Smolarkiewicz 1983

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One dimensional case

Upstream advection equation on staggered grid:

$$\psi_i^{N+1} = \psi_i^N - \Big(F\left(\psi_i^N, \psi_{i+1}^N, u_{i+1/2}^N\right) - F\left(\psi_{i-1}^N, \psi_i^N, u_{i-1/2}^N\right)\Big), \tag{1}$$

where

$$F\left(\psi_i^N, \psi_{i+1}^N, u_{i+1/2}^N\right) = \Big(\left(u_{i+1/2}^N + \left|u_{i+1/2}^N\right|\right)\psi_i^N + \left(u_{i+1/2}^N - \left|u_{i+1/2}^N\right|\right)\psi_{i+1}^N\Big)\frac{\Delta t}{2\Delta x}. \tag{2}$$

This gives

$$\begin{aligned}\psi_i^{N+1} = \psi_i^N - \frac{\Delta t}{2\Delta x}\Bigg(&\Big(\left(u_{i+1/2}^N + \left|u_{i+1/2}^N\right|\right)\psi_i^N + \left(u_{i+1/2}^N - \left|u_{i+1/2}^N\right|\right)\psi_{i+1}^N\Big) \\ &- \Big(\left(u_{i-1/2}^N + \left|u_{i-1/2}^N\right|\right)\psi_{i-1}^N + \left(u_{i-1/2}^N - \left|u_{i-1/2}^N\right|\right)\psi_i^N\Big)\Bigg).\end{aligned} \tag{3}$$

Collecting terms gives the method of lines representation

$$\begin{aligned}\psi_i^{N+1} = &\frac{\Delta t}{2\Delta x}\left(u_{i-1/2}^N + \left|u_{i-1/2}^N\right|\right)\psi_{i-1}^N \\ &+ \left(1 - \frac{\Delta t}{2\Delta x}\left(u_{i+1/2}^N + \left|u_{i+1/2}^N\right| - u_{i-1/2}^N + \left|u_{i-1/2}^N\right|\right)\right)\psi_i^N \\ &- \frac{\Delta t}{2\Delta x}\left(u_{i+1/2}^N - \left|u_{i+1/2}^N\right|\right)\psi_{i+1}^N\end{aligned} \tag{4}$$

Two dimensional case

Upstream advection equation on staggered grid:

$$\psi_{ij}^{N+1} = \psi_{ij}^N - \Big(F\left(\psi_{ij}^N, \psi_{i+1,j}^N, u_{i+1/2,j}^N\right) - F\left(\psi_{i-1,j}^N, \psi_{ij}^N, u_{i-1/2,j}^N\right) + F\left(\psi_{ij}^N, \psi_{i,j+1}^N, v_{i,j+1/2}^N\right) - F\left(\psi_{i,j-1}^N, \psi_{ij}^N, v_{i,j-1/2}^N\right) \Big), \quad (5)$$

where

$$F\left(\psi_{ij}^N, \psi_{i+1,j}^N, u_{i+1/2,j}^N\right) = \Big(\left(u_{i+1/2,j}^N + \left|u_{i+1/2,j}^N\right|\right) \psi_{ij}^N + \left(u_{i+1/2,j}^N - \left|u_{i+1/2,j}^N\right|\right) \psi_{i+1,j}^N \Big) \frac{\Delta t}{2\Delta x} \quad (6)$$

and

$$F\left(\psi_{ij}^N, \psi_{i,j+1}^N, v_{i,j+1/2}^N\right) = \Big(\left(v_{i,j+1/2}^N + \left|v_{i,j+1/2}^N\right|\right) \psi_{ij}^N + \left(v_{i,j+1/2}^N - \left|v_{i,j+1/2}^N\right|\right) \psi_{i,j+1}^N \Big) \frac{\Delta t}{2\Delta y}. \quad (7)$$

This gives

$$\begin{aligned}\psi_{ij}^{N+1} = \psi_{ij}^N - \Big(&\Big(\left(u_{i+1/2,j}^N + \left|u_{i+1/2,j}^N\right|\right) \psi_{ij}^N + \left(u_{i+1/2,j}^N - \left|u_{i+1/2,j}^N\right|\right) \psi_{i+1,j}^N \Big) \frac{\Delta t}{2\Delta x} \\ &- \Big(\left(u_{i-1/2,j}^N + \left|u_{i-1/2,j}^N\right|\right) \psi_{i-1,j}^N + \left(u_{i-1/2,j}^N - \left|u_{i-1/2,j}^N\right|\right) \psi_{ij}^N \Big) \frac{\Delta t}{2\Delta x} \\ &+ \Big(\left(v_{i,j+1/2}^N + \left|v_{i,j+1/2}^N\right|\right) \psi_{ij}^N + \left(v_{i,j+1/2}^N - \left|v_{i,j+1/2}^N\right|\right) \psi_{i,j+1}^N \Big) \frac{\Delta t}{2\Delta y} \\ &- \Big(\left(v_{i,j-1/2}^N + \left|v_{i,j-1/2}^N\right|\right) \psi_{i,j-1}^N + \left(v_{i,j-1/2}^N - \left|v_{i,j-1/2}^N\right|\right) \psi_{ij}^N \Big) \frac{\Delta t}{2\Delta y} \Big), \quad (8)\end{aligned}$$

Collecting terms gives the method of lines representation

$$
\begin{aligned}
\psi_{ij}^{N+1} = & \frac{\Delta t}{2\Delta y}\left(v_{i,j-1/2}^N + \left|v_{i,j-1/2}^N\right|\right)\psi_{i,j-1}^N \\
& + \frac{\Delta t}{2\Delta x}\left(u_{i-1/2,j}^N + \left|u_{i-1/2,j}^N\right|\right)\psi_{i-1,j}^N \\
& + \Big(1 - u_{i+1/2,j}^N - \left|u_{i+1/2,j}^N\right| + u_{i-1/2,j}^N - \left|u_{i-1/2,j}^N\right| - v_{i,j+1/2}^N - \left|v_{i,j+1/2}^N\right| \\
& + v_{i,j-1/2}^N - \left|v_{i,j-1/2}^N\right|\Big)\psi_i^N \\
& - \frac{\Delta t}{2\Delta x}\left(u_{i+1/2,j}^N - \left|u_{i+1/2,j}^N\right|\right)\psi_{i+1,j}^N \\
& - \frac{\Delta t}{2\Delta y}\left(v_{i,j+1/2}^N - \left|v_{i,j+1/2}^N\right|\right)\psi_{i,j+1}^N
\end{aligned}
\tag{9}
$$