# LOG111 Hand-in 3

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## 1

*Proof.* By construction, the set $\Gamma_1 \cup \Gamma_2$ is unsatisfiable, so by compactness, there is a finite unsatisfiable subset $\Delta \subseteq \Gamma_1 \cup \Gamma_2$.

Consider

$$\Delta_1 := \Delta \setminus \Gamma_2 \qquad\qquad \Delta_2 := \Delta \setminus \Gamma_1.$$

We claim that $\Delta_1$ and $\Delta_2$ respectively axiomatize $\mathsf{Th}(\Gamma_1)$ and $\mathsf{Th}(\Gamma_2)$. We prove that this is the case for $\Delta_1$; the argument for $\Delta_2$ is completely analogous.

We need to prove that for any formula $\varphi$, $\Gamma_1 \vdash \varphi$ iff $\Delta_1 \vdash \varphi$. To this end, it suffices to prove their semantic counterpart by soundness and completeness.

The if direction is an immediate consequence of monotonicity. In the other direction, suppose that $\Gamma_1 \vDash \varphi$ and let $M \vDash \Delta_1$. If $M \vDash \Gamma_1$ then we are done. On the other hand, if $M \nvDash \Gamma_1$ then it follows that $M \vDash \Delta_2$, but this means that $M \vDash \Delta_1 \cup \Delta_2$ contradicting the fact that $\Delta = \Delta_1 \cup \Delta_2$ is unsatisfiable. □

## 2

1. $$\dfrac{\dfrac{\dfrac{c_x < c_y \qquad [c_y < c_x]^1}{c_x < c_x}\,\mathrm{T}}{\perp}\,\mathrm{R}}{c_y \nless c_x}\,\mathrm{RAA}_2^1$$

2. $$(X, R) \nvDash \perp \qquad\qquad (X, R) \vDash c_x < c_y \text{ iff } (x, y) \in R$$

   $$(X, R) \vDash c_x \nless c_y \text{ iff } (x, y) \notin R$$

   $\Gamma \vDash \varphi$ iff for every $(X, R)$, if $(X, R)$ satisfies every formula in $\Gamma$ then $(X, R)$ satisfies $\varphi$.

3. To prove soundness, we can do an induction on the height of the derivation tree.

   The base case is immediate. In the induction case, we do a case analysis on the last applied rule. When the last applied rule is $\mathrm{RAA}_1$, the induction hypothesis yields $\Gamma, c_x \nless c_y \vDash \perp$. Thus, for any $(X, R)$ satisfying $\Gamma$, $(X, R)$ must satisfy $c_x < c_y$, i.e., $\Gamma \vDash c_x < c_y$.

4. *Proof.* We show that if $\Gamma$ is consistent then $\Gamma$ is satisfiable. First, $X \times X$ is countable because $X$ is, so the set of formulas is enumerable. It is sufficient to enumerate formulas of the form $c_x < c_y$ and their negation as we have no use for $\bot$. We write $c_{n_1} < c_{n_2}$ or $c_{n_1} \not< c_{n_2}$ for the $n$-th enumeration.

   Let $\Gamma$ be a consistent set of formulas. We extend $\Gamma$ as follows:

$$\Gamma^* = \bigcup \{\Gamma_n \mid n \in \mathbb{N}\} \qquad\qquad \Gamma_0 = \Gamma$$

$$\Gamma_{n+1} = \begin{cases} \Gamma_n, c_{n_1} < c_{n_2} & \text{if the resulting set is consistent,} \\ \Gamma_n, c_{n_1} \not< c_{n_2} & \text{otherwise.} \end{cases}$$

   Show that it is consistent...

   We include $c_x \not< c_y$ because we want this example to work in the next sub-question.

   We are now ready to define a partial order:

$$R = \{(x, y) \mid c_x < c_y \in \Gamma^*\}$$

   If $(x, x) \in R$, then by definition $c_x < c_x \in \Gamma^*$. But this means that $\Gamma^*$ is inconsistent as $\Gamma^* \vdash \bot$ using R. Thus, $R$ is irreflexive. $\square$