1

Eng Maths S 2019-2020 exam

LIAM BURKE, TCD

Q1

$$4\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 2, \quad t > 0$$

$$u(0,t) = u(2,t) = 0$$

$$u(x,0) = f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2-x & 1 \le x \le 2 \end{cases}$$

Show

$$u(x,t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)}{n^2} \sin\left(\frac{n\pi}{2}\right)x\, e^{-\left(\frac{n\pi}{2}\right)^2 t}$$

Ans, let $u(x,t) = F(x)G(t)$

- Sub into PDE

$$4FG' = F''G$$

$$\Rightarrow \frac{4FG'}{FG} = \frac{F''G}{FG} \Rightarrow \frac{4G'}{G} = \frac{F''}{F} = k$$

Const

ODE #1	ODE #2
$F'' - kF = 0$	$G' - \frac{1}{4}kG = 0$
$F(0) = F(2) = 0$	

Solve ODE #1 for non-trivial Solution check all possible values of k,

ie $k = 0$, $k < 0$, $k > 0$

(2)

$K = 0$

$$F'' = 0$$
$$F(x) = ax + b$$

$F(0) = 0 = b$, $F(2) = a2 = 0$

$\Rightarrow a = 0$

$\Rightarrow F(x) = 0$ trivial Sol

- $K = p^2 > 0$

$$F'' - p^2 F = 0$$

Constant Coefficient ODE. Assume $F(x) = e^{\lambda x}$

$\Rightarrow \lambda^2 e^{\lambda x} - p^2 e^{\lambda x} = 0$

$e^{\lambda x}(\lambda^2 - p^2) = 0 \Rightarrow$ only a Solution if $\lambda = \pm p$

Two linearly independent solutions

- e^{px} , e^{-px}

General Solution is linear combination

$F(x) = Ae^{px} + Be^{-px}$

$F(0) = 0 = A + B \Rightarrow A = -B$

$F(2) = Ae^{2p} - Ae^{-2p} = 0$

- multiply by e^{2p}

$Ae^{4p} - A = 0$

$A(e^{4p} - 1) = 0 \Rightarrow A = 0$

$F(x) = 0$ Trivial

3

$k = -p^2$

$F'' + p^2 F = 0$

$F = e^{\lambda x} \Rightarrow \lambda^2 e^{\lambda x} + p^2 e^{\lambda x} = 0$

$e^{\lambda x}(\lambda^2 + p^2) = 0$

$\lambda = \pm\sqrt{-p^2}$

General Sol

$F(x) = A\cos px + B\sin px$

$F(0) = A = 0$

$F(2) = B\sin(2p) = 0$

Either $B = 0$ (trivial Sol)

OR

$\sin(2p) = 0$ (non-trivial Sol)

Go with non-trivial

$\Rightarrow$ only possible when $2p = n\pi$

$n \in \mathbb{Z}$ (integer)

$\Rightarrow F_n(x) = B_n \sin\left(\frac{n\pi}{2}x\right) \Rightarrow p = \frac{n\pi}{2}$

Solve ODE # 2 with $k = -p^2 = -\frac{n^2\pi^2}{4}$

$G' + \frac{1}{4}p^2 G = 0$

$G = e^{\lambda t} \Rightarrow \lambda e^{\lambda t} + \frac{1}{4}p^2 e^{\lambda t} = 0$

$e^{\lambda t}\left(\lambda + \frac{1}{4}p^2\right) = 0$

$\Rightarrow \lambda = -\frac{1}{4}p^2 \Rightarrow G(t) = e^{-\frac{1}{4}p^2 t}$

④

$$\Rightarrow \quad G_n(t) = Be^{-\frac{1}{4}p^2 t}$$

$\Rightarrow$

$$U_n(x,t) = F_n(x)\,G_n(t)$$

$$= B_n \sin\left(\frac{n\pi}{2}x\right)e^{-\left(\frac{n\pi}{4}\right)^2 t}$$

$$\Rightarrow \quad u(x,t) = \sum_{n=0}^{\infty} B_n \sin\left(\frac{n\pi}{2}x\right)e^{-\left(\frac{n\pi}{4}\right)^2 t}$$

At $n=0$, $u(x,t)=0$, so we can start sum at $n=1$.

Apply BC

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{2}x\right)$$

True if B_n is Fourier coefficient of $f(x)$ on $[0,2]$

ie $$B_n = \frac{2}{2}\int_0^2 f(x)\sin\left(\frac{n\pi}{2}x\right)dx$$

$$= \underbrace{\int_0^1 x\sin\left(\frac{n\pi}{2}x\right)dx}_{1} + \underbrace{\int_1^2 (2-x)\sin\left(\frac{n\pi}{2}x\right)dx}_{2}$$

$u = x \qquad dv = \sin\left(\frac{n\pi}{2}x\right)dx$

$du = dx \qquad v = -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}x\right)$

1) $$\int_0^1 x\sin\left(\frac{n\pi}{2}x\right)dx = -\frac{2}{n\pi}x\cos\left(\frac{n\pi}{2}x\right)\Big|_0^1 - \int_0^1 -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}x\right)dx$$

① Cnt...

$$= -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{2}{n\pi}\int_0^1 \cos\left(\frac{n\pi}{2}x\right)dx$$

$$= -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{2}{n\pi}\cdot\frac{2}{n\pi}\left[\sin\left(\frac{n\pi}{2}x\right)\right]\Big|_0^1$$

$$= -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{4}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)$$

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Integral 2 ((

$$\int_1^2 (2-x)\sin\left(\frac{n\pi}{2}x\right)dx = 2\int_1^2 \sin\left(\frac{n\pi}{2}x\right)dx - \int_1^2 x\sin\left(\frac{n\pi}{2}x\right)dx$$

\#

$u = x \quad dv = \sin\left(\frac{n\pi}{2}x\right)dx$

$du = dx$

$v = -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}x\right)$

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$$= \frac{2\cdot 2}{n\pi}\cos\left(\frac{n\pi}{2}x\right)\Big|_1^2 - \left[x\left(-\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}x\right)\right)\Big|_1^2 - \int_1^2 -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}x\right)dx\right]$$

$$= -\frac{4}{n\pi}\cos(n\pi) + \frac{4}{n\pi}\cos\left(\frac{n\pi}{2}\right) - \left[-\frac{2}{n\pi}\cdot 2\cos(n\pi) + \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right)\right.$$

$$\left. + \frac{2}{n\pi}\cdot\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}x\right)\Big|_1^2\right]$$

$$= \cancel{-\frac{4}{n\pi}}\cancel{\cos(n\pi)} + \cancel{\frac{4}{n\pi}\cos\left(\frac{n\pi}{2}\right)} + \cancel{\frac{4}{n\pi}\cos(n\pi)} - \cancel{\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right)}$$ lx

6

Add both integrals to get

$$B_n = \frac{8}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow u(x,t) = \sum_{n=0}^{\infty} \frac{8}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi x}{2}\right) e^{-\left(\frac{n\pi}{4}\right)^2 t}$$

$\frac{8}{\pi^2}$ $\rightarrow$ Gives result.