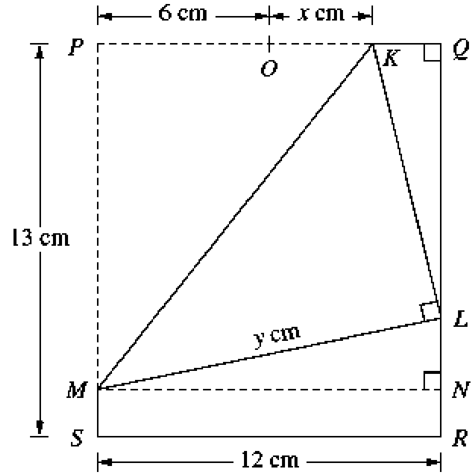
06 **10 b**

A rectangular piece of paper $PQRS$ has sides PQ = 12 cm and PS = 13 cm. The point O is the midpoint of PQ. The points K and M are to be chosen on OQ and PS respectively, so that when the paper is folded along KM, the corner that was at P lands on the edge QR at L. Let $OK = x$ cm and $LM = y$ cm.

Copy or trace the diagram into your writing booklet.



(i) Show that $QL^2 = 24x$. **1**

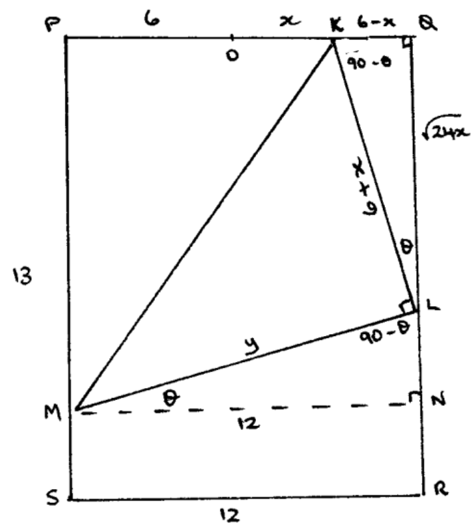
(ii) Let N be the point on QR for which MN is perpendicular to QR. By showing that $\Delta QKL\ |||\ \Delta NLM$, deduce that **3**

$$y = \frac{\sqrt{6}(6+x)}{\sqrt{x}}$$

(iii) Show that the area, A, of ΔKLM is given by $A = \dfrac{\sqrt{6}(6+x)^2}{2\sqrt{x}}$. **1**

(iv) Use the fact that $12 \le y \le 13$ to find the possible values of x. **2**

(v) Find the minimum possible area of ΔKLM. **3**



(i) $$\begin{aligned} QL^2 &= (6+x)^2 - (6-x)^2 \\ &= 36 + 12x + x^2 - [36 - 12x + x^2] \\ &= 24x \end{aligned}$$

(ii) Let $\angle QLK = \theta$

$\therefore \angle MLN = 180 - (90 + \theta)$ (straight $\angle$)

$= 90 - \theta$

$\therefore \angle LMN = 180 - (90 + 90 - \theta)$ ($\angle$ sum of Δ)

$= \theta$

$\therefore \angle MLN = \angle LMN$

and $\angle LQK = \angle MNL$ (given)

$\therefore \Delta QKL\ |||\ \Delta NLM$ (2 $\angle$s equal)

$\dfrac{y}{6+x} = \dfrac{12}{\sqrt{24x}}$ (matching sides of sim Δs in prop^n)

$$\begin{aligned} y &= \frac{12(6+x)}{2\sqrt{6x}} \\ &= \frac{6(6+x)}{\sqrt{6x}} \\ &= \frac{\sqrt{6}(6+x)}{\sqrt{x}} \end{aligned}$$

(iii) $$\begin{aligned} \text{Area} &= \frac{1}{2} \times MN \times LK \\ &= \frac{1}{2} \times y \times (6+x) \\ &= \frac{\sqrt{6}(6+x)^2}{2\sqrt{x}} \end{aligned}$$

(iv) Let $y = 12$

$$\frac{\sqrt{6}(6+x)}{\sqrt{x}} = 12$$
$$\sqrt{6}\,(6 + x) = 12\sqrt{x}$$
$$6(6 + x)^2 = 144x$$
$$36 + 12x + x^2 = 24x$$
$$x^2 - 12x + 36 = 0$$
$$(x - 6)^2 = 0$$
$$x = 6$$

Let $y = 13$

$$\frac{\sqrt{6}(6+x)}{\sqrt{x}} = 13$$
$$\sqrt{6}\,(6 + x) = 13\sqrt{x}$$
$$6(6 + x)^2 = 169x$$
$$6(36 + 12x + x^2) = 169x$$
$$6x^2 + 72x + 216 = 169x$$
$$6x^2 - 97x + 216 = 0$$
$$(3x - 8)(2x - 27) = 0$$
$$x = 2\tfrac{2}{3},\ 13\tfrac{1}{2}$$

But $0 \le x \le 6$, therefore, $x = 2\frac{2}{3}$.

As $12 \le y \le 13$, then $2\frac{2}{3} \le x \le 6$.

(v)
$$A = \frac{\sqrt{6}(6+x)^2}{2\sqrt{x}}$$

$$\frac{dA}{dx} = \frac{2\sqrt{x}.\sqrt{6}.2(6+x)^1.1 - \sqrt{6}(6+x)^2.2.\frac{1}{2}x^{-\frac{1}{2}}}{4x}$$

$$= \frac{4\sqrt{x}.\sqrt{6}.(6+x) - \sqrt{6}(6+x)^2.x^{-\frac{1}{2}}}{4x} = 0$$

$$\frac{\sqrt{6}x^{-\frac{1}{2}}.(6+x)[4x-(6+x)]}{4x} = 0$$

$$\frac{\sqrt{6}(6+x)(3x-6)}{4x\sqrt{x}} = 0$$

$$(x + 6)(3x - 6) = 0$$
$$x = -6, 2$$

But as $2\frac{2}{3} \le x \le 6$, then $x \ne -6$ and $x \ne 2$

The minimum must lie at endpoints: $x = 2\frac{2}{3}$ and $x = 6$.

Subs $x = 2\frac{2}{3}$ in $A = \dfrac{\sqrt{6}(6+x)^2}{2\sqrt{x}}$

$$= \frac{\sqrt{6}(6+2\frac{2}{3})^2}{2\sqrt{2\frac{2}{3}}}$$
$$= 56\tfrac{1}{3}$$

Subs $x = 6$ in $A = \dfrac{\sqrt{6}(6+x)^2}{2\sqrt{x}}$

$$= \frac{\sqrt{6}(6+6)^2}{2\sqrt{6}}$$
$$= 72$$

$\therefore$ The minimum area is $56\frac{1}{3}$ units^2

* These solutions have been provided by *projectmaths* and are not supplied or endorsed by the Board of Studies

Board of Studies: Notes from the Marking Centre

Having ten marks attached to part (b) seemed to deter a large number of candidates from attempting this part. Candidates are advised to look carefully at the mark value for sub-parts as an indication of the amount of work required to answer them.

(i) Most candidates failed to recognise that $KL = 6 + x$, others going through some elaborate calculations in order to prove it. Having correctly stated the Pythagorean relationship, a large number of candidates had difficulty expanding the terms to show that $QL^2 = 24x$.

(ii) Candidates should be advised to attempt geometry proofs even though they are late in the paper. A large percentage failed to attempt the similarity proof and moved straight to the proportionality section. When proofs were attempted, the demonstration of geometrical reasoning was poor, with some candidates indicating that three angles were 'supplementary' for example, and/or neglecting to state the similarity test used. Again, candidates are advised to make use of their sketched diagram to discover the matched angles required for the similarity.

(iii) The majority of candidates who attempted this part were able to gain the allotted mark. The most common error was that $A = \frac{1}{2}y^2$, while a number of candidates appeared to simply 'fudge' their response by rewriting the statement given. Some candidates recalculated the expression for KL in order to use it in their working.

(iv) The algebra involved in this part proved to be very difficult and even beyond most candidates. Those who looked at it as two distinct equality statements were able to make the best progress, but failure to link the fact that $0 < x < 6$ was prevalent, with $x = \frac{27}{2}$ still being tested and even the interval $6 \le x \le 13.5$ being stated as the final answer or part thereof.

(v) In order to gain full marks for this part, candidates were required to draw the link between the domain in part (iv) and their area. A common error was the use of the expression for y from part (ii) instead of the expression for A from part (iii). Candidates are advised to avoid the use of shortcuts when differentiation is involved, for example leaving the denominator out of the quotient rule because it was to be set equal to zero. A significant number of candidates obtained their only mark for part (b) by correctly differentiating. When testing for a minimum, values of the derivative on either side should be used instead of just stating $-\ 0\ +$ or $\backslash\ \_\ /$ in a table.

Source: http://www.boardofstudies.nsw.edu.au/hsc_exams/