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HSC Questions by Topic
Mathematics
2016 – 2005

Preliminary Course

Includes HSC 'Coordinate methods in geometry'
** Includes HSC 'Applications of geometrical properties'

HSC Course

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Basic Arithmetic and Algebra

Back

16	**11 c**	Solve $\|x - 2\| \leq 3$.	**2**	Solution
16	**11 e**	Find the points of intersection of $y = -5 - 4x$ and $y = 3 - 2x - x^2$.	**3**	Solution
15	**1**	What is 0.00523359 written in scientific notation, correct to 4 significant figures? (A) 5.2336×10^{-2} (B) 5.234×10^{-2} (C) 5.2336×10^{-3} (D) 5.234×10^{-3}	**1**	Solution
15	**11 a**	Simplify $4x - (8 - 6x)$	**1**	Solution
15	**11 b**	Factorise fully $3x^2 - 27$	**2**	Solution
15	**11 c**	Express $\dfrac{8}{2+\sqrt{7}}$ with a rational denominator.	**2**	Solution
14	**1**	What is the value of $\dfrac{\pi^2}{6}$, correct to 3 significant figures? (A) 1.64 (B) 1.65 (C) 1.644 (D) 1.645	**1**	Solution
14	**6**	Which expression is a factorisation of $8x^3 + 27$? (A) $(2x - 3)(4x^2 + 12x - 9)$ (B) $(2x + 3)(4x^2 - 12x + 9)$ (C) $(2x - 3)(4x^2 + 6x - 9)$ (D) $(2x + 3)(4x^2 - 6x + 9)$	**1**	Solution
14	**11 a**	Rationalise the denominator of $\dfrac{1}{\sqrt{5}-2}$.	**2**	Solution
14	**11 b**	Factorise $3x^2 + x - 2$.	**2**	Solution
13	**1**	What are the solutions of $2x^2 - 5x - 1 = 0$? (A) $x = \dfrac{-5 \pm \sqrt{17}}{4}$ (B) $x = \dfrac{5 \pm \sqrt{17}}{4}$ (C) $x = \dfrac{-5 \pm \sqrt{33}}{4}$ (D) $x = \dfrac{5 \pm \sqrt{33}}{4}$	**1**	Solution
12	**1**	What is 4.097 84 correct to three significant figures? (A) 4.09 (B) 4.10 (C) 4.097 (D) 4.098	**1**	Solution
12	**2**	Which of the following is equal to $\dfrac{1}{2\sqrt{5}-\sqrt{3}}$? (A) $\dfrac{2\sqrt{5}-\sqrt{3}}{7}$ (B) $\dfrac{2\sqrt{5}+\sqrt{3}}{7}$ (C) $\dfrac{2\sqrt{5}-\sqrt{3}}{17}$ (D) $\dfrac{2\sqrt{5}+\sqrt{3}}{17}$	**1**	Solution
12	**11 a**	Factorise $2x^2 - 7x + 3$.	**2**	Solution
12	**11 b**	Solve $\|3x - 1\| < 2$.	**2**	Solution
11	**1a**	Evaluate $\sqrt[3]{\dfrac{651}{4\pi}}$ correct to four significant figures.	**2**	Solution
11	**1b**	Simplify $\dfrac{n^2 - 25}{n - 5}$.	**1**	Solution

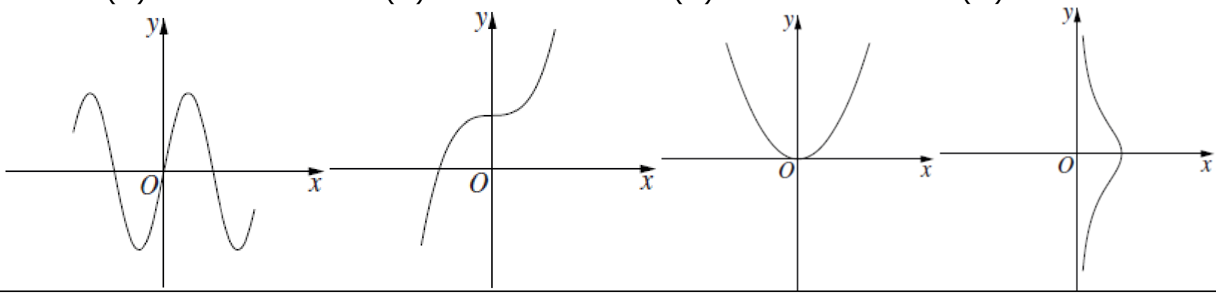
11	**1c**	Solve $2^{2x+1} = 32$.	**2**	Solution
11	**1e**	Solve $2 - 3x \le 8$.	**2**	Solution
11	**1f**	Rationalise the denominator of $\dfrac{4}{\sqrt{5}-\sqrt{3}}$. Give your answer in the simplest form.	**2**	Solution
11	**9d**	(i) Rationalise the denominator in the expression $\dfrac{1}{\sqrt{n}+\sqrt{n+1}}$, where n is an integer and $n \ge 1$.	**1**	Solution
		(ii) Using your result from part (i), or otherwise, find the value of the sum $\dfrac{1}{\sqrt{1}+\sqrt{2}} + \dfrac{1}{\sqrt{2}+\sqrt{3}} + \dfrac{1}{\sqrt{3}+\sqrt{4}} + \ldots + \dfrac{1}{\sqrt{99}+\sqrt{100}}$.	**2**	
10	**1a**	Solve $x^2 = 4x$.	**2**	Solution
10	**1b**	Find integers a and b such that $\dfrac{1}{\sqrt{5}-2} = a + b\sqrt{5}$.	**2**	Solution
10	**1d**	Solve $\|2x + 3\| = 9$.	**2**	Solution
09	**1b**	Solve $\dfrac{5x-4}{x} = 2$.	**2**	Solution
09	**1c**	Solve $\|x + 1\| = 5$.	**2**	Solution
08	**1b**	Factorise $3x^2 + x - 2$.	**2**	Solution
08	**1c**	Simplify $\dfrac{2}{n} - \dfrac{1}{n+1}$.	**2**	Solution
08	**1d**	Solve $\|4x - 3\| = 7$.	**2**	Solution
08	**1e**	Expand and simplify $(\sqrt{3} - 1)(2\sqrt{3} + 5)$.	**2**	Solution
07	**1a**	Evaluate $\sqrt{\pi^2 + 5}$ correct to two decimal places.	**2**	Solution
07	**1b**	Solve $2x - 5 > -3$ and graph the solution on a number line.	**2**	Solution
07	**1c**	Rationalise the denominator of $\dfrac{1}{\sqrt{3}-1}$.	**2**	Solution
07	**1e**	Factorise $2x^2 + 5x - 12$.	**2**	Solution
06	**1b**	Factorise $2x^2 + 5x - 3$.	**2**	Solution
06	**1e**	Solve $3 - 5x \le 2$.	**2**	Solution

05	**1a**	Evaluate $\sqrt{\dfrac{275.4}{5.2 \times 3.9}}$ correct to two significant figures.	**2**	Solution
05	**1b**	Factorise $x^3 - 27$.	**2**	Solution
05	**1d**	Express $\dfrac{(2x-3)}{2} - \dfrac{(x-1)}{5}$ as a single fraction in its simplest form.	**2**	Solution
05	**1e**	Find the values of x for which $\lvert x - 3\rvert \le 1$.	**2**	Solution

projectmaths **Real Functions of a Real Variable and Their Geometrical Representation** Back

16 **4** Which diagram best shows the graph of an odd function? **1** Solution

(A) (B) (C) (D)

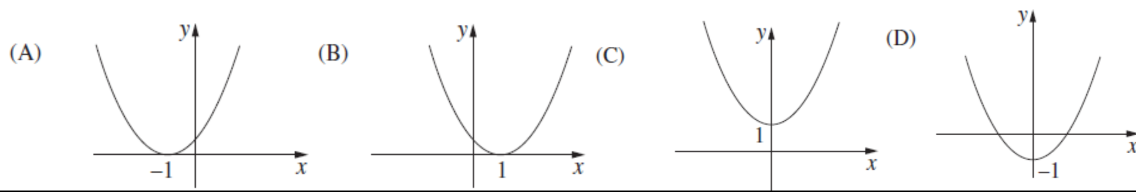


16 **11 a** Sketch the graph of $(x - 3)^2 + (y + 2)^2 = 4$. **2** Solution

15 **13 b** (i) Find the domain and range for the function $f(x) = \sqrt{9 - x^2}$. **2** Solution

(ii) On a number plane, shade the region where the points (x, y) satisfy both of the inequalities $y \leq \sqrt{9 - x^2}$ and $y \geq x$. **2**

14 **2** Which graph best represents $y = (x - 1)^2$? **1** Solution



13 **3** Which inequality defines the domain of the function $f(x) = \dfrac{1}{\sqrt{x + 3}}$? **1** Solution

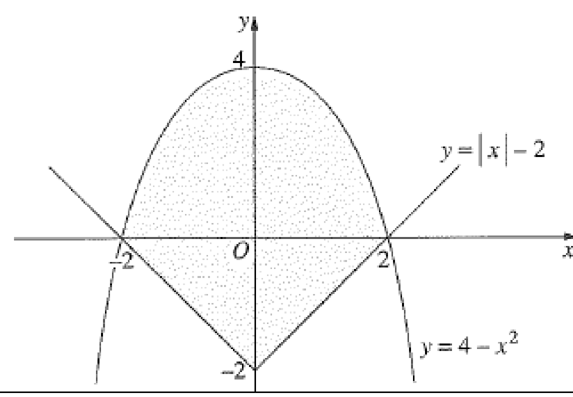
(A) $x > -3$ (B) $x \geq -3$ (C) $x < -3$ (D) $x \leq -3$

13 **11 g** Sketch the region defined by $(x - 2)^2 + (y - 3)^2 \geq 4$. **3** Solution

11 **4e** The diagram shows the graphs of $y = |x| - 2$ and $y = 4 - x^2$. **2** Solution

Write down the inequalities that together describe the shaded region.

Not to scale



10 **1c** Write down the equation of the circle with centre $(-1, 2)$ and radius 5. **1** Solution

10 **1g** Let $f(x) = \sqrt{x - 8}$. What is the domain of $f(x)$? **1** Solution

09 **3c** Shade the region in the plane defined by $y \geq 0$ and $y \leq 4 - x^2$. **2** Solution

06 **1c** Sketch the graph of $y = |x + 4|$. **2** Solution

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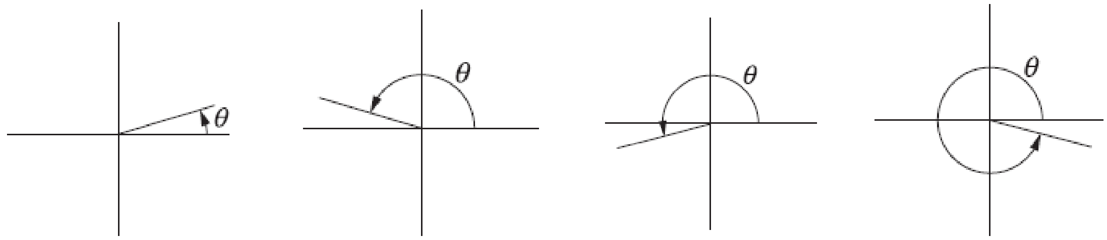
Trigonometric Ratios – Review and Some Preliminary Results

Back

16 **1** For the angle θ, $\sin\theta = \frac{7}{25}$ and $\cos\theta = -\frac{24}{25}$. **1** Solution

Which diagram best shows the angle θ?

(A) (B) (C) (D)

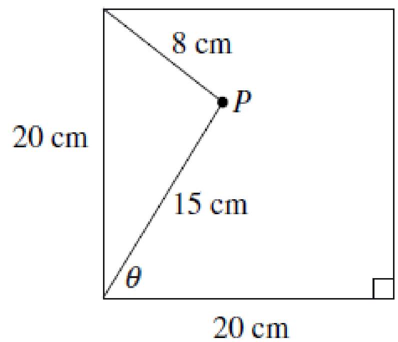


16 **12 c** Square tiles of side length 20 cm are being used to tile a bathroom. **3** Solution

The tiler needs to drill a hole in one of the tiles at a point P which is 8 cm from one corner and 15 cm from an adjacent corner.

To locate the point P the tiler needs to know the size of the angle θ shown in the diagram.

Find the size of the angle θ to the nearest degree.

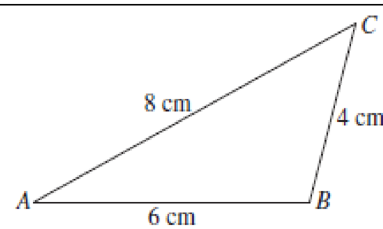


NOT TO SCALE

15 **13 a** The diagram shows $\triangle ABC$ with sides AB = 6 cm, BC = 4 cm and AC = 8 cm. Solution

(i) Show that $\cos A = \frac{7}{8}$. **1**

(ii) By finding the exact value of $\sin A$, determine the exact value of the area of $\triangle ABC$. **2**

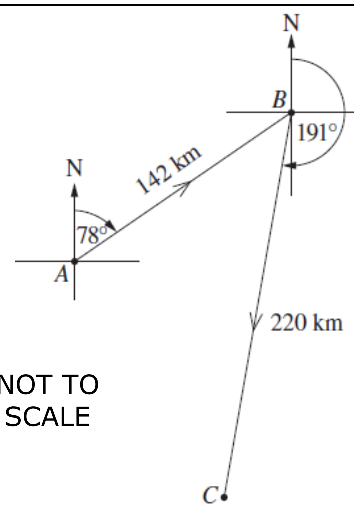


Not to scale

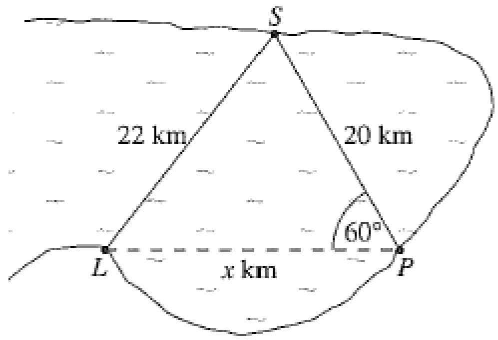
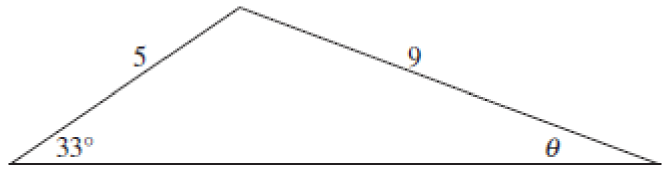
14 **13 d** Chris leaves island A in a boat and sails 142 km on a bearing of 078° to island B. Chris then sails on a bearing of 191° for 220 km to island C, as shown in the diagram. Solution

(i) Show that the distance from island C to island A is approximately 210 km. **2**

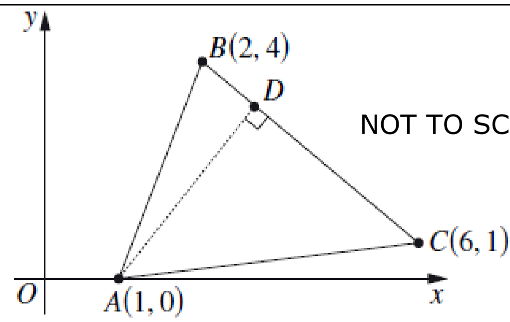
(ii) Chris wants to sail from island C directly to island A. On what bearing should Chris sail? Give your answer correct to the nearest degree. **3**



NOT TO SCALE

11	**8a**	In the diagram, the shop at S is 20 kilometres across the bay from the post office at P. The distance from the shop to the lighthouse at L is 22 kilometres and $\angle SPL$ is 60°. Let the distance PL be x kilometres.  (i) Use the cosine rule to show that $x^2 - 20x - 84 = 0$. (ii) Hence, find the distance from the post office to the lighthouse. Give your answer correct to the nearest kilometre.	 1 2	**Solution**
06	**1d**	Find the value of θ in the diagram. Give your answer to the nearest degree. 	2	**Solution**
05	**3b**	The lengths of the sides of a triangle are 7 cm, 8 cm and 13 cm. (i) Find the size of the angle opposite the longest side. (ii) Find the area of the triangle.	 2 1	**Solution**

16 **12 a** The diagram shows points $A(1, 0)$, $B(2, 4)$ and $C(6, 1)$. The point D lies on BC such that $AD \perp BC$. **Solution**



NOT TO SCALE

(i) Show that the equation of BC is $3x + 4y - 22 = 0$. **2**

(ii) Find the length of AD. **2**

(iii) Hence, or otherwise, find the area of $\triangle ABC$. **2**

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15 **2** What is the slope of the line with equation $2x - 4y + 3 = 0$? **1** **Solution**

(A) -2 (B) $-\frac{1}{2}$ (C) $\frac{1}{2}$ (D) 2

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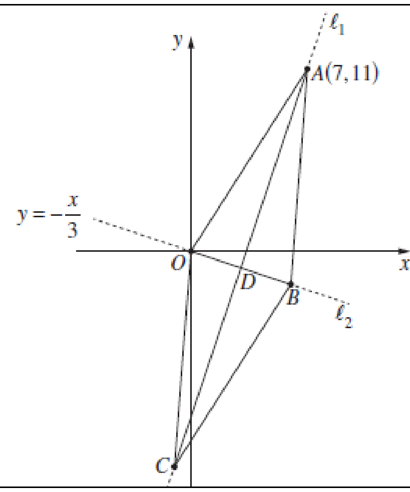
15 **12 b** The diagram shows the rhombus $OABC$. The diagonal from the point $A(7, 11)$ to the point C lies on the line ℓ_1. The other diagonal, from the origin O to the point B, lies on the line ℓ_2 which has equation $y = -\frac{x}{3}$. **Solution**



(i) Show that the equation of the line ℓ_1 is $y = 3x - 10$. **2**

(ii) The lines ℓ_1 and ℓ_2 intersect at the point D. Find the coordinates of D. **2**

Not to scale

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14 **5** Which equation represents the line perpendicular to $2x - 3y = 8$, passing through the point $(2, 0)$? **1** **Solution**

(A) $3x + 2y = 4$ (B) $3x + 2y = 6$ (C) $3x - 2y = -4$ (D) $3x - 2y = 6$

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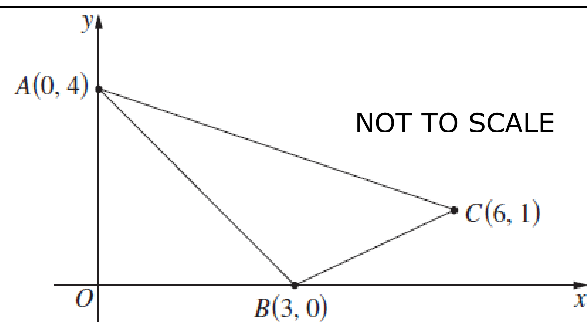
14 **12 b** The points $A(0, 4)$, $B(3, 0)$ and $C(6, 1)$ form a triangle, as shown in the diagram. **Solution**



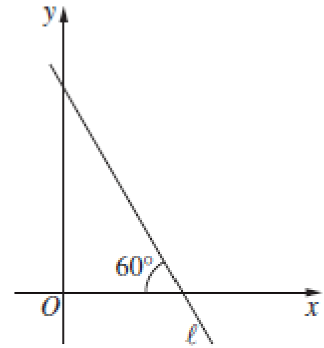
NOT TO SCALE

(i) Show that the equation of AC is $x + 2y - 9 = 0$. **2**

(ii) Find the perpendicular distance from B to AC. **2**

(iii) Hence, or otherwise, find the area of $\triangle ABC$. **2**

13 **2** The diagram shows the line ℓ. What is the slope of the line ℓ? **1** [Solution]



(A) $\sqrt{3}$

(B) $-\sqrt{3}$

(C) $\dfrac{1}{\sqrt{3}}$

(D) $-\dfrac{1}{\sqrt{3}}$

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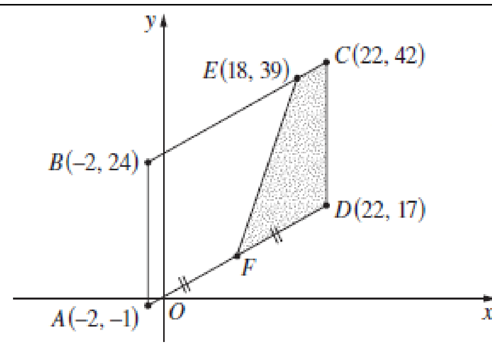
13 **12 b** The points $A(-2, -1)$, $B(-2, 24)$, $C(22, 42)$ and $D(22, 17)$ form a parallelogram as shown. The point $E(18, 39)$ lies on BC. The point F is the midpoint of AD. [Solution]



Not to scale

(i) Show that the equation of the line through A and D is $3x - 4y + 2 = 0$. **2**

(ii) Show that the perpendicular distance from B to the line through A and D is 20 units. **1**

(iii) Find the length of EC. **1**

(iv) Find the area of the trapezium $EFDC$. **2**

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13 **15 c** [Solution]

(i) Sketch the graph $y = |2x - 3|$. **1**

(ii) Using the graph from part (i), or otherwise, find all values of m for which the equation $|2x - 3| = mx + 1$ has exactly one solution. **2**

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12 **5** What is the perpendicular distance of the point $(2, -1)$ from the line $y = 3x + 1$? **1** [Solution]

(A) $\dfrac{6}{\sqrt{10}}$ (B) $\dfrac{6}{\sqrt{5}}$ (C) $\dfrac{8}{\sqrt{10}}$ (D) $\dfrac{8}{\sqrt{5}}$

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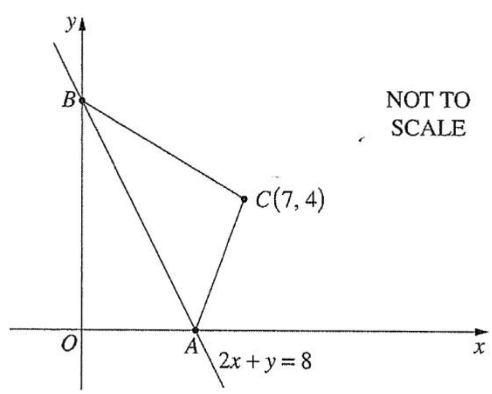
12 **13 a** The diagram shows a triangle ABC. The line $2x + y = 8$ meets the x and y axes at the points A and B respectively. The point C has coordinates $(7, 4)$. [Solution]



(i) Calculate the distance AB. **2**

(ii) It is known that $AC = 5$ and $BC = \sqrt{65}$. (Do NOT prove this.) **2**
Calculate the size of $\angle ABC$ to the nearest degree.

(iii) The point N lies on AB such that CN is perpendicular to AB. Find the coordinates of N. **3**

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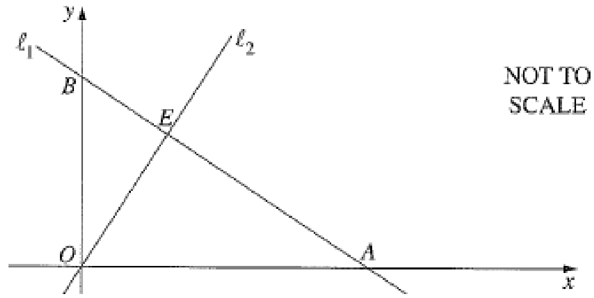
11 **3c** The diagram shows a line ℓ_1, with equation $3x + 4y - 12 = 0$, which intersects the y-axis at B. A second line ℓ_2, with equation $4x - 3y = 0$, passes through the origin O and intersects ℓ_1 at E. **Solution**



(i) Show that the co-ordinates of B are $(0, 3)$ **1**

(ii) Show that ℓ_1 is perpendicular to ℓ_2. **2**

(iii) Show that the perpendicular distance from O to ℓ_1 is $\frac{12}{5}$. **1**

(iv) Using Pythagoras' theorem, or otherwise, find the length of the interval BE. **1**

(v) Hence, or otherwise, find the area of ΔBOE. **1**

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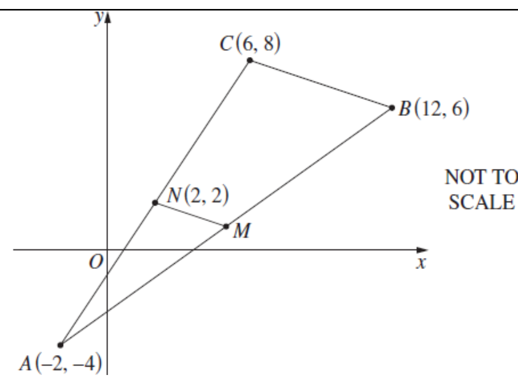
10 **3a** In the diagram, A, B and C are the points $(-2, -4)$, $(12, 6)$ and $(6, 8)$ respectively. The point $N(2, 2)$ is the midpoint of AC. The point M is the midpoint of AB. **Solution**



(i) Find the coordinates of M. **1**

(ii) Find the gradient of BC. **1**

(iii) Prove that ΔABC is similar to ΔAMN. **2**

(iv) Find the equation of MN. **2**

(v) Find the exact length of BC. **1**

(vi) Given that the area of ΔABC is 44 square units, find the perpendicular distance from A to BC. **1**

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09 **1a** Sketch the graph of $y - 2x = 3$, showing the intercepts on both axes. **2** **Solution**

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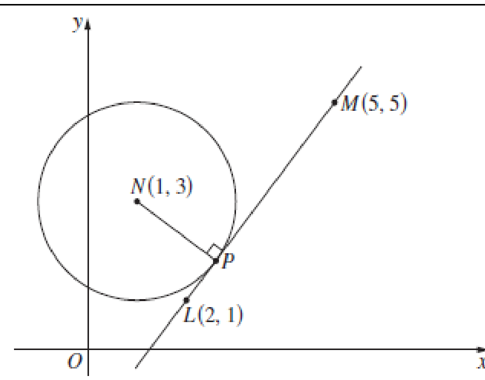
09 **3b** The circle in the diagram has centre N. The line LM is tangent to the circle at P. **Solution**



(i) Find the equation of LM is in the form $ax + by + c = 0$. **2**

(ii) Find the distance NP. **2**

(iii) Find the equation of the circle. **1**

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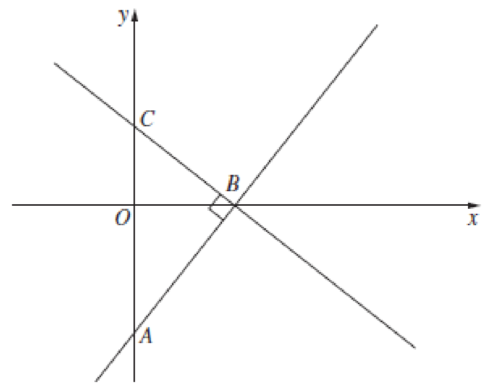
09 **5a** In the diagram, the points A and C lie on the y-axis and the point B lies on the x-axis. The line AB has equation $y = \sqrt{3}\,x - 3$.
The line BC is perpendicular to AB. [Solution]



(i) Find the equation of the line *BC.* **2**

(ii) Find the area of the triangle *ABC.* **2**

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08 **2b** Let M be the midpoint of $(-1,4)$ and $(5,8)$. **2** [Solution]
Find the equation of the line through M with gradient $-\frac{1}{2}$.

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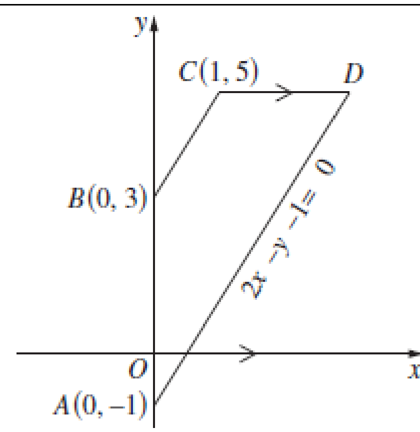
08 **3a** In the diagram $ABCD$ is a quadrilateral. The equation of the line AD is $2x - y - 1 = 0$. [Solution]



(i) Show that $ABCD$ is a trapezium by showing BC is parallel to AD. **2**

(ii) The line CD is parallel to the x-axis. Find the co-ordinates of D. **1**

(iii) Find the length of BC. **1**

(iv) Show that the perpendicular distance from B to AD is $\frac{4}{\sqrt{5}}$. **2**

(v) Hence, or otherwise, find the area of the trapezium *ABCD.* **2**

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07 **1f** Find the equation of the line that passes through the point $(-1, 3)$ and is perpendicular to $2x + y + 4 = 0$. **2** [Solution]

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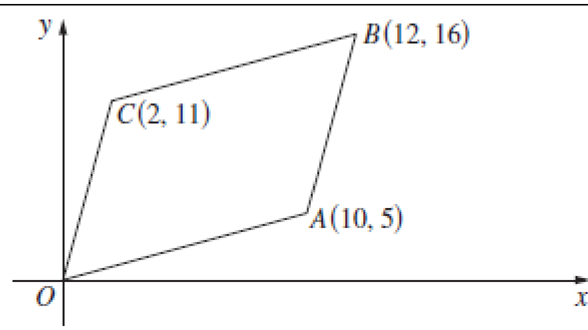
07 **3a** In the diagram, A, B and C are the points $(10, 5)$, $(12, 16)$ and $(2, 11)$ respectively. Copy or trace this diagram into your writing booklet. [Solution]



(i) Find the distance AC. **1**

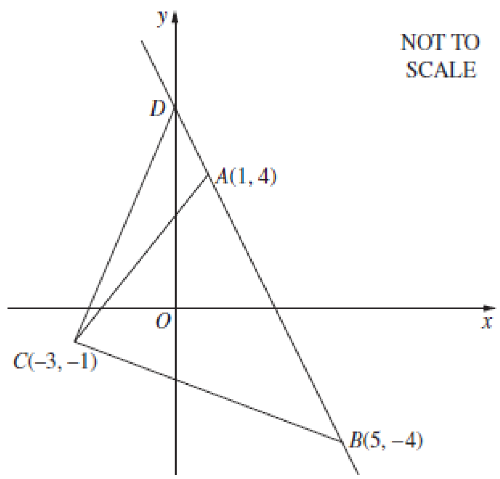
(ii) Find the midpoint of AC. **1**

(iii) Show that $OB \perp AC$. **2**

(iv) Find the midpoint of OB and hence explain why $OABC$ is a rhombus. **2**

(v) Hence, or otherwise, find the area of $OABC$. **1**

06 **3a** In the diagram, A, B and C are the points (1, 4), (5, −4) and (−3, −1) respectively. The line AB meets the y-axis at D. Solution



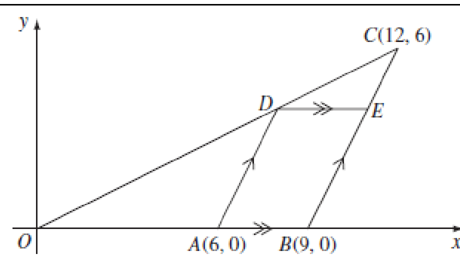
(i) Show that the equation of the line AB is $2x + y - 6 = 0$. **2**

(ii) Find the coordinates of the point D. **1**

(iii) Find the perpendicular distance of the point C from the line AB. **1**

(iv) Hence, or otherwise, find the area of the triangle ADC. **2**

05 **3c** In the diagram, A, B and C are the points (6, 0), (9, 0) and (12, 6) respectively. The equation of the line OC is $x - 2y = 0$. The point D on OC is chosen so that AD is parallel to BC. The point E on BC is chosen so that DE is parallel to the x-axis. Solution



(i) Show that the equation of the line AD is $y = 2x - 12$ **2**

(ii) Find the coordinates of the point D. **2**

(iii) Find the coordinates of the point E. **1**

(iv) Prove that $\Delta OAD \parallel\!| \Delta DEC$ **2**

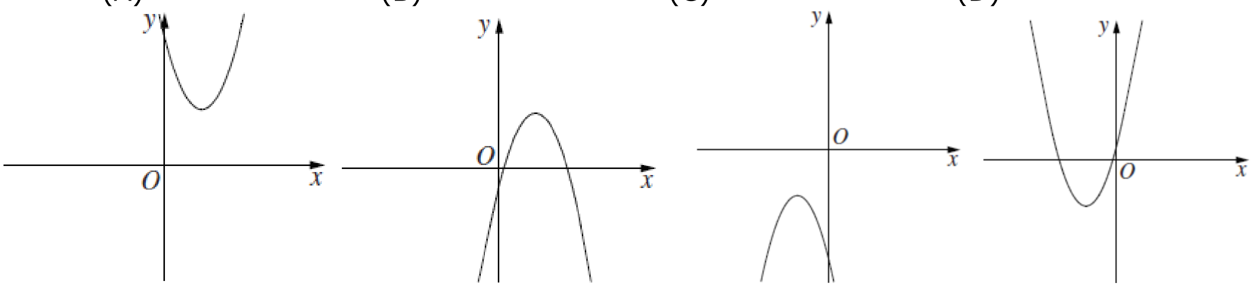
(v) Hence, or otherwise, find the ratio of the lengths AD and EC. **1**

The Quadratic Polynomial and the Parabola

Back

16 **3** Which diagram best shows the graph of the parabola $y = 3 - (x - 2)^2$? **1** Solution

(A) (B) (C) (D)



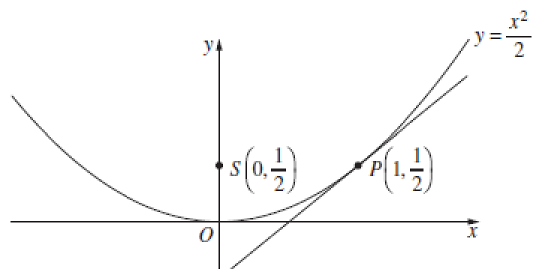
16 **13 b** Consider the parabola $x^2 - 4x = 12y + 8$. Solution

(i) By completing the square, or otherwise, find the focal length of the parabola. **2**

(ii) Find the coordinates of the focus. **1**

15 **12 d** For what values of k does the quadratic equation $x^2 - 8x + k = 0$ have real roots? **2** Solution

15 **12 e** The diagram shows the parabola $y = \dfrac{x^2}{2}$ with focus $S(0, \frac{1}{2})$. A tangent to the parabola is drawn at $P(1, \frac{1}{2})$. Solution



Not to scale

(i) Find the equation of the tangent at the point P. **2**

(ii) What is the equation of the directrix of the parabola? **1**

(iii) The tangent and directrix intersect at Q. Show that Q lies on the y-axis. **1**

(iv) Show that $\triangle PQS$ is isosceles. **1**

14 **14 b** The roots of the quadratic equation $2x^2 + 8x + k = 0$ are α and β. Solution

(i) Find the value of $\alpha + \beta$. **1**

(ii) Given that $\alpha^2\beta + \alpha\beta^2 = 6$, find the value of k. **2**

13 **7** A parabola has focus $(5, 0)$ and directrix $x = 1$. **1** Solution

What is the equation of the parabola?

(A) $y^2 = 16(x - 5)$ (B) $y^2 = 8(x - 3)$ (C) $y^2 = -16(x - 5)$ (D) $y^2 = -8(x - 3)$

12 **3** The quadratic equation $x^2 + 3x - 1 = 0$ has roots α and β. **1** Solution

What is the value of $\alpha\beta + (\alpha + \beta)$?

(A) 4 (B) 2 (C) −4 (D) −2

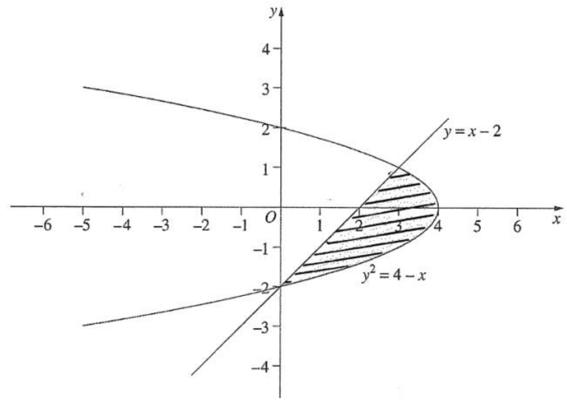
12 **8** The diagram shows the region enclosed by $y = x - 2$ and $y^2 = 4 - x$.
Which of the following pairs of inequalities describes the shaded region in the diagram?

(A) $y^2 \le 4 - x$ and $y \le x - 2$

(B) $y^2 \le 4 - x$ and $y \ge x - 2$

(C) $y^2 \ge 4 - x$ and $y \le x - 2$

(D) $y^2 \ge 4 - x$ and $y \ge x - 2$



1 Solution

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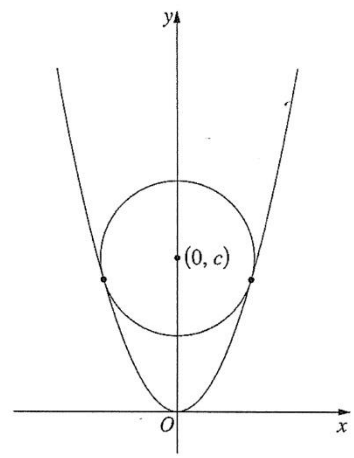
12 **11e** Find the coordinates of the focus of the parabola $x^2 = 16(y - 2)$. **2** Solution

---

12 **16c** The circle $x^2 + (y - c)^2 = r^2$, where $c > 0$ and $r > 0$, lies inside the parabola $y = x^2$.
The circle touches the parabola at exactly two points located symmetrically on opposite sides of the y-axis, as shown in the diagram.

(i) Show that $4c = 1 + 4r^2$. **2**

(ii) Deduce that $c > \frac{1}{2}$. **1**



Solution

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11 **2a** The quadratic equation $x^2 - 6x + 2 = 0$ has roots α and β.

(i) Find $\alpha + \beta$. **1**

(ii) Find $\alpha\beta$. **1**

(iii) Find $\frac{1}{\alpha} + \frac{1}{\beta}$. **1**

Solution

---

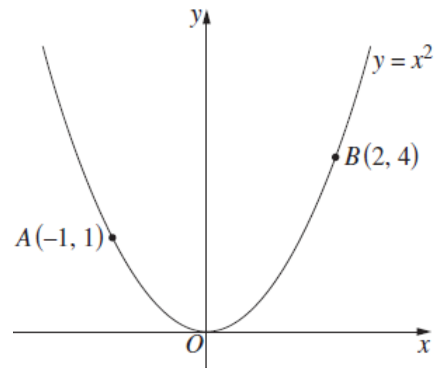
11 **3b** A parabola has focus (3, 2) and directrix $y = -4$.
Find the coordinates of the vertex. **2** Solution

---

11 **6b** A point $P(x, y)$ moves so that the sum of the square of its distance from each of the points $A(-1, 0)$ and $B(3, 0)$ is equal to 40.
Show that the locus of $P(x, y)$ is a circle, and state its radius and centre. **3** Solution

---

10 **2b** Solve the inequality $x^2 - x - 12 < 0$. **2** Solution

10	**7b**	The parabola shown in the diagram is the graph $y = x^2$. The points $A(-1, 1)$ and $B(2, 4)$ are on the parabola. (i) Find the equation of the tangent to the parabola at A. (ii) Let M be the midpoint of AB. There is a point C on the parabola such that the tangent at C is parallel to AB. Show that the line MC is vertical. (iii) The tangent at A meets the line MC at T. Show that the line BT is a tangent to the parabola. 	 2 2 2	Solution
09	**4b**	Find the values of k for which the quadratic equation $x^2 - (k + 4)x + (k + 7) = 0$ has equal roots.	3	Solution
08	**4c**	Consider the parabola $x^2 = 8(y - 3)$. (i) Write down the coordinates of the vertex. (i) Find the coordinates of the focus. (iii) Sketch the parabola.	 1 1 1	Solution
07	**7a**	(i) Find the coordinates of the focus, S, of the parabola $y = x^2 + 4$. (ii) The graphs of $y = x^2 + 4$ and the line $y = x + k$ have only one point of intersection, P. Show that the x-coordinate of P satisfies $x^2 - x + 4 - k = 0$. (iii) Using the discriminant, or otherwise, find the value of k. (iv) Find the coordinates of P. (v) Show that SP is parallel to the directrix of the parabola.	2 1 1 2 1	Solution
06	**7a**	Let α and β be the solutions of $x^2 - 3x + 1 = 0$. (i) Find $\alpha\beta$. (ii) Hence find $\alpha + \frac{1}{\alpha}$.	 1 1	Solution
06	**7c**	(i) Write down the discriminant of $2x^2 + (k - 2)x + 8$, where k is a constant. (ii) Hence, or otherwise, find the values of k for which the parabola $y = 2x^2 + kx + 9$ does not intersect the line $y = 2x + 1$.	1 2	Solution
06	**9a**	Find the coordinates of the focus of the parabola $12y = x^2 - 6x - 3$.	2	Solution
05	**1f**	Find the coordinates of the focus of the parabola $x^2 = 8(y - 1)$.	2	Solution

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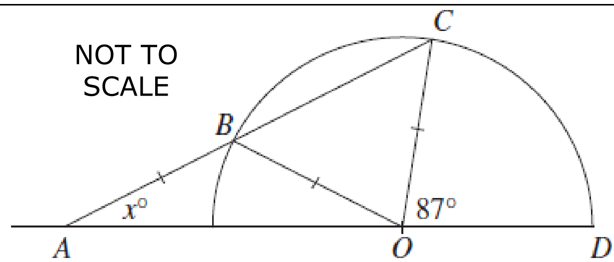
Plane Geometry

Back

16 **12 b** The diagram shows a semicircle with centre O. It is given that $AB = OB$, $\angle COD = 87^\circ$ and $\angle BAO = x^\circ$.



Solution

(i) Show that $\angle CBO = 2x^\circ$, giving reasons. **1**

(ii) Find the value of x, giving reasons. **2**

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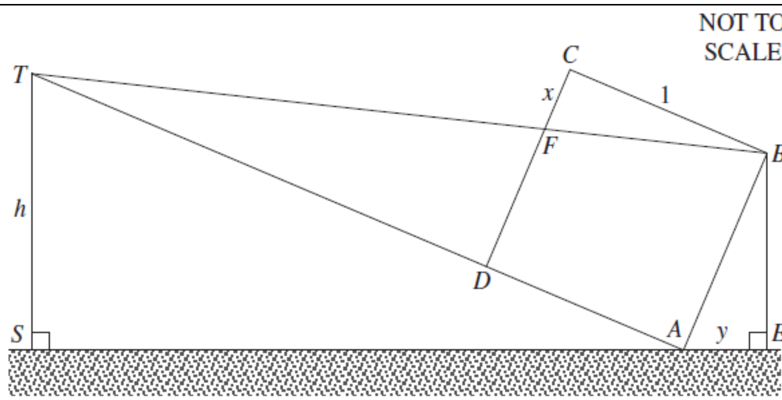
16 **15 c** Maryam wishes to estimate the height, h metres, of a tower, ST, using a square, $ABCD$, with side length 1 metre.



Solution

She places the point A on the horizontal ground and ensures that the point D lies on the line joining A to the top of the tower T.
The point F is the intersection of the line joining B and T and the side BC. The point E is the foot of the perpendicular from B to the ground. Let CF have length x metres and AE have length y metres.
Copy and trace the diagram into your writing booklet.

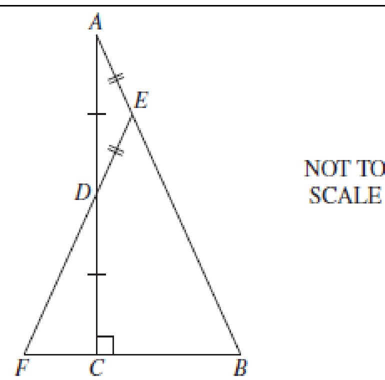
(i) Show that ΔFCB and ΔBAT are similar. **2**

(ii) Show that ΔTSA and ΔAEB are similar. **2**

(iii) Find h in terms of x and y. **2**

---

15 **15 b** The diagram shows ΔABC which has a right angle at C. The point D is the midpoint of the side AC. The point E is chosen on AB such that $AE = ED$. The line segment ED is produced to meet the line BC at F. Copy or trace the diagram into your writing booklet.



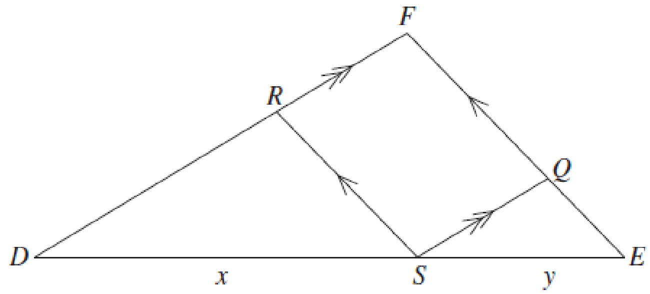
Solution

(i) Prove that ΔACB is similar to ΔDCF. **2**

(ii) Explain why ΔEFB is isosceles. **1**

(iii) Show that $EB = 3AE$. **2**

14 **15 b** In ΔDEF, a point S is chosen on the side DE. The length of DS is x, and the length of ES is y. The line through S parallel to DF meets EF at Q. The line through S parallel to EF meets DF at R. The area of ΔDEF is A. The areas of ΔDSR and ΔSEQ are A_1 and A_2 respectively. Solution



(i) Show that ΔDEF is similar to ΔDSR. **2**

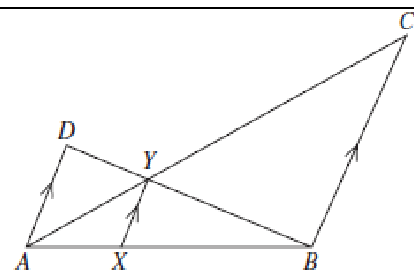
(ii) Explain why $\dfrac{DR}{DF} = \dfrac{x}{x+y}$. **1**

(iii) Show that $\sqrt{\dfrac{A_1}{A}} = \dfrac{x}{x+y}$. **2**

(iv) Using the result from part (iii) and a similar expression for $\sqrt{\dfrac{A_2}{A}}$, deduce that $\sqrt{A} = \sqrt{A_1} + \sqrt{A_2}$. **2**

---

13 **16 c** The diagram shows triangles ABC and ABD with AD parallel to BC. The sides AC and BD intersect at Y. The point X lies on AB such that XY is parallel to AD and BC. Solution

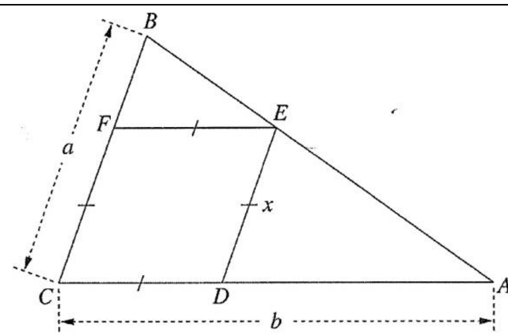


(i) Prove that ΔABC is similar to ΔAXY. **2**

(ii) Hence, or otherwise, prove that $\dfrac{1}{XY} = \dfrac{1}{AD} + \dfrac{1}{BC}$. **2**

---

12 **16 a** The diagram shows a triangle ABC with sides $BC = a$ and $AC = b$. The points D, E and F lie on the sides AC, AB and BC, respectively, so that $CDEF$ is a rhombus with sides of length x. Solution

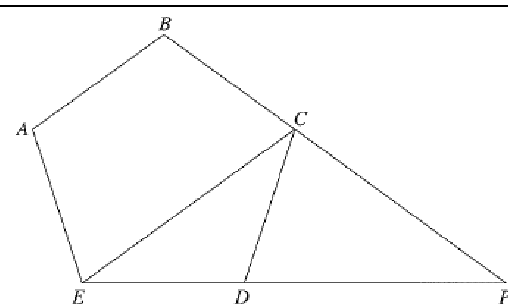


(i) Prove that ΔEBF is similar to ΔAED. **2**

(ii) Find an expression for x in terms of a and b. **2**

---

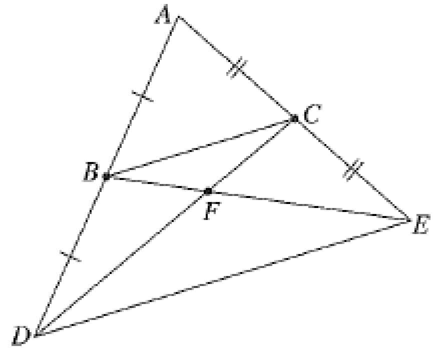
11 **6a** The diagram shows a regular pentagon $ABCDE$. Sides ED and BC are produced to meet at P. Copy or trace the diagram into your writing booklet. Solution



(i) Find the size of $\angle CDE$. **1**

(ii) Hence, show that ΔEPC is isosceles. **2**

11 **9a** The diagram shows ΔADE, where B is the midpoint of AD and C is the midpoint of AE. The intervals BE and CD meet at F. [Solution]

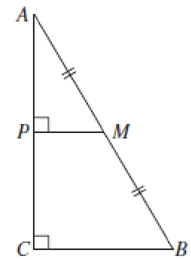


(i) Explain why ΔABC is similar to ΔADE. **1**

(ii) Hence, or otherwise, prove that the ratio $BF{:}FE = 1{:}2$. **2**

---

09 **4c** In the diagram, ΔABC is a right-angled triangle, with the right angle at C. [Solution]
The midpoint of AB is M, and $MP \perp AC$.



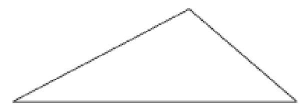
(i) Prove that ΔAMP is similar to ΔABC. **2**

(ii) What is the ratio of AP to AC? **1**

(iii) Prove that ΔAMC is isosceles. **2**

(iv) Show that ΔABC can be divided into two isosceles triangles. **1**

(v) Copy or trace this triangle into your writing booklet and show how to divide it into four isosceles triangles. **1**



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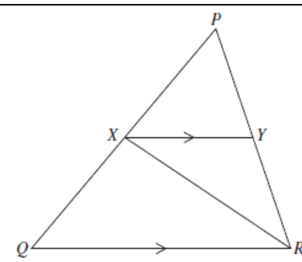
08 **4a** In the diagram, XR bisects $\angle PRQ$ and $XY \parallel QR$. **2** [Solution]



Copy or trace the diagram into your writing booklet.

Prove that ΔXYR is an isosceles triangle.

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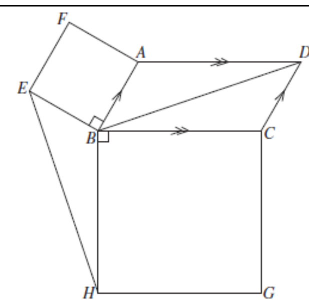
08 **8b** In the diagram, $ABCD$ is a parallelogram and ABEF and BCGH are both squares. [Solution]
Copy or trace the diagram into your writing booklet.



(i) Prove that $CD = BE$. **1**

(ii) Prove that $BD = EH$. **3**

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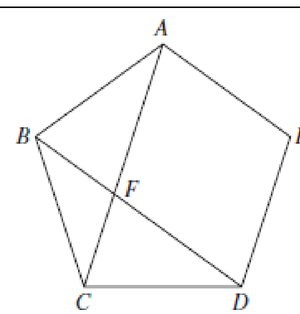
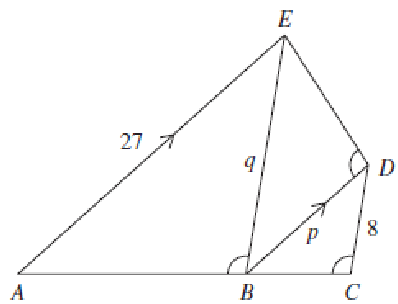
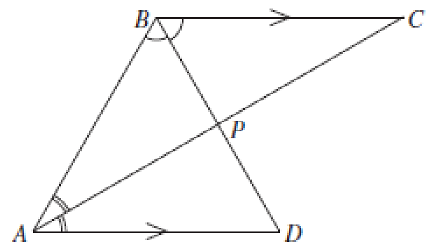
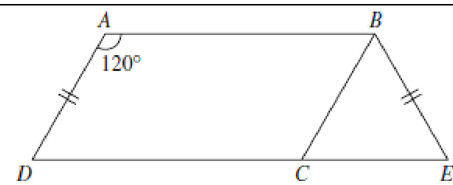
07 **5a** In the diagram, $ABCDE$ is a regular pentagon. The diagonals AC and BD intersect at F. Copy or trace this diagram into your writing booklet. [Solution]



(i) Show that the size of $\angle ABC$ is 108°. **1**

(ii) Find the size of $\angle BAC$. **2**
Give reasons for your answer.

(iii) By considering the sizes of angles, show that ΔABF is isosceles. **2**

07	**8b**	In the diagram, *AE* is parallel to *BD*, *AE* = 27, *CD* = 8, *BD* = *p*, *BE* = *q* and $\angle ABE$, $\angle BCD$ and $\angle BDE$ are equal. Copy or trace this diagram into your writing booklet. (i) Prove that $\Delta ABE \;\vert\vert\vert\; \Delta BCD$. (ii) Prove that $\Delta EDB \;\vert\vert\vert\; \Delta BCD$. 	 **2** **2**	Solution
06	**6a**	In the diagram, *AD* is parallel to *BC*, *AC* bisects $\angle BAD$ and *BD* bisects $\angle ABC$. The lines *AC* and *BD* intersect at *P*. Copy or trace the diagram into your writing booklet. (i) Prove that $\angle BAC = \angle BCA$. (ii) Prove that $\Delta ABP \equiv \Delta CBP$. (iii) Prove that *ABCD* is a rhombus. 	 **1** **2** **3**	Solution
05	**5b**	The diagram shows a parallelogram *ABCD* with $\angle DAB = 120°$. The side *DC* is produced to *E* so that *AD* = *BE*. Copy or trace the diagram into your writing booklet. Prove that Δ *BCE* is equilateral. 	**3**	Solution

The Tangent to a Curve and the Derivative of a Function

Back

16 **11 b** Differentiate $\dfrac{x+2}{3x-4}$. **2** Solution

15 **12 c** Find $f'(x)$, where $f(x) = \dfrac{x^2+3}{x-1}$. **2** Solution

14 **11 c** Differentiate $\dfrac{x^3}{x+1}$. **2** Solution

13 **11 b** Evaluate $\displaystyle\lim_{x\to 2} \frac{x^3-8}{x^2-4}$. **2** Solution

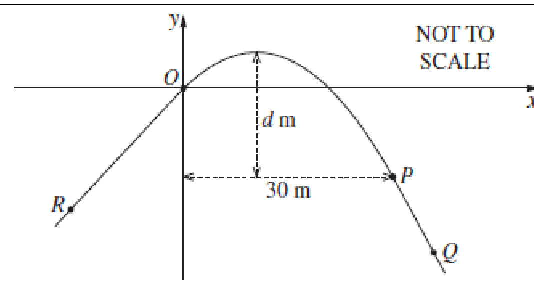
12 **11 c** Find the equation of the tangent to the curve $y = x^2$ at the point where $x = 3$. **2** Solution

11 **2c** Find the equation of the tangent to the curve $y = (2x + 1)^4$ at the point where $x = -1$. **3** Solution

10 **8d** Let $f(x) = x^3 - 3x^2 + kx + 8$, where k is a constant. Find the values of k for which $f(x)$ is an increasing function. **2** Solution

09 **1d** Find the gradient of the tangent to the curve $y = x^4 - 3x$ at the point (1, -2). **2** Solution

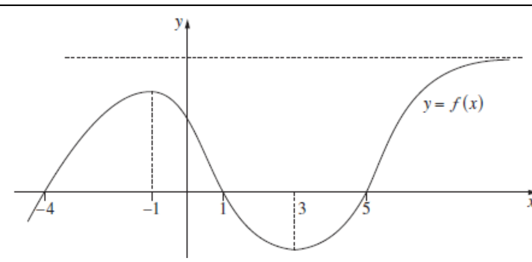
09 **6c** The diagram illustrates the design for part of a roller-coaster track. The section RO is a straight line with slope 1.2 and the section PQ is a straight line with slope -1.8. The section OP is a parabola $y = ax^2 + bx$. The horizontal distance from the y-axis to P is 30 m. In order that the ride is smooth, the straight sections must be tangent to the parabola at O and at P. Solution



(i) Find the values of a and b so that the ride is smooth. **3**

(ii) Find the distance d, from the vertex of the parabola to the horizontal line through P, as shown on the diagram. **2**

09 **8a** The diagram shows the graph of a function $y = f(x)$. Solution



(i) For which values of x is the derivative, $f'(x)$, negative? **1**

(ii) What happens to $f'(x)$ for large values of x? **1**

(iii) Sketch the graph of $y = f'(x)$. **2**

08 **2a** Differentiate with respect to x: (i) $(x^2 + 3)^9$ **2** Solution

05 **2b** Differentiate with respect to x: (ii) $\dfrac{x^2}{x-1}$ **2** Solution

Geometric Applications of Differentiation

Back

16 13 a Consider the function $y = 4x^3 - x^4$. Solution

(i) Find the two stationary points and determine their nature. **4**

(ii) Sketch the graph of the function, clearly showing the stationary points and the x and y intercepts. **2**

---

15 13 c Consider the curve $y = x^3 - x^2 - x + 3$. Solution

(i) Find the stationary points and determine their nature. **4**

(ii) Given that the point $P(\frac{1}{3}, \frac{70}{27})$ lies on the curve, prove that there is a point of inflexion at P. **2**

(iii) Sketch the curve, labelling the stationary points, point of inflexion and y-intercept. **2**

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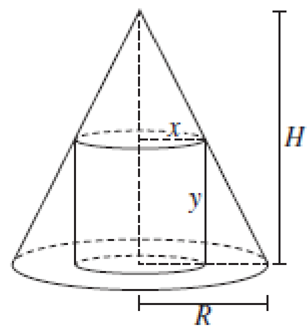
15 16 c The diagram shows a cylinder of radius x and height y inscribed in a cone of radius R and height H, where R and H are constants. Solution



The volume of a cone of radius r and height h is $\frac{1}{3}\pi r^2 h$.

The volume of a cylinder of radius r and height h is $\pi r^2 h$.

(i) Show that the volume, V, of the cylinder can be written as $V = \frac{H}{R}\pi x^2(R - x)$. **3**

(ii) By considering the inscribed cylinder of maximum volume, show that the volume of any inscribed cylinder does not exceed $\frac{4}{9}$ of the volume of the cone. **4**

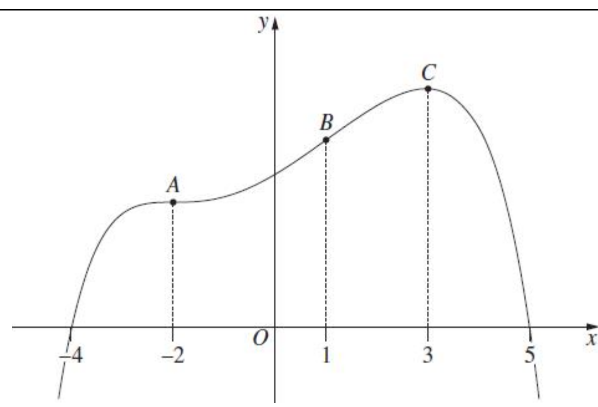
---

14 11 f The gradient function of a curve $y = f(x)$ is given by $f'(x) = 4x - 5$. The curve passes through the point (2, 3). Find the equation of the curve. **2** Solution

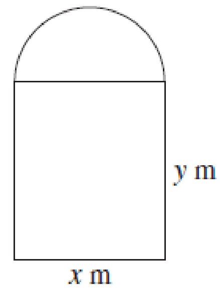
---

14 14 e The diagram shows the graph of a function $f(x)$. The graph has a horizontal point of inflexion at A, a point of inflexion at B and a maximum turning point at C. **3** Solution

Sketch the graph of the derivative $f'(x)$.



14 **16 c** The diagram shows a window consisting of two sections.
The top section is a semicircle of diameter x m. The bottom section is a rectangle of width x m and height y m.
The entire frame of the window, including the piece that separates the two sections, is made using 10 m of thin metal.
The semicircular section is made of coloured glass and the rectangular section is made of clear glass.
Under test conditions the amount of light coming through one square metre of the coloured glass is 1 unit and the amount of light coming through one square metre of the clear glass is 3 units. The total amount of light coming through the window under test conditions is L units.



Solution

(i) Show that $y = 5 - x\left(1 + \frac{\pi}{4}\right)$. **2**

(ii) Show that $L = 15x - x^2\left(3 + \frac{5\pi}{8}\right)$. **2**

(ii) Find the values of x and y that maximise the amount of light coming through the window under test conditions. **3**

---

13 **8** The diagram shows the points A, B, C and D on the graph $y = f(x)$. **1** Solution

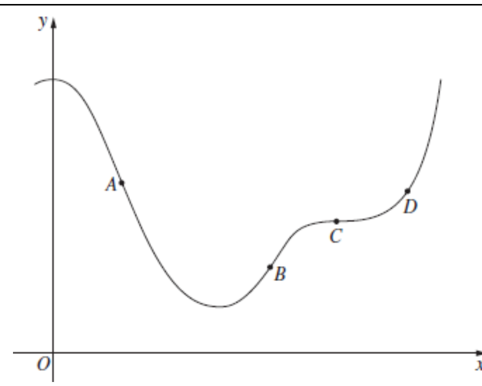
At which point is $f'(x) > 0$ and $f''(x) = 0$.

(A) A

(B) B

(C) C

(D) D



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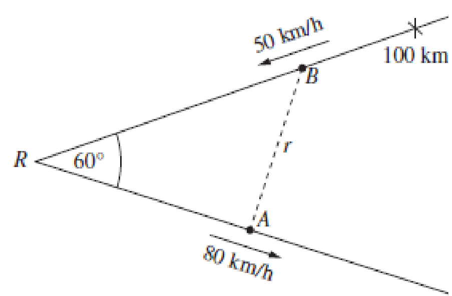
13 **12 a** The cubic $y = ax^3 + bx^2 + cx + d$ has a point of inflexion at $x = p$. **2** Solution

Show that $p = -\frac{b}{3a}$.

---

13 **14 b** Two straight roads meet at R at an angle of 60º.
At time $t = 0$ car A leaves R on one road, and car B is 100 km from R on the other road. Car A travels away from R at a speed of 80 km/h, and car B travels towards R at a speed of 50 km/h.
The distance between the cars at time t hours is r km.



Solution

(i) Show that $r^2 = 12\ 900t^2 - 18\ 000t + 10\ 000$. **2**

(ii) Find the minimum distance between the cars. **3**

---

13 **16 a** The derivative of a function $f(x)$ is $f'(x) = 4x - 3$. **3** Solution
The line $y = 5x - 7$ is tangent to the graph of $f(x)$.
Find the function $f(x)$.

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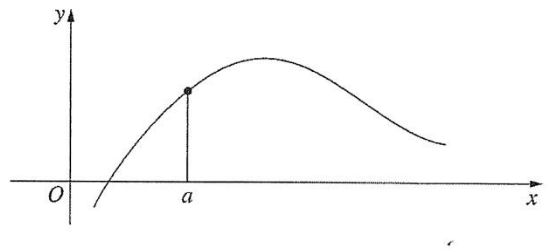
12 **4** The diagram shows the graph of $y = f(x)$. **1** [Solution]
Which of the following statements is true?



(A) $f'(a) > 0$ and $f''(a) < 0$

(B) $f'(a) > 0$ and $f''(a) > 0$

(C) $f'(a) < 0$ and $f''(a) < 0$

(D) $f'(a) < 0$ and $f''(a) > 0$

---

12 **14 a** A function is given by $f(x) = 3x^4 + 4x^3 - 12x^2$. [Solution]

(i) Find the nature of the stationary points of $f(x)$ and determine their nature. **3**

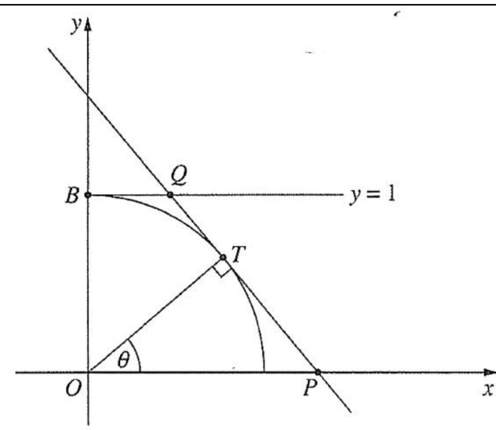
(ii) Hence, sketch the graph of $y = f(x)$ showing the stationary points. **2**

(iii) For what values of x is the function increasing? **1**

(iv) For what values of k will $3x^4 + 4x^3 - 12x^2 + k = 0$ have no solution? **1**

---

12 **16 b** The diagram shows a point T on the unit circle $x^2 + y^2 = 1$ at angle θ from the positive x-axis, where $0 < \theta < \dfrac{\pi}{2}$. The tangent to the circle at T is perpendicular to OT, and intersects the x-axis at P, and the line $y = 1$ at Q. The line $y = 1$ intersects the y-axis at B. [Solution]



(i) Show that the equation of the line PT is $x\cos\theta + y\sin\theta = 1$. **2**

(ii) Find the length of BQ in terms of θ. **1**

(iii) Show that the area, A, of the trapezium $OPQB$ is given by $A = \dfrac{2 - \sin\theta}{2\cos\theta}$. **2**

(iv) Find the angle θ that gives the minimum area of the trapezium. **3**

---

11 **4c** The gradient of a curve is given by $\dfrac{dy}{dx} = 6x - 2$. The curve passes through the point $(-1, 4)$. What is the equation of the curve? **2** [Solution]

---

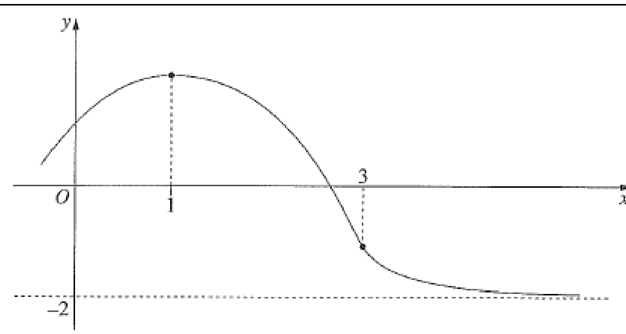
11 **7a** Let $f(x) = x^3 - 3x + 2$. [Solution]

(i) Find the coordinates of the stationary points of $y = f(x)$, and determine their nature. **3**

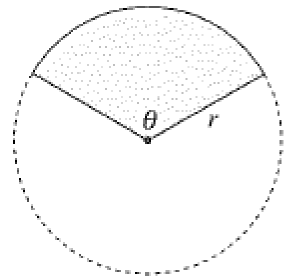
(ii) Hence, sketch the graph $y = f(x)$ showing all stationary points and the y-intercept. **2**

---

11 **9c** The graph $y = f(x)$ in the diagram has a stationary point when $x = 1$, a point of inflexion when $x = 3$, and a horizontal asymptote $y = -2$. **3** [Solution]
Sketch the graph $y = f'(x)$, clearly indicating its features at $x = 1$ and at $x = 3$, and the shape of the graph as $x \to \infty$.



11 **10 b** A farmer is fencing a paddock using P metres of fencing. The paddock is to be in the shape of a sector of a circle with radius r and sector θ in radians, as shown in the diagram. [Solution](#)



(i) Show that the length of fencing required to fence the perimeter of the paddock is $P = r(\theta + 2)$. **1**

(ii) Show that the area of the sector is $A = \frac{1}{2}Pr - r^2$. **1**

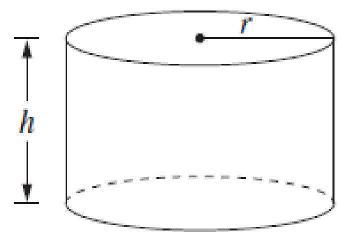
(iii) Find the radius of the sector, in terms of P, that will maximize the area of the paddock. **2**

(iv) Find the angle θ that gives the maximum area of the paddock. **1**

(v) Explain why it is only possible to construct a paddock in the shape of a sector if $\frac{P}{2(\pi + 1)} < r < \frac{P}{2}$. **2**

---

10 **5a** A rainwater tank is to be designed in the shape of a cylinder with radius r metres and height h metres. The volume of the tank is to be 10 cubic metres. Let A be the surface area of the tank, including its top and base, in square metres. [Solution](#)



(i) Given that $A = 2\pi r^2 + 2\pi rh$, show that $A = 2\pi r^2 + \frac{20}{r}$. **2**

(ii) Show that A has a minimum value and find the value of r for which the minimum occurs. **3**

---

10 **6a** Let $f(x) = (x + 2)(x^2 + 4)$. [Solution](#)

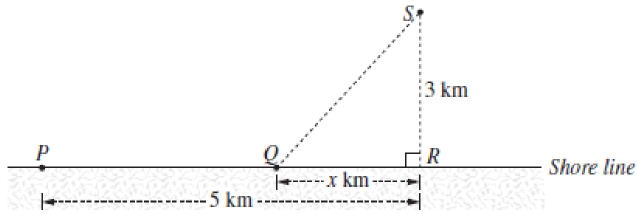
(i) Show that the graph of $y = f(x)$ has no stationary points. **2**

(ii) Find the values of x for which the graph $y = f(x)$ is concave down, and the values for which it is concave up. **2**

(iii) Sketch the graph of $y = f(x)$, indicating the values of the x and y intercepts. **2**

09 **9b** An oil rig, S, is 3 km offshore. A power station, P, is on the shore. A cable is to be laid from P to S. It costs \$1000 per kilometres to lay the cable along the shore and \$2600 per kilometre to lay the cable underwater from the shore to S. The point R is the point on the shore closest to S, and the distance PR is 5 km. The point Q is on the shore, at a distance of x km from R, as shown in the diagram. Solution

(i) Find the total cost of laying the cable in a straight line from P to R and then in a straight line from R to S. **1**



(ii) Find the cost of laying the cable in a straight line from P to S. **1**

(iii) Let \$$C$ be the total cost of laying the cable in a straight line from P to Q, and then in a straight line from Q to S. **2**

Show that $C = 1000(5 - x + 2.6\sqrt{x^2 + 9})$.

(iv) Find the minimum cost of laying the cable. **3**

(v) New technology means that the cost of laying the cable underwater can be reduced to \$1100 per kilometre. Determine the path for laying the cable in order to minimise the cost in this case. **1**

08 **8a** Let $f(x) = x^4 - 8x^2$. Solution

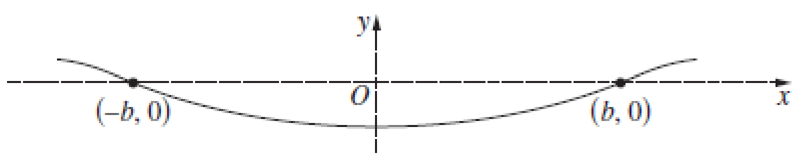
(i) Find the coordinates of the points where the graph of $y = f(x)$ crosses the axes. **2**

(ii) Show that $f(x)$ is an even function. **1**

(iii) Find the coordinates of the stationary points of $y = f(x)$ and determine their nature. **4**

(iv) Sketch the graph of $y = f(x)$. **1**

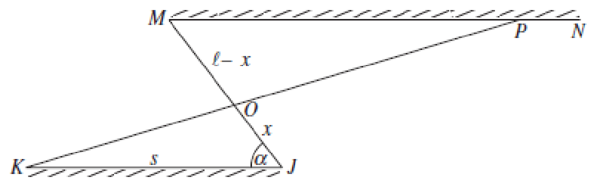
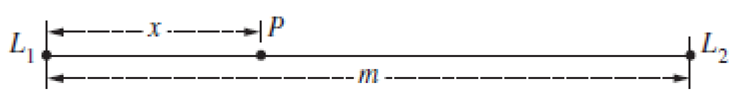
08 **9c** A beam is supported at $(-b, 0)$and $(b, 0)$as shown in the diagram. Solution



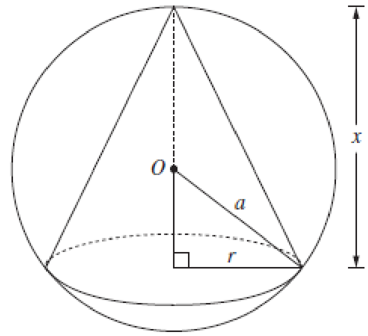
It is known that the shape formed by the beam has equation $y = f(x)$, where $f(x)$ satisfies $f''(x) = k(b^2 - x^2)$ (k is a positive constant) and $f'(-b) = -f'(b)$.

(i) Show that $f'(x) = k(b^2x - \frac{x^3}{3})$ **2**

(ii) How far is the beam below the x-axis at $x = 0$? **2**

08	10 b	The diagram shows two parallel brick walls KJ and MN joined by a fence from J to M. The wall KJ is s metres long and $\angle KJM = \alpha$. The fence JM is l metres long.  A new fence is to be built from K to a point P somewhere on MN. The new fence KP will cross the original fence JM at O. Let $OJ = x$ metres, where $0 < x < l$. (i) Show that the total area, A square metres, enclosed by ΔOKJ and ΔOMP is given by $A = s(x - l + \frac{l^2}{2x})\sin\alpha$. (ii) Find the value of x that makes A as small as possible. Justify the fact that this value of x gives the minimum value for A. (iii) Hence, find the length of MP when A is as small as possible.	 3 3 1	Solution
07	6b	Let $f(x) = x^4 - 4x^3$. (i) Find the coordinates of the points where the curve crosses the axes. (ii) Find the coordinates of the stationary points and determine their nature. (iii) Find the coordinates of the points of inflexion. (iv) Sketch the graph of $y = f(x)$, indicating clearly the intercepts, stationary points and points of inflexion.	 2 4 1 3	Solution
07	10 b	The noise level, N, at a distance d metres from a single sound source of loudness L is given by the formula $N = \frac{L}{d^2}$.  Two sound sources, of loudness L_1 and L_2 are placed m metres apart. The point P lies on the line between the sound sources and is x metres from the sound source with loudness L_1. (i) Write down a formula for the sum of the noise levels at P in terms of x. (ii) There is a point on the line between the sound sources where the sum of the noise levels is a minimum. Find an expression for x in terms of m, L_1 and L_2 if P is chosen to be this point.	 1 4	Solution
06	5a	A function $f(x)$ is defined by $f(x) = 2x^2(3 - x)$. (i) Find the coordinates of the turning points of $y = f(x)$ and determine their nature. (ii) Find the coordinates of the point of inflexion. (iii) Hence sketch the graph of $y = f(x)$, showing the turning points, the point of inflexion and the points where the curve meets the x-axis. (iv) What is the minimum value of $f(x)$ for $-1 \le x \le 4$?	 3 1 3 1	Solution

06 **9c** A cone is inscribed in a sphere of radius a, centred at O. The height of the cone is x and the radius of the base is r, as shown in the diagram. [Solution]



(i) Show that the volume, V, of the cone is given by **2**

$$V = \frac{1}{3}\pi(2ax^2 - x^3).$$

(ii) Find the value of x for which the volume of the cone is a maximum. You must give reasons why your value of x gives the maximum volume. **3**

05 **4b** A function $f(x)$ is defined by $f(x) = (x + 3)(x^2 - 9)$. [Solution]

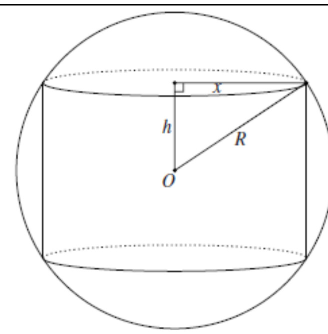
(i) Find all solutions of $f(x) = 0$. **2**

(ii) Find the coordinates of the turning points of the graph $y = f(x)$, and determine their nature. **3**

(iii) Hence sketch the graph of $y = f(x)$, showing the turning points and the points where the curve meets the x-axis. **2**

(iv) For what values of x is the graph of $y = f(x)$ concave down? **1**

05 **8a** A cylinder of radius x and height $2h$ is to be inscribed in a sphere of radius R centred at O as shown. [Solution]



(i) Show that the volume of the cylinder is given by $V = 2\pi h(R^2 - h^2)$. **1**

(ii) Hence, or otherwise, show that the cylinder has a maximum volume when $h = \dfrac{R}{\sqrt{3}}$. **3**

projectmaths

Integration

Back

16 **9** What is the value of $\int_{-3}^{2} |x+1|\,dx$? **1** Solution

(A) $\frac{5}{2}$ (B) $\frac{11}{2}$ (C) $\frac{13}{2}$ (D) $\frac{17}{2}$

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16 **11 d** Evaluate $\int_{0}^{1} (2x+1)^3\,dx$. **2** Solution

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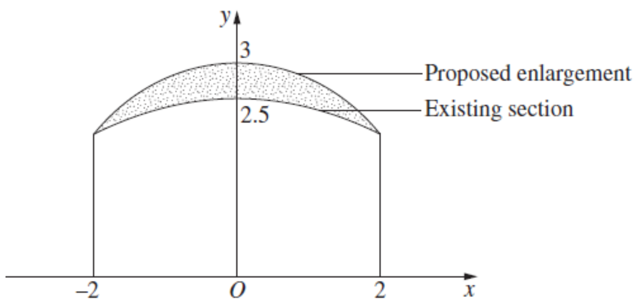
16 **14 a** The diagram shows the cross-section of a tunnel and a proposed enlargement. **3** Solution



The heights, in metres, of the existing section at 1 metre intervals are shown in Table *A*.

Table *A*: Existing heights

x	–2	–1	0	1	2
y	2	2.38	2.5	2.38	2

The heights, in metres, of the proposed enlargement are shown in Table *B*.

Table *B*: Proposed heights

x	–2	–1	0	1	2
y	2	2.78	3	2.78	2

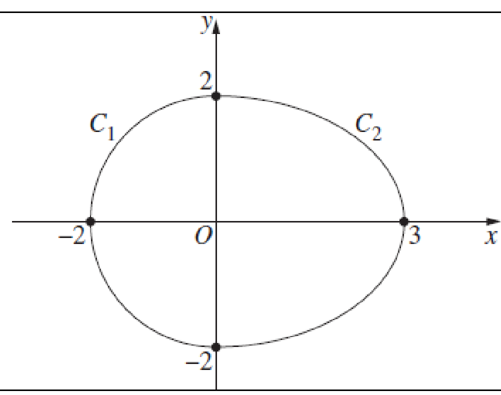
Use Simpson's rule with the measurements given to calculate the approximate increase in area.

---

16 **15 a** The diagram shows two curves C_1 and C_2. The curve C_1 is the semicircle $x^2 + y^2 = 4$, $-2 \le x \le 2$. The curve C_2 has equation **4** Solution

$$\frac{x^2}{9} + \frac{y^2}{4} = 1,\ 0 \le x \le 3.$$

An egg is modelled by rotating the curves about the x-axis to form a solid of revolution.
Find the exact value of the volume of the solid of revolution.



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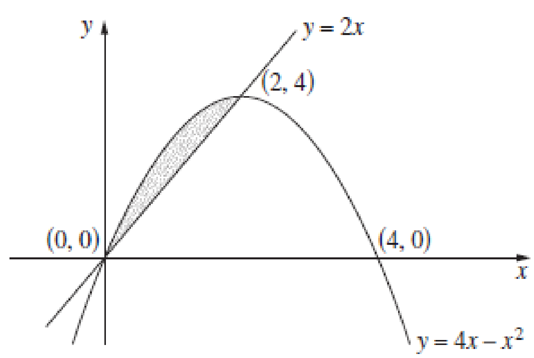
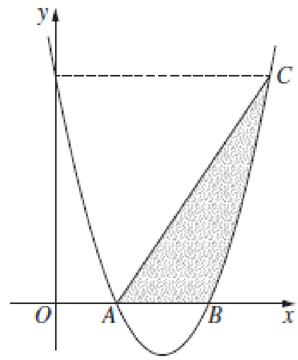
15 **7** The diagram shows the parabola $y = 4x - x^2$ meeting the line $y = 2x$ at (0, 0) and (2, 4). Which expression gives the area of the shaded region bounded by the parabola and the line? **1** Solution

(A) $\int_{0}^{2} x^2 - 2x\ dx$ (B) $\int_{0}^{2} 2x - x^2\ dx$

(C) $\int_{0}^{4} x^2 - 2x\ dx$ (D) $\int_{0}^{4} 2x - x^2\ dx$



15 **16 a** The diagram shows the curve with equation $y = x^2 - 7x + 10$. The curve intersects the x-axis at points A and. The point C on the curve has the y-coordinate as the y-intercept of the curve. Solution



(i) Find the x-coordinates of points A and B. **1**

(ii) Write down the coordinates of C. **1**

(iii) Evaluate $\int_0^2 (x^2 - 7x + 10 \; dx$. **1**

(iv) Hence, or otherwise, find the area of the shaded region. **2**

---

14 **11 d** Find $\int \frac{1}{(x+3)^2} \; dx$. **2** Solution

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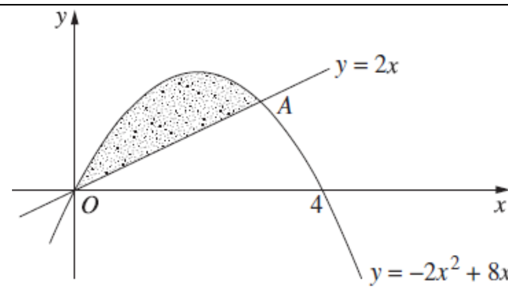
14 **12 d** The parabola $y = -2x^2 + 8x$ and the line $y = 2x$ intersect at the origin and at the point A. Solution

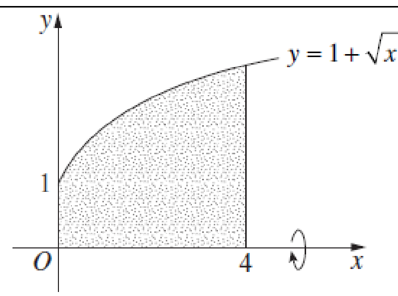


(i) Find the x-coordinate of the point A. **1**

(ii) Calculate the area enclosed by the parabola and the line. **3**

---

14 **14 c** The region bounded by the curve $y = 1 + \sqrt{x}$ and the x-axis between $x = 0$ and $x = 4$ is rotated about the x–axis to form a solid. Find the volume of the solid. **3** Solution



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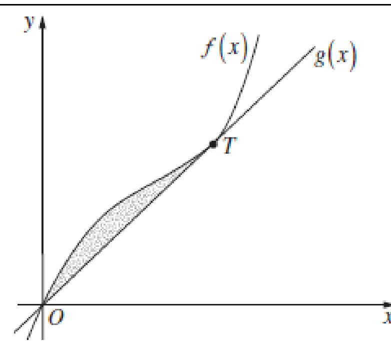
13 **13 b** The diagram shows the graphs of the functions $f(x) = 4x^3 - 4x^2 + 3x$ and $g(x) = 2x$. Solution



The graphs meet at O and at T.

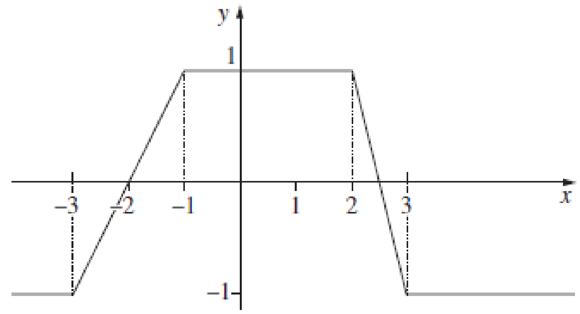
(i) Find the x-coordinate of T. **1**

(ii) Find the area of the shaded regions between the graphs of the functions $f(x)$ and $g(x)$. **3**

13 **14 d** The diagram shows the graph $y = f(x)$. **1** Solution

What is the value of a, where $a > 0$,

so that $\int_{-a}^{a} f(x)\, dx = 0$.



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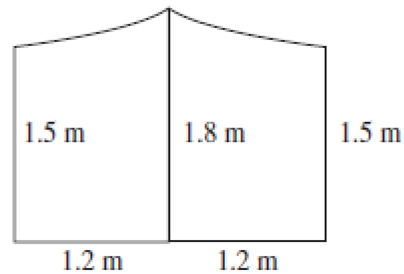
13 **15 a** The diagram shows the front of a tent supported by three vertical poles. The poles are 1.2 m apart. The height of each outer pole is 1.5 m, and the height of the middle pole is 1.8 m. The roof hangs between the poles. Solution

The front of the tent has area A m^2.

(i) Use trapezoidal rule to estimate A. **1**

(ii) Use Simpson's rule to estimate A. **1**

(iii) Explain why the trapezoidal rule gives the better estimate of A. **1**



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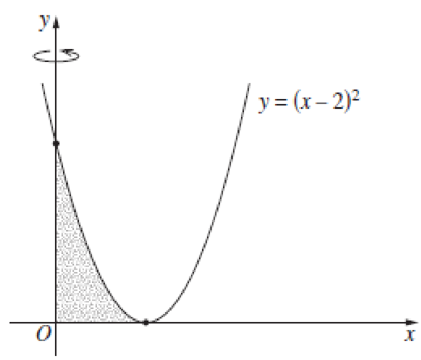
13 **15 b** The region bounded by the x-axis, the y-axis and the parabola $y = (x - 2)^2$ is rotated about the y-axis to form a solid. **4** Solution

Find the volume of the solid.



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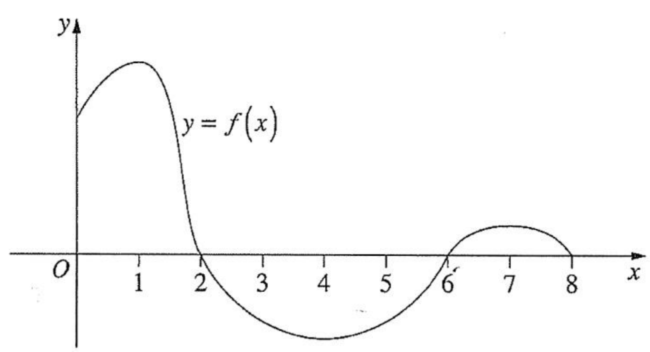
12 **10** The graph of $y = f(x)$ has been drawn to scale for $0 \leq x \leq 8$. **1** Solution

Which of the following integrals has the greatest value?

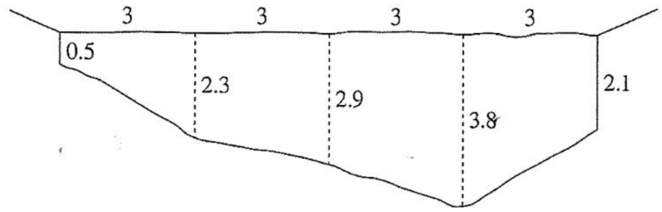
(A) $\int_0^1 f(x)\, dx$ (B) $\int_0^2 f(x)$

(C) $\int_0^7 f(x)$ (D) $\int_0^8 f(x)$



12 **12 d**

At a certain location a river is 12 metres wide. At this location the depth of the river, in metres, has been measured at 3 metre intervals. The cross-section is shown.



(i) Use Simpson’s rule with the five depth measurements to calculate the approximate area of the cross-section. **3**

(ii) The river flows at 0.4 metres per second. Calculate the approximate volume of water flowing through the cross-section in 10 seconds. **1**

Solution

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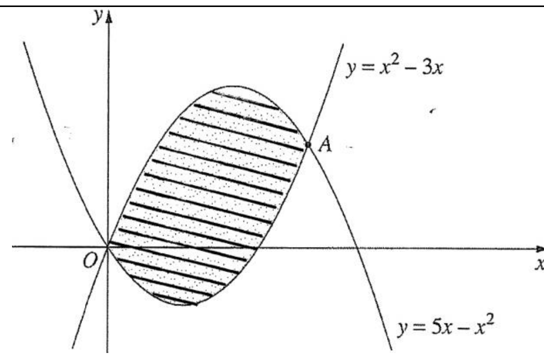
12 **13 b**

The diagram shows the parabolas $y = 5x - x^2$ and $y = x^2 - 3x$. The parabolas intersect at the origin O and the point A. The region between the two parabolas is shaded.



(i) Find the x-coordinate of the point A. **1**

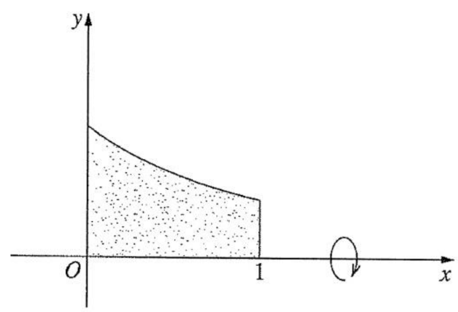
(ii) Find the area of the shaded region. **3**

Solution

---

12 **14 b** **3**

The diagram shows the region bounded by $y = \dfrac{3}{(x+2)^2}$, the x-axis, the y-axis, and the line $x = 1$.
The region is rotated about the x-axis to form a solid.
Find the volume of the solid.



Solution

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11 **2e**

Find $\displaystyle\int \frac{1}{3x^2}\,dx.$ **2**

Solution

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11 **4d**

(i) Differentiate $y = \sqrt{9 - x^2}$ with respect to x. **2**

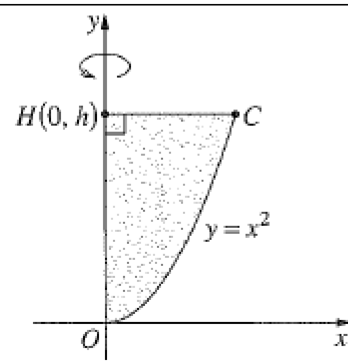
(ii) Hence, or otherwise, find $\displaystyle\int \frac{6x}{\sqrt{9 - x^2}}\,dx.$ **2**

Solution

11 **5c** The table gives the speed v of a jogger at time t in minutes over a 20-minute period. The speed v is measured in metres per minute, in intervals of 5 minutes. The distance covered by the jogger over the 20-minute period is given by $\int_0^{20} v\,dt$. Use Simpson's rule and the speed at each of the five time values to find the approximate distance the jogger covers in the 20-minute period. **3** Solution

t	0	5	10	15	20
v	173	81	127	195	168

11 **8b** The diagram shows the region enclosed by the parabola $y = x^2$, the y-axis and the line $y = h$, where $h > 0$. This region is rotated about the y-axis to form a solid called a paraboloid. The point C is the intersection of $y = x^2$ and $y = h$. The point H has coordinates $(0, h)$. Solution



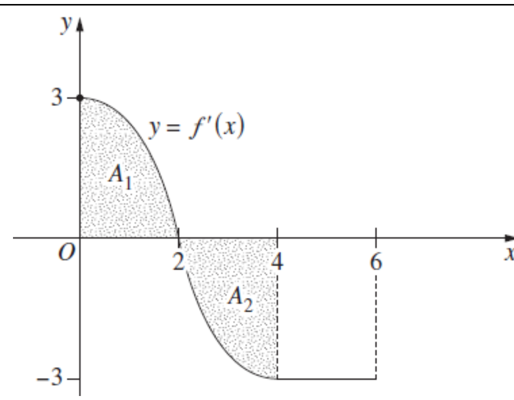
(i) Find the exact volume of the paraboloid in terms of h. **2**

(ii) A cylinder has radius HC and height h. What is the ratio of the volume of the paraboloid to the volume of the cylinder? **1**

10 **2d** (i) Find $\int \sqrt{5x+1}\,dx$. **2** Solution

10 **2e** Given that $\int_0^6 (x+k)\,dx = 30$, and k is a constant, find the value of k. **2** Solution

10 **9b** Let $y = f(x)$ be a function defined for $0 \le x \le 6$, with $f(0) = 0$. The diagram shows the graph of the derivative of f, $y = f'(x)$. The shaded region A_1 has area 4 square units. The shaded region A_2 has area 4 square units. Solution



(i) For which values of x is $f(x)$ increasing? **1**

(ii) What is the maximum value of $f(x)$? **1**

(iii) Find the value of $f(6)$. **1**

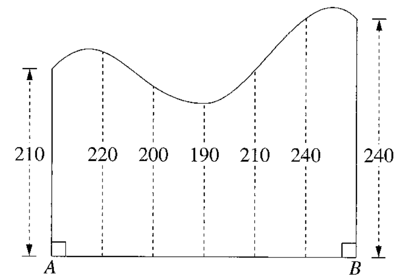
(iv) Draw a graph of $y = f(x)$ for $0 \le x \le 6$. **2**

09 **2b** (i) Find $\int 5\,dx$. **1** Solution

09 **2b** (ii) Find $\int \frac{3}{(x-6)^2}\,dx$. **2** Solution

09 **2b** (iii) Find $\int_1^4 x^2 + \sqrt{x}\,dx$. **3** Solution

09 **3d** The diagram shows a block of land and its dimensions, in metres. The block of land is bounded on one side by a river. Measurements are taken perpendicular to the line *AB*, from *AB* to the river, at equal intervals of 50 m.
Use Simpson’s rule with six subintervals to find an approximation to the area of the block of land.



(not to scale)

3 Solution

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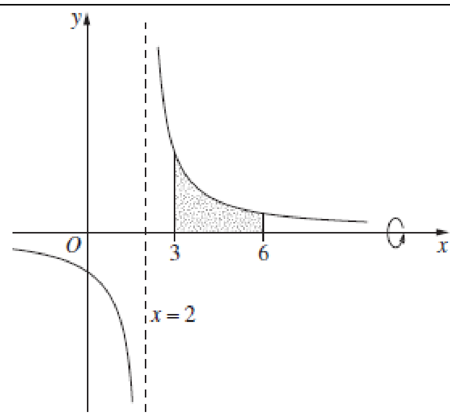
08 **4c** Consider the parabola $x^2 = 8(y - 3)$.
(iv) Calculate the area bounded by the parabola and the line $y = 5$.

3 Solution

---

08 **6c** The graph of $y = \dfrac{5}{x-2}$ is shown. The shaded region in the diagram is bounded by the curve $y = \dfrac{5}{x-2}$, the x-axis, and the lines $x = 3$ and $x = 6$.
Find the volume of the solid of revolution formed when the shaded region is rotated about the x-axis.



3 Solution

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07 **2b** ii. Evaluate $\displaystyle\int_1^4 \frac{8}{x^2}\,dx$.

3 Solution

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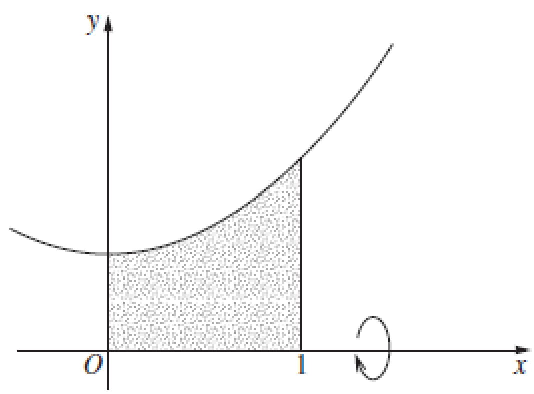
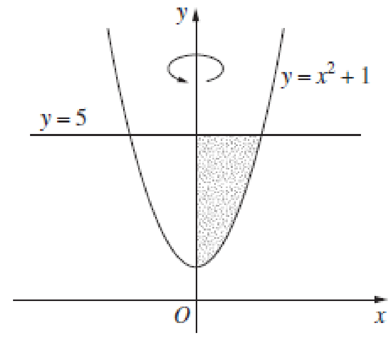
07 **9a** In the shaded region in the diagram is bounded by the curve $y = x^2 + 1$, the x-axis, and the lines $x = 0$ and $x = 1$.
Find the volume of the solid of revolution formed when the shaded region is rotated about the x-axis.



3 Solution

06 **4b** In the diagram, the shaded region is bounded by the parabola $y = x^2 + 1$, the y-axis and the line $y = 5$.

Find the volume of the solid formed when the shaded region is rotated about the y-axis.



3 Solution

05 **6a** Five values of the function $f(x)$ are shown in the table. Use Simpson's rule with the five values given in the table to estimate $\int_0^{20} f(x)\ dx$.

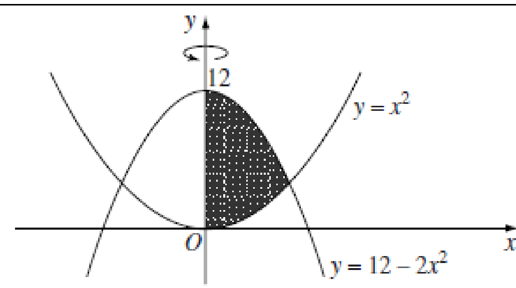
x	0	5	10	15	20
$f(x)$	15	25	22	18	10

3 Solution

05 **6c** The graphs of the curves $y = x^2$ and $y = 12 - 2x^2$ are shown in the diagram. Solution

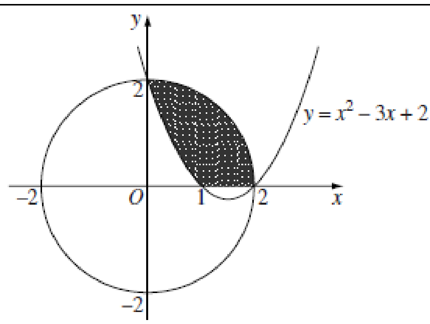
(i) Find the points of intersection of the two curves. **1**

(ii) The shaded region between the curves and the y-axis is rotated about the y-axis. By splitting the shaded region into two parts, or otherwise, find the volume of the solid formed. **3**



05 **8b** The shaded region in the diagram is bounded by the circle of radius 2, centred at the origin, the parabola $y = x^2 - 3x + 2$, and the x-axis.

By considering the difference of two areas, find the area of the shaded region.



3 Solution

The Trigonometric Functions

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16 6 What is the period of the function $f(x) = \tan(3x)$? **1** Solution

(A) $\frac{\pi}{3}$ (B) $\frac{2\pi}{3}$ (C) 3π (D) 6π

---

16 7 The circle centred at O has radius 5. Arc AB has length 7 as shown in the diagram. **1** Solution

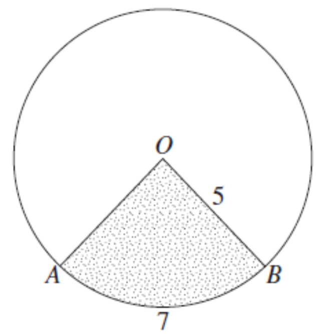
What is the area of the shaded sector OAB?

(A) $\frac{35}{2}$

(B) $\frac{35}{2}\pi$

(C) $\frac{125}{14}$

(D) $\frac{125}{14}\pi$



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16 8 How many solutions does the equation $|\cos(2x)| = 1$ have for $0 \le x \le 2\pi$? **1** Solution

(A) 1 (B) 3 (C) 4 (D) 5

---

16 11 f Find the gradient of the tangent to the curve $y = \tan x$ at the point where $x = \frac{\pi}{8}$. **2** Solution

Give your answer correct to 3 significant figures.

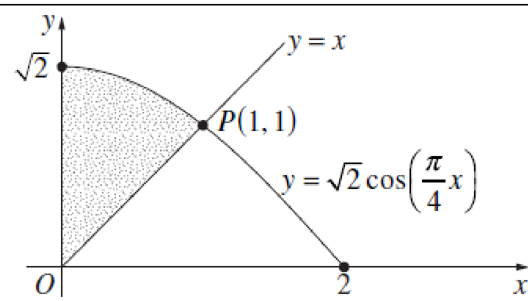
---

16 11 g Solve $\sin\left(\frac{x}{2}\right) = \frac{1}{2}$ for $0 \le x \le 2\pi$? **2** Solution

---

16 13 d The curve $y = \sqrt{2}\cos\left(\frac{\pi}{4}x\right)$ meets the line $y = x$ at $P(1, 1)$, as shown in the diagram. **3** Solution

Find the exact value of the shaded area.



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15 6 What is the value of the derivative of $y = 2\sin 3x - 3\tan x$ at $x = 0$? **1** Solution

(A) -1 (B) 0 (C) 3 (D) -9

---

15 11 g Evaluate $\int_0^{\frac{\pi}{4}} \cos 2x \; dx$. **2** Solution

---

15 12 a Find the solutions of $2\sin\theta = 1$ for $0 \le \theta \le 2\pi$. **2** Solution

---

14 7 How many solutions of the equation $(\sin x - 1)(\tan x + 2) = 0$ lie between 0 and 2π? **1** Solution

(A) 1 (B) 2 (C) 3 (D) 4

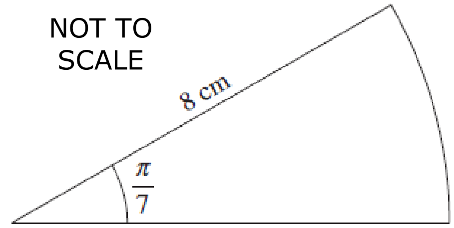
---

14 11 e Evaluate $\int_0^{\frac{\pi}{2}} \sin\frac{x}{2} \; dx$. **3** Solution

---

14 **11 g** The angle of a sector in a circle of radius 8 cm is $\frac{\pi}{7}$ radians, as shown in the diagram.

Find the exact value of the perimeter of the sector. **2** Solution



14 **13 a** (i) Differentiate $3 + \sin 2x$. **1** Solution

(ii) Hence, or otherwise, find $\int \frac{\cos 2x}{3+\sin 2x}\,dx$. **2**

14 **15 a** Find all solutions of $2\sin^2 x + \cos x - 2 = 0$, where $0 \le x \le 2\pi$. **3** Solution

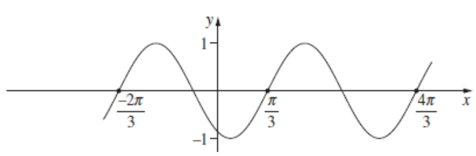
14 **16 a** Use Simpson's Rule with five function values to show that $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sec dx \approx \frac{\pi}{9}\left(3 + \frac{8}{\sqrt{3}}\right)$. **3** Solution

13 **4** What is the derivative of $\frac{x}{\cos x}$? **1** Solution

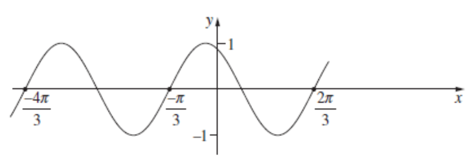
(A) $\frac{\cos x + x\sin x}{\cos^2 x}$ (B) $\frac{\cos x - x\sin x}{\cos^2 x}$ (C) $\frac{x\sin x - \cos x}{\cos^2 x}$ (D) $\frac{-x\sin x - \cos x}{\cos^2 x}$

13 **6** Which diagram shows the graph $y = \sin(2x + \frac{\pi}{3})$? **1** Solution

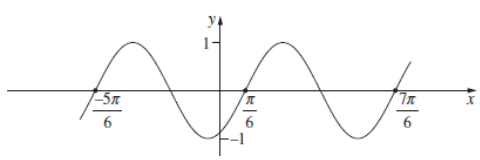
(A)



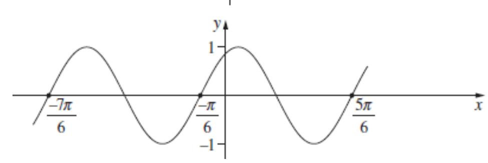
(B)



(C)



(D)



13 **11 c** Differentiate $(\sin x - 1)^8$. **2** Solution

13 **13 a** The population of a herd of wild horses is given by $P(t) = 400 + 50\cos\left(\frac{\pi}{6}t\right)$, where t is time in months. Solution

(i) Find all times during the first 12 months when the population equals 375 horses. **2**

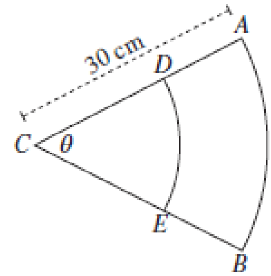
(ii) Sketch the graph of $P(t)$ for $0 \le t \le 12$. **2**

---

13 13c The region ABC is a sector of a circle with radius 30 cm, centred at C. The angle of the sector is θ. The arc DE lies on a circle also centred at C, as shown in the diagram.

The arc DE divides the sector ABC into two regions of equal area.

Find the exact length of the interval CD.



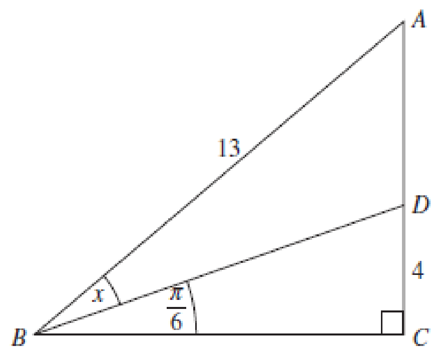
2 Solution

---

13 14c The right-angled triangle ABC has hypotenuse $AB = 13$.

The point D is on AC such that $DC = 4$, $\angle DBC = \dfrac{\pi}{6}$ and $\angle DBC = x$.

Using the sine rule, or otherwise, find the exact value of $\sin x$.



Not to scale

3 Solution

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12 6 What are the solutions of $\sqrt{3}\tan x = -1$ for $0 \le x \le 2\pi$?

(A) $\dfrac{2\pi}{3}$ and $\dfrac{4\pi}{3}$ (B) $\dfrac{2\pi}{3}$ and $\dfrac{5\pi}{3}$ (C) $\dfrac{5\pi}{6}$ and $\dfrac{7\pi}{6}$ (D) $\dfrac{5\pi}{6}$ and $\dfrac{11\pi}{6}$

1 Solution

---

12 11f The area of a sector of a circle of radius 6 cm is 50 cm^2. Find the length of the arc of the sector.

2 Solution

---

12 11g Find $\displaystyle\int_0^{\frac{\pi}{2}} \sec^2 \frac{x}{2}\, dx$.

3 Solution

---

12 12a Differentiate with respect to x:

(ii) $\dfrac{\cos x}{x^2}$

2 Solution

---

11 2b Find the exact values of x such that $2\sin x = -\sqrt{3}$, where $0 \le x \le 2\pi$.

2 Solution

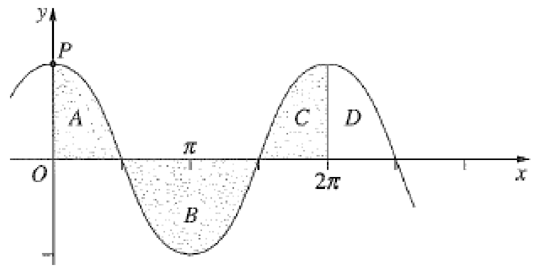
---

11 4a Differentiate $\dfrac{x}{\sin x}$ with respect to x.

2 Solution

---

11 **6c** The diagram shows the graph $y = 2\cos x$. [Solution]



(i) State the coordinates of P. **1**

(ii) Evaluate the integral **2**

$$\int_0^{\frac{\pi}{2}} 2\cos x\, dx.$$

(iii) Indicate which area in the diagram, A, B, C or D, is represented by the integral $\int_{\frac{3\pi}{2}}^{2\pi} 2\cos x \, \mathrm{dx}$. **1**

(iii) Using parts (ii) and (iii), or otherwise, find the area of the region bounded by the curve $y = 2\cos x$ and the x-axis, between $x = 0$ and $x = 2\pi$. **1**

(v) Using the parts above, write down the value of $\int_{\frac{\pi}{2}}^{2\pi} 2\cos x\, dx$. **1**

---

10 **1e** Differentiate $x^2 \tan x$ with respect to x. **2** [Solution]

---

10 **2a** Differentiate $\dfrac{\cos x}{x}$ with respect to x. **2** [Solution]

---

10 **5b** [Solution]

(i) Prove that $\sec^2 x + \sec x \tan x = \dfrac{1 + \sin x}{\cos^2 x}$. **1**

(ii) Hence prove that $\sec^2 x + \sec x \tan x = \dfrac{1}{1 - \sin x}$. **1**

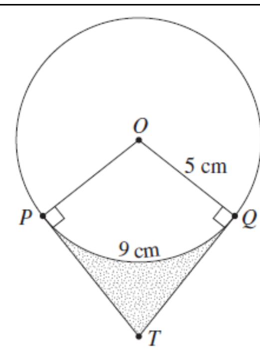
(iii) Hence, use the table of standard integrals to find the exact value of **2**

$$\int_0^{\frac{\pi}{4}} \frac{1}{1 - \sin x}\, dx.$$

---

10 **6b** The diagram shows a circle with centre O and radius 5 cm. [Solution]

The length of the arc PQ is 9 cm. Lines drawn perpendicular to OP and OQ at P and Q respectively meet at T.



(Not to scale)

(i) Find $\angle POQ$ in radians. **1**

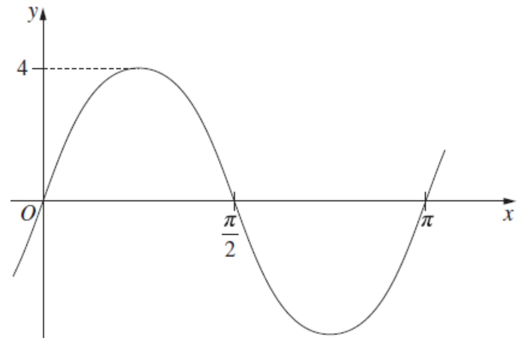
(ii) Prove that ΔOPT is congruent to ΔOQT. **2**

(iii) Find the length of PT. **1**

(iv) Find the area of the shaded region. **2**

10 **8c** Solution

The graph shown is $y = A\sin bx$.



(i) Write down the value of A. **1**

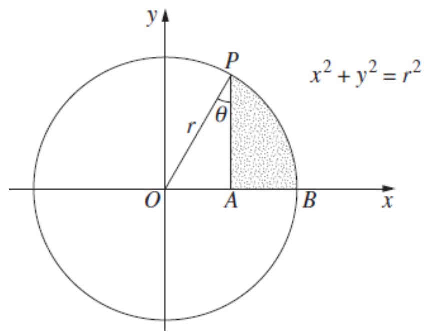
(ii) Find the value of b. **1**

(iii) Copy or trace the graph into your writing booklet. On the same set of axes, draw the graph $y = 3\sin x + 1$, for $0 \le x \le \pi$. **2**

---

10 **10b** Solution

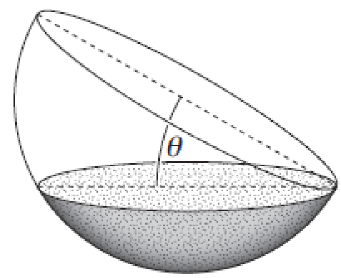
The circle $x^2 + y^2 = r^2$ has radius r and centre O. The circle meets the positive x-axis at B. The point A is on the interval OB. A vertical line through A meets the circle at P. Let $\theta = \angle OPA$.



(i) The shaded region bounded by the arc PB and the intervals AB and AP is rotated about the x-axis. Show that the volume, V, formed is given by **3**

$$V = \frac{\pi r^3}{3}(2 - 3\sin\theta + \sin^3\theta).$$

(ii) A container is in the shape of a hemisphere of radius r metres. The container is initially horizontal and full of water. The container is then tilted at an angle of θ to the horizontal so that some water spills out.



(1) Find θ so that the depth of water remaining is one half of the original depth. **1**

(2) What fraction of the original volume is left in the container? **2**

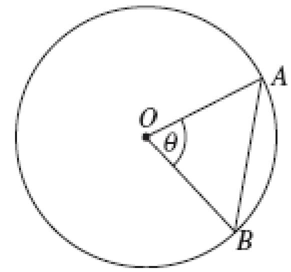
---

09 **1e** Find the exact value of θ such that $2\cos\theta = 1$, where $0 \le \theta \le \frac{\pi}{2}$. **2** Solution

---

09 **2a** (i) Differentiate with respect to x: $x\sin x$ **2** Solution

09 5c The diagram shows a circle with centre O and radius 2 centimetres. The points A and B lie on the circumference of the circle and $\angle AOB = \theta$. Solution



(Not to scale)

(i) There are two possible values of θ for which the area of Δ AOB is $\sqrt{3}$ square centimetres. One value is $\frac{\pi}{3}$. Find the other value. **2**

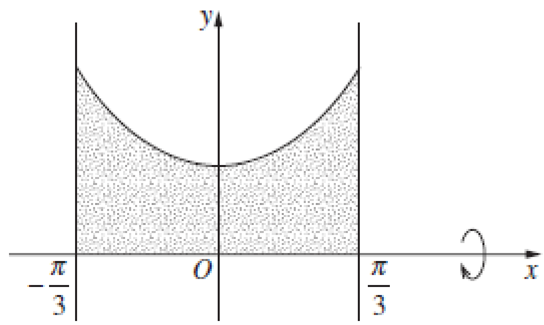
(ii) Suppose that $\theta = \frac{\pi}{3}$.

(1) Find the area of the sector AOB. **1**

(2) Find the exact length of the perimeter of the minor segment bounded by the chord AB and the arc AB. **2**

---

09 6a The diagram shows the region bounded by the curve $y = \sec x$, the lines $x = \frac{\pi}{3}$ and $x = -\frac{\pi}{3}$, and the x-axis. Solution



The region is rotated about the x-axis. Find the volume of the solid of revolution formed. **2**

---

09 7b Between 5 am and 5 pm on 3 March 2009, the height, h, of the tide in a harbour was given by $h = 1 + 0.7 \sin\frac{\pi}{6}t$ for $0 \le t \le 12$, where h is in minutes and t is in hours, with $t = 0$ at 5 am. Solution

(i) What is the period of the function h? **1**

(ii) What was the value of h at low tide, and at what time did low tide occur? **2**

(iii) A ship is able to enter the harbour only if the height of the tide is at least 1.35 m. Find all times between 5 am and 5 pm on 3 March 2009 during which the ship was able to enter the harbour. **3**

---

08 1a Evaluate $2\cos\frac{\pi}{5}$ correct to three significant figures. **2** Solution

---

08 2a (iii) Differentiate with respect to x: $\frac{\sin x}{x+4}$ **2** Solution

---

08 2c (ii) Evaluate $\int_0^{\frac{\pi}{12}} \sec^2 3x \; dx$. **3** Solution

---

08 3b (i) Differentiate $\log_e(\cos x)$ with respect to x. **2** Solution

(ii) Hence, or otherwise, evaluate $\int_0^{\frac{\pi}{4}} \tan x \; dx$. **2**

08 5a The gradient of a curve is given by $\frac{dy}{dx} = 1 - 6\sin 3x$. **3** [Solution]

The curve passes through the point (0, 7).
What is the equation of the curve?

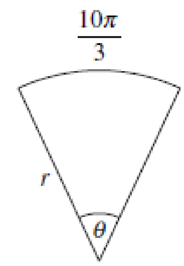
---

08 6a Solve $2\sin^2 \frac{x}{3} = 1$ for $-\pi \le x \le \pi$. **3** [Solution]

---

08 7b The diagram shows a sector with radius r and angle θ where $0 \le \theta \le 2\pi$. [Solution]

The arc length is $\frac{10\pi}{3}$.



(i) Show that $r \ge \frac{5}{3}$. **2**

(ii) Calculate the area of the sector when $r = 4$. **2**

---

07 2a (ii) Differentiate with respect to x: $(1 + \tan x)^{10}$. **2** [Solution]

---

07 2b (i) Find $\int (1 + \cos 3x)\ dx$. **2** [Solution]

---

07 2c The point $P(\pi, 0)$ lies on the curve $y = x \sin x$. Find the equation of the tangent to the curve at P. **3** [Solution]

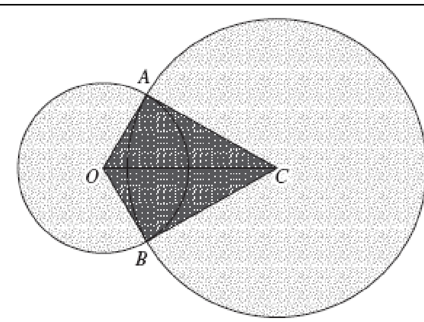
---

07 4a Solve $\sqrt{2}\sin x = 1$ for $0 \le x \le 2\pi$. **2** [Solution]

---

07 4c An advertising logo is formed from two circles, which intersect as shown in the diagram. The circles intersect at A and B and have centres at O and C. [Solution]

The radius of the circle centred at O is 1 metre and the radius of the circle centred at C is $\sqrt{3}$ metres. The length of OC is 2 metres.



Not to scale

(i) Use Pythagoras' theorem to show that $\angle OAC = \frac{\pi}{2}$. **1**

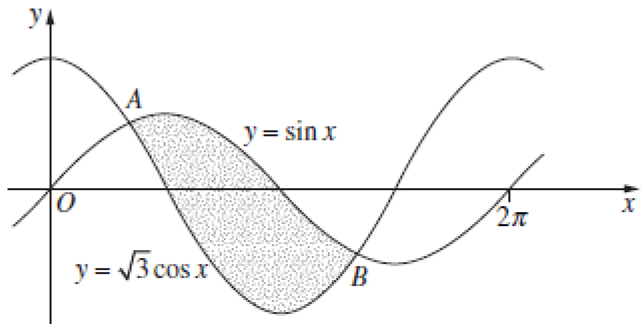
(ii) Find $\angle ACO$ and $\angle AOC$. **2**

(iii) Find the area of the quadrilateral AOBC. **1**

(iv) Find the area of the major sector ACB. **1**

(v) Find the total area of the logo (the sum of all the shaded areas). **2**

07 7b The diagram shows the graphs of $y = \sqrt{3}\cos x$ and $y = \sin x$. The first two points of intersection to the right of the y-axis are labelled A and B. Solution



(i) Solve the equation $\sqrt{3}\cos x = \sin x$ to find the x-coordinates of A and B. **2**

(ii) Find the area of the shaded region in the diagram. **3**

---

06 2a Differentiate with respect to x: Solution

(i) $x \tan x$ **2**

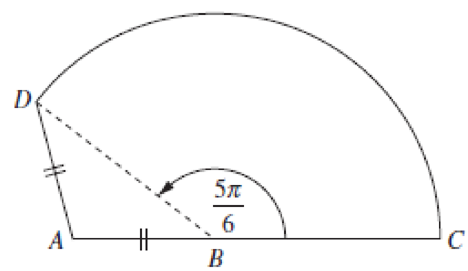
(ii) $\dfrac{\sin x}{x+1}$. **2**

---

06 2c Find the equation of the tangent to the curve $y = \cos 2x$ at the point whose x-coordinate is $\dfrac{\pi}{6}$. **3** Solution

---

06 4a In the diagram, $ABCD$ represents a garden. The sector BCD has centre B and $\angle DBC = \dfrac{5\pi}{6}$. The points A, B and C lie on a straight line and $AB = AD = 3$ metres. Copy or trace the diagram into your writing booklet. Solution



(i) Show that $\angle DAB = \dfrac{2\pi}{3}$. **1**

(ii) Find the length of BD. **2**

(iii) Find the area of the garden $ABCD$. **2**

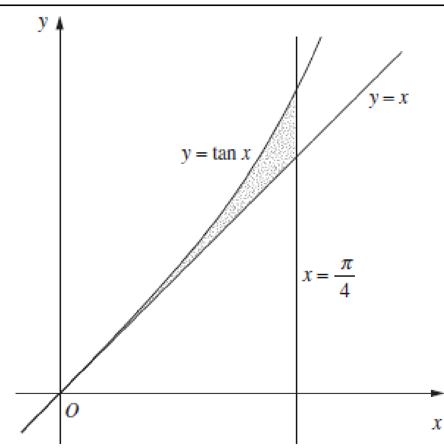
---

06 5b Solution

(i) Show that $\dfrac{d}{dx}\log_e(\cos x) = -\tan x$. **1**

(ii) The shaded region in the diagram is bounded by the curve $y = \tan x$ and the lines $y = x$ and $x = \dfrac{\pi}{4}$. **3**

Using the result of part (i), or otherwise, find the area of the shaded region.



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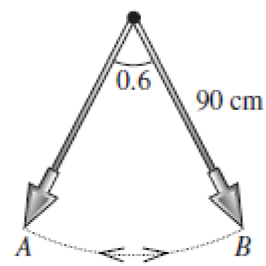
06 7b A function $f(x)$ is defined by $f(x) = 1 + 2\cos x$. Solution

(i) Show that the graph of $y = f(x)$ cuts the x-axis at $x = \dfrac{2\pi}{3}$. **1**

(ii) Sketch the graph of $y = f(x)$ for $-\pi \le x \le \pi$ showing where the graph cuts each of the axes. **3**

(iii) Find the area under the curve $y = f(x)$ between $x = -\dfrac{\pi}{2}$ and $x = \dfrac{2\pi}{3}$. **3**

05	**1c**	Find a primitive of $4 + \sec^2 x$.	**2**	Solution
05	**2a**	Solve $\cos\theta = \dfrac{1}{\sqrt{2}}$ for $0 \le \theta \le 2\pi$.	**2**	Solution
05	**2b**	Differentiate with respect to x: (i) $x \sin x$	**2**	Solution
05	**2c**	(ii) Evaluate $\displaystyle\int_0^{\frac{\pi}{6}} \cos 3x \; dx$.	**2**	Solution
05	**4a**	A pendulum is 90 cm long and swings through an angle of 0.6 radians. The extreme positions of the pendulum are indicated by the points A and B in the diagram. (i) Find the length of the arc AB. (ii) Find the straight-line distance between the extreme positions of the pendulum. (iii) Find the area of the sector swept out by the pendulum. 	**1** **2** **2**	Solution

projectmaths

Logarithmic and Exponential Functions

Back

---

16 **5** What is the derivative of $\ln(\cos x)$? **1** Solution

(A) $-\sec x$ (B) $-\tan x$ (C) $\sec x$ (D) $\tan x$

---

16 **10** Which expression is equivalent to $4 + \log_2 x$? **1** Solution

(A) $\log_2(2x)$ (B) $\log_2(16 + x)$ (C) $4\log_2(2x)$ (D) $\log_2(16x)$

---

16 **12 d** (i) Differentiate $y = xe^{3x}$. **1** Solution

(ii) Hence find the exact value of $\int_0^2 e^{3x}(3 + 9x)\,dx$. **2**

---

16 **14 e** Write $\log 2 + \log 4 + \log 8 + \ldots + \log 512$ in the form of $a \log b$ where a and b are integers greater than 1. **2** Solution

---

15 **5** Using the trapezoidal rule with 4 subintervals, which expression gives the approximate area under the curve $y = xe^x$ between $x = 1$ and $x = 3$? **1** Solution

(A) $\frac{1}{4}(e^1 + 6e^{1.5} + 4e^2 + 10e^{2.5} + 3e^3)$ (B) $\frac{1}{4}(e^1 + 3e^{1.5} + 4e^2 + 5e^{2.5} + 3e^3)$

(C) $\frac{1}{2}(e^1 + 6e^{1.5} + 4e^2 + 10e^{2.5} + 3e^3)$ (D) $\frac{1}{2}(e^1 + 3e^{1.5} + 4e^2 + 5e^{2.5} + 3e^3)$

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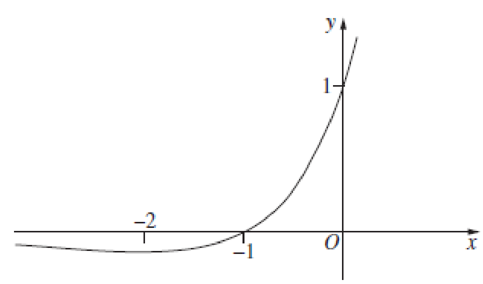
15 **8** The diagram shows the graph of $y = e^x(1 + x)$. **1** Solution



How many solutions are there to the equation $e^x(1 + x) = 1 - x^2$?

(A) 0 (B) 1

(C) 2 (D) 3

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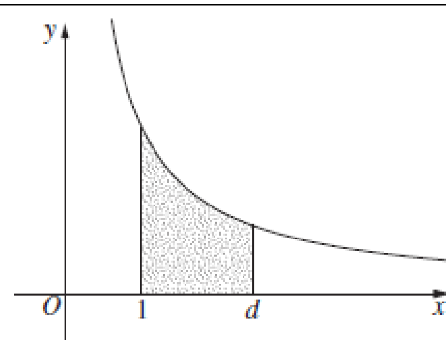
15 **10** The diagram shows the area under the curve $y = \frac{2}{x}$ from $x = 1$ to $x = d$. What value of d makes the shaded area equal to 2? **1** Solution



(A) e (B) $e + 1$

(C) $2e$ (D) e^3

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15 **11 e** Differentiate $(e^x + x)^5$. **2** Solution

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15 **11 f** Differentiate $y = (x + 4)\ln x$. **2** Solution

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15 **11 h** Find $\int \frac{x}{x^2 - 3}\,dx$. **2** Solution

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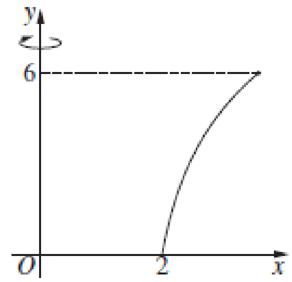
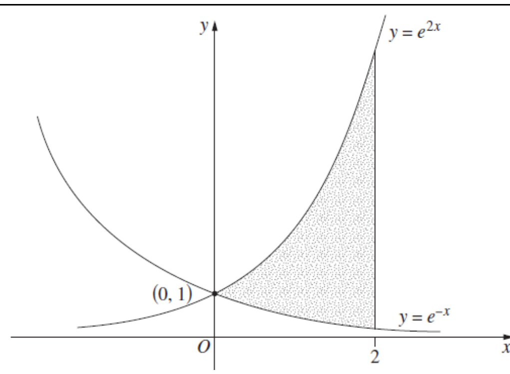
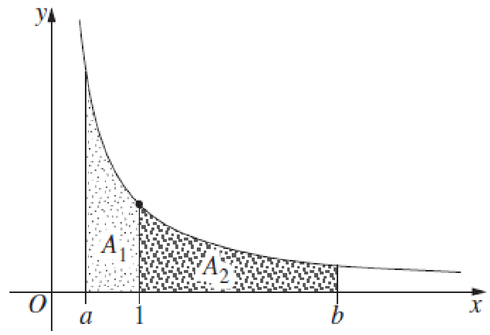
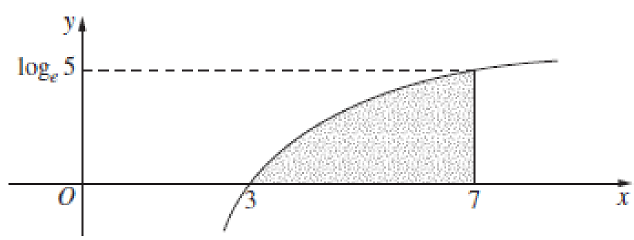
15	**16 b**	A bowl is formed by rotating the curve $y = 8\log_e(x - 1)$ about the y-axis for $0 \le y \le 6$. Find the volume of the bowl. Give your answer correct to 1 decimal place. Not to scale 	**3**	Solution
14	**3**	What is the solution to the equation $\log_2(x - 1) = 8$? (A) 4 (B) 17 (C) 65 (D) 257	**1**	Solution
14	**4**	Which expression is equal to $\int e^{2x}\,dx$? (A) $e^{2x} + c$ (B) $2e^{2x} + c$ (C) $\dfrac{e^{2x}}{2} + c$ (D) $\dfrac{e^{2x+1}}{2x+1} + c$	**1**	Solution
14	**14 a**	Find the coordinates of the stationary point on the graph $y = e^x - ex$ and determine its nature.	**3**	Solution
14	**15 c**	The line $y = mx$ is a tangent to the curve $y = e^{2x}$ at a point P. (i) Sketch the line and the curve on one diagram. (ii) Find the coordinates of P. (iii) Find the value of m.	 **1** **3** **1**	Solution
13	**9**	What is the solution of $5^x = 4$? (A) $x = \dfrac{\log_e 4}{5}$ (B) $x = \dfrac{4}{\log_e 5}$ (C) $x = \dfrac{\log_e 4}{\log_e 5}$ (D) $x = \log_e\left(\dfrac{4}{5}\right)$	**1**	Solution
13	**11 a**	Evaluate ln 3 correct to three significant figures.	**1**	Solution
13	**11 d**	Differentiate x^2e^x.	**2**	Solution
13	**11 e**	Find $\int e^{4x+1}\,dx$.	**2**	Solution
13	**11 f**	Evaluate $\displaystyle\int_0^1 \frac{x^2}{x^3+1}\,dx$.	**3**	Solution
12	**7**	Let $a = e^x$. Which expression is equal to $\log_e(a^2)$? (A) e^{2x} (B) e^{x^2} (C) $2x$ (D) x^2	**1**	Solution
12	**11 d**	Differentiate $(3 + e^{2x})^5$.	**2**	Solution
12	**12 a**	Differentiate with respect to x: (i) $(x - 1)\log_e x$	**2**	Solution

12	**12 b**	Find $\int \frac{4x}{x^2+6}\,dx$.	**2**	Solution
11	**1d**	Differentiate ln(5x + 2) with respect to x.	**2**	Solution
11	**2d**	Find the derivative of $y = x^2e^x$ with respect to x.	**2**	Solution
11	**4b**	Evaluate $\int_e^{e^3} \frac{5}{x}\,dx$.	**2**	Solution
10	**2c**	Find the gradient of the tangent to the curve $y = \ln(3x)$ at the point where $x = 2$.	**2**	Solution
10	**2d**	(ii) Find $\int \frac{x}{4+x^2}\,dx$.	**2**	Solution
10	**3b**	(i) Sketch the curve y = ln x. (ii) Use the trapezoidal rule with three function values to find an approximation to $\int_1^3 \ln x\,dx$. (iii) State whether the approximation found in (ii) is greater than or less than the exact value of $\int_1^3 \ln x\,dx$. Justify your answer.	**1** **2** **1**	Solution
10	**4b**	The curves $y = e^{2x}$ and $y = e^{-x}$ intersect at the point (0, 1) as shown in the diagram. Find the exact area enclosed by the curves and the line $x = 2$.  	**3**	Solution
10	**4d**	Let $f(x) = 1 + e^x$. Show that $f(x) \times f(-x) = f(x) + f(-x)$.	**2**	Solution

10	**5c**	The diagram shows the curve $y = \dfrac{1}{x}$, for $x > 0$. The area under the curve between $x = a$ and $x = 1$ is A_1. The area under the curve between $x = 1$ and $x = b$ is A_2. The area A_1 and A_2 are each equal to 1 square unit. Find the values of a and b. 	**3**	Solution
09	**1f**	Solve the equation $\ln x = 2$. Give your answer correct to four decimal places.	**2**	Solution
09	**2a**	Differentiate with respect to x: (ii) $(e^x + 1)^2$.	**2**	Solution
08	**2a**	(ii) Differentiate with respect to x: $x^2\log_e x$	**2**	Solution
08	**2c**	(i) Find $\displaystyle\int \frac{dx}{x+5}$	**1**	Solution
08	**7a**	Solve $\log_e x - \dfrac{3}{\log_e x} = 2$.	**3**	Solution
08	**10 a**	In the diagram, the shaded region is bounded by $y = \log_e(x - 2)$, the x-axis and the line $x = 7$. Find the exact value of the area of the shaded region. 	**5**	Solution
07	**2a**	(i) Differentiate with respect to x: $\dfrac{2x}{e^x + 1}$.	**2**	Solution
07	**6a**	Solve the following equation for x: $2e^{2x} - e^x = 0$.	**2**	Solution
06	**1a**	Evaluate $e^{-0.5}$ correct to three decimal places.	**2**	Solution
06	**2b**	(i) Find $\displaystyle\int 1 + e^{7x}\ dx$ (ii) Evaluate $\displaystyle\int_0^3 \frac{8x}{1+x^2}\,dx$.	**2** **3**	Solution
06	**10 a**	Use Simpson's rule with three function values to find an approximation to the value of $\displaystyle\int_{0.5}^{1.5} (\log_e x)^3\ dx$. Give your answer correct to three decimal places.	**2**	Solution
05	**2c**	(i) Find $\displaystyle\int \frac{6x^2}{x^3+1}\,dx$	**2**	Solution

05	**2d**	Find the equation of the tangent to $y = \log_e x$ at the point $(e, 1)$.	**2**	Solution
05	**5a**	Use the change of base formula to evaluate $\log_3 7$, correct to two decimal places.	**2**	Solution
05	**5c**	Find the coordinates of the point P on the curve $y = 2e^x + 3x$ at which the tangent to the curve is parallel to the line $y = 5x - 3$.	**3**	Solution

projectmaths

Applications of Calculus to the Physical World – Rates of Change

Back

16 **16 b** Some yabbies are introduced into a small dam. The size of the population, y, of yabbies can be modelled by the function $y = \dfrac{200}{1+19e^{-0.5t}}$, where t is the time in months after the yabbies are introduced into the dam. Solution

(i) Show that the rate of growth of the size of the population is $\dfrac{1900e^{-0.5t}}{(1+19e^{-0.5t})^2}$. **2**

(ii) Find the range of the function y, justifying your answer. **2**

(iii) Show that the rate of growth of the size of the population can be rewritten as $\dfrac{y}{400}(200-y)$. **1**

(iv) Hence, find the size of the population when it is growing at its fastest rate. **2**

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15 **15 c** Water is flowing in and out of a rock pool. The volume of water in the pool at time t hours is V litres. The rate of change of the volume is given by $\dfrac{dV}{dt} = 80\sin(0.5t)$. Solution

At time $t = 0$, the volume of water in the pool is 1200 litres and is increasing.

(i) After what time does the volume of water first start to decrease? **2**

(ii) Find the volume of water in the pool when $t = 3$. **2**

(iii) What is the greatest volume of water in the pool? **1**

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11 **9b** A tap releases liquid A into a tank at the rate of $\left(2+\dfrac{t^2}{t+1}\right)$ litres per minute, where t is time in minutes. A second tap releases liquid B into the same tank at the rate of $\left(1+\dfrac{1}{t+1}\right)$ litres per minute. The taps are opened at the same time and release the liquids into an empty tank. Solution

(i) Show that the rate of liquid A is greater than the rate of flow of liquid B by t litres per minute. **1**

(ii) The taps are closed after 4 minutes. By how many litres is the volume of liquid A greater than the volume of liquid B in the tank when the taps are closed? **2**

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06 **9b** During a storm, water flows into a 7000-litre tank at a rate of $\dfrac{dV}{dt}$ litres per minute, where $\dfrac{dV}{dt} = 120 + 26t - t^2$ and t is the time in minutes since the storm began. Solution

(i) At what times is the tank filling at twice the initial rate? **2**

(ii) Find the volume of water that has flowed into the tank since the start of the storm as a function of t. **1**

(iii) Initially, the tank contains 1500 litres of water. When the storm finishes, 30 minutes after it began, the tank is overflowing. How many litres of water have been lost? **2**

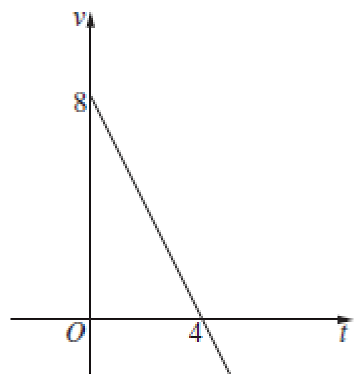
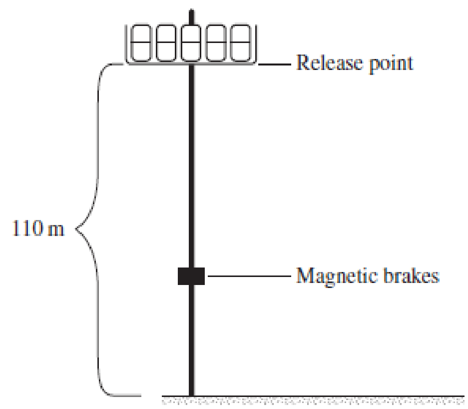
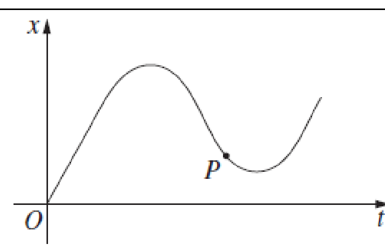
05 **6b** A tank initially holds 3600 litres of water. The water drains from the bottom of the tank. The tank takes 60 minutes to empty. A mathematical model predicts that the volume, V litres, of water that will remain in the tank after t minutes is given by

$V = 3600\left(1 - \dfrac{t}{60}\right)^2$, where $0 \le t \le 60$. **Solution**

(i) What volume does the model predict will remain after ten minutes? **1**

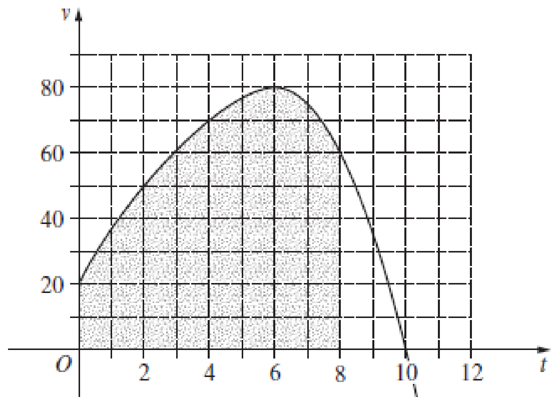
(ii) At what rate does the model predict that the water will drain from the tank after twenty minutes? **2**

(iii) At what time does the model predict that the water will drain from the tank at its fastest rate? **2**

16	**16 a**	A particle moves in a straight line. Its velocity v ms^{-1} at time t seconds is given by $v = 2 - \dfrac{4}{t+1}$. (i) Find the initial velocity. (ii) Find the acceleration of the particle when the particle is stationary. (iii) By considering the behavior of v for large t, sketch a graph of v against t for $t \geq 0$, showing any intercepts. (iv) Find the exact distance travelled by the particle in the first 7 seconds.	**1** **2** **2** **3**	**Solution**
15	**9**	A particle is moving along the x-axis. The graph shows its velocity v metres per second at time t seconds. When $t = 0$ the displacement x is equal to 2 metres. What is the maximum value of the displacement x?  (A) 8 m (B) 14 m (C) 16 m (D) 18 m	**1**	**Solution**
15	**14 a**	In a theme park ride, a chair is released from a height of 110 metres and falls vertically. Magnetic brakes are applied when the velocity of the chair reaches −37 metres per second. The height of the chair at time t seconds is x metres. The acceleration of the chair is given by $\ddot{x} = -10$. At the release point, $t = 0$, $x = 110$ and $\dot{x} = 0$.  Not to scale (i) Using calculus, show that $x = -5t^2 + 110$. (ii) How far has the chair fallen when the magnetic brakes are applied?	 **2** **2**	**Solution**
14	**9**	The graph shows the displacement x of a particle moving along a straight line as a function of time t. Which statement describes the motion of the particle at the point P?  (A) The velocity is negative and the acceleration is positive. (B) The velocity is negative and the acceleration is negative. (C) The velocity is positive and the acceleration is positive. (D) The velocity is positive and the acceleration is negative.	**1**	**Solution**

14	13 c	The displacement of a particle moving along the x-axis is given by $x = t - \frac{1}{1+t}$, where x is the displacement from the origin in metres, t is the time in seconds, and $t \geq 0$. (i) Show that the acceleration of the particle is always negative. (ii) What value does the velocity approach as t increases indefinitely?	 2 1	Solution
13	14 a	The velocity of a particle moving along the x-axis is given by $\dot{x} = 10 - 2t$, where x is the displacement from the origin in metres and t is the time in seconds. Initially the particle is 5 metres to the right of the origin. (i) Show that the acceleration of the particle is constant. (ii) Find the time when the particle is at rest. (iii) Show that the position of the particle after 7 seconds is 26 metres to the right of the origin. (iv) Find the distance travelled by the particle during the first 7 seconds.	 1 1 2 2	Solution
12	15 b	The velocity of a particle is given by $\dot{x} = 1 - 2\cos t$, where x is the displacement in metres and t is the time in seconds. Initially the particle is 3 m to the right of the origin. (i) Find the initial velocity of the particle. (ii) Find the maximum velocity of the particle. (iii) Find the displacement, x, of the particle in terms of t. (iv) Find the position of the particle when it is at rest for the first time.	 1 1 2 2	Solution
11	7b	The velocity of a particle moving along the x-axis is given by $\dot{x} = 8 - 8e^{-2t}$, where t is the time in seconds and x is the displacement in metres. (i) Show that the particle is initially at rest. (ii) Show that the acceleration of the particle is always positive. (iii) Explain why the particle is moving in the positive direction for all $t > 0$. (iv) As $t \to \infty$, the velocity of the particle approaches a constant. Find the value of this constant. (v) Sketch the graph of the particle's velocity as a function of time.	 1 1 2 1 2	Solution
10	7a	The acceleration of a particle is given by $\ddot{x} = 4\cos 2t$, where x is displacement in metres and t is time in seconds. Initially the particle is at the origin with a velocity of 1 m s^{-1}. (i) Show that the velocity of the particle is given by $\dot{x} = 2\sin 2t + 1$. (ii) Find the time when the particle first comes to rest. (iii) Find the displacement, x, of the particle in terms of t.	 2 2 2	Solution
09	7a	The acceleration of a particle is given by $\ddot{x} = 8e^{-2t} + 3e^{-t}$, where x is displacement in metres and t is time in seconds. Initially its velocity is -6 ms^{-1} and its displacement is 5 m. (i) Show that the displacement of the particle is given by $x = 2e^{-2t} + 3e^{-t} + t$. (ii) Find the time when the particle comes to rest. (iii) Find the displacement when the particle comes to rest.	 2 3 1	Solution

08 **6b** The graph shows the velocity of a particle, v metres per second, as a function of time, t seconds. [Solution]



(i) What is the initial velocity of the particle? **1**

(ii) When is the velocity of the particle equal to zero? **1**

(iii) When is the acceleration of the particle equal to zero? **1**

(iv) By using Simpson's Rule with five function values, estimate the distance travelled by the particle between t = 0 and t = 8. **3**

---

07 **5b** A particle is moving on the x-axis and is initially at the origin. Its velocity, v metres per second, at time t seconds is given by $v = \dfrac{2t}{16+t^2}$. [Solution]

(i) What is the initial velocity of the particle? **1**

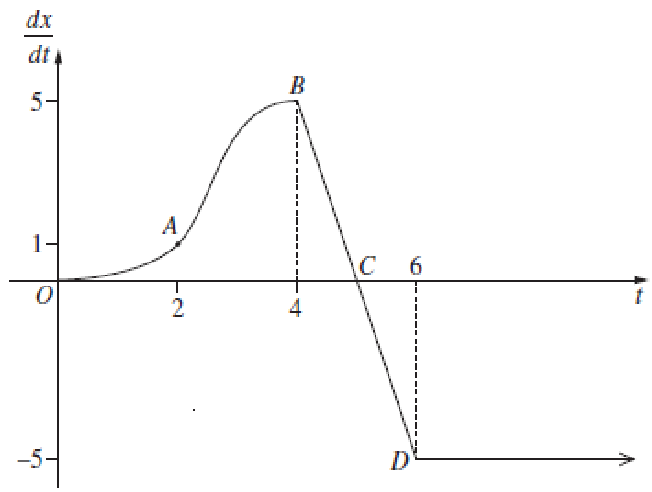
(ii) Find an expression for the acceleration of the particle. **2**

(iii) Find the time when the acceleration of the particle is zero. **1**

(iv) Find the position of the particle when t = 4. **3**

---

07 **10a** An object is moving on the x-axis. The graph shows the velocity, $\dfrac{dx}{dt}$, of the object, as a function of time, t. The coordinates of the points shown on the graph are A(2, 1), B(4, 5), C(5, 0) and D(6, -5). The velocity is constant for $t \geq 6$. [Solution]



(Not to scale)

(i) Using Simpson's rule, estimate the distance travelled between t = 0 and t = 4. **2**

(ii) The object is initially at the origin. During which time(s) is the displacement of the object decreasing? **1**

(iii) Estimate the time at which the object returns to the origin. Justify your answer. **2**

(iv) Sketch the displacement, x, as a function of time. **1**

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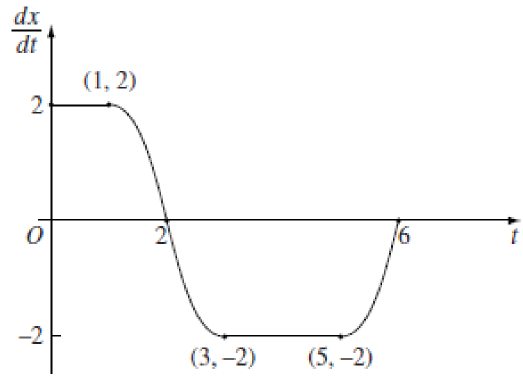
06 **8a** A particle is moving in a straight line. Its displacement, x metres, from the origin, O, at time t seconds, where $t \geq 0$, is given by $x = 1 - \dfrac{7}{t+4}$. [Solution]

(i) Find the initial displacement of the particle. **1**

(ii) Find the velocity of the particle as it passes through the origin. **3**

(iii) Show that the acceleration of the particle is always negative. **1**

(iv) Sketch the graph of the displacement of the particle as a function of time. **2**

05	**7b**	The graph shows the velocity, $\frac{dx}{dt}$, of a particle as a function of time. Initially the particle is at the origin. (i) At what time is the displacement, x, from the origin a maximum? (ii) At what time does the particle return to the origin? Justify your answer. (iii) Draw a sketch of the acceleration, $\frac{d^2x}{dt^2}$, as a function of time for $0 \le t \le 6$.		Solution **1** **2** **2**
05	**9a**	A particle is initially at rest at the origin. Its acceleration as a function of time, t, is given by $\ddot{x} = 4\sin 2t$. (i) Show that the velocity of the particle is given by $\dot{x} = 2 - 2\cos 2t$. (ii) Sketch the graph of the velocity for $0 \le t \le 2\pi$ AND determine the time at which the particle first comes to rest after $t = 0$. (iii) Find the distance travelled by the particle between $t = 0$ and the time at which the particle first comes to rest after $t = 0$.		Solution **2** **3** **2**

projectmaths **Apps of Calculus to Phys World – Exponential Growth & Decay** Back

16	**13 c**	A radioactive isotope of Curium has a half-life of 163 days. Initially there are 10 mg of Curium in a container. The mass $M(t)$ in milligrams of Curium, after t days, is given by $M(t) = Ae^{-kt}$, where A and k are constants. (i) State the value of A. (ii) Given that after 163 days only 5 mg of Curium remain, find the value of k.	 **1** **3**	**Solution**
15	**15 a**	The amount of caffeine, C, in the human body decreases according to the equation $\frac{dC}{dt} = -0.14C$, where C is measured in mg and t is the time in hours. (i) Show that $C = Ae^{-0.14t}$ is a solution to $\frac{dC}{dt} = -0.14C$, where A is a constant. When $t = 0$, there are 130 mg of caffeine in Lee's body. (ii) Find the value of A. (iii) What is the amount of caffeine in Lee's body after 7 hours? (iv) What is the time taken for the amount of caffeine in Lee's body to halve?	 **1** **1** **1** **2**	**Solution**
14	**13 b**	A quantity of radioactive material decays according to the equation $\frac{dM}{dt} = -kM$, where M is the mass of the material in kg, t is the time in years and k is a constant. (i) Show that $M = Ae^{-kt}$ is a solution to the equation, where A is a constant. (ii) The time for half of the material to decay is 300 years. If the initial amount of material is 20 kg, find the amount remaining after 1000 years.	 **1** **3**	**Solution**
13	**16 b**	Trout and carp are types of fish. A lake contains a number of trout. At a certain time 10 carp are introduced into the lake and start eating the trout. As a consequence, the number of trout, N, decreases according to $N = 375 - e^{0.04t}$, where t is the time in months after the carp are introduced. The population of carp, P, increases according to $\frac{dP}{dt} = 0.02P$. (i) How many trout were in the lake when the carp were introduced? (ii) When will the population of trout be zero? (iii) Sketch the number of trout as a function of time. (iv) When is the rate of increase of carp equal to the rate of decrease of trout? (v) When is the number of carp equal to the number of trout?	 **1** **1** **1** **3** **2**	**Solution**
12	**14 c**	Professor Smith has a colony of bacteria. Initially, there are 1000 bacteria. The number of bacteria, $N(t)$, after t minutes is given by $N(t) = 1000e^{kt}$. (i) After 20 minutes there are 2000 bacteria. Show that $k = 0.0347$ correct to four decimal places. (ii) How many bacteria are there when $t = 120$? (iii) What is the rate of change of the number of bacteria per minute, when $t = 120$? (iv) How long does it take for the number of bacteria to increase from 1000 to 100 000?	 **1** **1** **1** **2**	**Solution**

11	**10 a**	The intensity I, measured in watt/m^2, of a sound is given by $I = 10^{-12} \times e^{0.1L}$, where L is the loudness of the sound in decibels. (i) If the loudness of a sound at a concert is 110 decibels, find the intensity of the sound. Give your answer in scientific notation. (ii) Ear damage occurs if the intensity of a sound is greater than 8.1×10^{-9} watt/m^2. What is the maximum loudness of a sound so that no ear damage occurs? (iii) By how much will the loudness of a sound have increased if its intensity has doubled?	**1** **2** **2**	Solution
10	**8a**	Assume that the population, P, of cane toads in Australia has been growing at a rate proportional to P. That is, $\frac{dP}{dt} = kP$, where k is a positive constant. There were 102 cane toads brought to Australia from Hawaii in 1935. Seventy-five years later, in 2010, it is estimated that there are 200 million cane toads in Australia. If the population continues to grow at this rate, how many cane toads will there be in Australia in 2035?	**4**	Solution
09	**6b**	Radium decays at a rate proportional to the amount of radium present. That is, if $Q(t)$ is the amount of radium present at time t, then $Q = Ae^{-kt}$, where k is a positive constant and A is the amount present at $t = 0$. It takes 1600 years for an amount of radium to reduce by half. (i) Find the value of k. (ii) A factory site is contaminated with radium. The amount of radium on the site is currently three times the safe level. How many years will it be before the amount of radium reaches the safe level?	**2** **2**	Solution
08	**5c**	Light intensity is measured in lux. The light intensity at the surface of a lake is 6000 lux. The light intensity, I lux, a distance s metres below the surface of the lake is given by $I = Ae^{-ks}$ where A and k are constants. (i) Write down the value of A. (ii) The light intensity 6 metres below the surface of the lake is 1000 lux. Find the value of k. (iii) At what rate, in lux per metre, is the light intensity decreasing 6 metres below the surface of the lake?	**1** **2** **2**	Solution
07	**8a**	One model for the number of mobile phones in use worldwide is the exponential growth model, $N = Ae^{kt}$, where N is the estimate for the number of mobile phones in use (in millions), and t is the time in years after 1 January 2008. (i) It is estimated that at the start of 2009, when $t = 1$, there will be 1600 million mobile phones in use, while at the start of 2010, when $t = 2$, there will be 2600 million. Find A and k. (ii) According to the model, during which month and year will the number of mobile phones in use first exceed 4000 million?	**3** **2**	Solution

06	**6b**	A rare species of bird lives only on a remote island. A mathematical model predicts that the bird population, P, is given by $P = 150 + 300e^{-0.05t}$ where t is the number of years after observations began.		**Solution**
		(i) According to the model, how many birds were there when observations began?	**1**	
		(ii) According to the model, what will be the rate of change in the bird population ten years after observations began?	**2**	
		(iii) What does the model predict will be the limiting value of the bird population?	**1**	
		(iv) The species will become eligible for inclusion in the endangered species list when the population falls below 200. When does the model predict that this will occur?	**2**	

projectmaths

Probability

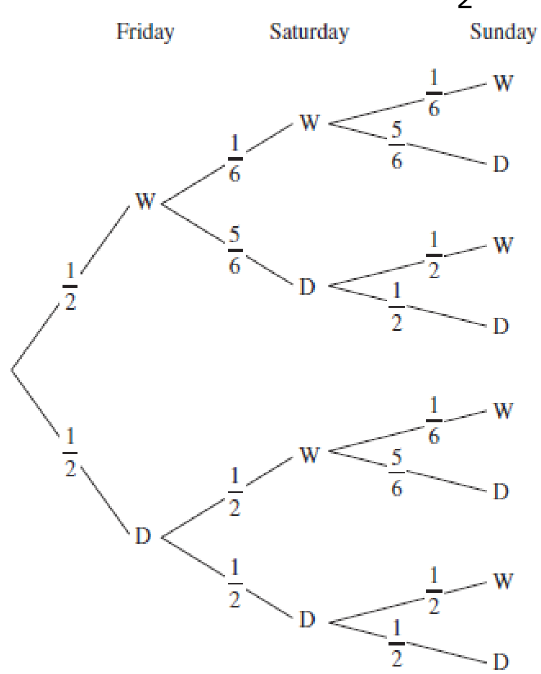
Back

16	**2**	In a raffle, 30 tickets are sold and there is one prize to be won. What is the probability that someone buying 6 tickets wins the prize? (A) $\frac{1}{30}$ (B) $\frac{1}{6}$ (C) $\frac{1}{5}$ (D) $\frac{1}{4}$	**1**	Solution
16	**15 b**	An eight-sided die is marked with numbers 1, 2, …, 8. A game is played by rolling the die until an 8 appears on the uppermost face. At this point the game ends. (i) Using a tree diagram, or otherwise, explain why the probability of the game ending before the fourth roll is $\frac{1}{8} + \frac{7}{8} \times \frac{1}{8} + \left(\frac{7}{8}\right)^2 \times \frac{1}{8}$. (ii) What is the smallest value of n for which the probability of the game ending before the nth roll is more than $\frac{3}{4}$?	**2** **3**	Solution
15	**4**	The probability that Mel's soccer team wins this weekend is $\frac{5}{7}$. The probability that Mel's rugby league team wins this weekend is $\frac{2}{3}$. What is the probability that neither team wins this weekend? (A) $\frac{2}{21}$ (B) $\frac{10}{21}$ (C) $\frac{13}{21}$ (D) $\frac{19}{21}$	**1**	Solution

15 **14 b** Weather records for a town suggest that: **2** **2** Solution

- if a particular day is wet (W), the probability of the next day being dry is $\frac{5}{6}$.
- if a particular day is dry (D), the probability of the next day being dry is $\frac{1}{2}$.

In a specific week Thursday is dry.
The tree diagram shows the possible outcomes for the next three days: Friday, Saturday and Sunday.



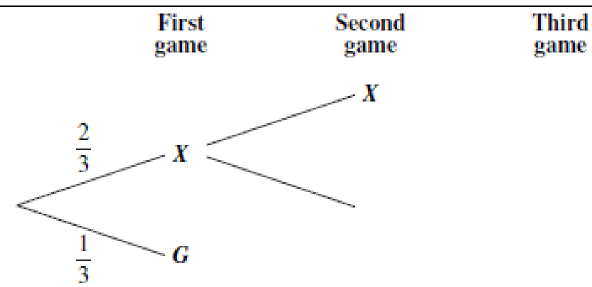
(i) Show that the probability of Saturday being dry is $\frac{2}{3}$. **1**

(ii) What is the probability of both Saturday and Sunday being wet? **2**

(iii) What is the probability of at least one of Saturday and Sunday being dry? **1**

14	**10**	Three runners compete in a race. The probabilities that the three runners finish the race in under 10 seconds are $\frac{1}{4}$, $\frac{1}{6}$ and $\frac{2}{5}$ respectively. What is the probability that at least one of the three runners will finish the race in under 10 seconds? (A) $\frac{1}{60}$ (B) $\frac{37}{60}$ (C) $\frac{3}{8}$ (D) $\frac{5}{8}$	**1**	Solution
14	**12 c**	A packet of lollies contains 5 red lollies and 14 green lollies. Two lollies are selected at random without replacement. (i) Draw a tree diagram to show the possible outcomes. Include the probability on each branch. (ii) What is the probability that the two lollies are of different colours?	 **2** **1**	Solution
13	**5**	A bag contains 4 red marbles and 6 blue marbles. Three marbles are selected at random without replacement. What is the probability that at least one of the marbles selected is red? (A) $\frac{1}{6}$ (B) $\frac{1}{2}$ (C) $\frac{5}{6}$ (D) $\frac{29}{30}$	**1**	Solution
13	**15 d**	Pat and Chandra are playing a game. They take turns throwing two dice. The game is won by the first player to throw a double six. Pat starts the game. (i) Find the probability that Pat wins the game on the first throw. (ii) What is the probability that Pat wins the game on the first or on the second throw? (iii) Find the probability that Pat eventually wins the game.	 **1** **2** **2**	Solution
12	**13 c**	Two buckets each contain red marbles and white marbles. Bucket A contains 3 red and 2 white marbles. Bucket B contains 3 red and 4 white marbles. Chris randomly chooses one marble from each bucket. (i) What is the probability that both marbles are red? (ii) What is the probability that at least one of the marbles is white? (iii) What is the probability that both marbles are the same colour?	 **1** **1** **2**	Solution
11	**1g**	A batch of 800 items is examined. The probability that an item from this batch is defective is 0.02. How many items from this batch are defective?	**1**	Solution
11	**5b**	Kim has three red shirts and two yellow shirts. On each of the three days, Monday, Tuesday and Wednesday, she selects one shirt at random to wear. Kim wears each shirt that she selects only once. (i) What is the probability that Kim wears a red shirt on Monday? (ii) What is the probability that Kim wears a shirt of the same colour on all three days? (iii) What is the probability that Kim does not wear a shirt of the same colour on consecutive days?	 **1** **1** **2**	Solution
10	**4c**	There are twelve chocolates in a box. Four of the chocolates have mint centres, four have caramel centres and four have strawberry centres. Ali randomly selects two chocolates and eats them. (i) What is the probability that the two chocolates have mint centres? (ii) What is the probability that the two chocolates have same centres? (iii) What is the probability that the two chocolates have different centres?	 **1** **1** **1**	Solution

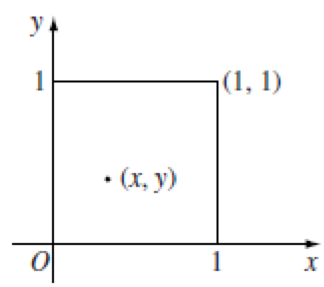
10	**8b**	Two identical biased coins are tossed together, and the outcome is recorded. After a large number of trials it is observed that the probability that both coins land showing heads is 0.36. What is the probability that both coins land showing tails?	**2**	Solution
09	**5b**	On each working day James parks his car in a parking station which has three levels. He parks his car on a randomly chosen level. He always forgets where he has parked so when he leaves work he chooses a level at random and searches for his car. If his car is not on that level, he chooses a different level and continues in this way until he finds his car.		Solution
		(i) What is the probability that his car is on the first level he searches?	**1**	
		(ii) What is the probability that he must search all three levels before he finds his car?	**1**	
		(iii) What is the probability that on every one of the five working days in a week, his car is not on the first level he searches?	**1**	
09	**9a**	Each week Van and Marie take part in a raffle at their respective workplaces. The probability that Van wins a prize in his raffle is $\frac{1}{9}$. The probability that Marie wins a prize in her raffle is $\frac{1}{16}$. What is the probability that, during the next three weeks, at least one of them wins a prize?	**2**	Solution
08	**7c**	Xena and Gabrielle compete in a series of games. The series finishes when one player has won two games. In any game, the probability that Xena wins is $\frac{2}{3}$ and the probability that Gabrielle wins is $\frac{1}{3}$.		Solution



		(i) Copy and complete the tree diagram.	**1**	
		(ii) What is the probability that Gabrielle wins the series?	**2**	
		(iii) What is the probability that three games are played in the series?	**2**	
08	**9a**	It is estimated that 85% of students in Australia own a mobile phone.		Solution
		(i) Two students are selected at random. What is the probability that neither of them owns a mobile phone?	**2**	
		(ii) Based on a recent survey, 20% of the students who own a mobile phone have used their mobile phone during class time. A student is selected at random. What is the probability that the student owns a mobile phone and has used it during classtime?	**1**	
07	**4b**	Two ordinary dice are rolled. The score is the sum of the numbers on the top faces.		Solution
		(i) What is the probability that the score is 10?	**2**	
		(ii) What is the probability that the score is not 10?	**1**	

07	**9b**	A pack of 52 cards consists of four suits with 13 cards in each suit. (i) One card is drawn from the pack and kept on the table. A second card is drawn and placed beside it on the table. What is the probability that the second card is from a different suit to the first? (ii) The two cards are replaced and the pack shuffled. Four cards are chosen from the pack and placed side by side on the table. What is the probability that these four cards are all from different suits?	 **1** **2**	Solution
06	**4c**	A chessboard has 32 black squares and 32 white squares. Tanya chooses three different squares at random. (i) What is the probability that Tanya chooses three white squares? (ii) What is the probability that the three squares Tanya chooses are the same colour? (iii) What is the probability that the three squares Tanya chooses are not the same colour?	 **2** **1** **1**	Solution
05	**5d**	A total of 300 tickets are sold in a raffle which has three prizes. There are 100 red, 100 green and 100 blue tickets. At the drawing of the raffle, winning tickets are NOT replaced before the next draw. (i) What is the probability that each of the three winning tickets is red? (ii) What is the probability that at least one of the winning tickets is not red? (iii) What is the probability that there is one winning ticket of each colour?	 **2** **1** **2**	Solution

05 **10 b** Xuan and Yvette would like to meet at a cafe on Monday. They each agree to come to the cafe sometime between 12 noon and 1 pm, wait for 15 minutes, and then leave if they have not seen the other person. Their arrival times can be represented by the point (x, y) in the Cartesian plane, where x represents the fraction of an hour after 12 noon that Xuan arrives, and y represents the fraction of an hour after 12 noon that Yvette arrives. Solution



Thus $\left(\frac{1}{3}, \frac{2}{5}\right)$ represents Xuan arriving at 12:20 pm and Yvette arriving at 12:24 pm.

Note that the point (x, y) lies somewhere in the unit square $0 \leq x \leq 1$ and $0 \leq y \leq 1$ as shown in the diagram.

(i) Explain why Xuan and Yvette will meet if $x - y \leq \frac{1}{4}$ or $y - x \leq \frac{1}{4}$. **1**

(ii) The probability that they will meet is equal to the area of the part of the region given by the inequalities in part (i) that lies within the unit square $0 \leq x \leq 1$ and $0 \leq y \leq 1$.
Find the probability that they will meet. **2**

(iii) Xuan and Yvette agree to try to meet again on Tuesday. They agree to arrive between 12 noon and 1 pm, but on this occasion they agree to wait for t minutes before leaving. For what value of t do they have a 50% chance of meeting? **2**

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Series and Applications

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16 14 b A gardener develops an eco-friendly spray that will kill harmful insects on fruit trees without contaminating the fruit. A trial is to be conducted with 100 000 insects. The gardener expects the spray to kill 35% of the insects each day and exactly 5000 new insects will be produced each day. **Solution**

The number of insects expected at the end of the nth day of the trial is A_n.

(i) Show that $A_2 = 0.65(0.65 \times 100\ 000 + 5000) + 5000$. **2**

(ii) Show that $A_n = 0.65^n \times 100\ 000 + 5000\dfrac{(1-0.65^n)}{0.35}$. **1**

(iii) Find the expected insect population at the end of the fourteenth day, correct to the nearest 100. **1**

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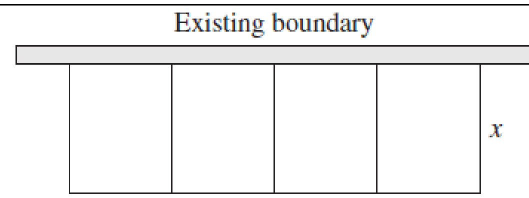
16 14 c A farmer wishes to make a rectangular enclosure of area 720 m^2. She uses an existing straight boundary as one side of the enclosure. She uses wire fencing for the remaining three sides and also to divide the enclosure into four equal rectangular areas of width x m as shown. **Solution**



(i) Show that the total length, ℓ m, of the wire fencing is given by $\ell = 5x + \dfrac{720}{x}$. **1**

(ii) Find the minimum length of wire fencing required, showing why this is the minimum length. **3**

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16 14 d By summing the geometric series $1 + x + x^2 + x^3 + x^4$, or otherwise, find $\lim\limits_{x\to 1}\dfrac{x^5-1}{x-1}$. **2** **Solution**

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15 3 The first three terms of an arithmetic series are 3, 7 and 11. **1** **Solution**

What is the 15th term of this series?

(A) 59 (B) 63 (C) 465 (D) 495

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15 11 d Find the limiting sum of the geometric series $1 - \dfrac{1}{4} + \dfrac{1}{16} - \dfrac{1}{64} + \ldots$ **2** **Solution**

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15 14 c Sam borrows \$100 000 to be repaid at a reducible interest rate of 0.6% per month. Let \$$A_n$ be the amount owing at the end of n months and \$$M$ be the monthly repayment. **Solution**

(i) Show that $A_2 = 100\ 000(1.006)^2 - M(1 + 1.006)$. **1**

(ii) Show that $A_n = 100\ 000(1.006)^n - M\left(\dfrac{(1.006)^n - 1}{0.006}\right)$. **2**

(iii) Sam makes monthly repayments of \$780. Show that after making 120 monthly repayments the amount owing is \$68 500 to the nearest \$100. **1**

(iv) Immediately after making the 120th repayment, Sam makes a one-off payment, reducing the amount owing to \$48 500. The interest rate and monthly repayment remain unchanged. After how many more months will the amount owing be completely repaid? **3**

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14 8 Which expression is a term of the geometric series $3x - 6x^2 + 12x^3 - \ldots$? **1** **Solution**

(A) $3072x^{10}$ (B) $-3072x^{10}$ (C) $3072x^{11}$ (D) $-3072x^{11}$

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14 **12 a** Evaluate the arithmetic series 2 + 5 + 8 + 11 + … + 1094. **2** [Solution]

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14 **14 d** At the beginning of every 8-hour period, a patient is given 10 mL of a particular drug. During each of these 8-hour periods, the patient's body partially breaks down the drug. Only $\frac{1}{3}$ of the total amount of the drug present in the patient's body at the beginning of each 8-hour period remains at the end of that period. [Solution]

(i) How much of the drug is in the patient's body immediately after the second dose is given? **1**

(ii) Show that the total amount of the drug in the patient's body never exceeds 15 mL. **2**

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14 **16 b** At the start of a month, Jo opens a bank account and makes a deposit of \$500. At the start of each subsequent month, Jo makes a deposit which is 1% more than the previous deposit. At the end of each month, the bank pays interest of 0.3% (per month) on the balance of the account. [Solution]

(i) Explain why the balance of the account at the end of the second month is $\$500(1.003)^2 + \$500(1.01)(1.003)$. **2**

(ii) Find the balance of the account at the end of the 60^{th} month, correct to the nearest dollar. **3**

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13 **12 c** Kim and Alex start jobs at the beginning of the same year. Kim's annual salary in the first year is \$30 000, and increases by 5% at the beginning of each subsequent year. Alex's annual salary in the first year is \$33 000, and increases by \$1500 at the beginning of each subsequent year. [Solution]

(i) Show that in the 10^{th} year Kim's annual salary is higher than Alex's annual salary. **2**

(ii) In the first 10 years how much, in total, does Kim earn? **2**

(iii) Every year, Alex saves $\frac{1}{3}$ of her annual salary. How many years does it take her to save \$87 500? **3**

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13 **13 d** A family borrows \$500 000 to buy a house. The loan is to be repaid in equal monthly instalments. The interest, which is charged at 6% per annum, is reducible and calculated monthly. The amount owing after n months, $\$A_n$, is given by [Solution]

$A_n = Pr^n - M(1 + r + r^2 + \ldots + r^{n-1})$, (Do NOT prove this)

where $\$P$ is the amount borrowed, $r = 1.005$ and $\$M$ is the monthly repayment.

(i) The loan is to be repaid over 30 years. Show that the monthly repayment is \$2998 to the nearest dollar. **2**

(ii) Show that the balance owing after 20 years is \$270 000 to the nearest thousand dollars. **1**

(iii) After 20 years the family borrows an extra amount, so that the family then owes a total of \$370 000. The monthly repayment remains \$2998, and the interest rate remains the same. How long will it take to repay the \$370 000? **2**

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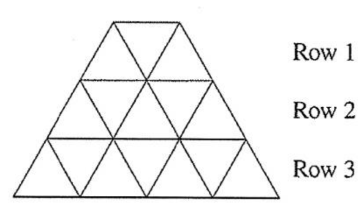
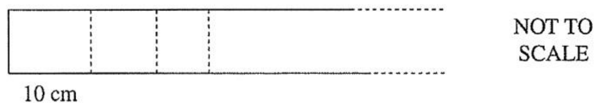
12 **12 c** Jay is making a pattern using triangular tiles. The pattern has 3 tiles in the first row, 5 tiles in the second row, and each successive row has 2 more tiles than the previous row. [Solution]



(i) How many tiles would Jay use in row 20? **2**

(ii) How many tiles would Jay use altogether to make the first 20 rows? **1**

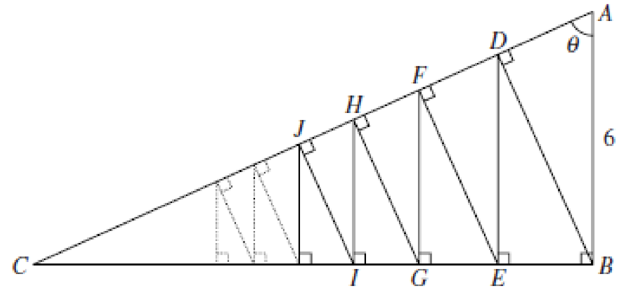
(iii) Jay has only 200 tiles. How many complete rows of the pattern can Jay make? **2**

12	15 a	Rectangles of the same height are cut from a strip and arranged in a row. The first rectangle has width 10 cm. The width of each subsequent rectangle is 96% of the width of the previous rectangle.  (i) Find the length of the strip required to make the first ten rectangles. (ii) Explain why a strip of length 3 m is sufficient to make any number of rectangles.	 2 1	Solution
12	15 c	Ari takes out a loan of \$360 000. The loan is to be repaid in equal monthly repayments, \$$M$, at the end of each month, over 25 years (300 months). Reducible interest is charged at 6% per annum, calculated monthly. Let \$$A_n$ be the amount owing after the nth repayment. (i) Write down an expression for the amount owing after two months, \$$A_2$. (ii) Show that the monthly repayment is approximately \$2319.50. (iii) After how many months will the amount owing, \$$A_n$, become less than \$180 000?	 1 2 3	Solution
11	3a	A skyscraper of 110 floors is to be built. The first floor to be built will cost \$3 million. The cost of building each subsequent floor will be \$0.5 million more than the floor immediately below. (i) What will be the cost of building the 25th floor? (ii) What will be the cost of building all 110 floors of the skyscraper?	 2 2	Solution
11	5a	The number of members of a new social networking site doubles every day. On Day 1 there were 27 members and on Day 2 there were 54 members. (i) How many members were there on Day 12? (ii) On which day was the number of members first greater than 10 million? (iii) The site earns 0.5 cents per member per day. How much money did the site earn in the first 12 days? Give your answer to the nearest dollar.	 1 2 2	Solution
11	8c	When Jules started working she began paying \$100 at the beginning of each month into a superannuation fund. The contributions are compounded monthly at an interest rate of 6% per annum. She intends to retire after having worked for 35 years. (i) Let \$$P$ be the final value of Jules's superannuation when she retires after 35 years (420 months). Show that \$$P$ = \$143 183 to the nearest dollar. (ii) Fifteen years after she started working Jules read a magazine article about retirement, and realized that she would need \$800 000 in her fund when she retires. At the time of reading the magazine article she had \$29 227 in her fund. For the remaining 20 years she intends to work, she decides to pay a total of \$$M$ into her fund at the beginning of each month. The contributions continue to attract the same interest rate of 6% per annum, compounded monthly. At the end of n months after starting the new contributions, the amount in the fund is \$$A_n$. (1) Show that $A_2 = 29\ 227 \times 1.005^2 + M(1.005 + 1.005^2)$. (2) Find the value of M so that Jules will have \$800 000 in her fund after the remaining 20 years (240 months).	 2 1 3	Solution
10	1f	Find the limiting sum of the geometric series $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \ldots$	2	Solution

10	**4a**	Susannah is training for a fun run by running every week for 26 weeks. She runs 1 km in the first week and each week after that she runs 750 m more than the previous week, until she reaches 10 km in a week. She then continues to run 10 km each week. (i) How far does Susannah run in the 9^{th} week? (ii) In which week does she first run 10 km? (iii) What is the total distance that Susannah runs in 26 weeks?	 **1** **1** **2**	Solution
10	**9a**	(i) When Chris started a new job, \$500 was deposited into his superannuation fund at the beginning of each month. The money was invested at 0.5% per month, compounded monthly. Let $\$P$ be the value of the investment after 240 months, when Chris retires. Show that $P = 232\ 175.55$ (ii) After retirement, Chris withdraws \$2000 from the account at the end of each month, without making any further deposits. The account continues to earn interest at 0.5% per month. Let $\$A_n$ be the amount left in the account n months after Chris's retirement. (1) Show that $A_n = (P - 400\ 000) \times 1.005^n + 400\ 000$. (2) For how many months after retirement will there be money left in the account?	**2** **3** **2**	Solution
09	**2c**	Evaluate $\sum_{k=1}^{4}(-1)^k k^2$.	**2**	Solution
09	**3a**	An arithmetic series has 21 terms. The first term is 3 and the last term is 53. Find the sum of the series.	**2**	Solution
09	**4a**	A tree grows from ground level to a height of 1.2 metres in one year. In each subsequent year, it grows $\frac{9}{10}$ as much as it did in the previous year. Find the limiting height of the tree.	**2**	Solution
09	**8b**	One year ago Daniel borrowed \$350 000 to buy a house. The interest rate was 9% per annum, compounded monthly. He agreed to repay the loan in 25 years with equal monthly repayments of \$2937. (i) Calculate how much Daniel owed after his monthly repayment. (ii) Daniel has just made his 12^{th} monthly repayment. He now owes \$346 095. The interest rate now decreases to 6% per annum, compounded monthly. The amount $\$A_n$, owing on the loan after the n^{th} monthly repayment is now calculated using the formula $A_n = 346\ 095 \times 1.005^n - 1.005^{n-1} M - \ldots - 1.005M - M$ where $\$M$ is the monthly repayment and $n = 1, 2, \ldots, 288$. (Do NOT prove this formula.) Calculate the monthly repayment if the loan is to be repaid over the remaining 24 years (288 months). (iii) Daniel chooses to keep his monthly repayments at \$2937. Use the formula in part (ii) to calculate how long it will take him to repay the \$346 095. (iv) How much will Daniel save over the term of the loan by keeping his monthly repayments at \$2937, rather than reducing his repayments to the amount calculated in part (ii)?	 **1** **3** **3** **1**	Solution

08	**1f**	Find the sum of the first 21 terms of the arithmetic series 3 + 7 + 11 + …	**2**	Solution
08	**4b**	The zoom function in a software package multiplies the dimensions of an image by 1.2. In an image, the height of a building is 50 mm. After the zoom function is applied once, the height of the building in the image is 60 mm. After a second application, its height is 72 mm. (i) Calculate the height of the building in the image after the zoom function has been applied eight times. Give your answer to the nearest mm. (ii) The height of the building in the image is required to be more than 400 mm. Starting from the original image, what is the least number of times the zoom function must be applied?	 **2** **2**	Solution
08	**5b**	Consider the geometric series $5 + 10x + 20x^2 + 40x^3 + \ldots$ (i) For what values of x does this series have a limiting sum? (ii) The limiting sum of this series is 100. Find the value of x.	 **2** **2**	Solution
08	**9b**	Peter retires with a lump sum of \$100 000. The money is invested in a fund which pays interest each month at a rate of 6% per annum, and Peter receives a fixed monthly payment of $\$M$ from the fund. Thus, the amount left in the fund after the first monthly payment is $\$(100\,500 - M)$. (i) Find a formula for the amount, $\$A_n$, left in the fund after n monthly payments. (ii) Peter chooses the value of M so that there will be nothing left in the fund at the end of the 12th year (after 144 payments). Find the value of M.	 **2** **3**	Solution
07	**1d**	Find the limiting sum of the geometric series $\frac{3}{4} + \frac{3}{16} + \frac{3}{64} + \ldots$	**2**	Solution
07	**3b**	Heather decides to swim every day to improve her fitness level. On the first day she swims 750 metres, and on each day after that she swims 100 metres more than the previous day. That is, she swims 850 metres on the second day, 950 metres on the third day and so on. (i) Write down a formula for the distance she swims on the nth day. (ii) How far does she swim on the 10th day? (iii) What is the total distance she swims in the first 10 days? (iv) After how many days does the total distance she has swum equal the width of the English Channel, a distance of 34 kilometres?	 **1** **1** **1** **2**	Solution

07	**9c**	Mr and Mrs Caine each decide to invest some money each year to help pay for their son's university education. The parents choose different investment strategies. (i) Mr Caine makes 18 yearly contributions of \$1000 into an investment fund. He makes his first contribution on the day his son is born, and his final contribution on his son's seventeenth birthday. His investment earns 6% compound interest per annum. Find the total value of Mr Caine's investment on his son's eighteenth birthday. (ii) Mrs Caine makes her contributions into another fund. She contributes \$1000 on the day of her son's birth, and increases her annual contribution by 6% each year. Her investment also earns 6% compound interest per annum. Find the total value of Mrs Caine's investment on her son's third birthday (just before she makes her fourth contribution). (iii) Mrs Caine also makes her final contribution on her son's seventeenth birthday. Find the total value of Mrs Caine's investment on her son's eighteenth birthday.	 **3** **2** **1**	Solution
06	**1f**	Find the limiting sum of the geometric series $\frac{13}{5} + \frac{13}{25} + \frac{13}{125} + \ldots$	**2**	Solution
06	**3b**	Evaluate $\sum_{n=2}^{4} \frac{1}{r}$.	**1**	Solution
06	**3c**	On the first day of the harvest, an orchard produces 560 kg of fruit. On the next day, the orchard produces 543 kg, and the amount produced continues to decrease by the same amount each day. (i) How much fruit is produced on the fourteenth day of the harvest? (ii) What is the total amount of fruit that is produced in the first 14 days of the harvest? (iii) On what day does the daily production first fall below 60 kg?	 **2** **1** **2**	Solution
06	**8b**	Joe borrows \$200 000 which is to be repaid in equal monthly instalments. The interest rate is 7.2% per annum reducible, calculated monthly. It can be shown that the amount, \$$A_n$, owing after the nth repayment is given by the formula: $A_n = 200\,000r^n - M(1 + r + r^2 + \cdots + r^{n-1})$, where $r = 1.006$ and \$$M$ is the monthly repayment. (Do NOT show this.) (i) The minimum monthly repayment is the amount required to repay the loan in 300 instalments. Find the minimum monthly repayment. (ii) Joe decides to make repayments of \$2800 each month from the start of the loan. How many months will it take for Joe to repay the loan?	 **3** **2**	Solution
05	**3a**	Evaluate $\sum_{n=3}^{5}(2n+1)$.	**1**	Solution
05	**7a**	Anne and Kay are employed by an accounting firm. Anne accepts employment with an initial annual salary of \$50 000. In each of the following years her annual salary is increased by \$2500. Kay accepts employment with an initial annual salary of \$50 000. In each of the following years her annual salary is increased by 4%. (i) What is Anne's annual salary in her thirteenth year? (ii) What is Kay's annual salary in her thirteenth year? (iii) By what amount does the total amount paid to Kay in her first twenty years exceed that paid to Anne in her first twenty years?	 **2** **2** **3**	Solution

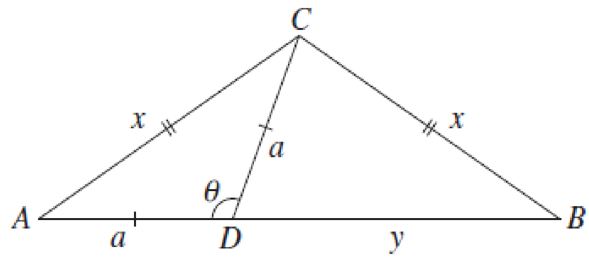
05	**8c**	Weelabarrabak Shire Council borrowed \$3 000 000 at the beginning of 2005. The annual interest rate is 12%. Each year, interest is calculated on the balance at the beginning of the year and added to the balance owing. The debt is to be repaid by equal annual repayments of \$480 000, with the first repayment being made at the end of 2005. Let A_n be the balance owing after the n-th repayment. (i) Show that $A_2 = (3 \times 10^6)(1.12)^2 - (4.8 \times 10^5)(1 + 1.12)$. (ii) Show that $A_n = 10^6[4 - (1.12)^n]$. (iii) In which year will Weelabarrabak Shire Council make the final repayment?	 **1** **2** **2**	**Solution**
05	**9b**	The triangle ABC has a right angle at B, $BAC = \theta$ and $AB = 6$. The line BD is drawn perpendicular to AC. The line DE is then drawn perpendicular to BC. This process continues indefinitely as shown in the diagram.  (i) Find the length of the interval BD, and hence show that the length of the interval EF is $6\sin^3\theta$. (ii) Show that the limiting sum $BD + EF + GH + \cdots$ is given by $6\sec\theta\tan\theta$.	 **2** **3**	**Solution**

Combined Topics

Back

10 **10 a** In the diagram, *ABC* is an isosceles triangle $AC = BC = x$. The point *D* on the interval *AB* is chosen so that $AD = CD$. Let $AD = a$, $DB = y$ and $\angle ADC = \theta$.

Solution



(i) Show that ΔABC is similar to ΔACD. **2**
(ii) Show that $x^2 = a^2 + ay$ **1**
(iii) Show that $y = a(1 - 2\cos\theta)$ **2**
(iv) Deduce that $y \le 3a$. **1**

09 **10** Let $f(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$.

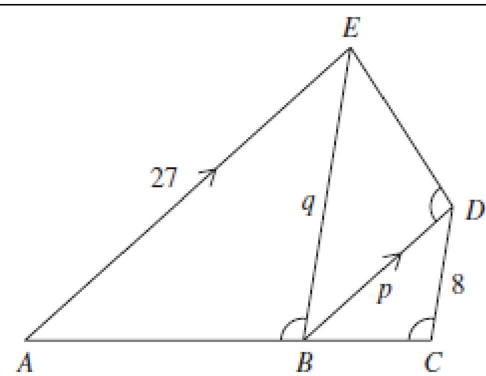
Solution

(a) Show that the graph of $y = f(x)$ has no turning points. **2**
(b) Find the point of inflexion of $y = f(x)$. **1**
(c) (i) Show that $1 - x + x^2 - \frac{1}{1+x} = \frac{x^3}{1+x}$ for $x \ne -1$. **1**
(ii) Let $g(x) = \ln(1 + x)$. **2**
Use the result in part (c) (i) to show that $f'(x) \ge g'(x)$ for all $x \ge 0$.
(d) On the same set of axes, sketch the graphs of $y = f(x)$ and $y = g(x)$ for $x \ge 0$. **2**
(e) Show that $\frac{d}{dx}[(1 + x)\ln(1 + x) - (1 + x)] = \ln(1 + x)$. **2**
(f) Find the area enclosed by the graphs of $y = f(x)$ and $y = g(x)$, and the straight line $x = 1$. **2**

07 **8b** In the diagram, *AE* is parallel to *BD*, $AE = 27$, $CD = 8$, $BD = p$, $BE = q$ and $\angle ABE$, $\angle BCD$ and $\angle BDE$ are equal. Copy or trace this diagram into your writing booklet.

Solution

(i) Prove that $\Delta ABE \;|||\; \Delta BCD$. **2**
(ii) Prove that $\Delta EDB \;|||\; \Delta BCD$. **2**
(iii) Show that 8, p, q, 27 are the first four terms of a geometric series. **1**
(iv) Hence find the values of p and q. **2**

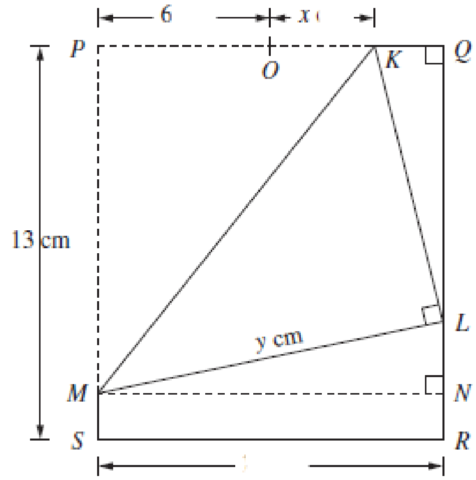


(Not to scale)

06 **10 b** A rectangular piece of paper $PQRS$ has sides PQ = 12 cm and PS = 13 cm. The point O is the midpoint of PQ. The points K and M are to be chosen on OQ and PS respectively, so that when the paper is folded along KM, the corner that was at P lands on the edge QR at L. Let $OK = x$ cm and $LM = y$ cm.

Solution

Copy or trace the diagram into your writing booklet.



(i) Show that $QL^2 = 24x$. **1**

(ii) Let N be the point on QR for which MN is perpendicular to QR. By showing that $\Delta QKL \,|||\, \Delta NLM$, deduce that $y = \dfrac{\sqrt{6}(6+x)}{\sqrt{x}}$ **3**

(iii) Show that the area, A, of ΔKLM is given by $A = \dfrac{\sqrt{6}(6+x)^2}{2\sqrt{x}}$. **1**

(iv) Use the fact that $12 \le y \le 13$ to find the possible values of x. **2**

(v) Find the minimum possible area of ΔKLM. **3**

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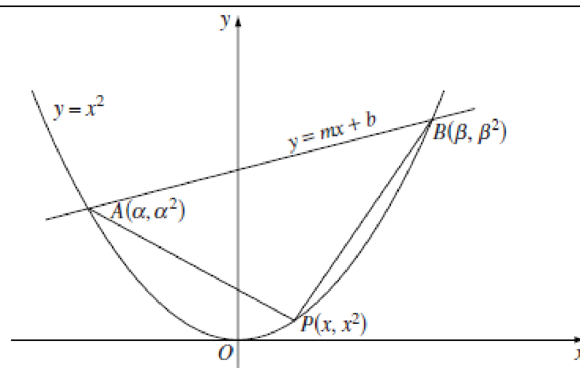
05 **10 a** The parabola $y = x^2$ and the line $y = mx + b$ intersect at the points $A(\alpha, \alpha^2)$ and $B(\beta, \beta^2)$ as shown in the diagram.

Solution



(i) Explain why $\alpha + \beta = m$ and $\alpha\beta = -b$. **1**

(ii) Given that $(\alpha - \beta)^2 + (\alpha^2 - \beta^2)^2 = (\alpha - \beta)^2\,[1 + (\alpha + \beta)^2]$ show that the distance $AB = \sqrt{(m^2 + 4b)(1 + m^2)}$. **2**

(iii) The point $P(x, x^2)$ lies on the parabola between A and B. Show that the area of the triangle ABP is given by $\frac{1}{2}(mx - x^2 + b)\sqrt{m^2 + 4b}$. **2**

(iv) The point P in part (iii) is chosen so that the area of the triangle ABP is a maximum. Find the coordinates of P in terms of m. **2**

BOSTES BOARD OF STUDIES TEACHING & EDUCATIONAL STANDARDS NSW

2016 HIGHER SCHOOL CERTIFICATE EXAMINATION

REFERENCE SHEET

– Mathematics –

– Mathematics Extension 1 –

– Mathematics Extension 2 –

Mathematics

Factorisation

$$a^2 - b^2 = (a+b)(a-b)$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Angle sum of a polygon

$$S = (n-2) \times 180^\circ$$

Equation of a circle

$$(x-h)^2 + (y-k)^2 = r^2$$

Trigonometric ratios and identities

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

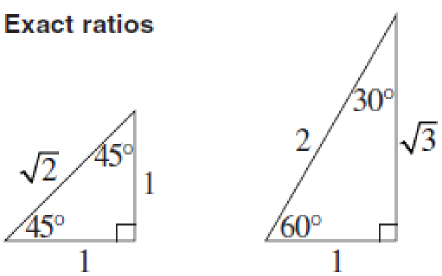
$$\sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1$$

Exact ratios



Sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Cosine rule

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Area of a triangle

$$\text{Area} = \frac{1}{2}ab\sin C$$

Distance between two points

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Perpendicular distance of a point from a line

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Slope (gradient) of a line

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Point-gradient form of the equation of a line

$$y - y_1 = m(x - x_1)$$

***n*th term of an arithmetic series**

$$T_n = a + (n-1)d$$

Sum to *n* terms of an arithmetic series

$$S_n = \frac{n}{2}[2a + (n-1)d] \quad \text{or} \quad S_n = \frac{n}{2}(a+l)$$

***n*th term of a geometric series**

$$T_n = ar^{n-1}$$

Sum to *n* terms of a geometric series

$$S_n = \frac{a(r^n - 1)}{r-1} \quad \text{or} \quad S_n = \frac{a(1-r^n)}{1-r}$$

Limiting sum of a geometric series

$$S = \frac{a}{1-r}$$

Compound interest

$$A_n = P\left(1 + \frac{r}{100}\right)^n$$

Mathematics (continued)

Differentiation from first principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Derivatives

If $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$

If $y = uv$, then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

If $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

If $y = F(u)$, then $\frac{dy}{dx} = F'(u)\frac{du}{dx}$

If $y = e^{f(x)}$, then $\frac{dy}{dx} = f'(x)e^{f(x)}$

If $y = \log_e f(x) = \ln f(x)$, then $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$

If $y = \sin f(x)$, then $\frac{dy}{dx} = f'(x)\cos f(x)$

If $y = \cos f(x)$, then $\frac{dy}{dx} = -f'(x)\sin f(x)$

If $y = \tan f(x)$, then $\frac{dy}{dx} = f'(x)\sec^2 f(x)$

Solution of a quadratic equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Sum and product of roots of a quadratic equation

$$\alpha + \beta = -\frac{b}{a} \qquad \alpha\beta = \frac{c}{a}$$

Equation of a parabola

$$(x-h)^2 = \pm 4a(y-k)$$

Integrals

$$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

$$\int e^{ax+b} \, dx = \frac{1}{a}e^{ax+b} + C$$

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$$

$$\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b) + C$$

Trapezoidal rule (one application)

$$\int_a^b f(x)dx \approx \frac{b-a}{2}\left[f(a) + f(b)\right]$$

Simpson's rule (one application)

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Logarithms – change of base

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Angle measure

$180^\circ = \pi$ radians

Length of an arc

$l = r\theta$

Area of a sector

$$\text{Area} = \frac{1}{2}r^2\theta$$

Mathematics Extension 1

Angle sum identities

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

$$\tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1-\tan\theta\tan\phi}$$

***t* formulae**

If $t = \tan\frac{\theta}{2}$, then

$$\sin\theta = \frac{2t}{1+t^2}$$

$$\cos\theta = \frac{1-t^2}{1+t^2}$$

$$\tan\theta = \frac{2t}{1-t^2}$$

General solution of trigonometric equations

$\sin\theta = a, \quad \theta = n\pi + (-1)^n \sin^{-1} a$

$\cos\theta = a, \quad \theta = 2n\pi \pm \cos^{-1} a$

$\tan\theta = a, \quad \theta = n\pi + \tan^{-1} a$

Division of an interval in a given ratio

$$\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$$

Parametric representation of a parabola

For $x^2 = 4ay$,

$$x = 2at, \quad y = at^2$$

At $(2at, at^2)$,

tangent: $y = tx - at^2$

normal: $x + ty = at^3 + 2at$

At (x_1, y_1),

tangent: $xx_1 = 2a(y+y_1)$

normal: $y - y_1 = -\frac{2a}{x_1}(x-x_1)$

Chord of contact from (x_0, y_0): $xx_0 = 2a(y+y_0)$

Acceleration

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

Simple harmonic motion

$x = b + a\cos(nt+\alpha)$

$\ddot{x} = -n^2(x-b)$

Further integrals

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

Sum and product of roots of a cubic equation

$$\alpha+\beta+\gamma = -\frac{b}{a}$$

$$\alpha\beta+\alpha\gamma+\beta\gamma = \frac{c}{a}$$

$$\alpha\beta\gamma = -\frac{d}{a}$$

Estimation of roots of a polynomial equation

Newton's method

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Binomial theorem

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k} = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$