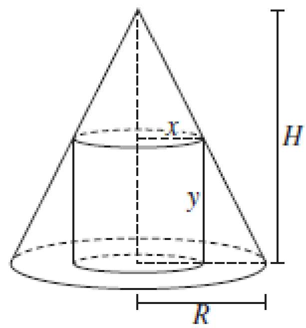
projectmaths

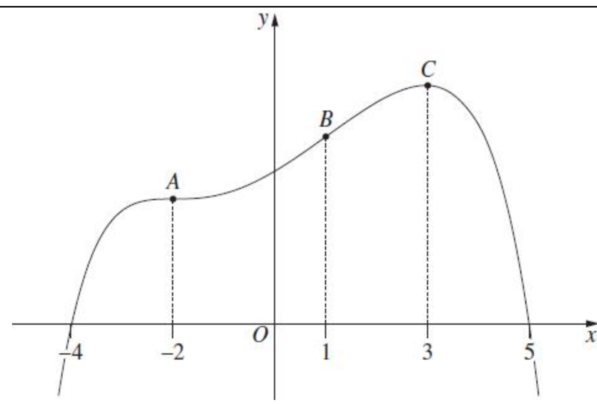
Geometric Applications of Differentiation

16	13 a	Consider the function $y = 4x^3 - x^4$. (i) Find the two stationary points and determine their nature. (ii) Sketch the graph of the function, clearly showing the stationary points and the x and y intercepts.	 4 2	Solution
15	13 c	Consider the curve $y = x^3 - x^2 - x + 3$. (i) Find the stationary points and determine their nature. (ii) Given that the point $P(\frac{1}{3}, \frac{70}{27})$ lies on the curve, prove that there is a point of inflexion at P. (iii) Sketch the curve, labelling the stationary points, point of inflexion and y-intercept.	 4 2 2	Solution

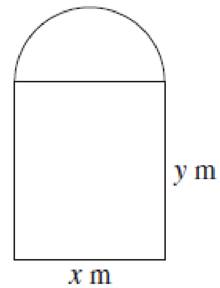
15	16 c	The diagram shows a cylinder of radius x and height y inscribed in a cone of radius R and height H, where R and H are constants. The volume of a cone of radius r and height h is $\frac{1}{3}\pi r^2 h$. The volume of a cylinder of radius r and height h is $\pi r^2 h$. (i) Show that the volume, V, of the cylinder can be written as $V = \frac{H}{R}\pi x^2(R - x)$. (ii) By considering the inscribed cylinder of maximum volume, show that the volume of any inscribed cylinder does not exceed $\frac{4}{9}$ of the volume of the cone.	 3 4	Solution



14	11 f	The gradient function of a curve $y = f(x)$ is given by $f'(x) = 4x - 5$. The curve passes through the point (2, 3). Find the equation of the curve.	2	Solution
14	14 e	The diagram shows the graph of a function $f(x)$. The graph has a horizontal point of inflexion at A, a point of inflexion at B and a maximum turning point at C. Sketch the graph of the derivative $f'(x)$.	3	Solution



14 **16 c** The diagram shows a window consisting of two sections. The top section is a semicircle of diameter x m. The bottom section is a rectangle of width x m and height y m.
The entire frame of the window, including the piece that separates the two sections, is made using 10 m of thin metal.
The semicircular section is made of coloured glass and the rectangular section is made of clear glass.
Under test conditions the amount of light coming through one square metre of the coloured glass is 1 unit and the amount of light coming through one square metre of the clear glass is 3 units. The total amount of light coming through the window under test conditions is L units. Solution



(i) Show that $y = 5 - x\left(1 + \dfrac{\pi}{4}\right)$. **2**

(ii) Show that $L = 15x - x^2\left(3 + \dfrac{5\pi}{8}\right)$. **2**

(ii) Find the values of x and y that maximise the amount of light coming through the window under test conditions. **3**

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13 **8** The diagram shows the points A, B, C and D on the graph $y = f(x)$. **1** Solution

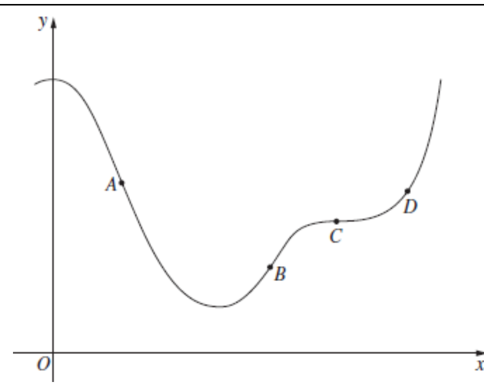
At which point is $f'(x) > 0$ and $f''(x) = 0$.

(A) A

(B) B

(C) C

(D) D



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13 **12 a** The cubic $y = ax^3 + bx^2 + cx + d$ has a point of inflexion at $x = p$. **2** Solution

Show that $p = -\dfrac{b}{3a}$.

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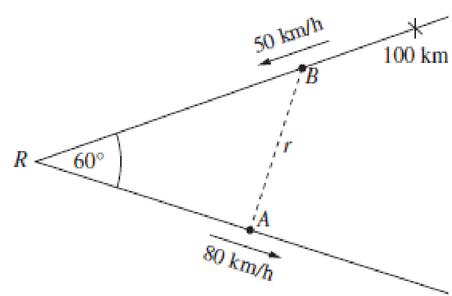
13 **14 b** Two straight roads meet at R at an angle of 60°. At time $t = 0$ car A leaves R on one road, and car B is 100 km from R on the other road. Car A travels away from R at a speed of 80 km/h, and car B travels towards R at a speed of 50 km/h. The distance between the cars at time t hours is r km. Solution

(i) Show that $r^2 = 12\,900t^2 - 18\,000t + 10\,000$. **2**

(ii) Find the minimum distance between the cars. **3**



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13 **16 a** The derivative of a function $f(x)$ is $f'(x) = 4x - 3$. **3** Solution

The line $y = 5x - 7$ is tangent to the graph of $f(x)$.
Find the function $f(x)$.

12 **4** The diagram shows the graph of $y = f(x)$. **1** Solution

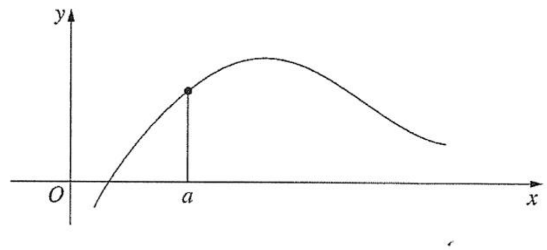
Which of the following statements is true?

(A) $f'(a) > 0$ and $f''(a) < 0$

(B) $f'(a) > 0$ and $f''(a) > 0$

(C) $f'(a) < 0$ and $f''(a) < 0$

(D) $f'(a) < 0$ and $f''(a) > 0$



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12 **14a** A function is given by $f(x) = 3x^4 + 4x^3 - 12x^2$. Solution

(i) Find the nature of the stationary points of $f(x)$ and determine their nature. **3**

(ii) Hence, sketch the graph of $y = f(x)$ showing the stationary points. **2**

(iii) For what values of x is the function increasing? **1**

(iv) For what values of k will $3x^4 + 4x^3 - 12x^2 + k = 0$ have no solution? **1**

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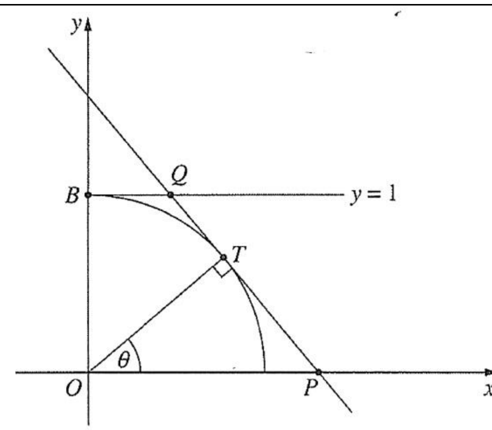
12 **16b** The diagram shows a point T on the unit circle $x^2 + y^2 = 1$ at angle θ from the positive x-axis, where $0 < \theta < \frac{\pi}{2}$. The tangent to the circle at T is perpendicular to OT, and intersects the x-axis at P, and the line $y = 1$ at Q. The line $y = 1$ intersects the y-axis at B. Solution



(i) Show that the equation of the line PT is $x\cos\theta + y\sin\theta = 1$. **2**

(ii) Find the length of BQ in terms of θ. **1**

(iii) Show that the area, A, of the trapezium $OPQB$ is given by $A = \dfrac{2 - \sin\theta}{2\cos\theta}$. **2**

(iv) Find the angle θ that gives the minimum area of the trapezium. **3**

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11 **4c** The gradient of a curve is given by $\dfrac{dy}{dx} = 6x - 2$. The curve passes through the point $(-1, 4)$. What is the equation of the curve? **2** Solution

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11 **7a** Let $f(x) = x^3 - 3x + 2$. Solution

(i) Find the coordinates of the stationary points of $y = f(x)$, and determine their nature. **3**

(ii) Hence, sketch the graph $y = f(x)$ showing all stationary points and the y-intercept. **2**

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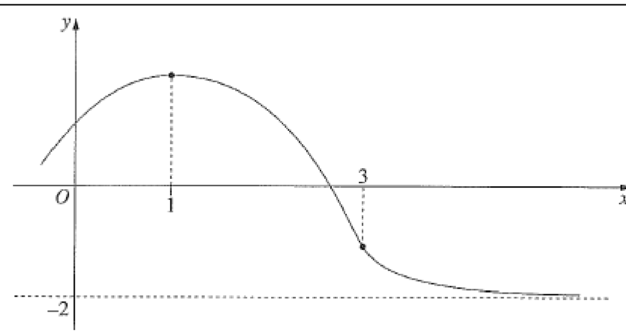
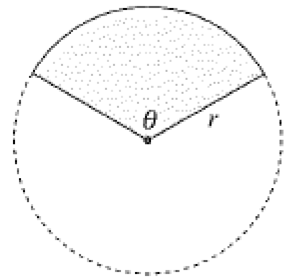
11 **9c** The graph $y = f(x)$ in the diagram has a stationary point when $x = 1$, a point of inflexion when $x = 3$, and a horizontal asymptote $y = -2$. Sketch the graph $y = f'(x)$, clearly indicating its features at $x = 1$ and at $x = 3$, and the shape of the graph as $x \to \infty$. **3** Solution



11 **10 b** A farmer is fencing a paddock using P metres of fencing. The paddock is to be in the shape of a sector of a circle with radius r and sector θ in radians, as shown in the diagram. [Solution]



(i) Show that the length of fencing required to fence the perimeter of the paddock is $P = r(\theta + 2)$. **1**

(ii) Show that the area of the sector is $A = \frac{1}{2}Pr - r^2$. **1**

(iii) Find the radius of the sector, in terms of P, that will maximize the area of the paddock. **2**

(iv) Find the angle θ that gives the maximum area of the paddock. **1**

(v) Explain why it is only possible to construct a paddock in the shape of a sector if $\frac{P}{2(\pi + 1)} < r < \frac{P}{2}$. **2**

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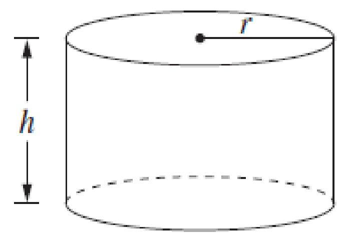
10 **5a** A rainwater tank is to be designed in the shape of a cylinder with radius r metres and height h metres. The volume of the tank is to be 10 cubic metres. Let A be the surface area of the tank, including its top and base, in square metres. [Solution]



(i) Given that $A = 2\pi r^2 + 2\pi rh$, show that $A = 2\pi r^2 + \frac{20}{r}$. **2**

(ii) Show that A has a minimum value and find the value of r for which the minimum occurs. **3**

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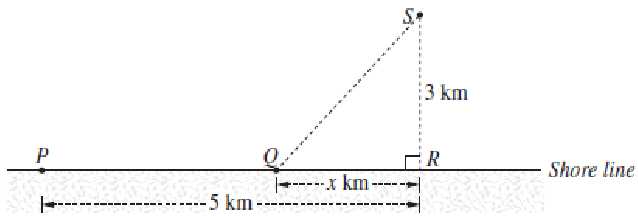
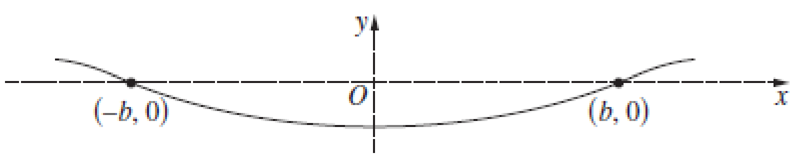
10 **6a** Let $f(x) = (x + 2)(x^2 + 4)$. [Solution]

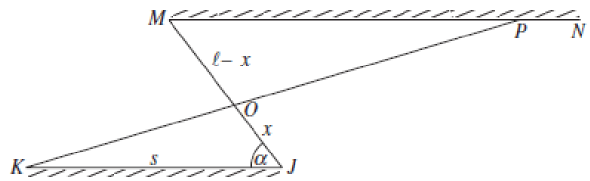
(i) Show that the graph of $y = f(x)$ has no stationary points. **2**

(ii) Find the values of x for which the graph $y = f(x)$ is concave down, and the values for which it is concave up. **2**

(iii) Sketch the graph of $y = f(x)$, indicating the values of the x and y intercepts. **2**

09	**9b**	An oil rig, S, is 3 km offshore. A power station, P, is on the shore. A cable is to be laid from P to S. It costs \$1000 per kilometres to lay the cable along the shore and \$2600 per kilometre to lay the cable underwater from the shore to S. The point R is the point on the shore closest to S, and the distance PR is 5 km. The point Q is on the shore, at a distance of x km from R, as shown in the diagram.		Solution
		(i) Find the total cost of laying the cable in a straight line from P to R and then in a straight line from R to S.	**1**	
				
		(ii) Find the cost of laying the cable in a straight line from P to S.	**1**	
		(iii) Let \$$C$ be the total cost of laying the cable in a straight line from P to Q, and then in a straight line from Q to S. Show that $C = 1000(5 - x + 2.6\sqrt{x^2 + 9})$.	**2**	
		(iv) Find the minimum cost of laying the cable.	**3**	
		(v) New technology means that the cost of laying the cable underwater can be reduced to \$1100 per kilometre. Determine the path for laying the cable in order to minimise the cost in this case.	**1**	
08	**8a**	Let $f(x) = x^4 - 8x^2$.		Solution
		(i) Find the coordinates of the points where the graph of $y = f(x)$ crosses the axes.	**2**	
		(ii) Show that $f(x)$ is an even function.	**1**	
		(iii) Find the coordinates of the stationary points of $y = f(x)$ and determine their nature.	**4**	
		(iv) Sketch the graph of $y = f(x)$.	**1**	
08	**9c**	A beam is supported at $(-b, 0)$and $(b, 0)$as shown in the diagram.		Solution
				
		It is known that the shape formed by the beam has equation $y = f(x)$, where $f(x)$ satisfies $f''(x) = k(b^2 - x^2)$ (k is a positive constant) and $f'(-b) = -f'(b)$.		
		(i) Show that $f'(x) = k(b^2x - \frac{x^3}{3})$	**2**	
		(ii) How far is the beam below the x-axis at $x = 0$?	**2**	

08 **10 b** The diagram shows two parallel brick walls KJ and MN joined by a fence from J to M. The wall KJ is s metres long and $\angle KJM = \alpha$. The fence JM is l metres long. **Solution**



A new fence is to be built from K to a point P somewhere on MN. The new fence KP will cross the original fence JM at O.
Let $OJ = x$ metres, where $0 < x < l$.

(i) Show that the total area, A square metres, enclosed by ΔOKJ and ΔOMP is given by $A = s(x - l + \dfrac{l^2}{2x})\sin \alpha$. **3**

(ii) Find the value of x that makes A as small as possible. Justify the fact that this value of x gives the minimum value for A. **3**

(iii) Hence, find the length of MP when A is as small as possible. **1**

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07 **6b** Let $f(x) = x^4 - 4x^3$. **Solution**

(i) Find the coordinates of the points where the curve crosses the axes. **2**

(ii) Find the coordinates of the stationary points and determine their nature. **4**

(iii) Find the coordinates of the points of inflexion. **1**

(iv) Sketch the graph of $y = f(x)$, indicating clearly the intercepts, stationary points and points of inflexion. **3**

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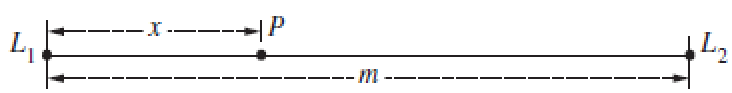
07 **10 b** The noise level, N, at a distance d metres from a single sound source of loudness L is given by the formula $N = \dfrac{L}{d^2}$. **Solution**



Two sound sources, of loudness L_1 and L_2 are placed m metres apart.

The point P lies on the line between the sound sources and is x metres from the sound source with loudness L_1.

(i) Write down a formula for the sum of the noise levels at P in terms of x. **1**

(ii) There is a point on the line between the sound sources where the sum of the noise levels is a minimum. Find an expression for x in terms of m, L_1 and L_2 if P is chosen to be this point. **4**

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06 **5a** A function $f(x)$ is defined by $f(x) = 2x^2(3 - x)$. **Solution**

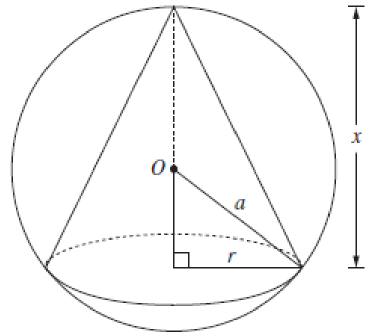
(i) Find the coordinates of the turning points of $y = f(x)$ and determine their nature. **3**

(ii) Find the coordinates of the point of inflexion. **1**

(iii) Hence sketch the graph of $y = f(x)$, showing the turning points, the point of inflexion and the points where the curve meets the x-axis. **3**

(iv) What is the minimum value of $f(x)$ for $-1 \le x \le 4$? **1**

06 **9c** A cone is inscribed in a sphere of radius a, centred at O. The height of the cone is x and the radius of the base is r, as shown in the diagram. Solution



(i) Show that the volume, V, of the cone is given by **2**

$$V = \frac{1}{3}\pi(2ax^2 - x^3).$$

(ii) Find the value of x for which the volume of the cone is a maximum. You must give reasons why your value of x gives the maximum volume. **3**

05 **4b** A function $f(x)$ is defined by $f(x) = (x + 3)(x^2 - 9)$. Solution

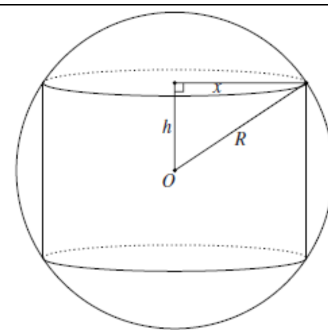
(i) Find all solutions of $f(x) = 0$. **2**

(ii) Find the coordinates of the turning points of the graph $y = f(x)$, and determine their nature. **3**

(iii) Hence sketch the graph of $y = f(x)$, showing the turning points and the points where the curve meets the x-axis. **2**

(iv) For what values of x is the graph of $y = f(x)$ concave down? **1**

05 **8a** A cylinder of radius x and height $2h$ is to be inscribed in a sphere of radius R centred at O as shown. Solution



(i) Show that the volume of the cylinder is given by $V = 2\pi h(R^2 - h^2)$. **1**

(ii) Hence, or otherwise, show that the cylinder has a maximum volume when $h = \dfrac{R}{\sqrt{3}}$. **3**