

LOG111 Hand-in 3

Frank Tsai

October 11, 2024

1

Proof. By construction, the set $\Gamma_1 \cup \Gamma_2$ is unsatisfiable, so by compactness, there is a finite unsatisfiable subset $\Delta \subseteq \Gamma_1 \cup \Gamma_2$.

Consider

$$\Delta_1 := \Delta \setminus \Gamma_2 \qquad \Delta_2 := \Delta \setminus \Gamma_1.$$

We claim that Δ_1 and Δ_2 respectively axiomatize $\text{Th}(\Gamma_1)$ and $\text{Th}(\Gamma_2)$. We prove that this is the case for Δ_1 ; the argument for Δ_2 is completely analogous.

We need to prove that for any formula φ , $\Gamma_1 \vdash \varphi$ iff $\Delta_1 \vdash \varphi$. To this end, it suffices to prove their semantic counterpart by soundness and completeness.

The if direction is an immediate consequence of monotonicity. In the other direction, suppose that $\Gamma_1 \models \varphi$ and let $M \models \Delta_1$. If $M \models \Gamma_1$ then we are done. On the other hand, if $M \not\models \Gamma_1$ then it follows that $M \models \Delta_2$, but this means that $M \models \Delta_1 \cup \Delta_2$ contradicting the fact that $\Delta = \Delta_1 \cup \Delta_2$ is unsatisfiable. \square

2

1.

$$\frac{\frac{\frac{c_x < c_y \quad [c_y < c_x]^1}{c_x < c_x} \text{R}}{\perp} \text{RAA}_2^1}{c_y \not< c_x} \text{T}$$

2.

$$(X, R) \not\models \perp \qquad (X, R) \models c_x < c_y \text{ iff } (x, y) \in R$$

$$(X, R) \models c_x \not< c_y \text{ iff } (x, y) \notin R$$

$\Gamma \models \varphi$ iff for every (X, R) , if (X, R) satisfies every formula in Γ then (X, R) satisfies φ .

3. To prove soundness, we can do an induction on the height of the derivation tree.

The base case is immediate. In the induction case, we do a case analysis on the last applied rule. When the last applied rule is RAA_1 , the induction hypothesis yields $\Gamma, c_x \not< c_y \models \perp$. Thus, for any (X, R) satisfying Γ , (X, R) must satisfy $c_x < c_y$, i.e., $\Gamma \models c_x < c_y$.

4. Given a formula φ that is not \perp , we shall use the informal notation $\neg\varphi$ to denote the opposite of φ , e.g., if $\varphi \equiv c_x \not< c_y$ then $\neg\varphi \equiv c_x < c_y$. Note that the set of formulas is enumerable because $X \times X$ is countable. We write $c_{n_1} < c_{n_2}$ or $c_{n_1} \not< c_{n_2}$ for the n -th enumeration.

Construction 1. Let Γ be consistent, we extend Γ by the following process and we name the resulting set Γ^* .

$$\Gamma^* = \bigcup \{\Gamma_n \mid n \in \mathbb{N}\} \quad \Gamma_0 = \Gamma$$

$$\Gamma_{n+1} = \begin{cases} \Gamma_n, c_{n_1} < c_{n_2} & \text{if the resulting set is consistent,} \\ \Gamma_n, c_{n_1} \not< c_{n_2} & \text{otherwise.} \end{cases}$$

Lemma 2. *If Γ is consistent, then the set Γ^* in Construction 1 is consistent and complete in the sense that for any $a, b \in X$, either $c_a < c_b \in \Gamma^*$ or $c_a \not< c_b \in \Gamma^*$.*

Proof. Completeness is evident: if the pair (n_1, n_2) names $(a, b) \in X \times X$, then either $c_a < c_b$ or $c_a \not< c_b$ is added at step $n + 1$.

Now, we show that each Γ_n is consistent. We proceed by induction on n . The base case is trivial. In the induction step, if $\Gamma_{n+1} = \Gamma_n, c_{n_1} < c_{n_2}$ then there is nothing left to prove.

If $\Gamma_{n+1} = \Gamma_n, c_{n_1} \not< c_{n_2}$, then $\Gamma_n, c_{n_1} < c_{n_2} \vdash \perp$ by construction. Suppose that $\Gamma_n, c_{n_1} \not< c_{n_2} \vdash \perp$. Then we have a derivation of $\Gamma_n \vdash c_{n_1} < c_{n_2}$ as follows.

$$\begin{array}{c} [c_{n_1} \not< c_{n_2}] \\ \vdots \\ \frac{\perp}{c_{n_1} < c_{n_2}} \text{ ARR} \end{array}$$

Cutting the derivations of $\Gamma_n, c_{n_1} < c_{n_2} \vdash \perp$ and $\Gamma_n \vdash c_{n_1} < c_{n_2}$ gives a derivation of $\Gamma_n \vdash \perp$. This contradicts the induction hypothesis.

Now suppose that $\Gamma^* \vdash \perp$ then there is an inconsistent subset $\Gamma' \subseteq \Gamma^*$, rendering Γ_n for some n inconsistent. This gives a contradiction. \square

Lemma 3. *If Γ is consistent then Γ has a model. Moreover, if $\Gamma \not\models \varphi$, then this model also satisfies $\neg\varphi$, where φ is not \perp .*

Proof. By Lemma 2, Γ^* is consistent and complete. Consider the following relation.

$$R = \{(x, y) \mid c_x < c_y \in \Gamma^*\}$$

We postpone the additional proof obligations for irreflexivity and transitivity.

To show that $(X, R) \models \Gamma$, it suffices to show that $(X, R) \models \Gamma^*$. Let $\varphi \in \Gamma^*$. Note that it must be either $c_x < c_y$ or $c_x \not< c_y$ because Γ^* is consistent. In the former case, there is nothing left to prove. In the latter case, suppose that $(x, y) \in R$. Then by definition $c_x < c_y \in \Gamma^*$. This yields $\Gamma^* \vdash \perp$ as follows.

$$\frac{c_x < c_y \quad c_x \not< c_y}{\perp}$$

This is a contradiction.

It remains to show the second half of the statement. Note that if $\Gamma \not\models \varphi$, then $\Gamma, \neg\varphi$ is consistent because if $\Gamma, \neg\varphi \vdash \perp$ then we would have a derivation of $\Gamma \vdash \varphi$.

$$\begin{array}{c} [\neg\varphi] \\ \vdots \\ \frac{}{\varphi} \text{RAA} \end{array}$$

Thus, $\neg\varphi$ has to be added to Γ^* at some point. Then it follows by construction that $(X, R) \models \neg\varphi$. \square

Proof of irreflexivity. If $(x, x) \in R$, then by definition $c_x < c_x \in \Gamma^*$. But this means that there is a derivation of $\Gamma^* \vdash \perp$.

$$\frac{c_x < c_x}{\perp} \text{R}$$

This contradicts the fact that Γ^* is consistent. \square

Proof of transitivity. Let $(x, y), (y, z) \in R$. We need to show that $(x, z) \in R$. By completeness, either $c_x < c_z \in \Gamma^*$ or $c_x \not< c_z \in \Gamma^*$. In the former case, there is nothing left to prove. On the other hand, if $c_x \not< c_z \in \Gamma^*$ then we have a derivation of $\Gamma^* \vdash \perp$.

$$\frac{\frac{c_x < c_y \quad c_y < c_z}{c_x < c_z} \text{T} \quad c_x \not< c_z}{\perp} \perp$$

This contradicts the fact that Γ^* is consistent. \square

Theorem 4 (Completeness). *If $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$.*

Proof. Suppose that $\Gamma \models \varphi$ and that $\Gamma \not\vdash \varphi$. The first assumption implies that Γ is consistent, so by Lemma 3, Γ has a model (X, R) which also satisfies $\neg\varphi$. But this is a contradiction because (X, R) must satisfy its opposite φ by the second assumption. \square

5. *Proof.* It suffices to repair the induction step of Lemma 2. The proof of the first half of Lemma 3 goes through without any problem.

There is nothing to repair in the first case of the induction step. In the second case, we know that $\Gamma_n, c_{n_1} < c_{n_2} \vdash_i \perp$. Suppose that $\Gamma_n, c_{n_1} \not< c_{n_2} \vdash_i \perp$. Let us consider the last rule of this derivation. Only R and \perp are immediately applicable.

The premise of R is a positive formula, so it must come from Γ_n . But this is impossible because Γ_n is consistent by the induction hypothesis. Thus, we can rule out R.

The only applicable rule is \perp . This rule demands $c_{n_1} < c_{n_2}$, so we must have a derivation of $\Gamma_n \vdash_i c_{n_1} < c_{n_2}$. But this means that Γ_n is not consistent after all. \square

6. Note that the intuitionistic fragment cannot derive negative formulas, so consider $c_x < c_y \vdash c_x \not< c_y$. This clearly holds. In fact, Problem A and soundness imply this.