① Introduction To Numerical Analysis (2022 Exam)

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Van der Houwen's Wray Butcher tableau

$$\begin{array}{c|ccc} 0 & & & \\ 8/15 & 8/15 & & \\ 2/3 & 1/4 & 5/12 & \\ \hline & 1/4 & 0 & 3/4 \end{array}$$

ODE $\quad y'(t) = -\dfrac{y(t)}{2} + \dfrac{y(t)}{t+1}\ ; \ y(0)=1$

Exact Sol $\quad y(t) = (t+1)e^{-t/2}$

Compute approximation to $y(0.2)$ using Van der Houwen's Wray for Step Size $h=0.2$ (one iteration) and $h=0.1$.

Compare to exact.

Ans, General form of Explicit! Runge Kutta methods

$$y_{N+1} = y_N + h\sum_{i=1}^{s} b_i k_i$$

Butcher Tableau

$$\begin{array}{c|ccccc} 0 & & & & & \\ c_2 & a_{21} & & & & \\ c_3 & a_{31} & a_{32} & & & \\ \vdots & & & & & \\ c_s & a_{s1} & a_{s2} & a_{s3} & \cdots & a_{s,s-1} \\ \hline & b_1 & b_2 & b_3 & \cdots & b_{s,s-1} \quad b_s \end{array}$$

$k_1 = f(t_N, y_N)$

$k_2 = f(t_N + c_2 h, y_N + (a_{21}k_1)h)$

$k_3 = f(t_N + c_3 h, y_N + (a_{31}k_1 + a_{33}k_2)h)$

$\vdots$

$k_s = f(t_N + c_s h, y_N + (a_{s1}k_1 + a_{s2}k_2 + \cdots + a_{s,s-1}k_{s-1}))$

(2) It is clear the Butcher tableau corresponds to update

$$y_{N+1} = y_N + \tfrac{1}{4} h \underbrace{f(t_N, y_N)}_{k_1} + \tfrac{3}{4} h k_3 \quad , \quad \text{where}$$

$$k_2 = f\left(t_N + \tfrac{8}{15}h,\ y_N + \tfrac{8}{15} f(t_N, y_N) h\right)$$

AND

$$k_3 = f\left(t_N + \tfrac{2}{3}h,\ y_N + \left(\tfrac{1}{4} f(t_N, y_N) + \tfrac{5}{12} k_2\right) h\right)$$

$\boxed{\text{Case } h = 0.2}$

$$k_1 = f(0,1) = -\tfrac{1}{2} + 1 = 0.5$$

$$k_2 = f\left(0 + \tfrac{8}{15}(0.2),\ 1 + \tfrac{8}{15}(0.5)(0.2)\right) = f(0.106, 0.053)$$

$$= -\frac{0.053}{2} + \frac{0.053}{0.106+1} = 0.021420$$

$$k_3 = f\left(0 + \tfrac{2}{3}(0.5),\ 1 + \left(\tfrac{1}{4}(0.5) + \tfrac{5}{12}(0.021420)\right)(0.2)\right)$$

$$= f\left(\tfrac{1}{3},\ 1 + (0.133926)(0.2)\right) = f\left(\tfrac{1}{3},\ 1.026786\right)$$

$$= -\frac{1.026786}{2} + \frac{1.026786}{\frac{1}{3}+1} = 0.256697$$

$$y(0.2) \approx 1 + \tfrac{1}{4}(0.2)(0.5) + \tfrac{3}{4}(0.2)(0.256697) = 1.063546$$

Exact $y(0.2) = (0.2+1)e^{-0.2/2} = 1.085805$