(1) Eng Math S

LIAM BURKE, TCD

23 (a) (i) 2018 2020 exam

$$f(x) = \begin{cases} 0 & -1 \le x \le 0 \\ x & 0 \le x \le 1 \end{cases} \quad \text{on } [-1,1]$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right]$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$a_0 = 1\int_{-1}^{1} f(x)\,dx = \int_{0}^{1} x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$a_n = \int_{-1}^{1} f(x)\cos(n\pi x) = \int_{0}^{1} x\cos(n\pi x)\,dx$$

$u = x \qquad dv = \cos(n\pi x)$

$du = dx \qquad v = \frac{1}{n\pi}\sin(n\pi x)$

$$\int_0^1 x\cos(n\pi x)\,dx = \frac{1}{n\pi}x\sin(n\pi x)\Big|_0^1 - \frac{1}{n\pi}\int_0^1 \sin(n\pi x)\,dx$$

$-\frac{1}{n\pi}\cos(n\pi x)$

$$= \frac{1}{n\pi}\sin(n\pi) + \frac{1}{n\pi}\frac{1}{n\pi}\cos(n\pi x)\Big|_0^1$$

$$= \frac{1}{(n\pi)^2}\left[\cos(n\pi) - 1\right]$$

(1) (2)

$$b_n = \int_{-1}^{1} f(x)\sin(n\pi x)\,dx = \int_{0}^{1} x\sin(n\pi x)\,dx$$

$u = x \qquad dv = \sin(n\pi x)$

$du = dx \qquad v = -\frac{1}{n\pi}\cos(n\pi x)$

$$= -\frac{1}{n\pi}x\cos(n\pi x)\Big|_0^1 - \left(\frac{-1}{n\pi}\right)\int_0^1 \cos(n\pi x)\,dx$$

$$= -\frac{1}{n\pi}\cos(n\pi) + \frac{1}{n\pi}\left[\frac{1}{n\pi}\sin(n\pi x)\right]\Big|_0^1$$

$$= -\frac{1}{n\pi}\cos(n\pi) \qquad 0$$

$$\text{So } f(x) = \frac{1}{4} + \sum_{n=1}^{\infty}\left[\frac{1}{(n\pi)^2}\left[\cos(n\pi)-1\right]\cos(n\pi x) - \frac{1}{n\pi}\cos(n\pi)\sin(n\pi x)\right]$$

(ii) Show $\frac{\pi^2}{8} = \sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}$

$x = 0 \Rightarrow 0 = \frac{1}{4} + \sum_{n=1}^{\infty}\frac{1}{(n\pi)^2}\underbrace{\left(\cos(n\pi)-1\right)}$

0 when n even

-2 when n odd

only interested in $n = 2m+1$, $m \in \mathbb{Z}$

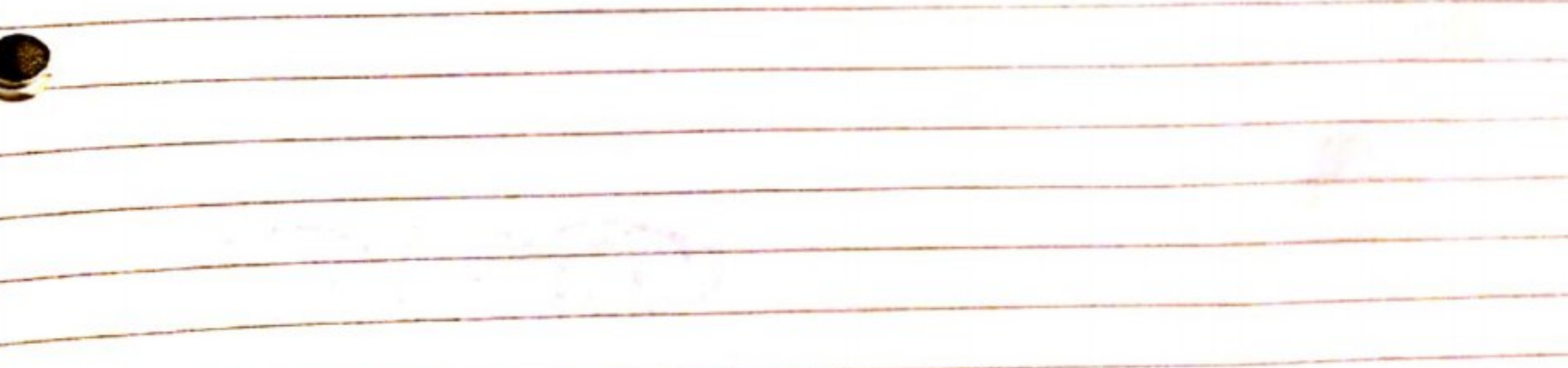
$n = 2m+1 \quad \Rightarrow \quad n=1=2m+1$

$\Rightarrow m=0$

$$-\frac{1}{4} = \sum_{m=0}^{\infty} \frac{0}{(2m+1)^2\pi^2} - 2$$

$$\Rightarrow \frac{\pi^2}{8} = \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2}$$ ✓

Question 3



$$f(x) = \begin{cases} 1 - |x| & -1 \le x \le 1 \\ 0 & x < -1,\ 1 < x \end{cases}$$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1 - |x|) e^{-i\omega x} dx$$

Euler $e^{ix} = \cos x + i \sin x$

$$e^{-i\omega x} = \cos(-\omega x) + i \sin(-\omega x)$$

$$e^{-i\omega x} = \cos(\omega x) - i \sin(\omega x)$$

cos even sin odd

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1 - |x|) \cos(\omega x) dx - \frac{i}{\sqrt{2\pi}} \int_{-1}^{1} (1 - |x|) \sin(\omega x)$$

$(1-|x|)$: even; $\cos(\omega x)$: even

$(1-|x|)$: even; $\sin(\omega x)$: odd

even $\cdot$ odd $= 0$

$$\frac{2}{\sqrt{2\pi}} \int_{0}^{1} \cos(\omega x) dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{1} (1 - x) \cos(\omega x) dx$$

$$= \frac{2}{\sqrt{2\pi}} \left[\int_{0}^{1} \cos(\omega x) dx - \int_{0}^{1} x \cos(\omega x) dx \right]$$

$u = x$ $dv = \cos(\omega x)$

$du = dx$ $v = \frac{1}{\omega} \sin(\omega x)$

$$= \frac{2}{\sqrt{2\pi}} \left[\frac{1}{\omega} \sin(\omega x) \Big|_0^1 - \left[\frac{1}{\omega} x \sin(\omega x) \Big|_0^1 - \frac{1}{\omega} \int_0^1 \sin(\omega x)\, dx \right] \right]$$

$$= \frac{2}{\sqrt{2\pi}} \left[\frac{1}{\omega} \sin(\omega) - \frac{1}{\omega} \sin(\omega) + \frac{1}{\omega} \left(-\frac{1}{\omega} \cos(\omega x) \right) \Big|_0^1 \right]$$

$$= \frac{2}{\sqrt{2\pi}} \left[-\frac{1}{\omega^2} \cos(\omega) - \frac{1}{\omega^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left(1 - \frac{\cos\omega}{\omega^2} \right)$$

$$\frac{2}{\sqrt{2\pi}} = \frac{\sqrt{4}}{\sqrt{2\pi}} = \sqrt{\frac{4}{2\pi}}$$

$$= \sqrt{\frac{2}{\pi}}$$