① Numerical Analysis Tutorial Week 4
Feb 2023

Recall model of floating point Arithmetic

$$fl(x \circ y) = (x \circ y)(1+\varepsilon), \quad |\varepsilon| \leq \varepsilon_{machine}$$

- Relative error of storing non-machine a number on a computer?

$$\left|\frac{fl(a) - a}{a}\right| = \left|\frac{a(1+\varepsilon_a) - a}{a}\right| = \varepsilon_a$$

- Adding two non-machine numbers a, b?

$$\left[a(1+\varepsilon_a) + b(1+\varepsilon_b)\right](1+\varepsilon_r)$$

$$\left[a + a\varepsilon_a + b + b\varepsilon_b\right](1+\varepsilon_r) = a + a\varepsilon_a + b + b\varepsilon_b + a\varepsilon_r + \underbrace{a\varepsilon_a\varepsilon_r}_{O(\varepsilon_{mach}^2)} + b\varepsilon_r + \underbrace{b\varepsilon_b\varepsilon_r}_{O(\varepsilon_{mach}^2)}$$

Neglect

$$= a + b + a\varepsilon_a + b\varepsilon_b + (a+b)\varepsilon_r$$

Relative error of a+b?

$$\left|\frac{fl(a+b) - (a+b)}{a+b}\right| = \left|\frac{(a+b) + a\varepsilon_a + b\varepsilon_b + (a+b)\varepsilon_r - (a+b)}{(a+b)}\right|$$

Can be problematic

$$= \frac{a}{a+b}\varepsilon_a + \frac{b}{a+b}\varepsilon_b + \varepsilon_r$$

— innocent term

② • Error ε_a from rounding a is amplified by a factor of $\frac{a}{a+b}$

• Error ε_b from rounding b is amplified by a factor of $\frac{b}{a+b}$

Multiplication of non-machine numbers

$$\left[a(1+\varepsilon_a)\,b(1+\varepsilon_b)\right](1+\varepsilon_r)$$

$$= \left[(a + a\varepsilon_a)(b + b\varepsilon_b)\right](1+\varepsilon_r)$$

$$= \left[a(b + b\varepsilon_b) + a\varepsilon_a(b + b\varepsilon_b)\right](1+\varepsilon_r)$$

$$= \left[ab + ab\varepsilon_b + ab\varepsilon_a + \underbrace{ab\varepsilon_a\varepsilon_b}_{\text{Neglect } O(\varepsilon_{mach}^2)}\right](1+\varepsilon_r)$$

∴ Neglecting $O(\varepsilon^2)$ terms

$$= ab + ab\varepsilon_b + ab\varepsilon_a + ab\varepsilon_r$$

Relative Error $\left|\frac{fl(ab) - ab}{ab}\right| = \left|\frac{ab + ab\varepsilon_b + ab\varepsilon_a + ab\varepsilon_r - ab}{ab}\right|$

$= \varepsilon_b + \varepsilon_a + \varepsilon_r$ } All error terms are "amplified" by a factor of 1.

↳ Harmless

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3 **Relative Error of Squaring a number:-**

Special case of multiplication $a \times b$ with $b = a$ and $\varepsilon_b = \varepsilon_a$

- From last part relative error

$$= \varepsilon_b + \varepsilon_a + \varepsilon_r$$

$$= 2\varepsilon_a + \varepsilon_r$$

Error associated to rounding a is amplified by a factor of 2

Horners Method

Numerically more stable means of evaluating a polynomial of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

by recursively factorizing

$$p(x) = a_0 + x\left(a_1 + a_2 x + \cdots + a_n x^{n-1}\right)$$

$$= a_0 + x\left(a_1 + x\left(a_2 + \cdots + a_n x^{n-2}\right)\right)$$

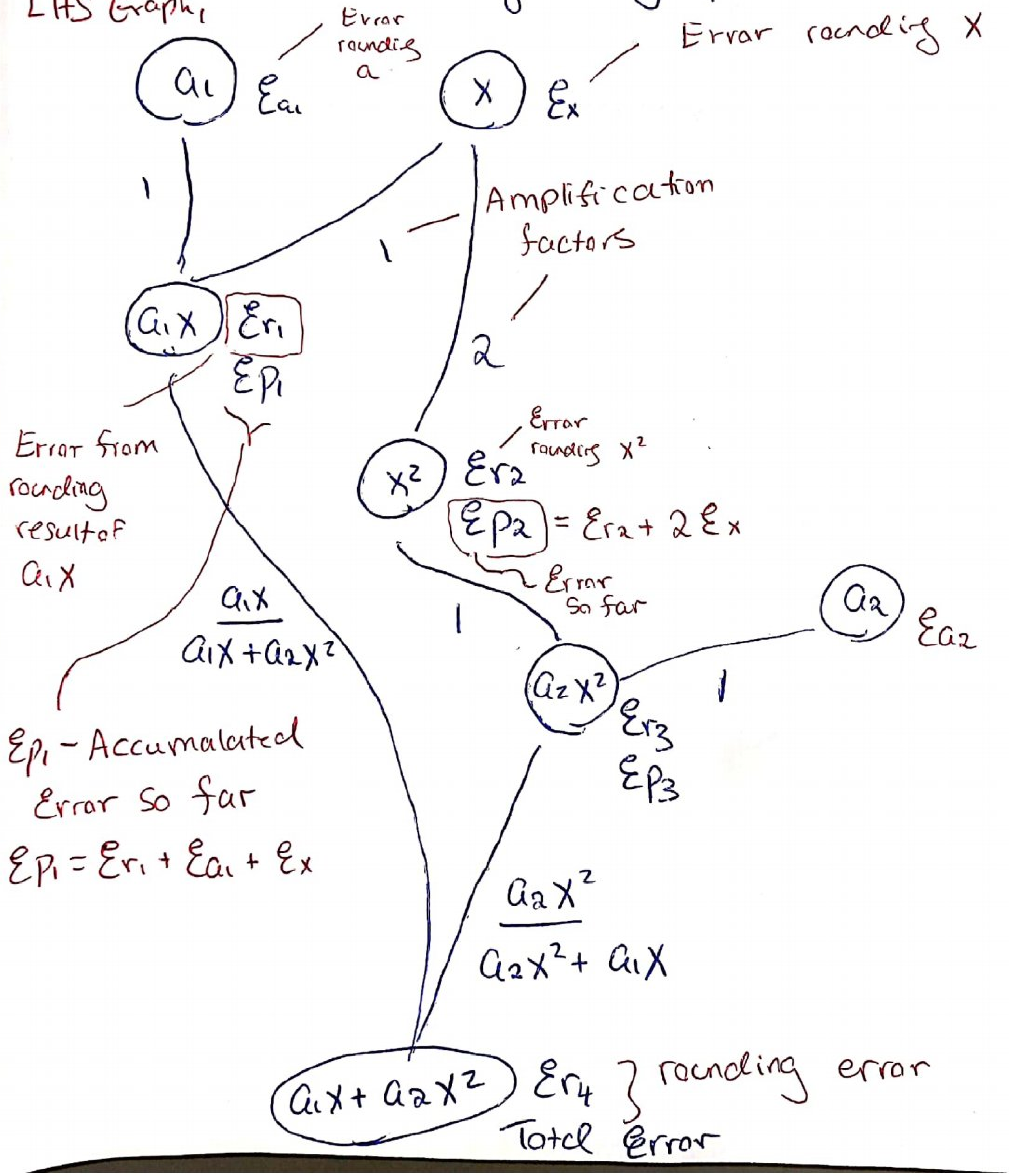
$$\vdots$$

(5) Simple Example

Evaluate $p(x) = \underbrace{a_1 x + a_2 x^2}_{\substack{\text{LHS} \\ \text{less stable}}} = \underbrace{x(a_1 + a_2 x)}_{\substack{\text{RHS} \\ \text{more stable}}}$

Error analysis of this problem can be made easier using a graph approach

LHS Graph,



(6) Total Error for LHS evaluation

$$\text{Total err} = \varepsilon_{r_4} + \frac{a_1 X}{a_1 X + a_2 X^2} \underline{\varepsilon_{p_1}} + \frac{a_2 X^2}{a_2 X^2 + a_1 X} \underline{\varepsilon_{p_3}}$$

ε_{p_1} and ε_{p_3} contain errors associated to earlier nodes

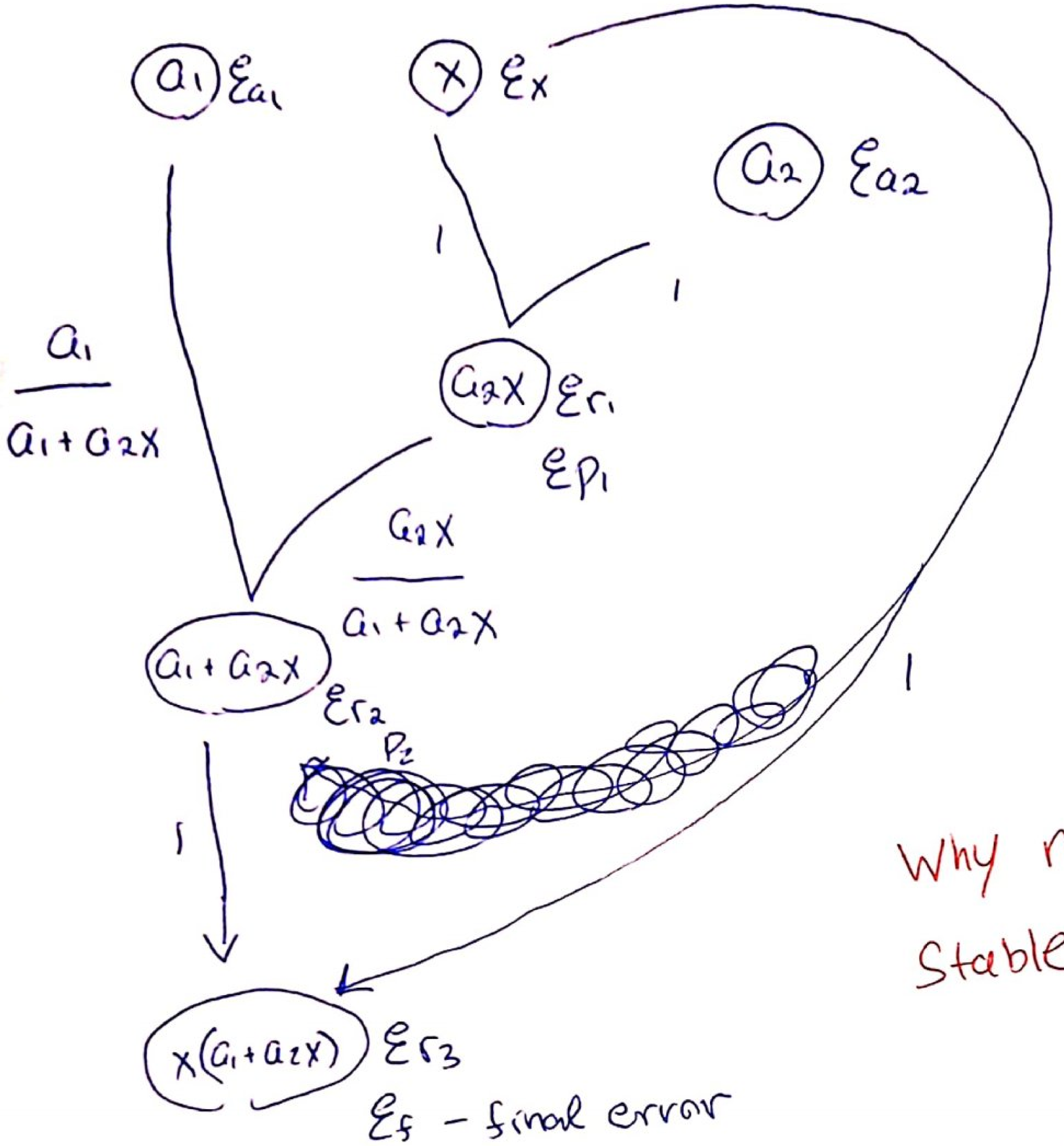
We can simplify

$$\text{Total err} = \frac{a_1}{a_1 X + a_2 X}(\varepsilon_{r_1} + \varepsilon_{a_1} + \varepsilon_X) + \frac{a_2 X}{a_2 X + a_1}(\varepsilon_{r_3} + \varepsilon_{p_2} + \varepsilon_{a_2}) + \varepsilon_{r_4}$$

$$= \varepsilon_{r_4} + \frac{a_1}{a_1 X + a_2 X}(\varepsilon_{r_1} + \varepsilon_{a_1} + \varepsilon_X) + \frac{a_2 X}{a_2 X + a_1}(\varepsilon_{r_3} + \varepsilon_{r_2} + 2\varepsilon_X + \varepsilon_{a_2})$$

- Expanding using information from other nodes is shown in red.

7 Error Analysis of RHS $x(a_1 + a_2 x)$



Why more stable?

$$\text{Total error} = \varepsilon_{r_3} + \varepsilon_x + P_2$$

$$= \varepsilon_{r_3} + \varepsilon_x + \left(\varepsilon_{r_2} + \frac{a_1}{a_1+a_2x}\varepsilon_{a_1} + \frac{a_2x}{a_1+a_2x}\varepsilon_{p_1}\right)$$

$$= \varepsilon_{r_3} + \varepsilon_x + \varepsilon_{r_2} + \frac{a_1}{a_1+a_2x}\varepsilon_{a_1} + \frac{a_2x}{a_1+a_2x}\left(\varepsilon_{r_1} + \varepsilon_x + \varepsilon_{a_2}\right)$$

$$= \varepsilon_{rx} + \varepsilon_x + \varepsilon_{r_2} + \frac{a_1}{a_1+a_2x}\varepsilon_{a_1} + \frac{a_2x}{a_1+a_2x}\left(\varepsilon_{r_1} + \varepsilon_x + \varepsilon_{a_2}\right)$$