① Eng Math 5

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$$f(x) = \begin{cases} 0 & -1 \le x \le 0 \\ x & 0 \le x \le 1 \end{cases} \quad \text{on } [-1,1]$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\,dx \qquad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$a_0 = 1 \int_{-1}^{1} f(x)\,dx = \int_{0}^{1} x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$a_n = \int_{-1}^{1} f(x) \cos(n\pi x) = \int_{0}^{1} x \cos(n\pi x)\,dx$$

$$u = x \qquad dv = \cos(n\pi x)$$

$$du = dx \qquad v = \frac{1}{n\pi} \sin(n\pi x)$$

$$\int_0^1 x \cos(n\pi x)\,dx = \frac{1}{n\pi} x \sin(n\pi x)\Big|_0^1 - \frac{1}{n\pi}\int_0^1 \sin(n\pi x)\,dx$$

$$-\frac{1}{n\pi}\cos(n\pi x)$$

$$= \frac{1}{n\pi} \sin(n\pi) + \frac{1}{n\pi}\frac{1}{n\pi} \cos(n\pi x)\Big|_0^1$$

$$= \frac{1}{(n\pi)^2}\left[\cos(n\pi) - 1\right]$$

(2)

$$b_n = \int_{-1}^{1} f(x)\sin(n\pi x)\,dx = \int_0^1 x\sin(n\pi x)\,dx$$

$u = x \qquad dv = \sin(n\pi x)$

$du = dx \qquad v = -\frac{1}{n\pi}\cos(n\pi x)$

$$= -\frac{1}{n\pi}x\cos(n\pi x)\Big|_0^1 - \left(\frac{-1}{n\pi}\right)\int_0^1 \cos(n\pi x)\,dx$$

$$= -\frac{1}{n\pi}\cos(n\pi) + \frac{1}{n\pi}\underbrace{\left[\frac{1}{n\pi}\sin(n\pi x)\right]\Big|_0^1}_{0}$$

$$= -\frac{1}{n\pi}\cos(n\pi)$$

So $f(x) = \frac{1}{4} + \sum_{n=1}^{\infty}\left[\frac{1}{(n\pi)^2}\left[\cos(n\pi) - 1\right]\cos(n\pi x) - \frac{1}{n\pi}\cos(n\pi)\sin(n\pi x)\right]$

(ii) Show $\frac{\pi^2}{8} = \sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}$

$x = 0 \Rightarrow \quad 0 = \frac{1}{4} + \sum_{n=1}^{\infty}\frac{1}{(n\pi)^2}\underbrace{(\cos(n\pi) - 1)}$

0 when n even

-2 when n odd

only interested in $n = 2m+1$, $m \in \mathbb{Z}$

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$n = 2m+1 \Rightarrow n = 1 = 2m+1$

$\Rightarrow m = 0$

$$-\frac{1}{4} = \sum_{m=0}^{\infty} \frac{-2}{(2m+1)^2\pi^2}$$

$$\Rightarrow \frac{\pi^2}{8} = \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2}$$

~~Question 3~~

$$f(x) = \begin{cases} 1 - |x| & -1 \le x \le 1 \\ 0 & x < -1 \, , \, 1 < x \end{cases}$$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|) e^{-i\omega x} dx$$

Euler $e^{ix} = \cos x + i \sin x$

$$e^{-i\omega x} = \cos(-\omega x) + i\sin(-\omega x)$$

$$e^{-i\omega x} = \cos(\omega x) - i\sin(\omega x)$$

Cos even Sin odd

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \underbrace{(1-|x|)}_{\text{even}} \underbrace{\cos(\omega x)}_{\text{even}} dx \underbrace{- \frac{i}{\sqrt{2\pi}} \int_{-1}^{1} \underbrace{(1-|x|)}_{\text{even}} \underbrace{\sin(\omega x)}_{\text{odd}}}_{0}$$

$$\cancel{= \frac{2}{\sqrt{2\pi}} \int_{-1}^{1} \cos(\omega x) dx}$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{1} (1-x) \cos(\omega x) dx$$

$$= \frac{2}{\sqrt{2\pi}} \left[\int_{0}^{1} \cos(\omega x) dx - \int_{0}^{1} x \cos(\omega x) dx \right]$$

$u = x \quad dv = \cos(\omega x)$

$du = dx \quad v = \frac{1}{\omega} \sin(\omega x)$

$$= \frac{2}{\sqrt{2\pi}} \left[\frac{1}{\omega} \sin(\omega x) \Big|_0^1 - \left[\frac{1}{\omega} x \sin(\omega x) \Big|_0^1 - \frac{1}{\omega} \int_0^1 \sin(\omega x)\, dx \right] \right]$$

$$= \frac{2}{\sqrt{2\pi}} \left[\frac{1}{\omega} \sin(\omega) - \frac{1}{\omega} \sin(\omega) + \frac{1}{\omega} \left(-\frac{1}{\omega} \cos(\omega x) \right) \Big|_0^1 \right]$$

$$= \frac{2}{\sqrt{2\pi}} \left[-\frac{1}{\omega^2} \cos(\omega) - \frac{1}{\omega^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{1 - \cos\omega}{\omega^2} \right)$$

$$\frac{2}{\sqrt{2\pi}} = \frac{\sqrt{4}}{\sqrt{2\pi}} = \sqrt{\frac{4}{2\pi}} = \sqrt{\frac{2}{\pi}}$$