Assignment 2: Cardinality (Draft)

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1. *Proof.* Consider the following functions

$$f(a,(b,c)) = ((a,b),c)$$
$$g((a,b),c) = (a,(b,c))$$

This pair of functions are inverses of each other since $f(g((a,b),c)) = f(a,(b,c)) = ((a,b),c)$ and $g(f(a,(b,c))) = g((a,b),c) = (a,(b,c))$, so we have the required bijection. □

2. *Proof.* Consider the following functions

$$\mathsf{curry}(f)(a)(b) = f(a,b)$$
$$\mathsf{uncurry}(g)(a,b) = g(a)(b)$$

This pair of functions are inverses of each other because $\mathsf{curry}(\mathsf{uncurry}(g))(a)(b) = \mathsf{uncurry}(g)(a,b) = g(a)(b)$. Functional extensionality implies that $\mathsf{curry}(\mathsf{uncurry}(g)) = g$. We also have $\mathsf{uncurry}(\mathsf{curry}(g))(a,b) = \mathsf{curry}(g)(a)(b) = g(a,b)$. Functional extensionality implies that $\mathsf{uncurry}(\mathsf{curry}(g)) = g$. We have the required bijection. □

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Proof. There is an obvious injection, namely the inclusion $A \hookrightarrow A \cup B$, so $\mathbb{N} \leq A \cup B$. It suffices to construct an injection $A \cup B \to \mathbb{N}$. Consider the following function

$$h(c) = \begin{cases} 2f(c) & \text{if } c \in A, \\ 2g(c)+1 & \text{o.w.} \end{cases}$$

blah blah blah so this is an injection. □

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Proof. It suffices to rule out $X \prec \mathbb{N}$. Suppose that $X \prec \mathbb{N}$. Let $f : X \to \mathbb{N}$ be an injection witnessing this fact. Consider $\mathsf{ran}(f) \subseteq \mathbb{N}$. If $\mathsf{ran}(f)$ is infinite, then we have $X \approx \mathsf{ran}(f) \approx \mathbb{N}$. This is a contradiction.

If $\mathsf{ran}(f)$ is finite, then X is finite, but this contradicts the assumption that X is infinite. □