① 2022 Exam - Techniques in Theoretical Physics

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Q2 (a) $x(x-1)y'' + 3xy' + y = 0$

Choose some point $x_0$ to expand solution around.

$x_0 = 0$

$$y'' + \frac{3x}{x(x-1)} + \frac{y}{x(x-1)} = 0$$

$$p(x) = \frac{3}{x-1} \qquad q(x) = \frac{1}{x(x-1)}$$

$p(x)$ ↓ Analytic @ $x_0 = 0$

$q(x)$ → Non analytic @ $x_0 = 0$

$x_0 = 0$ is singular point

$$x\,p(x) = \frac{3x}{x-1} \rightarrow \text{Analytic @ } x_0 = 0$$

$$x^2 q(x) = \frac{x}{x-1} \rightarrow \text{Analytic @ } x_0 = 0$$

$x_0 = 0$ is a regular singular point so we can assume a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Such that:

$$y'(x) = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

$$xy' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r}$$

$$x^2 y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r}$$

$$xy'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-1}$$

②

$$\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} - \underbrace{\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-1}}_{*}$$

$$+\sum_{n=0}^{\infty} 3a_n(n+r)x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

All terms involve $x^{n+r}$ except term (*)
we can transform (*) to contain $x^{n+r}$ using
the new index $m=n-1$

$$\sum_{m=-1}^{\infty} a_{m+1}(m+r+1)(m+r)x^{m+r}$$
$$= a_0 r(r-1)x^{r-1} + \sum_{n=0}^{\infty} a_{n+1}(n+r+1)(n+r)x^{n+r}$$

I just rellabeld "m" with "n".

Thus

$$-a_0 r(r-1)x^{r-1} + \sum_{n=0}^{\infty} a_n\left[(n+r)(n+r-1)+3(n+r)+1 - \frac{a_{n+1}(n+r+1)(n+r)}{a_n}\right]x^{n+r} = 0$$

Since $a_0 \neq 0$ $r(r-1)=0$ } indicial Equation

$r=0$ $r=1$

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③

$$a_n\left[(n+r)(n+r-1)-3(n+r)+1\right]-a_{n+1}(n+r+1)(n+r)=0$$

$$a_{n+1}=\frac{a_n\left[(n+r)(n+r-1)+3(n+r)+1\right]}{(n+r+1)(n+r)}$$

@ $r=0$ $\quad a_{n+1}=\frac{a_n\left[n(n-1)+3n+1\right]}{(n+1)n} \rightarrow$ No solution @ $n=0$

@ $r=1$

$$a_{n+1}=\frac{a_n\left[(n+1)n+3(n+1)+1\right]}{(n+2)(n+1)}$$

Set $a_0=1$

$$a_1=\frac{4}{2}a_0=2a_0$$

$$a_2=a_1\left[\frac{2+3(2)+1}{3(2)}\right]=\frac{9}{6}a_1=3a_0$$

⋮ So on..

④ $a_0 = 1$

$$y(x) = x\sum_{n=0}^{\infty} a_n x^n = x\left(1 + 2x + 3x^2 + \cdots\right) = \frac{x}{(1-x)^2}$$

ⓑ Closest singular point is $x=1$ So series converges for $x$ such that $|x|<1$

ⓒ Construct a second solution

Look for a second Solution $y_2$ of the form

$$y_2(x) = y(x)v(x) = \frac{x}{(1-x)^2} v(x)$$

- Sub $y_2$ into ODE
- First compute $y_2' = y'v + yv'$

$$y_2'' = y''v + y'v' + y'v' + yv''$$

$$x(x-1)\left[y''v + 2y'v' + yv''\right] + 3x(y'v + yv') + yv = 0$$

$$\underbrace{x(x-1)y''v + 3xy'v + yv}_{0 \text{ [original ODE]}} + x(x-1)\left[2y'v' + yv''\right] + 3xyv' = 0$$

$$\cancel{x(x-1)2\left[\frac{(1-x)^2 + x\,2(1-x)}{(1-x)^2}\right]v' + x(x-1)\frac{x}{(1-x)^2}v'' + \frac{3x^2}{(1-x)^2}v'} = 0$$

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⑤

$$x(x-1)2y'v' + x(x-1)yv'' + 3xyv' = 0$$

$$-x(1-x)2y'v' - x(1-x)yv'' + 3xyv' = 0$$

$$x(1-x)2y'v' + x(1-x)yv'' - 3xyv' = 0$$

$$v'' + \frac{2y'v'}{y} - \frac{3}{(1-x)}v' = 0$$

use $y'/y = \ln(y)'$

$$\ln(y) = \ln\left(\frac{x}{(1-x)^2}\right) = \ln(x) - 2\ln(1-x)$$

$$\frac{d}{dx}\ln(1-x) = \frac{1}{u}\cdot -1 = -\frac{1}{1-x}$$

$$\ln(y)' = \frac{1}{x} + \frac{2}{(1-x)} = \frac{1}{x} + \frac{2}{(1-x)}$$

$$v'' + \left(\frac{2}{x} + \frac{4}{(1-x)} - \frac{3}{(1-x)}\right)v' = 0$$

$$v'' + \left(\frac{2}{x} + \frac{1}{(1-x)}\right)v' = 0$$

First order ODE for $v'$ → use seperation of variables

$$\frac{dv'}{v'} = -\left(\frac{2}{x} + \frac{1}{(1-x)}\right)dx$$

$$\ln(v') = -2\ln(x) + \ln(x-1) + C_1$$

6) $\ln v' = -2\ln x + \ln(x-1) + C_1$

$= \ln x^{-2} + \ln(x-1) + \underbrace{C_1}_{\ln(C)} = \ln\left(C\,\frac{(x-1)}{x^2}\right)$

$\hookrightarrow$ Some other const

So $v' = C\,\frac{(x-1)}{x^2} = \frac{C}{x} - \frac{C}{x^2}$

Integrating

$v = C\ln(x) + \frac{C}{x} + C_2$

Then $y_2 = yv = \frac{x}{(1-x)^2}\left[C\left(\ln(x) + \frac{1}{x}\right) + C_2\right]$

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