MAU33E01: Engineering Mathematics V: Extra Question on Fourier Series

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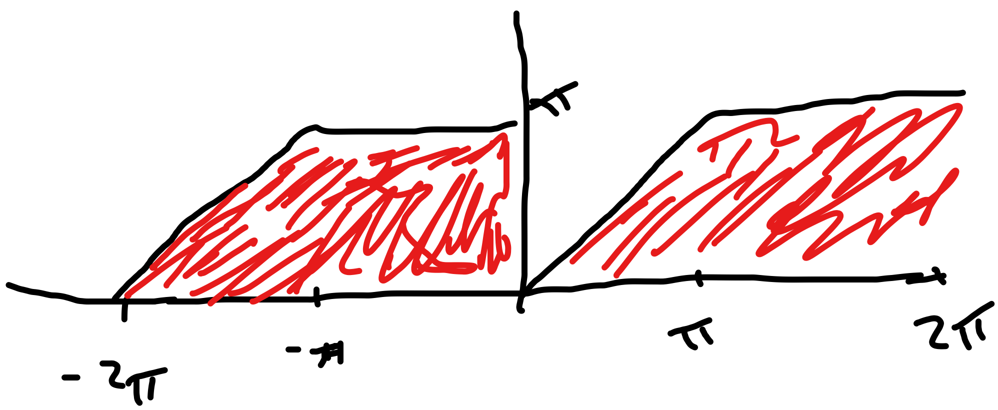
NB: This worksheet is an additional unseen exercise for MAU33E01. It is not for module credit and is intended only as an additional resource for exam preparation.

Question 1

$$f(x) = \begin{cases} x & 0 \leq x < \pi \\ \pi & \pi \leq x < 2\pi \end{cases}$$

(a) Assuming $f(x)$ satisfies $f(x) = f(x + 2\pi)$, sketch $f(x)$ in the interval $-2\pi \leq x \leq 2\pi$.
(b) Find the Fourier Series of the 2π periodic extension of $f(x)$.

(a)



(b)

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} \pi\,dx + \frac{1}{2\pi}\int_{0}^{\pi} x\,dx = \frac{3\pi}{4}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \underbrace{\frac{1}{\pi}\int_{-\pi}^{0}\pi\cos nx\,dx}_{\text{Integral vanishes } = 0} + \frac{1}{\pi}\int_{0}^{\pi} x\cos(nx)\,dx$$

Integrate by parts

$u = x \qquad dv = \cos nx\,dx$

$du = dx \qquad v = \frac{1}{n}\sin(nx)$

$$= \frac{1}{\pi}\left[\underbrace{\frac{1}{n}x\sin(nx)\Big|_{0}^{\pi}}_{\text{vanishes } = 0} - \int_{0}^{\pi}\frac{1}{n}\sin(nx)\,dx\right]$$

$$= -\frac{1}{\pi}\frac{1}{n}\int_{0}^{\pi}\sin(nx)\,dx = \frac{1}{n^2\pi^2}\cos(nx)\Big|_{0}^{\pi}$$

$$= \frac{1}{n^2\pi}\left(\cos(n\pi) - 1\right)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}\pi\sin(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi}x\sin(nx)\,dx$$

$$= -\frac{1}{n}\cos(nx)\Big|_{-\pi}^{0} + \frac{1}{\pi}\int_{0}^{\pi}x\sin(nx)\,dx$$

Cos even

$$= \frac{1}{n}\left(\cos(n\pi) - 1\right) - \frac{1}{n\pi}\pi\cos(n\pi)$$
$$+ \frac{1}{n\pi}\int_{0}^{\pi}\cos(nx)\,dx$$

$$= -\frac{1}{n} + \frac{1}{n\pi}\underbrace{\left[\frac{1}{n}\sin(nx)\right]_{0}^{\pi}}_{0} = -\frac{1}{n}$$

$$f(x) = \frac{3\pi}{4} + \sum_{n=1}^{\infty} \left(\frac{1}{n^2\pi} \left(\cos(n\pi) - 1 \right) \right.$$

$$\left. \times \cos(nx) - \frac{1}{n} \sin(nx) \right)$$

multiplication