① TCD MAU11404

Techniques In Theoretical Physics
Exam 2022 Solutions

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Q1 ⓐ Solve $2y' - y = 4\sin(3t)$

$$\to \quad \underbrace{y' - \tfrac{1}{2}}_{p(t)} y = \underbrace{2\sin(3t)}_{g(t)}$$

use integrating factor $v(t) = e^{\int p(t)\,dt}$

$$= e^{-\frac{1}{2}t + c_1} = Ce^{-1/2 t}$$

where $C = e^{c_1}$

Solution $y(t) = \frac{1}{v(t)}\left[\int v(t) g(t)\,dt\right] = \frac{1}{Ce^{-1/2 t}} \int Ce^{-1/2 t} 2\sin(3t)\,dt$

$$= 2e^{1/2 t} \underbrace{\int e^{-1/2 t}\sin(3t)\,dt}_{(\gamma)}$$

Treat integral (γ) seperatly using integration by parts (IBP) with

$u = e^{-1/2 t}$ $\qquad dv = \sin(3t)\,dt$

$du = -\frac{1}{2}e^{-1/2 t}\,dt$ $\qquad v = -\frac{1}{3}\cos(3t)$

$$\int e^{-1/2 t}\sin(3t)\,dt = e^{-1/2 t}\left(-\tfrac{1}{3}\cos(3t)\right) - \int -\tfrac{1}{3}\cos(3t)\left(-\tfrac{1}{2}e^{-1/2 t}\right)dt$$

$$= -\frac{1}{3}e^{-1/2 t}\cos(3t) - \frac{1}{6}\underbrace{\int e^{-1/2 t}\cos(3t)\,dt}_{\text{Apply IBP}}$$

Apply IBP $u = e^{-1/2 t}$, $du = -\frac{1}{2}e^{-1/2 t}\,dt$ | $dv = \cos(3t)\,dt$, $v = \frac{1}{3}\sin(3t)$

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$$\int e^{-\frac{1}{2}t}\sin(3t)\,dt =$$

$$-\frac{1}{3}e^{-\frac{1}{2}t}\cos(3t)-\frac{1}{6}\left[e^{-\frac{1}{2}t}\,\frac{1}{3}\sin(3t) - \int -\frac{1}{2}e^{-\frac{1}{2}t}\,\frac{1}{3}\sin(3t)\,dt\right]$$

$$= -\frac{1}{3}e^{-\frac{1}{2}t}\cos(3t) - \frac{1}{18}e^{-\frac{1}{2}t}\sin(3t) - \frac{1}{36}\underbrace{\int e^{-\frac{1}{2}t}\sin(3t)\,dt}_{\text{original integral}}$$

bring to other side of eqn

Dont forget cost

$$\int e^{-\frac{1}{2}t}\sin(3t)\,dt = \frac{-12e^{-\frac{1}{2}t}\cos(3t) - 2e^{-\frac{1}{2}t}\sin(3t)}{37} + C$$

Thus

$$y(t) = 2e^{\frac{1}{2}t}\left[\frac{-12e^{-\frac{1}{2}t}\cos(3t) - 2e^{-\frac{1}{2}t}\sin(3t)}{37}\right] + C2e^{\frac{1}{2}t}$$

$$= -\frac{4}{37}\left(6\cos(3t) + \sin(3t)\right) + 2Ce^{\frac{1}{2}t}$$

Techniques in TP 2022

Q1 (b) $y' = e^{-y}(2x-4) \qquad y(5) = 0$

$$\int e^{y}\,dy = \int (2x-4)\,dx$$

$$e^{y} = \frac{2x^2}{2} - 4x + C$$

$$\ln(e^{y}) = \ln\left(\frac{2x^2}{2} - 4x + c\right)$$

$$y = \ln\left(\frac{2x^2}{2} - 4x + c\right) = \ln(x^2 - 4x + c)$$

$$\cancel{y(5) = 0 = \ln(50 - 20 + c)}$$

$$\cancel{0 = \ln(30 + c)}$$

$$\cancel{30 + c = 1} \qquad \cancel{c = -29}$$

$$\cancel{y(x) = \ln(2x^2 - 4x - 29)}$$

Apply IC

$$y(5) = 0 = \ln(25 - 20 + c)$$

$$25 - 20 + c = 1$$

$$5 + c = 1 \qquad c = -4$$

$$y(x) = \ln(x^2 - 4x - 4)$$

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(c) $\dfrac{dr}{d\theta} = \dfrac{r^2}{\theta}$ $\qquad r(1) = 2$

$$\int \frac{dr}{r^2} = \int \frac{1}{\theta}\, d\theta$$

$$\int r^{-2}\, dr = \int \frac{1}{\theta}\, d\theta$$

$$-r^{-1} = \ln(|\theta|) + C$$

$$-\frac{1}{r} = \ln|\theta| + C$$

Apply IC

$$-\frac{1}{2} = \underbrace{\ln|1|}_{0} + C$$

$$C = -1/2$$

$$-\frac{1}{r} = \ln|\theta| - \frac{1}{2}$$

$$r = \frac{1}{\frac{1}{2} - \ln|\theta|}$$

$\rightarrow$ Issues to Conside

- Avoid $\theta = 0$
- Avoid values of θ for which $\frac{1}{2} - \ln|\theta| = 0$

ie $|\theta| = e^{1/2} \rightarrow$

$\theta = \pm\sqrt{e} \rightarrow$ four valid intervals

$-\infty < \theta < -\sqrt{e}$

$-\sqrt{e} < \theta < 0$

$0 < \theta < \sqrt{e} \rightarrow$ ONLY interval of interest from IC @ $\theta = 1$

$\sqrt{e} < \theta < \infty$

(5) 2022

Q1 (d) $y' + \frac{3y}{x} = \frac{e^x}{x^3}$

Already in correct form for integrating factor ~~with~~ $\frac{dy}{dx} + p(x)y(x) = g(x)$

with $p(x) = 3/x$, $g(x) = \frac{e^x}{x^3}$

Integrating factor $v(x) = e^{\int p(x)\,dx} = e^{3\int \frac{1}{x}\,dx}$

$= e^{3\ln|x| + c_1}$

$= \cancel{e^{x}} = e^{3\ln|x|}\underbrace{e^{c_1}}_{C}$

$= C|x|^3$

So

$y(x) = \frac{1}{v(x)}\left[\int v(x)g(x)\,dx\right] = \frac{1}{C|x|^3}\left[\int C|x|^3 \frac{e^x}{x^3}\,dx\right]$

$= \frac{\cancel{C}}{C|x|^3}\frac{x}{|x|}\int \frac{|x|}{x}|x|^3\frac{e^x}{x^3}\,dx = \frac{1}{x^3}\int e^x\,dx$

$= \frac{1}{x^3}\left[e^x + c\right] = \frac{e^x}{x^3} + \frac{C}{x}$

CS CamScanner

6) 2022

Q 1 e) Recall for LINEAR ODE's we use integrating factor. For Bernouilli ODE

$y' + p(x)y = q(x)y^n$, the equation is NON linear due to y^n term (assuming $n>1$).

Idea → Use the substitution $v = y^{1-n}$ to transform to a linear ODE.

In this Q we have

$$y' + \frac{4}{x}y = x^3 y^2 \qquad y(2) = -1 , x>0$$

with $v = y^{-1}$

$$v' = -y^{-2}\frac{dy}{dx} = -y^{-2}y'$$

- Divide ODE by y^2

$$y^{-2}y' + y^{-1}\frac{4}{x} = x^3$$

$$-v' + \frac{4}{x}v = x^3$$

$$v' - \frac{4}{x}v = -x^3$$

Correct form for Integrating factor with $p(x) = -\frac{4}{x}$

$g(x) = -x^3$

$$v' - \frac{4}{x} v = -x^3$$

Integraty factor $\omega(x)$

$$\omega(x) = e^{\int p(x)dx} = e^{-4\int \frac{1}{x} dx} = e^{-4(\ln|x| + c_1)}$$

$$= e^{-4\ln|x|} C, \quad C = e^{c_1}$$

$$= \frac{C}{|x|^4}$$

$$y(t) = \frac{|x|^4}{C}\left[\int \frac{C}{|x|^4}(-x)^3 \, dx\right] = \frac{Cx^4}{C}\left[-\int \frac{1}{x} \, dx\right]$$

$$= x^4\left[-\ln|x| + c\right] = -x^4\ln|x| + Cx^4$$

$$y(t) = \frac{1}{-x^4\ln|x| + Cx^4}$$

$$y(2) = -1 = \frac{1}{-(-2)^4\ln(2) + C(-2)^4} = \frac{1}{-16\ln(2) + 16C} = -1$$

$$16\ln(2) - 16C = 1 \rightarrow -16C = 1 - 16\ln(2)$$

$$C = -\frac{1}{16} + \ln(2)$$

$$y(t) = \frac{1}{x^4\left(\ln(2) - 1/16 - \ln|x|\right)}$$