Antidiffusion techniques to refine the numerical solution of the advection equation

Case study: Smolarkiewicz' iterative approach

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Outline

1. The advection equation in climate modelling
   - Importance
   - Diffusion
   - Antidiffusion methods
2. Case study: Smolarkiewicz
   - Analyzing the scheme
   - Implementation
   - Numerical Results
   - Progress

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Advection

Transport mechanism of a **substance** by a *fluid*, due to the fluids motion in a particular direction.

Examples in ocean, atmosphere and climate modelling

- Transport of **trace gasses** by *air* due to wind
- Transport of **heat** by *ocean water* due to currents
- Transport of **warm and moist air** over a colder surface by *air* due to wind: advection fog

Advection equation

Continuity equation describing the advection of a nondiffusive quantity in a flow field:

$$\frac{\partial \psi}{\partial t} + \operatorname{div}(V\psi) = 0 \tag{1}$$

where $\psi(x, y, z, t)$ is the nondiffusive scalar quantity, $V = (u, v, w)$ is the velocity vector and x, y, z, t are space and time independent variables.

Advection simulation

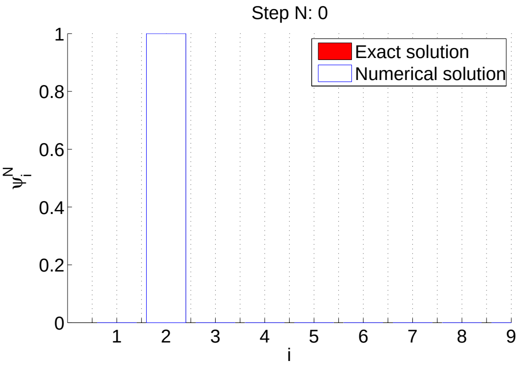
Constraints on advection simulation [?]

- Solutions should contain no unphysical overshoot or undershoot: positive definite schemes
- Methods should be volume preserving. No loss of matter
- The solutions should be local: the solution at any one point should not be influenced by what is going on far away from that point
- The numerical solution should not introduce new extrema in the solution, because the continuous form of the solution would not
- The method should be cost effective: memory and computational requirements should be sufficiently small so that practical problems may be solved

Problem: diffusion

- When simulating an advective process, we need to discretize space. When doing so, we introduce numerical constraints on the calculation and we get diffusion.
- The numerical solution spreads out.
- We get interactions, our simulations become less reliable.
- $\Longrightarrow$ example

Example of diffusion with an upstream scheme



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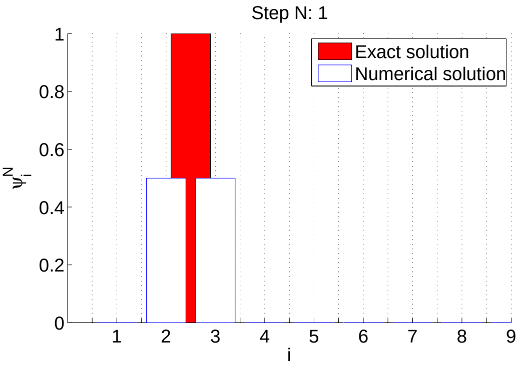

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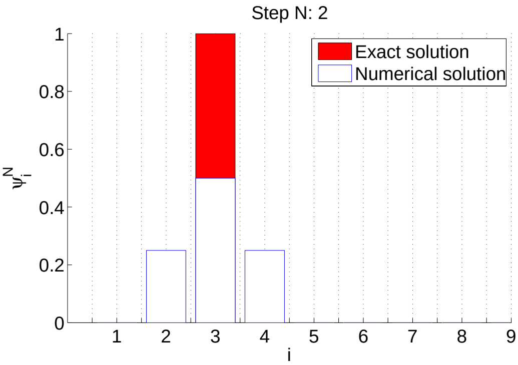

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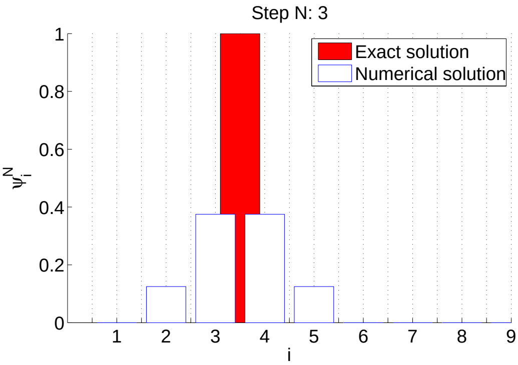

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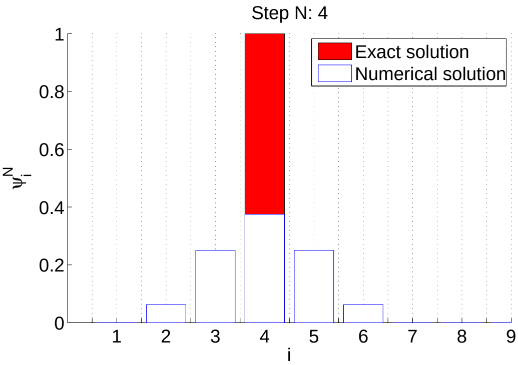

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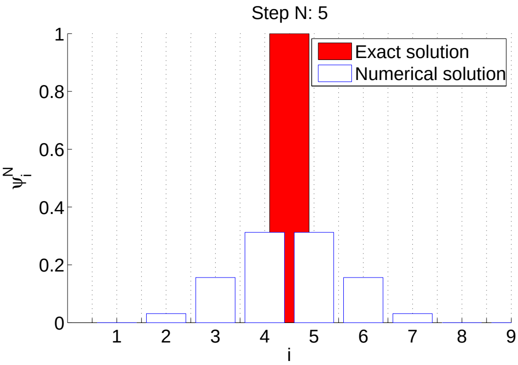

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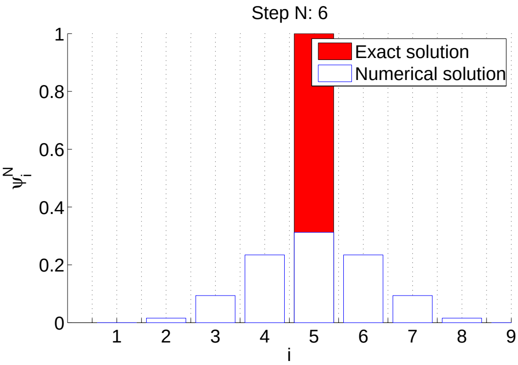

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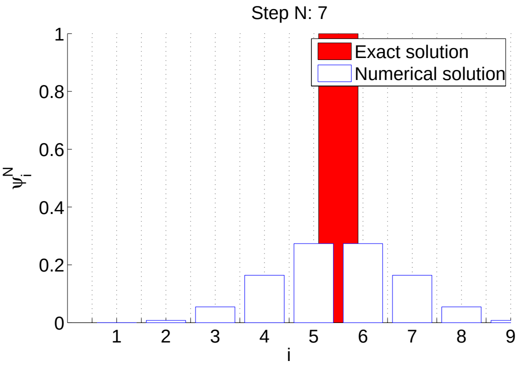

Antidiffusion methods

- Flux-corrected transport (FCT) method (Boris and Book, 1973),
- self-adjusting hybrid scheme (SAHS) (Harten en Zwas, 1972),

Both can be very accurate but require excessive computing time, better is the

- hybrid-type scheme based on a Crowley advection scheme (Clark and Hall, 1979),

with more diffusion but half the computation time.

Antidiffusion methods

Smolarkiewicz' iterative correction, 1983

- Less time consuming while results are comparable to those of the more complex hybrid schemes
- Positive semidefinite
- Iterative method
- Does not contain strong implicit diffusion

Basis: upstream on a staggered grid

We start with the following upstream advection equation on staggered grid:

$$\psi_i^{n+1} = \psi_i^n - \left(F\left(\psi_i^n, \psi_{i+1}^n, u_{i+1/2}^n\right) - F\left(\psi_{i-1}^n, \psi_i^n, u_{i-1/2}^n\right) \right),$$

where

$$F\left(\psi_i^n, \psi_{i+1}^n, u_{i+1/2}^n\right) =$$
$$\left(\left(u_{i+1/2}^n + \left|u_{i+1/2}^n\right|\right) \psi_i^n + \left(u_{i+1/2}^n - \left|u_{i+1/2}^n\right|\right) \psi_{i+1}^n \right) \frac{\Delta t}{2\Delta x}.$$

Method of lines 1

Writing the scheme out and collecting terms gives

$$\begin{aligned}\psi_i^{n+1} &= \frac{\Delta t}{2\Delta x}\left(u_{i-1/2}^n + \left|u_{i-1/2}^n\right|\right)\psi_{i-1}^n \\ &+ \left(1 - \frac{\Delta t}{2\Delta x}\left(u_{i+1/2}^n + \left|u_{i+1/2}^n\right| - u_{i-1/2}^n + \left|u_{i-1/2}^n\right|\right)\right)\psi_i^n \\ &- \frac{\Delta t}{2\Delta x}\left(u_{i+1/2}^n - \left|u_{i+1/2}^n\right|\right)\psi_{i+1}^n\end{aligned}$$

Method of lines 2

This can be rewritten to

$$\psi_i^{n+1} = \alpha_i \psi_{i-1}^n + \beta_i \psi_i^n + \gamma_i \psi_{i+1}^n, \quad \text{for } i = 1, \ldots, M-1,$$

where we have that

$$\alpha_i = \frac{\Delta t}{2\Delta x}\left(u_{i-1/2}^n + \left|u_{i-1/2}^n\right|\right),$$

$$\beta_i = \left(1 - \frac{\Delta t}{2\Delta x}\left(u_{i+1/2}^n + \left|u_{i+1/2}^n\right| - u_{i-1/2}^n + \left|u_{i-1/2}^n\right|\right)\right),$$

$$\gamma_i = -\frac{\Delta t}{2\Delta x}\left(u_{i+1/2}^n - \left|u_{i+1/2}^n\right|\right).$$

Method of lines 3

We can also write this in matrix form using Direchlet boundary conditions

- $y(0) = 0$ and $y(M) = 0$
- using periodic boundary conditions $y(0) = y(M)$

$$\begin{bmatrix} \psi_1^{n+1} \\ \psi_2^{n+1} \\ \vdots \\ \psi_{M-2}^{n+1} \\ \psi_{M-1}^{n+1} \end{bmatrix} = \begin{bmatrix} \beta_1 & \gamma_1 & & & 0 \\ \alpha_2 & \beta_2 & \gamma_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \alpha_{M-2} & \beta_{M-2} & \gamma_{M-2} \\ 0 & & & \alpha_{M-1} & \beta_{M-1} \end{bmatrix} \begin{bmatrix} \psi_1^{n} \\ \psi_2^{n} \\ \vdots \\ \psi_{M-2}^{n} \\ \psi_{M-1}^{n} \end{bmatrix}$$

So the values of ψ at timestep $N+1$ can be obtained by a sparse matrix-vector multiplication

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So the values of ψ at timestep $N+1$ can be obtained by a sparse matrix-vector multiplication

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Antidiffusion 1

To apply antidiffusion we need to redefine the scheme into

$$\psi_i^* = \psi_i^n - \left(F\left(\psi_i^n, \psi_{i+1}^n, u_{i+1/2}^n\right) - F\left(\psi_{i-1}^n, \psi_i^n, u_{i-1/2}^n\right) \right),$$

$$\psi_i^{n+1} = \psi_i^* - \left(F\left(\psi_i^*, \psi_{i+1}^*, \tilde{u}_{i+1/2}^n\right) - F\left(\psi_*^n, \psi_*^n, \tilde{u}_{i-1/2}^n\right) \right),$$

were the antidiffusion velocity $\tilde{u}_{i+1/2}$ is defined as

$$\tilde{u}_{i+1/2} = \frac{\left(\left|u_{i+1/2}\right| \Delta x - \Delta t u_{i+1/2}^2\right)\left(\psi_{i+1}^* - \psi_i^*\right)}{\left(\psi_i^* + \psi_{i+1}^* + \epsilon\right) \Delta x}$$

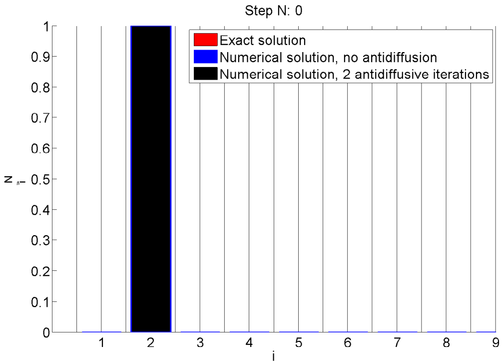
Since the second step is similar to the first we can also apply the 'method of lines' to the second step.

Algorithm

input : Initial state ψ^0, velocities u, iter, tsteps, Δx, Δt
output: Final state ψ^N

$\psi \leftarrow \psi^0$;
mat1 $\leftarrow$ `ComputeMatrix(`$u, \Delta x, \Delta t$`)`;
for $i \leftarrow 0$ **to** tsteps **do**
  $\psi \leftarrow$ `MatrixMultiplication(`mat1,ψ`)`;
  for $j \leftarrow 1$ **to** iter **do**
    $\widetilde{u} \leftarrow$ `ComputeAntidiffusionVelocity(`ψ, u`)`;
    mat2 $\leftarrow$ `ComputeMatrix(`$\widetilde{u}, \Delta x, \Delta t$`)`;
    $\psi \leftarrow$ `MatrixMultiplication(`mat2,ψ`)`;
  end
end
$\psi^N \leftarrow \psi$;

Example of antidiffusion with an upstream scheme



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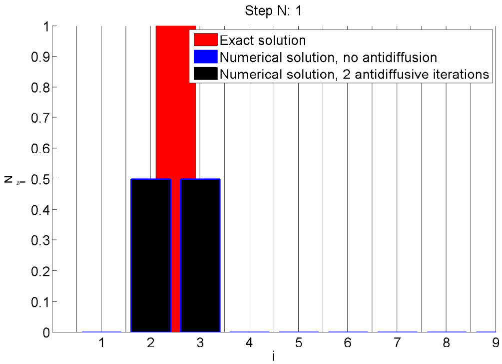

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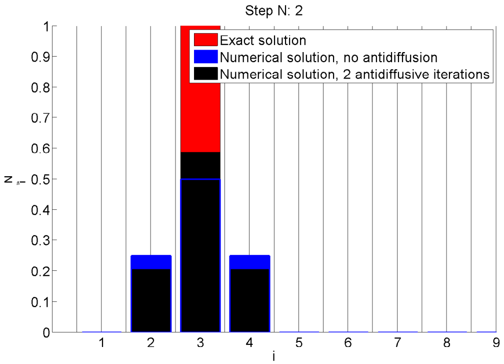

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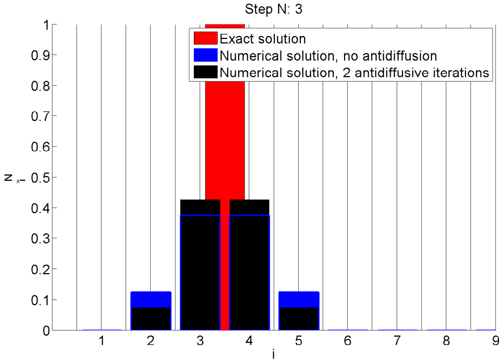

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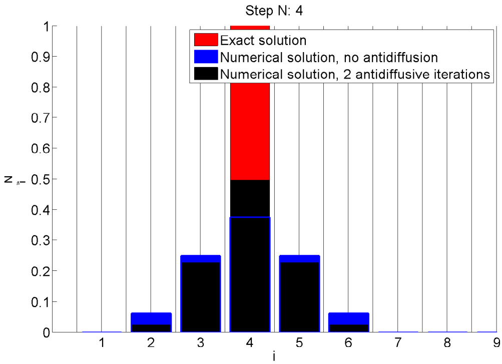

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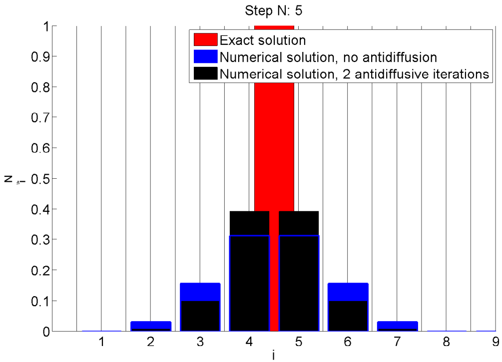

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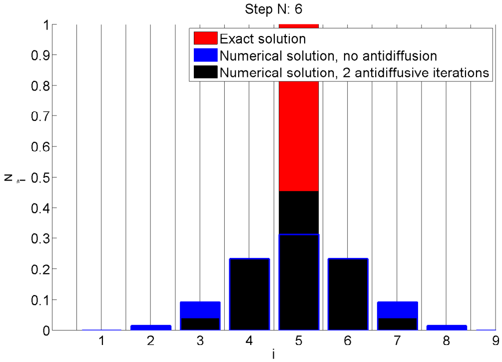

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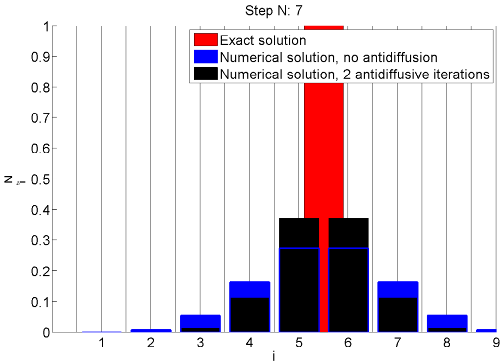
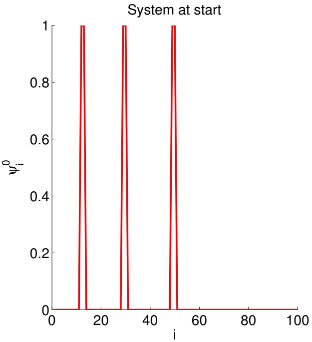
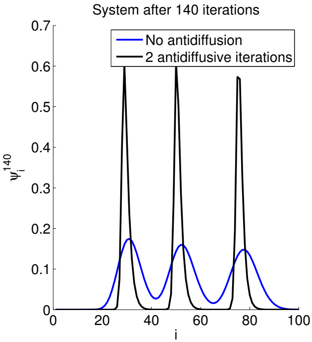


Less interaction

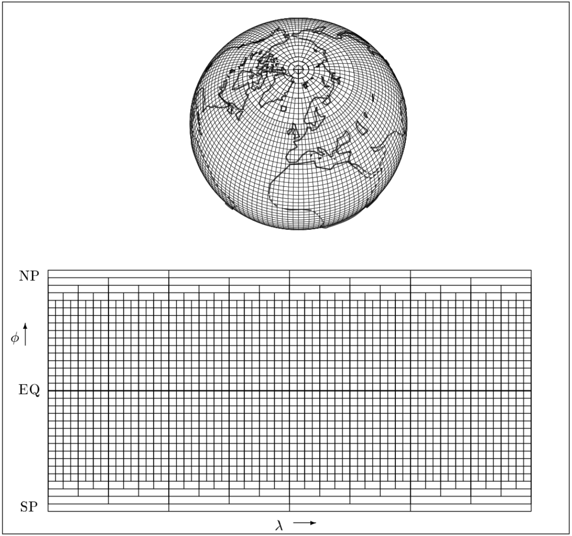
Simulations are more localized, less interaction between neighbouring pockets.





- Since 1983 a lot of progress has been made
- Advection calculations on the sphere pose new problems.

Reduced grid (Spee, 1991)



Spherical hexagonal-pentagonal grid (Lipscomb and Ringler, 2005)