IE 203 PS 7 17.04.2025

$$f(x_1, x_2) = x_1^3 + x_2^3$$

$$\nabla f(x) = \begin{pmatrix} 3x_1^2 \\ 3x_2^2 \end{pmatrix} \qquad H(x) = \begin{pmatrix} 6x_1 & 0 \\ 0 & 6x_2 \end{pmatrix}$$

$x_1 = (0,0)$ is the only stationary point.

Hessian is both positive semi-definite and negative semidefinite at $(0,0)$. We need to check further derivatives to make any conclusion of the point $(0,0)$

Read *Nonlinear Programming: Theory and Algorithms*, M.S Bazaraa, H.D. Sherali, C.M. Shetty, 3rd ed., 4.1.7. Example 2.

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Lagrangean Function : $L(x,\lambda,\mu) = f(x) + \sum_{j\in J} \lambda_j h_j(x) + \sum_{i\in I} \mu_i g_i(x)$

where

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t} \quad & g_i(x) \leq 0 \quad i \in I \\ & h_j(x) = 0 \quad j \in J \end{aligned}$$

<u>KKT conditions</u>:

i) Stationarity : $\nabla f(x) + \sum_{j\in J} \lambda_j \nabla h_j(x) + \sum_{i\in I} \mu_i g_i(x) = 0$

ii) Complementary Slackness : $\mu_i \, g_i(x) = 0 \quad \forall i \in I$

iii) Dual Feasibility : $\mu_i \geq 0 \quad \forall i \in I$

iv) Primal Feasibility : $g_i(x) \leq 0 \quad \forall i \in I$

$h_j(x) = 0 \quad \forall j \in J$

KKT conditions are necessary for global optimum. But they are not sufficient. If the NLP instance is a convex programming instance, KKT conditions are necessary and sufficient. An NLP is a convex programming instance if the feasible region is convex and the objective is minimization of a convex func. (or max. of concave func.).

(2)

$$L(x_1, x_2, \lambda, \mu_1, \mu_2) = x_1^2 + 2x_2^2 + \lambda(x_1 + x_2 - 2) + \mu_1(2x_1 - x_2 - 2) + \mu_2(x_2 - x_1)$$

i) Stationary Conditions:

$$\frac{\partial L(\dots)}{\partial x_1} = 2x_1 + \lambda + 2\mu_1 - \mu_2 = 0$$

$$\frac{\partial L(\dots)}{\partial x_2} = 4x_2 + \lambda - \mu_1 + \mu_2 = 0$$

ii) CSC:

$$\mu_1(2x_1 - x_2 - 2) = 0$$
$$\mu_2(x_2 - x_1) = 0$$

iii) Dual feas.

$$\mu_1 \geq 0$$
$$\mu_2 \geq 0$$

iv) Primal feas.

$$x_1 + x_2 = 2$$
$$2x_1 - x_2 \leq 2$$
$$x_1 \geq x_2$$

We can solve the equations case by case via "disecting" CSC.

Case I: $\mu_1 = 0$, $\mu_2 = 0$

from i) $\rightarrow$ $2x_1 + \lambda = 0$
$4x_2 + \lambda = 0$
iv $\rightarrow$ $x_1 + x_2 = 2$

$$\left.\begin{aligned} 2x_1 + \lambda &= 0 \\ 4x_2 + \lambda &= 0 \\ x_1 + x_2 &= 2 \end{aligned}\right\} \quad \begin{aligned} 2x_1 - 4x_2 &= 0 \\ x_1 + x_2 &= 2 \end{aligned} \quad \begin{aligned} x_1 &= 4/3 \\ x_2 &= 2/3 \end{aligned}$$

$\Rightarrow$ ii & iii are satisfied by assumption
i is satisfied by solving the system.
Check what is left in iv)

$2x_1 - x_2 \leq 2 \rightarrow 2 \leq 2$ ✓

$x_1 \geq x_2 \rightarrow 4/3 \geq 2/3$ ✓

$\Rightarrow$ $(4/3, 2/3)$ Satisfies all conditions $\Rightarrow$ $(4/3, 2/3)$ is a KKT point.

Case II: $\mu_1 = 0$, $x_2 - x_1 = 0$

from i) $\rightarrow$ $2x_1 + \lambda - \mu_2 = 0$
$4x_1 + \lambda + \mu_2 = 0$

$$\left.\begin{aligned} 2x_1 + \lambda - \mu_2 &= 0 \\ 4x_1 + \lambda + \mu_2 &= 0 \end{aligned}\right\} \quad \begin{aligned} \lambda - \mu_2 &= -2 \\ \lambda + \mu_2 &= -4 \end{aligned}$$

iv) $\rightarrow$ $x_1 + x_2 = 2 \rightarrow x_1 = x_2 = 1$

$\lambda = -3$

$\mu_2 = -1$

$\hookrightarrow$ violates dual feasibility

Case III: $2x_1 - x_2 - 2 = 0$, $\mu_2 = 0$

$$\text{iv)} \rightarrow \left.\begin{array}{l} 2x_1 - x_2 = 2 \\ x_1 + x_2 = 2 \end{array}\right\} \quad \begin{array}{l} x_1 = 4/3 \\ x_2 = 2/3 \end{array}$$

$$\text{i)} \rightarrow \begin{array}{l} 8/3 + \lambda + 2\mu_1 = 0 \\ 8/3 + \lambda - \mu_1 = 0 \\ \rightarrow \mu_1 = 0 \end{array}$$

$\Rightarrow$ Same point in Case I.

Case IV: $(2x_1 - x_2 - 2) = 0$, $(x_2 - x_1) = 0$

$$\left.\begin{array}{l} -x_1 + x_2 = 0 \\ 2x_1 - x_2 = 2 \end{array}\right\} \quad \left.\begin{array}{l} x_1 = 2 \\ x_2 = 2 \end{array}\right\} \text{ does not satisfy primal feasibility } (x_1 + x_2 = 2)$$

Only KKT point is $(4/3, 2/3)$. This is a local minimum. If we can conclude this NLP is a convex programming instance, then we can conclude that this point is global minimum.

Question 1: Is feasible region a convex set?

$x_1 + x_2 = 2$ is a line

$$\left.\begin{array}{r} 2x_1 - x_2 \leq 2 \\ x_1 \geq 2 \end{array}\right\} \text{ are half spaces}$$

The feasible region is an intersection of three convex sets, thus it is a convex set. It is actually a polyhedron since it is an intersection of half spaces.

Question 2: The NLP is a minimization. Is the objective function a convex function?

$$f(x) = x_1^2 + 2x_2^2$$

$$\nabla f(x) = \begin{pmatrix} 2x_1 \\ 4x_2 \end{pmatrix} \qquad H(x) = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

Hession is positive definite over the domain. Thus, f is strictly convex.

Q1 and Q2 answered "Yes" $\Rightarrow$ This NLP instance is a convex programming instance.

Therefore KKT conditions are necessary and <u>sufficient</u> to conclude that a KKT point is a global minimum.

$x = (4/3, 2/3)$ is a global minimum.

(3) Change obj. fnc. $\rightarrow$ min $-x_1 x_2$

$$L(x_1, x_2, \mu_1, \mu_2, \mu_3) = -x_1 x_2 + \mu_1 (x_1 + x_2^2 - 2) + \mu_2(-x_1) + \mu_3(-x_2)$$

i) Stationary Conditions:

$$\frac{\partial L(\dots)}{\partial x_1} = -x_2 + \mu_1 - \mu_2 = 0$$

$$\frac{\partial L(\dots)}{\partial x_2} = -x_1 + 2\mu_1 x_2 - \mu_3 = 0$$

ii) CSC:

$$\mu_1 (x_1 + x_2^2 - 2) = 0$$

$$\mu_2 x_1 = 0$$

$$\mu_3 x_2 = 0$$

iii) $\mu_1, \mu_2, \mu_3 \geq 0$

iv)

$$x_1 + x_2^2 \leq 2$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

Case I: $\mu_1 = \mu_2 = \mu_3 = 0$

$\rightarrow x_1 = x_2 = 0$ $\quad$ (0,0) KKT point

Case II: $\mu_1 = \mu_2 = x_2 = 0$

$\rightarrow -x_1 - \mu_3 = 0$

both $x_1 \geq 0, \mu_3 \geq 0$

$\rightarrow x_1 = 0 \rightarrow$ same point

Case III: $\mu_1 = x_1 = x_2 = 0$ $\quad$ same point

Case IV: $\mu_1 = x_1 = \mu_3 = 0$

i) $\rightarrow -x_2 - \mu_2 = 0$

$x_2 \geq 0, \mu_2 \geq 0$

$x_2 = 0, \mu_2 = 0$

Same point

Case V: $(x_1 + x_2^2 - 2) = \mu_2 = \mu_3 = 0$

$-x_2 + \mu_1 = 0 \rightarrow \mu_1 = x_2$

$-x_1 + 2\mu_1 x_2 = 0 \rightarrow -x_1 + 2x_2^2 = 0$

$x_1 + x_2^2 = 2$

$x_1 = 4/3 \quad x_2 = \sqrt{2/3} \quad \mu_1 = \sqrt{2/3}$

i) satisfied

ii) satisfied (CSC)

iii) satisfied $\mu_1 = \sqrt{2/3} \geq 0$, $\mu_2, \mu_3 \geq 0$

(iv) satisfied

$\Rightarrow \left(4/3, \sqrt{2/3}\right)$ KKT point

Case VI: $(x_1 + x_2^2 - 2) = x_1 = \mu_3 = 0$

$x_2 = \sqrt{2}$ ($x_2 = -\sqrt{2}$ not primal feasible)

i) $\rightarrow -\sqrt{2} + \mu_1 - \mu_2 = 0$

$0 + 2\sqrt{2}\mu_1 = 0 \quad \mu_1 = 0 \quad \mu_2 = -\sqrt{2}$

$\hookrightarrow$ not dual feasible

case VII: $(x_1 + x_2^2 - 2) = x_1 = x_2 = 0 \rightarrow$ Case Inconsistent

case VIII $(x_1 + x_2^2 - 2) = \mu_2 = x_2 = 0$

$x_1 = 2$

i) $\rightarrow \mu_1 = 0$

$-2 + 2\mu_1 \underset{0}{x_2} - \mu_3 = 0$

$\hookrightarrow \mu_3 = -2$ (not primal feasible)

Two KKT points: $(0,0)$, $\left(4/3, \sqrt{2/3}\right)$

Is this a convex programming instance?

$f(x_1, x_2) = -x_1 x_2$ convex?

$H(x) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \rightarrow$ indefinite $\rightarrow f$ neither convex nor concave

Then, No! A KKT point may or may not be a local minimum.