LOG 111 Hand-in 1

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1 Prop 1.6

Proof. Let us proceed by induction.

- Case p or $\bot$: In this case, the formula is atomic. Its only proper initial segment is the empty string ε, which is not a formula.
- Case $\neg\varphi$: Let σ be a proper initial segment of $\neg\varphi$. The ε case is immediate. In the non-ε case, suppose that $\sigma \equiv \neg\xi$ and that it is a formula. Then it follows that ξ is a formula by inverting the inductive definition. Since ξ is a proper initial segment of φ, this contradicts the induction hypothesis.
- Case $(\varphi \star \psi)$: In this case, $\star$ denotes any binary connective. The ε case is immediate. Suppose that σ is a proper initial segment of $(\varphi \star \psi)$ and that σ is a formula. Then by Prop 1.5, σ is balanced. This rules out $\sigma \equiv (\varphi$ and $\sigma \equiv (\varphi \star \psi$ because φ and ψ are also balanced. Thus, there are only two cases to consider:
  - Case $\sigma \equiv (\sigma'$: In this case, σ' is a proper initial segment of φ. By inversion, $\sigma' \equiv \xi \star' \zeta)$, where $\star'$ is any binary connective. Note that ξ is a proper initial segment of φ. This contradicts the induction hypothesis.
  - Case $\sigma \equiv (\varphi \star \sigma'$: In this case, σ' is a proper initial segment of ψ. By inversion, $\sigma' \equiv \xi)$. The rest of the argument mirrors the previous case. □

2 Prob 1.3

Let φ denote the given formula in each sub-problem.

1. $$\varphi \equiv p_0[\varphi/p_0].$$
2. $$\begin{aligned}\varphi &\equiv p_0[\varphi/p_0]\\ \varphi &\equiv (\neg p_0 \wedge p_1)[p_0/p_1].\end{aligned}$$
3. $$\varphi \equiv p_0[\varphi/p_0].$$

4.
$$\varphi \equiv p_0[\varphi/p_0].$$

5.
$$\varphi \equiv p_0[\varphi/p_0]$$
$$\varphi \equiv ((\neg p_0 \to p_1) \wedge p_2)[(p_0 \to p_1)/p_0, (p_0 \vee p_1)/p_1, \neg(p_0 \wedge p_1)/p_2].$$

3 Prob 1.6

φ	ψ	χ	$\diamond(\varphi, \psi, \chi)$
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	F

4 Prob 1.13

DNF: $p_0 \wedge \neg p_1 \wedge p_2$

CNF:

$$\begin{aligned}(p_0 \vee p_1 \vee p_2)\wedge \\ (p_0 \vee p_1 \vee \neg p_2)\wedge \\ (p_0 \vee \neg p_1 \vee \neg p_2)\wedge \\ (p_0 \vee \neg p_1 \vee p_2)\wedge \\ (\neg p_0 \vee p_1 \vee p_2)\wedge \\ (\neg p_0 \vee \neg p_1 \vee p_2)\wedge \\ (\neg p_0 \vee \neg p_1 \vee \neg p_2)\end{aligned}$$

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1. Let us define $\bigwedge \varnothing := \top$. This is sensible as for any valuation v, $v \models \varnothing$. For a given permutation $\sigma = \varphi_1, \dots, \varphi_n$, let us write $\bigwedge \sigma$ for the formula $\varphi_1 \wedge \cdots \wedge \varphi_n$.

2. *Proof.* It suffices to show that for any valuation v, $v \models \varphi_i$ for all i iff $v \models \psi_j$ for all j. Let us proceed by induction on the length of Γ. When $\Gamma = \varnothing$, $\top \approx \top$ is trivial. When Γ has length $n+1$, let $\varphi_1, \dots, \varphi_{n+1}$ and $\psi_1, \dots, \psi_{n+1}$ be two permutations of Γ and v be a valuation. Suppose that $\varphi_{n+1} \equiv \psi_j$. Then clearly $v \models \varphi_{n+1}$ iff $v \models \psi_j$. Excluding these formulas from their respective permutation yields two permutations of the same context of length n. The result now follows from the induction hypothesis. □

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A three-valued semantics can describe the local view of a process in a distributed system, where the valuations of some propositional variables cannot be locally determined because they rely on an external process.

φ	$\neg\varphi$
T	F
F	T
U	U

φ	ψ	$\varphi \vee \psi$
T	T	T
T	F	T
T	U	T
F	T	T
F	F	F
F	U	U
U	T	T
U	F	U
U	U	U

φ	ψ	$\varphi \wedge \psi$
T	T	T
T	F	F
T	U	U
F	T	F
F	F	F
F	U	F
U	T	U
U	F	F
U	U	U

φ	ψ	$\varphi \rightarrow \psi$
T	T	T
T	F	F
T	U	U
F	T	T
F	F	T
F	U	T
U	T	T
U	F	U
U	U	U

The Law of Excluded Middle is not a tautology in this semantics because $U \vee \neg U = U$, but every tautology in this semantics is automatically a tautology in classical logic (by deleting rows with U).