Assignment 1: Internal Models of ZF

LOG121: Set theory　　　　　　　　　　　　　　　　　　　　　　Semester H24

Instructions　The following exercises are not mandatory but should help you understand the course material better. Please *submit your solutions on Canvas*; both handwritten and typeset solutions are accepted. The deadline is on the evening of the *24th September*.

Exercises　Recall the following construction from the lecture:

$$\mathcal{U} := \bigcup \{\mathcal{U}_i \mid i \in \mathbb{N}\} \qquad \text{with} \qquad \mathcal{U}_0 := \mathbb{N} \qquad \mathcal{U}_{i+1} := \mathcal{P}(\mathcal{U}_i).$$

(a) Prove that $\mathcal{U}_i \subseteq \mathcal{U}_{i+1}$ for every $i \in \mathbb{N}$.

(b) Prove that every $\mathcal{U}_i$ is a *transitive set*, i.e. if $x \in y \in \mathcal{U}_i$ then $x \in \mathcal{U}_i$. Conclude that $y \subseteq \mathcal{U}_i$ for every $y \in \mathcal{U}_i$.

A formula A in the language of ZF is *relativized to a set* $\mathcal{V}$, by replacing each quantifier with one bounded by $\mathcal{V}$, that is

$$\exists x.\, B \rightsquigarrow \exists x \in \mathcal{V}.\, B \qquad \text{and} \qquad \forall x.\, B \rightsquigarrow \forall x \in \mathcal{V}.\, B$$

for each quantifier instance in A. For example, relativizing the empty set axiom to $\mathcal{U}$ yields:

$$\exists x. \forall y.\, y \notin x \qquad \rightsquigarrow \qquad \exists x \in \mathcal{U}. \forall y \in \mathcal{U}.\, y \notin x.$$

(c) Prove that $\mathcal{U}$ provides an internal model of ZF1–7,9. That is, prove each instance of said axioms, relativized to $\mathcal{U}$.

(d) Find an instance of ZF8, the axiom of replacement, relativized to $\mathcal{U}$, which is not true.

Bonus exercise　Recall Dominik's claim that ZF1–3 cannot prove that any sets with more than two elements exist. Find an internal model witnessing this. That is, find a set $\mathcal{B}$ in ZF such that

- the axioms ZF1–3, relativized to $\mathcal{B}$, are true and
- no $x \in \mathcal{B}$ has more than two elements.

What other axioms of ZF are satisfied by $\mathcal{B}$? Can you find an analogous internal model witnessing the fact that ZF1–6 cannot prove the existence of infinite sets?