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Q. 1

Given the sets A and B, prove that

$$(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$$

using set membership notation and logical equivalences. Show each step clearly.

S. 1

Prove the equation

$$(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$$

$$
\begin{aligned}
&(A \cup B) \setminus (A \cap B) \\
&= \{x \mid x \in (A \cup B) \land x \notin (A \cap B)\} && \text{(Definition of difference)} \\
&= \{x \mid (x \in A \lor x \in B) \land x \notin (A \cap B)\} && \text{(Definition of Union)} \\
&= \{x \mid (x \in A \lor x \in B) \land \neg(x \in (A \cap B))\} && \text{(Definition of } \notin\text{)} \\
&= \{x \mid (x \in A \lor x \in B) \land \neg(x \in A \land x \in B))\} && \text{(Definition of Intersection)} \\
&= \{x \mid (x \in A \lor x \in B) \land (\neg(x \in A) \lor \neg(x \in B))\} && \text{(De Morgan's for Logical Rules)} \\
&= \{x \mid (x \in A \lor x \in B) \land (x \notin A \lor x \notin B)\} && \text{(Definition of } \notin\text{)} \\
&= \{x \mid ((x \in A \lor x \in B) \land x \notin A) \lor ((x \in A \lor x \in B) \land x \notin B)\} && \text{(Definition of Distributive Law)} \\
&= \{x \mid ((x \in A \land x \notin A) \lor (x \in B \land x \notin A)) \lor \\
&\quad ((x \in A \land x \notin B) \lor (x \in B \land x \notin B))\} && \text{(Definition of Distributive Law)} \\
&= \{x \mid (\emptyset \lor (x \in B \land x \notin A)) \lor ((x \in A \land x \notin B) \lor \emptyset)\} && \text{(Domination Law)} \\
&= \{x \mid (x \in B \land x \notin A) \lor (x \in A \land x \notin B)\} && \text{(Identity Law)} \\
&= \{x \mid x \in (B \setminus A) \lor x \in (A \setminus B)\} && \text{(Definition of difference)} \\
&= (B \setminus A) \cup (A \setminus B) && \text{(Definition of Union)} \\
&= (A \setminus B) \cup (B \setminus A) && \text{(Commutative Law)}
\end{aligned}
$$

Q. 2

Prove that the set

$$\{f \mid f \subseteq \mathbb{N} \times \{0,1\}, \text{f is a function}\} \setminus \{f \mid f : \{0,1\} \to \mathbb{N}, \text{f is a function}\}$$

is uncountable.

S. 2

Proof 1

Proof of the fact that if A is an **uncountable** set and B is a **countable** set, then $A \setminus B$ is **uncountable**.

Suppose for an uncountable set A and a countable set B that $A \setminus B$ is countable. The union of countably many countable sets is countable; thus $(A \setminus B) \cup B$ is countable. But then A is a subset of $(A \setminus B) \cup B$ and thus must be countable itself, which is a **contradiction**.

Therefore, if A is an **uncountable** set and B is a **countable** set, then $A \setminus B$ is **uncountable**.

Proof 2

Let S_1 and S_2 be countable sets. From the definition of countable, there exists a **injection** from S_1 to $\mathbb{N}$, and from S_2 to $\mathbb{N}$. Hence, there exists an **injection** g from $S_1 \times S_2$ to $\mathbb{N}^2$.

Now let us investigate the **cardinality** of N2. From the **Fundamental Theorem of Arithmetic**, every natural number greater than 1 has a unique prime decomposition. Thus, if a number can be written as $2^n 3^m$, it can be done thus in only one way. So, consider the function $f : \mathbb{N}^2 \to \mathbb{N}$ defined by:

$$f(n,m) = 2^n 3^m$$

Now suppose $\exists m, n, r, s \in \mathbb{N}$ such that $f(n,m) = f(r,s)$. Then $2^n 3^m = 2^r 3^s$ so that $n = r$ and $m = s$. Thus f is an **injection**; hence, $\mathbb{N}^2$ is **countably infinite**.

Solution

Since Cartesian product of two countable sets is countable set **(Proof 2)**. $\mathbb{N} \times \{0,1\}$ is a infinitely countable set.

$$\{f \mid f \subseteq \mathbb{N} \times \{0,1\}, \text{f is a function}\} \text{ is power set of } \{\mathbb{N} \times \{0,1\}\}$$

Power set of a infinitely countable set is uncountable according to **Cantor's Theorem**. Therefore, $\{f \mid f \subseteq \mathbb{N} \times \{0,1\}, \text{f is a function}\}$ is an **uncountable** set.

Also, $\{f \mid f : \{0,1\} \to \mathbb{N}$ is a countable set because f is a function, whose domain is $\{0,1\}$ while its range is $\mathbb{N}$.

According to **Proof 1**,

$$\{f \mid f \subseteq \mathbb{N} \times \{0,1\}, \text{f is a function}\} \setminus \{f \mid f : \{0,1\} \to \mathbb{N}, \text{f is a function}\}$$

is **uncountable**.

Q. 3

Prove that the function $f(n) = 4^n + 5n^2 logn$ is **not** $O(2^n)$.

S. 3

f is a function. Since f contains $logn$, its domain needs to be the set $\mathrm{D} = \{n \in \mathbb{R} \mid n > 0\}$.

$$4^x > 2^x, \ \forall x \in \mathbb{R}(x > 0) \tag{1}$$

The Eq.1 can be proved by using graph of the functions.

Definition 1 in Book Chapter 3.2.2: Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $f(x)$ is $O(g(x))$ if there are constants C and k such that

$$|f(x)| \leq C|g(x)|$$

whenever $x > k$. [This is read as "$f(x)$ is big-oh of $g(x)$."]

We can prove that $f(n) = 4^n + 5n^2 logn$ is **not** $O(2^n)$ by using contradiction. If the function f is $O(2^n)$, then we can say that

$$|f(n)| \leq C|2^n| \qquad \forall n \in \mathbb{R}(n > k) \tag{2}$$

$$= |4^n + 5n^2 logn| \leq C|2^n| \qquad \forall n \in \mathbb{R}(n > k) \tag{3}$$

There is a contradiction because there is not specific C and k pair that satisfies the inequality $\forall n$. Therefore, the function $f(n)$ is **not** $O(2^n)$

Q. 4

Given two positive integers x and n such that $x > 2$ and $n > 2$, and the congruence relation

$$(2x-1)^n - x^2 \equiv -x - 1 \pmod{(x-1)}$$

determine the value of x.

S. 4

Corollary 1

$$\text{if } a \equiv b \bmod n, \text{ then } a^k \equiv b^k \bmod n \tag{4}$$

Solution

$$\begin{aligned}
(2x-1)^n - x^2 &\equiv -x - 1 \pmod{(x-1)} & \\
(2x-1)^n &\equiv x^2 - x - 1 \pmod{(x-1)} & \\
(2x-1)^n &\equiv -1 \pmod{(x-1)} & [x^2 - x - 1 \equiv -1 \pmod{(x-1)}] \\
(2x-1)^n &\equiv x - 2 \pmod{(x-1)} & [-1 \equiv x - 2 \pmod{(x-1)}] \\
1 &\equiv x - 2 \pmod{(x-1)} & [2x - 1 \equiv 1 \pmod{(x-1)} \text{ and by } \textbf{Corollary 1}] \\
3 &\equiv x \pmod{(x-1)} & \\
3 &\equiv 1 \pmod{(x-1)} &
\end{aligned}$$

$x - 1 = 2$. Therefore, $x = 3$.