LOG111 Hand-in 3

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1

Proof. By construction, the set $\Gamma_1 \cup \Gamma_2$ is unsatisfiable, so by compactness, there is a finite unsatisfiable subset $\Delta \subseteq \Gamma_1 \cup \Gamma_2$.

Consider

$$\Delta_1 := \Delta \setminus \Gamma_2 \qquad\qquad \Delta_2 := \Delta \setminus \Gamma_1.$$

We claim that Δ_1 and Δ_2 respectively axiomatize $\mathsf{Th}(\Gamma_1)$ and $\mathsf{Th}(\Gamma_2)$. We prove that this is the case for Δ_1; the argument for Δ_2 is completely analogous.

We need to prove that for any formula φ, $\Gamma_1 \vdash \varphi$ iff $\Delta_1 \vdash \varphi$. To this end, it suffices to prove their semantic counterpart by soundness and completeness.

The if direction is an immediate consequence of monotonicity. In the other direction, suppose that $\Gamma_1 \vDash \varphi$ and let $M \vDash \Delta_1$. If $M \vDash \Gamma_1$ then we are done. On the other hand, if $M \nvDash \Gamma_1$ then it follows that $M \vDash \Delta_2$, but this means that $M \vDash \Delta_1 \cup \Delta_2$ contradicting the fact that $\Delta = \Delta_1 \cup \Delta_2$ is unsatisfiable. □

2

1.
$$\dfrac{\dfrac{\dfrac{c_x < c_y \qquad [c_y < c_x]^1}{c_x < c_x}\,\mathrm{T}}{\bot}\,\mathrm{R}}{c_y \not< c_x}\,\mathrm{RAA}_2^1$$

2.
$$(X,R) \nvDash \bot \qquad\qquad (X,R) \vDash c_x < c_y \text{ iff } (x,y) \in R$$

$$(X,R) \vDash c_x \not< c_y \text{ iff } (x,y) \notin R$$

$\Gamma \vDash \varphi$ iff for every (X,R), if (X,R) satisfies every formula in Γ then (X,R) satisfies φ.

3. To prove soundness, we can do an induction on the height of the derivation tree.

The base case is immediate. In the induction case, we do a case analysis on the last applied rule. When the last applied rule is RAA_1, the induction hypothesis yields $\Gamma, c_x \not< c_y \vDash \bot$. Thus, for any (X,R) satisfying Γ, (X,R) must satisfy $c_x < c_y$, i.e., $\Gamma \vDash c_x < c_y$.

4. *Proof.* We show that if Γ is consistent then Γ is satisfiable. First, $X \times X$ is countable because X is, so the set of formulas is enumerable. It is sufficient to enumerate formulas of the form $c_x < c_y$ as we have no use for the other formulas. We write $c_{n_1} < c_{n_2}$ for the n-th enumeration.

Let Γ be a consistent set of formulas. We extend Γ as follows:

$$\Gamma^* = \bigcup \{\Gamma_n \mid n \in \mathbb{N}\} \qquad\qquad \Gamma_0 = \Gamma$$

$$\Gamma_{n+1} = \begin{cases} \Gamma_n, c_{n_1} < c_{n_2} & \text{if the resulting set is consistent,} \\ \Gamma_n & \text{otherwise.} \end{cases}$$

By a simple induction on n, one can show that each Γ_n is consistent. If $\Gamma^* \vdash \bot$, then there is a subset $\Gamma' \subseteq \Gamma^*$ such that $\Gamma' \vdash \bot$. Note that $\Gamma' \subseteq \Gamma_n$ for some n (each approximation of Γ^* is cumulative), but this contradicts the fact that Γ_n is consistent.

We are now ready to define a relation:

$$R = \{(x, y) \mid c_x < c_y \in \Gamma^*\}$$

If $(x, x) \in R$, then by definition $c_x < c_x \in \Gamma^*$. But this means that Γ^* is inconsistent as $\Gamma^* \vdash \bot$ via R. Thus, R is irreflexive.

Let $(a, b), (b, c) \in R$. We need to show that $(a, c) \in R$. Suppose that $c_a < c_c \notin \Gamma^*$. Then it follows by construction that $\Gamma^*, c_a < c_c$ is inconsistent. But this gives a derivation $\Gamma^* \vdash \bot$ as follows.

$$\dfrac{\dfrac{c_a < c_b \qquad c_b < c_c}{c_a < c_c}\,\mathrm{T} \qquad \dfrac{\begin{matrix}[c_a < c_c]^* \\ \vdots \\ \bot\end{matrix}}{c_a \not< c_c}\,\mathrm{RAA}_2^*}{\bot}\,\bot$$

This contradicts the fact that Γ^* is consistent. Thus, R is transitive.

R satisfies all the positive formulas in Γ^* by construction. It also satisfies all the negative formulas because if a negative formula $c_a \not< c_b$ is not satisfied then $(a, b) \in R$. By construction, this means that $c_a < c_b \in \Gamma^*$, rendering Γ^* inconsistent.

Now, suppose that $\Gamma \nvdash \varphi$ and that $\Gamma \vDash \varphi$. Then Γ is consistent, so we can use the construction above to extract a model. Since $\Gamma, \neg\varphi$ is also consistent, $\neg\varphi \in \Gamma^*$ (this relies on one of the RAA rules). But this means that (X, R) has to satisfy both φ and $\neg\varphi$, which is impossible. $\square$