Assignment 1: Internal Models of ZF

LOG121: Set theory — Semester H24

Instructions The following exercises are not mandatory but should help you understand the course material better. Please *submit your solutions on Canvas*; both handwritten and typeset solutions are accepted. The deadline is on the evening of the *24th September*.

Exercises Recall the following construction from the lecture:

$$\mathcal{U} := \bigcup \{\mathcal{U}_i \mid i \in \mathbb{N}\} \qquad \text{with} \qquad \mathcal{U}_0 := \mathbb{N} \qquad \mathcal{U}_{i+1} := \mathcal{P}(\mathcal{U}_i).$$

(a) Prove that $\mathcal{U}_i \subseteq \mathcal{U}_{i+1}$ for every $i \in \mathbb{N}$.

(b) Prove that every $\mathcal{U}_i$ is a *transitive set*, i.e. if $x \in y \in \mathcal{U}_i$ then $x \in \mathcal{U}_i$. Conclude that $y \subseteq \mathcal{U}_i$ for every $y \in \mathcal{U}_i$.

A formula A in the language of ZF is *relativized to a set* $\mathcal{V}$, by replacing each quantifier with one bounded by $\mathcal{V}$, that is

$$\exists x.\, B \rightsquigarrow \exists x \in \mathcal{V}.\, B \qquad \text{and} \qquad \forall x.\, B \rightsquigarrow \forall x \in \mathcal{V}.\, B$$

for each quantifier instance in A. For example, relativizing the empty set axiom to $\mathcal{U}$ yields:

$$\exists x. \forall y.\, y \notin x \qquad \rightsquigarrow \qquad \exists x \in \mathcal{U}. \forall y \in \mathcal{U}.\, y \notin x.$$

(c) Prove that $\mathcal{U}$ provides an internal model of ZF1–7,9. That is, prove each instance of said axioms, relativized to $\mathcal{U}$.

(d) Find an instance of ZF8, the axiom of replacement, relativized to $\mathcal{U}$, which is not true.

Bonus exercise Recall Dominik's claim that ZF1–3 cannot prove that any sets with more than two elements exist. Find an internal model witnessing this. That is, find a set $\mathcal{B}$ in ZF such that

- the axioms ZF1–3, relativized to $\mathcal{B}$, are true and
- no $x \in \mathcal{B}$ has more than two elements.

What other axioms of ZF are satisfied by $\mathcal{B}$? Can you find an analogous internal model witnessing the fact that ZF1–6 cannot prove the existence of infinite sets?

Internal models of ZF

Frank Tsai

September 21, 2024

Problem 1

1. *Proof.* Let us proceed by induction on i.

   - When $i = 0$, we need to show that $\mathbb{N} \subseteq \mathcal{P}(\mathbb{N})$, i.e., $n \in \mathbb{N} \Rightarrow n \in \mathcal{P}(\mathbb{N})$ for all $n \in \mathbb{N}$. Again, let us proceed by induction on n.
     - When $n = 0$, we clearly have $\varnothing \in \mathcal{P}(\mathbb{N})$.
     - When $n = s(k)$, note that $s(k) = k \cup \{k\}$ by definition. By the induction hypothesis, $k \subseteq \mathbb{N}$. And clearly, $\{k\} \subseteq \mathbb{N}$. Thus, it follows that $k \cup \{k\} \subseteq \mathbb{N}$, i.e., $k \cup \{k\} \in \mathcal{P}(\mathbb{N})$.
   - When $i = s(k)$, we need to show that $\mathcal{U}_{s(k)} \subseteq \mathcal{U}_{s(s(k))}$. Suppose that $x \in \mathcal{U}_{s(k)} = \mathcal{P}(\mathcal{U}_k)$, or equivalently, that $x \subseteq \mathcal{U}_k$. By the induction hypothesis, $\mathcal{U}_k \subseteq \mathcal{U}_{s(k)}$, so $x \subseteq \mathcal{U}_{s(k)}$. But this means that $x \in \mathcal{P}(\mathcal{U}_{s(k)}) = \mathcal{U}_{s(s(k))}$.

   □

2. *Proof.* Proceed by case analysis on i.

   - When $i = 0$, we need to show that if $x \in y$ and $y \in \mathbb{N}$ then $x \in \mathbb{N}$. Let us proceed by induction on y.
     - When $y = 0$, the thesis holds vacuously as $x \notin \varnothing$ for any x.
     - When $y = s(k)$, suppose that $x \in k \cup \{k\}$. If $x \in k$, then $x \in \mathbb{N}$ by the induction hypothesis. If $x \in \{k\}$ (i.e., $x = k$ by the axiom of pairing), then we are done as $k \in \mathbb{N}$.
   - When $i = s(k)$, suppose that $x \in y$ and $y \in \mathcal{U}_{s(k)} = \mathcal{P}(\mathcal{U}_k)$. Then since $y \subseteq \mathcal{U}_k$, $x \in \mathcal{U}_k$. By 1, $\mathcal{U}_k \subseteq \mathcal{U}_{s(k)}$, so $x \in \mathcal{U}_{s(k)}$.

   From 2, we can deduce that for any $y \in \mathcal{U}_i$ and any $x \in y$, we have $x \in \mathcal{U}_i$, i.e., $y \subseteq \mathcal{U}_i$. □

Problem 2

1. - Axiom of extensionality.

     Proof. Let $x, y \in \mathcal{U}$. The only if direction is trivial. For the if direction, suppose that $z \in x \Leftrightarrow z \in y$ for all $z \in \mathcal{U}$. To show $x = y$, we appeal to the "external" axiom of extensionality and show that the assumed hypothesis is sufficiently strong.

Let z' be an arbitrary set (not necessarily a priori in $\mathcal{U}$). We need to show that $z' \in x \Leftrightarrow z' \in y$. If $z' \in x$, then since $x \in \mathcal{U}_i$ for some i, $z' \in \mathcal{U}_i$ by transitivity, but this implies that $z' \in \mathcal{U}$. Thus, $z' \in y$ follows from the assumed hypothesis. The converse is completely analogous. □

In the proofs that follow, we follow the same strategy: we show that the universe $\mathcal{U}$ is closed under a given construction; then we can appeal to the external version of the relevant axiom to conclude the proof.

- Axiom of empty set.

  Proof. Note that $\varnothing \in \mathcal{U}_1 = \mathcal{P}(\mathbb{N})$. □

- Axiom of pairing.

  Proof. Let $x, y \in \mathcal{U}$. Since $\mathcal{U}_i \subseteq \mathcal{U}_{s(i)}$ for all i, it is legitimate to assume that $x, y \in \mathcal{U}_i$ for the same i. Then it follows that $\{x, y\} \subseteq \mathcal{U}_i$, i.e., $\{x, y\} \in \mathcal{U}_{s(i)}$. The rest of the statement follows from the external axiom of pairing. □

- Axiom schema of separation.

  Proof. Let φ be any formula in the language of ZF set theory with free variables among $z, p_1, \ldots, p_n$ and let $x, p_1, \ldots, p_n \in \mathcal{U}$. It suffices to show that $y := \{z \in x \mid \varphi(z, p_1, \ldots, p_n)\} \in \mathcal{U}$. Since $x \in \mathcal{U}_i$ for some i, $x \subseteq \mathcal{U}_i$ by Problem 1. Then since $y \subseteq x$, $y \subseteq \mathcal{U}_i$, i.e., $y \in \mathcal{U}_{s(i)}$. □

- Axiom of powerset.

  Proof. It suffices to show that the universe $\mathcal{U}$ is closed under the powerset construction. The rest of the argument follows from the external axiom of powerset.

  Let $x \in \mathcal{U}$. Then $x \in \mathcal{U}_i$ for some i, so $x \subseteq \mathcal{U}_i$ by Problem 1. Then it follows that $y \subseteq \mathcal{U}_i$ for each subset of x. In other words, each subset of x lives in $\mathcal{U}_{s(i)}$. Thus, the set containing exactly these sets, i.e., $\mathcal{P}(x)$, is a subset of $\mathcal{U}_{s(i)}$, so the powerset lives in $\mathcal{U}_{s(s(i))}$. □

- Axiom of union.

  Proof. It suffices to show that the universe $\mathcal{U}$ is closed under union. Let $x \in \mathcal{U}$, then $x \in \mathcal{U}_i$ for some i. For any arbitrary $y \in \bigcup x$, the external axiom of union implies that there is a set v such that $y \in v$ and $v \in x$. Thus, by transitivity, $v \in \mathcal{U}_i$ and (by transitivity again) $y \in \mathcal{U}_i$. In other words, $\bigcup x \subseteq \mathcal{U}_i$, so $\bigcup x \in \mathcal{U}_{s(i)}$. □

- Axiom of infinity.

  Proof. Note that $\mathbb{N} \in \mathcal{P}(\mathbb{N}) = V_1$. □

- Axiom of foundation.

  Proof. Let $x \in \mathcal{U}$ be a nonempty set. By the external axiom of foundation, there is a set $z \in x$ such that $x \cap z = \varnothing$, so it suffices to show that z lives in the universe $\mathcal{U}$. This follows immediately from transitivity: since $x \in \mathcal{U}_i$ for some i and $z \in x$ by assumption, we have that $z \in \mathcal{U}_i$. □

2. *Proof.* Consider $\varphi(x, y) := y = \mathcal{P}(x)$, or more precisely,

   $$\varphi(x, y) := \forall z.\, z \in y \Leftrightarrow (\forall a.\, a \in z \Leftrightarrow (\forall b.\, b \in a \Rightarrow b \in x)).$$

   It is a definable function because (1) the axiom of powerset guarantees existence and (2) the axiom of extensionality guarantees uniqueness.

   Applying the recursion scheme with $\mathbb{N}$ yields the definable function $n \mapsto \mathcal{U}_n$. The image of this function under $\mathbb{N}$ is the entire universe $\mathcal{U}$, which is too large to live in $\mathcal{U}$ by the axiom of foundation. □

Remark 1. Let us pause and reflect on the proofs we have written so far. Ignoring the axiom schema of replacement, every axiom except the axioms of empty set and infinity relies purely on 3 properties of the universe $\mathcal{U}$:

1. Every level of the hierarchy is monotone, i.e., $\mathcal{U}_i \subseteq \mathcal{U}_{s(i)}$ for all i.
2. Every level of the hierarchy is transitive.
3. Every level of the hierarchy is replete, i.e., $x \subseteq \mathcal{U}_i$ implies that $x \in \mathcal{U}_{s(i)}$ for all i.

This observation can be sharpened in two directions. First, monotonicity is in fact a consequence of transitivity and repleteness. To see this, observe that, for any $x \in \mathcal{U}_i$, we can build the following chain using repleteness.

$$x \in \{x\} \in \mathcal{U}_{s(i)}$$

Then transitivity yields the thesis.

Second, a replete universe cannot be empty since $\varnothing \subseteq \mathcal{U}_i$. In fact, this implies that the empty set must be in the universe $\mathcal{U}$. This provides a model for the axiom of empty set.

Problem 3

To model the axioms of extensionality, empty set, and foundation, it suffices to choose $\mathcal{B} = \{\varnothing\}$.

To model the axiom of pairing, we need to extend this universe. Let us construct a universe as follows:

$$\mathcal{B} := \bigcup\{\mathcal{B}_i \mid i \in \mathbb{N}\} \quad \text{with} \quad \mathcal{B}_0 := \{\varnothing\} \quad \mathcal{B}_{s(n)} := \{\{x, y\} \mid x, y \in \mathcal{B}_n\} \cup \mathcal{B}_n.$$

This universe is not replete (consider any 3 element subset of $\mathcal{B}_2$; it cannot live in $\mathcal{B}_3$), but it is monotone (by construction) and transitive (by simple induction).

Additionally, it contains the empty set and every set in $\mathcal{B}$ contains at most 2 elements.

Note that the proof for the axiom of pairing above does not work out of the box because it relies on repleteness, but it can be easily repaired because $\mathcal{B}$ is closed under pairing.

Repairing the proof for the axiom schema of separation requires more work. To make it work, we make the observation that every set in the universe $\mathcal{B}$ is either the empty set or a pair of sets from $\mathcal{B}$.

Proof. Let φ be any formula in the language of ZF set theory with free variables among $z, p_1, \ldots, p_n$ and let $x, p_1, \ldots, p_n \in \mathcal{B}$. It suffices to show that $y := \{z \in x \mid \varphi(z, p_1, \ldots, p_n)\} \in \mathcal{B}$.

Note that x is either the empty set or a pair of sets $\{a, b\}$, where $a, b \in \mathcal{B}_i$ for some i. In the former case, we have $y = \varnothing \in \mathcal{B}_0$. In the latter case, we have $y = \varnothing, \{a\}, \{b\},$ or $\{a, b\}$. Thus, it follows that $y \in \mathcal{B}_{s(i)}$. □

Thus, $\mathcal{B}$ provides a model for ZF1-4,9.

Now, consider the following universe.

$$C := \bigcup\{C_i \mid i \in \mathbb{N}\} \qquad \text{with} \qquad C_0 := \{\varnothing\} \qquad C_{s(n)} := \mathcal{P}(C_n).$$

Note that this universe is replete by construction. We can use the same argument in Problem 1 to prove that C is transitive. Thus, it provides a model for ZF1-6,9.

Now, observe that the powerset of a finite set is finite, so every set in $\mathcal{B}$ is finite. Thus, ZF1-6,9 cannot guarantee the existence of infinite sets.