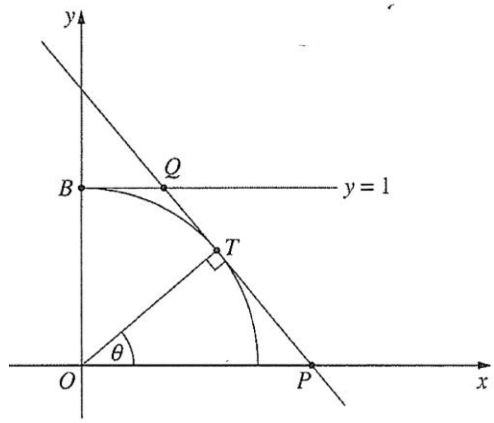
12 **16b**

The diagram shows a point T on the unit circle $x^2 + y^2 = 1$ at angle θ from the positive x-axis, where $0 < \theta < \frac{\pi}{2}$. The tangent to the circle at T is perpendicular to OT, and intersects the x-axis at P, and the line $y = 1$ at Q. The line $y = 1$ intersects the y-axis at B.

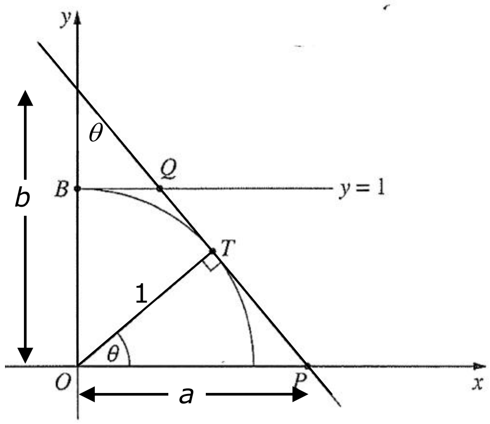


(i) Show that the equation of the line PT is $x\cos\theta + y\sin\theta = 1$. **2**

(ii) Find the length of BQ in terms of θ. **1**

(iii) Show that the area, A, of the trapezium $OPQB$ is given by $A = \dfrac{2-\sin\theta}{2\cos\theta}$. **2**

(iv) Find the angle θ that gives the minimum area of the trapezium. **3**



(i) Using $\dfrac{x}{a} + \dfrac{y}{b} = 1$:

In ΔOPT, is $\dfrac{OT}{OP} = \cos\theta$

$\dfrac{1}{a} = \cos\theta$

$a = \dfrac{1}{\cos\theta}$

In ΔORT, is $\dfrac{OT}{OR} = \cos\theta$

$\dfrac{1}{b} = \sin\theta$

$b = \dfrac{1}{\sin\theta}$

$\therefore$ equation is $\dfrac{x}{\frac{1}{\cos\theta}} + \dfrac{y}{\frac{1}{\sin\theta}} = 1$

$x\cos\theta + y\sin\theta = 1$ ......(1)

(ii) Subs $y = 1$ in (1):

$x\cos\theta + \sin\theta = 1$

$x\cos\theta = 1 - \sin\theta$

(iii) Area $= \dfrac{1}{2}h(a + b)$

$A = \dfrac{1}{2} \times 1 \times \left(\dfrac{1-\sin\theta}{\cos\theta} + \dfrac{1}{\cos\theta}\right)$

$= \dfrac{1}{2} \times \left(\dfrac{2-\sin\theta}{\cos\theta}\right)$

$= \dfrac{2-\sin\theta}{2\cos\theta}$

$\therefore A = \dfrac{2-\sin\theta}{2\cos\theta}$ units^2

State Mean:
0.39/2
0.30/1
0.47/2
0.58/3

(iv) $A = \dfrac{2-\sin\theta}{2\cos\theta}$

$$\frac{dA}{d\theta} = \frac{2\cos\theta(-\cos\theta) - (2-\sin\theta).-2\sin\theta}{4\cos^2\theta}$$

$$= \frac{-2\cos^2\theta + 4\sin\theta - 2\sin^2\theta}{4\cos^2\theta}$$

$$= \frac{4\sin\theta - 2(\sin^2\theta + \cos^2\theta)}{4\cos^2\theta}$$

$$= \frac{4\sin\theta - 2}{4\cos^2\theta}$$

$$= \frac{2(2\sin\theta - 1)}{4\cos^2\theta}$$

$$= \frac{(2\sin\theta - 1)}{2\cos^2\theta} = 0$$

$2\sin\theta - 1 = 0$

$\sin\theta = \dfrac{1}{2}$

$\theta = \dfrac{\pi}{6}$ (as $0 < \theta < \frac{\pi}{2}$)

HSC examination papers © Board of Studies NSW for and on behalf of the Crown in right of State of New South Wales

$$x = \frac{1 - \sin\theta}{\cos\theta}$$

$\therefore Q(\frac{1 - \sin\theta}{\cos\theta}, 1)$

As $B(0, 1)$, then $BQ = \dfrac{1 - \sin\theta}{\cos\theta}$

Test for min:

θ	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{3}$
$\frac{dA}{d\theta}$	< 0	0	> 0

$\therefore \ \theta = \frac{\pi}{6}$ for minimum

* These solutions have been provided by *projectmaths* and are not supplied or endorsed by the Board of Studies

Board of Studies: Notes from the Marking Centre

(i) In better responses, candidates who found the coordinates of the points T or P were generally able to progress towards the desired expression. In a number of responses, candidates struggled with finding the gradient of the tangent. The most common error was to assume that the gradient of OT was 1, meaning that the gradient of the tangent was -1. In most responses, candidates generally used the point-gradient formula of a line to derive the desired result, with only a small number using the two-point form. Candidates using the second approach had problems with the algebraic manipulation required.

(ii) In most responses, candidates realised that they needed to substitute $y = 1$ into the equation of the line PT to derive the co-ordinates of Q. In a number of responses, candidates first rearranged the equation PT in terms of y, and then used $y = 1$ to achieve the result. In a significant number of responses, candidates failed to realise that because BQ was horizontal, with B lying on the y-axis, the x-coordinate of Q was also the length of BQ. In these responses, candidates tended to use the distance formula with the coordinates of B and Q, involving a great deal of unnecessary working that often resulted in errors.

(iii) In most responses, candidates found the length of OP and used their answer from part (ii) to get the correct expression for the area of the trapezium. In weaker responses, candidates who encountered difficulties either used an incorrect length for BQ or an incorrect formula for the area of the trapezium.

HSC examination papers © Board of Studies NSW for and on behalf of the Crown in right of State of New South Wales

(iv) In the majority of responses, candidates recognised the need to use calculus in this question. The most common, and successful, method was to use the quotient rule and solve $\frac{dA}{d\theta} = 0$. In many responses, candidates used the breakdown for u, u', v and v' but few wrote the quotient rule. While a substantial number of candidates wrote the derivative expressions correctly, errors occurred in the simplifications. Common errors included incorrect signs when expanding, uv' first rather than $u'v$, having an incorrect denominator or failing to include the denominator in the derivative. In many responses, candidates did not solve the derivative equal to zero correctly due to poor algebraic skills. Quite often the test for the nature of the stationary point was omitted. Of those who attempted to test for a minimum, the change of sign of the first derivative was the most common method. Very few candidates who attempted to use the 'second derivative test' obtained the correct answer. Often, the second derivative was not correctly determined. Candidates are reminded to ensure that their solution fits into the given domain and that they explicitly show that it is a minimum.

HSC examination papers © Board of Studies NSW for and on behalf of the Crown in right of State of New South Wales

(ii) In most responses, candidates realised that they needed to substitute $y=1$ into the equation of the line PT to derive the co-ordinates of Q. In a number of responses, candidates first rearranged the equation PT in terms of y, and then used $y=1$ to achieve the result. In a significant number of responses, candidates failed to realise that because BQ was horizontal, with B lying on the y-axis, the x-coordinate of Q was also the length of BQ. In these responses, candidates tended to use the distance formula with the coordinates of B and Q, involving a great deal of unnecessary working that often resulted in errors.

(iii) In most responses, candidates found the length of OP and used their answer from part (ii) to get the correct expression for the area of the trapezium. In weaker responses, candidates who encountered difficulties either used an incorrect length for BQ or an incorrect formula for the area of the trapezium.

(iv) In the majority of responses, candidates recognised the need to use calculus in this question. The most common, and successful, method was to use the quotient rule and solve $\frac{dA}{d\theta}=0$. In many responses, candidates used the breakdown for u, u', v and v' but few wrote the quotient rule. While a substantial number of candidates wrote the derivative expressions correctly, errors occurred in the simplifications. Common errors included incorrect signs when expanding, uv' first rather than $u'v$, having an incorrect denominator or failing to include the denominator in the derivative. In many responses, candidates did not solve the derivative equal to zero correctly due to poor algebraic skills. Quite often the test for the nature of the stationary point was omitted. Of those who attempted to test for a minimum, the change of sign of the first derivative was the most common method. Very few candidates who attempted to use the 'second derivative test' obtained the correct answer. Often, the second derivative was not correctly determined. Candidates are reminded to ensure that their solution fits into the given domain and that they explicitly show that it is a minimum.

HSC examination papers © Board of Studies NSW for and on behalf of the Crown in right of State of New South Wales

(ii) In most responses, candidates realised that they needed to substitute $y = 1$ into the equation of the line PT to derive the co-ordinates of Q. In a number of responses, candidates first rearranged the equation PT in terms of y, and then used $y = 1$ to achieve the result. In a significant number of responses, candidates failed to realise that because BQ was horizontal, with B lying on the y-axis, the x-coordinate of Q was also the length of BQ. In these responses, candidates tended to use the distance formula with the coordinates of B and Q, involving a great deal of unnecessary working that often resulted in errors.

(iii) In most responses, candidates found the length of OP and used their answer from part (ii) to get the correct expression for the area of the trapezium. In weaker responses, candidates who encountered difficulties either used an incorrect length for BQ or an incorrect formula for the area of the trapezium.

(iv) In the majority of responses, candidates recognised the need to use calculus in this question. The most common, and successful, method was to use the quotient rule and solve $\frac{dA}{d\theta} = 0$. In many responses, candidates used the breakdown for u, u', v and v' but few wrote the quotient rule. While a substantial number of candidates wrote the derivative expressions correctly, errors occurred in the simplifications. Common errors included incorrect signs when expanding, uv' first rather than $u'v$, having an incorrect denominator or failing to include the denominator in the derivative. In many responses, candidates did not solve the derivative equal to zero correctly due to poor algebraic skills. Quite often the test for the nature of the stationary point was omitted. Of those who attempted to test for a minimum, the change of sign of the first derivative was the most common method. Very few candidates who attempted to use the 'second derivative test' obtained the correct answer. Often, the second derivative was not correctly determined. Candidates are reminded to ensure that their solution fits into the given domain and that they explicitly show that it is a minimum.

Source: http://www.boardofstudies.nsw.edu.au/hsc_exams/

HSC examination papers © Board of Studies NSW for and on behalf of the Crown in right of State of New South Wales