# Pseudodifferential Operators on $\mathbb{Q}_p$ and L-Series

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#### Abstract

We define a family of pseudodifferential operators on the Hilbert space  $L^2(\mathbb{Q}_p)$  of complex valued square-integrable functions on the p-adic number field  $\mathbb{Q}_p$ . The Riemann zeta-function and the related Dirichlet L-functions can be expressed as a trace of these operators on a subspace of  $L^2(\mathbb{Q}_p)$ . We also extend this to the L-functions associated with modular (cusp) forms. Wavelets on  $\mathbb{Q}_p$  are common sets of eigenfunctions of these operators.

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#### 1 Introduction

The Riemann zeta function [1] has a product representation in terms of prime numbers that allows one to connect to the p-adic number field  $\mathbb{Q}_p$ . This prompted us (in a larger collaboration) to construct a unitary matrix model (UMM) for the former by combining UMMs for the local factors corresponding to each prime [2]. The parameters that define the UMM for the Riemann zeta function is divergent, however, after a renormalisation, resultant UMM agrees with that constructed directly in Ref. [3]. Moreover, we show that the partition function of the UMM can be written as a trace of the (generalised) Vladimirov derivative [4] acting on a (subspace) of complex valued square-integrable functions on the p-adic number field  $\mathbb{Q}_p$ . The partition function is essentially the Riemann zeta function in this approach.

While integration of complex valued valued functions on  $\mathbb{Q}_p$  is quite straightforward, the totally disconnected topology of  $\mathbb{Q}_p$  does not allow for a naïve notion of a derivative. The Valdimirov derivative is, therefore, defined through an integral kernel. Moreover, Kozyrev [5] (see also [6–8]) constructed complex valued wavelets on  $\mathbb{Q}_p$  and demonstrated that these are eigenfunctions of the generalised Vladimirov derivative. These wavelets are rather like the complex valued generalised Haar wavelets on the real line  $\mathbb{R}$ . The eigenvalues are related to the scaling property of the wavelet. Enhancement of the manifest affine symmetry of the wavelets were studied in [9].

The zeta function of Riemann is but one member of an infinite family of Dirichlet L-functions, which are defined using arithmetic functions called Dirichlet characters. These are multiplicative characters, hence, all Dirichlet L-functions admit Euler product representations over prime numbers. It is, therefore, imperative that we generalise the notion of Vladimirov derivative further and seek pseudodifferential operators, the eigenvalues of which will involve the Dirichlet character.

Indeed, it turns out that one can do more. There are L-functions associated with modular forms, which we refer to as modular L-function for brevity. These functions too have both Dirichlet series and Euler product representations. We are able to define a

pair of pseudodifferential operators corresponding to each prime factor. We discuss the eigenvalues and eigenfunctions (which are the Kozyrev wavelets again) of these generalised pseudodifferential operators. The L-functions can be written in terms of the trace of these operators on a subspace of the Hilbert space spanned by the wavelets. We also comment on the realisation of the Hecke operators as traces of these operators after conjugation by raising-lowering operators.

#### 2 Dirichlet characters and Dirichlet L-functions

For a positive integer k, the Dirichlet character modulo k is a map  $\chi_k : \mathbb{Z} \to \mathbb{C}$  such that

- 1. For all  $m_1, m_2 \in \mathbb{Z}$ ,  $\chi_k(m_1 m_2) = \chi_k(m_1) \chi_k(m_2)$
- 2.  $\chi_k(m_1) = \chi_k(m_2)$  if  $m_1 \equiv m_2 \pmod{k}$
- 3.  $\chi_k(m) \neq 0$  if and only if m is relatively prime to k

There is a *trivial character* that assigns the value 1 to all integers (including 0). This corresponds to k = 1.

More precisely, the multiplicative group  $G(k) = (\mathbb{Z}/k\mathbb{Z})^*$  consisting of the *invertible* elements of  $\mathbb{Z}/k\mathbb{Z}$  is an abelian group. An element  $\chi_k \in \text{Hom}(G(k), \mathbb{C}^*)$  is called a character modulo k. It is a  $\mathbb{C}^*$ -valued function on the set of integers relatively prime to k, such that the property (i) above holds. It is convenient and often conventional to *extend* a character to all  $\mathbb{Z}$  by setting  $\chi_k(m) = 0$  for all m which are not coprime to k [10].

The periodicity k of the Dirichlet character does not specify the function completely. For any k there are  $\varphi(k)$  number<sup>1</sup> of inequivalent characters. There is a principal character  $\chi_{k,0}$  that assumes the value 1 for arguments coprime to k and vanishes otherwise. If all the values of a character is real, it is called a real character. Clearly, if k is a prime number, there are (k-1) inequivalent characters, the values of which are  $\varphi(k)$ -th roots of unity. Of these the character  $\chi_{k,0}(n)$  vanishes for all integers  $n \equiv 0 \pmod{k}$  and 1 otherwise. If k is not a prime, the character vanishes for all integers that share common (prime) factors with it.

The Dirichlet L-series corresponding to a Dirichlet character  $\chi$ , is the infinite series

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \tag{1}$$

which is convergent for Re(s) > 1. Its analytic continuation in the complex s-plane defines a meromorphic function called the Dirichlet L-function, also denoted by the same symbol. If  $\chi$  is chosen to be the trivial character, one gets the Riemann zeta function by

<sup>&</sup>lt;sup>1</sup>For any positive integer n, the Euler totient function  $\varphi(n)$  counts the number of positive integers 1 < m < n that are relatively prime to n.

the analytic continuation of

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in \text{primes}} \frac{1}{(1 - p^{-s})}$$
 (2)

where, the second expression is the Euler product form. Unlike the Riemann zeta function, the Dirichlet functions  $L(s,\chi)$  defined by the analytic continuation of Eq. (1) is an entire function, except when  $\chi$  is the principal character (or the trivial one), in which case there is a simple pole at s=1. Moreover, they have a set of trivial zeroes at negative even or odd integers (depending on whether  $\chi(-1)=\pm 1$ ) and a set of non-trivial zeroes which must lie symmetrically about the critical line  $\mathrm{Re}(s)=\frac{1}{2}$  in the critical strip  $0<\mathrm{Re}(s)<1$ . The assertion that the non-trivial zeroes are exactly on the critical line is the generalised Riemann hyposthesis.

Since the Dirichlet characters are multiplicative, the L-series Eq. (1) can be written as an infinite product over prime numbers

$$L(s,\chi) = \prod_{p \in \text{primes}} \frac{1}{(1 - \chi(p)p^{-s})}$$
 (3)

in the region of its convergence. We shall be interested in studying the *local* factor  $(1 - \chi(p)p^{-s})^{-1}$  at the prime p, which is the sum of an infinite geometric series.

Thanks to the multiplicative nature of a Dirichlet character,  $\chi_k(m^n) = (\chi_k(m))^n$  for  $n \geq 0$ . In other words, Dirichlet characters of a positive integer power of an integer is well defined. However, since negative powers of an integer is not an integer,  $\chi_k(m^{-n})$  is not defined. In order to define the pseudodifferential operators we are interested in, however, we shall need to *extend* the notion of a Dirichlet character to negative integer powers as well. Therefore, let us *define* the function  $\mathfrak{x}_k(m^n)$  as

$$\mathfrak{x}_{k}(m^{n}) = \begin{cases}
(\chi_{k}(m))^{n} & \text{if } \chi_{k}(m) \neq 0 \text{ and } n \geq 0 \\
(\chi_{k}(m))^{n} = \chi_{k}^{*}(m^{-n}) & \text{if } \chi_{k}(m) \neq 0 \text{ and } n < 0 \\
0 & \text{if } \chi_{k}(m) = 0 \text{ for all } n \neq 0
\end{cases}$$
(4)

Thus, if the Dirichlet character of an integer is zero, we take all its integer powers (positive or negative) to be zero as well. In this paper we shall refer to  $\mathfrak{x}_k$  as an extended Dirichlet character. In fact, for our purposes, it will be sufficient to define the extended character for integer powers of prime numbers<sup>2</sup> only. That is because, we need this extension not on all p-adic numbers, but only on their norms<sup>3</sup>. Consequently, if k contains p (of  $\mathbb{Q}_p$ ) as one of its factors, the extended character vanishes for all  $\xi \in \mathbb{Q}_p$ . On the other hand, if k

<sup>&</sup>lt;sup>2</sup>The extension we need is really quite minimal. The original Dirichlet character is a map  $\chi: \mathbb{Z} \to \mathbb{C}$ . This of course makes sense as a map  $\chi|_{(p)}: p^{\mathbb{N}} \to \mathbb{C}$ . We need to extend this restricted form to  $\mathfrak{x}: p^{\mathbb{Z}} \to \mathbb{C}$ . Moreover, only the non-zero values of the characters are precisely defined, the rest was by extension. Therefore, our definition should also be thought of as an extension in the same spirit.

<sup>&</sup>lt;sup>3</sup>It should also be noted that the extended character we need is really a combination of the 'norm function' on  $\mathbb{Q}_p$  ( $|\cdot|_p:\mathbb{Q}_p\to\mathbb{R}$ ) and an extended character on rational numbers. It may be possible to define an extended character on  $\mathbb{Q}_p$  itself, however, the present construction serves our purpose.

and p are relatively prime, its contribution is a constant phase on a subset of  $\mathbb{Q}_p$  with a fixed norm.

## 3 Twisted p-adic Gelfand-Graev-Tate Gamma functions

With the help of the extended Dirichlet character  $\mathfrak{x}$  in Eq. (4), let us define the Gelfand-Graev-Tate gamma function  $\Gamma_{\mathfrak{x}}$ , twisted by  $\mathfrak{x}_k$  as

$$\Gamma_{\mathfrak{x}}(s) = \int_{\mathbb{Q}_{p}^{\times}} \frac{d\xi}{|\xi|_{p}} e^{2\pi i \xi} |\xi|_{p}^{s} \mathfrak{x}_{k}(|\xi|_{p}^{-1}) = \int_{\mathbb{Q}_{p}} d\xi \, e^{2\pi i \xi} |\xi|_{p}^{s-1} \mathfrak{x}_{k}(|\xi|_{p}^{-1}), \quad p \nmid k$$
 (5)

If p divides k, then the function vanishes identically. The definitions above as well as in Eq. (31) to appear below are consistent with the definition of a generalised gamma function with a multiplicative character [11] (similar functions have been used in [12,13]). We shall now evaluate this integral.

Recall the 'Laurent expansion' of a p-adic number  $\xi = p^n (\xi_0 + \xi_1 p + \xi_2 p^2 + \cdots), p \in \mathbb{Z}, \xi_m \in \{0, 1, \dots, p-1\}$  but  $\xi_0 \neq 0$ . As usual, we split the integral into three parts corresponding to the regions

(i)  $|\xi|_p < 1$ . Hence  $e^{2\pi i \xi} = 1$ . The region can be further divided into 'circles'  $C_n = \{\xi : |\xi|_p = p^{-n}\}, n = 1, 2, \cdots$ , the measure of which is  $\frac{p-1}{p}p^{-n}$ . Thus the contribution to the integral from this region is

$$\frac{p-1}{p} \sum_{m=1}^{\infty} p^{-n} p^{-n(s-1)} \mathfrak{x}_k(p^n) = \frac{p-1}{p} \sum_{m=1}^{\infty} \left( \chi_k(p) p^{-s} \right)^n = \frac{p-1}{p} \frac{\chi_k(p)}{p^s - \chi_k(p)}$$

for non-zero  $\chi_k(p)$ .

- (ii)  $|\xi|_p = 1$ . Hence  $\mathfrak{x}_k(|\xi_p|) = \chi_k(|\xi_p|) = 1$  and  $e^{2\pi i \xi} = 1$ . The measure of this region is  $\frac{p-1}{p}$ . So the contribution to the integral is  $\frac{p-1}{p}$ .
- (iii)  $|\xi|_p > 1$ . It consists of 'circles'  $C_n = \{\xi : |\xi|_p = p^n\}$ ,  $n = 1, 2, \cdots$ . In the circle  $C_n$ ,  $\mathfrak{x}_k(|\xi|_p) = \mathfrak{x}_k(p^{-n}) = (\chi_k(p))^{-n}$  and the measure of  $C_n$  is  $\frac{p-1}{p}p^n$ . The circle  $C_n$  is divided into (p-1) subsets  $C_{n;1}, C_{n;2}, \cdots, C_{n;p-1}$  of equal measures. This, in turn is further divided into subsets such that

$$C_n = \bigcup_{m_0=1}^{p-1} C_{n;m_0} = \bigcup_{m_0,m_1=0}^{p-1} C_{n;m_0,m_1} = \dots = \bigcup_{m_0,\dots,m_{n-1}=0}^{p-1} C_{n;m_0,m_1,\dots,m_{n-1}=0}$$

In  $C_{n;m_0,m_1,\cdots,m_{n-1}}$ , the exponential  $e^{2\pi i\xi}=\omega_{p^n}^{\xi_0}\omega_{p^{n-1}}^{\xi_1}\cdots\omega_p^{\xi_{n-1}}$ . What is important is the last factor, which is a p-th roots of unity. When integrated, i.e., summed over all values  $m_{n-1}=0,1,\cdots,p-1$ , the sum over roots of units vanish (all subsets have equal measure). Therefore, the contribution to the integral from  $C_n$  is zero, except for n=1. In this case, the root 1 is missing, so the contribution to the integral is  $\frac{p-1}{p}\frac{p}{p-1}\left(\omega_p+\omega_p^2+\cdots+\omega_p^{p-1}\right)p^{s-1}\mathfrak{x}_k(p^{-1})=-\frac{p^{s-1}}{\chi_k(p)}$ , if  $\chi(p)\neq 0$ , and 0 otherwise.

Combining the three contributions

$$\Gamma_{\mathfrak{r}}(s) = \frac{p^{s} \left(\chi_{k}(p) - p^{s-1}\right)}{\chi_{k}(p) \left(p^{s} - \chi_{k}(p)\right)} = \frac{\chi_{k}(p) - p^{s-1}}{\chi_{k}(p) \left(1 - \chi_{k}(p)p^{-s}\right)} \tag{6}$$

For the trivial character this reduces to  $\Gamma(s) = \frac{1 - p^{s-1}}{1 - p^{-s}}$ , the standard Gelfand-Graev-Tate gamma function. In the general case, Eq. (5) further generalises the gamma function defined with multiplicative characters in, e.g, Ref. [12,13].

## 4 Vladimirov derivative twisted by a character

Let us recall that the generalised Vladimirov derivative, defined as an integral kernel

$$D^{\alpha}g(\xi) = \frac{1}{\Gamma(-\alpha)} \int d\xi' \frac{g(\xi') - g(\xi)}{|\xi' - \xi|_p^{\alpha+1}}$$

$$\tag{7}$$

is defined for any  $\alpha \in \mathbb{C}$  by analytic continuation. The eigenfunctions of this pseudo-differential operator

$$D^{\alpha}\psi_{n,m,j}^{(p)}(\xi) = p^{\alpha(1-n)}\psi_{n,m,j}^{(p)}(\xi) \tag{8}$$

with eigenvalue  $p^{\alpha(1-n)}$  are the Kozyrev wavelets

$$\psi_{n,m,j}^{(p)}(\xi) = p^{-\frac{n}{2}} e^{2\pi i j p^{n-1} \xi} \Omega^{(p)}(p^n \xi - m), \quad \xi \in \mathbb{Q}_p$$
 (9)

for  $n \in \mathbb{Z}$ ,  $m \in \mathbb{Q}_p/\mathbb{Z}_p$  and  $j \in \{1, 2, 3, \dots, p-1\}$ . These are Bruhat-Schwartz functions on  $\mathbb{Q}_p$ , consisting of the indicator function<sup>4</sup>  $\Omega^{(p)}(\xi)$  and an additive character, the exponential function.

As expected the eigenvalues depend only on the quantum number related to scaling n (and not on those related to translation and phase). Since we are primarily interested in the eigenvalues, we may as well restrict our attention to the set of eigenfunctions  $\psi_{n,0,1}^{(p)}(\xi) = p^{-\frac{n}{2}}e^{2\pi i p^{n-1}\xi}\Omega^{(p)}(p^n\xi)$ , which we shall often denote simply as  $|1-n\rangle$  labelled by the eigenvalues. It is natural to define the raising-lowering operators  $a_{\pm}$ 

$$a_{\pm}\psi_{n,0,1}(\xi) = \psi_{n\pm1,0,1}(\xi), \qquad a_{\pm}|n\rangle = |n\mp1\rangle$$
 (10)

that changes the scaling quantum number by one. These operators, together with  $\log_p D = \lim_{\alpha \to 0} \frac{D^{\alpha} - 1}{\alpha \ln p}$ , (formally) generate a large symmetry of the wavelets [9].

For now it suffices to note that

$$D^{\alpha} |n\rangle = p^{\alpha n} |n\rangle \tag{11}$$

which allows us to write the local factor  $(1-p^{-s})^{-1}$  in the Euler product form of the

<sup>&</sup>lt;sup>4</sup>The indicator function  $\Omega^{(p)}(\xi) = 1$  for  $|\xi|_p \leq 1$ , and 0 otherwise.

Riemann zeta function Eq. (2) as

$$\zeta_p(s) = \frac{1}{(1 - p^{-s})} = \sum_{m=0}^{\infty} p^{-sm} = \sum_{m=0}^{\infty} \langle m | D^{-s} | m \rangle = \text{Tr}_{\mathcal{H}_-} D^{-s}$$
(12)

i.e., a trace of the generalised Vladimirov derivative  $D^{-s}$  over a subspace  $\mathcal{H}_{-}$  spanned by the wavelets  $\{|m\rangle \sim \psi_{1-m,0,1}: m=1,2,\cdots\}$ . In other words, the subspace  $\mathcal{H}_{-}$  is spanned by wavelets on  $\mathbb{Z}_p$  with compact support.

Let us now define the generalised Vladimirov derivative, twisted by the character  $\mathfrak{x}$ , as

$$D_{\mathfrak{x}}^{\alpha}g(\xi) = \frac{1}{\Gamma_{\mathfrak{x}}(-\alpha)} \int_{\mathbb{Q}_p} d\xi' \frac{g(\xi') - g(\xi)}{|\xi' - \xi|_p^{\alpha + 1}} \mathfrak{x}_k(|\xi' - \xi|_p^{-1})$$
(13)

Once again, this operator is meaningful if  $p \nmid k$ . If, on the other hand, p divides k, the twisted character vanishes identically. consequently the numerator on the RHS is zero. However, the twisted gamma function  $\Gamma_{\mathfrak{p}}$  in the denominator also vanishes. In this case, we define  $D_{\mathfrak{p}} = 1$  if  $(p, k) \neq 1$ .

We shall now show that the Kozyrev wavelets Eq. (9) are also eigenfunctions of these operators. Actually due to the translation, scaling and the phase rotation properties, it is sufficient to establish that  $f(\xi) = \psi_{0,0,1} = e^{2\pi i \xi/p} \Omega^{(p)}(\xi)$  is an eigenfunction. The rest follows trivially. The proof is in fact very similar to the case of the generalised Vladimirov derivative with trivial character, so we shall be brief. We consider two cases:

- Case (i) The set  $|\xi|_p > 1$  is not in the support of  $\Omega^{(p)}(\xi)$ , therefore,  $f(\xi) = 0$ . Hence, the contribution to the integral is only from the set  $|\xi'|_p < 0$ . By the triangle inequality  $\mathfrak{x}_k(|\xi'-\xi|_p^{-1}) = \mathfrak{x}_k(|\xi|_p^{-1})$  and  $|\xi'-\xi|_p^{\alpha+1} = |\xi|_p^{\alpha+1}$  does not have any dependence on the variable of integration. Thus the integral, which is reduced to  $\int_{|\xi'|_p < 1} e^{2\pi i \xi'/p} d\xi'$ , is zero.
- Case (ii) Only those  $\xi$  for which  $|\xi|_p \leq 1$  need to be considered. The function  $f(\xi)$  is supported on this set, hence,  $\Omega^{(p)}(|\xi|_p) = 1$ , therefore, the numerator of the integrand involves  $e^{2\pi i \xi'/p} \Omega^{(p)}(\xi') e^{2\pi i \xi/p} = e^{2\pi i \xi/p} \left(e^{2\pi i (\xi'-\xi)/p} \Omega^{(p)}(\xi') 1\right)$ . The prefactor does not depend on the variable of integration. Due to the fact that every point in a p-adic disc could be considered to be its centre, we may write  $\Omega^{(p)}(\xi') = \Omega^{(p)}(\xi'-\xi)$  and change the variable of integration to  $z = \xi' \xi$  by a translation. Now we can split the contribution to the integral from the three sets  $|z|_p < 1$ ,  $|z|_p = 1$  and  $|z|_p > 1$ , of which the first one vanishes (since  $e^{2\pi i z/p} = 1$  hence the integrand is zero). From the other two sets, we get

$$D_{\mathfrak{x}}^{\alpha} f(\xi) = \frac{e^{2\pi i \xi/p}}{\Gamma_{\mathfrak{x}}(-\alpha)} \left[ \int_{|z|_{p}=1} \left( e^{2\pi i z/p} - 1 \right) dz - \int_{|z|_{p}>1} \frac{\mathfrak{x}_{k}(|z|_{p}^{-1})}{|z|_{p}^{\alpha+1}} dz \right]$$

$$= \frac{e^{2\pi i \xi/p}}{\Gamma_{\mathfrak{x}}(-\alpha)} \left[ \frac{1}{p} \left( \omega_{p} + \dots + \omega_{p}^{p-1} \right) - \frac{(p-1)}{p} - \sum_{n=1}^{\infty} \frac{\mathfrak{x}_{k}(p^{-n})}{p^{n(\alpha+1)}} \frac{p^{n}(p-1)}{p} \right]$$

$$= -\frac{e^{2\pi i\xi/p} \Omega^{(p)}(\xi)}{\Gamma_{\mathfrak{r}}(-\alpha)} \left(1 + \frac{p-1}{p} \frac{1}{p^{\alpha} \chi_{k}(p) - 1}\right)$$
$$= p^{\alpha} \chi_{k}(p) f(\xi) \tag{14}$$

where we have used the fact that  $\Omega^{(p)}(\xi) = 1$ .

This proves that  $f(\xi) = \psi_{0,0,1}(\xi) = e^{2\pi i \xi/p} \Omega^{(p)}(\xi)$  is an eigenfunction corresponding to the eigenvalue  $\chi_k(p)p^{\alpha}$ . Hence,

$$D_{\mathfrak{x}}^{\alpha}\psi_{n,m,j}^{(p)}(\xi) = (\chi_k(p)p^{\alpha})^{(1-n)}\psi_{n,m,j}^{(p)}(\xi)$$
or, 
$$D_{\mathfrak{x}}^{\alpha}|n\rangle = (\chi_k(p)p^{\alpha})^n|n\rangle$$
(15)

where the latter expression is in the notation of Eq. (11). We see that the family of pseudodifferential operators  $D_{\mathfrak{x}}$ , twisted by the multiplicative Dirichlet character  $\mathfrak{x}$ , all commute with each other, and their common eigenfunctions, with eigenvalues  $(\chi_k(p)p^{\alpha})^{(1-n)}$ , are the Kozyrev wavelets on  $\mathbb{Q}_p$ .

Repeating the arguments in Eq. (12) mutatis mutandis, we can write each local factor in the Dirichlet L-function as

$$L_p(s,\chi) = \frac{1}{(1-\chi(p)p^{-s})} = \sum_{m=0}^{\infty} (\chi(p)p^{-s})^m = \sum_{m=0}^{\infty} \langle m | D_{\mathfrak{x}}^{-s} | m \rangle = \text{Tr}_{\mathcal{H}_-} (D_{\mathfrak{x}}^{-s})$$
 (16)

i.e., a trace over a subspace  $\mathcal{H}_- = L^2(\mathbb{Z}_p)$  of  $L^2(\mathbb{Q}_p)$  spanned by the Kozyrev wavelets.

# 5 L-functions of modular forms and pseudodifferential operators

A more general class of L-functions are those associated with a modular form f [10,14–16]. The group  $SL(2,\mathbb{R})$  has a natural action on the upper half plane  $\mathbb{H} = \{z : Im(z) > 0\}$ . Consider its discrete subgroup<sup>5</sup>  $SL(2,\mathbb{Z})$  and its following congruence subgroups of finite indices:

$$\Gamma_0(N) = \{ \gamma \in \operatorname{SL}(2, \mathbb{Z}) \mid c \equiv 0 \bmod N \}$$

$$\Gamma_1(N) = \{ \gamma \in \operatorname{SL}(2, \mathbb{Z}) \mid a, d \equiv 1 \bmod c \equiv 0 \bmod N \}$$

$$\Gamma(N) = \{ \gamma \in \operatorname{SL}(2, \mathbb{Z}) \mid a, d \equiv 1 \bmod b, c \equiv 0 \bmod N \}$$

$$(17)$$

The conditions are empty for N=1, hence  $\Gamma(1)=\mathrm{SL}(2,\mathbb{Z})$  is the full modular group. Clearly  $\Gamma(N)\subset\Gamma_1(N)\subset\Gamma_0(N)\subset\Gamma(1)$ , moreover  $\Gamma(N)$ , the principal congruence subgroup of level N, is the kernel of the homomorphism  $\mathrm{SL}(2,\mathbb{Z})\to\mathrm{SL}(2,\mathbb{Z}/N\mathbb{Z})$ . Let  $\Gamma$  be a discrete subgroup  $\Gamma(N)\subset\Gamma\subset\Gamma(1)$  such that N is the smallest such integer. A fundamental domain is the closure of  $\mathbb{H}/\Gamma$ , e.g.,  $\mathbb{H}/\Gamma(1)=\left\{z\in\mathbb{H}\mid -\frac{1}{2}<\mathrm{Re}(z)<\frac{1}{2},|z|>1\right\}$ .

<sup>&</sup>lt;sup>5</sup>More precisely, the relevant groups are the projective special linear groups  $PSL(2,\mathbb{R}) = SL(2,\mathbb{R}/\{\pm\})$  and  $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\{\pm\}$ .

A modular form  $f: \mathbb{H} \to \mathbb{C}$  of weight k and level N is a holomorphic form on the upper half plane  $\mathbb{H}$  that transforms as

$$f\left(\frac{az+b}{cz+d}\right) = \chi_N(d)(cz+d)^k f(z), \quad \text{where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$
 (18)

under the action of a discrete subgroup  $\Gamma(N) \subset \Gamma \subset \Gamma(1)$ .

A modular form of  $SL(2,\mathbb{Z})$  (i.e., level 1) admits a Fourier expansion (q-expansion) in terms of  $q = e^{2\pi iz}$  as

$$f = \sum_{n=0}^{\infty} a(n)q^n = \sum_{n=0}^{\infty} a(n)e^{2\pi i nz}$$
 (19)

It is called a cusp form if f vanishes as  $\operatorname{Im}(z) \to i\infty$ , or equivalently at q = 0. This requires a(0) = 0. It is conventional, and convenient for many purposes, to normalise such that a(1) = 1. We shall assume this in what follows. These forms are related to scaling functions on  $\mathbb{C}/\Lambda$  where  $\Lambda$  is a lattice left invariant by the action of the modular group. The set of modular forms of weight k form a finite dimensional complex vector space  $M_k(\Gamma(1))$  and the set of cusp forms of weight k is a subspace  $S_k(\Gamma(1))$ . Similar notions exist for modular forms of subgroups  $\Gamma \subset \Gamma(1)$  of level N. The cusp forms of a more general modular group  $\Gamma$  vanish as z approaches certain rational points on  $\mathbb{R} = \partial \mathbb{H}$ . In the fundamental domain these are images of  $\operatorname{Im}(z) \to i\infty$  under  $\Gamma(1)/\Gamma(N)$ .

The Dirichlet series associated to a cusp form f is

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = 1 + \frac{a(2)}{2^s} + \frac{a(3)}{3^s} + \frac{a(4)}{4^s} + \cdots$$
 (20)

where  $s \in \mathbb{C}$  and the normalisation is such that a(1) = 1. The series converges uniformly to a holomorphic function of s in the region  $\text{Re}(s) > \sigma$  as long as the coefficients a(n) are bounded by some power  $n^{\sigma}$ . The corresponding L-function associated to the cusp form f is then defined by analytic continuation to the complex s-plane. If f is a cusp form of weight k, then the series above converges in  $\text{Re}(s) > 1 + \frac{k}{2}$ . The series can also be expressed as

$$L(s,f) = \mathcal{M}[f](s) = \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty dy \, y^{s-1} f(iy)$$
 (21)

i.e., as a Mellin transform of f(iy).

Remarkably, the L-function of a cusp form f (or modular L-function for brevity) Eq. (20) has an Euler product form

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \prod_{p \in \text{primes}} \frac{1}{(1 - a(p)p^{-s} + \chi(p)p^{k-1}p^{-2s})}$$
(22)

The coefficients a(n) have very interesting properties, to which we shall return in a moment. As an example, consider one of the well known modular L-functions related to the discriminant function  $\Delta(z) = \left(\sqrt{2\pi}\,\eta(z)\right)^{24} = (2\pi)^{12}\,q\prod_{n=1}^{\infty}(1-q^n)$ , where  $\eta(z)$  is the

Dedekind  $\eta$ -function. It is a holomorphic modular form of weight 12 (and level 1) that vanishes as  $z \to i\infty$ , i.e., it is a cusp form. The coefficients in its q-expansion

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n)q^n \tag{23}$$

define the function  $\tau: \mathbb{N} \to \mathbb{Z}$ , known as the Ramanujan  $\tau$ -function. They satisfy the following properties

$$\tau(mn) = \tau(m)\tau(n) \text{ if gcd } (m,n) = 1$$

$$\tau(p^{m+1}) = \tau(p)\tau(p^m) - p^{11}\tau(p^{m-1}) \text{ for } p \text{ a prime and } m > 0$$

$$|\tau(p)| \le 2p^{11/2}$$
(24)

as were conjectured by Ramanujan. The first two were proved by Mordell soon after the conjecture, while the bound was proved by Deligne after many years. More generally, the coefficients a(n) of a modular form of weight k and level N satisfy

$$a(mn) = a(m)a(n)$$
 if gcd  $(m, n) = 1$   
 $a(p^{m+1}) = a(p) a(p^m) - \chi(p)p^{k-1}a(p^{m-1})$  for  $p$  a prime and  $m > 0$  (25)

as follows from comparing the series and product formulas in Eq. (22). Therefore, the coefficients a(n) define a multiplicative character [10] on  $\mathbb{N}$ . The convergence of the series also puts a bound on the growth of the coefficients.

Let us consider a prime factor in Eq. (22) for a fixed p, and call it a *local* modular L-function at a prime p in analogy with the local zeta function Eq. (12) at p, and factorise it as follows

$$L_{(p)}(s,f) = \frac{1}{(1 - a(p)p^{-s} + \chi(p)p^{k-1}p^{-2s})} = \frac{1}{(1 - a_1(p)p^{-s})(1 - a_2(p)p^{-s})}$$
(26)

Evidently, consistency demands that the sum and the product of the coefficient functions of the factorised form gives

$$a_1(p) + a_2(p) = a(p)$$
 and  $a_1(p) a_2(p) = \chi(p) p^{k-1}$  (27)

Notice that  $a_1(p)$  and  $a_2(p)$  are only defined for the (fixed) prime p. For the positive powers of p we take  $a_i : p^{\mathbb{N}} \to \mathbb{C}$  by  $a_i(p^n) = (a_i(p))^n$ .

In the region of convergence, we can realise Eq. (26) as infinite geometric series

$$L_{(p)}(s,f) = \sum_{m=0}^{\infty} \sum_{\ell=0}^{m} (-1)^{\ell} {m \choose \ell} (a(p))^{m-\ell} (\chi(p)p^{k-1})^{\ell} p^{-(m+\ell)s}$$

$$= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} (a_1(p))^{m_1} (a_2(p))^{m_2} p^{-(m_1+m_2)s}$$
(28)

Hence, by comparing terms of powers of  $p^{-s}$ , we find

$$\sum_{n=0}^{m} (a_1(p))^{m-n} (a_2(p))^n = \sum_{\ell=0}^{\left\lfloor \frac{m}{2} \right\rfloor} (-1)^{\ell} {m-\ell \choose \ell} (a(p))^{m-2\ell} \left( \chi(p) p^{k-1} \right)^{\ell}$$
 (29)

which can be checked to be consistent with Eq. (27).

From here onwards, we can closely follow the steps in the extension of the Dirichlet character in Section 2, and define the functions  $\mathfrak{a}_1$  and  $\mathfrak{a}_2$  as follows:

For 
$$i = 1, 2$$
,  $\mathfrak{a}_i(p^n) = \begin{cases} (a_i(p))^n & \text{if } a_i(p) \neq 0 \text{ and } n \geq 0\\ (a_i(p))^n = \frac{1}{(a_i(p))^{|n|}} & \text{if } a_i(p) \neq 0 \text{ and } n < 0\\ 0 & \text{if } a_i(p) = 0 \end{cases}$  (30)

The extended functions are  $\mathfrak{a}_i:p^{\mathbb{Z}}\to\mathbb{C}$ . With the help of these functions, we define the gamma functions  $\Gamma_{\mathfrak{a}_1}(s,f)$  and  $\Gamma_{\mathfrak{a}_2}(s,f)$ , related to the modular function f as

$$\Gamma_{(\mathfrak{a}_{i},f)}(s) = \int_{\mathbb{Q}_{p}} d\xi \, e^{2\pi i \xi} |\xi|_{p}^{s-1} \, \mathfrak{a}_{i}(|\xi|_{p}^{-1}) \,, \quad \text{for } i = 1, 2$$

$$= \frac{a_{i}(p) - p^{s-1}}{a_{i}(p) \, (1 - a_{i}(p)p^{-s})}$$
(31)

for non-vanishing  $a_i(p)$ , otherwise for  $a_i(p) = 0$ , we take the gamma function to be zero. Finally, let us define a pair of generalised Vladimirov derivatives associated with the modular function f, as

$$D_{(\mathfrak{a}_{i},f)}^{\alpha}g(\xi) = \frac{1}{\Gamma_{(\mathfrak{a}_{i},f)}(-\alpha)} \int_{\mathbb{Q}_{p}} d\xi' \, \frac{g(\xi') - g(\xi)}{|\xi' - \xi|_{p}^{\alpha+1}} \, \mathfrak{a}_{i}(|\xi' - \xi|_{p}^{-1})$$
(32)

unless,  $\mathfrak{a}_i = 0$ , in which case we define  $D_{(\mathfrak{a}_i, f)} = 1$ . Repeating our previous arguments *mutatis mutandis*, it is straightforward to check that the Kozyrev wavelets are eigenfunctions of the derivative operators Eq. (32)

$$D_{(\mathfrak{a}_{i},f)}^{\alpha}\psi_{n,m,j}^{(p)}(\xi) = (a_{i}(p)p^{\alpha})^{(1-n)}\psi_{n,m,j}^{(p)}(\xi), \quad \text{for } i = 1, 2$$
(33)

with eigenvalues involving the factorised coefficients. In proving this, one needs to sum over an infinite geometric series which converges for  $Re(s) > 1 + \frac{k}{2}$ .

The equations above, together with Eq. (26) and Eq. (28), allow us to write the local factors of the modular L-function associated f as

$$L_{(p)}(s,f) = \left(\sum_{m_1=0}^{\infty} \left(a_1(p)p^{-s}\right)^{m_1}\right) \left(\sum_{m_2=0}^{\infty} \left(a_2(p)p^{-s}\right)^{m_2}\right)$$
$$= \sum_{m=0}^{\infty} \left(\sum_{m_1=0}^{m} \left(a_1(p)\right)^{m_1} \left(a_2(p)\right)^{m-m_1}\right) p^{-sm}$$

The trace is now over a subspace  $\mathcal{H}_{-} \otimes \mathcal{H}_{-}$  of the tensor product space  $L^{2}(\mathbb{Q}_{p}) \otimes L^{2}(\mathbb{Q}_{p})$ . The sum in the trace is over the wavelets with negative scaling quantum number, hence compactly supported in  $\mathbb{Z}_{p}^{2} \subset \mathbb{Q}_{p}^{2}$ . Therefore, the double sum above span the upper right quadrangle of the lattice  $\mathbb{Z}^{2}$  (see Fig. 1).

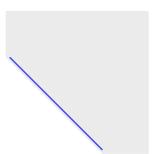


Figure 1: The sum in Eq.(34) is over the upper quadrangle of the lattice  $\mathbb{Z}^2$ , while the sum in Eq.(35) is over the region above and right of the blue line.

It is interesting to note that if we conjugate the operators Eq. (32) by the following combination of the raising/lowering operators in Eq. (10) and then take the trace<sup>6</sup>, we get

$$\operatorname{Tr}\left(\sum_{k=0}^{\ell} a_{+}^{k} D_{(\mathfrak{a}_{1},f)}^{-s} a_{-}^{k} \otimes a_{+}^{\ell-k} D_{(\mathfrak{a}_{2},f)}^{-s} a_{-}^{\ell-k}\right) = \sum_{m=\ell}^{\infty} \left(\sum_{m_{1}=0}^{m} (a_{1}(p))^{m_{1}} (a_{2}(p))^{m-m_{1}}\right) p^{-sm}$$

$$= \left(\sum_{k=0}^{\ell} (a_{1}(p))^{k} (a_{2}(p))^{\ell-k}\right) p^{-s\ell} L_{(p)}(s,f)$$

$$= a(p^{\ell}) p^{-s\ell} L_{(p)}(s,f) \tag{35}$$

In the above, the raising and lowering operators refer to their restrictions to the subspace  $\mathcal{H}_-$ , however, our use of the same notation will hopefully not cause any confusion. Therefore, the action of  $a_+$ , or more accurately that of  $a_+|_{\mathcal{H}_-}$ , on the 'ground state'  $|\Omega\rangle$  corresponding to n=1 (mother wavelet with scaling quantum number zero) is to annihilate it, i.e.,  $a_+|\Omega\rangle=0$ . The effect of the sum of the conjugated operators is to sum over a part of the lattice  $\mathbb{Z}^2$  satisfying  $m_1+m+2\geq \ell$  (see Fig. 1). This action mimics the action of the Hecke operator  $T_{p\ell}$  on the L-function.

Recall that the Hecke operators T(m),  $m \in \mathbb{N}$ , are a set of commuting operators whose action on the modular form is to return the coefficients in the q-expansion as eigenvalues

$$T(m)f(z) = a(m)f(z)$$
(36)

In other words a modular form is an eigenvector of the Hecke operators with the eigen-

<sup>&</sup>lt;sup>6</sup>Similarly, although not of much interest, the operators in Eqs. (12) and (16) may be conjugated to get  $\operatorname{Tr}_{\mathcal{H}_-}\left(a_+^\ell D_{\mathfrak{x}}^{-s} a_-^\ell\right) = \left(\chi(p) p^{-s}\right)^\ell L_p(s,\chi)$ , i.e, the  $\ell$ -th coefficient as a prefactor.

values as the coefficients in its q-expansion. They satisfy

$$T(m)T(n) = T(mn) \text{ for } m \nmid n$$
  

$$T(p)T(p^{\ell}) = T(p^{\ell+1}) + \chi(p)p^{k-1}T(p^{\ell-1})$$
(37)

A modular forms can also be understood [10,14] as a sum over the nodes of a lattice  $\Lambda$  in  $\mathbb{C}$ . From this point of view, the action of the Hecke operator T(n) involves sublattices  $\Lambda' \subset \Lambda$  of index n:  $T(n)\Lambda = \sum_{[\Lambda:\Lambda']=n} \Lambda'$ .

# 6 Summary

We have generalised the notion of the Vladimirov derivative, a pseudodifferential operator on Bruhat-Schwartz functions on the p-adic space  $\mathbb{Q}_p$ , by including the twist by multiplicative characters. The simplest of the multiplicative characters in this context are Dirichlet characters. We show that the wavelet functions, defined by Kozyrev, are eigenfunctions of the twisted operators with eigenvalues that depend on the character and the scaling properties of the wavelets. This allowed us to write the Dirichlet L-series (including the Riemann zeta function, which corresponds to the trivial character) as traces of appropriate twisted Vladimirov derivatives on a subspace of Bruhat-Schwartz functions supported on the compact subset of p-adic integers. Along the way, we have defined a generalised class of Gelfand-Graev-Tate gamma functions on  $\mathbb{Q}_p$  corresponding to twist by multiplicative characters. In Section 5, we have further generalised our construction to the L-series associated to the cusp forms of congruence subgroups of the modular group  $\mathrm{SL}(2,\mathbb{Z})$ . It would be interesting to realise these L-functions as the partition function of a 'statistical system'. We hope to report on the last point in the near future.

Acknowledgments: One of us (DG) presented the results in this paper in the VII-th International Conference p-Adic Mathematical Physics & its Applications held at the Universidade Beira Interior, Covilhã, Portugal during Sep 30–Oct 4, 2019, as well as in the National String Meeting 2019 held at IISER Bhopal, India during Dec 22–27, 2019. We thank the participants of these meetings, especially P. Bradley and W. Zuñiga-Galindo for their comments. We also thank V. Patankar for discussions.

**Note added:** Recently a paper (arXiv:2001.01721 [hep-th]) that discussed the construction of pseudodifferential operators of similar type appeared on the arXiv.

#### References

- [1] H. Edwards, *Riemann's zeta function*. Dover books on mathematics, Dover Publications, 2001.
- [2] A. Chattopadhyay, P. Dutta, S. Dutta, and D. Ghoshal, "Matrix Model for Riemann Zeta via its Local Factors," 2018, 1807.07342.
- [3] P. Dutta and S. Dutta, "Phase space distribution of Riemann zeros," J. Math. Phys., vol. 58, no. 5, p. 053504, 2017, arXiv:1610.07743 [hep-th].
- [4] V. Vladimirov, I. Volovic, and E. Zelenov, *p-adic analysis and mathematical physics*. Series On Soviet And East European Mathematics, World Scientific Publishing Company, 1994.
- [5] S. Kozyrev, "Wavelet theory as p-adic spectral analysis," Izv. Math., vol. 66, no. 2, p. 367—376, 2002, arXiv:math-ph/0012019.
- [6] A. Albeverio, A. Khrennikov, and V. Shelkovich, "Harmonic analysis in the *p*-adic Lizorkin spaces: fractional operators, pseudo-differential equations, *p*-adic wavelets, Tauberian theorems," *J. Fourier Anal. Appl.*, vol. 12, p. 393, 2006.
- [7] A. Albeverio and S. Kozyrev, "Frames of p-adic wavelets and orbits of the affine group," p-Adic Numbers Ultrametric Anal. Appl., vol. 1, p. 18, 2009, arXiv:0801.4713.
- [8] A. Albeverio, A. Khrennikov, and V. Shelkovich, "The Cauchy problems for evolutionary pseudo-differential equations over *p*-adic field and the wavelet theory," *J. Math. Anal. Appl.*, vol. 375, p. 82, 2011.
- [9] P. Dutta, D. Ghoshal, and A. Lala, "Enhanced Symmetry of the p-adic Wavelets," Phys. Lett., vol. B783, pp. 421–427, 2018, 1804.00958.
- [10] J. Serre, A course in arithmetic. Graduate texts in Mathematics, Springer, 1973.
- [11] I. Gelfand, M. Graev, and I. Piatetski-Shapiro, Representation theory and automorphic functions. Saunders Mathematics Books, Saunders, 1968.
- [12] D. Ghoshal and T. Kawano, "Towards p-adic string in constant B-field," Nucl. Phys., vol. B710, pp. 577–598, 2005, hep-th/0409311.
- [13] S. Gubser, M. Heydeman, C. Jepsen, S. Parikh, I. Saberi, B. Stoica, and B. Trundy, "Melonic theories over diverse number systems," *Phys. Rev.*, vol. D98, no. 12, p. 126007, 2018, 1707.01087.
- [14] N. Koblitz, *Introduction to Elliptic Curves and Modular Forms*. Graduate Texts in Mathematics, Springer, 1993.
- [15] E. Warner, "Modular forms and L-functions: a crash course,"
- [16] A. Sutherland, "Modular forms and L-functions," 2017.