IE 203 - Operations Research II
Quiz I - Solutions

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Duration: 30 minutes
This is a CLOSED BOOK exam.

Question I (40 pts.)

a. $k_1 = \log_2(15+1) = 4$ and $k_2 = \log_2(7+1) = 3$. Then, the formulation becomes

$$\max \quad z = 8\sum_{j=0}^{3} 2^j y_{1j} + 5\sum_{j=0}^{2} 2^j y_{2j} \tag{1}$$

$$\text{s.t.} \quad \sum_{j=0}^{3} 2^j y_{1j} + \sum_{j=0}^{2} 2^j y_{2j} \leq 8 \tag{2}$$

$$9\sum_{j=0}^{3} 2^j y_{1j} + 5\sum_{j=0}^{2} 2^j y_{2j} \leq 45 \tag{3}$$

$$\sum_{j=0}^{3} y_{1j} 2^j y_{2j} \leq 15 \tag{4}$$

$$\sum_{j=0}^{2} y_{2j} 2^j y_{2j} \leq 7 \tag{5}$$

$$y_{1j} = 0,1 \qquad j = 0,1,2,3 \tag{6}$$

$$y_{2j} = 0,1 \qquad j = 0,1,2. \tag{7}$$

$$\tag{8}$$

b. To have $x_1^* = 5$ and $x_2^* =$ as an optimal solution one must have all binary variables equal to zero, except $y_{10}^* = y_{12}^* = 1$.

Question II (60 pts.)

a. The feasible solution consist of the grid points reamaining within the polyhedron decribed by the inequalities

$$\begin{array}{rl} -3x_1 + 4x_2 & \leq 4 \\ 3x_1 + 2x_2 & \leq 11 \\ 2x_1 - x_2 & \leq 5 \\ x_1, x_2 & \geq 0 \end{array}.$$

Any noninteger point, even it satisfies the inequalities, is infeasible. The set of integer feasible solutions is illustrated in Figure 1. It consists of the points $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

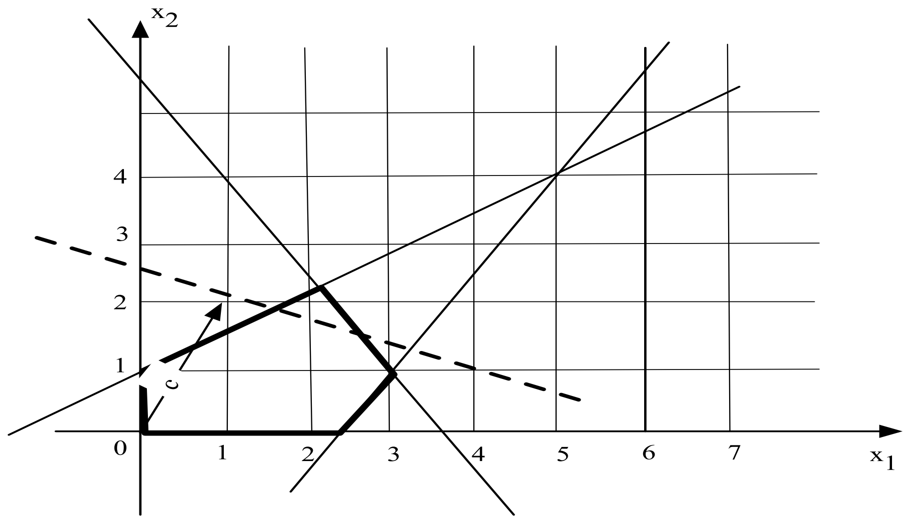

Figure 1: Feasible solutions: grid points in the polytope

b. $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ is the unique optimal solution. It is the last integer point where the contours of the objective function leaves the fasible solution set.

c. The convex hull of the feasible solution set is illustrated in Figure 2.

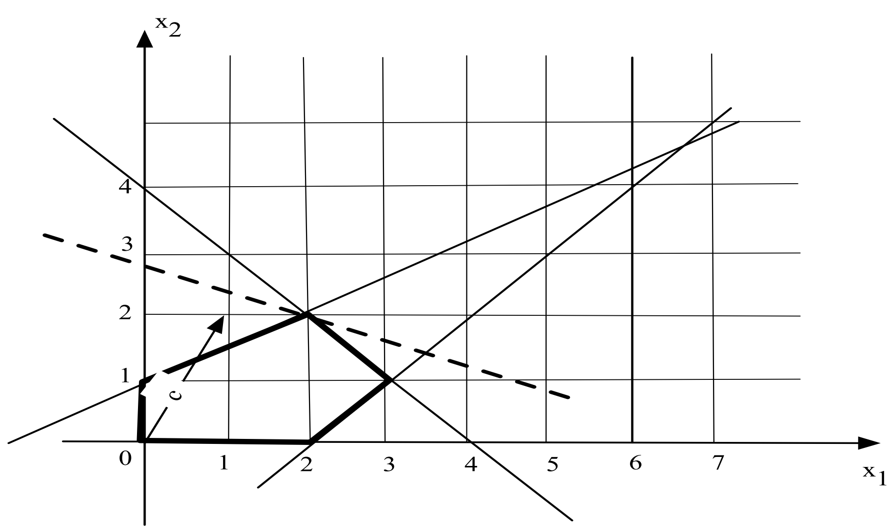

Figure 2: Convex hull of the feasible solutions

Observe that its extreme points are all integers and the point $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ becomes one of them. The algebraic description of the convex hull is

$$\begin{aligned} -2x_1 + 4x_2 &\le 4 \\ x_1 + x_2 &\le 4 \\ x_1 - x_2 &\le 2 \\ x_1, x_2 &\ge 0 \end{aligned} .$$

d. We first find $x^{(1)} = \begin{pmatrix} 2 \\ 2.5 \end{pmatrix}$. The upper bound is 7. This creates two subproblems either $x_2 \geq 3$ (Subproblem 1, which is infeasible and no further enumeration is needed), or $x_2 \leq 2$ (Subproblem 2). The optimal solution of Subproblem 2 is $x^{(2)} = \begin{pmatrix} 7/3 \\ 2 \end{pmatrix}$. This creates two subproblems: either $x_1 \leq 2$ (Subproblem 3) or $x_1 \geq 3$ (Subproblem 4). The optimal solution to Subproblem 3 is $x^{(3)} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, which is integer, so no further enumeration is needed. The lower bound is 6. The optimal solution to (Subproblem 4) is $x^{(4)} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, which is integer, so no further enumeration is needed. It gives the objective value 5, which is not better than the lower bound. $x^{(3)}$ is the optimal solution since it gives the best lower bound.