Student Information

Name : Burak Metehan Tunçel

ID : 2468726

Answer 1

a)

In this drawing, there are 8 possible mutually exclusive events. The question states "at least one" condition. So, the following probabilities need to be considered. (W: Choosing White, B: Choosing Black)

1. $\boldsymbol{P}\{WBB\}$
3. $\boldsymbol{P}\{BBW\}$
5. $\boldsymbol{P}\{WBW\}$
7. $\boldsymbol{P}\{WWW\}$

2. $\boldsymbol{P}\{BWB\}$
4. $\boldsymbol{P}\{WWB\}$
6. $\boldsymbol{P}\{BWW\}$

Since these events are mutually exclusive, the sum of the probabilities of these 7 seven events is asked. Instead of calculating and adding the above probabilities, we can calculate

$$(\boldsymbol{P}\{\Omega\} - \boldsymbol{P}\{BBB\}) = (1 - \boldsymbol{P}\{BBB\})$$

The reason is that only condition that violates the 'at least' statement is the event "BBB", so subtracting its probability from 1 is enough.

$$\boldsymbol{P}\{\Omega\} - \boldsymbol{P}\{BBB\} = 1 - \left(\frac{8}{10} \cdot \frac{11}{15} \cdot \frac{9}{12} \right) = \frac{14}{25} = 0.56$$

b)

Again there are 8 possible mutually exclusive events. However, this time, only one of them is asked. This event is "WWW". So the probability of this event,

$$\boldsymbol{P}\{WWW\} = \frac{2}{10} \cdot \frac{4}{15} \cdot \frac{3}{12} = \frac{1}{75} = 0.01\overline{3}$$

c)

I would choose the **BOX 2**. The reason for this choice is that the highest probability for asked condition is on BOX 2. The probabilities for each box:

- **P**{Drawing two white balls from BOX 1}: $\dfrac{2}{10} \cdot \dfrac{1}{9} = \dfrac{1}{45} \approx 0.022$

- **P**{Drawing two white balls from BOX 2}: $\dfrac{4}{15} \cdot \dfrac{3}{14} = \dfrac{2}{35} \approx 0.057$

- **P**{Drawing two white balls from BOX 3}: $\dfrac{3}{12} \cdot \dfrac{2}{11} = \dfrac{1}{22} \approx 0.045$

d)

First BOX 2, then BOX 3.

Probability of choosing white ball in first draw:

- From BOX 1: $\dfrac{2}{10} = 0.20$

- From BOX 2: $\dfrac{4}{15} = 0.2666666667$

- From BOX 3: $\dfrac{3}{12} = 0.25$

Since, I would choose the **BOX 2**, in the first draw, because the highest probability for drawing white ball is in BOX 2.

After choosing the first white ball from BOX 2, probability of choosing white ball in second draw:

- From BOX 1: $\dfrac{2}{10} = 0.20$

- From BOX 2: $\dfrac{3}{14} = 0.2142857143$

- From BOX 3: $\dfrac{3}{12} = 0.25$

When one white ball is drawn from BOX 2, the highest probability for drawing white balls is in BOX 3. Therefore, in the second draw, I would choose **BOX 3**.

e)

To calculate the expected value, we need to calculate the probabilities of the following:

- $\boldsymbol{P}\{0 \text{ White Balls}\} = \boldsymbol{P}\{BBB\}$

- $\boldsymbol{P}\{1 \text{ White Balls}\} = \boldsymbol{P}\{WBB\} + \boldsymbol{P}\{BWB\} + \boldsymbol{P}\{BBW\}$

- $\boldsymbol{P}\{2 \text{ White Balls}\} = \boldsymbol{P}\{WWB\} + \boldsymbol{P}\{WBW\} + \boldsymbol{P}\{BWW\}$

- $\boldsymbol{P}\{3 \text{ White Balls}\} = \boldsymbol{P}\{WWW\}$

After the calculations of these probabilities, we need to use the expactation formula for discrete case because this is a discrete case by using the following:

$$\mu = \mathbf{E}(X) = \sum_x xP(x)$$

Starting by calculating the probabilities of the events.

- $\boldsymbol{P}\{0 \text{ White Balls}\} = 0.440 = \dfrac{8}{10} \cdot \dfrac{11}{15} \cdot \dfrac{9}{12} = \dfrac{11}{25}$

- $\boldsymbol{P}\{1 \text{ White Balls}\} = 0.41\overline{6} = \dfrac{2}{10} \cdot \dfrac{11}{15} \cdot \dfrac{9}{12} + \dfrac{8}{10} \cdot \dfrac{4}{15} \cdot \dfrac{9}{12} + \dfrac{8}{10} \cdot \dfrac{11}{15} \cdot \dfrac{3}{12} = \dfrac{5}{12}$

- $\boldsymbol{P}\{2 \text{ White Balls}\} = 0.130 = \dfrac{2}{10} \cdot \dfrac{4}{15} \cdot \dfrac{9}{12} + \dfrac{2}{10} \cdot \dfrac{11}{15} \cdot \dfrac{3}{12} + \dfrac{8}{10} \cdot \dfrac{4}{15} \cdot \dfrac{3}{12} = \dfrac{13}{100}$

- $\boldsymbol{P}\{3 \text{ White Balls}\} = 0.01\overline{3} = \dfrac{2}{10} \cdot \dfrac{4}{15} \cdot \dfrac{3}{12} = \dfrac{1}{75}$

By using this probabilities, we calculate the expected value:

$$\mu = 0 \cdot \frac{11}{25} + 1 \cdot \frac{5}{12} + 2 \cdot \frac{13}{100} + 3 \cdot \frac{1}{75} = \frac{43}{60} = 0.71\overline{6}$$

f)

We are asked $\boldsymbol{P}\{B|W\}$ if the following denotations are used,

- $B = \{\text{Choosing a ball from BOX 1}\}$

- $W = \{\text{Choosing a white ball}\}$

Intuitively, the following notion can be used

Suppose that all the balls are in a giant box, and all the balls have a number on them that indicates which box they are from. If we know that drawn ball is white the probability that the ball has number 1 on it is $\frac{2}{9}$.

Instead, we can also use "*Bayes Rule*" and "*Law of Total Probability*" to show that intuition was not wrong.

Before, note that we can acquire the followings:

- $\boldsymbol{P}\{B\} = \frac{10}{37}$: There are 37 balls in total and 10 of them are in BOX 1.

- $\boldsymbol{P}\{\overline{B}\} = \frac{27}{37}$: There 37 balls in total and 27 of them are not in BOX 1

- $\boldsymbol{P}\{W|B\} = \frac{2}{10}$: If a ball is drawn from BOX 1, the probability of it should be 2/10.

- $\boldsymbol{P}\{W|\overline{B}\} = \frac{7}{27}$: There are 27 balls that are not in BOX 1, and 7 of them are white.

Now, we can start using *Bayes Rule* and *Law of Total Probability*:

$$
\begin{aligned}
\boldsymbol{P}\{B|W\} &= \frac{\boldsymbol{P}\{W|B\}\boldsymbol{P}\{B\}}{\boldsymbol{P}\{W\}} && \textit{(Bayes Rules)} \\
&= \frac{\boldsymbol{P}\{W|B\}\boldsymbol{P}\{B\}}{\boldsymbol{P}\{W|B\}\boldsymbol{P}\{B\} + \boldsymbol{P}\{W|\overline{B}\}\boldsymbol{P}\{\overline{B}\}} && \textit{(Law of Total Probability)} \\
&= \frac{\frac{2}{10} \cdot \frac{10}{37}}{\frac{2}{10} \cdot \frac{10}{37} + \frac{7}{27} \cdot \frac{27}{37}} \\
&= \frac{2}{9} \approx 0.22
\end{aligned}
$$

The answer is "$\frac{2}{9}$"

Answer 2

For this question the following event denotations will be used.

$$D = \{\text{The ring is destroyed}\} \quad F = \{\text{Frodo is corrupted}\} \quad S = \{\text{Sam is corrupted}\}$$

a)

The followings are given:

- $\boldsymbol{P}\{D|\overline{S}\} = 0.9$
- $\boldsymbol{P}\{D|S\} = 0.5$
- $\boldsymbol{P}\{S\} = 0.1$

and the following is asked
$\boldsymbol{P}\{S|D\} = ?$

By using *Bayes Rule*, we can calculate the $\boldsymbol{P}\{S|D\}$:

$$\boldsymbol{P}\{S|D\} = \frac{\boldsymbol{P}\{D|S\}\boldsymbol{P}\{S\}}{\boldsymbol{P}\{D\}}$$

(We can use the **Law of Total Probability** to calculate the $\boldsymbol{P}\{D\}$)

$$= \frac{\boldsymbol{P}\{D|S\}\boldsymbol{P}\{S\}}{\boldsymbol{P}\{D|S\}\boldsymbol{P}\{S\} + \boldsymbol{P}\{D|\overline{S}\}\boldsymbol{P}\{\overline{S}\}}$$

$$= \frac{(0.5)(0.1)}{(0.5)(0.1) + (0.9)(0.9)} = 0.05813953488 \approx 0.058$$

b)

The followings are given:

- $\boldsymbol{P}\{F\} = 0.25$
- $\boldsymbol{P}\{D|(F \setminus S)\} = 0.2$
- $\boldsymbol{P}\{D|(\overline{F} \cap \overline{S})\} = 0.9$

- $\boldsymbol{P}\{S\} = 0.1$
- $\boldsymbol{P}\{D|(S \setminus F)\} = 0.5$
- $\boldsymbol{P}\{D|(F \cap S)\} = 0.05$

- The corruption of Frodo and Sam are independent events.

and the following is asked
$\boldsymbol{P}\{(F \cap S)|D\} = ?$

Since we know the independence from question, we can calculate $\boldsymbol{P}\{F \cap S\}$

$$\boldsymbol{P}\{F \cap S\} = \boldsymbol{P}\{F\}\boldsymbol{P}\{S\} = (0.25)(0.1) = 0.025 \tag{1}$$

Since F is a union of disjoint events $F \setminus S$ and $F \cap S$, and -similarly- S is a union of disjoint events $S \setminus F$ and $F \cap S$, we can compute,

$$\boldsymbol{P}\{F \setminus S\} = \boldsymbol{P}\{F\} - \boldsymbol{P}\{F \cap S\} = 0.25 - 0.025 = 0.225 \tag{2}$$

$$\boldsymbol{P}\{S \setminus F\} = \boldsymbol{P}\{S\} - \boldsymbol{P}\{F \cap S\} = 0.1 - 0.025 = 0.075 \tag{3}$$

Event $(F \setminus S)$, $(S \setminus F)$, $F \cap S$, and $\overline{(F \cup S)}$ form a partition Ω, because they are mutually exclusive and exhaustive. The last of them is the event of no corruption. Notice that F, S, and $(F \cap S)$ are neither mutually exclusive nor exhaustive, so they cannot be used for the Bayes Rules. Now orginze the data,

Corruption	Destroy
$\boldsymbol{P}\{F \setminus S\} = 0.225$	$\boldsymbol{P}\{D|(F \setminus S)\} = 0.2$
$\boldsymbol{P}\{S \setminus F\} = 0.075$	$\boldsymbol{P}\{D|(S \setminus F)\} = 0.5$
$\boldsymbol{P}\{F \cap S\} = 0.025$	$\boldsymbol{P}\{D|(F \cap S)\} = 0.05$
$\boldsymbol{P}\{\overline{F \cup S}\} = 0.675$	$\boldsymbol{P}\{D|(\overline{F \cup S})\} = 0.9$

Combining the *Bayes Rules* and the *Law of Total Probability*,

$$\boldsymbol{P}\{(F \cap S)|D\} = \frac{\boldsymbol{P}\{D|(F \cap S)\}\boldsymbol{P}\{F \cap S\}}{\boldsymbol{P}\{D\}}$$

where

$$\boldsymbol{P}\{D\} = \boldsymbol{P}\{D|(F \setminus S)\}\boldsymbol{P}\{F \setminus S\} + \boldsymbol{P}\{D|(S \setminus F)\}\boldsymbol{P}\{S \setminus F\}$$
$$+ \boldsymbol{P}\{D|(F \cap S)\}\boldsymbol{P}\{F \cap S\} + \boldsymbol{P}\{D|(\overline{F \cup S})\}\boldsymbol{P}\{\overline{F \cup S}\}$$

Then

$$\boldsymbol{P}\{(F \cap S)|D\} = \frac{(0.05)(0.025)}{(0.2)(0.225) + (0.5)(0.075) + (0.05)(0.025) + (0.9)(0.675)}$$
$$= 0.001808318264 \approx 0.0018$$

Answer 3

By using the data provided in PDF, we can construct a table by organizing the joint pmf of X and Y. So, in this way, we get the marginal pmf. (This table is similar to the table in example 3.6 in book.)

$P_{(A,I)}(a,i)$		a			$P_I(i)$
		1	**2**	**3**	
i	**1**	0.18	0.3	0.12	0.6
	2	0.12	0.2	0.08	0.4
$P_A(a)$		0.3	0.5	0.2	1.0

Table 1

a)

There are 2 cases that satisfy the conditon *four snowy days*:

- $a = 2,\ i = 2$
- $a = 3,\ i = 1$

So we need to calculate the followings:

- $P_{(A,I)}(2,2)$
- $P_{(A,I)}(3,1)$

From Table 1, we can read these probabilities.

$$\boldsymbol{P}\{\text{Four Snowy Days}\} = P_{(A,I)}(2,2) + P_{(A,I)}(3,1) = 0.2 + 0.12 = 0.32$$

The answer is "**0.32**".

b)

To decide on the independence of A and I, we need to check if their joint pmf factors into a product of marginal pmfs. If there is at least one pair that violates the equality, it showing violation is enough to porve the dependence.

Therefore, we need to check all pairs.

- $P_{(A,I)}(1,1) = 0.3 \cdot 0.6 = 0.18$
- $P_{(A,I)}(1,2) = 0.3 \cdot 0.4 = 0.12$
- $P_{(A,I)}(2,1) = 0.5 \cdot 0.6 = 0.30$

- $P_{(A,I)}(2,2) = 0.5 \cdot 0.4 = 0.20$
- $P_{(A,I)}(3,1) = 0.2 \cdot 0.6 = 0.12$
- $P_{(A,I)}(3,2) = 0.2 \cdot 0.4 = 0.08$

All pairs satisfy the condition for independence. There is no pair to violate the equality mentioned above.

Answer is "**Independent**".