(2) Let $(x_1, y_1), (x_2, y_2) \in S$ and $0 \leq \lambda \leq 1$

Let $(x_3, y_3)$ be the convex comb. of $(x_1, y_1)$ and $(x_2, y_2)$

$$\lambda^2(x_1^2+y_1^2) \leq 4\lambda^2 \rightarrow \underbrace{\lambda^2 x_1^2 + \lambda^2 y_1^2 \leq 4\lambda^2}_{A}$$

$$(1-\lambda)^2 (x_2^2+y_2^2) \leq 4(1-\lambda)^2 \quad \underbrace{(1-\lambda)^2 x_2^2 + (1-\lambda^2) y_2^2 \leq 4(1-\lambda)^2}_{B}$$

$$(x_1 - x_2)^2 \geq 0 \Rightarrow 2x_1x_2 \leq x_1^2 + x_2^2$$

$$2\lambda(1-\lambda)x_1x_2 \leq \lambda(1-\lambda)(x_1^2+x^2)$$

$$(y_1 - y_2)^2 \geq 0 \Rightarrow 2y_1y_2 \leq y_1^2 + y_2^2$$

$$2\lambda(1-\lambda)y_1y_2 \leq \lambda(1-\lambda)(y_1^2+y_2^2)$$

$$\underbrace{2\lambda(1-\lambda)x_1x_2 + 2\lambda(1-\lambda)y_1y_2 \leq \lambda(1-\lambda)(x_1^2+y_1^2+x_2^2+y_2^2)}_{C}$$

$$\underline{\leq \lambda(1-\lambda)8}$$

Summing inequalities A, B, C:

$$\underbrace{\lambda^2x_1^2 + 2\lambda(1-\lambda)x_1x_2 + (1-\lambda)x_2^2}_{\underbrace{(\lambda x_1 + (1-\lambda)x_2)^2}_{x_3^2}} + \underbrace{\lambda^2 y_1 + 2\lambda(1-\lambda)y_1y_2 + (1-\lambda)^2 y_2}_{\underbrace{(\lambda y_1 + (1-\lambda)y_2)^2}_{y_3^2}} \leq \underbrace{4\lambda^2 + 8\lambda(1-\lambda) + 4(1-\lambda)^2}_{\underbrace{4(\lambda+(1-\lambda))^2}_{4}}$$

$$x_3^2 + y_3^2 \leq 4 \quad \checkmark$$

$x \geq 1$ also convex. Intersection of two convex sets is also convex. $S$ is convex.

③

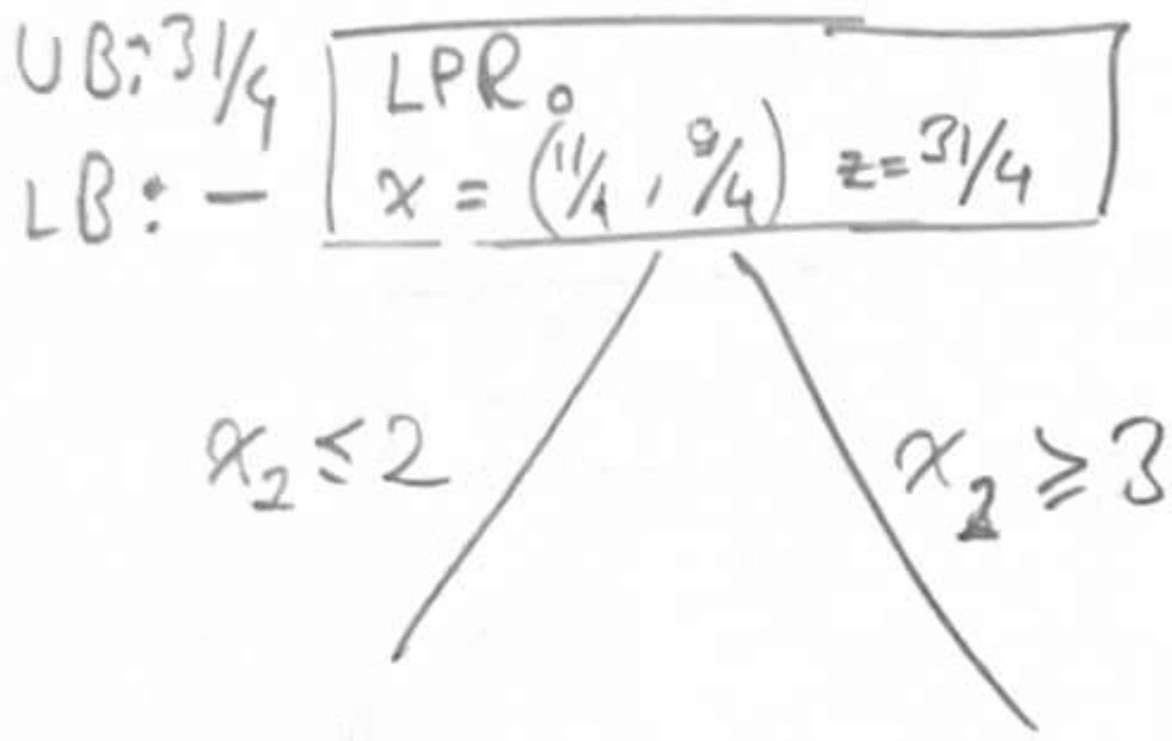
$LPR_0$: $\max\ 2x_1 + x_2$

s.t. $x_1 + x_2 \le 5$

$3x_1 - x_2 \le 6$

$x_1, x_2 \ge 0$

| | $x_1$ | $x_2$ | $S_1$ | $S_2$ | RHS |
|---|---|---|---|---|---|
| $z$ | $-2$ | $-1$ | 0 | 0 | 0 |
| $s_1$ | 1 | 1 | 1 | 0 | 5 |
| $s_2$ | 3 | $-1$ | 0 | 1 | 6 → |
| $z$ | 0 | $-5/3$ | 0 | $2/3$ | 4 |
| $s_1$ | 0 | $4/3$ | 1 | $-1/3$ | 3 → |
| $x_1$ | 1 | $-1/3$ | 0 | $1/3$ | 2 |
| $z$ | 0 | 0 | $5/4$ | $1/4$ | $31/4$ |
| $x_2$ | 0 | 1 | $3/4$ | $-1/4$ | $9/4$ |
| $x_1$ | 1 | 0 | $1/4$ | $1/4$ | $11/4$ |




$LPR_1$: $\max\ 2x_1 + x_2$

s.t. $x_1 + x_2 \le 5$

$3x_1 - x_2 \le 6$

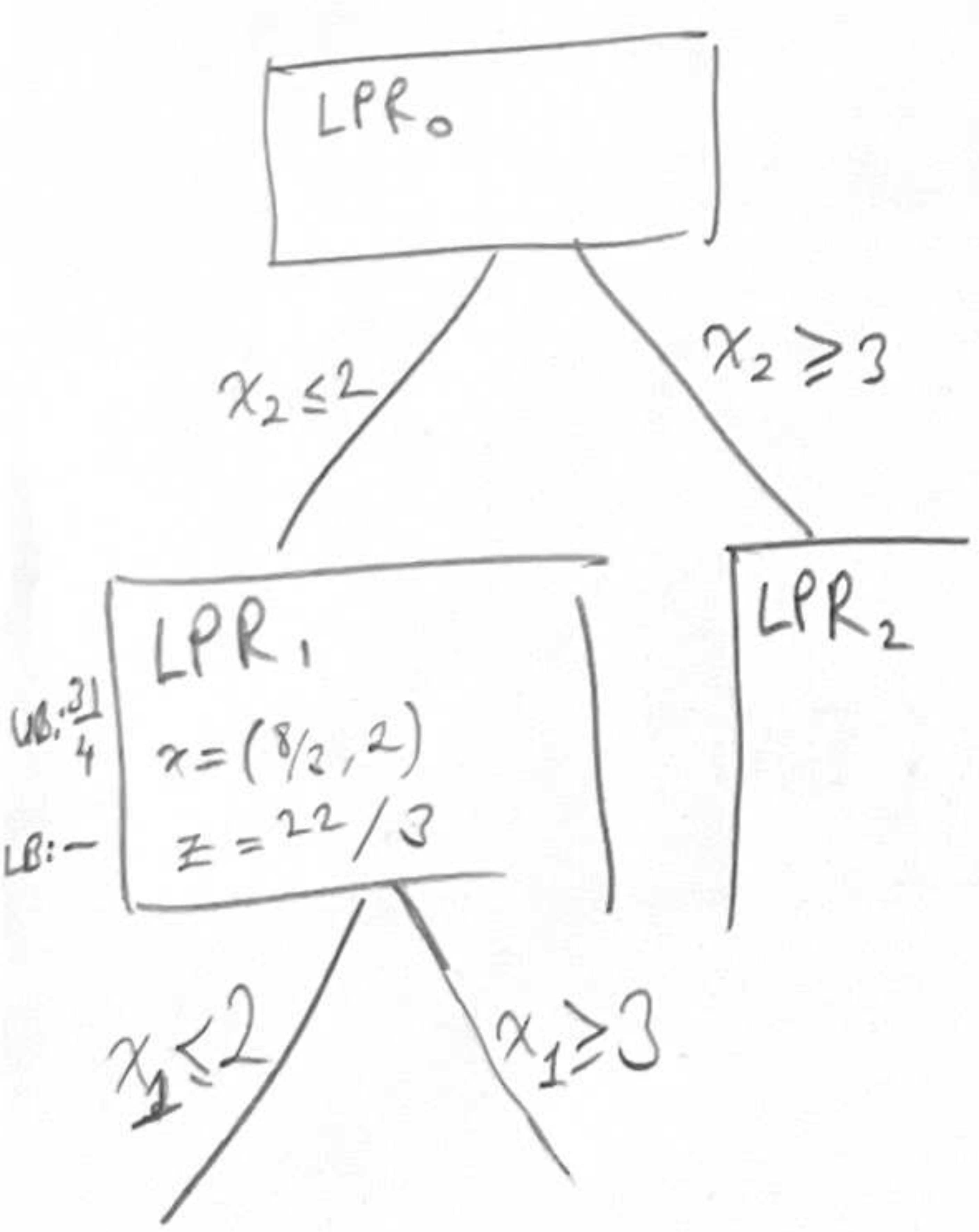
$x_2 \le 2$

$x_1, x_2 \ge 0$

Dual Simplex:

| | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $S_3$ | RHS |
|---|---|---|---|---|---|---|
| $z$ | 0 | 0 | $5/4$ | $1/4$ | 0 | $31/4$ |
| $x_2$ | 0 | 1 | $3/4$ | $-1/4$ | 0 | $9/4$ |
| $x_1$ | 1 | 0 | $1/4$ | $1/4$ | 0 | $11/4$ |
| $S_3$ | 0 | 1 | 0 | 0 | 1 | 2 |
| $z$ | 0 | 0 | $5/4$ | $1/4$ | 0 | $31/4$ |
| $x_2$ | 0 | 1 | $3/4$ | $-1/4$ | 0 | $9/4$ |
| $x_1$ | 1 | 0 | $1/4$ | $1/4$ | 0 | $11/4$ |
| $S_3$ | 0 | 0 | $-3/4$ | $1/4$ | 1 | $-1/4$ → |
| $z$ | 0 | 0 | 0 | $2/3$ | $5/3$ | $22/3$ |
| $x_2$ | 0 | 1 | 0 | 0 | 1 | 2 |
| $x_1$ | 1 | 0 | 0 | $1/3$ | $1/3$ | $8/3$ |
| $S_1$ | 0 | 0 | 1 | $-1/3$ | $-4/3$ | $1/3$ |

Not in canonical form



$LPR_2$: $\max\ 2x_1 + x_2$

st $x_1 + x_2 \le 5$

$3x_1 - x_2 \le 6$

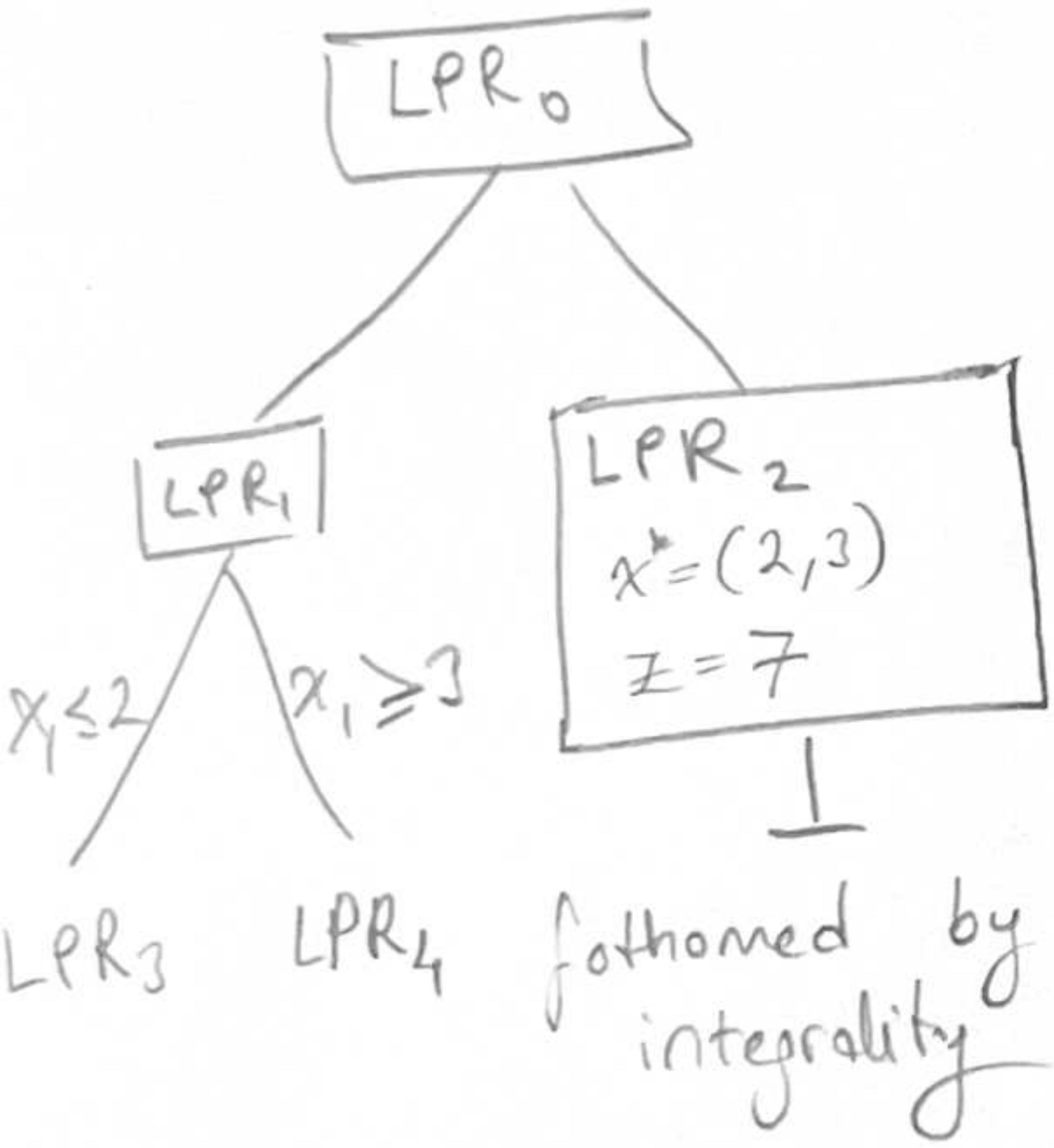
$x_2 \ge 3$

$x_1, x_2 \ge 0$

| | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | RHS |
|---|---|---|---|---|---|---|
| $Z$ | 0 | 0 | 5/4 | 1/4 | 0 | 31/4 |
| $x_2$ | 0 | 1 | 3/4 | -1/4 | 0 | 9/4 |
| $x_1$ | 1 | 0 | 1/4 | 1/4 | 0 | 11/4 |
| $s_3$ | 0 | -1 | 0 | 0 | 1 | -3 |
| $Z$ | 0 | 0 | 5/4 | 1/4 | 0 | 31/4 |
| $x_2$ | 0 | 1 | 3/4 | -1/4 | 0 | 9/4 |
| $x_1$ | 1 | 0 | 1/4 | 1/4 | 0 | 11/4 |
| $s_3$ | 0 | 0 | 3/4 | -1/4 | 1 | $-3/4 \rightarrow$ |
| $Z$ | 0 | 0 | 2 | 0 | 1 | 7 |
| $x_2$ | 0 | 1 | 0 | 0 | -1 | 3 |
| $x_1$ | 1 | 0 | 1 | 0 | 1 | 2 |
| $s_2$ | 0 | 0 | -3 | 1 | -4 | 3 |

UB: 22/3

LB: 7




$LPR_3$: $x^* = (2,2)$

$Z = 6$

Fathom by integrality

UB: 22/3 LB: 7

$LPR_4$: Inf.

Fathom by infeasibility

Terminate, B&B tree explored

$x^*_{IP} = (2,3) \quad Z^*_{IP} = 7$

5

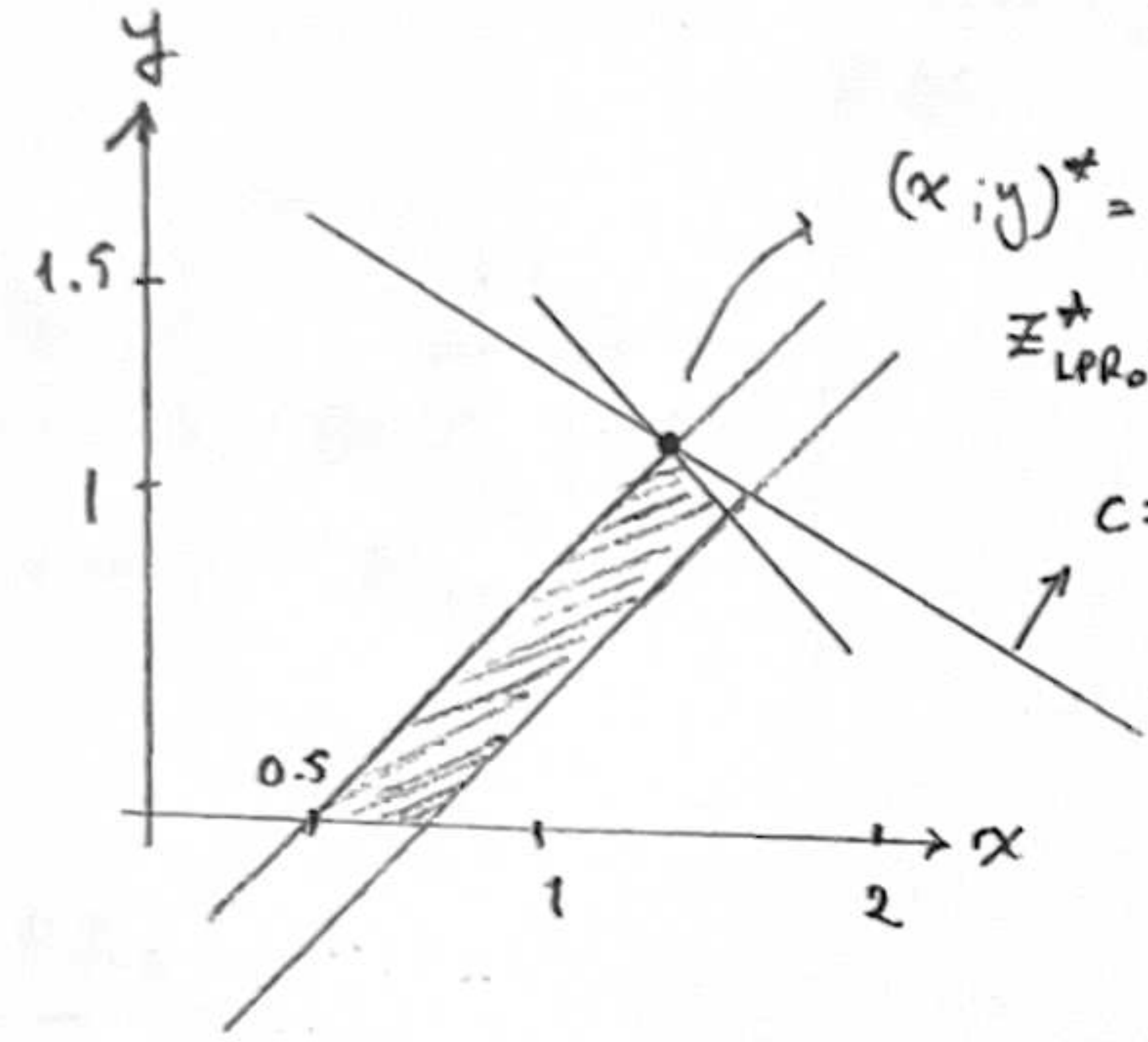
$LPR_0$: max $x + 2y$

s.t $x - y \le 3/4$

$x - y \ge 1/2$

$x + y \le 3$

$x, y \ge 0$




$(x;y)^* = \left(\frac{7}{4}, \frac{5}{4}\right)$

$Z^*_{LPR_0} = 17/4$

$c = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

B&B tree:

UB = 4 (*)

LB = —

Incumbent sol. = —

(*) : C vector is integral

$\Rightarrow$ : $Z^*_{IP}$ cannot be fractional

$\boxed{\begin{array}{l} LPR_0: \\ (x,y)^* = \left(\frac{7}{4}, \frac{5}{4}\right) \\ z^* = 17/4 \end{array}}$

$x \le 1$ → $LPR_1$

$x \ge 2$ → $LPR_2$

$LPR_1$:

max $x + 2y$

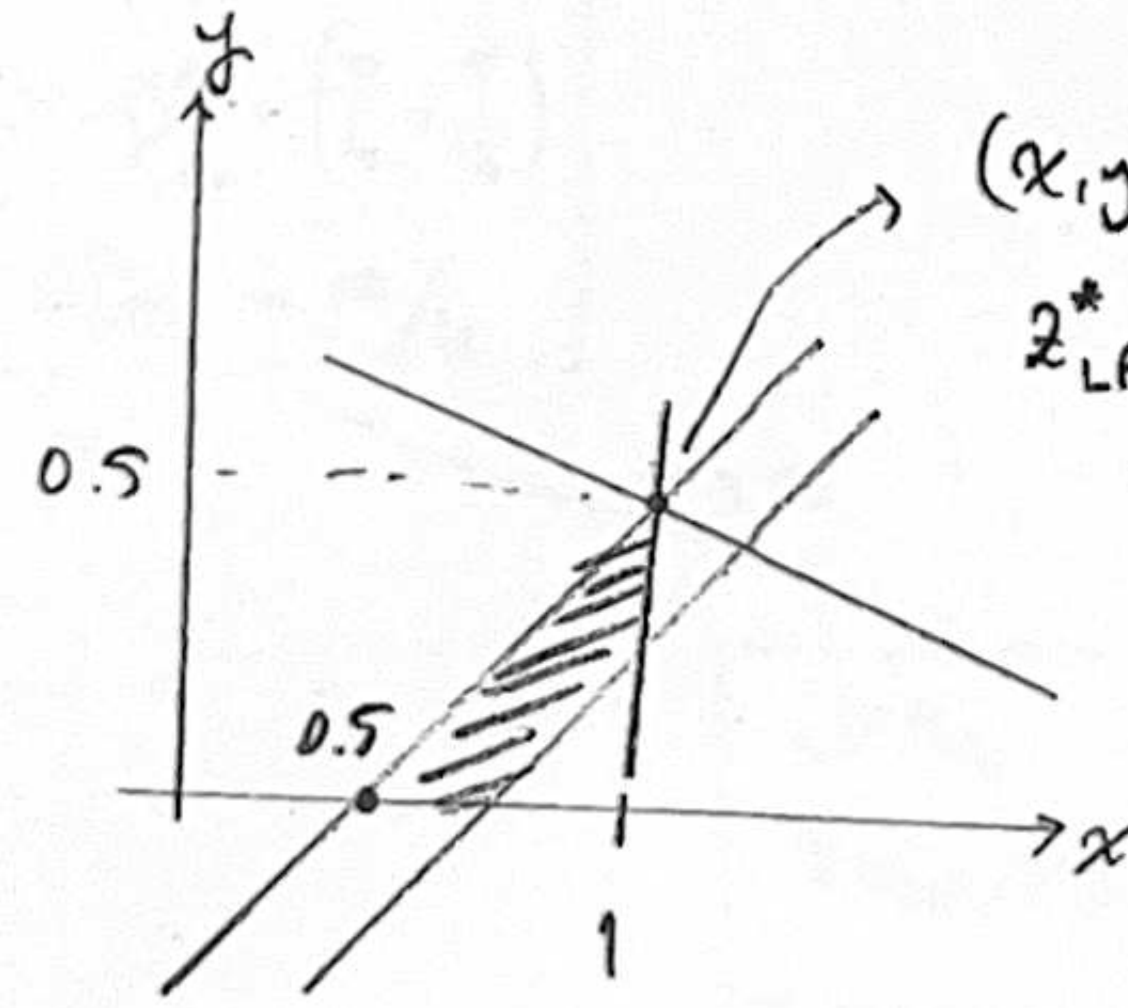
s.t $x - y \le 3/4$

$x - y \ge 1/2$

$x + y \le 3$

$x \le 1$

$x, y \ge 0$




$(x,y)^* = (1, 1/2)$

$z^*_{LP} = 2$

2.1

## B&B tree:

UB = 4
LB = —
Inc. Sol = —

**LPR$_0$:**
$(x,y)^* = \left(\frac{7}{4}, \frac{5}{4}\right)$
$z^*_{LPR_0} = 17/4$

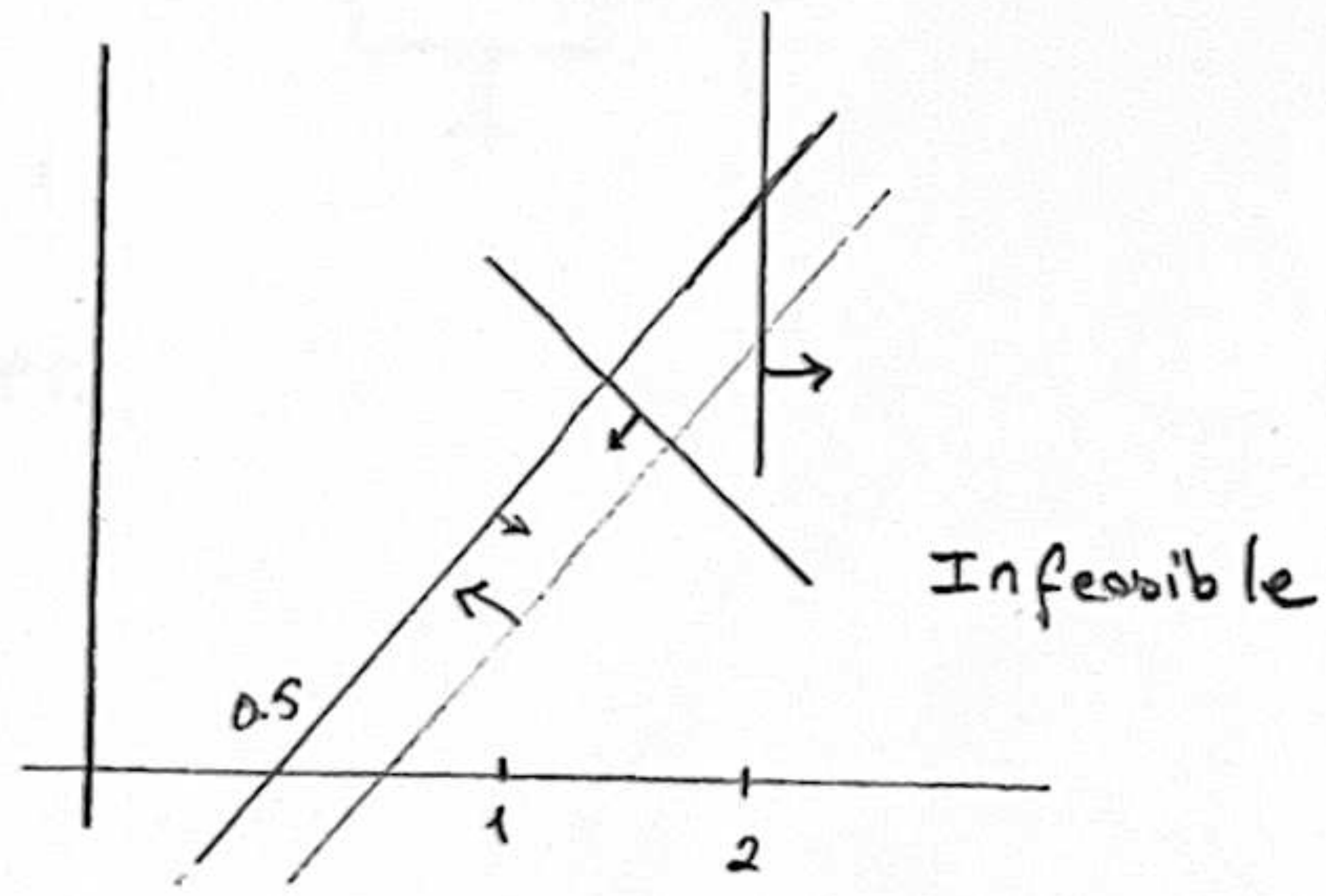
$x \le 1$ → LPR$_1$ ; $x \ge 2$ → LPR$_2$:

UB = 2
LB = —
Inc. Sol. = —

**LPR$_1$:**
$(x,y)^*_{LPR_1} = (1, 1/2)$
$z^*_{LPR_1} = 2$

**LPR$_2$:**

$$\begin{aligned} \max \quad & x+2y \\ \text{s.t} \quad & x-y \le 3/4 \\ & x-y \ge 1/2 \\ & x+y \le 3 \\ & x \ge 2 \\ & x,y \ge 0 \end{aligned}$$




## B&B tree:

UB = 4
LB = –
Inc. Sol. = –

**LPR$_0$:**
$(x,y)^*_{LPR_0} = \left(\frac{7}{4}, \frac{5}{4}\right)$
$z^*_{LPR_0} = 17/4$

$x \le 1$ → LPR$_1$ ; $x \ge 2$ → LPR$_2$

UB = 2
LB =

**LPR$_1$:**
$(x,y)^*_{LPR_1} = (1, 1/2)$
$z^*_{LPR_1} = 2$

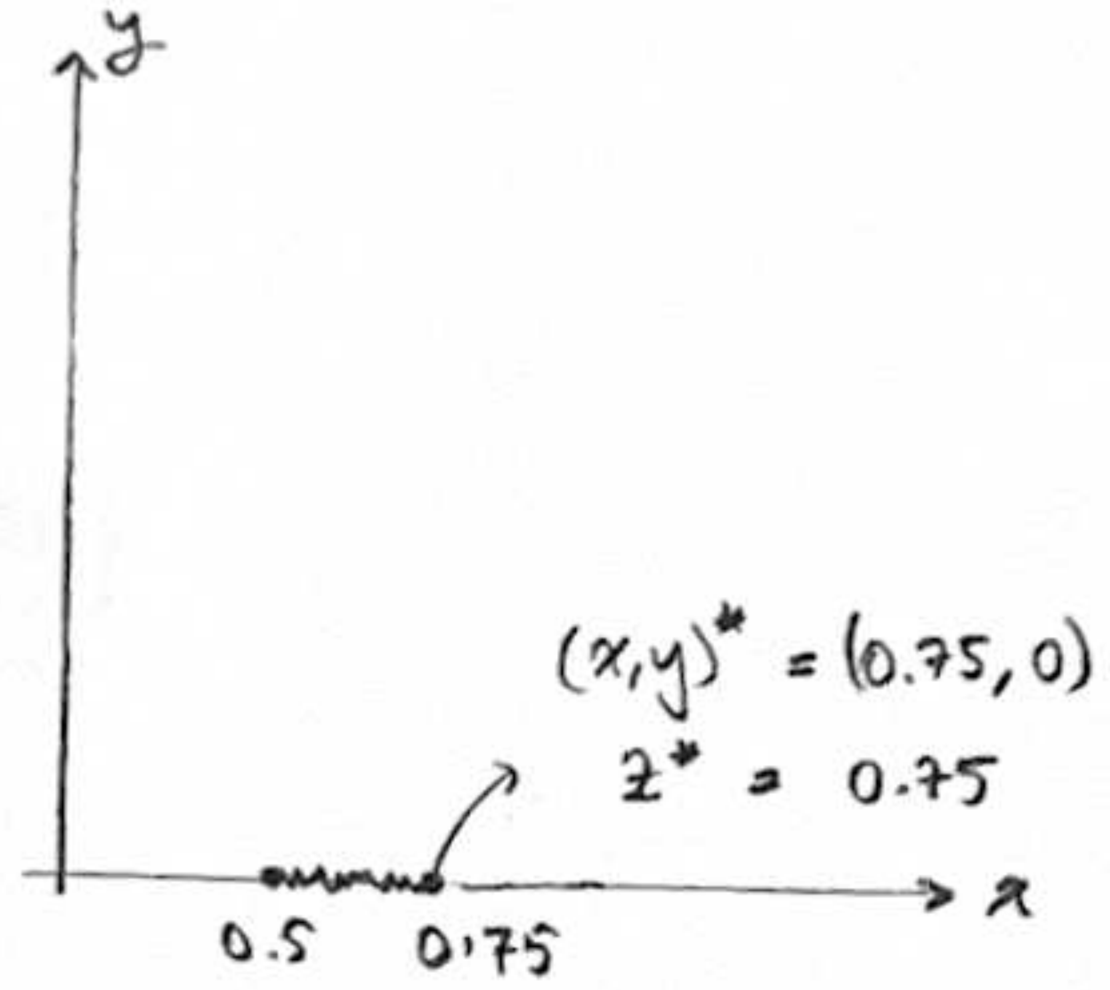
$y \le 0$ → LPR$_3$ ; $y \ge 1$ → LPR$_4$

**LPR$_2$:**
Infeasible

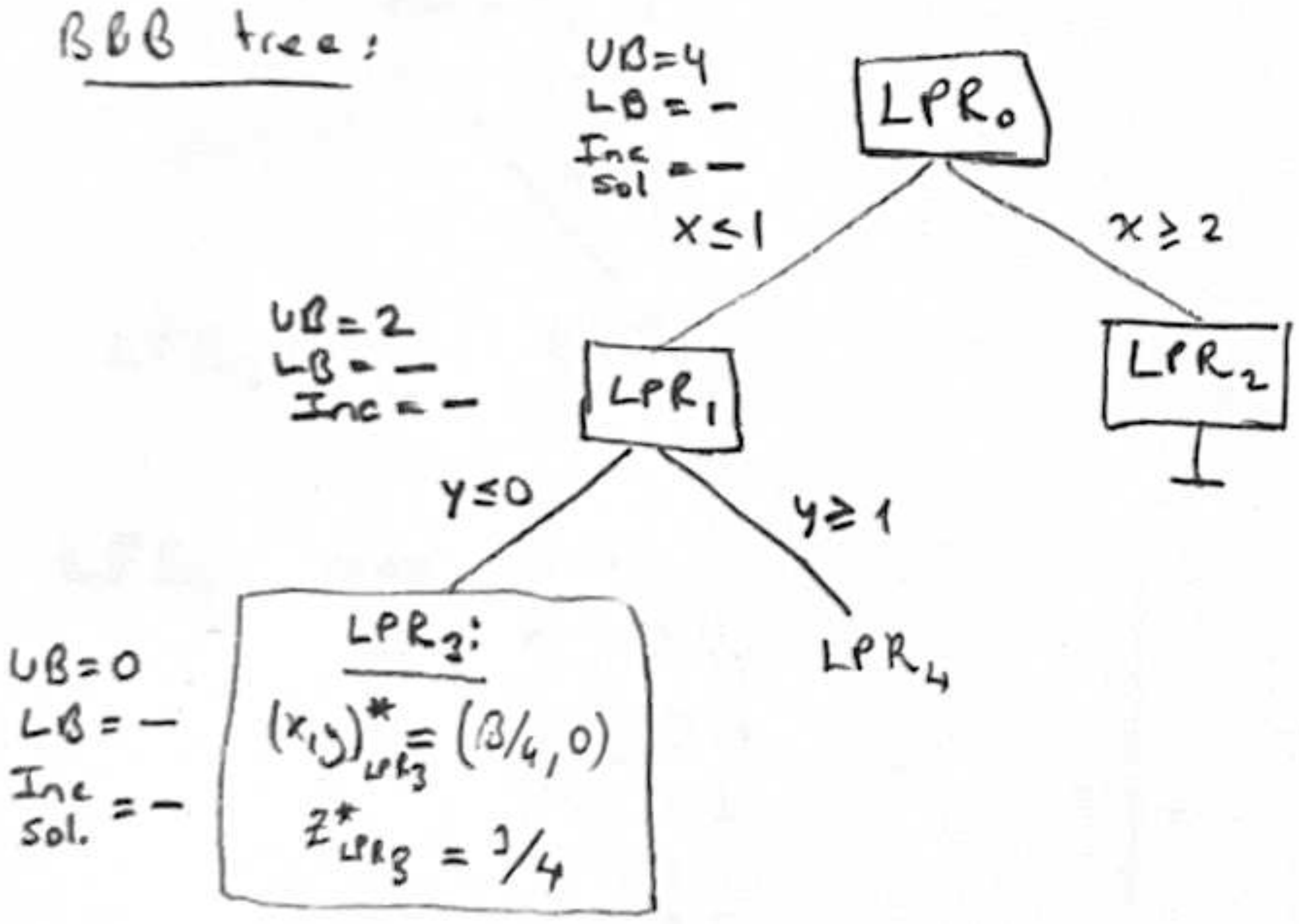
Fathom

$\underline{LPR_3:}$

$$\begin{aligned} \max \quad & x+2y \\ \text{s.t} \quad & x-y \le 3/4 \\ & x-y \ge 1/2 \\ & x+y \le 3 \\ & x \le 1 \\ & y \le 0 \\ & x, y \ge 0 \end{aligned}$$

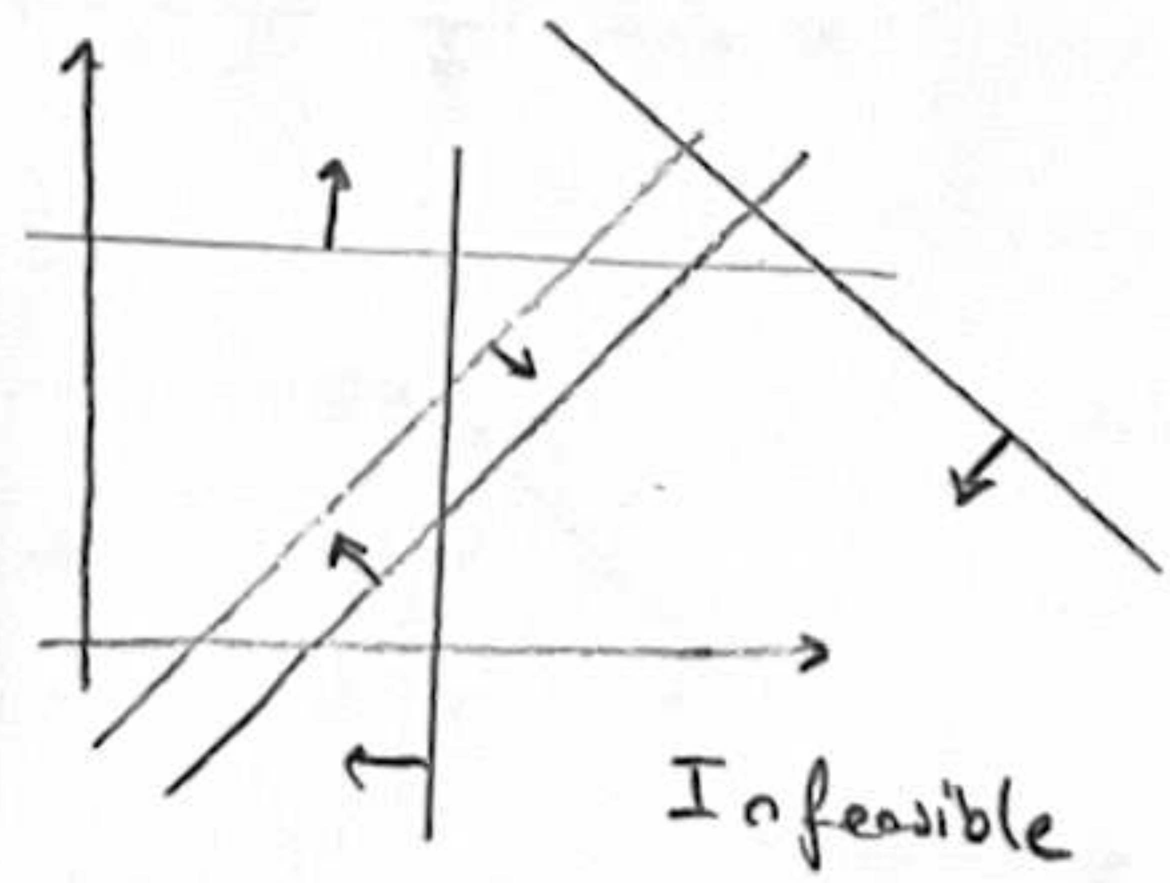



BBB tree:

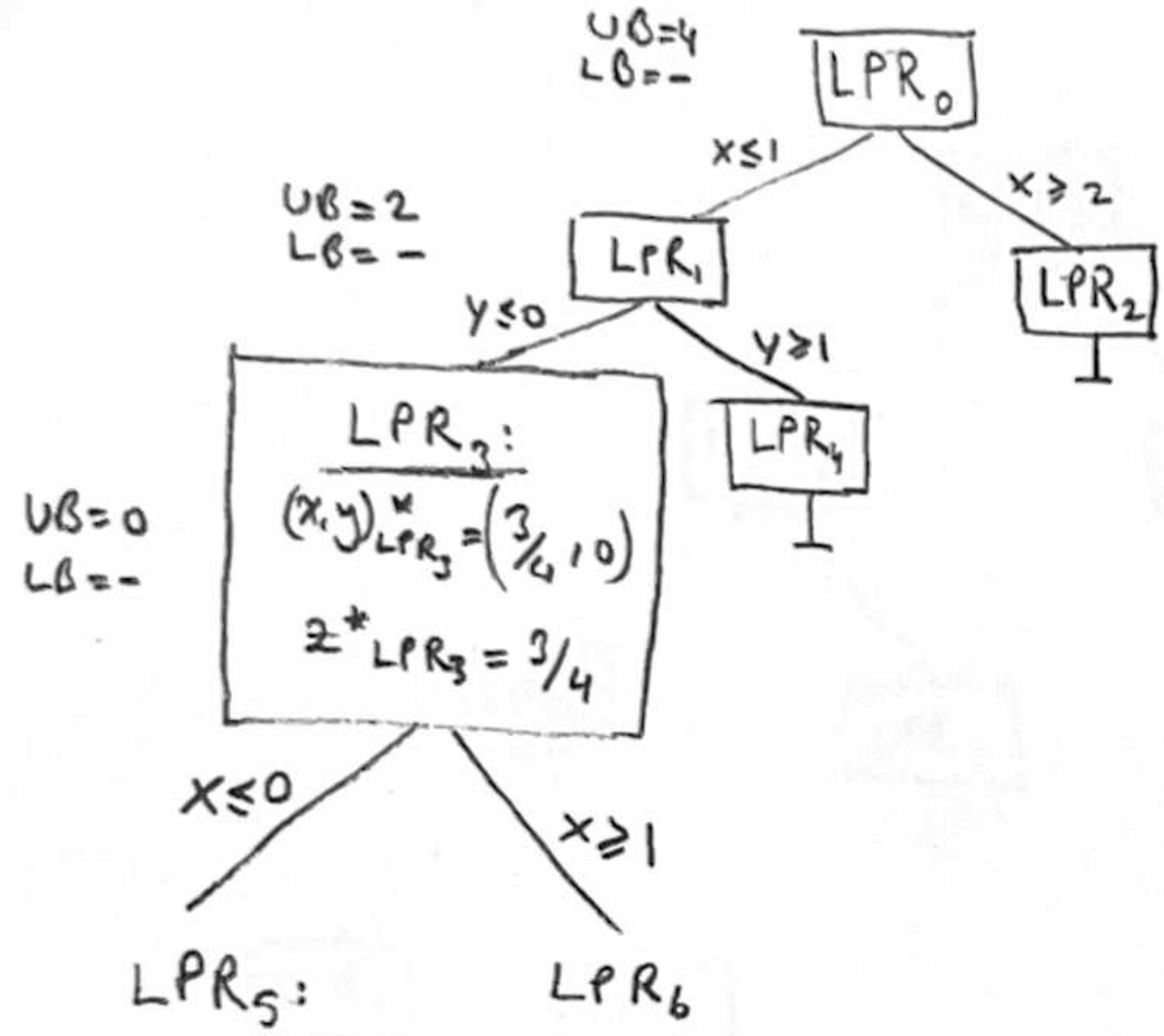



$\underline{LPR_4:}$

$$\begin{aligned} \max \quad & x+2y \\ \text{s.t} \quad & x-y \le 3/4 \\ & x-y \ge 1/2 \\ & x+y \le 3 \\ & x \le 1 \\ & y \ge 1 \\ & x, y \ge 0 \end{aligned}$$



B&B tree:




LPR$_5$: max $x+2y$

s.t.

$x - y \le 3/4$

$x - y \ge 1/2$

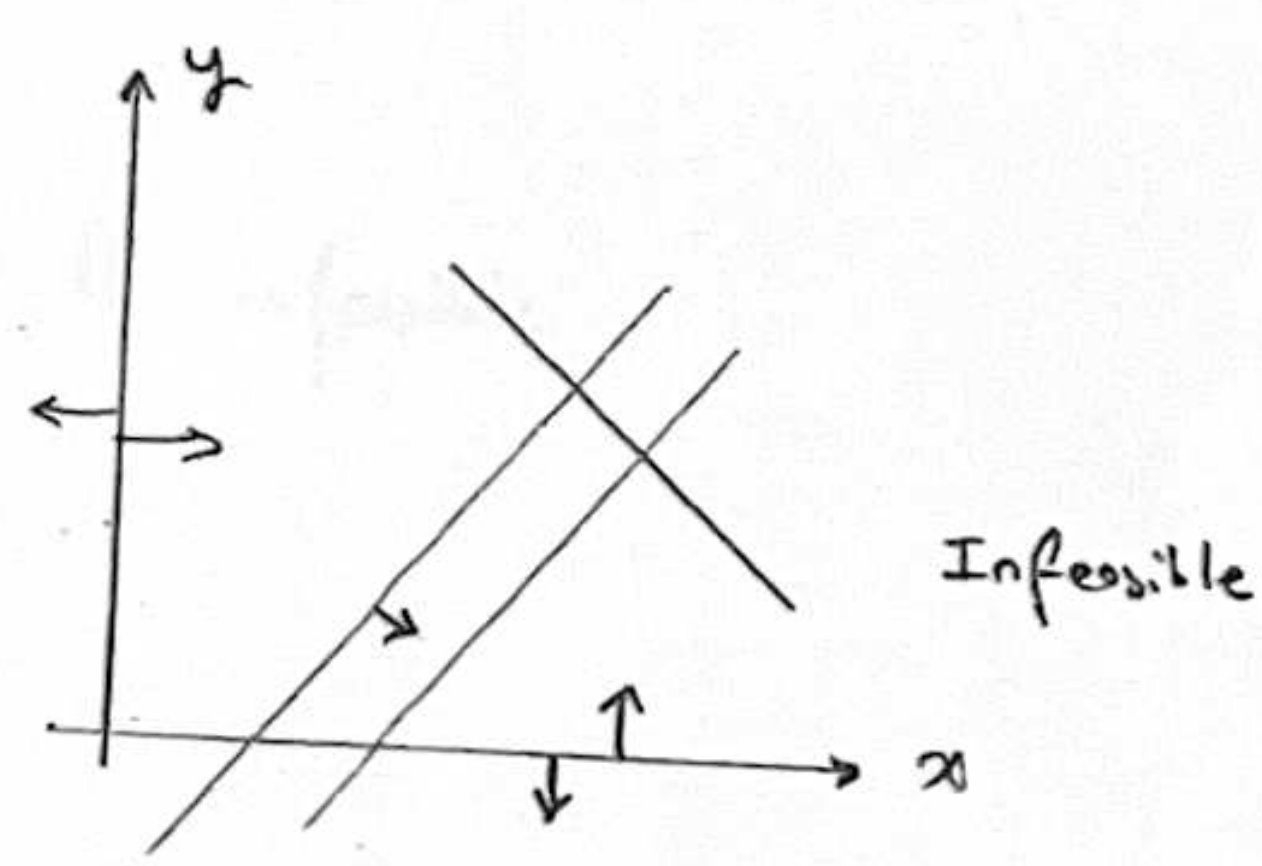
$x + y \le 3$

$x \le 1$

$x \le 0$

$y \le 0$

$x, y \ge 0$




LPR$_6$: max $x+2y$

s.t.

$x - y \le 3/4$

$x - y \ge 1/2$

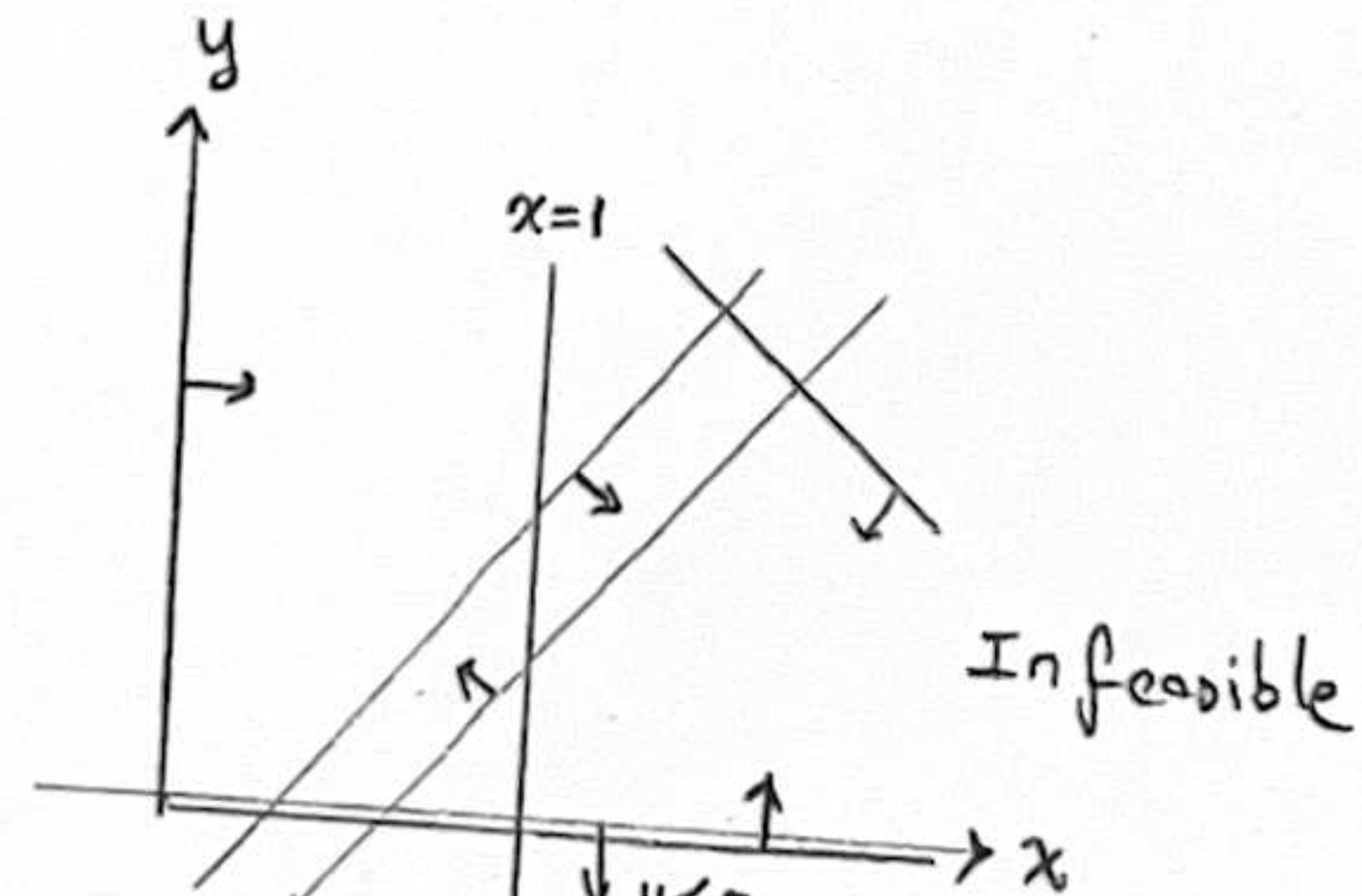
$x + y \le 3$

$x \le 1$

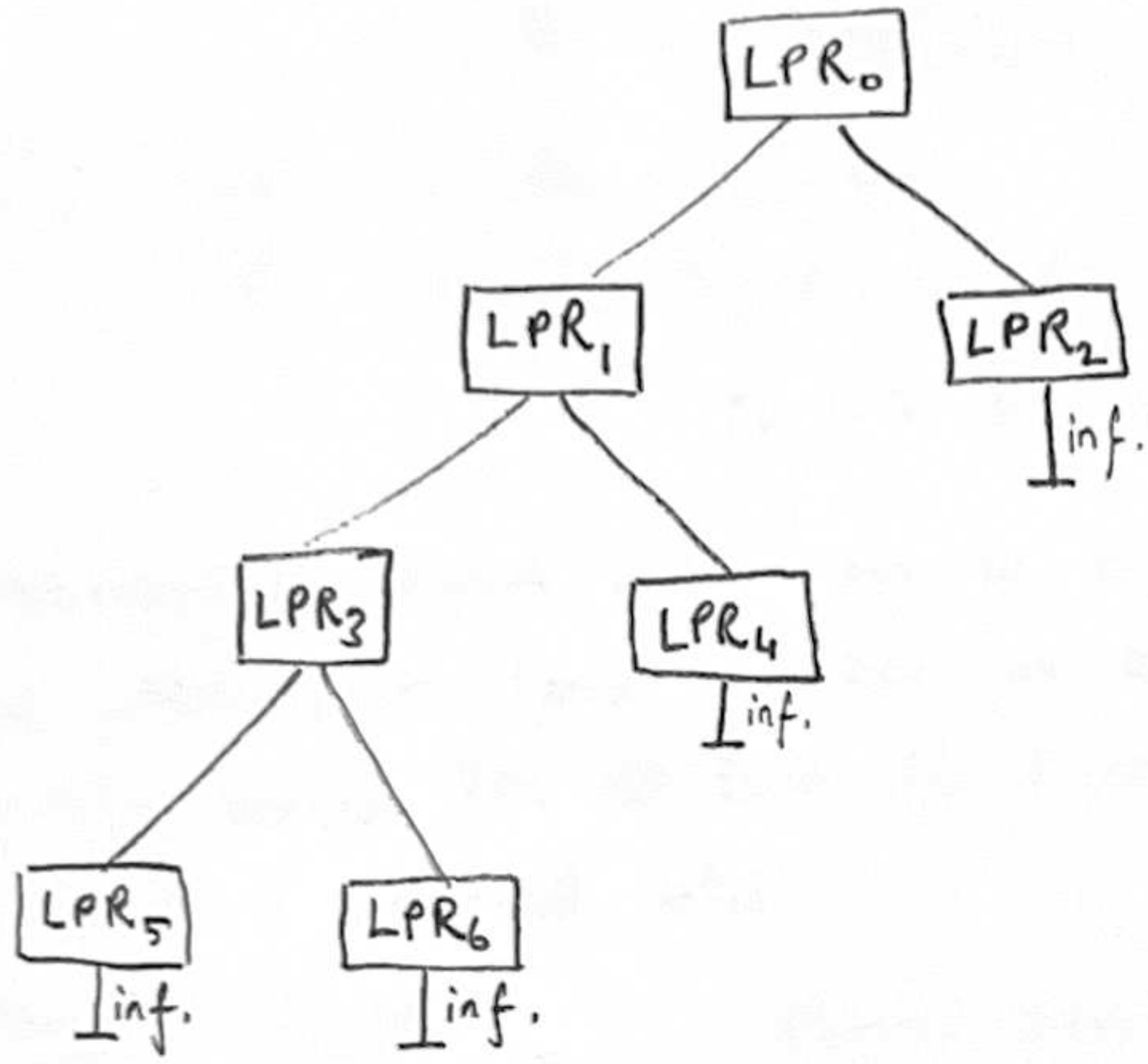
$x \ge 1$

$y \le 0$

$x, y \ge 0$



B&B tree:




No integer solution exists $\Rightarrow$ IP infeasible

D.V.s: $x_i : \begin{cases} 1 & \text{if item } i \text{ is selected} \\ 0 & \text{o/w} \end{cases}$ $i = 1,2,3,4$

BIP: $\max \; 5x_1 + 8x_2 + 3x_3 + 7x_4$

s.t. $3x_1 + 5x_2 + 2x_3 + 4x_4 \le 6$

$x_i \in \{0,1\} \;\; \forall i \in \{1,2,3,4\}$

6

LPR$_0$: $\max \; 5x_1 + 8x_2 + 3x_3 + 7x_4$

s.t. $3x_1 + 5x_2 + 2x_3 + 4x_4 \le 6$

$0 \le x_i \le 1 \;\; \forall i \in \{1,2,3,4\}$

* LP relaxation of Knapsack problem can be solved by a greedy heuristic which takes items in their non-decreasing order of $\frac{\text{profit}}{\text{weight}}$ as capacity allows. The last taken item's corresponding dec. var. may get assigned a fractional value.

| Items | $p_i / w_i$ |
|---|---|
| 1 | $5/3 = 1.\overline{6}$ |
| 2 | $8/5 = 1.6$ |
| 3 | $3/2 = 1.5$ |
| 4 | $7/4 = 1.75$ |

Favored order: 4, 1, 2, 3

Solution of LPR$_0$:

| Item: | 4 | 1 | 2 | 3 |
|---|---|---|---|---|
| $x_i$ : | 1 | 2/3 | 0 | 0 |
| Left cap: | 2 | 0 | 0 | 0 |

B&B tree:

UB = 10
LB = –

LPR$_0$:
$x^*_{LPR_0} = (2/3, 0, 0, 1)$
$z^* = 31/3$

$x_1 = 0$ → LPR$_1$

$x_1 = 1$ → LPR$_2$

**Sol of $LPR_1$:** $x_1 = 0$

| Items : | 4 | 2 | 3 |
|---|---|---|---|
| $x_i$ : | 1 | 2/5 | 0 |
| Left Cap : | 2 | 0 | 0 |

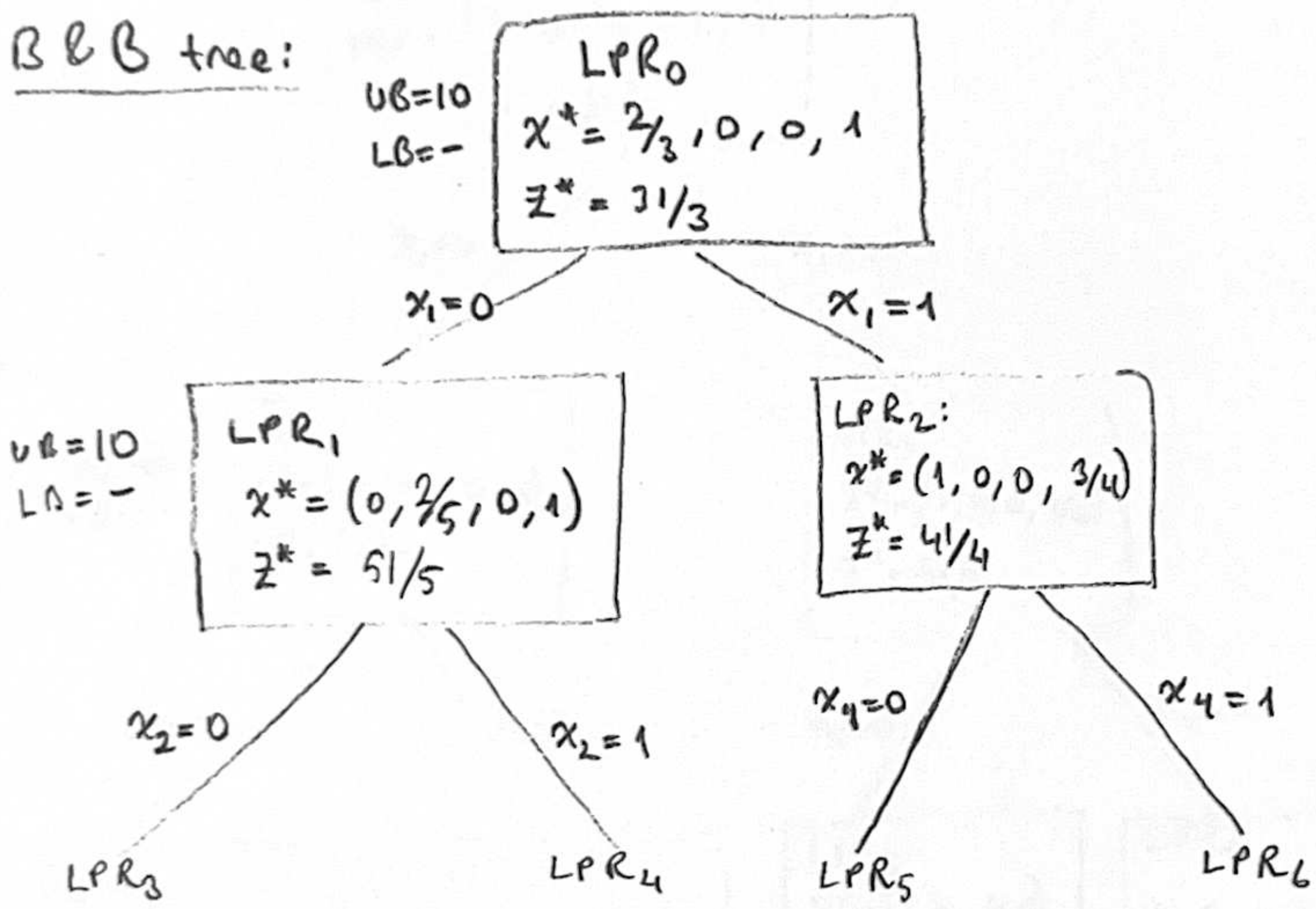
$z^* = 51/5$

**Sol of $LPR_2$:** $x_1 = 1$

| Items : | 1 | 4 | 2 | 3 |
|---|---|---|---|---|
| $x_i$ : | 1 | 1/4 | 0 | 0 |
| Cap. : | 3 | 0 | 0 | 0 |

$z^* = 41/5$

**B&B tree:**




**$LPR_3$:** $x_1 = 0$, $x_2 = 0$

| Items : | 4 | 3 |
|---|---|---|
| $x_i$ : | 1 | 1 |
| left cap. : | 2 | 0 |

$z^* = 10 = UB$

can stop here

**$LPR_4$:** $x_1 = 0$, $x_2 = 1$

| Items : | 2 | 4 | 3 |
|---|---|---|---|
| $x_i$ : | 1 | 1/4 | 0 |
| left cap : | 1 | 0 | 0 |

$z^* = 33/4$

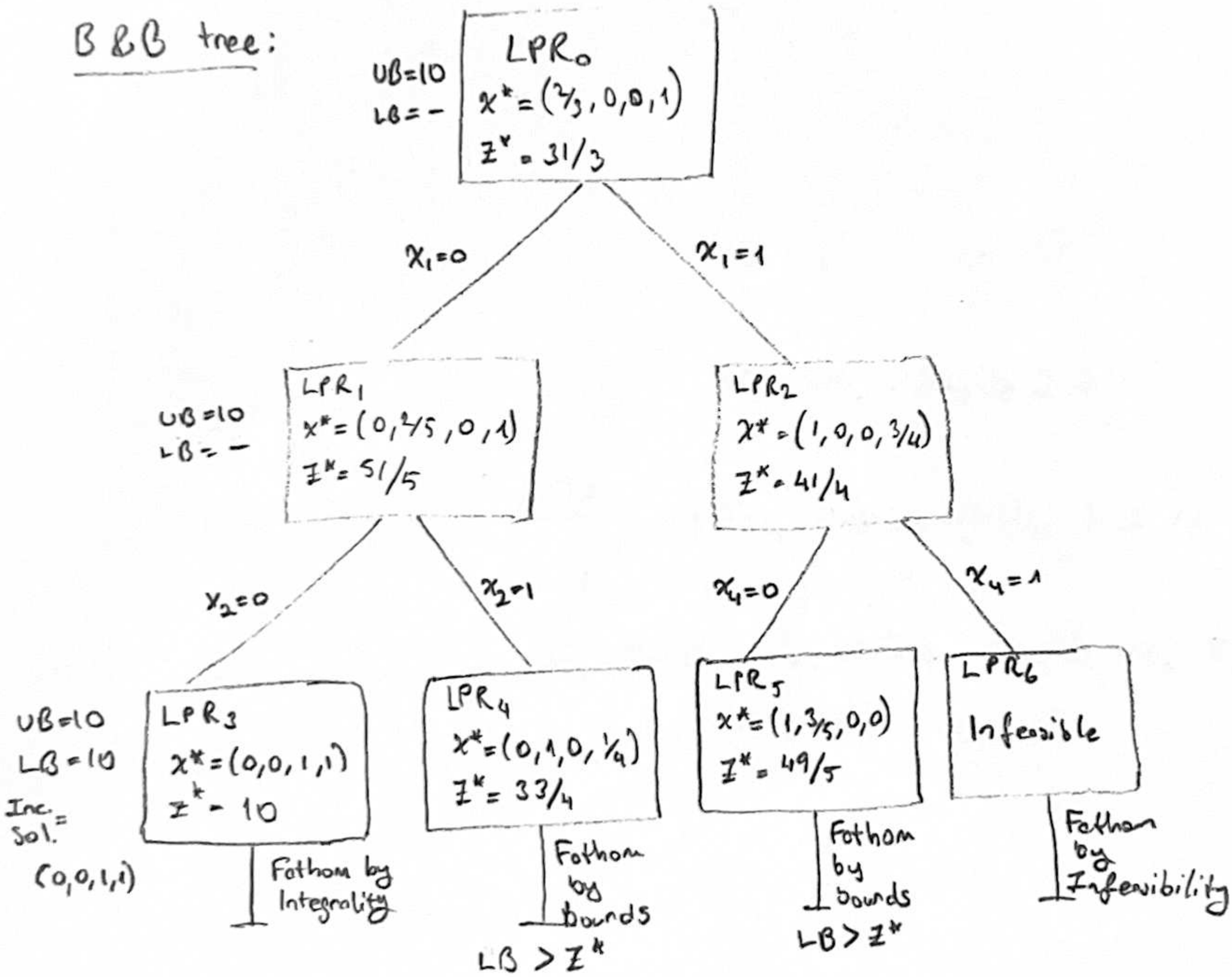
$LPR_5$: $x_1 = 1$, $x_4 = 0$

| Items: | 1 | 2 | 3 |
|---|---|---|---|
| $x_i$: | 1 | 3/5 | 0 |
| left cap: | 3 | 0 | 0 |

$Z^* = \frac{49}{5}$

$LPR_6$: $x_1 = 1$, $x_4 = 1$ Infeasible

B&B tree:




– Could stop when soltion of $LPR_3$ were found.

$x^*_{IP} = (0, 0, 1, 1)$ $Z^*_{IP} = 10$