Deriving the error metric caused by the rotational error

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Many error metrics defined for the registration, such as RMSE, is the summation of element-wise displacement error and does not yield any useful information for the regions not included in the registration process. The proposed error metric addresses this issue and establishes an upper bound for the displacement error caused by the rotational error.

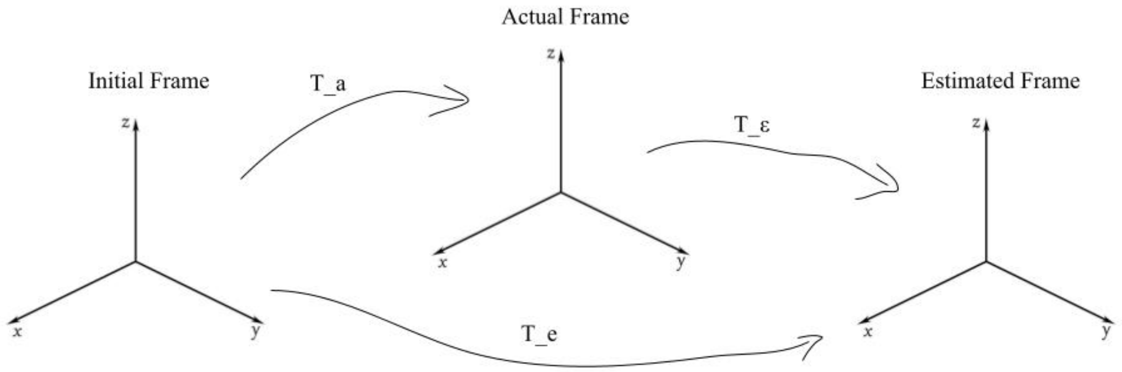

Figure 1: The relation between initial, actual and estimated frames

The registration process can be modeled as in Figure 1 and the following equation expresses the relation

$$T_e = T_\varepsilon T_a, \tag{1}$$

where T_e is the estimated transformation the algorithm yields, T_a is the actual transformation that maps the initial frame onto actual frame and T_ε is the transformation caused by the registration error. Rigid body transformations are one-to-one mapping of two Cartesian spaces and thus always invertible so, the transformation error can be obtained as the following

$$T_\varepsilon = T_e T_a^{-1}. \tag{2}$$

Let $\boldsymbol{R_\varepsilon}$ be the rotational error and $\boldsymbol{P_\varepsilon}$ be the positional error satisfying

$$T_\varepsilon = \begin{bmatrix} \boldsymbol{R}_\varepsilon & \boldsymbol{P}_\varepsilon \\ \boldsymbol{0} & 1 \end{bmatrix}. \tag{3}$$

$\boldsymbol{R_\varepsilon}$ is usually expressed in terms of Euler angles consisting of 3 consecutive rotations along the main axes, which would yield 3 angular error terms. Instead, if one convert it to an axis-angle representation, it will yield the axis of rotation $\boldsymbol{V}$ and a single angular term θ_ε

$$\boldsymbol{R_\varepsilon} = [\alpha_\varepsilon, \beta_\varepsilon, \gamma_\varepsilon] = [\boldsymbol{V}, \theta_\varepsilon]. \tag{4}$$

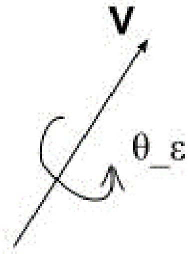

Figure 2: Any rotation can be represented by an axis of rotation and a rotation angle

For any vector $\boldsymbol{L}$, the displacement error will not occur if it is in the same direction as $\boldsymbol{V}$ and maximum error will occur if it is perpendicular as in Figure 3.

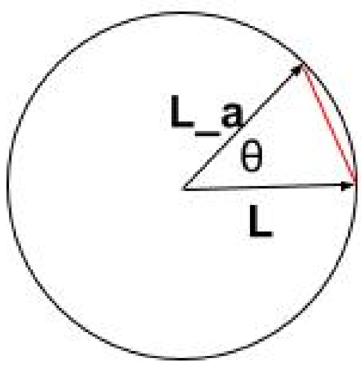

Figure 3: Any intentional vector **L** *will arrive at* $\boldsymbol{L_a}$ *(top view)*

The maximum displacement error caused by the rotation will be approximately $L\theta_\varepsilon$. Thus, the displacement error d_ε will obey the following inequality

$$d_\varepsilon \leq L\theta_\varepsilon + \left||\boldsymbol{P_\varepsilon}|\right|. \tag{5}$$

The error is at its maximum when it is in the same direction as $\boldsymbol{P_\varepsilon}$. For a standardized measure, length L is taken as 10cm, which is a common length for a biopsy needle. Thus, the direction dependent maximum displacement error in 10cm (MDE10) can be the upper bound for d_ε as follows

$$d_\varepsilon \leq (10cm)\theta_\varepsilon + \left||\boldsymbol{P_\varepsilon}|\right| = MDE10. \tag{6}$$

It guarantees that any needle, length of 10cm, inserted into a certain point will end up within the sphere whose center is the intended point and radius is MDE10.

It should be noted that this error metric is expected to work well only in a limited region since it takes its origin as where the registration is performed. As needle insertion occurs in a farther place, the error will increase proportionally to the distance due to the angular term.