IE 203 PS 8 15.05.2025

Stochastic Process $\{X_t\}_{t=0}^{\infty}$

Time t can be discrete or continous.

X_t indicates the state of the process at time t.

$X_t \in S$ where S is the state space.

Ex: $S = \{$sunny, rainy, snowy, cloudy$\}$

State space can be infinite or finite.

<u>Markov Chains</u>

Markovian Property: Future states depends solely on the current state. The past has no influence on the future. (Memoryless)

For a discrete time markov chain:

$$P(X_{t+1} = i_{t+1} \mid X_t = i_t, X_{t-1} = i_{t-1}, \ldots, X_0 = i_0)$$
$$= P(X_{t+1} = i_{t+1} \mid X_t = i_t) \qquad t = 0, 1, 2, \ldots$$

Stationary Property: One step transition probabilities do not change over time.

$$P(X_{t+1} = j \mid X_t = i) = P(X_1 = j \mid X_0 = i) \qquad t = 0, 1, 2, \ldots$$

<u>State Diagram:</u>

$$\text{state } i \xrightarrow{P_{ij}} \text{state } j$$

P_{ij}: probability of being in state j in the next step given that the current state is i

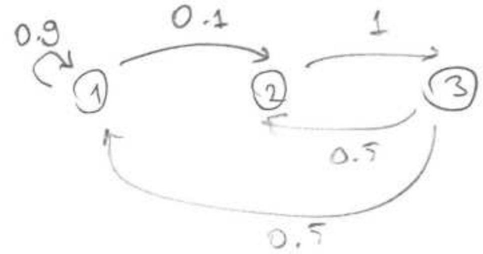
<u>One step transition matrix:</u>

$$P = \begin{array}{cc} & \text{next state} \\ \text{current state} & \begin{pmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ \vdots & & \ddots & \\ P_{n1} & & & P_{nn} \end{pmatrix} \end{array}$$

★ Entries in a row must sum up to 1.

Example :

$$P = \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0 & 0 & 1 \\ 0.5 & 0.5 & 0 \end{bmatrix}$$



k-step transition probabilities are given by $P^{(k)} = \underbrace{P \cdot P \cdot \ldots \cdot P}_{k}$

What happens when $k \to \infty$? Observe in the matrix above

Observe in $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

(You may use numpy.matmul( ) in Python)

We will see why the first case converges but the second does not.

Communication

If state j is reachable (accessible) from state i ($p_{ij}^{(k)} > 0$ for some k) and state i is reachable from state j, then states i and j are communicating.

If states i and j are communicating and states j and k are communicating, then states i and k are communicating.

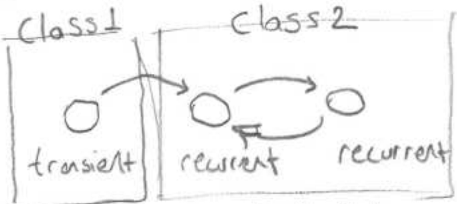
If all states in a MC are communicating, then the markov chain is irreducible. A maximal set of communicating states is called a class.

A state i is transient if there exist another state j which is reachable from i but state i is not reachable from state j. (At one point, we leave and never come back)

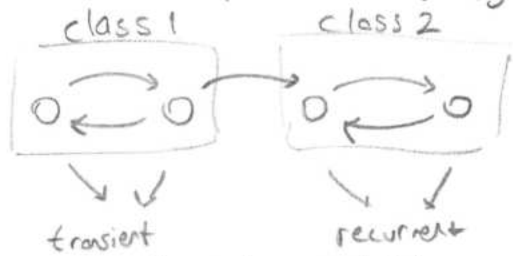
A state i is recurrent if the process eventually re-visits state i. In other words, given that the process is at i, the probability of the process being at i in some step in future is 1.

A state i is absorbing if the process never leaves i. ($p_{ii} = 1$)

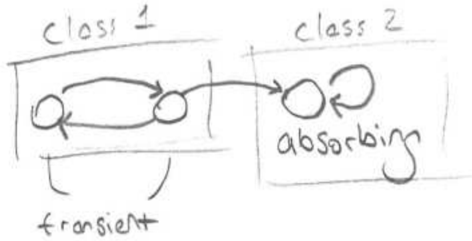
Example: Arcs indicate positive probability. If there is no arc from i to j, $p_{ij} = 0$.

1. 

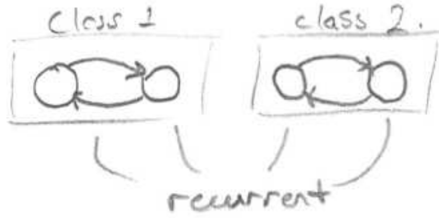
MC not irreducible

2. 

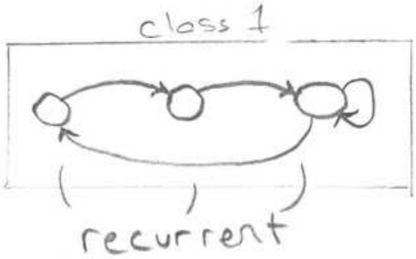
MC not irreducible

3. 

MC not irreducible

4. 

MC not irreducible

5. 

MC irreducible
(all state pairs are communicating)

★ Transiency and recurrency are class properties, meaning that all states in the same class are of the same type.

Periodicity

period of a state $d(i) = \gcd \{ n \in \mathbb{N}_+ : P_{ii}^n > 0 \}$

gcd → greatest common divisor.

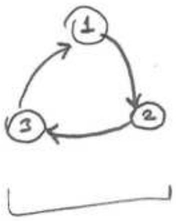
The set of time steps at which the process can return to the state with positive probability.

A state i is periodic if $d(i) > 1$.

A state i is aperiodic if $d(i) = 1$.

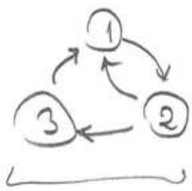
Periodicity is a class property.

★ A markov chain is aperiodic if all states are aperiodic.

Ex: 

$P_{11}^1 = 0$
$P_{11}^2 = 0$
$P_{11}^3 = 1$
$P_{11}^4 = 0$
$P_{11}^5 = 0$
$P_{11}^6 = 1$
$\vdots$

$d(1) = d(2) = d(3) = 3$
periodic

Ex: 

$P_{11}^1 = 0$
$P_{11}^2 > 0$
$P_{11}^3 > 0$

$\gcd \{2, 3, \dots\} = 1$
$d(1) = d(2) = d(3) = 1$
Aperiodic

In a finite state markov chain, recurrent states that are aperiodic are called <u>ergodic states</u>. A markov chain is <u>ergodic</u> if its all states are ergodic

Steady State Probabilities

For any <u>irreducible</u> and <u>ergodic</u> markov chain, (all states communicate and all states are aperiodic) $\lim_{n \to \infty} p_{ij}^{(n)}$ exists and is independent of state i. Furthermore, they are unique.

$$\lim_{n \to \infty} p_{ij}^{(n)} = \pi_j > 0$$

π_j : probability of being at state j in the long run.

Steady state equations:

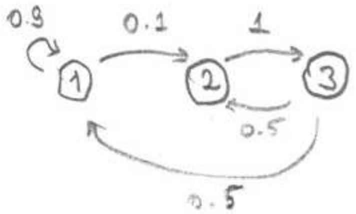
$$\pi_j = \sum_{i=1}^{M} \pi_i p_{ij} \qquad j = 1, \dots, M \qquad (1)$$

* (In vector form : $\Pi = \Pi P$)

$$\sum_{j=1}^{M} \pi_j = 1 \qquad (2)$$

Note that there are M unknowns and $M+1$ many equations. Since the steady state probabilities are unique one of the equations are redundant. However, we cannot remove Equation (2). Otherwise $x_j = 0 \;\; \forall j \in S$ and $x_j = c\,\pi_j$ (where c is some constant and π_j are the unique steady state probabilities) solve the Equation Set (1), which are clearly not the unique steady state probabilities.

Ex: $P = \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0 & 0 & 1 \\ 0.5 & 0.5 & 0 \end{bmatrix}$



all states are communicating (one class) → MC irreducible

$p_{33}^{(2)} > 0,\ p_{33}^{(3)} > 0 \rightarrow d(3) = 1 = d(1) = d(2)$
→ MC aperiodic

⇒ steady state probabilities exist

$$[\pi_1\ \pi_2\ \pi_3] = [\pi_1\ \pi_2\ \pi_3] \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0 & 0 & 1 \\ 0.5 & 0.5 & 0 \end{bmatrix}$$

$$\left.\begin{aligned} \pi_1 &= 0.9\pi_1 + 0.5\pi_3 \\ \pi_2 &= 0.1\pi_1 + 0.5\pi_3 \\ \pi_3 &= \pi_2 \end{aligned}\right\}$$ one of these are redundant, say we throw away the second.

$$\pi_1 + \pi_2 + \pi_3 = 1$$

$$\pi_1 = 5\pi_3$$

$$5\pi_3 + \pi_3 + \pi_3 = 1 \rightarrow \pi_3 = 1/7 = \pi_2$$

$$\pi_1 = 5/7$$

$$\pi = [\,5/7\ \ 1/7\ \ 1/7\,]$$
$$\cong [0.7143\ \ 0.1429\ \ 0.1429]$$

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IE 203 PS 8

Dynamic Programming

1- Find the optimal solution of the following knapsack instance with weight capacity 10 using dynamic programming.

i	**1**	**2**	**3**	**4**
Value	10	40	30	50
Weight	5	4	6	3

2- You have a machine that is t years old. $R(t)$ is the expected revenue from a machine for a year when the machine is t years old at the beginning of that year. Similarly, $C(t)$ is the operating cost for a year and, $S(t)$ is the salvage value if the machine is sold when it is t years old. I is the fixed cost of purchase of a new machine. We are interested in maximizing the net income for N years of operation.

a) Formulate this problem as a dynamic programming problem.

b) Let $N = 4, I = 10$ and R, C, S given as below. We start with a new machine. Solve this problem using dynamic programming.

$\boldsymbol{t}$	**0**	**1**	**2**	**3**
$\boldsymbol{R}$	5	4	3	2
$\boldsymbol{C}$	0	0	1	1
$\boldsymbol{S}$	8	7	6	3

Markov Chains

3- **Random Walk.** Imagine a person walking in an unusual manner. They either take a step to their left or to their right randomly with equal probability. Model this stochastic process as a Markov chain.

4- **Gambler's Ruin.** Consider two gamblers Alexei and Baboulinka playing a fair coin flipping game. Alexey has 60 kopecks and Baboulinka has 100. At each turn of the game, loser pays 5 kopecks to the winner. Game ends when one of them loses all their money. Model this stochastic process as a discrete time Markov chain.

5- Below, a one step transition probability matrix of a discrete time Markov chain is given. Answer the questions accordingly. State indices start with 1.

$$P = \begin{matrix} 0 & 0.6 & 0.4 & 0 & 0 & 0 & 0 \\ 0.7 & 0 & 0.3 & 0 & 0 & 0 & 0 \\ 0.1 & 0.1 & 0 & 0.3 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & 0.4 & 0.1 \end{matrix}$$

a) Draw the state diagram.
b) Identify the classes.
c) Identify the types of states (recurrent, transient, absorbing).
d) Calculate $P(X_2 = 1 | X_0 = 1)$.
e) Calculate $P(X_3 = 4 | X_0 = 1)$.
f) Explain the long run behavior of the Markov chain
g) Show that a subset of the state space is itself an ergodic and irreducible Markov chain.
h) Calculate the steady state probabilities of the Markov chain found in part g.

IE 203 PS 8

Dynamic Programming

(1) Knapsack

$f(i, w)$: max value

i → item pointer

w → Left capacity

```
# Top down Approach Knapsack DP
def f(i,w):
    if data[i][w] != NULL: # if this config. has calculated before
        return data[i][w]
    if i<=0 or w<=0: # Base conditions
        result = 0
    else if weight[i] > w: # Cannot pick item i
        result = f(i-1, w)
    else if:
        result = max(f(i-1, w), value[i] + f(i-1, w - w_i))
```

$$\text{result} = \max\Big(\underbrace{f(i-1, w)}_{\text{case where } i \text{ is not picked}}, \underbrace{\text{value}[i] + f(i-1, w-w_i)}_{\text{case where } i \text{ is picked}}\Big)$$

```
    data[i][w] = result
    return result
```

	i	1	2	3	4
① Data :	Value	10	40	30	50
	weight	5	4	6	3

capacity : 10

$f(4,10)$ will give the answer.

$$f(4,10) = \max \{ \underbrace{f(3,10)}_{50}, \underbrace{50 + \underbrace{f(3,7)}_{40}} \} = 90 //$$

$$f(3,10) = \max \{ \underbrace{f(2,10)}_{50}, 30 + \underbrace{f(2,4)}_{40} \} = 70$$

$$f(2,10) = \max \{ \underbrace{f(1,10)}_{10}, 40 + \underbrace{f(1,6)}_{10} \} = 50$$

$$f(1,10) = \max \{ \underbrace{f(0,10)}_{0}, 10 + \underbrace{f(0,5)}_{0} \} = 10$$

$$f(1,6) = \max \{ \underbrace{f(0,6)}_{0}, 10 + \underbrace{f(0,1)}_{0} \} = 10$$

$$f(2,4) = \max \{ \underbrace{f(1,4)}_{0}, 40 + \underbrace{f(1,0)}_{0} \} = 40$$

$$f(1,4) = f(0,4) = 0$$
(cannot take 1)

$$f(3,7) = \max \{ \underbrace{f(2,7)}_{40}, 30 + \underbrace{f(2,1)}_{0} \} = 40$$

$$f(2,7) = \max \{ \underbrace{f(1,7)}_{10}, \underbrace{40 + \underbrace{f(1,3)}_{0}} \} = 40$$

$$f(1,7) = \max \{ \underbrace{f(0,7)}_{0}, 10 + \underbrace{f(0,2)}_{0} \} = 10$$

$$f(1,3) = f(0,3) = 0$$

$$f(2,1) = f(1,1) = f(0,1) = 0$$

$$\boxed{\begin{array}{l}\text{optimal sol} = (0,1,0,1) \\ z^* = 90\end{array}}$$

②

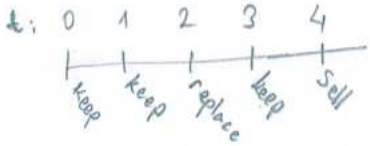
$$f(t,t_m) = \max\Big\{ \underbrace{f(t+1,t_m+1) + R(t_m) - C(t_m)}_{\text{keep the machine}},\ \underbrace{f(t+1,1) + R(0) + S(t_m) - C(0) - I}_{\text{replace the machine in the start of year } t} \Big\}$$

t_m ↓ machine age at start of t

$f(t,t_m)$ → max profit that can be achieved in the future

Base Cases : $f(N,t_m) = S(t_m)$

Question : $f(0,0) = ?$



$$f(0,0) = \max\{ \underbrace{f(1,1)}_{15} + 5 ,\ \underbrace{f(1,1)}_{15} + 3 \} = 20$$

$$f(1,1) = \max\{ \underbrace{f(2,2)}_{11} + 4 ,\ \underbrace{f(2,1)}_{12} + 2 \} = 15$$

$$f(2,2) = \max\{ \underbrace{f(3,3)}_{5} + 2 ,\ \underbrace{f(3,1)}_{10} + 1 \} = 11$$

$$f(3,3) = \max\{ \underbrace{f(4,4)}_{<3} + 2 ,\ \underbrace{f(4,1)}_{7} - 2 \} = 5$$

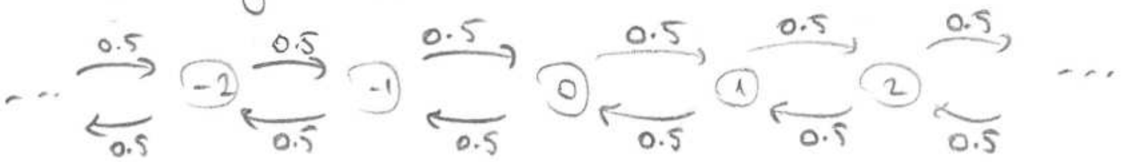
$$f(3,1) = \max\{ \underbrace{f(4,2)}_{6} + 4 ,\ \underbrace{f(4,1)}_{7} + 2 \} = 10$$

$$f(2,1) = \max\{ \underbrace{f(3,2)}_{8} + 4 ,\ \underbrace{f(3,1)}_{10} + 2 \} = 12$$

$$f(3,2) = \max\{ \underbrace{f(4,3)}_{3} + 2 ,\ \underbrace{f(4,1)}_{7} + 1 \} = 8$$

(3) $S = \{\dots, -2, -1, 0, 1, 2, \dots\}$

State Diagram:



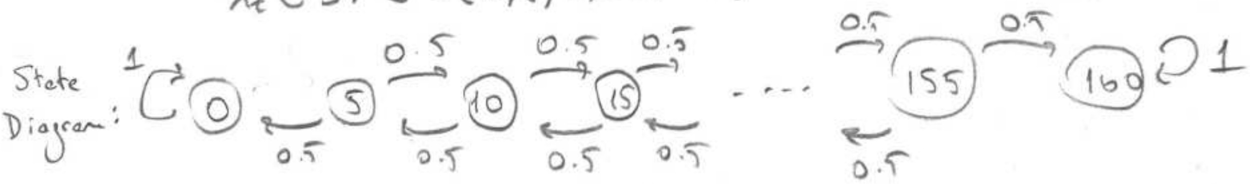
One step transition probabilities:

$$P(X_{t+1} = j \mid X_t = i) = \begin{cases} 1/2 & \text{if } j = i+1 \\ 1/2 & \text{if } j = i-1 \\ 0 & \text{for other } j \end{cases}$$

(4) X_t : The number of Kopecks that Alexei has

$X_t \in S$. $S = \{0, 5, 10, \dots, 160\}$

State Diagram:

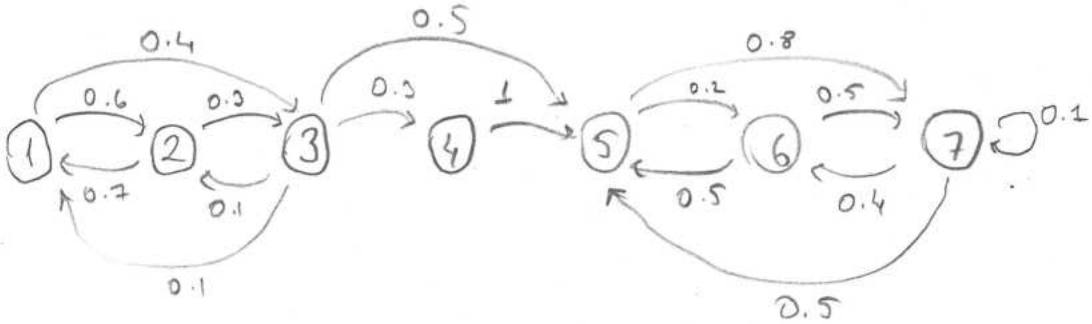


One step transition probabilities:

$$P(X_{t+1} = j \mid X_t = i) = \begin{cases} 0.5 & \text{if } i \in S \setminus \{0, 160\} \text{ and } j = i \pm 5 \\ 1 & \text{if } i = j = 0 \\ 1 & \text{if } i = j = 160 \\ 0 & \text{for other } i, j \end{cases}$$

Initial State: $X_0 = 60$

(5) a.



b) → 1 is reachable from 2, 2 is reachable from 1. 1 and 2 are communicating.

→ $P_{13}^{(1)} > 0$ and $P_{31}^{(1)} > 0$, thus 1 and 3 are communicating.

→ 1 and 2 are communicating and 1 and 3 are communicating, thus 2 and 3 are communicating.

Similarly, 5, 6 and 7 communicate with each other.

State 4 does not communicate with any other state.

Class 1 = {1, 2, 3}
Class 2 = {4}
Class 3 = {5, 6, 7}

c) Starting from state 1, the process eventually hits 4 or 5 and never revisits 1. Thus, 1 is transient. The states in the same class have to be of the same type, therefore 2 and 3 are also transient.

Starting from 4, the process never revisits 4. 4 is transient.

Starting from 5, the process will revisit 5 infinitely many times.

5, 6, 7 are recurrent.

d) $P_{11}^{(2)} = ?$

$$P_{11}^{(2)} = P_{12}P_{21} + P_{13}P_{31} + \underbrace{P_{14}P_{41} + P_{15}P_{51} + P_{16}P_{61} + P_{17}P_{71} + P_{11}P_{11}}_{0}$$

$$= 0.6 \times 0.7 + 0.4 \times 0.1$$

$$= 0.46$$

e) $P_{14}^{(3)} = ?$

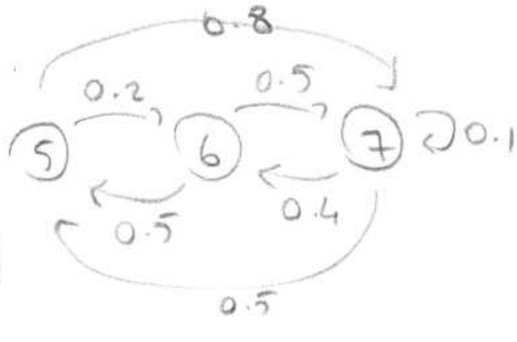
$$P_{14}^{(3)} = P_{13}^{(2)} P_{34}$$

$P_{13}^{(2)} = P_{12}P_{23}$ (one possible path)

$$= P_{12}P_{23}P_{34}$$

$$= 0.6 \times 0.3 \times 0.3 = 0.054$$

f) Regardless of the initial state, the process is absorbed by Class 3. This class is itself an ergodic and irreducible discrete-time Markov chain (will see at part g.). Therefore, unique steady state probabilities exist.



g) Markov chain induced by Class 3:

$$\text{period of } 5 = \gcd\{n \in \mathbb{N}_+ : p_{55}^n > 0\}$$
$$= \gcd\{2, 3, 4, \dots\} = 1$$

Thus, this MC is aperiodic. All states are recurrent as we have shown. Thus, this MC is ergodic.

We have shown all states (5, 6, 7) communicate with each other. Therefore, this MC is irreducible.

This MC is irreducible and ergodic, therefore unique steady state probabilities exist.

h) $$[\pi_5 \ \pi_6 \ \pi_7] = [\pi_5 \ \pi_6 \ \pi_7] \begin{bmatrix} 0 & 0.2 & 0.8 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.4 & 0.1 \end{bmatrix}$$

$$\left.\begin{aligned} \pi_5 &= 0.5\,\pi_6 + 0.5\,\pi_7 \\ \pi_6 &= 0.2\,\pi_5 + 0.4\,\pi_7 \\ \pi_7 &= 0.8\,\pi_5 + 0.5\,\pi_6 + 0.1\,\pi_7 \end{aligned}\right\} \text{one of them is redundant}$$

$$\pi_5 + \pi_6 + \pi_7 = 1$$

$$\rightarrow \pi = \begin{bmatrix} \frac{14}{42} & \frac{10}{42} & \frac{6}{14} \end{bmatrix}$$
$$\quad\ \ \pi_5 \quad \pi_6 \quad \pi_7$$

Due to the structure of MC:

$$\pi_1 = \pi_2 = \pi_3 = \pi_4 = 0$$

* We were able to "reduce" the whole MC due to its structure. For example steady state probabilities don't exist in the MC to the right, since initial state matters.