

13.1 Index laws

You are familiar with the following index laws:

$$1. a^m \times a^n = a^{m+n}$$

$$2. \frac{a^m}{a^n} = a^{m-n}$$

$$3. (a^m)^n = a^{mn}$$

$$4. (ab)^n = a^n \cdot b^n$$

$$5. a^0 = 1$$

$$6. a^{-n} = \frac{1}{a^n}$$

$$7. a^{\frac{m}{n}} = \sqrt[n]{a^m} \text{ or } (\sqrt[n]{a})^m$$

$$8. a^{-\frac{m}{n}} = \frac{1}{\sqrt[n]{a^m}}$$

EXAMPLE 1

$$(d) \frac{3^{-2} \times 6^3 \times 12^{-2}}{9^{-3} \times 2^{-1}}$$

$$\begin{aligned} (d) \quad 3^{-2} &= \frac{1}{3^2}; \quad 6^3 = (2 \times 3)^3 = 2^3 \cdot 3^3 \\ 12^{-2} &= \frac{1}{12^2} = \frac{1}{(2^2 \cdot 3)^2} = \frac{1}{2^4 \cdot 3^2} \\ 9^{-3} &= \frac{1}{9^3} = \frac{1}{(3^2)^3} = \frac{1}{3^6} \\ \text{Thus } \frac{3^{-2} \times 6^3 \times 12^{-2}}{9^{-3} \times 2^{-1}} &= \frac{1 \times 2^3 \cdot 3^3 \times 1 \times 3^6 \times 2}{3^2 \cdot 2^4 \cdot 3^2} \\ &= \frac{2^4 \cdot 3^9}{2^4 \cdot 3^8} \\ &= 3^1 \\ &= 3 \end{aligned}$$

EXAMPLE 2

Simplify $\frac{x^{-1} + 1}{x^{-1} - x}$, expressing the answer with positive indices.

$$\begin{aligned} x^{-1} + 1 &= \frac{1}{x} + 1 = \frac{1+x}{x} \\ x^{-1} - x &= \frac{1}{x} - x = \frac{1-x^2}{x} \\ \frac{x^{-1} + 1}{x^{-1} - x} &= \frac{1+x}{x} \times \frac{x}{1-x^2} \\ &= \frac{1+x}{1-x^2}, \quad x \neq 0 \\ &= \frac{1+x}{(1-x)(1+x)} \\ &= \frac{1}{1-x}, \quad x \neq -1 \end{aligned}$$

EXERCISES 13(a)

1. Simplify the following:

$$(a) x^2 \cdot x^5 \cdot x^3$$

$$(c) (p^2q)^4 \times (q^2p)^5$$

$$(e) (2x^2)^5 \times (4x^3)^2$$

$$(b) 2^3 \times 4^2 \times 8^2$$

$$(d) a^3b^{-2} \times (a^2b^2)^4$$

$$(f) m^2p^3 \times (m^3n^2)^3 \times (p^{-1})^2$$

2. Simplify the following:

$$(a) \frac{(x^2y^3)^4 \times (xy)^{-2}}{xy}$$

$$(c) \frac{(ab^2)^3 \times (a^2b)^2}{(a^2b^2)^2}$$

$$(e) \frac{5x^5y^2 \times 3(xy^3)^2}{15x^{-2}y}$$

$$(b) 2^n \times 2^{2n} \times 2^{3n}$$

$$(d) \frac{(2m^2n)^3}{(mn^3)^2 \times (4m^2)^2}$$

$$(f) \frac{(a^2b)^2 \cdot (ab)^4}{(a^2b)^3}$$

3. Simplify the following:

$$(a) \frac{(-2xy)^2 \times 2(x^2y^{-1})^3}{8(xy)^{-3}}$$

$$(c) \frac{ab^2 \times (a^2b^{-1})^2}{(a^{-2}b)^{-1}}$$

$$(e) \frac{x^2yz^{-2} \times 2(x^2y^{-1}z)^2}{xyz}$$

$$(b) m^2n^2p^{-2} \times (mnp^2)^{-3}$$

$$(d) \frac{p^2q^3r^{-3}}{p^3q^{-1}r}$$

$$(f) 2^3 \cdot (2^n)^2 \cdot 2^{-n}$$

4. Simplify the following, expressing them with positive indices:

(a) $\frac{x^2 \cdot x^3 \cdot x^4}{x^5}$

(b) $a^{p+q} \cdot a^{p-q}$

(c) $\frac{2^n \cdot 2^{n-3}}{2^3}$

(d) $\frac{3^{-2}x^{-1}}{3^{-3}x^{-2}}$

(e) $2^n \cdot 4^n \cdot 8^n$

(f) $\frac{a^2b^3c^{-4}}{a^4b^{-1}c^{-5}}$

(g) $\frac{x^{m+n} \cdot y^{3m-n}}{x^{-n} \cdot y^{3m}}$

(h) $\left(\frac{3xy^2}{4x^4y}\right)^{-3}$

5. Write down each of the following as a negative power of 2:

(a) $\frac{1}{4}$

(b) $\frac{1}{16}$

(c) $\frac{1}{32}$

(d) 0.125

(e) $\frac{1}{64}$

(f) $\frac{1}{128}$

(g) 0.25

(h) 8^{-3}

6. Write down each of the following as a power of 10:

(a) 100

(b) 10

(c) 1

(d) 0.1

(e) 0.01

(f) 0.001

(g) $\frac{1}{1000}$

(h) 0.0001

(i) $\frac{1}{100,000}$

7. Write down the value of:

(a) $9^2 \times 3^{-4}$

(b) $2^0 \times 5^0$

(c) $8^{-3} \times 2^7$

(d) $a^{-5} \times a^{-3} \times a^0 \times a^6$

(e) $6^{-3} \times 2^5 \times 3^3$

8. Simplify the following, expressing them with positive indices:

(a) $\frac{3^{2n} \times 25^{2n-1}}{15^{n-1}}$

(b) $(x^{-1} + y^{-1})(x^{-1} - y^{-1})$

(c) $\frac{2^n \cdot 4^{n+1}}{8^{n-2}}$

(d) $(x^{-2} + x^{-1})^3$

(e) $\frac{x - 5 + 6x^{-1}}{1 - 2x^{-1}}$

(f) $\frac{x^2 + 8x^{-1}}{1 + 2x^{-1}}$

9. Simplify the following, expressing them with positive indices:

(a) $4^{-2} \times 6^3 \times 8^4 \times 12^{-2}$

(b) $\frac{15^{n+1} \times 25 \times 5^{3n-4}}{9^{n-1} \times 25^{n-2}}$

(c) $(-4)^{-2} \times (-2)^3$

10. Show that, for any positive integer n ,

$(-1)^n = 1$ when n is even, and

$(-1)^n = -1$ when n is odd.

11. Find the value of x that makes each of the following true:

(a) $2^x = 8$

(b) $3^{x-1} = 27$

(c) $x^3 = -125$

(d) $x^{-2} = 81$

(e) $\frac{2^{x-3}}{4^{1-x}} = 1$

(f) $4^x = 32$

(g) $9^x = 27$

(h) $3^x + 5 \cdot 3^x = 54$

(i) $\frac{3 \cdot 5^x - 1}{5^x + 2} = 2$

(j) $3^x = \frac{1}{9}$

(k) $2^{-x} = \frac{1}{64}$

(l) $5^x = \frac{1}{125}$

12. Expand and simplify the following, expressing the results with positive indices:

(a) $(a^{-1} + b)(a^{-1} - b)$

(b) $(x^{-1} + y)(x + y^{-1})$

(c) $(x^{-2} + y^{-2})(x^{-2} - y^{-2})$

(d) $(a^2 - 2b^{-1})(a^{-2} - b)$

(e) $\frac{a^{-1} + b^{-1}}{a + b}$

(f) $\frac{y^{-1} + y}{1 + y^2}$

EXAMPLE 3Simplify (a) $32^{2/5}$ (b) $125^{-2/3}$ (c) $x^{5/2} \cdot x^{-3/4}$ (d) $\left(\frac{9}{49}\right)^{-1/2}$ (e) $8^{2/3} \times 9^{-3/2}$

(a) $32^{2/5} = (2^5)^{2/5} = 2^2 = 4$

or $32^{2/5} = (\sqrt[5]{32})^2 = 2^2 = 4$

(b) $125^{-2/3} = (5^3)^{-2/3} = 5^{-2} = \frac{1}{25}$

or $125^{-2/3} = (\sqrt[3]{125})^{-2} = 5^{-2} = \frac{1}{25}$

(c) $x^{5/2} \cdot x^{-3/4} = x^{5/2-3/4} = x^{7/4}$

(d) $\left(\frac{9}{49}\right)^{-1/2} = \left(\frac{49}{9}\right)^{1/2} = \frac{7}{3}$

(e) $8^{2/3} \times 9^{-3/2} = (2^3)^{2/3} \times (3^2)^{-3/2} = 2^2 \times 3^{-3} = \frac{4}{27}$

or $8^{2/3} \times 9^{-3/2} = (\sqrt[3]{8})^2 \times (\sqrt{9})^{-3} = 2^2 \times 3^{-3} = \frac{4}{27}$

EXAMPLE 4Simplify (a) $\frac{5^{1/4} \times \sqrt{10} \times \sqrt[4]{2}}{20^{3/4}}$ (b) $\frac{3^{n-2} \times 9^{n+1}}{81^{n-1}}$ (c) $(x^{1/2} - x^{-1/2})^2$

$$\begin{aligned}
 \text{(a)} \quad \frac{5^{1/4} \times \sqrt{10} \times \sqrt[4]{2}}{20^{3/4}} &= \frac{5^{1/4} \times (2 \times 5)^{1/2} \times 2^{1/4}}{(2^2 \times 5)^{3/4}} \\
 &= \frac{5^{1/4} \times 2^{1/2} \times 5^{1/2} \times 2^{1/4}}{2^{3/2} \times 5^{3/4}} \\
 &= \frac{5^{3/4} \times 2^{3/4}}{2^{3/2} \times 5^{3/4}} \\
 &= \frac{1}{2^{3/4}}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \frac{3^{n-2} \times 9^{n+1}}{81^{n-1}} &= \frac{3^{n-2} \times (3^2)^{n+1}}{(3^4)^{n-1}} \\
 &= \frac{3^{n-2} \times 3^{2n+2}}{3^{4n-4}} \\
 &= \frac{3^{3n}}{3^{4n-4}} \\
 &= 3^{4-n} \text{ or } \frac{1}{3^{n-4}}
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad (x^{1/2} - x^{-1/2})^2 &= (x^{1/2})^2 - 2x^{1/2}x^{-1/2} + (x^{-1/2})^2 \\
 &= x^1 - 2x^0 + x^{-1} \\
 &= x - 2 + \frac{1}{x}
 \end{aligned}$$

EXAMPLE 5

Solve the equations

(i) $5^{2x} = 125^{1/2}$

(ii) $(3^x - 1)\left(2^{2x} - \frac{1}{16}\right) = 0$

$$(i) \quad 5^{2x} = 125^{1/2} = (5^3)^{1/2} = 5^{3/2}$$

$$\therefore 2x = \frac{3}{2}$$

$$x = \frac{3}{4}$$

$$(ii) \quad 3^x - 1 = 0 \quad \text{or} \quad 2^{2x} - \frac{1}{16} = 0$$

$$\text{i.e.} \quad 3^x = 1 = 3^0 \quad 2^{2x} = \frac{1}{16} = \frac{1}{2^4} = 2^{-4}$$

$$\therefore \quad x = 0 \quad 2x = -4$$

$$x = -2$$

EXERCISES 13(b)

1. Evaluate the following:

(a) $64^{2/3}$

(b) $49^{-1/2}$

(c) $(9^3)^{1/2}$

(d) $\left(\frac{1}{3}\right)^{-2}$

(e) $2^{2/3} \times 4^{1/6}$

(f) $\left(\frac{1}{125}\right)^{-1/3}$

(g) $\left(\frac{1}{16}\right)^{-3/2}$

(h) $243^{-2/5}$

(i) $\sqrt[3]{27} \times \sqrt[5]{32}$

(j) $\sqrt[4]{64} \times \sqrt[3]{8}$

(k) $\sqrt[5]{8} \times \sqrt[5]{4}$

2. Express each of the following in simplest indicial form:

(a) $\sqrt[4]{36}$

(b) $\sqrt[8]{32}$

(c) $\sqrt[3]{4} \cdot \sqrt[6]{16}$

(d) $\sqrt{3} \sqrt[3]{81}$

(e) $\frac{a^2 \times \sqrt[3]{a}}{a^{2/3} \times \sqrt[4]{a}}$

(f) $\sqrt[3]{2^{3/2}}$

(g) $\frac{\sqrt[3]{a^4}}{\sqrt[6]{a^5}}$

(h) $\frac{\sqrt{x^3 y^2}}{\sqrt[3]{x^2 y^3}}$

(i) $8^{-2/3} \times 4^{3/2}$

(j) $(5^{1/2})^3 \times \sqrt{5}$

(k) $16^{3/4} \times 4^{-1/2}$

3. Simplify the following, expressing the results with positive indices:

(a) $x^{2/3} \cdot x^{3/2}$

(b) $(a^{-1}b)^2 \times \left(\frac{1}{b^{-2}}\right)^{1/2}$

(c) $(x^{1/2})^2 - (x^{-2})^{1/2}$

(d) $(x^{1/3})^2 \times (x^{-1}y^3)^{-1} \times x^{-5/3}y^2$

(e) $(x^{1/2} + y^{1/2})(x^{1/2} - y^{1/2})$

(f) $(x^{1/2} + y^{1/2})^2$

(g) $(9x^2)^{3/2} \times (8x^3)^{-2/3}$

(h) $a^{3/4} \times a^{5/4} \times a^{-2}$

4. Simplify the following:

(a) $\sqrt[6]{x^2 y^3} \times \frac{x^{1/3}}{y^{-1/2}}$

(b) $(y^{2/3})^{3/4} \times (y^{1/5})^{-5/3}$

(c) $\sqrt[3]{x^2 y} \div \sqrt{xy^3}$

(d) $\sqrt{a^3 b^2} \times \sqrt[3]{ab}$

(e) $\frac{54^{1/4}}{6^{3/4} \times 12^{-1/2}}$

(f) $(8x^6)^{1/3} \times (2x)^{-3}$

(g) $\frac{3x \times 9y^{1/2}}{(9xy)^{1/2}}$

(h) $\frac{(x^{m+1})^n \times x^{m+n}}{(x^m)^{n+1} \times x^{2n}}$

5. Simplify $\frac{x^{-3}\sqrt{xy^3}}{y^{-5}\sqrt{x^3y}}$ and evaluate for $x = \sqrt{2}$, $y = \sqrt{6}$.6. If $(3x^n)^3 \times (3x)^{n-6} = ax^2$, find the value of n and a .

7. Evaluate the following:

(a) $\left(\frac{64}{125}\right)^{2/3}$

(b) $0.001^{-2/3}$

(c) $0.81^{1/2}$

(d) $\left(\frac{9}{16}\right)^{-3/2}$

(e) $2 \times 4^{-1/2}$

(f) $3^{-1/2} \times 27^{1/2}$

8. Simplify the following, expressing the results with positive indices:

(a) $(a^2b^{-1})^{-2} \div (a^{-1}b^2)^2$

(b) $(x^{1/2} - y^{-1/2})^2$

(c) $(x^{1/3} + y^{1/3})(x^{1/3} - y^{1/3})$

(d) $\sqrt[3]{x^2} \times \sqrt[3]{x^4}$

9. Simplify the following:

(a) $\frac{1 - x^{-2}}{1 - x^{-1}}$

(b) $(a^2 + 2ab + b^2)^{1/2}$

when $a = 4$ and $b = 2$

(c) $(x^2 - 2x + 1)^{-1/2}$ when $x = 3$

(d) $a^{-1/3} \times (a^4)^{1/3}$

10. Simplify the following:

(a) $2^n \times 4^n \times 8^n$

(b) $\frac{25^{2n} \times 5^{n-1}}{5^{2n+1}}$

(c) $\frac{3^n \times 9^{n+1}}{27^n}$

(d) $\frac{6^{2n} \times 3^{2n}}{18^n}$

(e) $\frac{2^{2n} \times 4^{n+1}}{4^n \times 2^{2n+2}}$

(f) $\frac{a^{n-1} \times b^3}{a^{n-3} \times b^2}$

11. Simplify the following:

(a) $\frac{a^{1/2} \times b^{3/4} \times (a^2)^{1/4}}{(ab)^{1/4}}$

(b) $10^{-1} \times 100^{0.5} \times 1000$

(c) $2\sqrt{3} \times 3\sqrt{2} \times 6^{1/2}$

(d) $\sqrt{6} \times 4 \times 3^{1/2}$

12. Find the value of x for which

(a) $9^x = \frac{1}{3}$

(b) $a^{x-3} = 1$

(c) $2^x = \frac{1}{8}$

(d) $2^x \cdot 4^x \cdot 8^x = 2^{-3}$

(e) $3^x \cdot 2^x = 1$

(f) $5^x = \frac{1}{125}$

13. Solve the following for x :

(a) $5^x = 125$

(b) $16^x = 128$

(c) $8^{-x} = \frac{1}{32}$

(d) $\left(2^x - 1\right)\left(3^x - \frac{1}{9}\right) = 0$

(e) $(3^x - 9)(5^x - 1) = 0$

(f) $3^{2x+1} = \frac{1}{27}$

EXERCISES 13(a)

1. (a) x^{10} (b) 2^{13} (c) $p^{13}q^{14}$ (d) $a^{11}b^6$ (e) 2^9x^{16} (f) $m^{11}n^6p$
2. (a) x^5y^9 (b) 2^{6n} (c) a^3b^4 (d) $\frac{1}{2n^3}$ (e) x^9y^7 (f) a^2b^3
3. (a) $x^{11}y^2$ (b) $\frac{1}{mnp^8}$ (c) a^3b (d) $\frac{q^4}{pr^4}$ (e) $\frac{2x^5}{y^2z}$ (f) 2^{n+3}
4. (a) x^4 (b) a^{2p} (c) 2^{2n-6} (d) $3x$ (e) 2^{6n} (f) $\frac{b^4c}{a^2}$ (g) $\frac{x^{m+2n}}{y^n}$ (h) $\frac{64x^9}{27y^3}$
5. (a) 2^{-2} (b) 2^{-4} (c) 2^{-5} (d) 2^{-3} (e) 2^{-6} (f) 2^{-7} (g) 2^{-2} (h) 2^{-9}
6. (a) 10^2 (b) 10^1 (c) 10^0 (d) 10^{-1} (e) 10^{-2} (f) 10^{-3} (g) 10^{-3} (h) 10^{-4} (i) 10^{-5}
7. (a) 1 (b) 1 (c) $\frac{1}{4}$ (d) a^{-2} (e) 4
8. (a) $3^{n+1}5^{3n-1}$ (b) $\frac{1}{x^2} - \frac{1}{y^2}$ (c) 2^8 (d) $\frac{1}{x^6} + \frac{3}{x^5} + \frac{3}{x^4} + \frac{1}{x^3}$ (e) $x - 3$ (f) $x^2 - 2x + 4$
9. (a) $3 \cdot 2^7$ (b) $\frac{5^{2n+3}}{3^{n-3}}$ (c) $-\frac{1}{2}$
11. (a) 3 (b) 4 (c) -5 (d) $\pm \frac{1}{9}$ (e) $1\frac{2}{3}$ (f) $2\frac{1}{2}$ (g) $1\frac{1}{2}$ (h) 2 (i) 1 (j) -2 (k) 6 (l) -3
12. (a) $\frac{1}{a^2} - b^2$ (b) $2 + xy + \frac{1}{xy}$ (c) $\frac{1}{x^4} - \frac{1}{y^4}$ (d) $3 - a^2b - \frac{2}{a^2b}$ (e) $\frac{1}{ab}$ (f) $\frac{1}{y}$

EXERCISES 13(b)

1. (a) 16 (b) $\frac{1}{7}$ (c) 27 (d) 9 (e) 2 (f) 5 (g) 64 (h) $\frac{1}{9}$ (i) 6 (j) 5 (k) 2
2. (a) $6^{1/2}$ (b) $2^{5/8}$ (c) $2^{4/3}$ (d) $3^{11/6}$ (e) $a^{17/12}$ (f) $2^{1/2}$ (g) $a^{1/2}$ (h) $x^{5/6}$ (i) 2 (j) 5^2 (k) 2^2
3. (a) $x^{13/6}$ (b) $\frac{b^3}{a^2}$ (c) $x - \frac{1}{x}$ (d) $\frac{1}{y}$ (e) $x - y$ (f) $x - 2x^{1/2}y^{1/2} + y$ (g) $\frac{27}{4}x$ (h) 1
4. (a) $x^{2/3}y$ (b) $y^{1/6}$ (c) $\left(\frac{x}{y^7}\right)^{1/6}$ (d) $a^{11/6}b^{4/3}$ (e) $6^{1/2}$ (f) $\frac{1}{4x}$ (g) $9x^{1/2}$ (h) 1
5. $\frac{y^6}{x^4}, 54$ 6. $n = 2, a = \frac{1}{3}$ 7. (a) $\frac{16}{25}$ (b) 100 (c) 0.9 (d) $\frac{64}{27}$ (e) 1 (f) 3
8. (a) $\frac{1}{a^2b^2}$ (b) $x - 2\left(\frac{x}{y}\right)^{1/2} + \frac{1}{y}$ (c) $x^{2/3} - y^{2/3}$ (d) x^2
9. (a) $\frac{x+1}{x}$ (b) 6 (c) $\frac{1}{2}$ (d) a
10. (a) 2^{6n} (b) 5^{3n-2} (c) 9 (d) 18^n (e) 1 (f) a^2b
11. (a) $a^{3/4}b^{1/2}$ (b) 1000 (c) 36 (d) $12\sqrt{2}$
12. (a) $-\frac{1}{2}$ (b) 3 (c) -3 (d) $-\frac{1}{2}$ (e) 0 (f) -3
13. (a) 3 (b) $1\frac{3}{4}$ (c) $1\frac{2}{3}$ (d) 0, -2 (e) 2, 0 (f) -2