Student Information

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Answer 1

a)

*They are **not independent***. Independence formula for continuous variables is

$$f_{(X,Y)}(x,y) = f_X(x)f_Y(y)$$

So, we need to find equations for $f_X(x)$ and $f_Y(y)$.

Finding $f_X(x)$

$$f_X(x) = \int_{-\infty}^{\infty} f_{(X,Y)}(x,y)dy$$

We can rewrite the function $f_{(X,Y)}(x,y)$ as follows,

$$f_{(X,Y)}(x,y) = \begin{cases} \dfrac{1}{\pi} & \text{if } -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2} \\ 0 & \text{otherwise} \end{cases}$$

One can say that for otherwise part, the integral is 0. For other part,

$$\begin{aligned} f_X(x) &= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f_{(X,Y)}(x,y)dy \\ &= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}dy \\ &= \frac{2\sqrt{1-x^2}}{\pi} & -1 \leq x \leq 1 \end{aligned}$$

Finding $f_Y(y)$

Similarly, we can rewrite the function $f_{(X,Y)}(x,y)$ as follows,

$$f_{(X,Y)}(x,y) = \begin{cases} \dfrac{1}{\pi} & \text{if } -\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2} \\ 0 & \text{otherwise} \end{cases}$$

From here,

$$\begin{aligned}
f_Y(y) &= \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f_{(X,Y)}(x,y)dx \\
&= \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi}dx \\
&= \frac{2\sqrt{1-y^2}}{\pi} && -1 \le y \le 1
\end{aligned}$$

In the end, we have

$$f_X(x) = \begin{cases} \dfrac{2\sqrt{1-x^2}}{\pi} & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \dfrac{2\sqrt{1-y^2}}{\pi} & -1 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

So back to formula

$$\left[f_{(X,Y)}(x,y) = \begin{cases} \dfrac{1}{\pi} & x^2+y^2 \le 1 \\ 0 & \text{otherwise} \end{cases} \right] \neq \left[f_X(x)f_Y(y) = \begin{cases} \dfrac{2\sqrt{1-x^2}}{\pi} \cdot \dfrac{2\sqrt{1-y^2}}{\pi} & -1 \le x \le 1, -1 \le y \le 1 \\ 0 & \text{otherwise} \end{cases} \right.$$

Also, we can show that they are dependent if we find at least 1 pair that violates the independecy equation. Let $x = y = \dfrac{\sqrt{3}}{2}$, then

$$f_{(X,Y)}\left(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right) = \frac{1}{\pi} \neq \frac{1}{\pi^2} = f_X\left(\frac{\sqrt{3}}{2}\right) f_Y\left(\frac{\sqrt{3}}{2}\right)$$

b)

From part a,

$$f_X(x) = \begin{cases} \dfrac{2\sqrt{1-x^2}}{\pi} & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \dfrac{2\sqrt{1-y^2}}{\pi} & -1 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

c)

Since our pdf is piecewise, we need the calculate each piece one by one,

$$
\begin{aligned}
\mathbf{E}(X) = \mu &= \int_{-\infty}^{\infty} x f_X(x) dx \\
&= \int_{-\infty}^{-1} x f_X(x) dx + \int_{-1}^{1} x f_X(x) dx + \int_{1}^{\infty} x f_X(x) dx \\
&= \int_{-\infty}^{-1} x 0 dx + \int_{-1}^{1} x \frac{2\sqrt{1-x^2}}{\pi} dx + \int_{1}^{\infty} x 0 dx \\
&= \int_{-\infty}^{-1} 0 dx + \int_{-1}^{1} x \frac{2\sqrt{1-x^2}}{\pi} dx + \int_{1}^{\infty} 0 dx \\
&= 0 + \left[-\frac{2}{\pi} \cdot \frac{1}{3} \left(1-x^2\right)^{\frac{3}{2}} \right] \Bigg|_{-1}^{1} + 0 \\
&= \left[-\frac{2}{\pi} \cdot \frac{1}{3} \left(1-1^2\right)^{\frac{3}{2}} \right] - \left[-\frac{2}{\pi} \cdot \frac{1}{3} \left(1-(-1)^2\right)^{\frac{3}{2}} \right] \\
&= 0 - 0 = 0
\end{aligned}
$$

d)

Since our pdf is piecewise, we need the calculate each piece one by one,

$$
\begin{aligned}
\operatorname{Var}(X) &= \mathbf{E}\left(X - \mu^2\right)^2 \\
&= \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x) dx \qquad (\mu = 0) \\
&= \int_{-\infty}^{\infty} x^2 f_X(x) dx \\
&= \int_{-\infty}^{-1} x^2 f_X(x) dx + \int_{-1}^{1} x^2 f_X(x) dx + \int_{1}^{\infty} x^2 f_X(x) dx \\
&= 0 + \int_{-1}^{1} x^2 \frac{2\sqrt{1-x^2}}{\pi} dx + 0 \\
&= \left[\frac{1}{16\pi} \left(4\arcsin(x) - \sin(4\arcsin(x))\right) \right] \Bigg|_{-1}^{1} \\
&= \left[\frac{1}{16\pi} \left(4\arcsin(1) - \sin(4\arcsin(1))\right) \right] - \left[\frac{1}{16\pi} \left(4\arcsin(-1) - \sin(4\arcsin(-1))\right) \right] \\
&= \left(\frac{1}{8}\right) - \left(-\frac{1}{8}\right) \\
&= \frac{2}{8} = \frac{1}{4}
\end{aligned}
$$

Answer 2

a)

Since T_A and T_B are uniformly distributed

$$f_{T_A}(t_A) = \frac{1}{100-0} = \frac{1}{100}$$
$$f_{T_B}(t_B) = \frac{1}{100-0} = \frac{1}{100}$$

Also, since T_A and T_B are indenpendent,

$$f_{(T_A,T_B)}(t_A, t_B) = f_{T_A}(t_A) f_{T_B}(t_B) = \frac{1}{100} \cdot \frac{1}{100} = \frac{1}{10000} = 10^{-4}$$

We found joinst density function. Now we can find joint cdf. Let $u = t_A$, $v = t_B$, then

$$\begin{aligned}
F_{(X,Y)}(x,y) &= \int_0^y \int_0^x f_{(X,Y)}(u,v)dudv \\
&= \int_0^y \int_0^x 10^{-4} dudv \\
&= \int_0^y x \cdot 10^{-4} dv \\
&= xy \cdot 10^{-4}
\end{aligned}$$

b)

In joint cumulative function of two random variables X and Y, $F_{(X,Y)}(x,y) = F_X(x)F_Y(y)$ if X and Y are independent. Also, notice that it happened in this question.

$$F_{T_A}(t_a) = \int_0^{t_a} f_{T_A}(u)du = t_a \cdot 10^{-2}$$
$$F_{T_B}(t_b) = \int_0^{t_b} f_{T_B}(u)du = t_B \cdot 10^{-2}$$
$$F_{(T_A,T_B)}(t_A, t_B) = F_{T_A}(t_a) \cdot F_{T_B}(t_b) = t_a t_b \cdot 10^{-4}$$

This fact will be used in the following part.

We are asked the following probability

$$\boldsymbol{P}\left\{0 \le T_A \le 10,\ 90 \le T_B \le 100\right\}$$

We can write it as follows,

$$\boldsymbol{P}\left\{0 \le T_A \le 10,\ 90 \le T_B \le 100\right\} = F_{(T_A,T_B)}(t_A, t_B) = F_{T_A}(t_a) \cdot F_{T_B}(t_b)$$

Calculating $F_{T_A}(t_a)$

$$F_{T_A}(t_a) = \int_0^{10} f_{T_A}(u)du = \int_0^{10} 10^{-2}du = 10^{-2} \cdot 10 = 10^{-1}$$

Calculating $F_{T_B}(t_b)$

$$F_{T_B}(t_b) = \int_{90}^{100} f_{T_B}(u)du = \int_{90}^{100} 10^{-2}du = 10^{-2} \cdot 10 = 10^{-1}$$

Then,

$$F_{T_A}(t_a) \cdot F_{T_B}(t_b) = 10^{-1} \cdot 10^{-1} = 10^{-2}$$

So the answer is $\dfrac{1}{100}$

c)

If we think like the way mentioned in the hint. We are asked the ratio of the purple area to all area $[100 \times 100]$ calculate the probability of the area of the Figure 1. The area is the following

$$\boldsymbol{P}\left\{(T_A - T_B \leq 20) \cap (T_B \leq T_A)\right\}$$

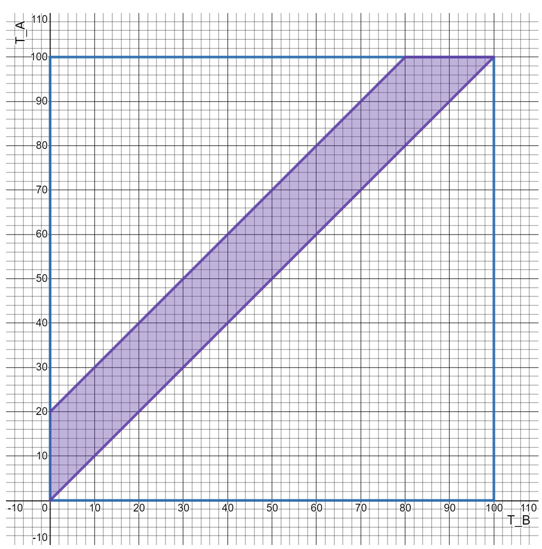

Figure 1

- The area of purple part $= 1800$
- The area of the whole part $= 10^4$

So, the ratio of the purple area to whole area is the answer and it is $\dfrac{18}{100} = 0.18$.

d)

Again, if we think like the way mentioned in the hint. We are asked the ratio of the purple area to all area [100 × 100] calculate the probability of the area of the Figure 2. The area is the following

$$\boldsymbol{P}\left\{(|T_A - T_B| \leq 30)\right\}$$

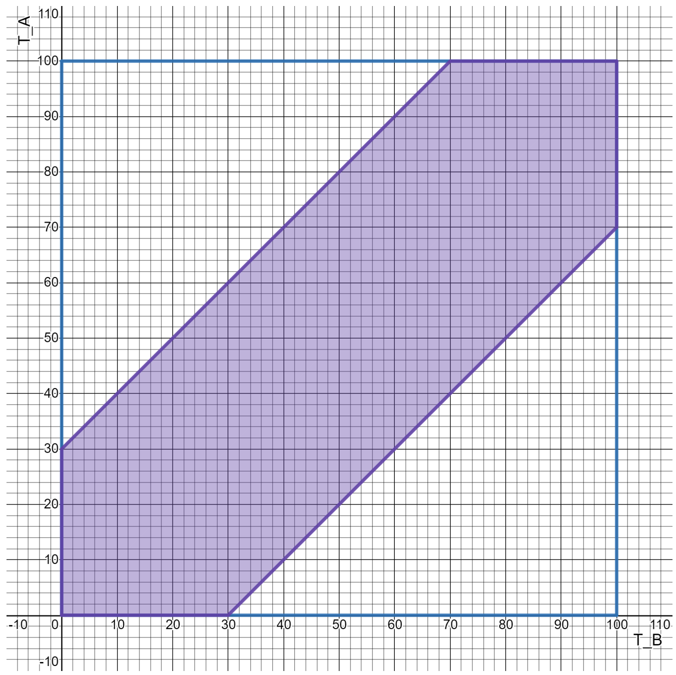

Figure 2

- The area of purple part = 5100
- The area of the whole part = 10^4

So, the ratio of the purple area to whole area is the answer and it is $\dfrac{51}{100} = 0.51$.

Answer 3

a)

For a random variable X_i, it has cdf

$$F_{X_i}(x) = \boldsymbol{P}\{X_i \leq x\} = 1 - e^{-\lambda_i x_i} \qquad\qquad x > 0$$

for $i = 1, 2, ..., N$. Also, $T = \min(X_1, X_2, ..., X_N)$. Then the CDF of T is

$$\begin{aligned}
F_T(t) &= \boldsymbol{P}\{T \leq t\} \\
&= 1 - \boldsymbol{P}\{T \geq t\} \\
&= 1 - \boldsymbol{P}\{\min(X_1, X_2, ..., X_N) \geq t\} \\
&= 1 - \boldsymbol{P}\{X_1 \geq t, X_2 \geq t, ..., X_N \geq t\} \\
&= 1 - \boldsymbol{P}\{X_1 \geq t\}\boldsymbol{P}\{X_2 \geq t\} \cdots \boldsymbol{P}\{X_N \geq t\} \\
&= 1 - e^{\lambda_1 t} e^{\lambda_2 t} \cdots e^{\lambda_N t} \\
&= 1 - e^{-\sum_{i=1}^{N} \lambda_i t} \qquad t > 0
\end{aligned}$$

The answer is

$$F_T(t) = 1 - e^{-(\lambda_1 + \cdots + \lambda_N)t}$$

b)

From question, there are 10 $\mathbf{E}(X_i)$'s:

- $\mathbf{E}(X_1) = \frac{10}{1}$
- $\mathbf{E}(X_2) = \frac{10}{2}$
- $\mathbf{E}(X_3) = \frac{10}{3}$
- $\mathbf{E}(X_4) = \frac{10}{4}$
- $\mathbf{E}(X_5) = \frac{10}{5}$
- $\mathbf{E}(X_6) = \frac{10}{6}$
- $\mathbf{E}(X_7) = \frac{10}{7}$
- $\mathbf{E}(X_8) = \frac{10}{8}$
- $\mathbf{E}(X_9) = \frac{10}{9}$
- $\mathbf{E}(X_{10}) = \frac{10}{10}$

Since $\lambda_i = \frac{1}{\mathbf{E}(X_i)}$, from question, there are 10 λ's:

- $\lambda_1 = \frac{1}{10}$
- $\lambda_2 = \frac{2}{10}$
- $\lambda_3 = \frac{3}{10}$
- $\lambda_4 = \frac{4}{10}$
- $\lambda_5 = \frac{5}{10}$
- $\lambda_6 = \frac{6}{10}$
- $\lambda_7 = \frac{7}{10}$
- $\lambda_8 = \frac{8}{10}$
- $\lambda_9 = \frac{9}{10}$
- $\lambda_{10} = \frac{10}{10}$

We found the CDF in part a. By using that CDF, we can calculate $\mathbf{E}(X)$ and we reach

$$\mathbf{E}(X) = \frac{1}{\lambda_1 + \cdots + \lambda_N}$$

So, when we use $\mathbf{E}(X)$ for 10 variables in question

$$\mathbf{E}(X) = \frac{1}{\frac{1}{10} + \frac{2}{10} + \frac{3}{10} + \frac{4}{10} + \frac{5}{10} + \frac{6}{10} + \frac{7}{10} + \frac{8}{10} + \frac{9}{10} + \frac{10}{10}} = \frac{2}{11}$$

Answer 4

According to Central Limit Theorem, if n is sufficiently large and p satisfies the condition $0.05 \leq p \leq 0.95$, then all distribution can be thought as Normal distribution. Since all variables are Binomial, we can use the following

$$\text{Binomial}(n, p) \approx \text{Normal}\left(\mu = np, \sigma = \sqrt{np(1-p)}\right)$$

a)

We need to determine n, p and, by using these, we need to find μ, σ

- $n = 100$
- $p = 0.74$
- $\mu = np = 74$
- $\sigma = \sqrt{np(1-p)} \approx 4.39$

And we are asked the following, ($X =$ the number of undergraduate students in the group.)

$$\begin{aligned}
\boldsymbol{P}\{X \geq 70\} &= \boldsymbol{P}\{X \geq 69.5\} && (\textit{Continuity correction}) \\
&= 1 - \boldsymbol{P}\{X \leq 69.5\} \\
&= 1 - \boldsymbol{P}\left\{\frac{X-\mu}{\sigma} \leq \frac{69.5-\mu}{\sigma}\right\} \\
&= 1 - \boldsymbol{P}\left\{\frac{X-74}{4.38634244} \leq \frac{69.5-74}{4.38634244}\right\} \\
&= 1 - \boldsymbol{P}\left\{Z \leq \frac{69.5-74}{4.38634244}\right\} \\
&= 1 - \boldsymbol{P}\{Z \leq -1.025911693\} \\
&= 1 - \Phi(-1.025911693) \\
&= 1 - 0.1525 = 0.8475 \approx 0.85
\end{aligned}$$

b)

We need to determine n, p and, by using these, we need to find μ, σ

- $n = 100$
- $p = 0.10$
- $\mu = np = 10$
- $\sigma = \sqrt{np(1-p)} = 3$

And we are asked the following, (X = the number of people pursuing a doctoral degree in the group.)

$$\begin{aligned}
\boldsymbol{P}\left\{X \leq 5\right\} &= \boldsymbol{P}\left\{X \leq 5.5\right\} && (\textit{Continuity correction}) \\
&= \boldsymbol{P}\left\{\frac{X-\mu}{\sigma} \leq \frac{5.5-\mu}{\sigma}\right\} \\
&= \boldsymbol{P}\left\{\frac{X-10}{3} \leq \frac{5.5-10}{3}\right\} \\
&= \boldsymbol{P}\left\{Z \leq -1.50\right\} \\
&= \Phi(-1.50) \\
&= 0.066807 \approx 0.07
\end{aligned}$$