Series and Applications

16 **14 b** A gardener develops an eco-friendly spray that will kill harmful insects on fruit trees without contaminating the fruit. A trial is to be conducted with 100 000 insects. The gardener expects the spray to kill 35% of the insects each day and exactly 5000 new insects will be produced each day.
The number of insects expected at the end of the nth day of the trial is A_n.

(i) Show that $A_2 = 0.65(0.65 \times 100\,000 + 5000) + 5000$. **2**

(ii) Show that $A_n = 0.65^n \times 100\,000 + 5000\dfrac{(1 - 0.65^n)}{0.35}$. **1**

(iii) Find the expected insect population at the end of the fourteenth day, correct to the nearest 100. **1**

Solution

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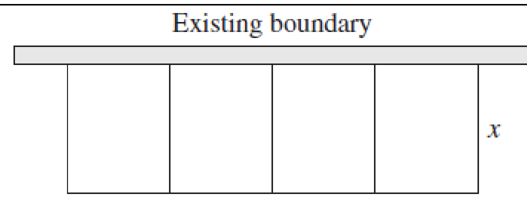
16 **14 c** A farmer wishes to make a rectangular enclosure of area 720 m^2. She uses an existing straight boundary as one side of the enclosure. She uses wire fencing for the remaining three sides and also to divide the enclosure into four equal rectangular areas of width x m as shown.



(i) Show that the total length, ℓ m, of the wire fencing is given by $\ell = 5x + \dfrac{720}{x}$. **1**

(ii) Find the minimum length of wire fencing required, showing why this is the minimum length. **3**

Solution

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16 **14 d** By summing the geometric series $1 + x + x^2 + x^3 + x^4$, or otherwise, find $\displaystyle\lim_{x \to 1} \frac{x^5 - 1}{x - 1}$. **2**

Solution

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15 **3** The first three terms of an arithmetic series are 3, 7 and 11.
What is the 15th term of this series? **1**

(A) 59 (B) 63 (C) 465 (D) 495

Solution

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15 **11 d** Find the limiting sum of the geometric series $1 - \dfrac{1}{4} + \dfrac{1}{16} - \dfrac{1}{64} + \ldots$ **2**

Solution

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15 **14 c** Sam borrows \$100 000 to be repaid at a reducible interest rate of 0.6% per month. Let $\$A_n$ be the amount owing at the end of n months and $\$M$ be the monthly repayment.

(i) Show that $A_2 = 100\,000(1.006)^2 - M(1 + 1.006)$. **1**

(ii) Show that $A_n = 100\,000(1.006)^n - M\left(\dfrac{(1.006)^n - 1}{0.006}\right)$. **2**

(iii) Sam makes monthly repayments of \$780. Show that after making 120 monthly repayments the amount owing is \$68 500 to the nearest \$100. **1**

(iv) Immediately after making the 120th repayment, Sam makes a one-off payment, reducing the amount owing to \$48 500. The interest rate and monthly repayment remain unchanged. After how many more months will the amount owing be completely repaid? **3**

Solution

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14 **8** Which expression is a term of the geometric series $3x - 6x^2 + 12x^3 - \ldots$? **1**

(A) $3072x^{10}$ (B) $-3072x^{10}$ (C) $3072x^{11}$ (D) $-3072x^{11}$

Solution

14 **12 a** Evaluate the arithmetic series $2 + 5 + 8 + 11 + \ldots + 1094$. **2** Solution

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14 **14 d** At the beginning of every 8-hour period, a patient is given 10 mL of a particular drug. During each of these 8-hour periods, the patient's body partially breaks down the drug. Only $\frac{1}{3}$ of the total amount of the drug present in the patient's body at the beginning of each 8-hour period remains at the end of that period. Solution

(i) How much of the drug is in the patient's body immediately after the second dose is given? **1**

(ii) Show that the total amount of the drug in the patient's body never exceeds 15 mL. **2**

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14 **16 b** At the start of a month, Jo opens a bank account and makes a deposit of \$500. At the start of each subsequent month, Jo makes a deposit which is 1% more than the previous deposit. At the end of each month, the bank pays interest of 0.3% (per month) on the balance of the account. Solution

(i) Explain why the balance of the account at the end of the second month is $\$500(1.003)^2 + \$500(1.01)(1.003)$. **2**

(ii) Find the balance of the account at the end of the 60^{th} month, correct to the nearest dollar. **3**

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13 **12 c** Kim and Alex start jobs at the beginning of the same year. Kim's annual salary in the first year is \$30 000, and increases by 5% at the beginning of each subsequent year. Alex's annual salary in the first year is \$33 000, and increases by \$1500 at the beginning of each subsequent year. Solution

(i) Show that in the 10^{th} year Kim's annual salary is higher than Alex's annual salary. **2**

(ii) In the first 10 years how much, in total, does Kim earn? **2**

(iii) Every year, Alex saves $\frac{1}{3}$ of her annual salary. How many years does it take her to save \$87 500? **3**

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13 **13 d** A family borrows \$500 000 to buy a house. The loan is to be repaid in equal monthly instalments. The interest, which is charged at 6% per annum, is reducible and calculated monthly. The amount owing after n months, $\$A_n$, is given by Solution

$A_n = Pr^n - M(1 + r + r^2 + \ldots + r^{n-1})$, (Do NOT prove this)

where $\$P$ is the amount borrowed, $r = 1.005$ and $\$M$ is the monthly repayment.

(i) The loan is to be repaid over 30 years. Show that the monthly repayment is \$2998 to the nearest dollar. **2**

(ii) Show that the balance owing after 20 years is \$270 000 to the nearest thousand dollars. **1**

(iii) After 20 years the family borrows an extra amount, so that the family then owes a total of \$370 000. The monthly repayment remains \$2998, and the interest rate remains the same. How long will it take to repay the \$370 000? **2**

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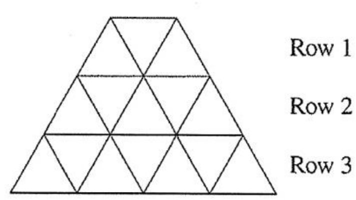
12 **12 c** Jay is making a pattern using triangular tiles. The pattern has 3 tiles in the first row, 5 tiles in the second row, and each successive row has 2 more tiles than the previous row. Solution



(i) How many tiles would Jay use in row 20? **2**

(ii) How many tiles would Jay use altogether to make the first 20 rows? **1**

(iii) Jay has only 200 tiles. How many complete rows of the pattern can Jay make? **2**

12 **15 a** Rectangles of the same height are cut from a strip and arranged in a row. The first rectangle has width 10 cm. The width of each subsequent rectangle is 96% of the width of the previous rectangle. Solution


10 cm

NOT TO SCALE

(i) Find the length of the strip required to make the first ten rectangles. **2**

(ii) Explain why a strip of length 3 m is sufficient to make any number of rectangles. **1**

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12 **15 c** Ari takes out a loan of \$360 000. The loan is to be repaid in equal monthly repayments, $\$M$, at the end of each month, over 25 years (300 months). Reducible interest is charged at 6% per annum, calculated monthly. Let $\$A_n$ be the amount owing after the nth repayment. Solution

(i) Write down an expression for the amount owing after two months, $\$A_2$. **1**

(ii) Show that the monthly repayment is approximately \$2319.50. **2**

(iii) After how many months will the amount owing, $\$A_n$, become less than \$180 000? **3**

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11 **3a** A skyscraper of 110 floors is to be built. The first floor to be built will cost \$3 million. The cost of building each subsequent floor will be \$0.5 million more than the floor immediately below. Solution

(i) What will be the cost of building the 25th floor? **2**

(ii) What will be the cost of building all 110 floors of the skyscraper? **2**

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11 **5a** The number of members of a new social networking site doubles every day. On Day 1 there were 27 members and on Day 2 there were 54 members. Solution

(i) How many members were there on Day 12? **1**

(ii) On which day was the number of members first greater than 10 million? **2**

(iii) The site earns 0.5 cents per member per day. How much money did the site earn in the first 12 days? Give your answer to the nearest dollar. **2**

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11 **8c** When Jules started working she began paying \$100 at the beginning of each month into a superannuation fund. The contributions are compounded monthly at an interest rate of 6% per annum. She intends to retire after having worked for 35 years. Solution

(i) Let $\$P$ be the final value of Jules's superannuation when she retires after 35 years (420 months). **2**
Show that $\$P = \$143\ 183$ to the nearest dollar.

(ii) Fifteen years after she started working Jules read a magazine article about retirement, and realized that she would need \$800 000 in her fund when she retires. At the time of reading the magazine article she had \$29 227 in her fund. For the remaining 20 years she intends to work, she decides to pay a total of $\$M$ into her fund at the beginning of each month. The contributions continue to attract the same interest rate of 6% per annum, compounded monthly. At the end of n months after starting the new contributions, the amount in the fund is $\$A_n$.

(1) Show that $A_2 = 29\ 227 \times 1.005^2 + M(1.005 + 1.005^2)$. **1**

(2) Find the value of M so that Jules will have \$800 000 in her fund after the remaining 20 years (240 months). **3**

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10 **1f** Find the limiting sum of the geometric series $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \ldots$ **2** Solution

10 **4a** Susannah is training for a fun run by running every week for 26 weeks. She runs 1 km in the first week and each week after that she runs 750 m more than the previous week, until she reaches 10 km in a week. She then continues to run 10 km each week. Solution

(i) How far does Susannah run in the 9^{th} week? **1**
(ii) In which week does she first run 10 km? **1**
(iii) What is the total distance that Susannah runs in 26 weeks? **2**

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10 **9a** (i) When Chris started a new job, \$500 was deposited into his superannuation fund at the beginning of each month. The money was invested at 0.5% per month, compounded monthly. **2** Solution
Let \$$P$ be the value of the investment after 240 months, when Chris retires.
Show that P = 232 175.55

(ii) After retirement, Chris withdraws \$2000 from the account at the end of each month, without making any further deposits. The account continues to earn interest at 0.5% per month.
Let \$$A_n$ be the amount left in the account n months after Chris's retirement.
(1) Show that $A_n = (P - 400\,000) \times 1.005^n + 400\,000$. **3**
(2) For how many months after retirement will there be money left in the account? **2**

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09 **2c** Evaluate $\sum_{k=1}^{4}(-1)^k k^2$. **2** Solution

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09 **3a** An arithmetic series has 21 terms. The first term is 3 and the last term is 53. Find the sum of the series. **2** Solution

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09 **4a** A tree grows from ground level to a height of 1.2 metres in one year. **2** Solution
In each subsequent year, it grows $\frac{9}{10}$ as much as it did in the previous year. Find the limiting height of the tree.

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09 **8b** One year ago Daniel borrowed \$350 000 to buy a house. The interest rate was 9% per annum, compounded monthly. He agreed to repay the loan in 25 years with equal monthly repayments of \$2937. Solution

(i) Calculate how much Daniel owed after his monthly repayment. **1**
(ii) Daniel has just made his 12^{th} monthly repayment. He now owes \$346 095. The interest rate now decreases to 6% per annum, compounded monthly. The amount \$$A_n$, owing on the loan after the n^{th} monthly repayment is now calculated using the formula **3**
$A_n = 346\,095 \times 1.005^n - 1.005^{n-1} M - \ldots - 1.005M - M$ where \$$M$ is the monthly repayment and n = 1, 2, …, 288. (Do NOT prove this formula.)
Calculate the monthly repayment if the loan is to be repaid over the remaining 24 years (288 months).
(iii) Daniel chooses to keep his monthly repayments at \$2937. Use the formula in part (ii) to calculate how long it will take him to repay the \$346 095. **3**
(iv) How much will Daniel save over the term of the loan by keeping his monthly repayments at \$2937, rather than reducing his repayments to the amount calculated in part (ii)? **1**

08 **1f** Find the sum of the first 21 terms of the arithmetic series **2** Solution
$3 + 7 + 11 + \ldots$

08 **4b** The zoom function in a software package multiplies the dimensions of an image by 1.2. In an image, the height of a building is 50 mm. After the zoom function is applied once, the height of the building in the image is 60 mm. After a second application, its height is 72 mm. Solution

(i) Calculate the height of the building in the image after the zoom function has been applied eight times. Give your answer to the nearest mm. **2**

(ii) The height of the building in the image is required to be more than 400 mm. Starting from the original image, what is the least number of times the zoom function must be applied? **2**

08 **5b** Consider the geometric series $5 + 10x + 20x^2 + 40x^3 + \ldots$ Solution

(i) For what values of x does this series have a limiting sum? **2**

(ii) The limiting sum of this series is 100. Find the value of x. **2**

08 **9b** Peter retires with a lump sum of \$100 000. The money is invested in a fund which pays interest each month at a rate of 6% per annum, and Peter receives a fixed monthly payment of $\$M$ from the fund. Thus, the amount left in the fund after the first monthly payment is $\$(100\,500 - M)$. Solution

(i) Find a formula for the amount, $\$A_n$, left in the fund after n monthly payments. **2**

(ii) Peter chooses the value of M so that there will be nothing left in the fund at the end of the 12th year (after 144 payments). Find the value of M. **3**

07 **1d** Find the limiting sum of the geometric series $\frac{3}{4} + \frac{3}{16} + \frac{3}{64} + \ldots$ **2** Solution

07 **3b** Heather decides to swim every day to improve her fitness level. On the first day she swims 750 metres, and on each day after that she swims 100 metres more than the previous day. That is, she swims 850 metres on the second day, 950 metres on the third day and so on. Solution

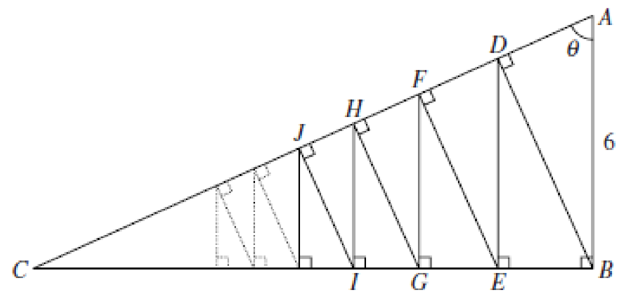
(i) Write down a formula for the distance she swims on the nth day. **1**

(ii) How far does she swim on the 10th day? **1**

(iii) What is the total distance she swims in the first 10 days? **1**

(iv) After how many days does the total distance she has swum equal the width of the English Channel, a distance of 34 kilometres? **2**

07	**9c**	Mr and Mrs Caine each decide to invest some money each year to help pay for their son's university education. The parents choose different investment strategies. (i) Mr Caine makes 18 yearly contributions of $1000 into an investment fund. He makes his first contribution on the day his son is born, and his final contribution on his son's seventeenth birthday. His investment earns 6% compound interest per annum. Find the total value of Mr Caine's investment on his son's eighteenth birthday. (ii) Mrs Caine makes her contributions into another fund. She contributes $1000 on the day of her son's birth, and increases her annual contribution by 6% each year. Her investment also earns 6% compound interest per annum. Find the total value of Mrs Caine's investment on her son's third birthday (just before she makes her fourth contribution). (iii) Mrs Caine also makes her final contribution on her son's seventeenth birthday. Find the total value of Mrs Caine's investment on her son's eighteenth birthday.	**3** **2** **1**	**Solution**
06	**1f**	Find the limiting sum of the geometric series $\frac{13}{5} + \frac{13}{25} + \frac{13}{125} + \ldots$	**2**	**Solution**
06	**3b**	Evaluate $\sum_{n=2}^{4} \frac{1}{r}$.	**1**	**Solution**
06	**3c**	On the first day of the harvest, an orchard produces 560 kg of fruit. On the next day, the orchard produces 543 kg, and the amount produced continues to decrease by the same amount each day. (i) How much fruit is produced on the fourteenth day of the harvest? (ii) What is the total amount of fruit that is produced in the first 14 days of the harvest? (iii) On what day does the daily production first fall below 60 kg?	**2** **1** **2**	**Solution**
06	**8b**	Joe borrows $200 000 which is to be repaid in equal monthly instalments. The interest rate is 7.2% per annum reducible, calculated monthly. It can be shown that the amount, $$A_n$, owing after the nth repayment is given by the formula: $A_n = 200\,000r^n - M(1 + r + r^2 + \cdots + r^{n-1})$, where $r = 1.006$ and $$M$ is the monthly repayment. (Do NOT show this.) (i) The minimum monthly repayment is the amount required to repay the loan in 300 instalments. Find the minimum monthly repayment. (ii) Joe decides to make repayments of $2800 each month from the start of the loan. How many months will it take for Joe to repay the loan?	**3** **2**	**Solution**
05	**3a**	Evaluate $\sum_{n=3}^{5} (2n+1)$.	**1**	**Solution**
05	**7a**	Anne and Kay are employed by an accounting firm. Anne accepts employment with an initial annual salary of $50 000. In each of the following years her annual salary is increased by $2500. Kay accepts employment with an initial annual salary of $50 000. In each of the following years her annual salary is increased by 4%. (i) What is Anne's annual salary in her thirteenth year? (ii) What is Kay's annual salary in her thirteenth year? (iii) By what amount does the total amount paid to Kay in her first twenty years exceed that paid to Anne in her first twenty years?	**2** **2** **3**	**Solution**

05	**8c**	Weelabarrabak Shire Council borrowed \$3 000 000 at the beginning of 2005. The annual interest rate is 12%. Each year, interest is calculated on the balance at the beginning of the year and added to the balance owing. The debt is to be repaid by equal annual repayments of \$480 000, with the first repayment being made at the end of 2005. Let A_n be the balance owing after the n-th repayment. (i) Show that $A_2 = (3 \times 10^6)(1.12)^2 - (4.8 \times 10^5)(1 + 1.12)$. **1** (ii) Show that $A_n = 10^6[4 - (1.12)^n]$. **2** (iii) In which year will Weelabarrabak Shire Council make the final repayment? **2**		**Solution**
05	**9b**	The triangle ABC has a right angle at B, $BAC = \theta$ and $AB = 6$. The line BD is drawn perpendicular to AC. The line DE is then drawn perpendicular to BC. This process continues indefinitely as shown in the diagram.  (i) Find the length of the interval BD, and hence show that the length of the interval EF is $6\sin^3\theta$. **2** (ii) Show that the limiting sum $BD + EF + GH + \cdots$ is given by $6\sec\theta\tan\theta$. **3**		**Solution**