Assignment 2: Cardinality

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6.13

1. *Proof.* Consider the following functions

$$f(a,(b,c)) = ((a,b),c)$$
$$g((a,b),c) = (a,(b,c))$$

By direct computation, we have

$$f(g((a,b),c)) = f(a,(b,c)) = ((a,b),c)$$

and

$$g(f(a,(b,c))) = g((a,b),c) = (a,(b,c)),$$

so we have the required bijection. □

2. *Proof.* Consider the following functions

$$\mathsf{curry}(f)(a)(b) = f(a,b)$$
$$\mathsf{uncurry}(g)(a,b) = g(a)(b)$$

By direct computation, we have

$$\mathsf{curry}(\mathsf{uncurry}(g))(a)(b) = \mathsf{uncurry}(g)(a,b) = g(a)(b)$$

and

$$\mathsf{uncurry}(\mathsf{curry}(f))(a,b) = \mathsf{curry}(f)(a)(b) = f(a,b).$$

Then functional extensionality implies that we have the required bijection. □

6.33

Proof. Let $i\colon A \to \mathbb{N}$ and $j\colon B \to \mathbb{N}$ be two injections witnessing the fact that $A \le \mathbb{N}$ and $B \le \mathbb{N}$ respectively. Consider the function $f\colon A \cup B \to \mathbb{N}$ defined by

$$f(x) = \begin{cases} 2 \cdot i(x) & \text{if } x \in A, \\ 2 \cdot j(x) + 1 & \text{otherwise.} \end{cases}$$

Suppose that $f(x) = f(x')$. If both $x, x' \in A$, then $x = x'$ because $2 \cdot -$ and i are injective. Similarly, if $x, x' \in B \setminus A$, then $x = x'$ because $2 \cdot -$, j, and $- + 1$ are all injective. Only the interesting cases remain. Without loss of generality, assume that $x \in A$ and $x' \in B \setminus A$. Then by assumption, $2 \cdot i(x) = 2 \cdot j(x') + 1$. This is not possible because the left-hand side is even while the right-hand side is odd. Thus, f is an injection witnessing $A \cup B \preceq \mathbb{N}$, i.e., $A \cup B$ is countable. □

6.37

Proof. It suffices to rule out $X \prec \mathbb{N}$. Let $f : X \to \mathbb{N}$ be an injection witnessing $X \prec \mathbb{N}$. Note that f restricts to a bijection $f' : X \to \mathsf{ran}(f)$ because it is injective, i.e., $X \approx \mathsf{ran}(f)$.

Since $\mathsf{ran}(f) \subseteq \mathbb{N}$, it is either finite or $\mathsf{ran}(f) \approx \mathbb{N}$. The former case implies that X is finite, which is a contradiction. The latter case implies that $X \approx \mathbb{N}$, which is also a contradiction. □

6.55

Lemma 1. *For any natural number $n \geq 2$, there is a function $\overline{(-)} : n \to n$ with no fixed points.*

Proof. By induction on n. In the base case, consider the function defined by

$$\overline{0} = 1$$
$$\overline{1} = 0$$

This is clearly a function with no fixed points.

In the induction step, we have a function $\overline{(-)} : k \to k$ with no fixed points. Consider $\overline{(-)}_{k+1} : k + 1 \to k + 1$ defined as follows.

$$\overline{m}_{k+1} = \begin{cases} \overline{m} & \text{if } m < k, \\ 0 & \text{otherwise.} \end{cases}$$

Let $m \in k + 1$. If $m < k$ then it cannot be a fixed point because $\overline{(-)}$ does not have any fixed point. If $m = k$ then $\overline{k}_{k+1} = 0$. Since k is at least 2, it cannot be a fixed point either. □

1. This is not true: let X be the singleton set 1; $1 \approx 1 \times 1$ but 1 is clearly not infinite. Let us additionally assume that X is not the singleton set. We show that X is infinite.

   Proof. If X is finite, then $X \approx n$ for some natural number n. Thus, $n \times n \approx X \times X \approx X \approx n$. We show that this is impossible for any $n \geq 2$.

Suppose that there is a bijection $f\colon n \to n \times n$. Let $\hat{f}\colon n \to \underbrace{n \times \cdots \times n}_{n \text{ times}}$ be the composite of the following $n-1$ bijections.

$$n \xrightarrow{f} n \times n \xrightarrow{1\times f} n \times n \times n \xrightarrow{1\times(1\times f)} \cdots \longrightarrow \underbrace{n \times \cdots \times n}_{n \text{ times}}$$

By Lemma 1, there is a function $\overline{(-)}\colon n \to n$ with no fixed points. Consider the pair $\delta := (\overline{\hat{f}(0).0}, \overline{\hat{f}(1).1}, \ldots, \overline{\hat{f}(n-1).n-1})$, where $(-).i$ is the i-th projection.

Suppose that $\hat{f}(k) = \delta$. Then $\hat{f}(k).k = \delta.k = \overline{\hat{f}(k).k}$, but this is a fixed point of $\overline{(-)}$ so we have a contradiction. □

2. *Proof.* This statement is not true for singleton sets since $2^1 \approx 2$, but there is a unique function from any set to the singleton set, i.e., $1^1 \approx 1$. Let us additionally assume that X is not the singleton set.

   If X is $\varnothing$ the statement is trivial: there is a unique function from 0 to any set, i.e., $2^0 \approx 0^0 \approx 1$.

   If X is infinite, there is an evident injection $2^X \to X^X$. For the other direction, consider the function $j\colon X^X \to \mathcal{P}(X \times X)$ defined by

   $$j(f) = \{(x, f(x)) \mid x \in X\}.$$

   If $j(f) = j(g)$ then for any $x \in X$, we have $(x, f(x)) = (x, g(x))$, so $f(x) = g(x)$. Then functional extensionality implies that $f = g$. Thus, the following composition gives the required injection.

   $$\begin{tikzcd} X^X \arrow[d, tail, "j"'] \arrow[drrrr, tail, dashed] & & & & \\ \mathcal{P}(X \times X) & \approx & \mathcal{P}(X) & \approx & 2^X \end{tikzcd}$$

   □

3. *Proof.* This is not true if X is the singleton set. Indeed, $1 + 1 \approx 2$.

   The statement is trivial when $X = \varnothing$ since $\varnothing + \varnothing = \varnothing$.

   If X is infinite. Let $m\colon 2 \to X$ be any injection witnessing $2 \preceq X$.

   There is an evident injection $i\colon X \to X + X$ defined by $i(x) = (x, 0)$. For the other direction, define $j\colon X + X \to X \times X$ by $j(x, 0) = (x, m(0))$ and $j(x, 1) = (x, m(1))$.

   If $j(x, 0) = j(y, 0)$ or $j(x, 1) = j(y, 1)$ then the pair property gives what we want. However, if we ever have $j(x, 0) = j(y, 1)$ then the pair property gives $m(0) = m(1)$. Since m is injective, we have a contradiction. Thus, the following composite gives the required injection.

$$\begin{tikzcd}
X + X \arrow[d, tail, "j"'] \arrow[dr, tail, dashed] & \\
X \times X \arrow[r, "\approx"] & X
\end{tikzcd}$$

$\square$