CENG 384 - Signals and Systems for Computer Engineers
Spring 2024
Homework 3

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1. Given a continuous-time periodic signal $x(t)$ with a period $T = 4$, whose Fourier series coefficients a_k are defined as:

$$a_k = \begin{cases} -1, & \text{if } k \text{ is even,} \\ 1, & \text{if } k \text{ is odd.} \end{cases}$$

The Fourier series representation of $x(t)$ is given by the formula:

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos\left(\frac{2\pi kt}{T}\right) + b_k \sin\left(\frac{2\pi kt}{T}\right)$$

For our specific case, the coefficients a_k are given as -1 for even k and 1 for odd k, with all b_k coefficients being zero. Therefore, the series simplifies to:

$$x(t) = a_0 + \sum_{k=1}^{\infty} (-1)^{k+1} \cos\left(\frac{2\pi kt}{4}\right)$$

This is an alternating series with cosine terms where the sign changes with the parity of k. Calculating the first few terms for an illustration, the series begins with $a_0 = -1$:

$$x(t) = -1 + \cos\left(\frac{\pi t}{2}\right) - \cos(\pi t) + \cos\left(\frac{3\pi t}{2}\right) - \cos(2\pi t) + \cos\left(\frac{5\pi t}{2}\right) - \cdots$$

This pattern will continue indefinitely with the a_k coefficient alternating sign for each term in the series. This series represents a signal that is a combination of cosine waves with amplitudes that alternate between -1 and 1 for even and odd values of k, respectively.

2. (a) Consider a continuous-time periodic signal described in one period $T = 4$ as:

$$x(t) = \begin{cases} 2t, & 0 < t < 2, \\ 4 - t, & 2 \leq t < 4. \end{cases}$$

The Fourier series coefficients are given by the equations:

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt,$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos\left(\frac{2\pi kt}{T}\right) dt,$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin\left(\frac{2\pi kt}{T}\right) dt.$$

For the given signal, the coefficients will be calculated as follows:

$$a_0 = \frac{1}{4} \left(\int_0^2 2t\, dt + \int_2^4 (4 - t)\, dt \right),$$

$$a_k = \frac{1}{2} \left(\int_0^2 2t \cos\left(\frac{\pi kt}{2}\right) dt + \int_2^4 (4 - t) \cos\left(\frac{\pi kt}{2}\right) dt \right), \quad k \geq 1,$$

$$b_k = \frac{1}{2} \left(\int_0^2 2t \sin\left(\frac{\pi kt}{2}\right) dt + \int_2^4 (4 - t) \sin\left(\frac{\pi kt}{2}\right) dt \right), \quad k \geq 1.$$

(b) According to the differentiation property of Fourier series, if $x(t)$ has a Fourier series representation with coefficients a_n and b_n, then the derivative $\frac{dx}{dt}$ has a Fourier series representation with coefficients given by:

$$\frac{dx(t)}{dt} = \sum [n\omega_0 b_n \cos(n\omega_0 t) - n\omega_0 a_n \sin(n\omega_0 t)].$$

In this case, $x(t)$ has spectral coefficients a_n and b_n as calculated in part (a). Therefore, differentiating $x(t)$ with respect to t, we have:

$$\frac{dx}{dt} = 2 \text{ for } 0 \leq t < 2,$$
$$\frac{dx}{dt} = -1 \text{ for } 2 \leq t < 4.$$

Hence, the spectral coefficients of $\frac{dx}{dt}$ are:
$a_0 = 0$, (since $\frac{dx}{dt}$ is a constant),
$a_n = 0$ for all values of n,
$b_n = 0$ for all values of n.

3. (a)

$$x[n] = sin(\frac{\pi}{2}n)$$

We choose ω_0 as 2π.Using Euler's relation, we have

$$x[n] = \frac{1}{2j}e^{j(\pi/2)n} - \frac{1}{2j}e^{-j(\pi/2)n}$$

$$a_0 = 0, a_1 = \frac{1}{2j}, a_{-1} = \frac{-1}{2j} \text{ All other } a_k\text{'s } = 0.$$

$$y[n] = cos(\frac{\pi}{2}n)$$

We choose ω_0 as 2π.

$$x[n] = \frac{1}{2}e^{j(\pi/2)n} + \frac{1}{2}e^{-j(\pi/2)n}$$

$$a_0 = 0, a_1 = a_{-1} = \frac{1}{2} \text{ All other } a_k\text{'s } = 0.$$

$$z[n] = x[n]y[n]$$
$$z[n] = sin(\frac{\pi}{2}n).cos(\frac{\pi}{2}n)$$
$$z[n] = \frac{1}{2}sin(\pi n)$$

Since

$$sin(\pi n)$$

is always zero

$$\text{All } a_k\text{'s } = 0.$$

(b) Multiplication Property is as follows

$$x(t) \longleftrightarrow a_k \text{ and } y(t) \longleftrightarrow b_k$$

then,

$$a_k * b_k \longleftrightarrow \sum_{\forall l} a_l b_{k-l}$$

$$a_k * b_k \rightarrow a_0 b_k + a_1 b_{k-1} + a_2 b_{k-2} + a_3 b_{k-3}$$

The first and third term are 0 since a_0 and a_2 are zero.

Since $a_1 = 1/2j$ and $b_{-1} = 1/2$ and $a_{-1} = -1/2j$ and $b_{-3} = b_1 = 1/2$ the result is 0

Comparing a_k with the a_k from the part (a), we see that both are the same.

4. The Fourier series coefficients of $x[n]$, which is periodic with period N, are given by

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk(2\pi/N)n}$$

For $N = 24$,

$$a_k = \frac{1}{24}\sum_{n=0}^{23} x[n]e^{-jk(\pi/12)n}$$

$$a_k = cos(\frac{4k\pi}{12}) + sin(\frac{3k\pi}{12})$$

$$a_k = \frac{1}{2}e^{j(\pi k/3)} + \frac{1}{2}e^{-j(\pi k/3)} + \frac{1}{2j}e^{j(\pi k/4)} - \frac{1}{2j}e^{j(\pi k/4)}$$

Hence,

$$x[n] = 12\delta[n-4] + 12\delta[n+4] - 12j\delta[n+4] + 12j\delta[n-4], 0 \leq n \leq 23$$

5. (a) The fundamental period of the signal $x[n] = \sin\left(\frac{6\pi n}{13} + \frac{\pi}{2}\right)$ is found by determining the smallest positive integer N that satisfies $\frac{6\pi N}{13} = 2\pi m$, where m is an integer. The fundamental period N is 13 since $N = \frac{13m}{3}$ and the smallest m to make N an integer is 3.

(b) Given the discrete-time signal

$$x[n] = \sin\left(\frac{6\pi n}{13} + \frac{\pi}{2}\right),$$

we can simplify this expression using the trigonometric identity:

$$\sin\left(\theta + \frac{\pi}{2}\right) = \cos(\theta).$$

Thus, the signal simplifies to

$$x[n] = \cos\left(\frac{6\pi n}{13}\right).$$

This is a cosine signal with a frequency of $\frac{6\pi}{13}$ radians per sample. The periodicity of a discrete cosine signal $\cos\left(\frac{2\pi kn}{N}\right)$ is N, where k and N are integers and N is the period. Here, the coefficient $\frac{6}{13}$ indicates the signal's periodic nature with a period of $N = 13$, because:

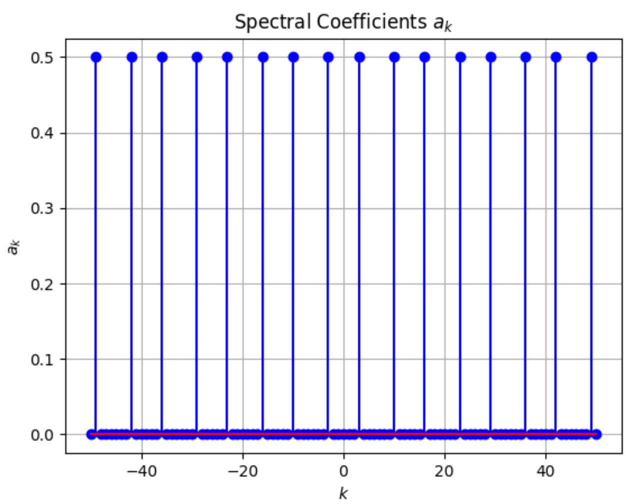
$$\cos\left(\frac{6\pi n}{13}\right) = \cos\left(\frac{6\pi(n+13)}{13}\right).$$

Since we know for a periodic discrete $\cos\omega_0 n$ the spectral coefficients is shown like this

$$a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$$

For our specific case

$$a_k = \begin{cases} \frac{1}{2}, & k = \pm 3, \pm 3 \pm 13, \pm 3 \pm 26, \ldots \\ 0, & \text{otherwise} \end{cases}$$



6. (a) The frequency response is given as:

$$H(j\omega) = \frac{1}{4j\omega + 3}$$

to find the impulse response we will perform inverse Fourier transform.

We can rewrite $H(j\omega)$ in the standard form:

$$H(j\omega) = \frac{1}{3 + 4j\omega}$$

Using the inverse Fourier transform formula for the first order system:

$$\mathcal{F}^{-1}\left\{\frac{1}{a + j\omega}\right\} = e^{-at}u(t)$$

where $a > 0$ and $u(t)$ is the unit step function, we apply it to our function:

$$h(t) = e^{-\frac{3}{4}t}u(t)$$

(b) Let us evaluate the Fourier transform of the given function:

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t)e^{-j\omega t}\, dt$$

where,

$$y(t) = e^{-5t}u(t) - e^{-10t}u(t)$$

Applying the Fourier transform:

$$= \int_{-\infty}^{\infty} \left(e^{-5t}u(t) - e^{-10t}u(t)\right) e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \left(e^{-t(5+j\omega)} - e^{-t(10+j\omega)}\right) u(t)\, dt$$

Since $u(t)$ is zero for $t < 0$:

$$= \int_{0}^{\infty} \left(e^{-t(5+j\omega)} - e^{-t(10+j\omega)}\right) dt$$

$$= \int_{0}^{\infty} e^{-t(5+j\omega)}\, dt - \int_{0}^{\infty} e^{-t(10+j\omega)}\, dt$$

These integrals simplify to:

$$Y(j\omega) = \left[\frac{-e^{-t(5+j\omega)}}{5 + j\omega}\right]_0^{\infty} - \left[\frac{-e^{-t(10+j\omega)}}{10 + j\omega}\right]_0^{\infty}$$

Solving these,

$$= \frac{1}{5 + j\omega} - \frac{1}{10 + j\omega}$$

$$= \frac{(10 + j\omega) - (5 + j\omega)}{(5 + j\omega)(10 + j\omega)}$$

$$= \frac{5}{(5 + j\omega)(10 + j\omega)}$$

Now, the Impulse response is given by:

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

$$\Rightarrow X(j\omega) = \frac{Y(j\omega)}{H(j\omega)}$$

$$= \frac{5}{(5 + j\omega)(10 + j\omega)} \times \frac{4j\omega + 3}{1}$$

$$= \frac{37}{10 + j\omega} - \frac{17}{5 + j\omega}$$

Now, by applying inverse Fourier transform on $X(j\omega)$, we will find $x(t)$

$$\mathcal{F}^{-1}\{X(j\omega)\} = \mathcal{F}^{-1}\left\{\frac{37}{10 + j\omega} - \frac{17}{5 + j\omega}\right\}$$

The inverse Fourier transform of $\frac{1}{a+j\omega}$, where $a > 0$, is $e^{-at}u(t)$. Applying this to each term:

1. $\frac{37}{10+j\omega}$:
$$\mathcal{F}^{-1}\left\{\frac{37}{10+j\omega}\right\} = 37e^{-10t}u(t)$$

2. $\frac{17}{5+j\omega}$:
$$\mathcal{F}^{-1}\left\{\frac{17}{5+j\omega}\right\} = 17e^{-5t}u(t)$$

Combining these, the inverse Fourier transform of the entire function is:

$$h(t) = 37e^{-10t}u(t) - 17e^{-5t}u(t)$$

7.
```
import numpy as np
import matplotlib.pyplot as plt

# Define the time variable
T = 6  # Fundamental period
omega_0 = 2 * np.pi / T  # Fundamental angular frequency
t = np.linspace(0, T, 1000, endpoint=False)

# Define the function x(t)
x_t = np.cos(np.pi * t / 3) - 2 * np.sin(np.pi * t)

# Define the number of terms in the Fourier series to compute
N = 30  # Compute coefficients up to the 30th

# Compute the Fourier coefficients using numerical integration
a_0 = 2 / T * np.trapz(x_t, t)
a_n = [2 / T * np.trapz(x_t * np.cos(n * omega_0 * t), t) for n in range(1, N+1)]
b_n = [2 / T * np.trapz(x_t * np.sin(n * omega_0 * t), t) for n in range(1, N+1)]

# Plotting the magnitude and phase of the coefficients
n_vals = np.arange(1, N+1)
magnitudes = np.sqrt(np.array(a_n)**2 + np.array(b_n)**2)
phases = np.arctan2(b_n, a_n)

plt.figure(figsize=(14, 5))

# Plotting the magnitude of coefficients
plt.subplot(1, 2, 1)
plt.stem(n_vals, magnitudes, basefmt=" ", use_line_collection=True)
plt.title('Magnitude of Fourier Coefficients')
plt.xlabel('n (Harmonic number)')
plt.ylabel('Magnitude')

# Plotting the phase of coefficients
plt.subplot(1, 2, 2)
plt.stem(n_vals, phases, basefmt=" ", use_line_collection=True)
plt.title('Phase of Fourier Coefficients')
plt.xlabel('n (Harmonic number)')
plt.ylabel('Phase (radians)')

plt.tight_layout()
plt.show()

# Output the fundamental period and the simplified Fourier series coefficients
print(f"Fundamental Period, T: {T}")
print(f"a_0: {a_0}")
print("First few a_n coefficients:", a_n[:5])
print("First few b_n coefficients:", b_n[:5])
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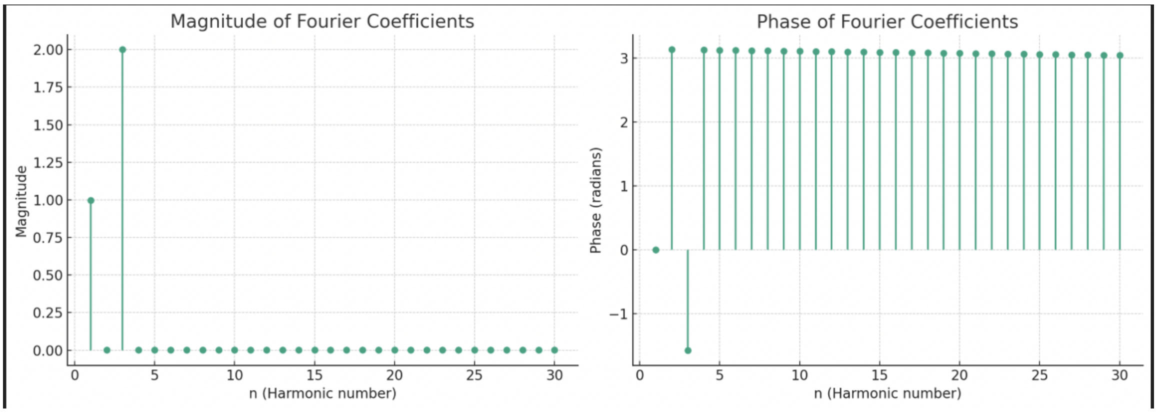
Magnitude of Fourier Coefficients
Magnitude
n (Harmonic number)
Phase of Fourier Coefficients
Phase (radians)
n (Harmonic number)