Antidiffusion techniques to refine the numerical solution of the advection equation

Case study: Smolarkiewicz' iterative approach

Gerhard Burger Jelmer Wolterink

Scientific Computing
Department of Mathematics
Utrecht University

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Outline

Advection

Transport mechanism of a **substance** by a *fluid*, due to the fluids motion in a particular direction.

Examples in ocean, atmosphere and climate modelling

- Transport of **trace gasses** by *air* due to wind
- Transport of **heat** by *ocean water* due to currents
- Transport of **warm and moist air** over a colder surface by *air* due to wind: advection fog

Advection equation

Continuity equation describing the advection of a nondiffusive quantity in a flow field:

$$\frac{\partial \psi}{\partial t} + \text{div}(V\psi) = 0 \tag{1}$$

where $\psi(x, y, z, t)$ is the nondiffusive scalar quantity, $V = (u, v, w)$ is the velocity vector and x, y, z, t are space and time independent variables.

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Advection simulation

Constraints on advection simulation [**?**]

- Solutions should contain no unphysical overshoot or undershoot: positive definite
- Methods should be volume preserving. No loss of matter
- The solutions should be local: the solution at any one point should not be influenced by what is going on far away from that point
- The numerical solution should not introduce new extrema in the solution, because the continuous form of the solution would not
- The method should be cost effective: memory and computational requirements should be sufficiently small so that practical problems may be solved

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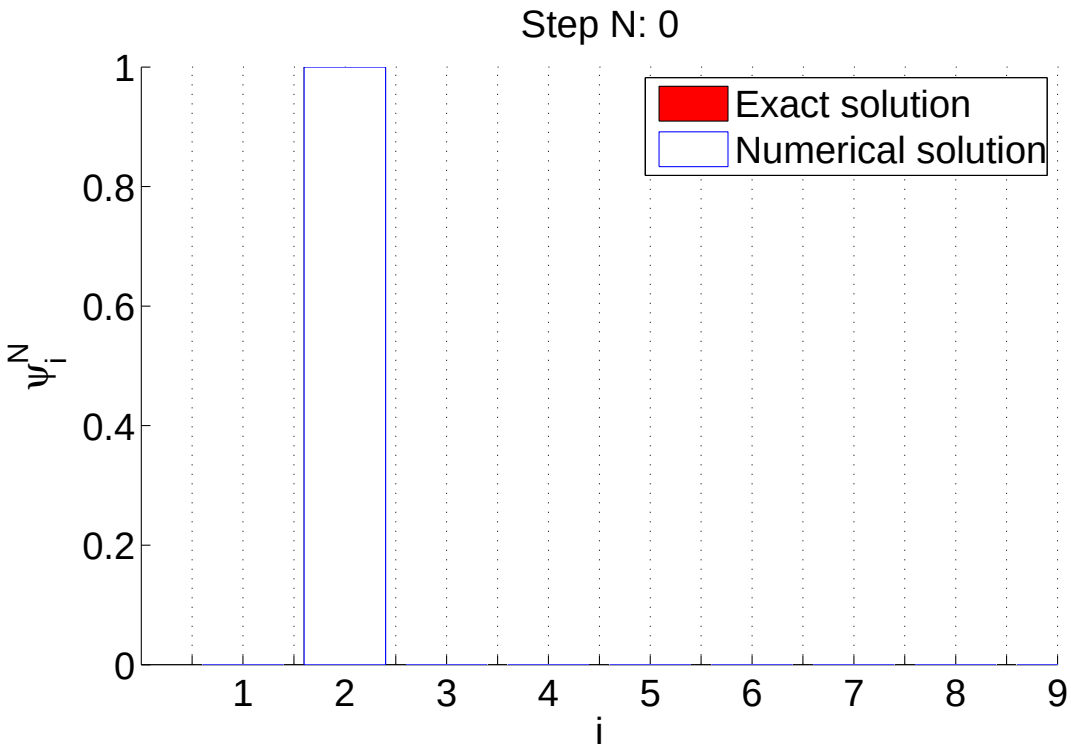
Problem: diffusion

- When simulating an advective process, we need to discretize space. When doing so, we introduce numerical constraints on the calculation and we get diffusion.
- The numerical solution spreads out.
- We get interactions, our simulations become less reliable.
- $\Longrightarrow$ example

Example of diffusion with an upstream scheme



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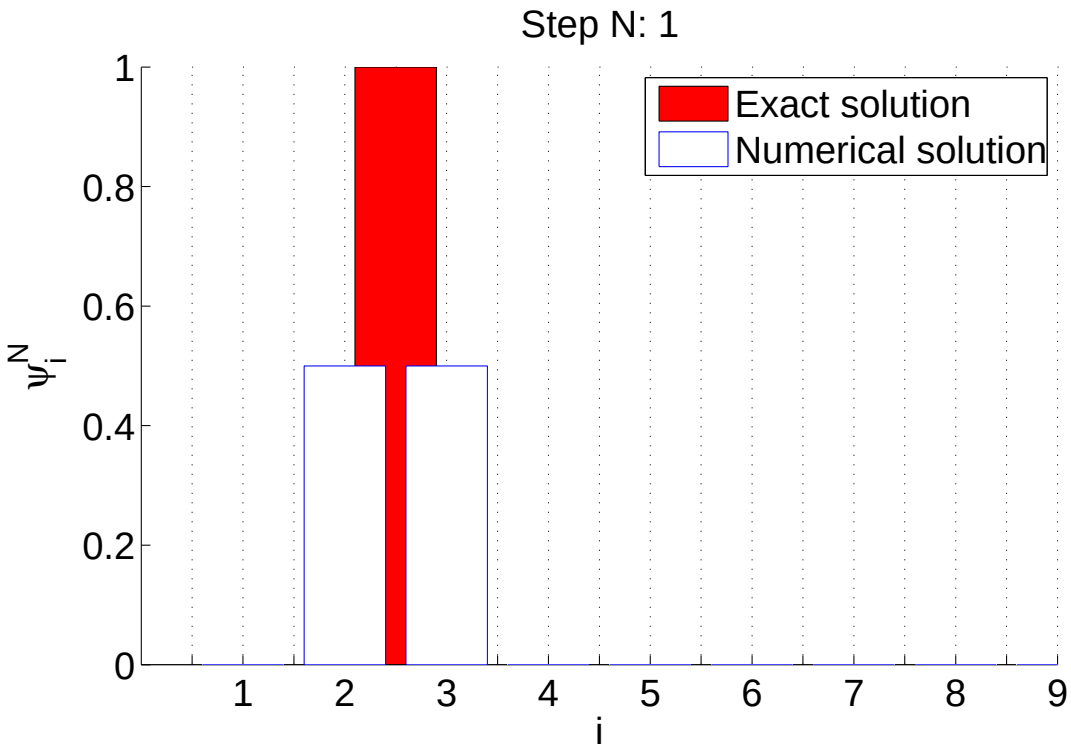

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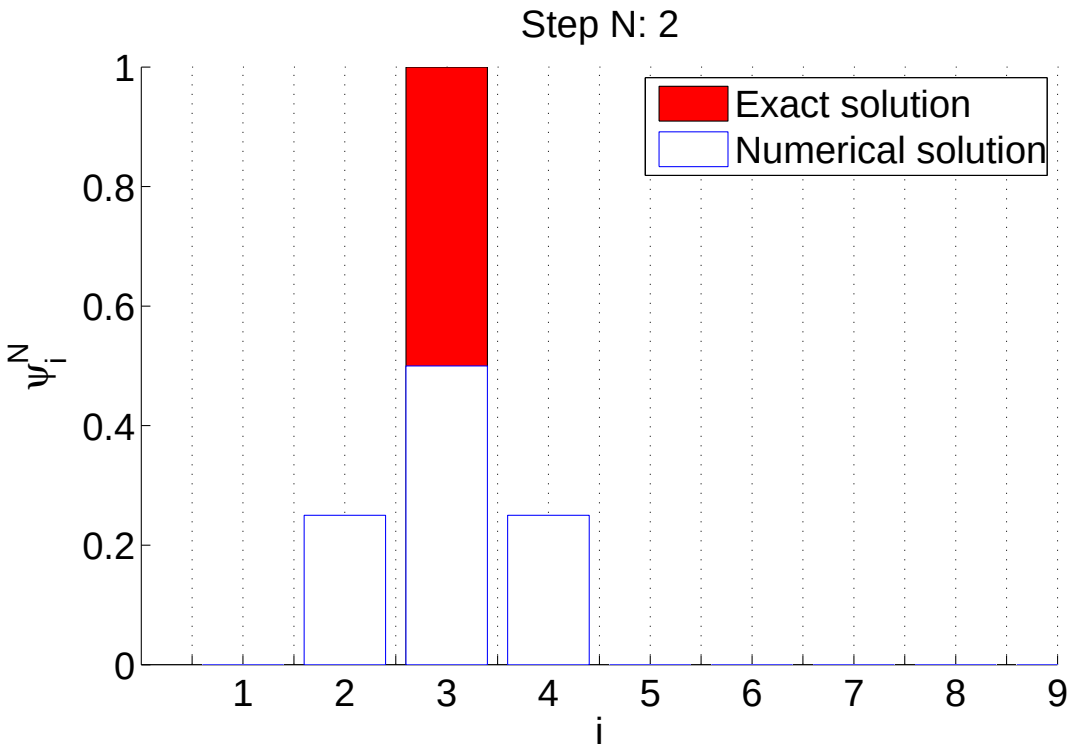

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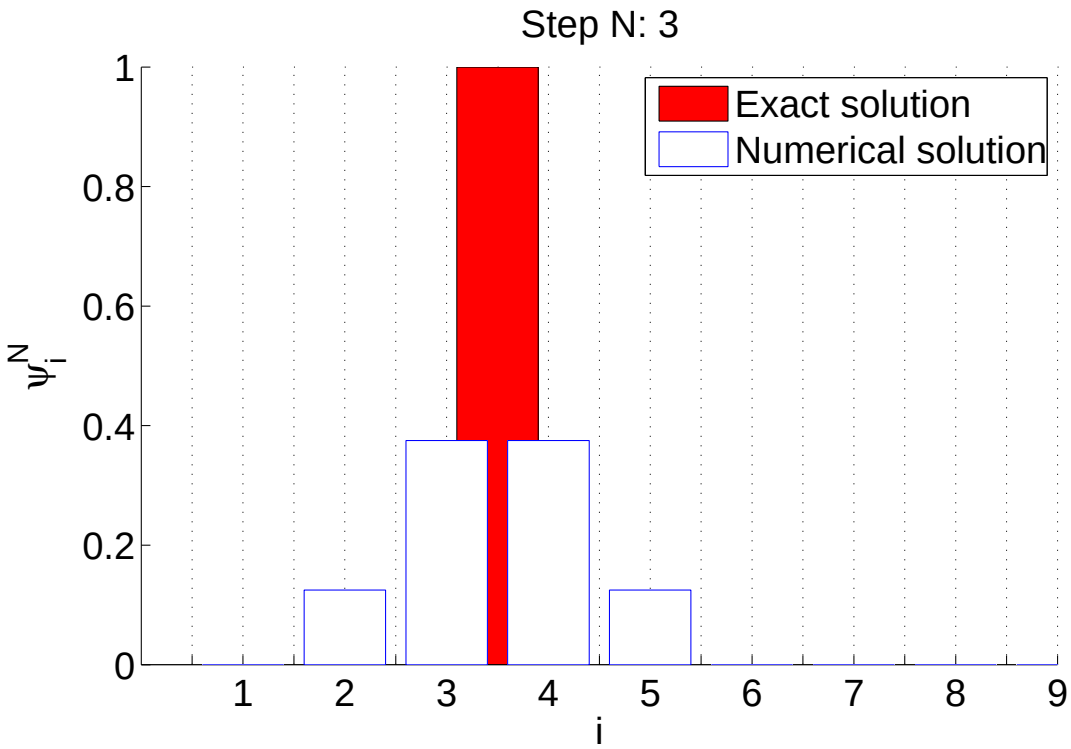

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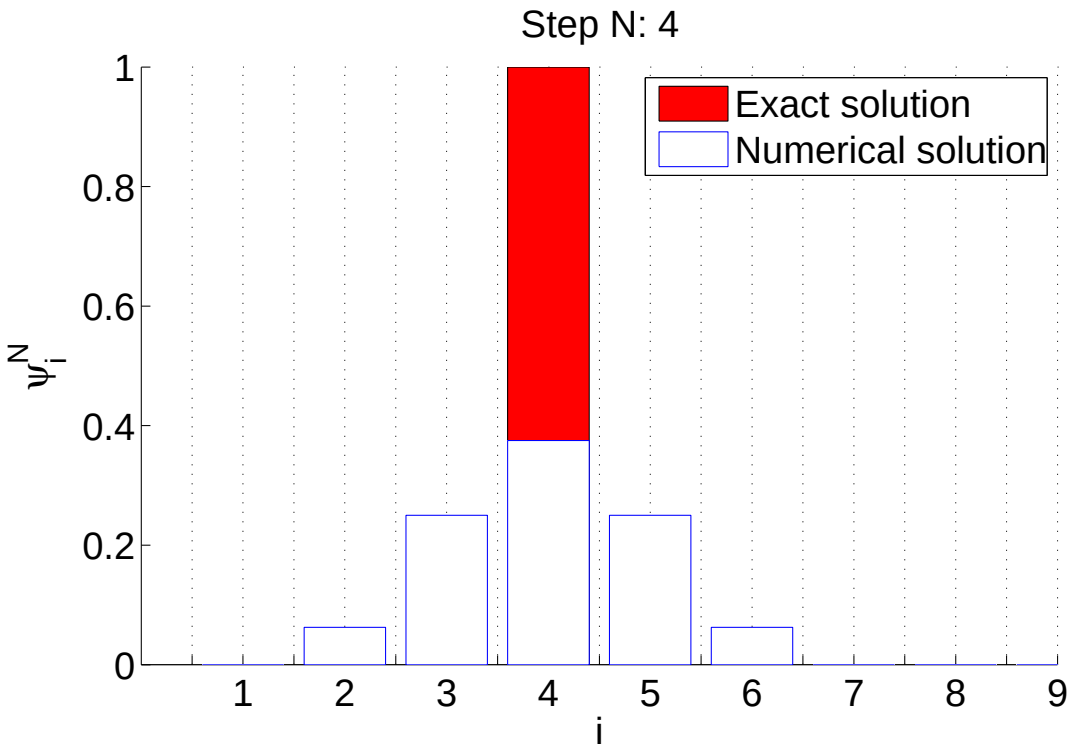

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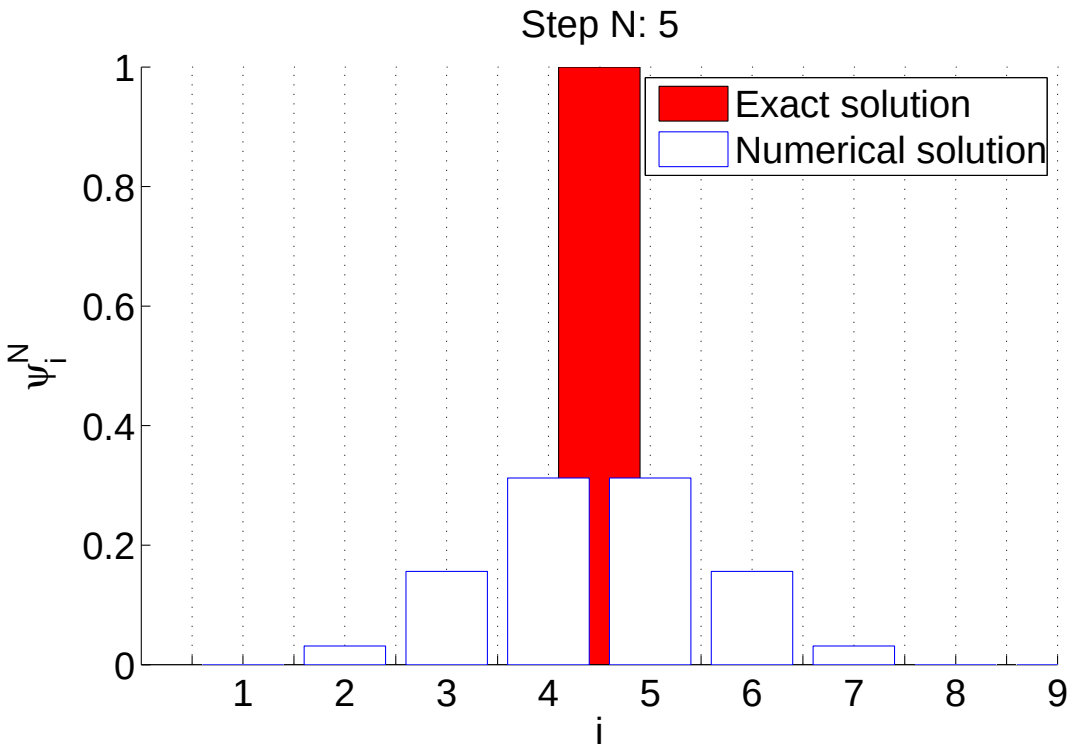

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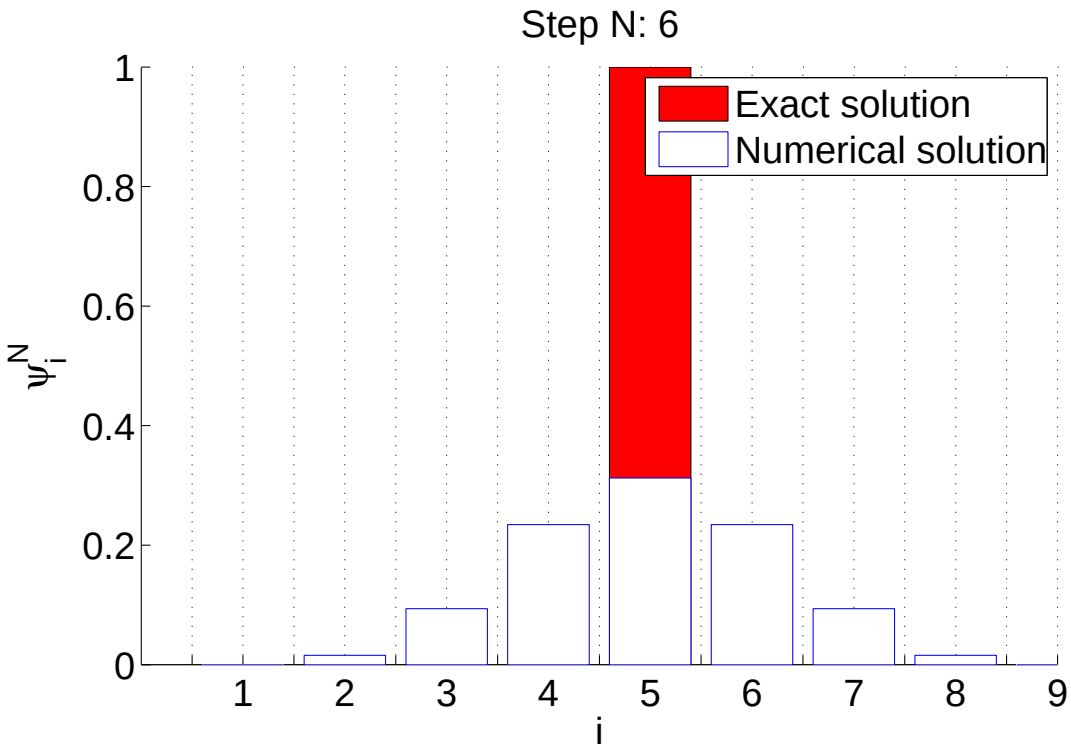

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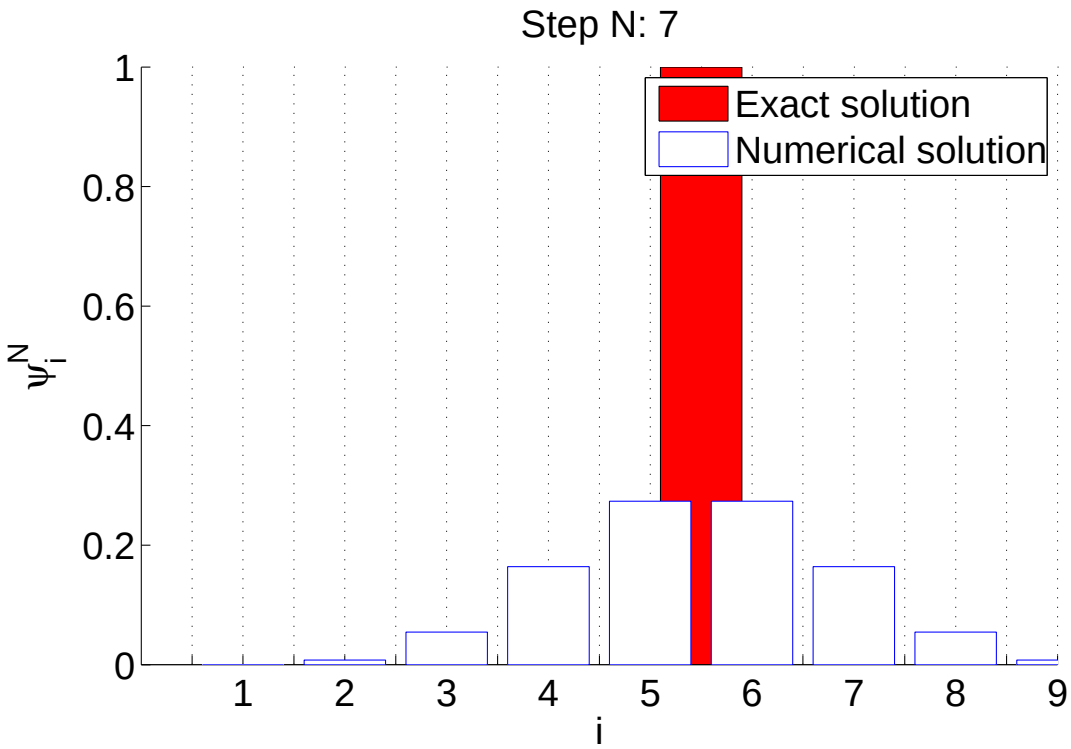

Antidiffusion methods 1

- Flux-corrected transport (FCT) method (Boris and Book, 1973),
- self-adjusting hybrid scheme (SAHS) (Harten en Zwas, 1972),

both can be very accurate but require excessive computing time, better is the

- hybrid-type scheme based on a Crowley advection scheme (Clark and Hall, 1979),

with more diffusion but half the computation time.

Antidiffusion methods 2

- Smolarkiewicz' iterative correction

less time consuming while results are comparable to those of the more complex hybrid schemes

Basis: upstream on a staggered grid

We start with the following upstream advection equation on staggered grid:

$$\psi_i^{N+1} = \psi_i^N - \left(F\left(\psi_i^N, \psi_{i+1}^N, u_{i+1/2}^N\right) - F\left(\psi_{i-1}^N, \psi_i^N, u_{i-1/2}^N\right) \right),$$

where

$$F\left(\psi_i^N, \psi_{i+1}^N, u_{i+1/2}^N\right) = \left(\left(u_{i+1/2}^N + \left| u_{i+1/2}^N \right| \right) \psi_i^N + \left(u_{i+1/2}^N - \left| u_{i+1/2}^N \right| \right) \psi_{i+1}^N \right) \frac{\Delta t}{2\Delta x}.$$

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Method of lines 1

Writing the scheme out and collecting terms gives

$$\begin{aligned}\psi_i^{N+1} = &\frac{\Delta t}{2\Delta x}\left(u_{i-1/2}^N + \left|u_{i-1/2}^N\right|\right)\psi_{i-1}^N \\ &+\left(1 - \frac{\Delta t}{2\Delta x}\left(u_{i+1/2}^N + \left|u_{i+1/2}^N\right| - u_{i-1/2}^N + \left|u_{i-1/2}^N\right|\right)\right)\psi_i^N \\ &-\frac{\Delta t}{2\Delta x}\left(u_{i+1/2}^N - \left|u_{i+1/2}^N\right|\right)\psi_{i+1}^N\end{aligned}$$

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Method of lines 2

This can be rewritten to

$$\psi_i^{N+1} = \alpha_i \psi_{i-1}^N + \beta_i \psi_i^N + \gamma_i \psi_{i+1}^N, \quad \text{for } i = 1, \ldots, M-1,$$

where we have that

$$\alpha_i = \frac{\Delta t}{2\Delta x}\left(u_{i-1/2}^N + \left|u_{i-1/2}^N\right|\right),$$

$$\beta_i = \left(1 - \frac{\Delta t}{2\Delta x}\left(u_{i+1/2}^N + \left|u_{i+1/2}^N\right| - u_{i-1/2}^N + \left|u_{i-1/2}^N\right|\right)\right),$$

$$\gamma_i = -\frac{\Delta t}{2\Delta x}\left(u_{i+1/2}^N - \left|u_{i+1/2}^N\right|\right).$$

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Method of lines 3

We can also write this in matrix form using Direchlet boundary conditions

- $y(0) = 0$ and $y(M) = 0$
- using periodic boundary conditions $y(0) = y(M)$

$$\begin{bmatrix} \psi_1^{N+1} \\ \psi_2^{N+1} \\ \vdots \\ \psi_{M-2}^{N+1} \\ \psi_{M-1}^{N+1} \end{bmatrix} = \begin{bmatrix} \beta_1 & \gamma_1 & & & & 0 \\ \alpha_2 & \beta_2 & \gamma_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \alpha_{M-2} & \beta_{M-2} & \gamma_{M-2} & \\ 0 & & & \alpha_{M-1} & \beta_{M-1} & \end{bmatrix} \begin{bmatrix} \psi_1^{N} \\ \psi_2^{N} \\ \vdots \\ \psi_{M-2}^{N} \\ \psi_{M-1}^{N} \end{bmatrix}$$

So the new values of ψ can be obtained by a sparse matrix-vector multiplication

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Antidiffusion 1

To apply antidiffusion we need to redefine the scheme into

$$\psi_i^* = \psi_i^N - \left(F\left(\psi_i^N, \psi_{i+1}^N, u_{i+1/2}^N\right) - F\left(\psi_{i-1}^N, \psi_i^N, u_{i-1/2}^N\right) \right),$$
$$\psi_i^{N+1} = \psi_i^* - \left(F\left(\psi_i^*, \psi_{i+1}^*, \tilde{u}_{i+1/2}^N\right) - F\left(\psi_*^N, \psi_*^N, \tilde{u}_{i-1/2}^N\right) \right),$$

were the antidiffusion velocity $\tilde{u}_{i+1/2}$ is defined as

$$\tilde{u}_{i+1/2} = \frac{\left(\left| u_{i+1/2} \right| \Delta x - \Delta t u_{i+1/2}^2 \right) \left(\psi_{i+1}^* - \psi_i^* \right)}{\left(\psi_i^* + \psi_{i+1}^* + \epsilon \right) \Delta x}$$

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Antidiffusion 2

The Method of lines can be applied in the same way to the second step in the scheme,
so the whole scheme can be executed with two sparse matrix-vector multiplications and a matrix update.

Algorithm

input : Initial configuration ψ^0, velocities u, number of iterations iter
output: Final configuration ψ^N

$\psi \leftarrow \psi^0$;
mat1 $\leftarrow$ ComputeMatrix(u);
for $i \leftarrow 0$ **to** iter **do**
    $\psi \leftarrow$ MatrixMultiplication(mat1,ψ);
    for $j \leftarrow 1$ **to** iter **do**
        $\widetilde{u} \leftarrow$ ComputeAntidiffusionVelocity(ψ, u);
        mat2 $\leftarrow$ ComputeMatrix($\widetilde{u}$);
        $\psi \leftarrow$ MatrixMultiplication(mat2,ψ);
    end
end


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