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Answer 1

I will use the following definitions:

A function $'f'$ from set A to set B is called a surjective function if for each $b \in B$ there exists at least one $a \in A$ such that $f(a) = b$.

A function $'f'$ from set A to set B is called injective if the image of distinct elements for A under f are distinct. i.e., for every $a_1, a_2 \in A$ there exists distinct $b_1, b_2 \in B$ such that $f(a_1) = b_1, f(a_2) = b_2$.

a) f_1 is not surjective because there does not exist a number $a \in R$ and $f(a) = b$ for $b \in R^-$. It is injective since for $f(a) = a^2 \in R$, there exist two different elements, such as $-a, a \in R$.

b) f_2 is not surjective because there does not exist a number $a \in R^+$ and $f(a) = b$ for $b \in R^-$. It is injective since for $f(a) = a^2 \in R$, there exists only one element, such as $a > 0 \in R^+$.

c) f_3 is surjective because exists a number $a \in R$ and $f(a) = b$ for $b \in R^+$. It is not injective since for $f(a) = a^2 \in R^+$, there exist two different elements, such as $-a, a \in R$.

d) f_4 is surjective because exists a number $a \in R^+$ and $f(a) = b$ for $b \in R^+$. It is injective since for $f(a) = a^2 \in R^+$, there exists only one element, such as $a > 0 \in R^+$.

Answer 2

a) Take any function $f : Z \to R$. Pick some $x_0 \in A$ and let $\epsilon > 0$. Then we need to find a δ to make the rest of definition true. An easy choice is $\delta = 1/3$.

Let $x \in A$ and suppose that $|x - x_0| < \delta = 1/3$.Since A is a subset of Z and since the only integer within $1/3$ distance of x_0 is itself, we can easily observe that $x = x_0$. Hence $f(x) = f(x_0)$ and $|f(x) - f(x_0)| = 0$ which is certainly less than ϵ.

This shows that f is continuous at x_0 since x_0 has chosen arbitrary, f is continuous everywhere on its domain.

b) I will use the intermediate value theorem from calculus. We can prove f is constant by contradiction:

Assume f is not constant; so we have different integers a and b in the range of f. Then $a = f(x)$ and $b = f(y)$ for some real numbers x and y.

Now there exists a real number c, not an integer, which is strictly between a and b. By the IVT and the assumption that f is continuous, there is a real number α, between x and y, such that $c = f(\alpha)$. But this contradicts the assumption that f takes only integer values.

Hence for a function $f : R \to Z$ to be continuous f must be a constant function.

Answer 3

a) Let $A_1, A_2, \ldots, A_n$ be countable sets.

We prove the statement by induction.

For $n = 2$, the statement clearly holds, as A_1 and A_2 are countable so $A_1 \times A_2$ is also countable.

Now suppose that $B := A_1 \times A_2 \times \ldots A_{n-1}$ is countable.

We have:

$$B \times A_n = \{(b, a) | b \in B, a \in A_n\} = \bigcup_{a \in A_n} \{(b, a) | b \in B\}$$

and $\{(b, a) | b \in B\}$ is countable for a fixed $a \in A_n$, since the function $f_a : B \to B \times \{a\} : b \to (b, a)$ is a bijection, and B is countable by induction hypothesis. Because the union of countable sets remains countable, we have proven that $(A_1 \times \ldots A_{n-1}) \times A_n$ is countable, and because $f : (A_1 \times \ldots A_{n-1}) \times B \to A_1 \times \ldots A_{n-1} \times A_n : ((a_1, \ldots, a_{n-1}), a_n) \mapsto (a_1, \ldots, a_{n-1}, a_n)$ is a bijection, the result follows.

b) By the definition of Cartesian product of sets,

$$\S = \Pi_{n \in N} \{f \colon N \to \cup_{n \in\ N} E_n \mid \forall n, f(n) \in E_n\}$$

If $E_n = \{0, 1\}$, then

$$S_{01} = \Pi_{n \in N} \{0, 1\} = E^N$$

where $E = \{0, 1\}$

We will use Cantor's diagonal trick.

Suppose S is countable. Let $(F_n : n \in N)$ be an enumeration of S. For each n, pick two points $a_n, b_n \in E_n$. Then define a new function $F \in S$ as follows:

$$F(m) = \begin{cases} b_m & \text{if } F_m(m) = a_m \\ a_m & \text{otherwise} \end{cases}$$

It follows that $F \in S$ but it is different for all F_n's which is a contradiction.

So by the contradiction we can observe that an infinite countable product of the set X with itself is uncountable.

Answer 4

I will use the limit comparison test from calculus for $f \in O(g)$ g is the big-O of f if the following can be said $\lim_{x\to\infty} \frac{f(x)}{g(x)} < \infty$.

a) For comparing $\log x.\sqrt{x}$ and $(\log x)^2$ we calculate $\lim_{x\to\infty} \frac{(\log x)^2}{\log x.\sqrt{x}} = 0 < \infty$ so the former function is the big-O of the latter.

b) For comparing x^{50} and $\log x.\sqrt{x}$ we calculate $\lim_{x\to\infty} \frac{\log x.\sqrt{x}}{x^{50}} = 0 < \infty$ so the former function is the big-O of the latter.

c) For comparing $x^{51} + x^{49}$ and x^{50} we calculate $\lim_{x\to\infty} \frac{x^{50}}{x^{51}+x^{49}} = 0 < \infty$ so the former function is the big-O of the latter.

d) For comparing 2^x and $x^{51} + x^{49}$ we calculate $\lim_{x\to\infty} \frac{x^{51}+x^{49}}{2^x} = 0 < \infty$ so the former function is the big-O of the latter.

e) For comparing 5^x and 2^x we calculate $\lim_{x\to\infty} \frac{2^x}{5^x} = 0 < \infty$ so the former function is the big-O of the latter.

f) For comparing $(x!)^2$ and 5^x we calculate $\lim_{x\to\infty} \frac{(5^x}{(x!)^2} = 0 < \infty$ so the former function is the big-O of the latter.

So we can say that for big-O comparisons,

$$(x!)^2 > 5^x > 2^x > x^{51} + x^{49} > x^{50} > \log x.\sqrt{x} > (\log x)^2$$

Answer 5

a)
Step 1: Divide 134 by 94 and get the remainder, since remainder 40 is greater than zero, we continue to divide.

Step 2: Divide 94 by 40 and get the remainder, since remainder 14 is greater than zero, we continue to divide.

Step 3: Divide 40 by 14 and get the remainder, since remainder 12 is greater than zero, we continue to divide.

Step 4: Divide 14 by 12 and get the remainder, since remainder 2 is greater than zero, we continue to divide.

Step 5: Divide 12 by 2 and get the remainder, since the remainder ≥ 0 the gcd is the last divisor 2.

b)
$k \in N^+ - \{1\}$for p_1, p_2 prime numbers the conjecture states that

$$2k = p_1 + p_2$$

$n \in N^+ - \{1, 2, 3, 4, 5\}$for p_1, p_2, p_3 prime numbers the statement states that

$$n = p_1 + p_2 + p_3$$

To prove the equivalence of the conjecture and statements we have two different cases:

Suppose n is even.

$$n = 2t \wedge t \geq 3$$

Observe that $(n - 2)$ is also even. Applying the conjecture

$$(n - 2) = p_1 + p_2$$

thus

$$n = p_1 + p_2 + 2$$

which is a sum of three prime numbers.

Suppose n is odd.

$$n = 2t + 1 \wedge t \geq 3$$

Observe that $(n-3) = 2t - 2 = 2(n-1)$ is even. Applying the conjecture

$$(n-3) = p_1 + p_2$$

thus

$$n = p_1 + p_2 + 3$$

which is a sum of three prime numbers.