LOG111 Hand-in 3

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Proof. By construction, the set $\Gamma_1 \cup \Gamma_2$ is unsatisfiable, so by compactness, there is a finite unsatisfiable subset $\Delta \subseteq \Gamma_1 \cup \Gamma_2$.

Consider

$$\Delta_1 := \Delta \setminus \Gamma_2 \qquad\qquad \Delta_2 := \Delta \setminus \Gamma_1.$$

We claim that Δ_1 and Δ_2 respectively axiomatize $\mathsf{Th}(\Gamma_1)$ and $\mathsf{Th}(\Gamma_2)$. We prove that this is the case for Δ_1; the argument for Δ_2 is completely analogous.

We need to prove that for any formula φ, $\Gamma_1 \vdash \varphi$ iff $\Delta_1 \vdash \varphi$. To this end, it suffices to prove their semantic counterpart by soundness and completeness.

The if direction is an immediate consequence of monotonicity. In the other direction, suppose that $\Gamma_1 \vDash \varphi$ and let $M \vDash \Delta_1$. If $M \vDash \Gamma_1$ then we are done. On the other hand, if $M \nvDash \Gamma_1$ then it follows that $M \vDash \Delta_2$, but this means that $M \vDash \Delta_1 \cup \Delta_2$ contradicting the fact that $\Delta = \Delta_1 \cup \Delta_2$ is unsatisfiable. □

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1.
$$\dfrac{\dfrac{\dfrac{c_x < c_y \qquad [c_y < c_x]^1}{c_x < c_x}\,\mathrm{T}}{\bot}\,\mathrm{R}}{c_y \not< c_x}\,\mathrm{RAA}_2^1$$

2.
$$(X, R) \nvDash \bot \qquad\qquad (X, R) \vDash c_x < c_y \text{ iff } (x, y) \in R$$
$$(X, R) \vDash c_x \not< c_y \text{ iff } (x, y) \notin R$$
   $\Gamma \vDash \varphi$ iff for every (X, R), if (X, R) satisfies every formula in Γ then (X, R) satisfies φ.

3. To prove soundness, we can do an induction on the height of the derivation tree.

   The base case is immediate. In the induction case, we do a case analysis on the last applied rule. When the last applied rule is RAA_1, the induction hypothesis yields $\Gamma, c_x \not< c_y \vDash \bot$. Thus, for any (X, R) satisfying Γ, (X, R) must satisfy $c_x < c_y$, i.e., $\Gamma \vDash c_x < c_y$.

4. We show that if Γ is consistent then Γ is satisfiable.