IE 203 - Operations Research II
Final - Solutions

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Duration: 2 hours
This is a CLOSED BOOK exam.

Question I (25 pts.)

a, b, f. Graphical representation of the feasible solutions and their convex hull is given in Figure 2

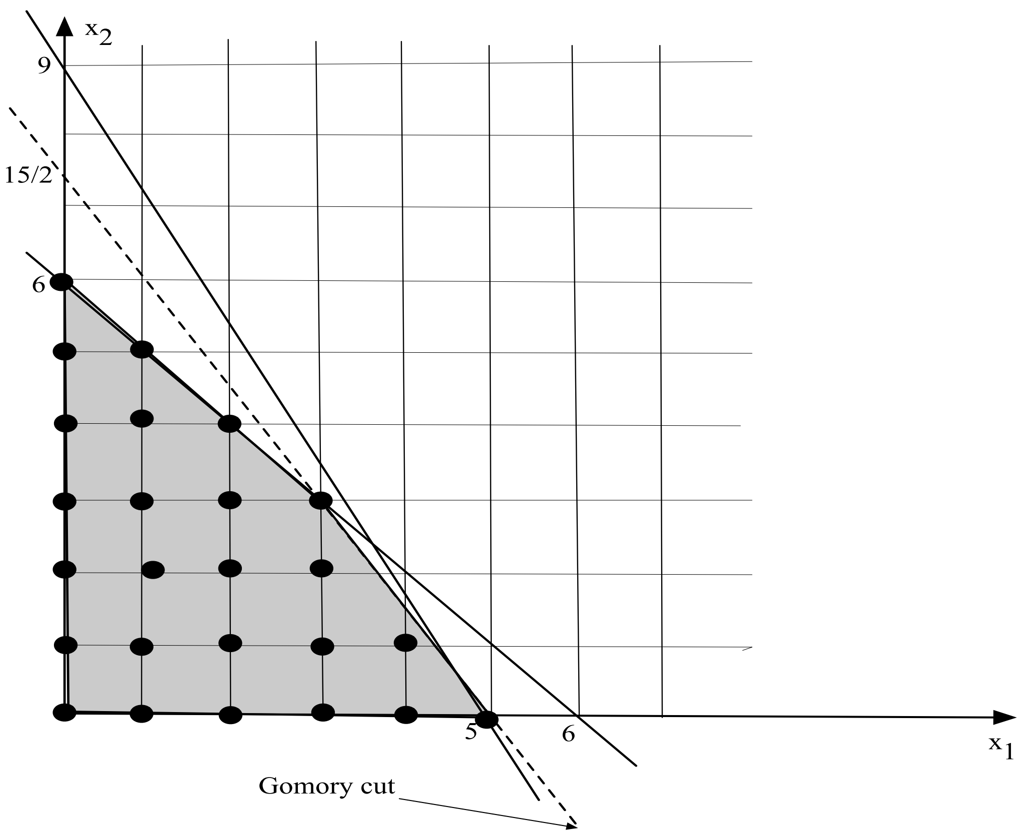

Figure 1: Feasible solutions, their convex hull and Gomory cut for Question I

The algebraic description of the convex hull:

$$\begin{aligned} x_1 + x_2 &\leq 6 \\ 3x_1 + 2x_2 &\leq 15 \\ x_1, x_2 &\geq 0 \end{aligned}$$

c. Optimal tableau:

	z	x_1	x_2	x_3	x_4	rhs
z	1	0	0	1.25	0.75	41.25
x_2	0	0	1	2.25	-0.25	2.25
x_1	0	1	0	-1.25	0.25	3.75

Here, x_3 and x_4 are respectively the slack variables of the first and second inequalities.

d. In the optimal solution $x_2 = 2.25$ and $x_1 = 3.75$. Hence, x_1 has the largest fractional part. Then, from the corresponding row we can write

$$(1+0)x_1 + (-2+0.75))x_3 + (0+0.25)x_4 = (3+0.75)$$

from which Gomory cut

$$0.75x_3 + 0.25x_4 \geq 0.75$$

can be obtained. It is clearly equivalent to

$$-0.75x_3 - 0.25x_4 \leq -0.75.$$

e. After adding Gomory cut the optimal tableau becomes

	z	x_1	x_2	x_3	x_4	x_5	rhs
z	1	0	0	1.25	0.75	0	41.25
x_2	0	0	1	2.25	-0.25	0	2.25
x_1	0	1	0	-1.25	0.25	0	3.75
x_5	0	0	0	-0.75	-0.25	1	-0.75

Then, applying dual simplex on x_5 results in the new optimal tableau:

	z	x_1	x_2	x_3	x_4	x_5	rhs
z	1	0	0	0	0.33	1.67	40
x_2	0	0	1	0	-1	3	0
x_1	0	1	0	0	0.67	-1.67	5
x_4	0	0	0	1	0.33	-1.33	1

As can be noticed this solution is also integer optimal with value 40.

f. To be able to show the generated Gomory cut

$$-0.75x_3 - 0.25x_4 \leq -0.75$$

it must be expressed equivalently in x_1 and x_2. For this purpose we use $x_3 = 6 - x_1 - x_2$ and $x_4 = 45 - 5x_1 - x_4$ in it to obtain

$$3x_1 + 2x_2 \leq 15,$$

which is shown in Figure 2 with the dashed line. Observe that it describes a facet of $conv(S)$.

Question II (25 pts.)

Concider the following optimization problem:

$$\min \ f(\mathbf{x}) = e^{x_1+x_2-2} - \frac{x_1^2 + x_2^2}{4}$$
$$\text{s.t.} - x_1 \leq 0, -x_2 \leq 0$$

The first order KKT conditions can be written as follows:

$$e^{x_1+x_2-2} - \frac{x_1}{2} - \mu_1 = 0$$
$$e^{x_1+x_2-2} - \frac{x_2}{2} - \mu_2 = 0$$
$$\mu_1 x_1 = 0$$
$$\mu_2 x_2 = 0$$
$$\mu_1 \geq 0$$
$$\mu_2 \geq 0$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

There are four possible cases:

1. $x_1 > 0$ and $x_2 > 0$: Then $\mu_1 = \mu_2 = 0$ and $x_1 = x_2$ follows as a consequence. However, $e^{2x_1-2} - \frac{x_1}{2} > 0$ and $e^{2x_2-2} - \frac{x_2}{2} > 0$ is always true for $x_1 > 0$ and $x_2 > 0$. In order to see this consider the function $f(x) = e^{2x-2} - \frac{x}{2}$ having $f'(x) = 2e^{2x-2} - \frac{1}{2}$ and $f''(x) = 4e^{2x-2}$. Notice that $f''(x) > 0$ for all $x \in \mathbb{R}$ and thus $f(x)$ is strictly convex it has a unique stationary point which is also global minimum. Then, $f'(x) = 0$ implies $e^{2x-2} = \frac{1}{4}$, from which $x^* = \frac{1}{2}(\ln \frac{1}{4} + 2)$ follows as the unique global minimum. As a result the minimum value of the function is $f(x^*) = \frac{1}{4}(\ln 4 - 1) = 0.096575$. At sum, $f(x) > 0$ for all $x \in \mathbb{R}$. Therefore, it is not possible to solve the system
$$e^{x_1+x_2-2} - \frac{x_1}{2} = 0$$
$$e^{x_1+x_2-2} - \frac{x_2}{2} = 0$$
and this case is not possible.
2. $x_1 = x_2 = 0$: Then $\mu_1 = \mu_2 = e^{-2} > 0$ and this is a stationary point.
3. $x_1 > 0$ and $x_2 = 0$: Then $\mu_1 = 0$. Besides $x_1 = 2$ and $\mu_2 = 1 > 0$ is a stationary point.
4. $x_2 > 0$ and $x_1 = 0$: Similar to the previous case. Now, $x_2 = 2$, $\mu_2 = 0$ and $\mu_1 = 1 > 0$.

These are the only stationary points. As it can be observed $f(0,0) = e^{-2}$, $f(2,0) = f(0,2) = 0$. Thus the minimum is reached at both $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ with 0 optimal value. Therefore
$$e^{x_1+x_2-2} - \frac{x_1^2 + x_2^2}{4} \geq 0$$
for all $x_1 \geq 0$ and $x_2 \geq 0$, which proves the assertion.

Question III (30 pts.)

Let Subject 1 = French, Subject 2 = English, and Subject 3 = Statistics. Then stage $n = 1, 2, 3$ correspond to subject $n = 1, 2, 3$, and the state s_n is the number of hours available for studying subjects $n, n+1, \ldots, 3$. The decision variables x_n, $n = 1, 2, 3$ are the number of hours of study allocated to subject n. Let $p_n(x_n)$ be the probability of passing subject n when x_n hours of study are allocated.

Our problem is to maximize the probability of passing at all 3 subjects. $f_n(s_n, x_n)$ is the probability of passing at subjects $n, n+1, \ldots, 3$ when s_n hours of study can be allocated to the study of subjects $n, n+1, \ldots, 3$, given that x_n hours of study are allocated to the study of subject n. The decision variables are x_n $n = 1, 2, 3$ and
$$f_n(s_n, x_n) = p_n(x_n) \cdot f^*_{n+1}(s_n - x_n)$$
where $f^*_n(s_n)$ is the maximum probability of passing at subjects $n, n+1, \ldots, 3$ when s_n hours can be allocated to all these subjects. Then
$$\begin{aligned} f^*_n(s_n) &= \max_{0 \leq x_n \leq s_n} \{f(s_n, x_n)\} \\ &= \max_{0 \leq x_n \leq s_n} \left\{p_n(x_n) \cdot f^*_{n+1}(s_n - x_n)\right\} \quad n = 1, 2, 3. \end{aligned}$$

Boundary conditions are $f^*_{N+1}(s_N - x_N) = 1$, namely $f^*_4(s_3 - x_3) = 1$ for all possible values of $s_3 - x_3$, and Zeynep is searching for $f^*_1(s_1 = 2)$.

$n = 1:$

$f^*_1(2) = \max_{x_1=0,1,2} \{p_1(x_1) \cdot f^*_2(2 - x_1)\} = \max \{0.6 f^*_2(2), 0.8 f^*_2(1), 0.85 f^*_2(0)\} = 0.18$ $\boxed{x^*_1 = 0}$

$n = 2:$

$f^*_2(2) = \max_{x_2=0,1,2} \{p_2(x_2) \cdot f^*_3(2 - x_2)\} = \max \{0.4 f^*_3(2), 0.6 f^*_3(1), 0.8 f^*_3(0)\} = 0.3$ $\boxed{x^*_2 = 1}$

$f^*_2(1) = \max_{x_2=0,1} \{p_2(x_2) \cdot f^*_3(1 - x_3)\} = \max \{0.4 f^*_3(1), 0.6 f^*_3(0)\} = 0.2$ $x^*_2 = 0$

$f^*_2(0) = \max_{x_2=0} \{p_2(x_2) \cdot f^*_3(0 - x_2)\} = \max \{0.4 f^*_3(0)\} = 0.08$ $x^*_2 = 0$

$n = 3:$

$f^*_3(2) = \max_{x_3=0,1,2} \{p_3(x_3) \cdot f^*_4(2 - x_3)\} = \max \{0.2 f^*_4(2), 0.5 f^*_4(1), 0.7 f^*_4(0)\} = 0.7$ $x^*_3 = 2$

$f^*_3(1) = \max_{x_3=0,1} \{p_3(x_3) \cdot f^*_4(1 - x_3)\} = \max \{0.2 f^*_4(1), 0.5 f^*_4(0)\} = 0.5$ $\boxed{x^*_3 = 1}$

$f^*_3(0) = \max_{x_3=0} \{p_3(x_3) \cdot f^*_4(0 - x_3)\} = \max \{0.2 f^*_4(0)\} = 0.2$ $x^*_3 = 0$

Zeynep has to spend 0 hour studying French, 1 hour studying German, and 1 hour studying Statistics. This optimal policy maximizes her chance of passing all three courses, which is 0.18.

Question IV (20 pts.)

a. Let the two states be 0 = Rain and 1 = Clear. Then, the (one-step) transition matrix is

$$\mathbf{P} = \begin{pmatrix} 0.5 & 0.5 \\ 0.1 & 0.9 \end{pmatrix}.$$

b. State transition diagram is given in Figure 2 below.

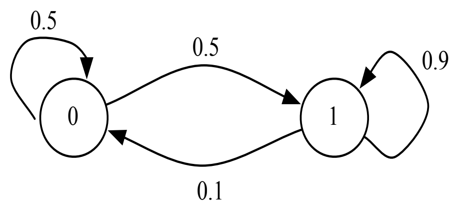

Figure 2: State Transition Diagram

c. Two-step transition matrix is

$$\mathbf{P}^{(2)} = \mathbf{PP} = \begin{pmatrix} 0.5 & 0.5 \\ 0.1 & 0.9 \end{pmatrix}\begin{pmatrix} 0.5 & 0.5 \\ 0.1 & 0.9 \end{pmatrix} = \begin{pmatrix} 0.3 & 0.7 \\ 0.14 & 0.86 \end{pmatrix}.$$

$\Pr\{\text{Rain 2 days from now} \setminus \text{Rain today}\} = p_{11}^{(2)} = 0.3$

$\Pr\{\text{Rain 2 days from now} \setminus \text{Clear today}\} = p_{21}^{(2)} = 0.14$

If the probability it will rain today is 0.5, then $\Pr\{\text{Rain 2 days from now}\} = 0.5p_{11}^{(2)} + 0.5p_{21}^{(2)} = 0.5 \times 0.3 + 0.5 \times 0.14 = 0.22$.

d. Five step transition matrix is

$$\mathbf{P}^{(5)} = \mathbf{PP}^{(2)}\mathbf{P}^{(2)} = \begin{pmatrix} 0.5 & 0.5 \\ 0.1 & 0.9 \end{pmatrix}\begin{pmatrix} 0.3 & 0.7 \\ 0.14 & 0.86 \end{pmatrix}\begin{pmatrix} 0.3 & 0.7 \\ 0.14 & 0.86 \end{pmatrix} = \begin{pmatrix} 0.1767 & 0.8248 \\ 0.16496 & 0.83504 \end{pmatrix}.$$

$\Pr\{\text{Clear 5 days from now} \setminus \text{Rain today}\} = p_{12}^{(5)} = 0.8248$

$\Pr\{\text{Clear 5 days from now} \setminus \text{Clear today}\} = p_{22}^{(5)} = 0.83504$

If the probability it will not rain today is 0.3, then $\Pr\{\text{Clear 5 days from now}\} = 0.7p_{12}^{(5)} + 0.3p_{22}^{(5)} = 0.7 \times 0.8248 + 0.3 \times 0.83504 = 0.827872$.