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Answer 1

$a_n = a_{n-1} + 2^n, n \geq 1$ and $a_0 = 1$ are given. It is not hard to see that

$$a_n - a_{n-1} = 2^n \qquad\qquad n \geq 1 \tag{1}$$

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function for the sequence $\{a_n\}$. Also note that,

$$xG(x) = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n$$

Using the recurrence relation, we see that,

$$\begin{aligned}
G(x) - xG(x) &= \sum_{n=0}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n \\
&= a_0 + \sum_{n=1}^{\infty} (a_n - a_{n-1}) x^n \\
&= 1 + \sum_{n=1}^{\infty} 2^n x^n && \text{From (1)} \\
&= \sum_{n=0}^{\infty} 2^n x^n && 1 + \sum_{n=1}^{\infty} 2^n x^n = 2x + 2^2x^2 + \cdots \\
&= \frac{1}{1-2x} && \text{From Table 1 in Chapter 8 in textbook}
\end{aligned}$$

So, we have the following,

$$\begin{aligned}
G(x) - xG(x) &= \frac{1}{1-2x} \\
G(x)(1-x) &= \frac{1}{1-2x} \\
G(x) &= \frac{1}{(1-2x)(1-x)} \qquad (2)
\end{aligned}$$

Also, note that

$$\frac{1}{(1-2x)(1-x)} = \frac{2}{1-2x} - \frac{1}{1-x} \tag{3}$$

By combining the (2) and (3), we get the following:

$$G(x) = \frac{2}{1-2x} - \frac{1}{1-x} \tag{4}$$

According to **Table 1 in Chapter 8 in textbook**

$$\frac{2}{1-2x} = 2\sum_{n=0}^{\infty} 2^n x^n = \sum_{n=0}^{\infty} 2^{n+1} x^n \tag{5}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \tag{6}$$

When we insert the (5) and (6) into (4), we get

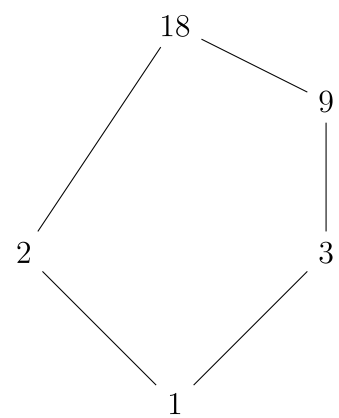
$$G(x) = \sum_{n=0}^{\infty} 2^{n+1} x^n - \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} (2^{n+1} - 1) x^n \tag{7}$$

From (7), it can be said that,

$$a_n = 2^{n+1} - 1$$

Answer 2

a)



b)

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

c)

It is a lattice because every pair of elements has both a least upper bound and a greatest lower bound. There is not pair to cause contrary to that fact.

d)

R_S, the symmetric closure of R, is,

$$R_S = R \cup R^{-1} = \{(a,\ b) \mid a \text{ divides } b\} \cup \{(a,\ b) \mid b \text{ divides } a\}$$

So, the matrix representation of the R_S as follows

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

e)

The integers 2 and 9 are not comparable because $2 \nmid 9$ which means 2 does not divide 9. However, 3 and 18 are comparable because $3|18$ which means 3 divides 18.

Answer 3

a)

Answer: $2^n \cdot 3^{(n^2-n)/2}$

Explanation:
The number of antisymmetric binary relations possible in A is $2^n \cdot 3^{(n^2-n)/2}$. For antisymmetric relation, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ when $a, b \in A$.

When matrix representation of relation is considered, antisymmetric means that if there is (a_i, a_j) from the lower triangle of the matrix, then (a_j, a_i) from the upper triangle should not be present in R and vice versa. Therefore, there are three possibilities for each (a_i, a_j).

That is, either (a_i, a_j) is in the relation or (a_j, a_i) is in the relation, or none of the $(a_i, a_j), (a_j, a_i)$ is in the relation. There are $(n^2 - n)/2$ pairs for (a_i, a_j) such that $i \neq j$. Therefore, there are $3^{(n^2-n)/2}$ antisymmetric binary relations.

Furthermore, any subset of the diagonal elements is also an antisymmetric relation. Therefore the number of antisymmetric binary relations is $2^n \cdot 3^{(n^2-n)/2}$.

b)

Answer: $3^{(n^2-n)/2}$

The number of binary relations which are both reflexive and antisymmetric in the set A is $3^{(n^2-n)/2}$.

All diagonal elements are part of the reflexive relation and there are 3 possibilities for each of the remaining $(n^2 - n)/2$ elements. Thus, we get $3^{(n^2-n)/2}$ binary relations which are reflexive and antisymmetric.

In other words, the antisymmetric part of this question is explained in the previous sub-question. In the last part of the previous sub-question, we think the any subset of the diagonal elements; however, only one subset of the diagonal elements is our concern because of the reflexive property. Therefore, there is $3^{(n^2-n)/2}$ binary relations which are reflexive and antisymmetric.