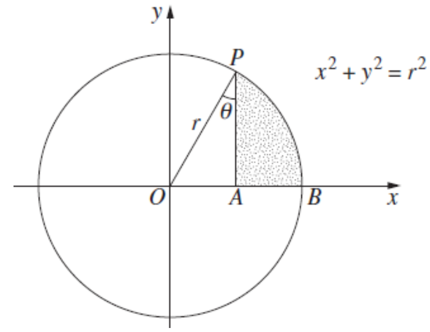
10 **10 b**

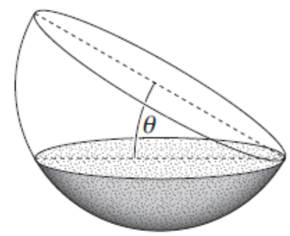
The circle $x^2 + y^2 = r^2$ has radius r and centre O. The circle meets the positive x-axis at B. The point A is on the interval OB. A vertical line through A meets the circle at P.

Let $\theta = \angle OPA$.



(i) The shaded region bounded by the arc PB and the intervals AB and AP is rotated about the x-axis. Show that the volume, V, formed is given by $V = \dfrac{\pi r^3}{3}(2 - 3\sin\theta + \sin^3\theta)$. **3**

(ii) A container is in the shape of a hemisphere of radius r metres. The container is initially horizontal and full of water. The container is then tilted at an angle of θ to the horizontal so that some water spills out.

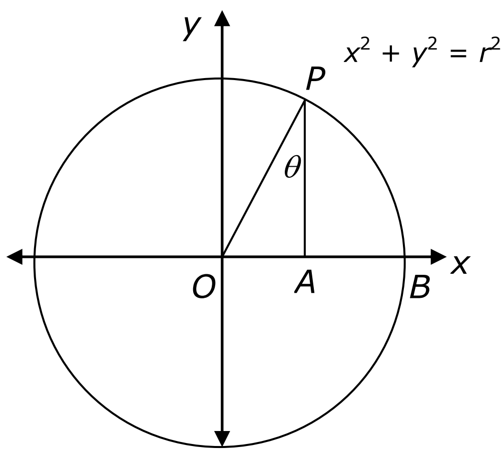


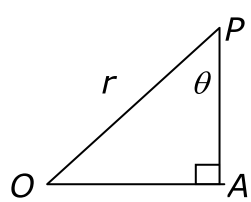
(1) Find θ so that the depth of water remaining is one half of the original depth. **1**

(2) What fraction of the original volume is left in the container? **2**

State Mean:
0.48/3
0.03/1
0.04/2

(i)





Using $\triangle PAO$, $\dfrac{OA}{r} = \sin\theta$

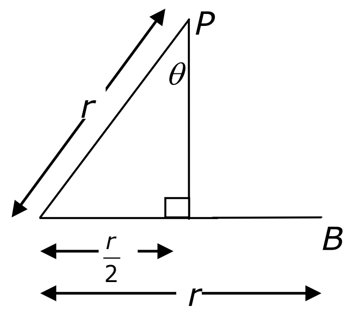
$OA = r\sin\theta$

Also, as $x^2 + y^2 = r^2$

$\therefore\ y^2 = r^2 - x^2$

Now, $V = \pi \displaystyle\int_{r\sin\theta}^{r} r^2 - x^2 \ dx$

(ii) (1)



As $OB = r$, then $OA = \dfrac{r}{2}$ (half-depth)

$$\therefore\ \sin\theta = \frac{\frac{r}{2}}{r}$$

$$= \frac{r}{2} \div r$$

$$= \frac{1}{2}$$

$$\therefore\ \theta = 30^\circ$$

(2) Vol. of hemisphere $= \dfrac{1}{2} \times \dfrac{4}{3}\pi r^3$

$= \dfrac{2}{3}\pi r^3$

$\therefore$ volume of original hemisphere is $\dfrac{2}{3}\pi r^3$ units^3

When water is original depth:
subs $\theta = 30^\circ$ in V:

$= \pi \left[r^2 x - \frac{x^3}{3} \right]_{r\sin\theta}^{r}$ $= \pi \, [r^3 - \frac{r^3}{3} - (r^3 \sin\theta - \frac{r^3 \sin^3\theta}{3})]$ $= \pi \, [\frac{2r^3}{3} - r^3 \sin\theta + \frac{r^3 \sin^3\theta}{3}]$ $= \frac{\pi r^3}{3} [2 - 3\sin\theta + \sin^3\theta]$	$V = \frac{\pi r^3}{3} [2 - 3\sin 30^\circ + \sin^3 30^\circ]$ $= \frac{\pi r^3}{3} [2 - \frac{3}{2} + \frac{1}{8}]$ $= \frac{5\pi r^3}{24}$ $\therefore$ new volume is $\frac{5\pi r^3}{24}$ units^3. $\therefore$ Fraction $= \frac{5\pi r^3}{24} \div \frac{2}{3}\pi r^3$ $= \frac{5\pi r^3}{24} \times \frac{3}{2\pi r^3}$ $= \frac{5}{16}$

* These solutions have been provided by *projectmaths* and are not supplied or endorsed by the Board of Studies

Board of Studies: Notes from the Marking Centre

Although most candidates realised the relevance of the formula $V = \pi \int y^2 dx$ in part (i), many could not find the correct limits of integration or an appropriate primitive. Common errors were to integrate r^2 to $\frac{1}{3}r^3$ rather than $r^2 x$ or to claim that the lower limit of integration was $r - r\sin(\theta)$. Once again the crucial connection between parts (i) and (ii) was used by many candidates.

A common error in part (ii)(1) was to confuse depth with volume with the resulting equations quickly spiralling out of control. Responses which simply stated a value for θ or measured this angle off the diagram could not be rewarded.

Candidates could still gain full marks in part (ii)(2) by correctly implementing an incorrect angle from part (ii)(1); however, full marks at this stage of the paper were rare, with many candidates clearly running out of time.

Source: http://www.boardofstudies.nsw.edu.au/hsc_exams/