EXERCISE SHEET 2

Exercise 1: Advanced shuffling

In lecture we discussed the shuffling formulas

- $\langle a, b, cd\rangle \subseteq \langle a, bc, d\rangle$,
- $\langle a, b, c\rangle d = a\langle b, c, d\rangle$ and
- $\langle a, b, c\rangle + \langle b, c, a\rangle + \langle c, a, b\rangle = 0$

and the situations in which they apply. In this exercise you will prove two additional shuffling formulas.

Let $\mathcal{C}$ be a stable category. Let

$$U \xrightarrow{a} V \xrightarrow{b} W \xrightarrow{c} X \xrightarrow{d} Y$$

be a sequence of four composable maps in $\mathcal{C}$ and suppose that all ab and bc are nullhomotopic.

(a) Show that

$$\langle a, b, c\rangle d \subseteq \langle a, b, cd\rangle.$$

(b) Identify which nullhomotopies one should use to obtain a equality.

Let

$$U \xrightarrow{a} V \xrightarrow{b} W \xrightarrow{c} X \xrightarrow{d} Y \xrightarrow{e} Z$$

be a sequence of five composable maps in $\mathcal{C}$ and let $e_2 : bc \leftrightarrow 0$ and $e_3 : cd \leftrightarrow 0$ be nullhomotopies. Suppose further that all of the following maps are nullhomotopic

$$ab, \quad de, \quad a\langle b, {}^{e_2}c, {}^{e_3}d\rangle, \quad \langle b, {}^{e_2}c, {}^{e_3}d\rangle e.$$

Consider the following expression:

$$\langle\langle a, b, c\rangle, d, e\rangle \pm \langle a, \langle b, c, d\rangle, e\rangle \pm \langle a, b, \langle c, d, e\rangle\rangle \tag{1}$$

(c) Confirm that the expression in (1) is defined, adding in the omitted suspensions as necessary.
(d) Determine an orientation of the signs so that (1) contains zero.
(e) Identify a collection of nullhomotopies so that the expression is exactly zero.

Background

The first 13 stable homotopy groups of the 2-local sphere $\mathbb{S}_{(2)}$ can be computed using only the Serre spectral sequence. These groups (and the ring structure) are given in the table below:

stem	group	generator	relations
0	$\mathbb{Z}$	1	
1	$\mathbb{Z}/2$	η	
2	$\mathbb{Z}/2$	η^2	
3	$\mathbb{Z}/8$	ν	$\eta^3 = 4\nu$
4	0		
5	0		
6	$\mathbb{Z}/2$	ν^2	
7	$\mathbb{Z}/16$	σ	
8	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\eta\sigma$, ϵ	
9	$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\eta^2\sigma$, $\eta\epsilon$, μ_9	$\nu^3 = \eta^2\sigma + \eta\epsilon$
10	$\mathbb{Z}/2$	$\eta\mu_9$	
11	$\mathbb{Z}/8$	ζ	$\eta^2\mu_9 = 4\zeta$
12	0		
13	0		

You will want to use this information as you do the next four exercises.

Exercise 2: Evaluating Toda brackets

Toda proved the following formula[1] for Toda brackets of elements in $\pi_*\mathbb{S}$: Given $\alpha, \beta \in \pi_*\mathbb{S}$ such that $\alpha\beta = 0$

- If $|\alpha| = k$ is even (or k is odd and $2\alpha = 0$), there is an element $Q_1(\alpha) \in \pi_{2k+1}\mathbb{S}$ such that
$$\langle \alpha, \beta, \alpha \rangle = Q_1(\alpha)\beta + \alpha\blacksquare.$$
Here $\blacksquare$ is an ideterminate used for keeping track of indeterminacy.
- If $|\alpha| = k$ is odd, then
$$\langle \alpha, \beta, \alpha \rangle \quad \text{and} \quad \langle \beta, \alpha, 2\alpha \rangle.$$
have non-empty intersection.

For elements in even degree $Q_1(-)$ is a (not necessarily additive) operation of signature
$$\pi_k\mathbb{S} \to \pi_{2k+1}\mathbb{S}.$$
In fact, for $\alpha_1, \alpha_2 \in \pi_k\mathbb{S}$ with k even we have
$$Q_1(\alpha_1 + \alpha_2) = Q_1(\alpha_1) + Q_1(\alpha_2) + \eta\alpha_1\alpha_2$$
and for $\alpha_1 \in \pi_{k_1}\mathbb{S}$ and $\alpha_2 \in \pi_{k_2}\mathbb{S}$ with k_1, k_2 even we have
$$Q_1(\alpha_1\alpha_2) = \alpha_1^2 Q_1(\alpha_2) + Q_1(\alpha_1)\alpha_2^2$$

Using the table above, shuffling rules and Toda's formula evaluate the following Toda brackets. Remember to keep indeterminacy in mind.

(0) Compute $Q_1(n)$ for each $n \in \mathbb{Z} \cong \pi_0\mathbb{S}$.
(a) $\langle 2, \eta, 2 \rangle$
(b) $\langle \eta, 2, \eta \rangle$
(c) $\langle 2, \eta, \nu \rangle$
(d) $\langle \eta, \nu, \eta \rangle$
(e) $\langle \nu, \eta, \nu \rangle$

[1] This formula comes from the theory of power operations on $\mathbb{S}$ and we will return to this subject at the end of the course.

(f) $\langle \eta^2, \eta^2, \eta^2 \rangle$
(g) $\langle \eta, 2, \nu^2 \rangle$
(h) $\langle \mu_9, 2, \eta \rangle$

1. **Exercise 3: Cells structures**

Like spaces, spectra can be given skeletal filtrations.

Definition 1.1. A skeletal filtration on X is a sequence

$$\cdots \to X_{-2} \to X_{-1} \to X_0 \to X_1 \to X_2 \to \cdots$$

such that

(1) $\mathrm{cof}(X_{i-1} \to X_i)$ is a sum of copies of $\mathbb{S}^i$,
(2) $0 \cong X_{-\infty} \coloneqq \lim X_i$ and
(3) $X \cong X_\infty \coloneqq \mathrm{colim}\, X_i$.

Alternatively, we can rotate the cofiber sequences in a skeletal filtration and view the pushout

$$\begin{tikzcd} \oplus \mathbb{S}^{i-1} \arrow[r] \arrow[d] & X_{i-1} \arrow[d] \\ 0 \arrow[r] & X_i \end{tikzcd}$$

as specifying a collection of cell attachments that build X_i from X_{i-1}.[2]

(a) Prove that every 2-local finite spectrum X can be given a skeletal filtration with exactly $\dim_{\mathbb{F}_2}(H_i(X;\mathbb{F}_2))$ many i-cells. We call a choice of such a skeletal filtration a minimal skeletal filtration. (Hint: use the Hurewicz theorem)
(b) Extend the results of part (a) to spectra which are of finite type.[3]

The homotopy and homology rings of the spectrum $\mathbb{F}_2$ representing cohomology with $\mathbb{F}_2$ coefficients are given by

$$\pi_*\mathbb{F}_2 \cong \begin{cases} \mathbb{F}_2 & * = 0 \\ 0 & * \neq 0 \end{cases} \qquad H_*(\mathbb{F}_2;\mathbb{F}_2) \coloneqq \pi_*(\mathbb{F}_2 \otimes \mathbb{F}_2) \cong \mathbb{F}_2[\zeta_1, \zeta_2, \dots]$$

where $|\zeta_i| = 2^i - 2$.

(c) Determine the attaching maps in the minimal 0-skeleton of $\mathbb{F}_2$.
(d) Determine the attaching maps in the minimal 1-skeleton of $\mathbb{F}_2$.
(e) Determine the attaching maps in the minimal 2-skeleton of $\mathbb{F}_2$.
(f) Determine the attaching maps in the minimal 3-skeleton of $\mathbb{F}_2$.

The homotopy and homology rings of the spectrum $\mathbb{Z}_{(2)}$ representing cohomology with $\mathbb{Z}_{(2)}$ coefficients are given by

$$\pi_*\mathbb{F}_2 \cong \begin{cases} \mathbb{Z}_{(2)} & * = 0 \\ 0 & * \neq 0 \end{cases} \qquad H_*(\mathbb{Z}_{(2)};\mathbb{F}_2) \coloneqq \pi_*(\mathbb{Z}_{(2)} \otimes \mathbb{F}_2) \cong \mathbb{F}_2[\zeta_1^2, \zeta_2, \dots]$$

where $|\zeta_1^2| = 2$ and $|\zeta_i| = 2^i - 2$ for $i \geq 2$. The name of the class ζ_1^2 comes from the fact that the reduction mod 2 map

$$\mathbb{Z}_{(2)} \to \mathbb{F}_2$$

[2] Recall that discs are contractible, which is why the bottom left is zero.
[3] X is of finite type if it is bounded below and $H_i(X;\mathbb{Z}_{(2)})$ is finitely generated for all i.

induces the natural injective map of polynomial algebras

$$\mathbb{F}_2[\zeta_1^2, \zeta_2, \dots] \to \mathbb{F}_2[\zeta_1, \zeta_2, \dots]$$

on $\mathbb{F}_2$-homology.

(g) Determine the attaching maps in the minimal 2-skeleton of $\mathbb{Z}_{(2)}$.
(h) Determine the attaching maps in the minimal 3-skeleton of $\mathbb{F}_2$.

2. Exercise 4: Characterizing KU

KU is an $\mathbb{E}_\infty$-algebra in Sp whose associated cohomology theory classifies stable complex vector bundles. The ring spectrum $\mathrm{KU}_{(2)}$ satisfies the following properties

(1) $\mathrm{KU}_{(2)}$ is 2-periodic. This means that there is an invertible class β in $\pi_2\mathrm{KU}_{(2)}$.
(2) The connective cover of $\mathrm{KU}_{(2)}$ is typically called $\mathrm{ku}_{(2)}$. The $\mathbb{E}_\infty$-algebra $\mathrm{ku}_{(2)}$ is finite type and its $\mathbb{F}_2$-homology is
$$H_*(\mathrm{ku}_{(2)}; \mathbb{F}_2) \coloneqq \pi_*(\mathbb{F}_2 \otimes \mathrm{ku}_{(2)}) \cong \mathbb{F}_2[\zeta_1^2, \zeta_2^2, \zeta_3, \zeta_4, \dots]$$
where $|\zeta_1^2| = 2$, $|\zeta_2^2| = 6$ and $|\zeta_i| = 2^i - 2$ for $i \geq 3$.
(3) There is a map of $\mathbb{E}_\infty$-algebras $\mathrm{ku}_{(2)} \to \mathbb{Z}_{(2)}$ which induces the natural inclusion
$$\mathbb{F}_2[\zeta_1^2, \zeta_2^2, \zeta_3, \zeta_4, \dots] \to \mathbb{F}_2[\zeta_1^2, \zeta_2, \dots]$$
on $\mathbb{F}_2$-homology.

(a) Show that the localization of $\mathrm{ku}_{(2)}$ at β is $\mathrm{KU}_{(2)}$.
(b) Determine the cells and attaching maps of a minimal 4 skeleton of $\mathrm{ku}_{(2)}$.
(c) Using (b), compute $\pi_*\mathrm{ku}_{(2)}$ for $* = 0, 1, 2$.
(d) Determine the ring $\pi_*\mathrm{KU}_{(2)}$.
(e) Describe β as a map into the 2-skeleton of $\mathrm{ku}_{(2)}$.
(f) Conclude that properties (1), (2) and (3) above characterize $\mathrm{KU}_{(2)}$.
(g) How much can you weaken properties (1), (2) and (3) while still being able to prove (f)?