THE 4 Solutions

Answer 1

(30 pts) Let $G(x)$ be the generating function for the sequence $\{a_n\}$, that is, $G(x) = \sum_{n=0}^{\infty} a_n x^n$.

First note that

$$xG(x) = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n.$$

Using the recurrence relation, we see that

$$\begin{aligned} G(x) - xG(x) &= \sum_{n=0}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n \\ &= a_0 + \sum_{n=1}^{\infty} (a_n - a_{n-1}) x^n \\ &= 1 + \sum_{n=1}^{\infty} 2^n x^n, \end{aligned}$$

because $a_0 = 1$ and $a_n - a_{n-1} = 2^n$.

Also we know that $1/(1 - ax) = \sum_{n=0}^{\infty} a^n x^n$.

Therefore, we get

$$\begin{aligned} G(x) - xG(x) &= 1 + \sum_{n=0}^{\infty} 2^n x^n - 1 \\ &= \sum_{n=0}^{\infty} 2^n x^n \\ &= \frac{1}{1 - 2x}. \end{aligned}$$

Solving for $G(x)$ shows that

$$G(x) = \frac{1}{(1 - 2x)(1 - x)}.$$

Expanding the right-hand side of this equation into partial fractions gives

$$G(x) = \frac{2}{1 - 2x} - \frac{1}{1 - x}.$$

Using identities $1/(1 - ax) = \sum_{n=0}^{\infty} a^n x^n$ and $1/(1 - x) = \sum_{n=0}^{\infty} x^n$ gives

$$G(x) = \sum_{n=0}^{\infty} (2^{n+1} - 1) x^n.$$

Consequently, we have shown that

$$a_n = 2^{n+1} - 1.$$

Answer 2

a)

(8 pts)

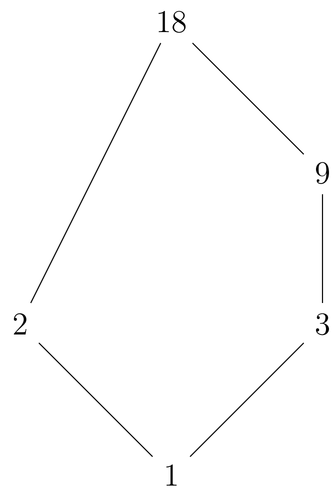

Figure 1: Hasse Diagram.

b)

(8 pts)

$$R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 9 \\ 18 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 9 & 18 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

c)

(8 pts) Yes, for every pair of elements there is a least upper bound and a greatest lower bound.

d)

(8 pts) We need to add $(2,1)$,$(3,1)$,$(9,1)$,$(18,1)$,$(18,2)$,$(9,3)$,$(18,3)$ and $(18,9)$ to produce R_s.

$$R_s = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 9 \\ 18 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 9 & 18 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

e)

(8 pts) The integers 2 and 9 are incomparable, because $2 \nmid 9$ and $9 \nmid 2$. The integers 3 and 18 are comparable, because $3 \mid 9$.

Answer 3

a)

(15 pts) $A = \{a_1, a_2, \cdots, a_n\}$

$$R = \begin{array}{c c} & \begin{array}{ccccc} a_1 & & \cdots & & a_n \end{array} \\ \begin{array}{c} a_1 \\ \\ \vdots \\ \\ a_n \end{array} & \begin{bmatrix} x & & & & \\ & x & & e_{i,j} & \\ & & \ddots & & \\ & e_{j,i} & & \ddots & \\ & & & & x \end{bmatrix} \end{array}$$

The diagonal elements x's can be 0 or 1, so there are 2^n possibilities.

For the other cells, there are three possibilities for each pair, because $(e_{i,j}, e_{j,i})$ can be equal to $(0,0)$, $(0,1)$ or $(1,0)$. The number of ways to select a pair from n elements is equal to $\binom{n}{2} = \frac{n(n-1)}{2}$. So for this condition we get $3^{\frac{n(n-1)}{2}}$ possibilities.

Therefore, the number of different **anti-symmetric** binary relations on A is $2^n \times 3^{\frac{n(n-1)}{2}}$.

b)

(15 pts)

$$R = \begin{array}{c c} & \begin{array}{ccccc} a_1 & & \cdots & & a_n \end{array} \\ \begin{array}{c} a_1 \\ \\ \vdots \\ \\ a_n \end{array} & \begin{bmatrix} 1 & & & & \\ & 1 & & e_{i,j} & \\ & & \ddots & & \\ & e_{j,i} & & \ddots & \\ & & & & 1 \end{bmatrix} \end{array}$$

All the diagonal elements must be 1 in order for the relation to be reflexive, so we don't apply 2^n as we do in part a).

The rest is same as part a). For the other cells, there are three possibilities for each pair, because $(e_{i,j}, e_{j,i})$ can be equal to $(0,0)$, $(0,1)$ or $(1,0)$. The number of ways to select a pair from n elements is equal to $\binom{n}{2} = \frac{n(n-1)}{2}$. So for this condition we get $3^{\frac{n(n-1)}{2}}$ possibilities.

Therefore, the number of different **reflexive** and **anti-symmetric** binary relations on A is $3^{\frac{n(n-1)}{2}}$.

Answer 4(self-study, ungraded)

Start with the following identity:

$$< 1, 1, 1, 1, 1, \cdots > \;\leftrightarrow\; \frac{1}{1-x}$$

Take derivative:

$$< 1, 2, 3, 4, 5, \cdots > \;\leftrightarrow\; \frac{1}{(1-x)^2}$$

Shift right:

$$< 0, 1, 2, 3, 4, \cdots > \;\leftrightarrow\; \frac{x}{(1-x)^2}$$

Scale with 3:

$$< 0, 3, 6, 9, 12, \cdots > \;\leftrightarrow\; \frac{3x}{(1-x)^2}$$

Add the first identity we used to the last one we found:

$$< 1, 4, 7, 10, 13, \cdots > \;\leftrightarrow\; \frac{2x+1}{(1-x)^2}$$