LOG111 Hand-in 3

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Proof. By construction, the set $\Gamma_1 \cup \Gamma_2$ is unsatisfiable, so by compactness, there is a finite unsatisfiable subset $\Delta \subseteq \Gamma_1 \cup \Gamma_2$.

Consider

$$\Delta_1 := \Delta \setminus \Gamma_2 \qquad\qquad \Delta_2 := \Delta \setminus \Gamma_1.$$

We claim that Δ_1 and Δ_2 respectively axiomatize $\mathsf{Th}(\Gamma_1)$ and $\mathsf{Th}(\Gamma_2)$. We prove that this is the case for Δ_1; the argument for Δ_2 is completely analogous.

We need to prove that for any formula φ, $\Gamma_1 \vdash \varphi$ iff $\Delta_1 \vdash \varphi$. To this end, it suffices to prove their semantic counterpart by soundness and completeness.

The if direction is an immediate consequence of monotonicity. In the other direction, suppose that $\Gamma_1 \vDash \varphi$ and let $M \vDash \Delta_1$. If $M \vDash \Gamma_1$ then we are done. On the other hand, if $M \nvDash \Gamma_1$ then it follows that $M \vDash \Delta_2$, but this means that $M \vDash \Delta_1 \cup \Delta_2$ contradicting the fact that $\Delta = \Delta_1 \cup \Delta_2$ is unsatisfiable. □

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1. $$\dfrac{\dfrac{\dfrac{c_x < c_y \qquad [c_y < c_x]^1}{c_x < c_x}\,\mathrm{T}}{\bot}\,\mathrm{R}}{c_y \nless c_x}\,\mathrm{RAA}_2^1$$

2. $$(X,R) \nvDash \bot \qquad\qquad (X,R) \vDash c_x < c_y \text{ iff } (x,y) \in R$$

   $$(X,R) \vDash c_x \nless c_y \text{ iff } (x,y) \notin R$$

   $\Gamma \vDash \varphi$ iff for every (X,R), if (X,R) satisfies every formula in Γ then (X,R) satisfies φ.

3. To prove soundness, we can do an induction on the height of the derivation tree.

   The base case is immediate. In the induction case, we do a case analysis on the last applied rule. When the last applied rule is RAA_1, the induction hypothesis yields $\Gamma, c_x \nless c_y \vDash \bot$. Thus, for any (X,R) satisfying Γ, (X,R) must satisfy $c_x < c_y$, i.e., $\Gamma \vDash c_x < c_y$.

4. Given a formula φ that is not $\bot$, we shall use the informal notation $\neg\varphi$ to denote the opposite of φ, e.g., if $\varphi \equiv c_x \not< c_y$ then $\neg\varphi \equiv c_x < c_y$. Note that the set of formulas is enumerable because $X \times X$ is countable. We write $c_{n_1} < c_{n_2}$ or $c_{n_1} \not< c_{n_2}$ for the n-th enumeration.

Construction 1. Let Γ be consistent, we extend Γ by taking the fixed point of the following process and we name the resulting set Γ^*.

$$\Gamma^* = \bigcup\{\Gamma_n \mid n \in \mathbb{N}\} \qquad\qquad \Gamma_0 = \Gamma$$

$$\Gamma_{n+1} = \begin{cases} \Gamma_n, c_{n_1} < c_{n_2} & \text{if the resulting set is consistent,} \\ \Gamma_n, c_{n_1} \not< c_{n_2} & \text{otherwise.} \end{cases}$$

Lemma 2. *If Γ is consistent, then the set Γ^* as in Construction 1 is consistent and complete in the sense that for any $a, b \in X$, either $c_a < c_b$ or $c_a \not< c_b$.*

Proof. Completeness is evident: if the pair (n_1, n_2) enumerates (a, b), then either $c_a < c_b$ or $c_a \not< c_b$ is added at step n.

Now, we show that each Γ_n is consistent. We proceed by induction on n. The base case is trivial. In the induction step, if $\Gamma_{n+1} = \Gamma_n, c_{n_1} < c_{n_2}$ then there is nothing left to prove.

If $\Gamma_{n+1} = \Gamma_n, c_{n_1} \not< c_{n_2}$, then $\Gamma_n, c_{n_1} < c_{n_2} \vdash \bot$. Suppose that $\Gamma_n, c_{n_1} \not< c_{n_2} \vdash \bot$. Then we have a derivation of $\Gamma_n \vdash c_{n_1} < c_{n_2}$ as follows.

$$\dfrac{\begin{matrix}[c_{n_1} \not< c_{n_2}] \\ \vdots \\ \bot\end{matrix}}{c_{n_1} < c_{n_2}}\ \text{ARR}$$

But this gives a derivation of $\Gamma_n \vdash \bot$.

$$\dfrac{\dfrac{\begin{matrix}[c_{n_1} \not< c_{n_2}] \\ \vdots \\ \bot\end{matrix}}{c_{n_1} < c_{n_2}}\ \text{ARR} \\ \vdots}{\bot}$$

This contradicts the induction hypothesis. If $\Gamma^* \vdash \bot$ then there is an inconsistent subset $\Gamma' \subseteq \bot$. This subset lies in Γ_n for some n, rendering this set inconsistent. But this is a contradiction. ◻

Lemma 3. *If Γ is consistent then Γ has a model. Moreover, if $\Gamma \nvdash \varphi$, then this model also satisfies $\neg\varphi$, where φ is not $\bot$.*

Proof. By Lemma 2, Γ^* is consistent and complete. Consider the following relation.

$$R = \{(x, y) \mid c_x < c_y \in \Gamma^*\}$$

We postpone the additional proof obligations for irreflexivity and transitivity.

To show that $(X, R) \vDash \Gamma$, it suffices to show that $(X, R) \vDash \Gamma^*$. Let $\varphi \in \Gamma^*$. Note that it must be either $c_x < c_y$ or $c_x \nless c_y$ because Γ^* is consistent. In the former case, there is nothing left to prove. In the latter case, suppose that $(x, y) \in R$. Then by definition $c_x < c_y \in \Gamma^*$. This yields $\Gamma^* \vdash \bot$ as follows.

$$\dfrac{c_x < c_y \qquad c_x \nless c_y}{\bot}\ \bot$$

This is a contradiction.

It remains to show the second half of the statement. Note that if $\Gamma \nvdash \varphi$, then $\Gamma, \neg\varphi$ is consistent because if $\Gamma, \neg\varphi \vdash \bot$ then we have a derivation of $\Gamma \vdash \varphi$.

$$\begin{array}{c} [\neg\varphi] \\ \vdots \\ \dfrac{\bot}{\varphi}\ \mathrm{RAA} \end{array}$$

Thus, $\neg\varphi$ has to be added to Γ^* at some point. Then it follows by construction that $(X, R) \vDash \neg\varphi$. □

Proof of irreflexivity. If $(x, x) \in R$, then by definition $c_x < c_x \in \Gamma^*$. But this means that there is a derivation of $\Gamma^* \vdash \bot$.

$$\dfrac{c_x < c_x}{\bot}\ \mathrm{R}$$

This contradicts the fact that Γ^* is consistent. □

Proof of transitivity. Let $(x, y), (y, z) \in R$. We need to show that $(x, z) \in R$. By completeness, either $c_x < c_z \in \Gamma^*$ or $c_x \nless c_z \in \Gamma^*$. In the former case, there is nothing left to prove. On the other hand, if $c_x \nless c_z \in \Gamma^*$ then we have a derivation $\Gamma^* \vdash \bot$.

$$\dfrac{\dfrac{c_x < c_y \qquad c_y < c_z}{c_x < c_z}\ \mathrm{T} \qquad c_x \nless c_z}{\bot}\ \bot$$

This contradicts the fact that Γ^* is consistent. □

Theorem 4 (Completeness)**.** *If* $\Gamma \vDash \varphi$ *then* $\Gamma \vdash \varphi$*.*

Proof. Suppose that $\Gamma \nvdash \varphi$ and that $\Gamma \vDash \varphi$. The first assumption implies that Γ is consistent, so by Lemma 3, Γ has a model (X, R) which also satisfies $\neg\varphi$. But this is a contradiction because (X, R) must satisfy its opposite φ by the second assumption. □

5. *Proof.* It suffices to repair the induction step of Lemma 2. The proof of the first half of Lemma 3 goes through without any problem.

   There is nothing to repair in the first case of the induction step. In the second case, we know that $\Gamma_n, c_{n_1} < c_{n_2} \vdash_i \bot$. Suppose that $\Gamma_n, c_{n_1} \nless c_{n_2} \vdash_i \bot$. Let us consider the last rule of this derivation. Only R and $\bot$ are immediately applicable.

The premise of R is a positive formula, so it must come from Γ_n. But this is impossible because Γ_n is consistent by the induction hypothesis. Thus, we can rule out R.

The only applicable rule is $\perp$. This rule demands $c_{n_1} < c_{n_2}$, so $\Gamma_n \vdash_i c_{n_1} < c_{n_2}$. But this means that Γ_n is not consistent after all. $\square$

6. Note that the intuitionistic fragment cannot derive negative formulas, so consider $c_x < c_y \vDash c_x \not< c_y$. This clearly holds. In fact, Problem A and soundness imply this.