EXERCISE SHEET 3

Background

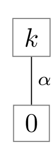
The first 13 stable homotopy groups of the 2-local sphere $\mathbb{S}_{(2)}$ can be computed using only the Serre spectral sequence. These groups (and the ring structure) are given in the table below:

stem	group	generator	relations
0	$\mathbb{Z}$	1	
1	$\mathbb{Z}/2$	η	
2	$\mathbb{Z}/2$	η^2	
3	$\mathbb{Z}/8$	ν	$\eta^3 = 4\nu$
4	0		
5	0		
6	$\mathbb{Z}/2$	ν^2	
7	$\mathbb{Z}/16$	σ	
8	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\eta\sigma$, ϵ	
9	$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\eta^2\sigma$, $\eta\epsilon$, μ_9	$\nu^3 = \eta^2\sigma + \eta\epsilon$
10	$\mathbb{Z}/2$	$\eta\mu_9$	
11	$\mathbb{Z}/8$	ζ	$\eta^2\mu_9 = 4\zeta$
12	0		
13	0		

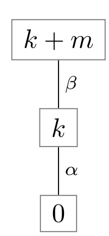
1. Exercise 1: Constructing finite spectra

In this exercise we will focus on constructing finite spectra. We begin with the general theory.

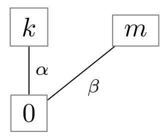
(a) Given a map $\alpha : \mathbb{S}^k \to \mathbb{S}^0$ when can we construct a finite spectrum with a cell structure as depicted below?



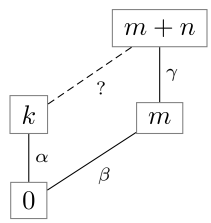
(b) Given a pair of classes $\alpha \in \pi_k\mathbb{S}$ and $\beta \in \pi_m\mathbb{S}$ when can we construct a finite spectrum with a cell structure as depicted below?



(c) Given a pair of classes $\alpha \in \pi_k\mathbb{S}$ and $\beta \in \pi_m\mathbb{S}$ when can we construct a finite spectrum with a cell structure as depicted below?

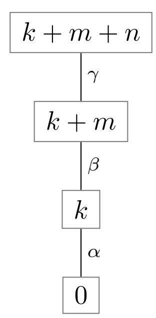


(d) Given three classes $\alpha \in \pi_k\mathbb{S}$, $\beta \in \pi_m\mathbb{S}$ and $\gamma \in \pi_n\mathbb{S}$ when can we construct a finite spectrum with a cell structure as depicted below?

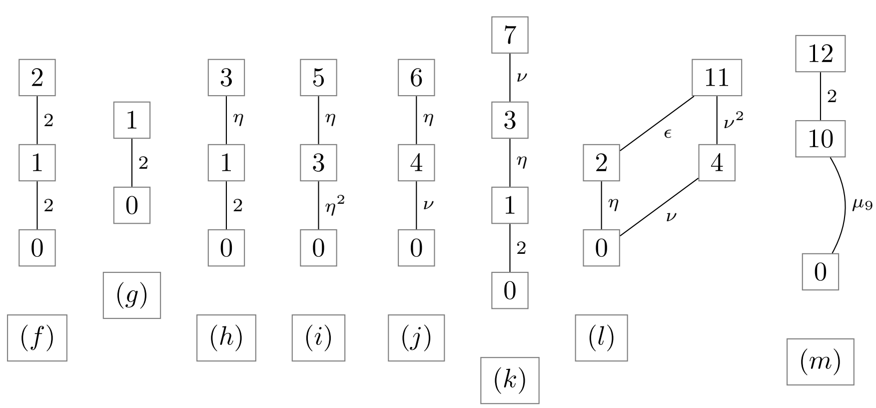


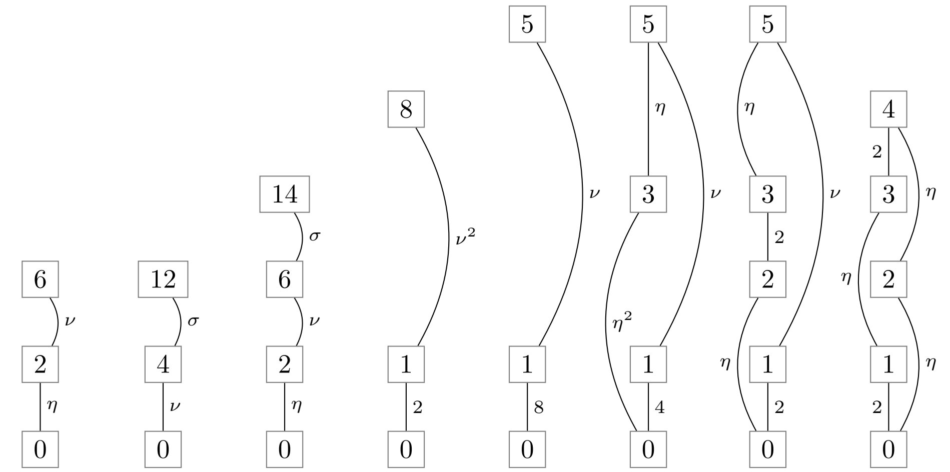
What requirements are there on the dashed attaching map?

(e) Given three classes $\alpha \in \pi_k\mathbb{S}$, $\beta \in \pi_m\mathbb{S}$ and $\gamma \in \pi_n\mathbb{S}$ when can we construct a finite spectrum with a cell structure as depicted below?

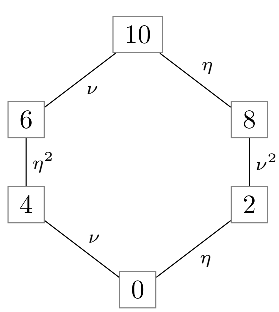


For the remaining parts of this question determine whether finite spectra with the cell structures depicted below exist.

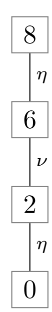




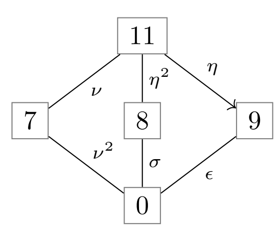
(n) (o) (p) (q) (r) (s) (t) (u)



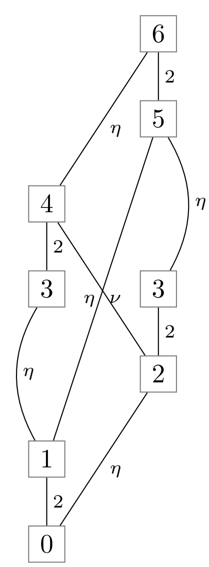
(v)



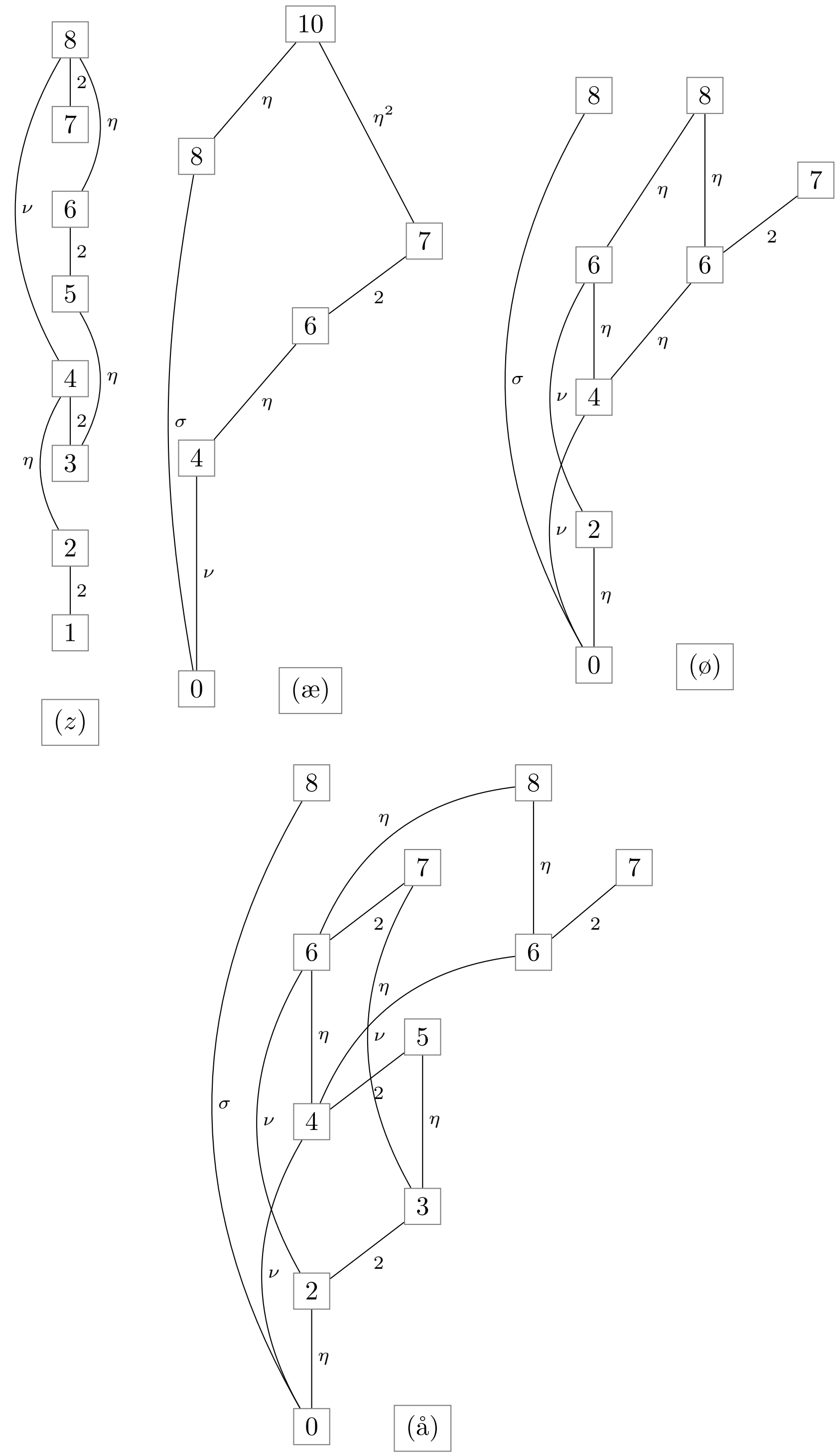
(w)



(x)



(y)



(z) (æ) (ø) (å)

Hint: part (æ) should be more difficult. Cross-reference this with Ex 3.b.

2. Exercise 2: Homotopy groups of finite spectra

(a) Determine $\pi_* X$ for $* \leq 10$ where X is the spectrum depicted below.

$$\begin{tikzcd} \boxed{1} \arrow[d, dash, "2"] \\ \boxed{0} \end{tikzcd}$$

(b) Determine $\pi_* X$ for $* \leq 10$ where X is the spectrum depicted below.

$$\begin{tikzcd} \boxed{2} \arrow[d, dash, "\eta"] \\ \boxed{0} \end{tikzcd}$$

(c) Determine $\pi_* X$ for $* \leq 10$ where X is the spectrum depicted below.

$$\begin{tikzcd} \boxed{4} \arrow[d, dash, "\nu"] \\ \boxed{0} \end{tikzcd}$$

(d) Determine $\pi_* X$ for $* \leq 8$ where X is the spectrum depicted below.

$$\begin{tikzcd} \boxed{3} \arrow[d, dash, "2"] \\ \boxed{2} \arrow[d, dash, "\eta"] \\ \boxed{0} \end{tikzcd}$$

(e) Determine $\pi_* X$ for $* \leq 7$ where X is the spectrum from Ex. 1.æ.
(f) Determine $\pi_* X$ for $* \leq 7$ where X is the spectrum from Ex. 1.å.

3. Exercise 3: The homotopy of KO (cellular approach)

KO is an $\mathbb{E}_\infty$-algebra in Sp whose associated cohomology theory classifies stable vector bundles. The ring spectrum $\mathrm{KO}_{(2)}$ satisfies the following properties:

(1) $\mathrm{KO}_{(2)}$ is 8-periodic. This means that there is an invertible class β in $\pi_8\mathrm{KU}_{(2)}$.
(2) The connective cover of $\mathrm{KO}_{(2)}$ is typically called $\mathrm{ko}_{(2)}$. The $\mathbb{E}_\infty$-algebra $\mathrm{ko}_{(2)}$ is finite type and its $\mathbb{F}_2$-homology is

$$H_*(\mathrm{ku}_{(2)};\mathbb{F}_2) := \pi_*(\mathbb{F}_2 \otimes \mathrm{ku}_{(2)}) \cong \mathbb{F}_2[\zeta_1^4, \zeta_2^2, \zeta_3, \zeta_4, \dots]$$

where $|\zeta_1^4| = 4$, $|\zeta_2^2| = 6$ and $|\zeta_i| = 2^i - 1$ for $i \geq 3$.
(3) There are maps of $\mathbb{E}_\infty$-algebras $\mathrm{ko}_{(2)} \to \mathrm{ku}_{(2)} \to \mathbb{Z}_{(2)}$ which induces the natural inclusions

$$\mathbb{F}_2[\zeta_1^4, \zeta_2^2, \zeta_3, \zeta_4, \dots] \to \mathbb{F}_2[\zeta_1^2, \zeta_2^2, \zeta_3, \zeta_4, \dots] \to \mathbb{F}_2[\zeta_1^2, \zeta_2, \dots]$$

on $\mathbb{F}_2$-homology.

Using properties (1), (2) and (3) you will now calculate the homotopy groups of $\mathrm{ko}_{(2)}$.

(a) Show that the localization of $\mathrm{ko}_{(2)}$ at β is $\mathrm{KO}_{(2)}$.
(b) Determine the cells and attaching maps of a minimal 8-skeleton of $\mathrm{ko}_{(2)}$.
(c) Compute $\pi_*\mathrm{ko}_{(2)}$ for $* \leq 7$.

(d) Determine the ring $\pi_*\mathrm{KO}_{(2)}$.
(e) Let α be a generator of $\pi_4\mathrm{ko}_{(2)}$. Describe α as a map into the 8-skeleton of $\mathrm{ko}_{(2)}$.
(f) Describe β as a map into the 8-skeleton of $\mathrm{ko}_{(2)}$.

Hint: For part (b) you will want to start by thinking about a minimal 8-skeleton of $\mathbb{Z}_{(2)}$ and the map $\mathrm{ko}_{(2)} \to \mathbb{Z}_{(2)}$.

4. **Exercise 4: The Wood cofiber sequence**

KO is an $\mathbb{E}_\infty$-algebra in Sp whose associated cohomology theory classifies stable vector bundles. Complexification gives a map of $\mathbb{E}_\infty$-algebras

$$\mathrm{KO} \to \mathrm{KU}.$$

(a) Prove that $\mathrm{ko}_{(2)} \otimes \mathrm{Cof}(\eta) \cong \mathrm{ku}_{(2)}$.