Finite satisfiability

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Lemma 1. *Every finitely satisfiable set* Γ *can be extended to a complete and finitely satisfiable set* Γ^*.

Proof. Fix an arbitrary enumeration of formulas. Define Γ^* as follows.

$$\Gamma_0 = \Gamma \qquad \Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\varphi_n\} & \text{if the resulting set is finitely satisfiable,} \\ \Gamma_n \cup \{\neg\varphi_n\} & \text{otherwise.} \end{cases}$$

$$\Gamma^* = \bigcup \Gamma_n$$

Complete: every formula φ is named by a natural number n, so either φ or $\neg\varphi$ is added at step $n+1$.

Finitely satisfiable: by induction on n, we show that each Γ_n is finitely satisfiable. In the base case, we have $\Gamma_0 = \Gamma$, which is finitely satisfiable by hypothesis. In the induction step, the induction hypothesis tells us that Γ_n is finitely satisfiable. It remains to verify that Γ_{n+1} is also finitely satisfiable. There are 2 cases. If $\Gamma_{n+1} = \Gamma_n \cup \{\varphi_n\}$, then Γ_{n+1} is finitely satisfiable by construction. If $\Gamma_{n+1} = \Gamma_n \cup \{\neg\varphi_n\}$, then $\Gamma_n \cup \{\varphi_n\}$ is not finitely satisfiable. Suppose that $\Gamma_n \cup \{\neg\varphi_n\}$ is not finitely satisfiable either. Then consider any two unsatisfiable finite subsets $\Delta \subseteq \Gamma_n \cup \{\varphi_n\}$ and $\Xi \subseteq \Gamma_n \cup \{\neg\varphi_n\}$. Note that $\varphi_n \in \Delta$ and $\neg\varphi_n \in \Xi$ since any finite subset of Γ_n is satisfiable.

We claim that $\Pi := (\Delta \setminus \{\varphi_n\}) \cup (\Xi \setminus \{\neg\varphi_n\})$ is an unsatisfiable subset of Γ_n. Suppose the contrary. Then there is a structure M such that $M \vDash \Pi$. Then we have $M \vDash \Delta \setminus \{\varphi_n\}$ and $M \vDash \Xi \setminus \{\neg\varphi_n\}$. Since Δ is unsatisfiable, we must have that $M \nvDash \varphi_n$, but this means that $M \vDash \neg\varphi_n$, so $M \vDash \Xi$. This is a contradiction. Thus, we have found an unsatisfiable finite subset Π of Γ_n, contradicting the induction hypothesis.

It remains to show that Γ^* is finitely satisfiable. To this end, let $A \subseteq \Gamma^*$ be an unsatisfiable finite subset. By construction $\Gamma_n \subseteq \Gamma_{n+1}$ for all n, so A must lie in Γ_k for some k, but this means that Γ_k is not finitely satisfiable, which is a contradiction. □