  or  $(\Lambda)$  in MeV

# Eigenvector Continuation for scattering with local chiral NN and optical potentials

## Summary

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buqeye.github.io

1

### Presented an efficient emulator using EC and complex KVP

- with local chiral nucleon-nucleon interactions and optical potentials
- KVPs for the  $T$  matrix and  $S$  matrix can reproduce well phase shifts and cross sections
- EC is efficient in providing trial wave functions for interpolation and extrapolation

2

### Studied diagnostic tools for Kohn anomalies

- Kohn anomalies can be encountered in variational calculations
- Consistency checks and variable basis sizes are key to Bayesian UQ
- An improved method for finding stationary points reduces numerical noise substantially

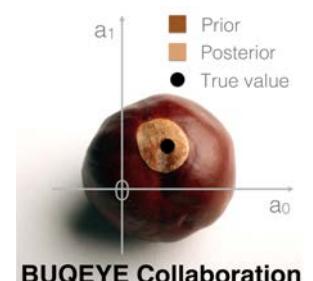
3

### Set a new standard for UQ in infinite-matter calculations

- Need for *statistically* robust comparisons between theory, observation, and experiment
- Correlations within *and* between observables are crucial for reliable UQ
- Efficiently quantify and propagate EOS uncertainties to derived quantities

Thanks to my collaborators:

P. Giuliani A. Lovell F. Nunes M. Quinonez  
R. Furnstahl J. Melendez K. McElvain D. Phillips



# Eigenvector Continuation for scattering with local chiral NN and optical potentials

Some topics for the discussion session

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?

## Convergence of Eigenvector Continuation

Avik Sarkar & Dean Lee, PRL 126, 032501 | Can scattering provide general insights?

?

## Extension to three-body scattering with EC

See Xilin Zhang's talk on Wednesday

?

## Improved variational principles: insights from pre-EC era?

e.g., Viviani, Kievsky, and Rosati, FBS 30, 3 and Zhang *et al.*, JChP 88, 6233

?

## Improving nuclear forces with novel fitting strategies

How would the workflow look like in practice?