IE 203 - Operations Research II
Midterm - Solutions

İ. Kuban Altınel

April 08, 2025

Duration: 1.5 hours
This is a CLOSED BOOK exam.

Question I (15 pts.)

a. This is the intersection of the triangle $\{\mathbf{x} \in \mathbb{R}^2 : x_1 + x_2 \leq \sqrt{2}, x_1 \geq 0, x_2 \geq 0\}$ with the set $\{\mathbf{x} \in \mathbb{R}^2 : x_1^2 + x_2^2 \geq 1\}$, and is not a polyhedron. One way of proving this formally is to notice that every vector on the boundary of S_1 satisfies the definition of a vertex; on the other hand, we know that polyhedra can only have finitely many vertices. Observe also that S_1 is not even a convex set.

b. This is the set of all x such that $(x-1)(x-2) \leq 0$, which is the interval $[1, 2]$, and therefore a polyhedron. It is the intersection of the closed halfspaces $\{x : x \leq 2\}$ and $\{x : x \geq 1\}$.

c. The empty set can be expressed as $\{x \in \mathbb{R} : x \leq 0,\ x \geq 1\}$.

Question II (25 pts.)

Given four items with weights 4, 7, 5 and 3 kg. and values 40, 42, 25 and 12 TL., and a knapsack with 10 kg. which items should be put into the knapsack so that total value is maximized?

a.

$$\begin{array}{rl} \max & 40x_1 + 42x_2 + 25x_3 + 12x_4 \\ \text{s.t} & 4x_1 + 7x_2 + 5x_3 + 3x_4 \leq 10 \\ & x_1, x_2, x_3, x_4 = 0, 1. \end{array}$$

b. The optiimum solution is $x_1^* = x_3^* = 1$, $x_2^* = x_4^* = 0$ with value 65.

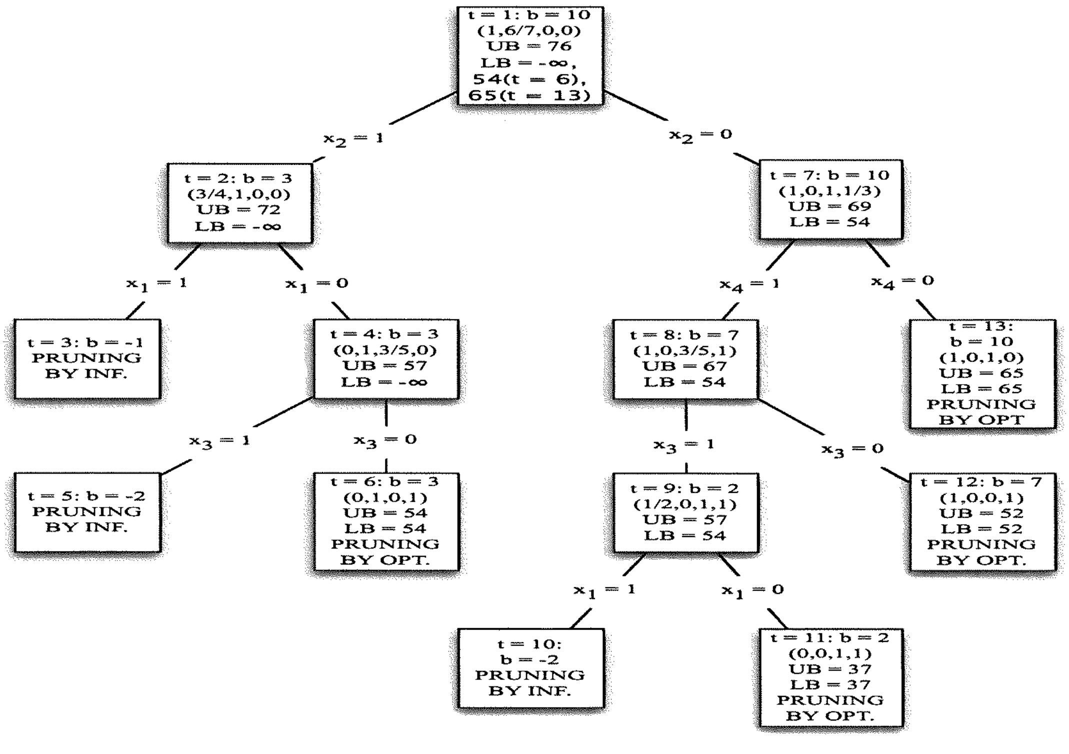

Figure 1: The branch-and-bound search tree for problem II

Question III (25 pts.)

Consider the following function

$$f(x_1, x_2) = 2x_1^4 - 5x_1^3 + 2x_2^3 - 9x_2^2 + 10x_2 - 3.$$

a. The gradient and Hessian of the function are respectively

$$\nabla f(x_1, x_2) = \begin{pmatrix} 8x_1^3 - 15x_1^2 & 6x_2^2 - 18x_2 + 10 \end{pmatrix} \text{ and } \nabla^2 f(x_1, x_2) = \begin{pmatrix} 24x_1^2 - 30x_1 & 0 \\ 0 & 12x_2 - 18 \end{pmatrix}.$$

The Hessian matrix is indefinite on the domain. For example, for $x_1 = 1$ and $x_2 = 0$ the eigenvalues are -6 and -18 and for $x_1 = 2$ and $x_2 = 2$ the eigenvalues are 36 and 6. In short, the eigenvalues are not nonnegative for all $\mathbf{x} \in \mathbb{R}^2$, which is the domain of the function. Hence, the function is neither convex, nor concave on its domain.

As can be seen by using the definition the two eigenvalues of the Hessian matrix are respectively $6x_1(4x_1 - 5)$ and $6(2x_2 - 3)$. They are both nonnegative, i.e., the function is convex, for $x_1 \geq 5/4$ and $x_2 \geq 3/2$ or $x_1 \leq 0$ and $x_2 \geq 3/2$.

b. The first order necessary conditions requires the solution of the system $8x_1^3 - 15x_1^2 = 0$, $3x_2^2 - 9x_2 + 5 = 0$, which result in four stationary points: $\begin{pmatrix} 0 \\ 2.26376 \end{pmatrix}, \begin{pmatrix} 0 \\ 0.73624 \end{pmatrix}, \begin{pmatrix} 1.875 \\ 2.26376 \end{pmatrix}, \begin{pmatrix} 1.875 \\ 0.73624 \end{pmatrix}$. Based on the analysis made in part (a) the eigenvalues of the Hessian matrix are nonnegative for the first and third one. At the second point, the eigenvalues are nonpositive, which makes the Hessian matrix negative semi-definite and thus the second point a local maximum. As for the last point, the first eigenvalue is positive while the second one is negative, which makes the Hessian matrix indefinite and the point a saddle point.

Question IV (25 pts.)

Consider the single facility location problem and suppose that the squared Euclidean distance is used to measure the distance between the unknown facility location $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and known customer locations $\mathbf{a}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \end{pmatrix}$ for $j = 1, 2, \ldots, n$, and known customer demands h_j for $j = 1, 2, \ldots, n$.

a.

$$\min f(x_1, x_2) = \sum_{j=1}^{n} h_j \left[(x_1 - a_{1j})^2 + (x_2 - a_{2j})^2 \right]$$

b. The gradient and Hessian of the of the objective function are respectively

$$\nabla f(x_1, x_2) = \begin{pmatrix} 2\sum_{j=1}^n h_j(x_1 - a_{1j}) & 2\sum_{j=1}^n h_j(x_2 - a_{2j}) \end{pmatrix}, \nabla^2 f(x_1, x_2) = \begin{pmatrix} 2\sum_{j=1}^n h_j & 0 \\ 0 & 2\sum_{j=1}^n h_j \end{pmatrix}.$$

As can be noticed easily the two eigenvalues of the Hessian matrix are $2\sum_{j=1}^n h_j$, which are clearly nonnegative since customer demands are nonnegative values in practice. In fact, they are positive since having only zero demands for all customers makes any location optimal. In short, the objective function is strictly convex in practice. Therefore, the first order necessary conditions are also sufficient.

Then, the first order necessary conditions gives the following stationary point, which is the unique global minimum of the objective function.

$$x_k^* = \frac{\sum_{j=1}^n h_j a_{kj}}{\sum_{j=1}^n h_j} \qquad k = 1, 2.$$

c. The coordinate of the optimum location are $x_1^* = 215/62 = 3.468$ and $x_2^* = 47/31 = 1.516$.