MAU11601:Introduction To Programming - Tutorial 5
Kronecker products and the Poisson matrix

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Introduction

For the $m \times m$ tridiagonal matrix $\mathbf{T}_m \in \mathbb{R}^{m \times m}$ defined by

$$\mathbf{T}_m = \text{tridiag}\{-1, 2, -1\} = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix},$$

and the $m \times m$ identity matrix $\mathbf{I}_m$, we define the $n \times n$ $(n = m^2)$ *Poisson matrix* $\mathbf{P}_n$ as

$$\mathbf{P}_n = \begin{bmatrix} \mathbf{T}_m + 2\mathbf{I}_m & -\mathbf{I}_m & & & & \\ -\mathbf{I}_m & \mathbf{T}_m + 2\mathbf{I}_m & -\mathbf{I}_m & & & \\ & -\mathbf{I}_m & \mathbf{T}_m + 2\mathbf{I}_m & -\mathbf{I}_m & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\mathbf{I}_m & \mathbf{T}_m + 2\mathbf{I}_m & -\mathbf{I}_m \\ & & & & -\mathbf{I}_m & \mathbf{T}_m + 2\mathbf{I}_m \end{bmatrix}.$$

For example, the 9×9 Poisson matrix is defined as

$$\mathbf{P}_9 = \begin{bmatrix} 4 & -1 & & -1 & & & & & \\ -1 & 4 & -1 & & -1 & & & & \\ & -1 & 4 & & & -1 & & & \\ -1 & & & 4 & -1 & & -1 & & \\ & -1 & & -1 & 4 & -1 & & -1 & \\ & & -1 & & -1 & 4 & & & -1 \\ & & & -1 & & & 4 & -1 & \\ & & & & -1 & & -1 & 4 & -1 \\ & & & & & -1 & & -1 & 4 \end{bmatrix}.$$

Definition 0.1. *Given a matrix $\boldsymbol{A} \in \mathbb{C}^{m \times n}$ and matrix $\boldsymbol{B} \in \mathbb{C}^{p \times q}$, we define the Kronecker product between $\boldsymbol{A}$ and $\boldsymbol{B}$, denoted $\boldsymbol{A} \otimes \boldsymbol{B}$, as the $pm \times qn$ block matrix $\boldsymbol{C}$ defined by*

$$\boldsymbol{C} = \boldsymbol{A} \otimes \boldsymbol{B} = \begin{pmatrix} a_{11}\boldsymbol{B} & \dots & a_{1n}\boldsymbol{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\boldsymbol{B} & \dots & a_{mn}\boldsymbol{B} \end{pmatrix}.$$

Questions

- (a) Show that the $n \times n$ Poisson matrix $\mathbf{P}_n$ can be constructed from the Kronecker sum

  $$\mathbf{P}_n = \mathbf{T}_m \otimes \mathbf{I}_m + \mathbf{I}_m \otimes \mathbf{T}_m$$

  (We will do this on the blackboard in class.)

- (b) Given an arbitrary vector $\mathbf{b} \in \mathbb{R}^n$ composed of m block vectors $\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_m \in \mathbb{R}^m$. ie

  $$\mathbf{b}^T = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & ... & \mathbf{b}_m \end{bmatrix},$$

  write down an expression for the vector $\mathbf{v} \in \mathbb{R}^n$ defined by the matrix-vector product $\mathbf{v} = \mathbf{P}_n \mathbf{b}$.

  (We will do this on the blackboard in class.)

- (c) Write a MATLAB function `generate_Poisson` which inputs an integer m and returns an $m^2 \times m^2$ Poisson matrix $\mathbf{P}$. Your function can make use of the built in MATLAB `gallery` function to construct the matrix $\mathbf{T}_m$ via

  ```
  T = full(gallery("tridiag",m,-1,2,-1));
  ```

  To construct $\mathbf{P}$, you may then use the built in function `kron` which takes in two matrices and returns their Kronecker product.

  Test your code works by generating the Poisson matrix directly from MATLAB via

  ```
  C = full(gallery("poisson",m))
  ```

  and checking that the 2 norm of the matrix `C - P` is zero.

- (d) Write a MATLAB function `mat_vec` which inputs an $n \times n$ matrix $\mathbf{A}$ and vector $\mathbf{b}$ and returns the matrix-vector product $\mathbf{v} = \mathbf{A}\mathbf{b}$.

  Test your function with the Poisson matrix and a randomly generated vector $\mathbf{b}$ by showing that the 2 norm of the vector $\mathbf{v} - \mathbf{A}^*\mathbf{b}$ is zero.

- (e) Using your answer from part (b), write a second function `mat_vec2` which takes in **only** a vector $\mathbf{b} \in \mathbb{R}^n$ and **implicitly** computes the matrix-vector product $\mathbf{v} = \mathbf{P}_n\mathbf{b}$ with the $n \times n$ Poisson matrix $\mathbf{P}_n$. Your code must **not** explicitly construct or store the Poisson matrix. Test your function using the same approach as part (d).