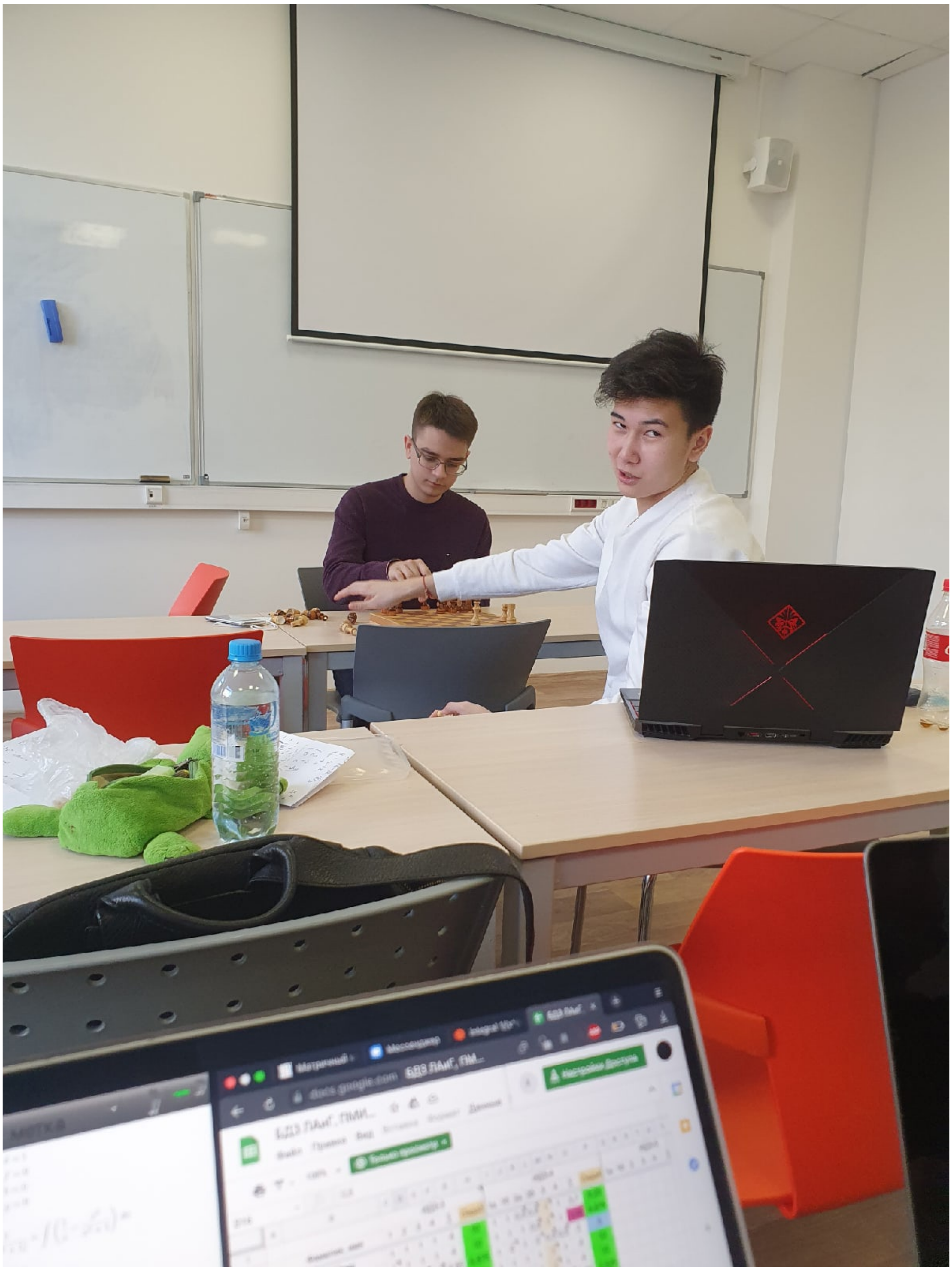
Бурмашев Григорий. 208. Матан – 7



Номер 6

3)

$$\int \frac{x^2+3x-2}{(x-1)(x^2+x+1)^2}\,dx$$

По методу Остроградского

$$\int \frac{x^2+3x-2}{(x-1)(x^2+x+1)^2}\,dx = \frac{ax+b}{x^2+x+1} + \int \frac{\text{что-то}}{(x-1)(x^2+x+1)}\,dx =$$

$$= \frac{ax+b}{x^2+x+1} + \int \left[\frac{\text{c}}{(x-1)} + \frac{dx+e}{x^2+x+1}\right]dx =$$

Продифференцируем:

$$\frac{x^2+3x-2}{(x-1)(x^2+x+1)^2} = \frac{a(x^2+x+1)-(2x+1)(ax+b)}{(x^2+x+1)^2} + \frac{c}{x-1} + \frac{dx+e}{x^2+x+1}$$

Тогда:

$$x^2+3x-2 =$$
$$= (x-1)\left[a(x^2+x+1)-(2x+1)(ax+b)\right] +$$
$$+c(x^2+x+1)^2 + (dx+e)(x-1)(x^2+x+1) =$$
$$= -ax^3+ax^2+ax-a-2bx^2+bx+b+cx^4+2cx^3+3cx^2+2cx+c+dx^4-dx+ex^3-e =$$
$$= (c+d)x^4+(-2+2c+e)x^3+(a-2b+3c)x^2+(a+b+2c-d)x+(-a+b+c-e)$$

Получаем систему:

$$\begin{cases} c+d=0 \\ -a+2c+e=0 \\ a-2b+3c=1 \\ a+b+2c-d=3 \\ -a+b+c-e=-2 \end{cases}$$

Надо (к сожалению) её решать, приведем к матричному виду:

$$\left(\begin{array}{ccccc|c} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & 5 & 0 & 1 & 1 \\ 1 & -2 & 3 & 0 & 0 & 1 \\ 1 & 1 & 2 & -1 & 0 & 3 \\ -1 & 1 & 1 & 0 & -1 & -2 \end{array}\right) = \left(\begin{array}{ccccc|c} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & 5 & 0 & 1 & 1 \\ 0 & -3 & 1 & 1 & 0 & -2 \\ 1 & 1 & 2 & -1 & 0 & 3 \\ -1 & 1 & 1 & 0 & -1 & -2 \end{array}\right) =$$

$$
= \left(\begin{array}{ccccc|c}
0 & 0 & 1 & 1 & 0 & 0 \\
0 & -2 & 5 & 0 & 1 & 1 \\
0 & -3 & 1 & 1 & 0 & -2 \\
1 & 1 & 2 & -1 & 0 & 3 \\
0 & 2 & 3 & -1 & -1 & 1
\end{array}\right) =
$$

$$
= \left(\begin{array}{ccccc|c}
1 & 1 & 2 & -1 & 0 & 3 \\
0 & 0 & 8 & -1 & 0 & 2 \\
0 & -1 & 4 & 0 & -1 & -1 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 2 & 3 & -1 & -1 & 1
\end{array}\right) =
$$

$$
= \left(\begin{array}{ccccc|c}
1 & 1 & 2 & -1 & 0 & 3 \\
0 & 1 & -4 & 0 & 1 & 1 \\
0 & 0 & 0 & -9 & 0 & 2 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 11 & -1 & -3 & -1
\end{array}\right) = \left(\begin{array}{ccccc|c}
1 & 1 & 2 & -1 & 0 & 3 \\
0 & 1 & -4 & 0 & 1 & 1 \\
0 & 0 & 0 & -9 & 0 & 2 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & -12 & -3 & -1
\end{array}\right) =
$$

$$
= \left(\begin{array}{ccccc|c}
1 & 1 & 2 & -1 & 0 & 3 \\
0 & 1 & -4 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & -9 & 0 & 2 \\
0 & 0 & 0 & -3 & -3 & -3
\end{array}\right) = \left(\begin{array}{ccccc|c}
1 & 1 & 2 & -1 & 0 & 3 \\
0 & 1 & -4 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 9 & 11 \\
0 & 0 & 0 & 1 & 1 & 1
\end{array}\right) =
$$

$$
= \left(\begin{array}{ccccc|c}
1 & 1 & 2 & -1 & 0 & 3 \\
0 & 1 & -4 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 9 & 11
\end{array}\right) = \left(\begin{array}{ccccc|c}
1 & 0 & 0 & 0 & 0 & \frac{5}{3} \\
0 & 1 & 0 & 0 & 0 & \frac{2}{3} \\
0 & 0 & 1 & 0 & 0 & \frac{2}{9} \\
0 & 0 & 0 & 1 & 0 & -\frac{2}{9} \\
0 & 0 & 0 & 0 & 1 & \frac{11}{9}
\end{array}\right)
$$

А значит:

$$
\begin{cases}
a = \frac{5}{3} \\
b = \frac{2}{3} \\
c = \frac{2}{9} \\
d = -\frac{2}{9} \\
e = \frac{11}{9}
\end{cases}
$$

$$
\frac{ax+b}{x^2+x+1} + \int \left[\frac{c}{(x-1)} + \frac{dx+e}{x^2+x+1} \right] dx =
$$

$$
= \frac{\frac{5}{3}x + \frac{2}{3}}{x^2+x+1} + \int \left[\frac{\frac{2}{9}}{(x-1)} + \frac{-\frac{2}{9}x + \frac{11}{9}}{x^2+x+1} \right] dx
$$

Посчитаем интеграл по отдельности:

$$\int \left[\frac{\frac{2}{9}}{(x-1)} + \frac{-\frac{2}{9}x + \frac{11}{9}}{x^2+x+1}\right] dx = \frac{2}{9}\int \frac{1}{x-1}\,dx - \frac{1}{9}\int \frac{2x-11}{x^2+x+1}\,dx =$$

$$= \frac{2}{9}\ln|x-1| - \frac{1}{9}\int \frac{2x-11}{x^2+x+1}\,dx$$

$$\int \frac{2x-11}{x^2+x+1}\,dx = \int \frac{(2x+1)-12}{x^2+x+1}\,dx = \int \frac{2x+1}{x^2+x+1}\,dx - \int \frac{12}{x^2+x+1}\,dx =$$

$$= \frac{2}{9}\ln|x-1| - \frac{1}{9}\int \frac{2x-11}{x^2+x+1}\,dx$$

$$\int \frac{2x-11}{x^2+x+1}\,dx = \int \frac{(2x+1)-12}{x^2+x+1}\,dx = \int \frac{2x+1}{x^2+x+1}\,dx - \int \frac{12}{x^2+x+1}\,dx =$$

$$= \int \frac{(2x+1)}{(x^2+x+1)(2x+1)}\,d(x^2+x+1) - 12\int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx =$$

$$= \ln(x^2+x+1) - 12\int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx =$$

$$= \ln(x^2+x+1) - \frac{12}{\sqrt{\frac{3}{4}}}\cdot \arctan\left(\frac{x+\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right) + C =$$

$$= \ln(x^2+x+1) + \frac{20}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

Тогда возвращаемся к исходному выражению:

$$\frac{\frac{5}{3}x+\frac{2}{3}}{x^2+x+1} + \frac{2}{9}\ln|x-1| - \frac{1}{9}\ln(x^2+x+1) + \frac{8\sqrt{3}}{9}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

Ответ:

$$\frac{\frac{5}{3}x+\frac{2}{3}}{x^2+x+1} + \frac{2}{9}\ln|x-1| - \frac{1}{9}\ln(x^2+x+1) + \frac{8\sqrt{3}}{9}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

4)

$$\int \frac{dx}{x(x^3+1)^2}$$

По методу Остроградского:

$$\int \frac{dx}{x(x^3+1)^2} = \frac{ax^2+bx+c}{x^3+1} + \int \frac{\text{что-то}}{x(x^3+1)}\,dx = \frac{ax^2+bx+c}{x^3+1} + \int \left(\frac{d}{x} + \frac{ex^2+fx+g}{x^3+1}\right) dx$$

Продифференцируем:

$$\frac{1}{x(x^3+1)^2} = \frac{(2ax+b)(x^3+1)-(3x^2)(ax^2+bx+c)}{(x^3+1)^2} + \frac{d}{x} + \frac{ex^2+fx+g}{x^3+1}$$

$$1 = x((ax+b)(x^3+1)-3x^2(ax^2+bx+c)+d(x^3+1)^2+(ex^2+fx+g)(x^2+1)x) =$$
$$= -ax^5+2ax-2x^4+bx-3ex^3+dx^6+2dx^3+d+ex^6+ex^3+fx^5+fx^2+gx^4+gx$$

$$\begin{cases} d+e=0 \\ -a+f=0 \\ -2b+g=0 \\ -3c+2d+e=0 \\ f=0 \\ 2a+b+g=0 \\ d=1 \end{cases}$$

$$\begin{cases} e=-1 \\ a=0 \\ c=\frac{1}{3} \\ d=1 \\ f=0 \\ b=0 \\ g=0 \end{cases}$$

Тогда:

$$\int \frac{dx}{x(x^3+1)^2} = \frac{1}{3(x^3+1)} + \int \left(\frac{1}{x} - \frac{x^2}{x^3+1}\right) dx$$

Посчитаем интеграл отдельно:

$$\int \left(\frac{1}{x} - \frac{x^2}{x^3+1}\right) dx = \int \frac{dx}{x} - \int \frac{x^2}{x^3+1}\, dx =$$

$$= \ln|x| - \frac{1}{3}\int \frac{x^2}{(x^3+1)x^2}\, d(x^3+1) = \ln|x| - \frac{1}{3}\int \frac{d(x^3+1)}{x^3+1} =$$

$$= \ln|x| - \frac{1}{3}\ln|x^3+1| + C$$

$$\int \frac{dx}{x(x^3+1)} = \frac{1}{3(x^3+1)} + \ln|x| - \frac{1}{3}\ln|x^3+1| + C$$

Ответ:

$$\frac{1}{3(x^3+1)} + \ln|x| - \frac{1}{3}\ln|x^3+1| + C$$

Номер 7

1)

$$\int \frac{dx}{3+\sin x}$$

Универсальная триг.подстановка:

$$\tan\frac{x}{2} = t$$

$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2t}{1+t^2}$$

$$dx = \frac{2dt}{1+t^2}$$

Тогда получаем:

$$\int \frac{dx}{3+\sin x} = \int \frac{2dt}{(3+\frac{2t}{1+t^2})(1+t^2)} =$$

$$= \int \frac{1+t^2}{3(1+t^2)+2t} \cdot \frac{2dt}{1+t^2} = 2\int \frac{dt}{3t^2+2t+3} = \frac{2}{3}\int \frac{dt}{t^2+\frac{2}{3}t+1} =$$

$$= \frac{2}{3}\int \frac{dt}{(t^2+\frac{1}{3})^2+\frac{8}{9}} = \frac{3}{4}\cdot\int \frac{dt}{\left(\frac{3t+1}{2\sqrt{2}}\right)^2+1} = \frac{\arctan\left(\frac{3\tan\frac{x}{2}+1}{2\sqrt{2}}\right)}{\sqrt{2}} + C$$

Ответ:

$$\frac{\arctan\left(\frac{3\tan\frac{x}{2}+1}{2\sqrt{2}}\right)}{\sqrt{2}} + C$$