Mathematical Modeling of Blood Flow

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Abstract—**The interface between mathematics and biology has initiated and fostered new mathematical areas, where the ideas from mathematics and biology are synergistically applied. Study of fluid dynamics plays a significant role in fluid flow inside the human body, and modeling of blood flow is an important field in cardiovascular physics. However, models have been developed so far are very complex with three-dimensional analysis. This paper presents a novel and simple mathematical model of the blood flow and the blood pressure. The main fluid component of the cardiovascular system is the blood which flows through the different blood vessels in the body. Although blood is the non-Newtonian fluid, in many cases, it behaves like a Newtonian fluid which is governed by the Navier–Stokes equations. With the help of continuity equation and the Navier–Stokes equations, a simple differential equation was derived under some assumption, which is called as the cardiovascular system equation. Then by applying the logical assumptions on this Cardiovascular System equation, the general mathematical model of the normal blood flow was developed. Then this model was extended for normal blood pressure using the Poisuelli's equation. At the end of this study, some analysis had been performed to determine the validity of the proposed model. The analysis showed that the model can satisfy both the different properties of blood flow and blood pressure.**

Keywords- mathematical model; cardiovascular system equation; non-newtonian fluid; differential equation; blood pressure

I. INTRODUCTION

A mathematical model uses mathematical language to describe a system in the real world. The process of developing a mathematical model is termed as mathematical modeling or modeling. Cardiovascular system is the blood distribution network in the body. The cardiovascular system in the body consists of three components: blood, heart and blood vessels. When blood flows through the vessels, pressure is detected on the wall which is termed as the blood pressure. Blood pressure depends mainly on flow rate and size of the vessels and on the pressure gradient. There are three major types of blood vessels: arteries, capillaries and veins [1]. Arteries are large blood vessels that carry blood away from the heart to all regions of the body [1]. The arterioles further divide into smaller vessels called capillaries. Capillaries are the anatomic units that connect the arterial and venous circulatory system. The veins form a low-pressure collecting system to return the oxygen-poor blood to the heart [1]. In this analysis, all the vessels are assumed to be same in nature excluding their size, length and cross-sectional area.

Many studies have been published in this field. There is an important literature on the functional imaging and of the heart [2], [3]. Some studies have been performed on the measurement of electrical activity, deformation, flows, fiber orientation, and modeling of the heart [4]–[9], and on the modeling of the electrical and mechanical activity of the heart [10]–[12]. This study gives the mathematical modeling of the blood flow and also represents some simulation results based on the developed model.

To develop the model of the blood flow and blood pressure, some assumptions have been considered. These include that the blood vessels are the cylindrical, deformable components with circular cross-sections. They change their size as the blood flows through it. The blood is considered to be the Newtonian fluid which is governed by the Navier-Stokes equation and by the continuity equation. Although the blood needs the help of the lungs for the supply of oxygen, its properties remain unchanged by the addition of that oxygen. Another assumption is required, and that is the blood has both the radial and axial flow in only one direction, i.e. z-direction in a three-dimensional system. So, the other two components (x-direction and y-direction) are vanished.

II. MATHEMATICAL MODELING

A. Developing the Cardiovascular System Equation

The velocity components in the *x, y,* and *z* directions are typically named *u, v,* and *w* respectively. Let ρ be the density of blood, P be the blood pressure and μ is the kinematic viscosity of blood. Then neglecting the orientation of gravity inside the body, the Navier-Stokes equation in the Cartesian co-ordinates is given by the following equations:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) \tag{1}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right) = -\frac{\partial P}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \tag{2}$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right) = -\frac{\partial P}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right) \tag{3}$$

With the assumption of no tangential velocity and no x and y components of velocity, a change of variables on the Cartesian equations will yield the following system of equation in the cylindrical co-ordinate system [14]:

$$\frac{\partial w}{\partial t}+f\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}\right) \quad (4)$$

$$\frac{\partial f}{\partial t}+f\frac{\partial f}{\partial t}+w\frac{\partial f}{\partial t}=-\frac{1}{\rho}\frac{\partial P}{\partial z}+\mu\left(\frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{\partial f}{\partial r}+\frac{\partial^2 f}{\partial z^2}+\frac{f}{r^2}\right) \quad (5)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rf)+\frac{\partial w}{\partial z}=0 \quad (6)$$

where $f(r,z,t)$ be the radial flow component, $w(r,z,t)$ be the axial flow component in z-direction.

And the continuity equation reads:

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho w)}{\partial z}=0 \quad (7)$$

Now, define a new variable γ as $\gamma=\frac{r}{R(z,t)}$, where $R(z,t)$ is the radius of the blood vessels, obviously the cylindrical coordinate (r,z,t) is now replaced by coordinate (γ,z,t)

Again, the velocity profile in the axial direction, $w(\eta,z,t)$, is assumed to have the expression in the polynomial form:[15]

$$w(\gamma,z,t)=\sum_{k=1}^{N} q_k(\gamma^{2k}-1) \quad (8)$$

where, $q(z,t)$ is a another variable which to be determined later. For simplification let, N=1. Then

$$w(\gamma,z,t)=q(z,t)(\gamma^2-1) \quad (9)$$

And the velocity profile in the radial direction, $w(\eta,z,t)$, is assumed to have the expression in the polynomial form: [15]

$$f(\gamma,z,t)=\frac{\partial R}{\partial z}\gamma f+\frac{\partial R}{\partial t}\gamma-\frac{\partial R}{\partial t}\frac{\gamma}{N}\sum_{k=1}^{N}\frac{1}{k}(\gamma^{2k}-1) \quad (10)$$

Again for simplification let N=1,

$$f(\gamma,z,t)=\frac{\partial R}{\partial z}\gamma f+\frac{\partial R}{\partial t}\gamma-\frac{\partial R}{\partial t}\gamma(\gamma^{2k}-1) \quad (11)$$

Using the help of equations of axial and radial velocity profile, radial coordinate and the continuity equation, the Navier–Stokes equations get the forms as below to determine the variable $q(z,t)$ and $R(z,t)$:

$$\frac{\partial q}{\partial t}-\frac{4q}{R}\frac{\partial R}{\partial t}-\frac{2q^2}{R}\frac{\partial R}{\partial z}+\frac{4\mu}{R^2}q+\frac{1}{\rho}\frac{\partial P}{\partial z}=0 \quad (12)$$

$$2R\frac{\partial R}{\partial t}+\frac{R^2}{2}\frac{\partial q}{\partial z}+q\frac{\partial R}{\partial z}=0 \quad (13)$$

Now, let to introduce the desired variable, the cross-sectional area of the blood vessel as

$$S=\pi R^2 \quad (14)$$

where R is the radius of the blood vessels.

And blood flow rate is given as the surface integral of w and $\partial\gamma$ [16]. Thus

$$Q=\iint w\partial\gamma=\frac{1}{2}q\pi R^2 \quad (15)$$

From (14) and (15) $\frac{\partial q}{\partial t}, \frac{\partial q}{\partial z}, \frac{\partial R}{\partial t}, and\ \frac{\partial R}{\partial z}$ can be found.

After inserting the value of $\frac{\partial q}{\partial t}, \frac{\partial q}{\partial z}, \frac{\partial R}{\partial t}, and\ \frac{\partial R}{\partial z}$ in (12) and (13), another two differential equations are obtained as:

$$\frac{\partial Q}{\partial t}+\frac{3Q}{S}\frac{\partial Q}{\partial z}-\frac{2Q^2}{S^2}\frac{\partial S}{\partial z}+\frac{4\pi\mu}{S}Q+\frac{S}{2\rho}\frac{\partial P}{\partial z}=0 \quad (16)$$

$$\frac{\partial S}{\partial t}+\frac{\partial Q}{\partial z}=0 \quad (17)$$

After combining (16) and (17) a simple differential equation is obtained as follows:

$$\frac{\partial Q}{\partial t}-\frac{3Q}{S}\frac{\partial S}{\partial t}-\frac{2Q^2}{S^2}\frac{\partial S}{\partial z}+\frac{4\pi\mu}{S}Q+\frac{S}{2\rho}\frac{\partial P}{\partial z}=0 \quad (18)$$

Equation (18) is now called as the 'master equation'. The model of the blood flow rate and blood pressure can be now got by applying some assumptions on this master equation which is performed in the next sections.

B. Modeling of the Blood Flow Rate

To develop the model of the blood flow it is assumed that the cross-section area of the blood vessel remains unchanged with time and it is also assumed to be constant over distance and the pressure gradient is assumed to be constant over the distance.

Applying these assumptions on (18), the master equation reduces to:

$$\frac{\partial Q}{\partial t}+\frac{4\pi\mu}{S}Q+\frac{S}{2\rho}\frac{\partial P}{\partial z}=0 \quad (19)$$

This is the one dimensional mathematical model of the blood flow rate. The required boundary condition and the values of the other parameters to solve this equation can be obtained from the past works in this field. Such as:

Pressure gradient, $\frac{\partial P}{\partial z}$ =100 to 40 mmHg [16]

Initial value of Q = 1 to 5.4 liter/minute [17]

Kinematic viscosity of blood, μ = 0.035 cm2/s [18]

Density of blood, ρ = 1.043 to 1.057 g/cm3 [18]

C. Modeling of the Blood pressure

To develop the mathematical model of the blood pressure Poisuelli's equation is considered. Poisuelli's equation determines the relation between blood flow rate and the pressure which is given as:

$$Q=\frac{\pi R^4}{8L\mu}P \quad (20)$$

where, L is the length and R is the radius of vessel.

After inserting (20) into (19), the new equations are obtained as follows:

$$\frac{\pi R^4}{8L\mu}\frac{dP}{dt} + \frac{4\pi\mu}{S}\frac{\pi R^4}{8L\mu}P + \frac{S}{2\rho}\frac{\partial P}{\partial z} = 0 \quad (21)$$

$$\frac{dP}{dt} + \frac{4\mu}{R^2}P + \frac{4L\mu}{\rho R^2}\frac{\partial P}{\partial z} = 0 \quad (22)$$

Equation (22) is the mathematical model of the blood pressure in the body. The required boundary condition and the values of the other parameters to solve this equation can be obtained from the past works in this field. Such as:

Pressure gradient, $\frac{\partial P}{\partial z}$ =100 to 40 mmHg [16]

Kinematic viscosity of blood, μ = 0.035 cm2/s [18]

Density of blood, ρ = 1.043 to 1.057 g/cm3 [18]

L varies for arteries, capillaries and veins, i.e. L is different for different types of blood vessels.

III. RESULTS

In this section of this paper, some analysis has been performed to verify the validity of the proposed model.

A. Analysis of Blood Flow Rate

The equation of the mathematical model of blood flow rate can be solved using the MATLAB function 'dsolve'. Fig. 1 represents this solution for different cross-sectional area. The significance of this plot is that the blood flow rate increases with the cross-sectional area. This result supports fig. 2 which is derived from the Poisuelli's equation.

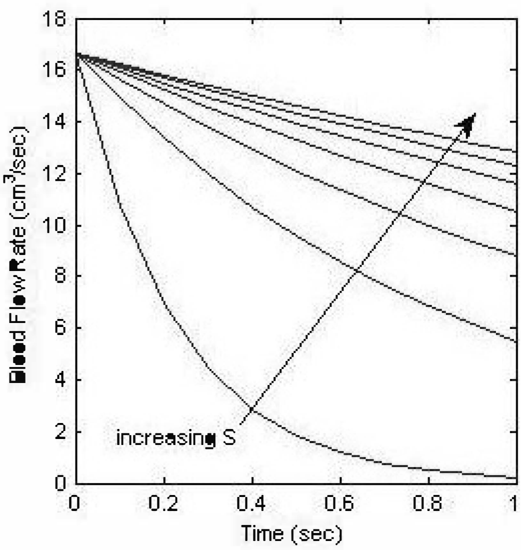

Figure 1. Blood flow rate for different cross-sectional area (from 0.1 to 2 cm^2)

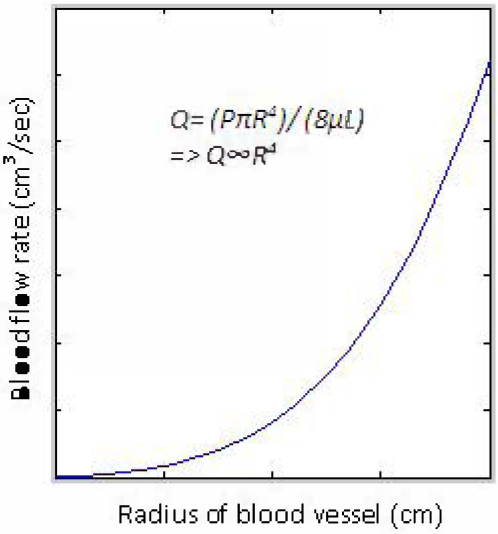

Figure 2. Variation of blood flow rate with vessel radius using Poisuelli's equation

In fig. 3, the solution is plotted for various pressure gradients. Accordingly, pressure is higher at the beginning than at the end of vessel, establishing a pressure gradient. The greater the pressure gradient forcing bloods through a vessel, the greater the rate of flow through the vessel [18]. Fig. 3 also shows that for a given pressure gradient, the blood flow rate decreases with time and as the pressure gradient increases the blood flow rate also increases.

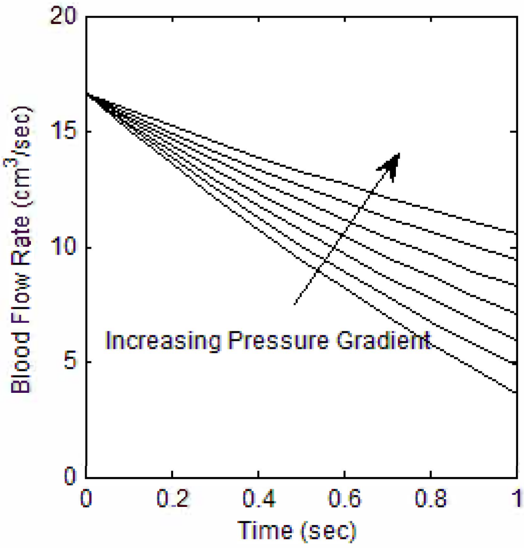

Figure 3. Blood flow rate for different pressure gradients (from 40 to 200mmHg)

B. Analysis of Blood Pressure

The equation of the mathematical model of blood pressure can be solved using the MATLAB function 'dsolve'. Fig. 4 represents this solution for different cross-sectional area. The significance of this plot is that the blood pressure decreases with the cross-sectional area. This result supports fig. 5 which is derived from the Poisuelli's equation.

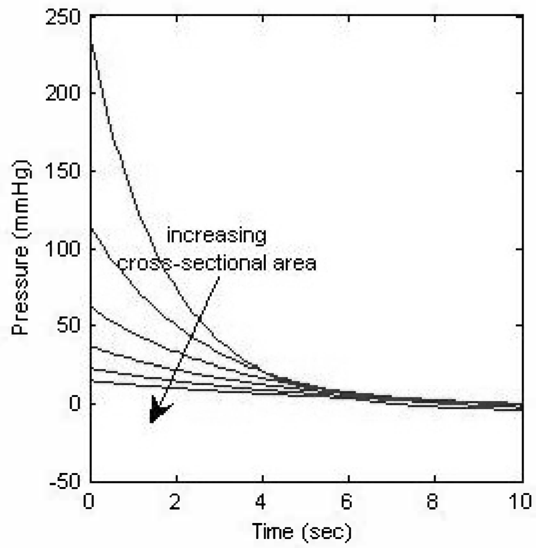

Figure 4: Blood pressure for different cross-sectional area of vessels

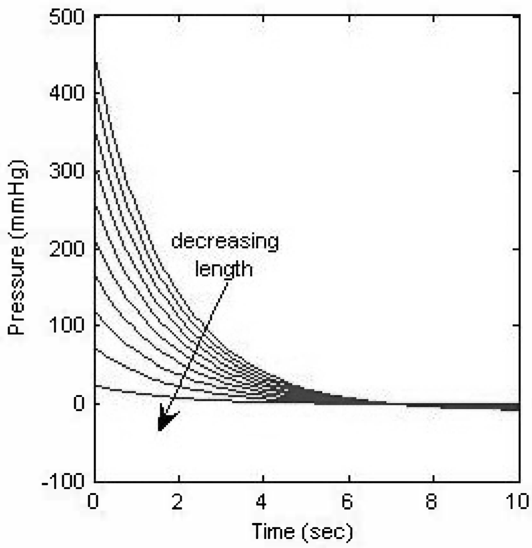

Figure. 6: Blood pressure for different length of vessels

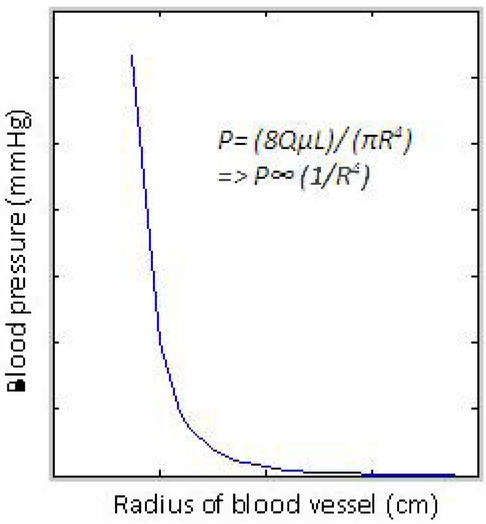

Figure 5: Variation of blood pressure with vessel radius using Poisuelli's equation

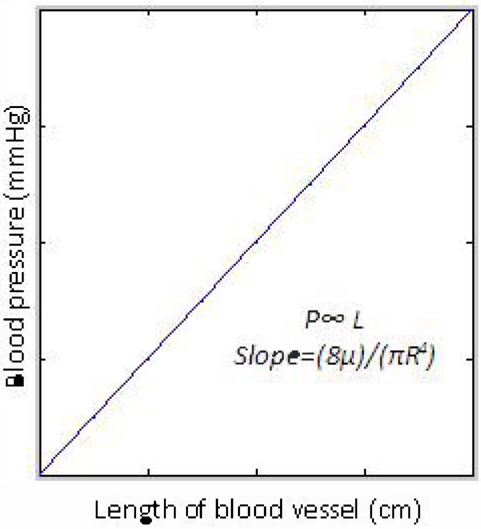

Fig. 7: Variation of blood pressure with vessel length using Poisuelli's equation

In fig. 6, the solution is plotted for various length of the blood vessel. This analysis indicates the increment of blood pressure with the increment of length of the blood vessels. Higher pressure at the beginning and lower pressure at the end and the difference between this two highly varies with length of the blood vessels. This result supports fig. 7 which is derived from the Poisuelli's equation.

Systolic (maximum) blood pressure in the normal adult is in the range of 95 to 140 mm Hg, with 120 mm Hg being average. These figures are subject to much variation with age, climate, eating habits, and other factors. Normal diastolic blood pressure (lowest pressure between beats) ranges from 60 to 90 mm Hg, 80 mm Hg being about average. This pressure is usually measured in the brachial artery in the arm [19]. Fig. 4 and fig. 6 show that the blood pressure ranges support the normal values for the artery in the arm with corresponding cross-sectional area and length.

IV. CONCLUSION

The design of computational models of human organs is a new research field which opens new possibilities for medical image analysis and therapy simulation [20]. The goal of this paper was to represent a mathematical model of the blood flow. A limited number of internal parameters were considered in developing the model. So, possible improvements of the study would include the integration of more anatomical structure (valve, exact size of the heart chambers), more realistic model and a more complex constitutive law. However, the objective of this research was not to build the more complex and faithful heart model ever. Instead, we want to adapt the complexity of the model but can show the sensitivity of the change in cross-sectional area, pressure gradient and length of blood vessel on the blood flow rate and on the blood pressure.

Although a large numbers of assumptions have been considered, the model can be treated as a valid one, because, this model is able to show that the blood flow rate is influenced by the change of cross-sectional area and the pressure gradient. The model can also show that the blood pressure is influenced by both of the cross-sectional area and the length of the blood vessel.

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