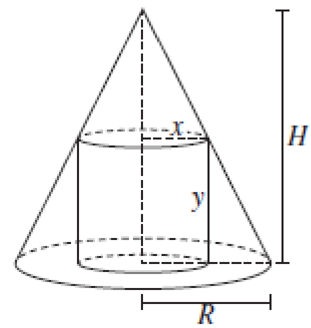
2015 16 c The diagram shows a cylinder of radius x and height y inscribed in a cone of radius R and height H, where R and H are constants.



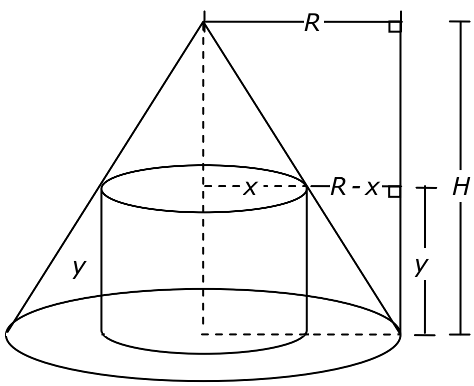
The volume of a cone of radius r and height h is $\frac{1}{3}\pi r^2 h$.

The volume of a cylinder of radius r and height h is $\pi r^2 h$.

(i) Show that the volume, V, of the cylinder can be written as $V = \frac{H}{R}\pi x^2(R - x)$. **3**

(ii) By considering the inscribed cylinder of maximum volume, show that the volume of any inscribed cylinder does not exceed $\frac{4}{9}$ of the volume of the cone. **4**

(i)



Consider sides of similar triangles:

$$\frac{y}{H} = \frac{R - x}{R} \quad \text{(matching sides of similar } \Delta \text{ s)}$$

$$y = \frac{H(R - x)}{R}$$

Now, volume of cylinder:

$$V = \pi x^2 y$$

$$= \pi x^2 \frac{H(R - x)}{R}$$

$$= \frac{H}{R}\pi x^2(R - x)$$

State Mean: **0.49**

(ii)

$$V = \frac{H}{R}\pi x^2(R - x)$$

$$\frac{dV}{dx} = \frac{H\pi}{R}[2x(R - x) + x^2(-1)]$$

$$= \frac{H\pi}{R}[2xR - 2x^2 - x^2]$$

$$\frac{dV}{dx} = \frac{H\pi}{R}[2xR - 3x^2] = 0$$

$$\therefore \ 2xR - 3x^2 = 0$$

$$x(2R - 3x) = 0$$

$$x = \frac{2R}{3} \quad (\text{as } x > 0)$$

$$\frac{d^2V}{dx^2} = \frac{H\pi}{R}[2R - 6x]$$

$$\frac{d^2V}{dx^2}\left[\frac{2R}{3}\right] = \frac{H\pi}{R}\left[2R - 6\left(\frac{2R}{3}\right)\right] < 0 \quad (H > 0,\ R > 0)$$

$\therefore$ maximum volume when $x = \frac{2R}{3}$.

Substitute $x = \frac{2R}{3}$ in $V = \frac{H}{R}\pi x^2(R - x)$:

$$V = \frac{H}{R}\pi\left(\frac{2R}{3}\right)^2\left(R - \frac{2R}{3}\right)$$

$$= \frac{H\pi}{R} \times \frac{4R^2}{9} \times \frac{R}{3}$$

$$\therefore V_{\text{cylinder}} = \frac{4H\pi R^2}{27}$$

Now, volume of cone with radius R and height H:

$$V_{\text{cone}} = \frac{1}{3}\pi R^2 H$$

But, $\frac{4}{9} \times \frac{1}{3}\pi R^2 H = \frac{4H\pi R^2}{27}$.

$\therefore$ vol of cylinder does not exceed $\frac{4}{9}$ of vol of cone.

State Mean: **0.85**

* These solutions have been provided by *projectmaths* and are not supplied or endorsed by BOSTES.

Board of Studies: Notes from the Marking Centre

(c)(i) This part was found to be challenging. In the better responses, candidates used relationships of similar triangles.

Common problems were:

- working backwards from the equation given in the question to arrive at an expression for y and then using it to 'show' the given expression
- not recognising the correct matching sides of similar triangles
- subtracting the volume of the cylinder from the volume of the cone
- using Pythagoras's theorem
- assuming that $R = 2x$ and/or $H = 2y$.

(c)(ii) This part was found to be challenging. Candidates are reminded that expanding the expression, where possible, before differentiating, is often easier than using the product rule.

Common problems were:

- using the product rule incorrectly to differentiate
- incorrectly solving $\frac{dV}{dx} = 0$
- omitting a test to establish a maximum
- not comparing the volumes of the cylinder and the cone.