CENG 223

Fall 2021-2022

THE3 - Sample Solutions

Question 1

By the well-ordering property, the set of positive integers has a least element, m. First we note that m cannot be larger than 1: 1 itself is a positive integer. Also m cannot be equal to or smaller than 0: that would make m a non-positive integer. Then we have $0 < m < 1$. Multiplying all sides by positive integer powers of m, we get $0 < m^{n+1} < m^n < m^{n-1} < \cdots < m < 1$. Since m is assumed to be a positive integer, by the provided hint, all of its positive integer powers are positive integers as well. Then there should exist an infinite number of positive integers ($\{m^{n+1} \,|\, n \in \mathbb{Z}^+ - \{1\}\}$) that are smaller than m. Hence, a contradiction.

Note that showing the above for only $n = 1$ is also acceptable.

Question 2

$S(1,1)$ declares number of different solutions to $x_1 = 1$ is 1, which is true as any $x_1 \neq 1$ would violate the equality. So we have the basis step established for both $S(m,1)$ and $S(1,n)$.

Let us first prove $S(m,1)$. Assume $S(m-1,1)$ is true. Then there are

$$f(m-1,1) = \frac{(m-1+1-1)!}{1!(m-1-1)!} = m-1$$

solutions to $x_1 + \cdots + x_{m-1} = 1$. For $S(m,1)$, we have $x_1 + \cdots + x_m = 1$. Taking $x_m = 0$ yields $f(m-1,1)$-many solutions, and $x_m = 1$ renders all other x_i vanish, thus, yields only 1 solution. Therefore, $f(m,1) = f(m-1,1) + 1 = (m-1) + 1 = m = \frac{m!}{1!(m-1)!}$.

Next is $S(1,n)$. Assume $S(1,n-1)$ is true, *i.e.*, there are $f(1,n-1) = 1$ solution to $x_1 = n-1$. Then $S(1,n)$ states that number of solutions to $x_1 = n$ should be $f(1,n) = 1$. For the purpose of induction, convert this problem to $x_1 = (n-1)+1$ and define $x_1' = x_1 - 1$. Then $x_1' = n-1$ is the resulting problem which by the induction assumption admits only one solution. Since x_1 is completely determined by x_1', indeed we have $f(1,n) = 1$, and as such $S(1,n)$ is proved by induction as well.

For the above part, solutions that did not properly apply mathematical induction will also be accepted.

We now turn to prove $S(m+1,n+1)$. Assume $S(m,n+1)$ and $S(m+1,n)$ are true. That is,

$$x_1 + \cdots + x_m = n+1 \quad \texttt{has} \quad f(m,n+1) = \frac{(n+m)!}{(n+1)!\,(m-1)!} \texttt{-many solutions},$$

$$\texttt{and} \quad x_1 + \cdots + x_{m+1} = n \quad \texttt{has} \quad f(m+1,n) = \frac{(n+m)!}{n!\,m!} \texttt{-many solutions}.$$

For $x_1 + \cdots + x_m + x_{m+1} = n+1$ we can divide the problem into two parts for the sake of induction: $x_{m+1} = 0$ and $x_{m+1} > 0$.

The first case corresponds to $S(m, n+1)$ and we have $f(m, n+1)$-many solutions.

For the second case we can rewrite the problem as $x_1 + \cdots + (x'_{m+1} + 1) = n+1$ where $x'_{m+1} = x_{m+1} - 1$. That is, $x_1 + \cdots + x'_{m+1} = n$. By doing so we inherently satisfied the condition of $x_{m+1} > 0$ and converted the problem into the one stated by $S(m+1, n)$. As such we get $f(m+1, n)$-many solutions.

Then, the total number of solutions to $x_1 + \cdots + x_{m+1} = n+1$ is

$$f(m, n+1) + f(m+1, n) = \frac{(n+m)!}{(n+1)!\,(m-1)!} + \frac{(n+m)!}{n!\,m!} = \frac{(n+m)!\,m}{(n+1)!\,m!} + \frac{(n+m)!\,(n+1)}{(n+1)!\,m!}$$

$$= \frac{(n+m)!\,(n+1+m)}{(n+1)!\,m!} = \boxed{\frac{(n+m+1)!}{(n+1)!\,m!} = f(m+1, n+1)}.$$

Question 3

a. In each square that do not encapsulate a grid point, we have 4 congruent triangles to the one provided in the figure. As there are $N = 6$ rows of squares with n squares in the nth row, there a total of $N\,(N+1)/2 = 21$ squares, meaning 84 triangles are present. There are also congruent triangles appearing at the end of each row of squares, and one at the top. Thus we have $N+1$ triangles more to count. This gives us a total of $84 + 7 = 91$ triangles.

b. Using the inclusion-exclusion principle, number of onto functions are given by $\sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m$ where $m = 6$ and $n = 4$. Then there are

$$\binom{4}{0} 4^6 - \binom{4}{1} 3^6 + \binom{4}{2} 2^6 - \binom{4}{3} 1^6 + \binom{4}{4} 0^6 = 1560$$

Solutions that use Stirling numbers are also accepted.

Question 4

a. Let a_n denote the number of strings of length n that contain two consecutive symbols that are the same. Then the number of strings of length n that do not satisfy this property is given by $3^n - a_n$.

To acquire a recurrence relation for a_n we can either concatenate any of $\{0, 1, 2\}$ to the strings counted in a_{n-1}, or we can concatenate the last digit of a string counted in $3^{n-1} - a_{n-1}$. With this we establish

$$a_n = 3a_{n-1} + (3^{n-1} - a_{n-1}) = 2a_{n-1} + 3^{n-1}.$$

b. The question of number of strings satisfying the given property makes sense for $n \geq 0$ where $a_0 = 0$, $a_1 = 0$, $a_2 = 3$ and so on. Note that the logic of the established recurrence relation requires a_{n-1} to have at least a digit, so that it can be repeated. Thus, we expect $a_0 = 0$ and $a_1 = 0$ to be sufficient as initial conditions, and the rest of the numbers can be acquired by the recurrence relation. So formal description of the recurrence relation is

$$a_n = 2a_{n-1} + 3^{n-1} \texttt{ for } n \geq 2, \quad a_0 = a_1 = 0.$$

c. For the homogeneous solution we have $a_n = 2a_{n-1}$, characteristic equation of which yields $r = 2$. Then $a_n^{(h)} = A\,2^n$ where A is some constant to be determined later on.

For the particular solution we guess the form of solution as $a_n^{(p)} = B\,3^n$. Then

$$B3^n = 2B\,3^{n-1} + 3^{n-1} \;\leftrightarrow\; 3B\,3^{n-1} = 2B\,3^{n-1} + 3^{n-1} \;\leftrightarrow\; 3B = 2B + 1 \;\leftrightarrow\; B = 1.$$

Thus, $a_n^{(p)} = 3^n$.

Then our solution $a_n = a_n^{(h)} + a_n^{(p)} = A\,2^n + 3^n$ can be determined with the initial condition $a_1 = 0$.

$$a_1 = A\,2^1 + 3^1 = 2A + 3 = 0 \;\leftrightarrow\; A = -3/2 \;\leftrightarrow\; \boxed{a_n = -\frac{3}{2}\,2^n + 3^n}.$$

Solutions that use a_2, a_3 etc. to determine the undetermined constant A are also accepted.