Sparse Sets in Triangle-free Graphs

Tınaz Ekim[1], Burak Nur Erdem[1], and John Gimbel[2]

[1]Department of Industrial Engineering, Boğaziçi University, 34342, Bebek, Istanbul, Turkey.

[2]Mathematics and Statistics, University of Alaska, Fairbanks, AK, 99775-6660, USA.

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Abstract

A set of vertices is k-sparse if it induces a graph with a maximum degree of at most k. In this missive, we consider the order of the largest k-sparse set in a triangle-free graph of fixed order. We show, for example, that every triangle-free graph of order 11 contains a 1-sparse 5-set; every triangle-free graph of order 13 contains a 2-sparse 7-set; and every triangle-free graph of order 8 contains a 3-sparse 6-set. Further, these are all best possible.

For fixed k, we consider the growth rate of the largest k-sparse set of a triangle-free graph of order n. Also, we consider Ramsey numbers of the following type. Given i, what is the smallest n having the property that all triangle-free graphs of order n contain a 4-cycle or a k-sparse set of order i. We use both direct proof techniques and an efficient graph enumeration algorithm to obtain several values for defective Ramsey numbers and a parameter related to largest sparse sets in triangle-free graphs, along with their extremal graphs.

Keywords: defective Ramsey numbers; k-dense; k-sparse; k-dependent; extremal graphs.

1 Introduction

Given positive integers i and j, the Ramsey number $R(i, j)$ is the smallest natural number such that every graph of order at least $R(i, j)$ has a clique of order i or an independent set of order j. These so called classical Ramsey numbers along with a number of variations are extensively studied in the literature. Among various generalizations we find so called defective Ramsey numbers that have been the focus of several research papers [2, 5, 6, 7, 8, 9, 10]. This variation relaxes the notions of cliques and independent sets in the following way. A *k-sparse set* is a set of vertices that induces a graph with maximum degree k or less. A *k-dense set* is the complement of a k-sparse set. In other words, each vertex in a k-dense set "misses" at most k other vertices in its neighborhood. A k-sparse (k-dense) *j-set* is a k-sparse (k-dense) set of order j. A set is *k-defective* (or *k-uniform*) if it is a k-sparse or k-dense set. The *defective Ramsey number* $R_k^{\mathcal{G}}(i, j)$ is the least n such that all graphs of order n in the class $\mathcal{G}$ have either a k-dense i-set or a k-sparse j-set.

In [6], some specific 1-defective Ramsey numbers are derived under a somewhat different terminology. In [5, 9], additional 1-defective and 2-defective Ramsey numbers are found using direct proofs. Further, several bounds are displayed for defective Ramsey numbers. It seems direct proofs have reached their limits in finding new values of defective Ramsey numbers. Indeed, this is rather not surprising given the great difficulty in computing specific defective Ramsey numbers. Having observed this fact, some computer based generation methods are used in [2, 5, 8] to improve known bounds on defective Ramsey numbers (and certain other defective parameters).

Noting that computing exact Ramsey numbers is extremely unlikely, various approaches are adopted in mathematical literature to partially deal with this problem. One tactic is to consider restricted graph families. In [20] for example, all classical Ramsey numbers in planar graphs are found. In [13, 12, 19], the authors compute several Ramsey numbers for graphs with bounded degree. Ramsey numbers for claw-free graphs are discussed in [16]. After all these studies dating back to 1980's and 1990's, this approach seems to have become popular again. In [3], we find a systematic study of Ramsey numbers in various graph classes. It seems that computation for claw-free graphs is as difficult as it is for arbitrary graphs. Further, [3] exhibits all classical Ramsey numbers for perfect graphs and some well-known subclasses of claw-free graphs. We note another work [4] focuses on the complexity of the coloring problem where each color class is a k-sparse set (called the k-defective coloring problem) when restricted to subclasses of perfect graphs.

Recently, the approach of considering Ramsey numbers in various graph classes has been applied to defective Ramsey numbers. In [10], Ekim et al. present small 1-defective Ramsey numbers for perfect graphs. In [7], Demirci et al. study k-defective Ramsey numbers (for any k) and provide exact formulas for forests, cacti, bipartite graphs, split graphs and cographs. They provide conjectures for the few exceptions left as open questions. In both of these studies [10] and [7], the authors observe that the limits of direct proof techniques seem to be reached. As such, Demirci et al. focuses more recently in [8] on the computation of defective Ramsey numbers by combining their efficient graph generation algorithm, called Sub-extremal, with classical proof techniques. They provide new defective Ramsey numbers in perfect graphs, bipartite graphs, and chordal graphs.

In this paper, we investigate defective Ramsey numbers in triangle-free graphs. Our contributions are two-fold: we provide both direct proofs and computer assisted results using an efficient implementation of the Algorithm Sub-extremal from [8] adapted for triangle-free graphs. Some simple observations show that dense sets in triangle-free graphs are very restricted. This implies that we rather focus on the existence of sparse sets of given size. From [1] we know that every triangle-free graph of order n contains an independent set of order at least $\sqrt{n \log n}$ and by [14] this is asymptotically best possible. In this work, we extend this notion and consider large sparse sets in triangle-free graphs. We work out some specific values and produce computer assisted proofs of others. All defective Ramsey numbers in triangle-free graphs obtained with direct proof techniques in Sections 3 and 4, as well as those obtained using an efficient graph generation approach in Section 5 are summarized in Tables 1, 2, 3, 4, 5. We postpone a more detailed description of our results until the end of Section 2, after all formal definitions, notations and preliminary remarks are introduced.

2 Definitions and preliminary remarks

Let $G = (V, E)$ be a graph. We will denote the order $|V|$ of a graph by n. A subgraph $H \subseteq G$ is a graph on $V' \subseteq V$ and $E' \subseteq E$ with both end-vertices of each edge of E' in V'. If all edges with both end-vertices in V' are in E', then H is said to be an *induced* subgraph of G. In our context, whenever we say that a graph contains a subgraph, we always mean as a partial subgraph, unless stated otherwise. For a vertex $x \in V$, we denote by $N(x)$ the set of *neighbors* of x, that is, vertices adjacent to x. The *degree* of a vertex x is $d(x) = |N(x)|$. We also have $N[x] = N(x) \cup \{x\}$. For a vertex $x \in V$ and a subgraph $H \subseteq G$, we denote by $N_H(x)$ the set of neighbors of x in H, that is $N(x) \cap V(H)$. Similarly, the degree of x in H is $d_H(x) = |N_H(x)|$. For a subset of vertices $X \subset V$, the *neighborhood* of X, denoted by $N(X)$, is defined as $N(X) = (\cup_{x \in X} N(x)) \setminus X$.

For a graph G and a subgraph H, we use the notation $G \setminus H$ to mean the subgraph of G induced by all vertices in $V(G) \setminus V(H)$. We also use the same notation when we remove a set of vertices from a graph. For graphs H and G, we say that G is *H-free* if it does not contain H as an induced subgraph. A *path* on n vertices is denoted by P_n, and a *cycle* on n vertices, also called an *n-cycle*, is denoted by C_n. A complete bipartite graph on p and ℓ vertices in each part is denoted by $K_{p,\ell}$. The *distance* between two vertices is the length of a shortest path between them. The *girth* of a graph G, denoted by $g(G)$, is the length of a shortest induced cycle in it. A set of vertices is called *independent* if all vertices in it are pairwise non-adjacent. Generalizing the notation for the size of a largest independent set $\alpha(G)$, we adopt the notation $\alpha_k(G)$ to denote the size of a largest k-sparse set of a graph G. An *extremal* graph for $R_k^{\mathcal{G}}(i, j)$ is a graph in the class $\mathcal{G}$ on $R_k^{\mathcal{G}}(i, j) - 1$ vertices containing neither a k-dense i-set nor a k-sparse j-set. We use the notation $R_k^{\Delta}(i, j)$ to denote the k-defective Ramsey numbers in triangle-free graphs.

In this work, sparse sets in triangle-free graphs will be our main focus. This is justified by the following remarks. First, let us note that in general, finding such sets is difficult; we know from [9] that finding a largest k-sparse set for any fixed k is NP-complete even in restricted cases:

Theorem 2.1. *[9] For fixed $k \geq 2$, given a graph G and an integer t, the problem of deciding if $\alpha_k(G) \geq t$ is NP-complete. The result holds when restricted to planar graphs with maximum degree $k + 1$ and girth g where g is arbitrarily large.*

Moreover, Theorem 2.1 holds for $k = 1$ when the maximum degree is three [9]. For $k = 0$, deciding if there is an independent set of size at least t is NP-complete in several restricted cases including triangle-free graphs [18].

We proceed with some observations on the absence of large dense sets in triangle-free graphs. This motivates the study of sparse sets in triangle-free graphs in further sections.

Remark 2.2. *In a triangle-free graph G, a 1-dense 4-set can only be a C_4. Morover, G does not admit 1-dense i-set for any $i \geq 5$.*

In a similar way, we can show that the only 2-dense 5-sets are C_5 and $K_{2,3}$, and the only 2-dense 6-set is $K_{3,3}$. These observations can be generalized for $k \geq 2$ as follows.

Proposition 2.3. *In a triangle-free graph, there is no k-dense i-set for $i \geq 2k + 3$. Moreover, this bound is best possible and the unique k-dense triangle-free graph on $2k + 2$ vertices is $K_{k+1,k+1}$.*

Proof. Let G be a triangle-free graph and assume to the contrary it contains a k-dense set, say A, having at least $2k+3$ vertices. Let x be a vertex of A. Note x can miss at most k other vertices of A. Hence x is adjacent to at least $k+2$ other vertices of A. Since G is triangle-free, $N(x) \cap A$ is an independent set. But then, a vertex in $N(x) \cap A$ misses the other $k+1$ vertices of $N(x) \cap A$, contradicting the fact that A is k-dense.

Note, a k-dense graph of order $2k+2$ contains at least $(k+1)^2$ edges. By Turan's Theorem [21], there is only one triangle-free graph of order $2k+2$ on $(k+1)^2$ edges, namely $K_{k+1,k+1}$. Further, if a graph of order $2k+2$ contains more than $(k+1)^2$ edges, it must contain a triangle. Hence, our result is best possible. □

Since there is no k-dense i-set for $i \geq 2k+3$ in a triangle-free graph, Proposition 2.3 implies that for each k, we have $R_k^{\Delta}(i,j) = R_k^{\Delta}(i',j)$ for all $i, i' \geq 2k+3$. Without k-dense sets, it makes sense to focus on k-sparse sets. This suggests the following notation. Let $T_k(j)$ be the minimum order n such that every triangle-free graph of order n has a k-sparse set of size j. We would say an *extremal* graph for $T_k(j)$ is a triangle-free graph with $T_k(j)-1$ vertices having no k-sparse set of order j. With this notation, we have $R_k^{\Delta}(i,j) = T_k(j)$ for all $i \geq 2k+3$. Motivated by this, we proceed by proving some exact values for $T_k(j)$ in Section 3. Then, in Section 4, we focus on $R_1^{\Delta}(3,j)$ for $j \geq 3$, and $R_1^{\Delta}(4,j)$ for $j \geq 4$, the only 1-defective Ramsey numbers of interest for triangle-free graphs. Both Sections 3 and 4 contain results shown by classical proof techniques. In Section 5, we compute several new values by efficient computer enumeration techniques. Based on these results, we conjecture that $T_k(k+i) = k+2i-1$ for all i and k such that $2 \leq i \leq k$. All of our codes and the extremal graphs we obtain are available online at [11].

3 Sparse sets in triangle-free graphs

The following lower bound on $\alpha_k(G)$ allows us to derive some values of $T_k(n)$. Note that the following lower bound is for general graphs, not restricted to triangle-free graphs.

Proposition 3.1. *For a graph G and fixed k, we have $\alpha_k(G) \geq \frac{n}{\left\lceil \frac{\Delta(G)+1}{k+1} \right\rceil}$.*

Proof. We rely on a proof technique found in [15]. Set $j = \left\lceil \frac{\Delta(G)+1}{k+1} \right\rceil$. Color the vertices of G with j colors so that the number of monochromatic edges (those edges having the same color on both end-vertices) is minimized. We claim that every color class is a k-sparse set. Assume this does not hold, that is, there is a vertex x with at least $k+1$ neighbors of the same color as x. Then one of the remaining $j-1$ colors, say c, occurs at most k times in the neighborhood of x, since otherwise $d(x) \geq \Delta(G)+1$, a contradiction. By recoloring x with c we obtain a coloring of G with fewer monochromatic edges, a contradiction. Now, by the Pigeonhole Principle, one of the color classes has at least $\frac{n}{\left\lceil \frac{\Delta(G)+1}{k+1} \right\rceil}$ vertices, and the proof is complete. □

Before investigating specific values of $T_k(n)$, it is worth noting the case where $k=0$, in which, the sparse set in question is an independent set. Consequently, the parameter $T_0(j)$ is equivalent to the classical Ramsey number $R(3,j)$. Moving onto the sparse sets, the first non-trivial value is $T_1(3) = 5$. Note C_4 is an extremal graph on 4 vertices. Suppose G is a triangle-free graph of order 5. If G is bipartite, it contains an independent set on

3 vertices. So, suppose G is not bipartite. Note G contains an odd cycle which is not a triangle. Hence G contains a 5-cycle without chord. This graph contains a 1-sparse 3-set.

In the following, we will repeatedly use (without explicitly mentioning it) the observation that any open neighborhood in a triangle-free graph is independent.

Theorem 3.2. *With the preceeding notation, $T_1(4) = 7$.*

Proof. $K_{3,3}$ is a triangle-free graph of order 6 which does not contain a 1-sparse 4-set. Thus, $T_1(4) \geq 7$. Let G be a triangle-free graph of order 7. If $\Delta(G) \geq 4$, then $N(x)$ contains a 1-sparse 4-set. If $\Delta(G) \leq 3$, by Proposition 3.1, the cardinality of a 1-sparse set is at least 4. Consequently, every triangle-free graph with 7 vertices includes a 1-sparse 4-set. Hence, the desired result. □

Theorem 3.3. *With the preceeding notation, $T_1(5) = 11$.*

Proof. The blow-up of a C_5 where every vertex is replaced with two independent vertices is a graph or order 10 which contains no 1-sparse 5-set. Thus, $T_1(5) \geq 11$.

Let G be a triangle-free graph of order 11. If G contains a vertex x of degree at least 5, then $N(x)$ contains a 1-sparse 5-set. So suppose $\Delta(G) \leq 4$. If G has a vertex x of degree 3, then $V \setminus N[x]$ has 7 vertices and contains a 1-sparse 4-set, say A, by Theorem 3.2. Now, $A \cup \{x\}$ is a 1-sparse 5-set. So, suppose G has no vertex of degree three and similarly no vertex of degree less than three.

So, assume G is a 4-regular triangle-free graph of order 11. For some vertex x, let $A = N(x)$ and $B = V \setminus N[x]$. Note A is independent. We have $|A| = 4$, $|B| = 6$ where each vertex of A is adjacent with exactly three vertices in B. Thus, there are exactly 12 edges with one end-vertex in A and the other in B. If there is a vertex $b \in B$ having at most one neighbor in A, then $A \cup \{b\}$ is a 1-sparse 5-set. Otherwise, every vertex of B has exactly 2 neighbors in A. This implies that the graph induced by B is 2-regular; thus a 6-cycle (since triangles are forbidden). Taking a 1-sparse 4-set in this 6-cycle together with x yields a 1-sparse 5-set. Hence, the desired result. □

In the sequel, we study 2-sparse sets in triangle-free graphs. We start with the first non-trivial value of the $T_k(j)$ for all $k \geq 2$.

Theorem 3.4. *With the preceeding notation, $T_k(k+2) = k+3$, for $k \geq 2$ with $K_{1,k+1}$ as the unique extremal graph.*

Proof. $K_{2,k+2}$ is a graph which do not include any triangles nor any k-sparse $(k+2)$-sets. Therefore, $T_k(k+2) \geq k+3$.

Let G be a triangle-free graph of order $k+3$ and let x be a vertex of maximum degree. If $\Delta(G) \leq k$ then G is k-sparse. If $\Delta(G) \geq k+2$, then $N(x)$ is independent and hence contains a k-sparse $(k+2)$-set. Suppose $\Delta(G) = k+1$, and let y be the vertex in $V \setminus N[x]$. Since $k \geq 2$, the set $\{x, y\}$ together with any k vertices from $N(x)$ is a k-sparse $(k+2)$-set. Thus, $T_k(k+2) = k+3$.

Let us now show that $K_{1,k+1}$ is the unique extremal graph. Indeed, by the previous observation, an extremal graph with $k+2$ vertices has maximum degree $k+1$ or else it has a k-sparse $(k+2)$-set. Since it is a triangle-free graph, it can only be a $K_{1,k+1}$. □

Theorem 3.4 implies in particular that $T_2(4) = 6$. We proceed with the next values.

Theorem 3.5. *With the preceeding notation, $T_2(5) = 9$.*

Proof. $K_{4,4}$ contains no 2-sparse 5-set. Thus, $T_2(5) \geq 9$. Let G be a triangle-free graph of order 9. If $\Delta(G) \geq 5$, the neighborhood of a maximum degree vertex contains a 2-sparse 5-set. If $\Delta(G) \leq 4$, there exists a 2-sparse set of size at least 5, by Proposition 3.1. Hence, the desired result. □

Theorem 3.6. *With the preceeding notation, $T_2(6) = 11$.*

Proof. $K_{5,5}$ contains no 2-sparse 6-set. Thus, $T_2(6) \geq 11$. Let G be a triangle-free graph of order 11. If $\Delta(G) \geq 6$, the neighborhood of a maximum degree vertex contains a 2-sparse 6-set. If $\Delta(G) \leq 5$, there exists a 2-sparse set of size at least 6, by Proposition 3.1. Hence, the desired result. □

Theorem 3.7. *With the preceeding notation, $T_2(7) = 13$.*

Proof. $K_{6,6}$ is a triangle-free graph of order 12 which has no 2-sparse 7-set. Thus $T_2(7) \geq 13$. Let G be a triangle-free graph of order 13. If $\Delta(G) \geq 7$, the neighborhood of a maximum degree vertex contains a 2-sparse 7-set. If $\Delta(G) \leq 5$, there exists a 2-sparse set of size at least 7, by Proposition 3.1. So assume $\Delta(G) = 6$ and let x be a vertex of degree six. Let $N(x) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $U = V \setminus N[x] = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ as shown in Figure 1. If U is a 2-sparse 6-set, then $\{x\} \cup U$ is a 2-sparse 7-set. Otherwise, there exists a vertex in U, say u_1, which is adjacent to at least three other vertices in U. Let $\{u_2, u_3, u_4\} \subseteq N(u_1)$. If u_1 has at most two neighbors in $N(x)$, then $\{u_1\} \cup N(x)$ is a 2-sparse 7-set. So assume u_1 is adjacent to at least three vertices in $N(x)$, say without loss of generality v_1, v_2, v_3. Note that the sets $\{u_2, u_3, u_4, v_1, v_2, v_3\} \subseteq N(u_1)$ and $N(x)$ are independent since G is triangle-free. Accordingly, we claim that the $\{u_2, u_3, v_1, v_2, v_3, v_4, v_5\}$ shown in Figure 1 is a 2-sparse 7-set. Indeed, the graph induced by this set can only have edges between vertices in $\{v_4, v_5\}$ and $\{u_2, u_3\}$; yielding at most two neighbors for any vertex. Consequently, there exists a 2-sparse set of size 7 in every triangle-free graph of order 13. Hence, the desired result.

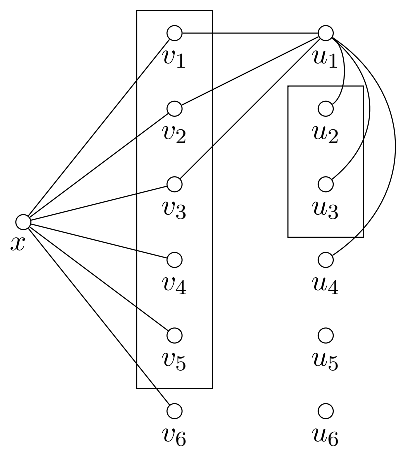

Figure 1: Illustration for the proof of $T_2(7) = 13$ in Theorem 3.7.

□

As for $k=3$, the first non-trivial value $T_3(5)=6$ is implied by Theorem 3.4. The next value for all $k \geq 3$, namely $T_k(k+3)$, is provided by the following theorem. For $k=3$, it implies $T_3(6)=8$.

Theorem 3.8. *With the preceeding notation, $T_k(k+3)=k+5$, for $k \geq 3$ with $K_{2,k+2}$ and $K_{2,k+2}$ with a missing edge as the only two extremal graphs.*

Proof. $K_{2,k+2}$ and $K_{2,k+2}$ with a missing edge are two graphs which do not include any triangles nor any k-sparse $(k+3)$-sets. Therefore, $T_k(k+3) \geq k+5$.

Let G be a triangle-free graph of order $k+5$ and let x be a vertex of maximum degree. If $\Delta(G) \leq k$ then G is k-sparse. If $\Delta(G) \geq k+3$, then $N(x)$ is independent and hence contains a k-sparse $(k+3)$-set.

Suppose $\Delta(G)=k+2$, and let y_1 and y_2 be the vertices in $V \setminus N[x]$. If $y_1y_2 \notin E$ then, since $k \geq 3$, the set $\{x, y_1, y_2\}$ together with any k vertices from $N(x)$ is a k-sparse $(k+3)$-set. If $y_1y_2 \in E$ then no vertex $v \in N(x)$ is adjacent to both y_1 and y_2, since G is triangle-free. Note, $|N(x)|=k+2$, it follows that at least one of y_1 and y_2 has at most k neighbors in $N(x)$, say y_1. Then $\{y_1\} \cup N(x)$ is a k-sparse $(k+3)$-set.

So suppose $\Delta(G)=k+1$, and let y_1, y_2 and y_3 be the vertices in $V \setminus N[x]$. Since G is triangle-free, there exist a non-edge between two of these three vertices. Without loss of generality, say $y_1y_2 \notin E$. Since $k \geq 3$, the set $\{x, y_1, y_2\}$ together with any k vertices from $N(x)$ is a k-sparse $(k+3)$-set. Thus, $T_k(k+3)=k+5$.

Now, let us show that there is no extremal graph other than $K_{2,k+2}$ and $K_{2,k+2}$ with a missing edge. So suppose G is an extremal graph of order $k+4$ and G is not $K_{2,k+2}$ nor $K_{2,k+2}$ minus an edge. Let again x be a vertex of maximum degree. It follows from our previous observations that we can assume $k+1 \leq \Delta(G) \leq k+2$ or else there is a trivial k-sparse $(k+3)$-set, contradicting the fact that G is extremal.

So suppose $\Delta(G)=k+1$. Let u and w be the two vertices not in $N[x]$. Suppose $uw \notin E$, then removing an element of $N(x)$ from G creates a k-sparse $(k+3)$-set. So suppose $uw \in E$. Then u and w share no common neighbors since G is triangle-free. Thus one of them, say u, is adjacent with at most $k-1$ vertices in $N(x)$. If w is adjacent with some vertex, say y, in $N(x)$ then removing y from G produces a k-sparse $(k+3)$-set. If w is not adjacent with anything in $N(x)$ then it has degree 1. Thus, removing x produces a k-sparse $(k+3)$-set.

So suppose $\Delta(G)=k+2$. Let u be the vertex not in $N[x]$. If the degree of u is $k+1$ or $k+2$ then G is a graph forbidden above. So suppose the degree of u is at most k. Then removing x produces a k-sparse $(k+3)$-set, completing the proof. □

Next, we prove the analogous of Theorems 3.4 and 3.8 for $k \geq 4$.

Theorem 3.9. *With the preceeding notation, $T_k(k+4)=k+7$, for $k \geq 4$ with $K_{3,k+3}$ as an extremal graph.*

Proof. Observe that $K_{3,k+3}$ does not include any triangles nor any k-sparse $(k+4)$-sets. Therefore, $T_k(k+3) \geq k+7$.

Let G be a triangle-free graph of order $k+7$ and let x be a maximum degree vertex of G. Similarly to the proof of Theorem 3.8, if $\Delta(G) \geq k+4$ then $N(x)$ includes a k-sparse $(k+4)$-set. If $\Delta(G) \leq k$ then G is k-sparse.

Suppose $\Delta(G) = k + 3$, and denote the remaining vertices in $V \setminus N[x]$ by $Y = \{y_1, y_2, y_3\}$. If Y is an independent set, then G is bipartite with $\{x\} \cup Y$ as one independent set and $N(x)$ as the other. Consequently, k vertices from $N(x)$ together with $\{x\} \cup Y$ is a k-sparse $(k+4)$-set, since $k \geq 4$. If Y is not an independent set, say $y_1y_2 \in E$, then a vertex from $N(x)$ cannot be adjacent to both y_1 and y_2, since G is triangle-free. For $k \geq 4$, either y_1 or y_2 has at most $k-1$ neighbors in $N(x)$, say it is y_1. Then $\{y_1\} \cup N(x)$ is a k-sparse $(k+4)$-set.

So suppose $\Delta(G) = k + 2$. Denote the vertices in $V \setminus N[x]$ by $Y = \{y_1, y_2, y_3, y_4\}$. If $\alpha(G[Y]) \geq 3$, noting $k \geq 4$, an independent set of $G[Y]$ of size 3, together with x and k vertices from $N(x)$ is a k-sparse $(k+4)$-set. If $\alpha(G[Y]) \leq 2$, then since G is triangle-free $G[Y]$ has $2K_2$ as a subgraph. Without loss of generality, say $\{y_1y_2, y_3y_4\} \in E$. Similar to the reasoning before, a vertex from $N(x)$ cannot be adjacent to both y_1 and y_2. Consequently, at least one of y_1 and y_2 has at most $k-1$ neighbors in $N(x)$, say y_1 has this property. By symmetry, we can also assume y_3 has at most $k-1$ neighbors in $N(x)$. Then, $\{y_1, y_3\} \cup N(x)$ is a k-sparse $(k+4)$-set.

So suppose $\Delta(G) = k+1$, and denote the vertices in $V \setminus N[x]$ by $Y = \{y_1, y_2, y_3, y_4, y_5\}$. If $G[Y]$ is bipartite, then $\alpha(G[Y]) \geq 3$, since $k \geq 4$, an independent set of size 3, together with x and k vertices from $N(x)$ is a k-sparse $(k+4)$-set. If $G[Y]$ is not bipartite, then it induces a C_5 with vertices y_1, y_2, y_3, y_4, y_5 in order, since G is triangle-free. Note a vertex of C_5 can have at most $k-1$ neigbors in $N(x)$, since $\Delta(G) = k+1$. As a result, for $k \geq 4$, the set $\{y_1, y_2, y_4, x\}$ and k vertices from $N(x)$ is a k-sparse $(k+4)$-set. $\square$

We stop proving exact values of $T_k(j)$ and leave the computation of further values using a computer enumeration algorithm for Section 5. We conclude this section with the following result that establishes the growth rate of $T_k(n)$.

Theorem 3.10. *For fixed k, we have $T_k(n) = \Theta(\frac{n^2}{\log n})$.*

Proof. Fix $k \geq 1$. We know that $c_1 \frac{n^2}{\log n} \leq R(3, n) \leq c_2 \frac{n^2}{\log n}$ for some positive constants c_1 and c_2. The first bound is established in [14] and the second in [1]. Note, if G is a triangle-free graph of order at least $c_2 \frac{n^2}{\log n}$ then G contains an independent set of order n. Thus, it contains a k-sparse set of order n. Accordingly, $T_k(n) \leq c_2 \frac{n^2}{\log n}$.

So set $j = \left\lfloor c_1 \frac{n^2}{\log n} \right\rfloor$ and let H be a triangle-free graph of order j which contains no independent set of order n. Let H' be the lexicographic product of H with an empty graph of order $2k$. Informally, we can think of blowing up each vertex of H with $2k$ isolated vertices, while preserving adjacencies. For a vertex v in H, let S_v be the "blown up" vertices of H' that correspond with v. Thus, for each v, the set S_v is independent and if uv is an edge of H, then every vertex of S_u is adjacent with each vertex of S_v. Further, an independent set having order n in H corresponds with an independent set in H' with order $2kn$. Note also that H' contains no triangle.

Let T be a k-sparse set of H' having maximum order. Note, $|T| \geq 2k\alpha(H)$. Suppose uv are adjacent in H and T meets both S_u and S_v. Say x and v are in T and $x \in S_u$ and $y \in S_v$. Note, at most k elements of S_u belong to T. Further, x can be adjacent to at most k elements of T. So take all members if T adjacent to x and remove them. Replace them with the vertices in S_u. This is a k-sparse set also of maximum order. Further, when the vertices of T are "shrunk" to H, an independent set is formed. Accordingly,

$|T| = 2k\alpha(H)$. Hence $\alpha_k(H') = 2k\alpha(H)$. As $\alpha(H) < n$, we note that H' is a graph of order $2kj$ which contains no k-sparse set of order $2kn$.

Thus, $T_k(2kn) > 2kj$. As k is fixed we are allowed a change of variable and note $T_k(n) \geq c_3 \frac{n^2}{\log n}$, for some positive constant c_3, and thus our desired conclusion. □

4 Some defective Ramsey numbers in triangle-free graphs

As noted earlier, the only interesting 1-defective Ramsey numbers in triangle-free graphs are $R_1^\Delta(3, j)$ for $j \geq 3$, and $R_1^\Delta(4, j)$ for $j \geq 4$ since there is no 1-dense i-set for $i \geq 5$ in a triangle-free graph.

Let us first deal with $R_1^\Delta(3, j)$ for $j \geq 3$, and more generally with $R_k^\Delta(k+2, j)$ for $j \geq k+2$. It is enough to note that the proof for $R_k(k+2, j) = j$ for all $j \geq k+2$ in general graphs given in [9] is also valid in triangle-free graphs. Thus, we have the following, which is also certified by computer enumeration in Section 5 (see Tables 2, 3, 4, 5).

Remark 4.1. *With the preceeding notation, $R_k^\Delta(k+2, j) = j$ for $j \geq k+2$.*

In what follows, we investigate $R_1^\Delta(4, j)$ for $j \geq 4$. Recall that C_4 is the only triangle-free 1-dense 4-set. In this section, we show $R_1^\Delta(4, 4) = 6, R_1^\Delta(4, 5) = 8, R_1^\Delta(4, 6) = 10$ and $R_1^\Delta(4, 7) = 13$. We provide extremal graphs for each result. Uniqueness will be established in Section 5 using computer enumeration.

Theorem 4.2. *With the preceeding notation, $R_1^\Delta(4, 4) = 6$ with the unique extremal graph being C_5.*

Proof. Note that C_5 is triangle-free which does not contain any 1-dense 4-set nor 1-sparse 4-set. Thus, $R_1^\Delta(4, 4) \geq 6$. Consider a triangle-free graph G of order 6. If G has a C_4, then it is a 1-dense 4-set. So assume that G does not contain C_4. If G contains C_5, the vertex x that is not on the C_5 can only be adjacent to a single vertex from the C_5, otherwise there would be a triangle or a C_4. Denote the vertices on the cycle by v_1, v_2, v_3, v_4, v_5 in order, with x being possibly adjacent to one vertex, say wothout loss of generality v_1, and no other vertex. Note the set $\{x, v_1, v_3, v_4\}$ is a 1-sparse 4-set, whether or not x is adjacent to v_1. So suppose that G has a C_6. Then G is a C_6 and two opposing edges create a 1-sparse 4 set. Lastly, consider the case where G has no cycles, that is G is a forest. In this case, G is actually a bipartite graph. If it is an unbalanced bipartite graph, meaning that one of the independent sets is of size at least 4, then that set is a 1-sparse 4-set. If G is a balanced bipartite graph, there is a vertex v that has at most 1 neighbor in the other independent set, say U, since all forests contain a vertex of degree at most 1. Then, $\{v\} \cup U$ is a 1-sparse 4-set. In conclusion, every triangle-free graph of order 6 has either a 1-dense 4-set or a 1-sparse 4-set. □

Theorem 4.3. *With the preceeding notation, $R_1^\Delta(4, 5) = 8$ with the unique extremal graph being C_7.*

Proof. Note that C_7 is a triangle-free graph of order 7 which does not contain any 1-dense 4-set nor 1-sparse 5-set. Thus, $R_1^\Delta(4, 5) \geq 8$. Let G be a triangle-free graph of order 8. If it has a 4-cycle, then it has a 1-dense 4-set. If G has no C_4, then we will show that it contains a 1-sparse 5-set.

If G has girth 5, then, the three vertices outside a 5-cycle C, denote by v_1, v_2, v_3, each can be adjacent to at most 1 vertex from the cycle. Otherwise, a triangle or a C_4 would exist. Call x_1, x_2, x_3, x_4, x_5 the vertices of C in order. If a vertex from C, say x_1, is adjacent to 2 or 3 vertices in $\{v_1, v_2, v_3\}$, then the set $(N(x_1) \setminus C) \cup \{x_2, x_3, x_5\}$ contains a 1-sparse 5-set. Now, assume that all vertices from C have at most 1 neighbor from $\{v_1, v_2, v_3\}$. Under these conditions, we can choose two vertices in $\{v_1, v_2, v_3\}$, say without loss of generality v_1 and v_2, such that $N(\{v_1, v_2\})$ do not contain two vertices of C which are adjacent. Then, we can choose a set A in $C \setminus N(\{v_1, v_2\})$ which is a 1-sparse 3-set. Then, $A \cup \{v_1, v_2\}$ is a 1-sparse 5-set.

Now, assume that G has girth 6 and let C be a 6-cycle. Then, call v_1 and v_2 the two vertices that are not on C. Since the girth is 6, each one of v_1 and v_2 has at most one neighbor in C. So, there exists a set A in $C \setminus N(\{v_1, v_2\})$ that is a 1-sparse 3-set. Then, $A \cup \{v_1, v_2\}$ is a 1-sparse 5-set. Hence, the girth is at least 7.

If G has girth 7, then it is a C_7 with vertices $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ and a remaining vertex outside the cycle, say v. Indeed, v can be adjacent to at most 1 vertex, say x_1, from the C_7, since girth is 7. The set $\{v, x_2, x_3, x_5, x_6\}$ is a 1-sparse 5-set whether v is adjacent to x_1 or not. Lastly, if G has girth 8 and is a C_8 with vertices $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$, then the set $\{x_1, x_3, x_4, x_6, x_7\}$ is a 1-sparse 5 set.

Finally the case where G is a forest remains. If G is a forest, then it is bipartite. If it is an unbalanced bipartite graph, then the independent set with higher size includes a 1-sparse 5-set. Assume that G is a balanced bipartite graph. There must exist a vertex with degree at most 1, since G is also a forest. This pendent vertex and an independent set it does not belong to together create a 1-sparse 5-set. Therefore, every traingle-free graph of order 8 includes either a 1-dense 4-set or a 1-sparse 5-set. □

Theorem 4.4. *With the preceeding notation, $R_1^{\Delta}(4, 6) = 10$ with the unique extremal graph being the graph given in Figure 2.*

Proof. Consider the graph in Figure 2. It is is a triangle-free graph which does not contain any 1-dense 4-set nor 1-sparse 6-set. So, $R_1^{\Delta}(4, 6) \geq 10$. Let G be a triangle-free graph of order 10. If it has a 4-cycle, then it has a 1-dense 4-set. So assume G has no C_4, then we will show that it contains a 1-sparse 6-set.

If $\Delta(G) \geq 6$, then the neighborhood of a maximum degree vertex includes a 1-sparse 6-set. So, assume $\Delta(G) \leq 5$. Suppose G has a vertex x of degree 5. Note every vertex outside $N[x]$ is adjacent to at most 1 vertex from $N(x)$, or else a 4-cycle is formed. Then $N(x)$ and a vertex outside $N_G[x]$ is a 1-sparse 6-set. So assume $\Delta(G) \leq 4$.

Suppose G has a vertex x of degree 4 and let $N(x) = \{v_1, v_2, v_3, v_4\}$ and $U = V \setminus N[x] = \{u_1, u_2, u_3, u_4, u_5\}$. If U is a 1-sparse 5-set, then $U \cup \{x\}$ is a 1-sparse 6-set. So assume U is not 1-sparse, thus there is a vertex from U, say u_1, which is adjacent two other vertices in U, say u_2 and u_3, without loss of generality. Note any vertex from U can be adjacent to at most one vertex in $N(x)$, or else a C_4 is formed. Moreover, $u_2u_3 \notin E$ since G is triangle-free. Likewise, a vertex from $N(x)$ cannot be adjacent to both u_2 and u_3, otherwise a C_4 is induced by that vertex and $\{u_1, u_2, u_3\}$. Consequently, $N(x) \cup \{u_2, u_3\}$ is a 1-sparse 6-set. So, we may assume $\Delta(G) \leq 3$.

If there is a vertex x of degree 1 in G, then $V \setminus N[x]$ has a 1-sparse 5-set S by $R_1^{\Delta}(4, 5) = 8$; thus $\{x\} \cup S$ is a 1-sparse 6-set. So assume every vertex in G has degree at least 2.

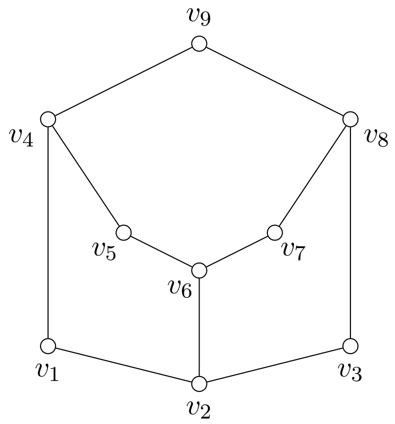

Figure 2: The unique extremal graph for $R_1^{\Delta}(4,6) = 10$.

Assume there is a vertex x of degree 3 and $U = V \setminus N[x] = \{u_1, u_2, u_3, u_4, u_5, u_6\}$. If U is 1-sparse, then it is a 1-sparse 6-set. Suppose U is not 1-sparse, thus there exists a vertex in U, say u_1, such that $d_U(u_1) \geq 2$. If $d_U(u_1) = 3$, then $N(x) \cup N(u_1)$ is a 1-sparse 6-set. So assume every vertex $u \in U$ has $d_U(u) \leq 2$. Thus, U induces a collection of disjoint paths and cycles. Note G has no triangle, nor C_4. Also, $G[U]$ cannot have a C_5 since the remaining vertex in U would be adjacent to two vertices in $N(x)$ (since $\delta(G) \geq 2$) forming a C_4. So the only cycle in $G[U]$ can be a 6-cycle. In this case, take an independent set I of 3 vertices in this 6-cycle; then $I \cup N(x)$ is a 1-sparse 6-set (any vertex in I has at most one neighbor in $N(x)$ and vice versa, or else a C_4 is formed). So assume $G[U]$ is a collection of paths. Observe every pendant vertex in $G[U]$ has to be adjacent to at least one vertex in $N(x)$ (since $\delta(G) \geq 2$); indeed it is adjacent to exactly one vertex in $N(x)$ or else a C_4 is formed. Thus, there is at most 3 pendant vertices in $G[U]$. By the Handshaking Lemma, $G[U]$ has exactly 2 pendant vertices; thus $G[U]$ is a P_6. Note there is a vertex in $N(x)$, say v, which is not adjacent to the end-vertices of the P_6. Clearly, v has at most two neighbors in P_6. Moreover, it is possible to chose a 1-sparse 4-set in $P_6 \setminus N(v)$ which together with x and v forms a 1-sparse 6-set. So we may assume G is 2-regular. Thus it is either a C_{10} or two disjoint copies of C_5. In both cases, there is a 1-sparse 6-set.

□

Theorem 4.5. *With the preceeding notation, $R_1^{\Delta}(4,7) = 13$ with exactly 2 extremal graphs given in Figure 3.*

Proof. The graphs in Figure 3 are both triangle-free graphs which do not contain any 1-dense 4-set nor 1-sparse 7-set. So, $R_1^{\Delta}(4,7) \geq 13$.

Suppose to the contrary, there is some triangle-free graph of order 13 containing no 1-dense 4-set nor 1-sparse 7-set. Let G be such a graph. Note G contains no 4-cycle induced or otherwise.

If $\delta(G) \leq 2$ then remove a minimum degree vertex x along with its neighborhood. Note a graph on at least 10 vertices remains. By $R_1^{\Delta}(4,6) = 10$, the remaining graph has a 1-dense 4-set, a contradiction; or a 1-sparse 6-set, which together with x, forms a 1-sparse 7-set. So, assume every vertex has degree at least 3.

If $\Delta(G) \geq 7$, then the neighborhood of a maximum degree vertex includes a 1-sparse 7-set. So we may assume $\Delta(G) \leq 6$. If $\Delta(G) = 6$, let x be a vertex of degree 6 and consider a vertex $y \in V \setminus N[x]$. If y has two neighbors in $N(x)$, then these two neighbors together with x and y form a C_4. So assume y has at most one neighbor in $N(x)$, then $N(x) \cup \{y\}$ is a 1-sparse 7-set. So we may assume $\Delta(G) \leq 5$.

So suppose G has a vertex x of degree 5. Then $V \setminus N[x]$ induces a triangle-free graph on 7 vertices; since $R(3,3) = 6$, it has an independent set A of size 3. If no vertex of A is adjacent to more than one vertex in $N(x)$, then $A \cup N(x)$ contains a 1-sparse 7-set. Note no vertex of A is adjacent to more than one vertex in $N(x)$, or else C_4 is present. If all three vertices of A are adjacent to the same vertex $y \in N(x)$, then $A \cup (N(x) \setminus y)$ is a 1-sparse 7-set. If there are two vertices, say $u, v \in A$ are adjacent to the same vertex of $N(x)$. Then $(A \setminus \{u\} \cup N(x))$ is a 1-sparse 7-set. If every vertex in $N(x)$ is adjacent to at most one vertex in A, $A \cup N(x)$ contains a 1-sparse 7-set. Thus, G has no vertex of degree 5.

So, suppose the maximum degree of G is 4. Let x be a vertex of degree four. Suppose also that G has a second vertex, say y, of degree 4. Let us consider the case where x and y are non-adjacent. Note, x and y cannot have two common neighbors, for otherwise G contains a 4-cycle. So, x and y have at most one common neighbor and hence, $N(x) \cup N(y)$ is a 1-sparse set on at least 7 vertices. Thus, all vertices of degree 4 are adjacent with x.

Pick y, a non-neighbor of x. Note, y has degree exactly 3 since $\delta(G) \geq 3$. Suppose $N(x)$ and $N(y)$ don't meet. Then their union is a 1-sparse 7-set because of the absence of 4-cycles. So let us assume these neighborhoods meet and z belongs to both. We note there can be no other vertex belonging to both. As G contains no vertices of degree 2, we note z is adjacent to some other vertex and this vertex is outside $N[x] \cup N[y]$. Call one such vertex w. Note, w cannot be adjacent with anything in $N(x) \cup N(y)$ other than z, for otherwise a 4-cycle is present in G. Thus, $N(x) \cup N(y) \cup \{w\}$ is a 1-sparse 7-set. Thus, G contains no vertex of degree 4.

Accordingly, G is 3-regular. But this is impossible; by the Handshaking Lemma, there is no 3-regular graph of order 13.

□

5 Computer enumeration

In this work, we obtain several defective Ramsey numbers in triangle-free graphs using proofs "by hand". Whenever classical proof techniques hit limits due to the highly combinatorial nature of the extremal graphs and Ramsey numbers, we also make use of a computer based search. We use an adaptation of the Algorithm Sub-extremal given in [8] for triangle-free graphs as described here in Algorithm 1. Algorithm 1 computes new defective Ramsey numbers and enumerates related extremal graphs for triangle-free graphs. It also serves as a checking mechanism for the proofs made by hand in earlier sections. All of our codes and the extremal graphs we obtain are available online at [11].

Let us denote by $\mathcal{T}_n^{\Delta}(k,i,j)$ the set of all triangle-free graphs of order n containing no k-dense i-set nor k-sparse j-set. Given $\mathcal{T}_n^{\Delta}(k,i,j)$, we call a k-dense i-set or a k-sparse j-set *a forbidden k-defective set.* Note that the set of all extremal graphs for $R_k^{\Delta}(i,j)$ is the set $\mathcal{T}_n^{\Delta}(k,i,j)$ for $n = R_k^{\Delta}(i,j) - 1$. Accordingly, a graph in $\mathcal{T}_n^{\Delta}(k,i,j)$ for $n < R_k^{\Delta}(i,j) - 1$

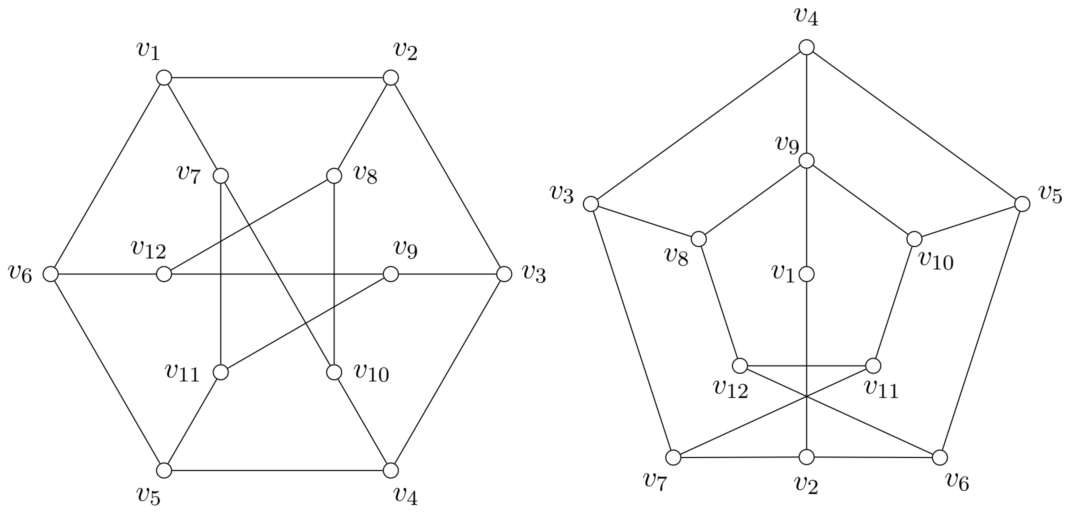

Figure 3: The two extremal graphs for $R_1^{\Delta}(4, 7) = 13$.

is called a *sub-extremal* graph for $R_k^{\Delta}(i, j)$.

Algorithm Sub-extremal in [8] computes a defective Ramsey number $R_k^{\mathcal{G}}(i, j)$ and all its extremal graphs for some parameters k, i, j such that $i, j \geq k + 2$ and for some graph class $\mathcal{G}$. In its generic form, it checks whether the generated graphs belong to the desired graph class $\mathcal{G}$ or not at the very end of the algorithm, and eliminates those not in $\mathcal{G}$. Our adaptation given in Algorithm 1 differs from Algorithm Sub-extremal only for checking the presence of triangles. Instead of cheking this at the end, we make sure that the generated graphs are always triangle-free by the way we add a new vertex in line 4.

Algorithm 1 is based on the fact that being triangle-free, (k-dense i-set)-free, and (k-sparse j-set)-free are hereditary properties. Given a sub-extremal graph G of order n, all graphs of order $n + 1$ that have G as an induced sub-graph is produced by adding a new vertex with all possible adjacency combinations to the vertices of G. If a new graph created by this procedure is also triangle-free and contains no forbidden k-defective set for $\mathcal{T}_{n+1}^{\Delta}(k, i, j)$, it belongs to the set of (sub-)extremal graphs of order $n + 1$. Taking $\mathcal{T}_n^{\Delta}(k, i, j)$ as input, the set $\mathcal{T}_{n+1}^{\Delta}(k, i, j)$ is generated by this method. We start with the one vertex graph K_1 as input. We run Algorithm 1 iteratively giving the output of one iteration as the input of the next itaration. We stop when the output set is empty and declare $R_k^{\Delta}(i, j) = n$ where n is the number of vertices for which the algorithm does not return a graph. That means all triangle-free graphs with the present order n (or larger) contain either a k-dense i-set or a k-sparse j-set. We conclude that the last non-empty output set of graphs with $R_k^{\Delta}(i, j) - 1$ vertices is the complete list of all extremal graphs for $R_k^{\Delta}(i, j)$.

The following observation allows us to avoid explicitly checking triangle-free graphs by guaranteeing that the generation precedure in line 4 provides all desired triangle-free graphs. Indeed, a triangle-free graph of order $n + 1$ containing a triangle-free graph G of order n can be obtained by taking G and adding a new vertex adjacent to an independent set of G. If this process is repeated for every independent set of G, all triangle-free graphs of order $n + 1$ containing G are obtained.

Algorithm 1: Sub-extremal for Triangle-free Graphs

Input: $\mathcal{T}_n^{\Delta}(k,i,j)$ for some k,i,j such that $i,j \geq k+2$
Output: $\mathcal{T}_{n+1}^{\Delta}(k,i,j)$
1 Let $K = \emptyset$.
2 **foreach** $G \in \mathcal{T}_n^{\Delta}(k,i,j)$ **do**
3   **forall** $S \subseteq V(G)$ **do**
4   **if** *S is an independent set* **then**
5   Take the graph G_S that is formed by adding a new vertex v into G that is adjacent to all vertices in S.
6   Let *add* = **TRUE**.
7   **forall** *$I \subseteq V(G_S)$ such that $v \in I$ and $|I| \in \{i,j\}$* **do**
8   **if** *$|I| = i$ and $G[I]$ is k-dense* **then**
9   *add* = **FALSE** and **BREAK**
10   **if** *$|I| = j$ and $G[I]$ is k-sparse* **then**
11   *add* = **FALSE** and **BREAK**
12   **if** *add* = ***TRUE*** **then**
13   Add G_S into K.
14 Return a maximal non-isomorphic set of graphs in K.

Having guaranteed the absence of triangles, all we need to check is whether one of the forbidden k-defective sets has been formed. Indeed, since the input graphs have no forbidden k-defective sets, if a newly generated graph G_S contains a forbidden k-defective set, then this must contain the new vertex v. Accordingly, it is sufficient to check all subsets including the new vertex v for forbidden k-defective sets in lines 7 to 11. Checking the existance of a k-dense i-set, in lines 8 to 9, is included in the search for $R_k^{\Delta}(4,j)$ values. However, the k-dense set checking mechanism is omitted for $T_k(j)$ values which only consider sparse sets.

The nature of Algorithm 1 allows for parallel computing, thus, to improve the runtime efficiency of the algorithm, both the graph generation and isomorphism checks are implemented to execute in parallel. In the graph generation, each thread works with a separate graph from the set of input graphs in line 2. All generated and valid graphs are pooled together in an array. In this pool, isomorphic copies of graphs exist and getting rid of isomorphic copies of a graph is a challenge for this algorithm. The isomorphism checks are carried out by comparing graphs by their canonical labelings which are calculated using the *nauty* program [17]. This isomorphism checking part is also programmed to execute in parallel with each thread checking a different graph and utilizing mutex locks to work on shared data structures.

The program is implemented in C++ and executed on a personal computer with 8 gigabytes of RAM and Apple M1 chip which has 8 cores and maximum CPU clock rate of 3.2 GHz. For the relatively small defective numbers, the runtime is trivially quick. However, as the graph sizes increase, runtimes grow exponentially both in generation and isomorphism checks phases. The longest runtime encountered for a defective Ramsey number, which is $R_4^\Delta(9,11) = 18$, is approximately 2.5 hours.

We found several defective Ramsey numbers that we weren't able to prove by hand as well as the number of the extremal graphs using Algorithm 1. Table 1 displays $T_k(j)$ values computed by Algorithm 1 as well as the corresponding number of extremal graphs for each number computed. In Table 1, the missing numbers are due to insufficient memory. Tables 2, 3, 4, 5 display similar results obtained for defective Ramsey numbers $R_k^\Delta(i,j)$ for $k = 1,2,3,4$. Missing numbers in these tables could not be obtained due to insufficient memory to store the subextremal graphs.

Table 1: $T_k(j)$ values and corresponding number of extremal graphs.

		j									
		3	4	5	6	7	8	9	10	11	12
k	1	5 (1)	7 (2)	11 (1)	13 (16)	18 (1)					
	2	3 (2)	5(1)	9 (2)	11 (6)	13 (288)	16 (281)				
	3	3 (2)	4 (3)	6 (1)	8 (2)	13 (5)	15 (40)	17 (9713)			
	4	3 (2)	4 (3)	5 (7)	7 (1)	9 (2)	11 (7)	17 (19)	19 (606)		
	5	3 (2)	4 (3)	5 (7)	6 (14)	8 (1)	10 (2)	12 (7)	14 (46)	21 (112)	
	6	3 (2)	4 (3)	5 (7)	6 (14)	7 (38)	9 (1)	11(2)	13 (7)	15 (46)	17 (723)
	7	3 (2)	4 (3)	5 (7)	6 (14)	7 (38)	8 (107)	10 (1)	12 (2)	14 (7)	16 (46)

Table 2: $R_1^\Delta(i,j)$ values and corresponding number of extremal graphs.

		j						
		3	4	5	6	7	8	9
i	3	3 (2)	4 (2)	5 (3)	6 (3)	7 (4)	8 (4)	9 (5)
	4	4 (1)	6 (1)	8 (1)	10 (1)	13 (2)	15 (3)	18 (4)

Table 3: $R_2^\Delta(i,j)$ values and corresponding number of extremal graphs.

		j						
		4	5	6	7	8	9	10
i	4	4 (3)	5 (3)	6 (3)	7 (3)	8 (3)	9 (3)	10 (3)
	5	5 (1)	6 (4)	8 (1)	10 (2)	11 (62)	15 (2)	17 (4)
	6	5 (1)	7 (3)	9 (6)	12 (5)	15 (3)		

Table 4: $R_3^\Delta(i,j)$ values and corresponding number of extremal graphs.

		j							
		5	6	7	8	9	10	11	12
i	5	5 (7)	6 (7)	7 (8)	8 (8)	9 (9)	10 (9)	11 (10)	12 (10)
	6	6 (1)	7 (5)	9 (1)	10 (8)	12 (2)	13 (25)	15 (7)	16 (144)
	7	6 (1)	8 (2)	10 (1)	12 (3)	15 (2)			
	8	6 (1)	8 (2)	10 (10)	13 (2)	15 (551)			

Lastly, we suggest the following generalization of Theorems 3.4, 3.8 and 3.9 as a conjecture.

Table 5: $R_4^{\Delta}(i,j)$ values and corresponding number of extremal graphs.

		j							
		6	7	8	9	10	11	12	13
i	6	6 (14)	7 (14)	8 (14)	9 (14)	10 (14)	11 (14)	12 (14)	13 (14)
	7	7 (1)	8 (6)	10 (1)	11 (7)	12 (36)	13 (194)	14 (959)	16 (41)
	8	7 (1)	9 (2)	11 (1)	12 (44)	14 (20)	15 (3115)		
	9	7 (1)	9 (2)	11 (7)	13 (19)	15 (146)	18 (255)		
	10	7 (1)	9 (2)	11 (7)	13 (70)	16 (123)			

Conjecture 5.1. *With the preceeding notation, $T_k(k+i) = k+2i-1$ for $2 \leq i \leq k$, with $K_{i-1,k+i-1}$ an extremal graph.*

Clearly, the complete bipartite graph $K_{i-1,k+i-1}$ does not contain a k-sparse set of size $k+i$ for $2 \leq i \leq k$. This implies $T_k(k+i) \geq k+2i-1$. For $k \geq 2$, Conjecture 5.1 claims all $T_k(j)$ values where $k+2 \leq j \leq 2k$. This suggests that for large (but fixed) k, there are $k-1$ values of $T_k(j)$ that grow linearly. Note that this trend does not continue as n grows since we have $T_k(n) = \Theta(\frac{n^2}{\log n})$ for fixed k by Theorem 3.10.
Referring to Table 1, each colored diagonal corresponds to $T_k(k+i)$ values for a fixed i. We note that the values of $T_k(k+i)$ grow linearly as k goes to infinity for fixed i. Moreover, the extremal graph count and their structures are the same along a diagonal (for fixed i). The non-colored values on said diagonals (which fall out of the range $2 \leq i \leq k$) do not carry the observed regularity. Theorems 3.4, 3.8 and 3.9 prove Conjecture 5.1 for $i = 2$ (orange), $i = 3$ (yellow) and $i = 4$ (blue) respectively. Furthermore, the values on the green diagonal ($i = 5$) carries the regularity and supports the conjecture. We suspect that this unexpected pattern on extremal graphs continues for larger k. The last value we could compute is for $i = 6$ which is $T_6(12) = 17$ with 723 extremal graphs. Computing, $T_7(13)$ was not possible due to the need of higher computer memory and longer runtime.

6 Conclusion

In the search for defective Ramsey numbers in triangle-free graphs, we have looked into two parameters which are $R_k^{\Delta}(i,j)$ and $T_k(j)$. Some defective Ramsey numbers for specific configurations of parameters (i,j,k) are obtained with direct proof techniques, whereas some values are obtained by computer enumeration. Further values can be developed with the aid of novel structural results for triangle-free graphs and a streamlining of our algorithms.

Growth rates of these parameters, relative to one other, is also of interest. We do not know if, for fixed k, whether $T_k(m) - T_{k+1}(m)$ is bounded, let alone if the difference goes to infinity. Similarly, we do not know the behavior of $\frac{T_k(m)}{T_{k+1}(m)}$. We suspect this ratio moves towards 1, but cannot prove it. Along these same lines, we do not know if there is a small k and a large m where $T_k(m) = T_{k+1}(m)$. Similarly, we do not know if there is a large m and small k where $T_k(m) = T_k(m+1)$.

As a future work, one could investigate Conjecture 5.1. This would most probably require techniques other than the one used in proving Theorems 3.4, 3.8 and 3.9. Note that the number of cases for possible maximum degree values to be considered in these proofs will increase with i, making it inconvenient to obtain a proof for all i and k such that $2 \leq i \leq k$ using this approach.

In general, we think that the interaction between efficient computer enumeration methods and classical proof techniques is a promising research direction for computing defective Ramsey numbers (and/or related parameters) in various graph classes.

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