IE 203 - Operations Research II
Quiz II - Solutions

İ. Kuban Altınel **March 08, 2025**

Duration: 40 minutes
This is a CLOSED BOOK exam.

Question I (60 pts.)

a,b. Graphical representation of the feasible solutions and their convex hull is given in Figure 1

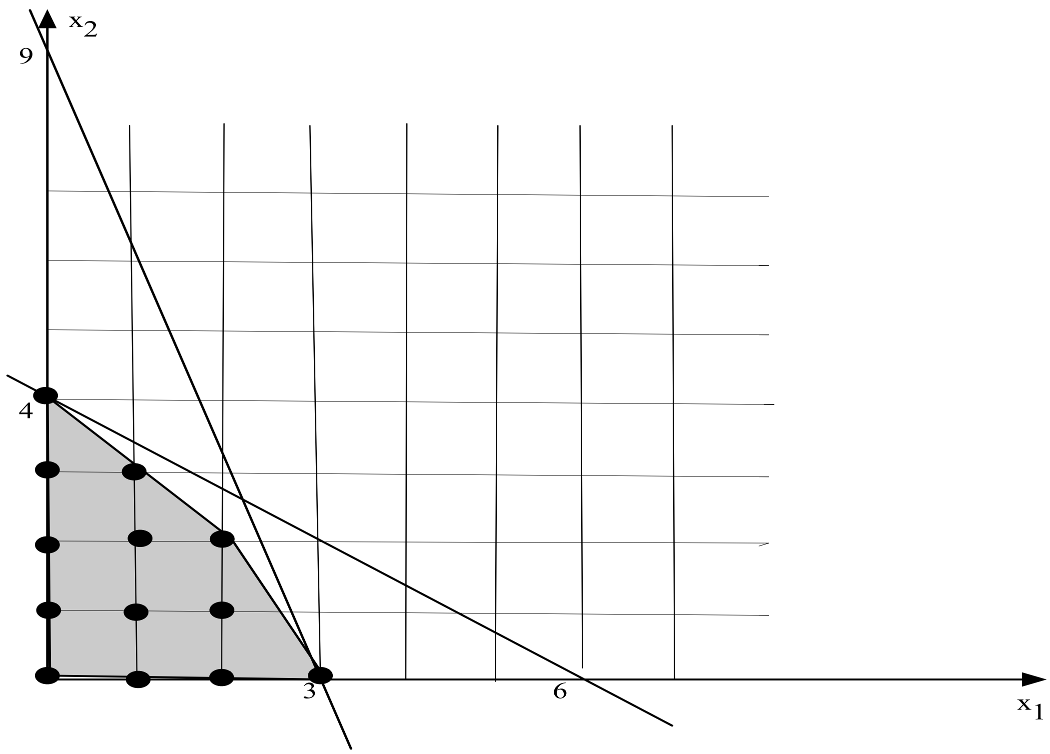

Figure 1: Feasible solutions and their convex hull

The algebraic description of the convex hull:

$$
\begin{aligned}
x_1 + x_2 &\leq 4 \\
2x_1 + x_2 &\leq 6 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

c. Optimal tableau:

	z	x_1	x_2	x_3	x_4	rhs
z	1	0	0	110/7	20/7	1500/7
x_2	0	0	1	3/7	$-2/7$	18/7
x_1	0	1	0	$-1/7$	3/7	15/7

Here, x_3 and x_4 are respectively the slack variables of the first and second inequalities.

d. In the optimal solution $x_2 = 18/7 = 2 + 4/7$ and $x_1 = 15/7 = 2 + 1/7$. Hence, x_2 has the largest fractional part. Then, from the corresponding row we can write

$$(1 + 0)x_2 + (0 + 3/7x_3 + (-1 + 5/7)x_4 = (2 + 4/7)$$

from which Gomory cut

$$3/7x_3 + 5/7x_4 \geq 4/7$$

can be obtained. It is clearly equivalent to

$$-3/7x_3 - 5/7x_4 \leq -4/7.$$

e. After adding Gomory cut the optimal tableau becomes

	z	x_1	x_2	x_3	x_4	x_5	rhs
z	1	0	0	110/7	20/7	0	1500/7
x_2	0	0	1	3/7	$-2/7$	0	18/7
x_1	0	1	0	$-1/7$	3/7	0	15/7
x_5	0	0	0	$-3/7$	$-5/7$	1	$-4/7$

Then, applying dual simplex on x_5 results in the new optimal tableau:

	z	x_1	x_2	x_3	x_4	x_5	rhs
z	1	0	0	14	0	4	212
x_2	0	0	1	3/5	0	$-2/5$	14/5
x_1	0	1	0	$-2/5$	0	3/5	9/5
x_4	0	0	0	3/5	1	$-7/5$	4/5

As can be noticed this solution is not integer optimal and more iterations are required.

f. To be able to show the generated Gomory cut

$$-3/7x_3 - 5/7x_4 \leq -4/7$$

it must be expressed equivalently in x_1 and x_2. For this purpose we use $x_3 = 12 - 2x_1 - 3x_2$ and $x_4 = 9 - 3x_1 - x_2$ in it to obtained

$$3x_1 + 2x_2 \leq 11,$$

which is plot in Figure 2.

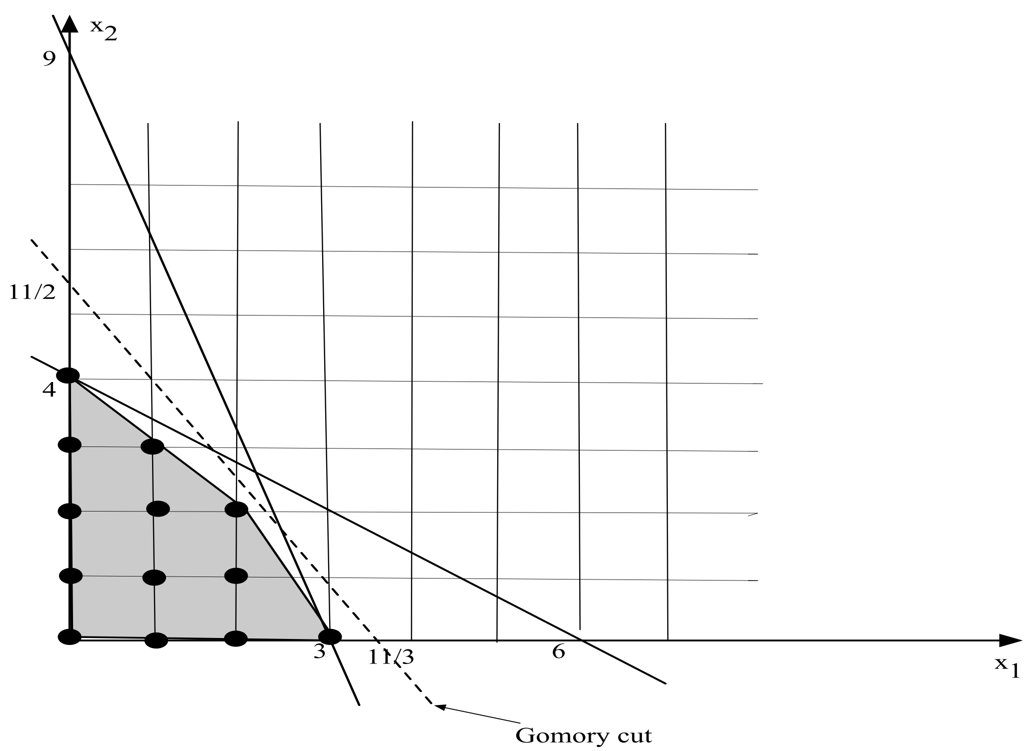

Figure 2: Gomory cut

Question II (40 pts.)

The problem can be formulated as

$$\begin{array}{ll} \min & -xy(x-y) \\ \text{s. t.} & x+y=8 \\ & x \geq 0,\ y \geq 0. \end{array}$$

The Lagrangean function is

$$\pounds(x,y,\lambda,\mu_1,\mu_2) = -xy(x-y) + \lambda(x+y-8) - \mu_1 x - \mu_2 y.$$

The first order KKT conditions:

$$\begin{aligned} -2xy + y^2 + \lambda - \mu_1 &= 0 \\ -x^2 + 2xy + \lambda - \mu_2 &= 0 \\ \mu_1 x &= 0 \\ \mu_2 y &= 0 \\ \mu_1 &\geq 0 \\ \mu_2 &\geq 0 \\ \lambda & \text{ unrestricted} \\ x+y-8 &= 0 \\ x &\geq 0 \\ y &\geq 0 \quad . \end{aligned}$$

We can find solutions to this system by considering cases based on the complementary slackness conditions.

Case 1. For $\mu_1 = \mu_2 = 0$, using the fact that $x + y = 8$

$$\begin{aligned} -2(8-y)y + y^2 + \lambda &= 0 \\ -64 - y^2 + 16y + 2(8-y)y + \lambda &= 0 \quad , \end{aligned}$$

and

$$\begin{aligned} -16y + 2y^2 + y^2 + \lambda &= 0 \\ -64 - y^2 + 16y + 16y - 2y^2 + \lambda &= 0 \end{aligned}$$

can be written. The last equalities are equivalent to

$$\begin{aligned} 3y^2 - 16y + \lambda &= 0 \\ -3y^2 + 32y - 64 + \lambda &= 0 \quad . \end{aligned}$$

They give

$$6y^2 - 48y + 64 = 0$$

resulting in

$$3y^2 - 24y + 32 = 0,$$

which is a second order equation.

When it is solved it is possible to see that it has the two real roots

$$y = \frac{1}{3}\left(12 \pm \sqrt{144-96}\right) = 4 \pm \frac{4}{\sqrt{3}},$$

and they give

as stationary points.

$$
\begin{aligned}
y &= 4 + \frac{4}{\sqrt{3}} \qquad & x &= 8 - y = 4 - \frac{4}{\sqrt{3}} \\
y &= 4 - \frac{4}{\sqrt{3}} \qquad & x &= 8 - y = 4 + \frac{4}{\sqrt{3}}.
\end{aligned}
$$

Case 2. When $x = \mu_2 = 0$, $y = 8$ from $x + y = 8$. Using stationary conditions, $\lambda = 0$ and $\mu_1 = 64$. These assignments satisfy KKT conditions. Thus, $(x, y) = (0, 8)$ is a KKT point.

Case 3. When $y = \mu_1 = 0$, $x = 8$ from $x + y = 8$. From stationary conditions $\lambda = 0$ and $\mu_2 = -64$. The assignment $\mu_2 = -64$ violates a dual slackness condition. Thus, we do not obtain a KKT point in this case.

Case 4. $x = y = 0$ clearly violates the primal feasibility condition $x+y = 8$. A KKT point is not obtained.

In summary, there are three KKT points which are

$$
\begin{aligned}
x &= 4 - \frac{4}{\sqrt{3}} \quad & y &= 4 + \tfrac{4}{\sqrt{3}}, \\
x &= 4 + \frac{4}{\sqrt{3}} \quad & y &= 4 - \tfrac{4}{\sqrt{3}}, \\
x &= 0 \quad & y &= 8.
\end{aligned}
$$