# Exercise session: modal analysis on an externally forced swirling flame

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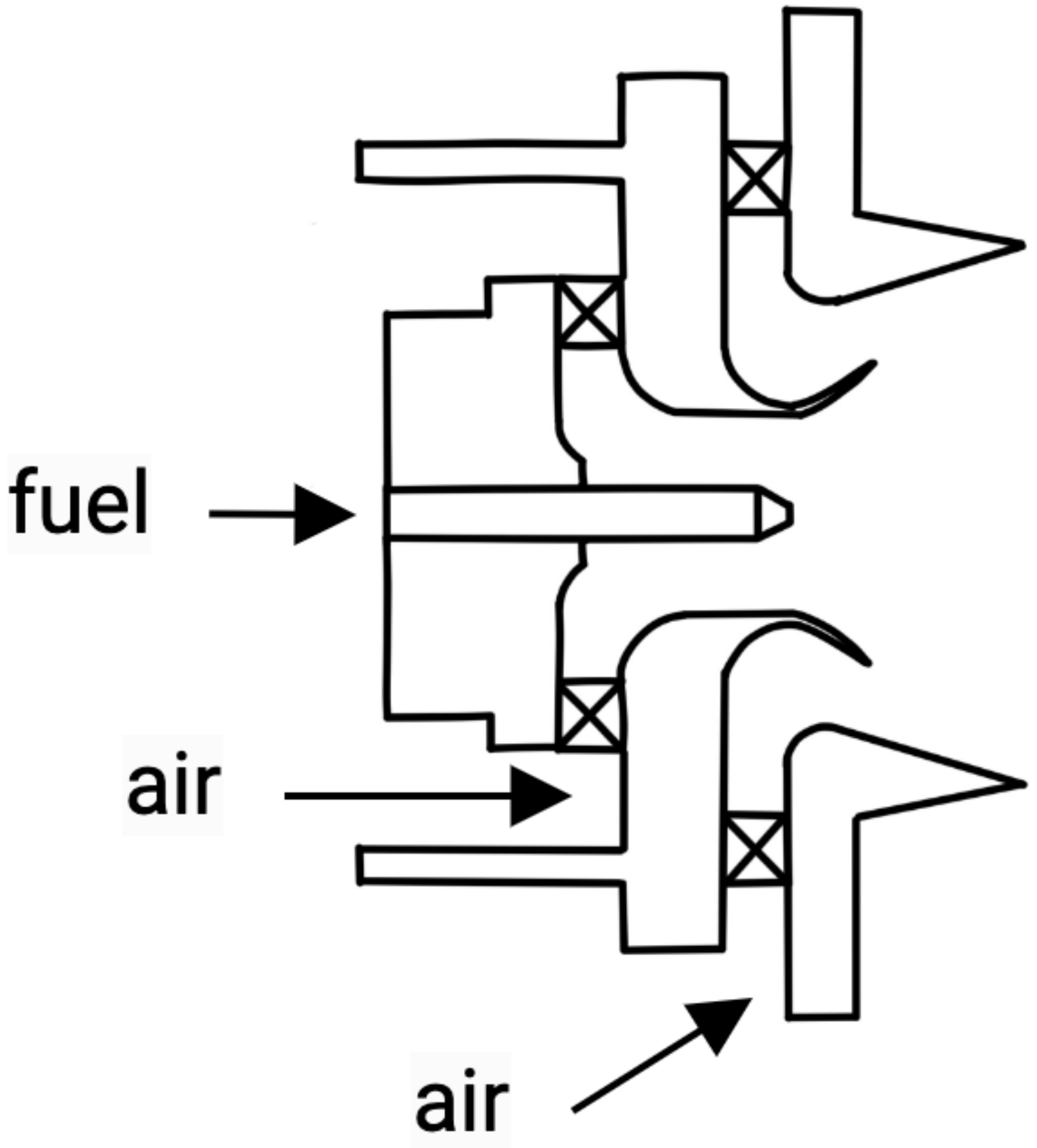
# Application

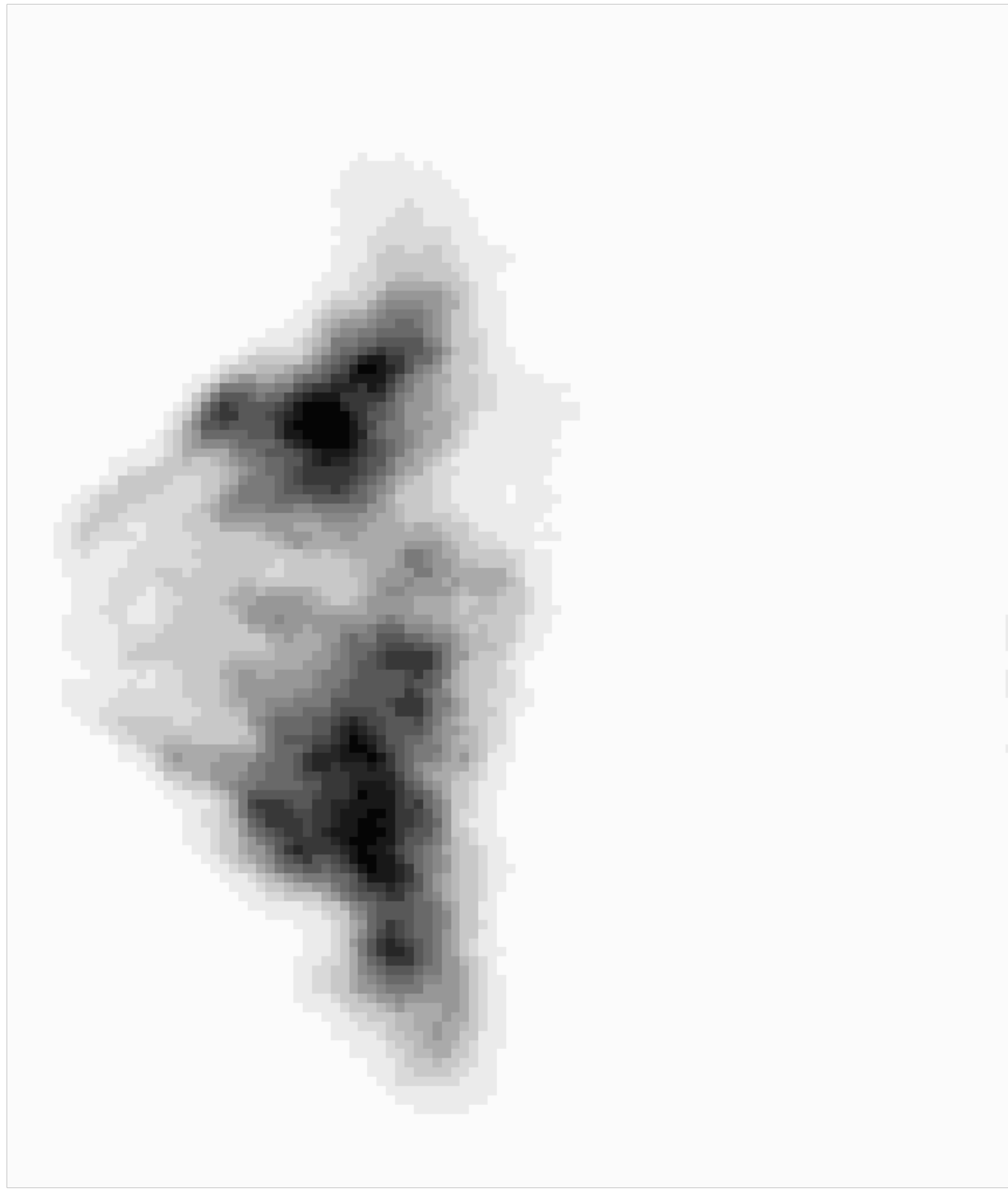
We want to **analyse** the **dynamical** behaviour of a **swirling flame** forced at 100 Hz.

The **excitation** is achieved by **modulating** the **air flow**.

This is the BIMER [1] experimental apparatus and the experiment was carried out at the Lab EM2C (CentraleSupelec).




CH* emission



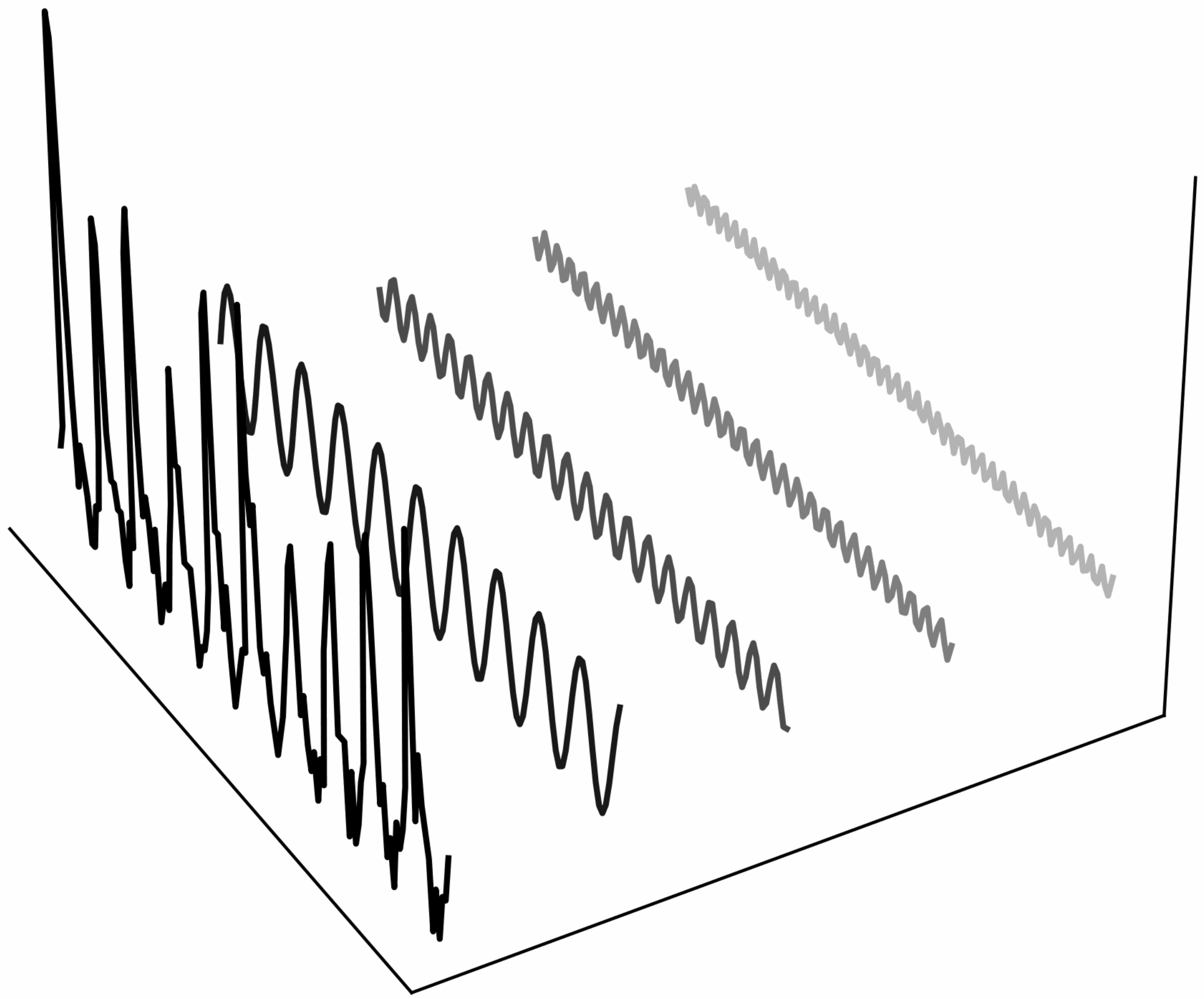
[1] Renaud, A., Ducruix, S., and Zimmer, L. (October 21, 2019). "Experimental Study of the Precessing Vortex Core Impact on the Liquid Fuel Spray in a Gas Turbine Model Combustor." ASME. *J. Eng. Gas Turbines Power*. November 2019; 141(11): 111022.

# Modal analysis

The **objective** of modal analysis is to **identify** the most important structures (**modes**) in the flow.

In general, the objective is to **decompose** the signal into the sum of **fundamental blocks**, ordered by importance.

$$\mathbf{x} = \sum_{i=1}^{q} \boldsymbol{\phi}_i \psi_i$$



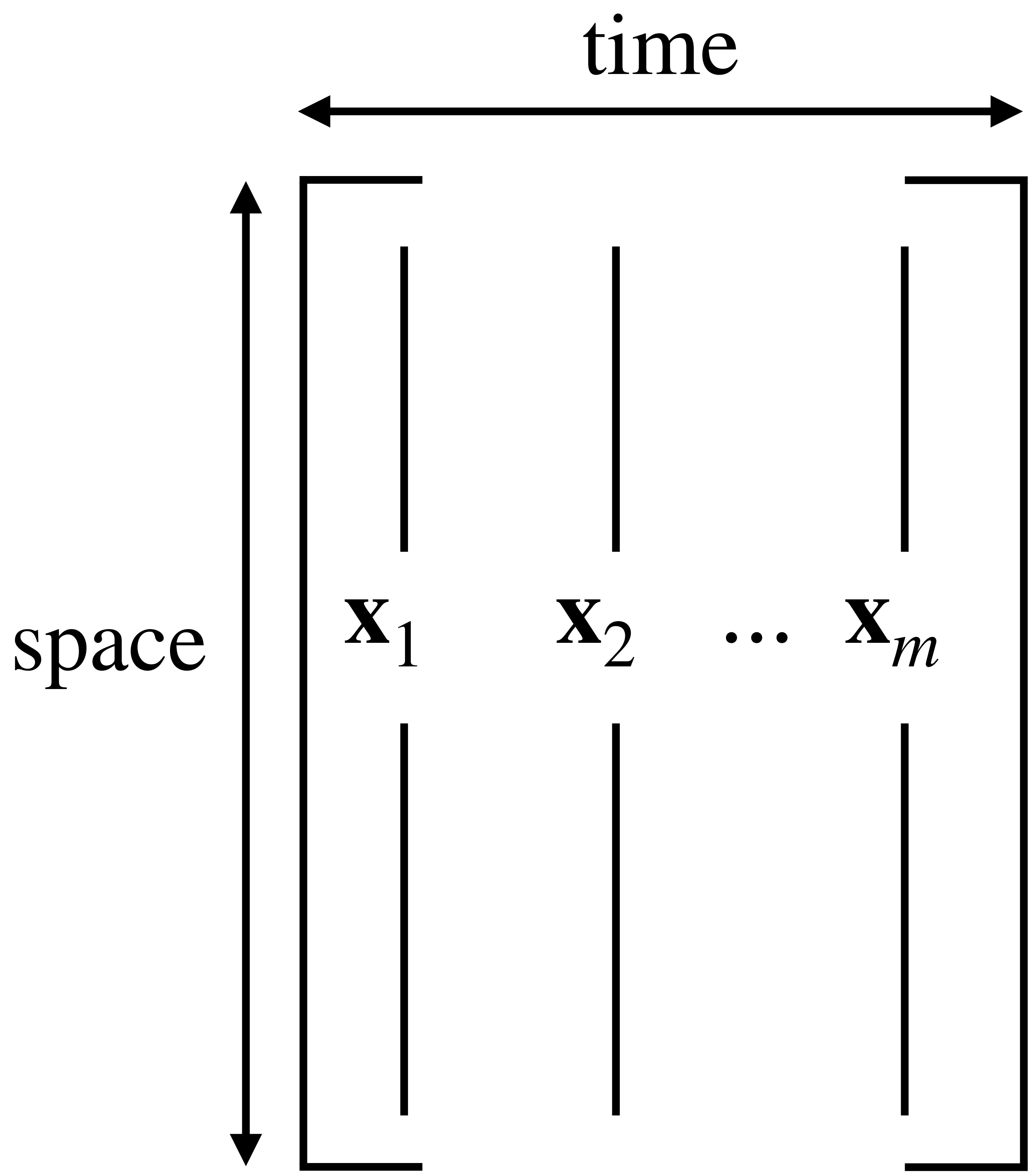
This can also be used to reduce the dimensionality.

# Data preparation

The dataset is composed of $1024$ images of the flame with resolution $64 \times 64$.

Each **image** is a reshaped as a **column vector** of dimensions $n = 4096$. The data matrix is then $\mathbf{X} \in \mathbb{R}^{n,m}$ where $m = 1024$.

Basically, each **column** represents an **image**, and each **row** represents a **point in time**.



# Exercise 1: Proper Orthogonal Decomposition

In the first exercise we use POD to decompose and analyse the dataset.

The POD algorithm (snapshots' method) is:

- Center the data matrix by removing the time-averaged field: $\mathbf{X}_0(\mathbf{r}, t) = \mathbf{X}(\mathbf{r}, t) - \bar{\mathbf{x}}(\mathbf{r})$
- Compute the covariance matrix: $\mathbf{K}(t) = 1/(m-1)\mathbf{X}_0^T\mathbf{X}_0$
- Compute the eigendecomposition of the covariance matrix: $\mathbf{K} = \mathbf{V}\mathbf{L}\mathbf{V}^T$
- Compute the POD modes: $\mathbf{Z}(\mathbf{r}) = \mathbf{X}_0\mathbf{V}(t)$

or, using the SVD:

- Center the data matrix: $\mathbf{X}_0(\mathbf{r}, t) = \mathbf{X}(\mathbf{r}, t) - \bar{\mathbf{x}}(\mathbf{r})$
- Compute the SVD: $\mathbf{X}_0(\mathbf{r}, t) = \mathbf{U}(\mathbf{r})\boldsymbol{\Sigma}\mathbf{V}^T(t)$ where $\mathbf{L} = \boldsymbol{\Sigma}^2$.

In the end, a single snapshot can be represented as: $\mathbf{x}_i = \sum_{j=1}^{q} \mathbf{z}_j v_{i,j}$ or $\mathbf{x}_i = \sum_{j=1}^{q} \mathbf{u}_j \sigma_j v_{i,j}$

# Exercise 1: POD's properties

The POD is widely used for modal analysis because it has some important properties:

- The POD **decouples** the **spatial** information in $\mathbf{U}$ from the **temporal** information in $\mathbf{V}$.
- Both $\mathbf{U}$ and $\mathbf{V}$ are **orthonormal**.
- The **eigenvalues** represent the (relative) **information content**, and are thus used to identify the most important flow structures.
- The **less important modes** can be **discarded** without loosing much information:

$$\mathbf{X}_0 \approx \mathbf{U}_q \boldsymbol{\Sigma}_q \mathbf{V}_q^T$$

# Exercise 2: Dynamic Mode Decomposition

The objective of DMD is to find a linear approximation $\mathbf{A}$ of the dynamical system such that $\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k$. In theory, we could find this operator by solving the least-squares regression problem $\mathbf{A} = \mathbf{X}_2\mathbf{X}_1^+$, however this is generally infeasible.

The DMD algorithm (exact DMD) is:

- Center the data matrix by removing the time-averaged field: $\mathbf{X}_0(\mathbf{r}, t) = \mathbf{X}(\mathbf{r}, t) - \bar{\mathbf{x}}(\mathbf{r})$
- Split the data matrix: $\mathbf{X}_{0,1} = \{\mathbf{x}_{0,1}, \ldots, \mathbf{x}_{0,m-1}\}$ and $\mathbf{X}_{0,2} = \{\mathbf{x}_{0,2}, \ldots, \mathbf{x}_{0,m}\}$
- Compute the SVD: $\mathbf{X}_{0,1} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$
- Compute the approximation $\tilde{\mathbf{A}} = \mathbf{U}^T\mathbf{X}_{0,2}\mathbf{V}\boldsymbol{\Sigma}^{-1}$
- Compute its eigendecomposition: $\tilde{\mathbf{A}} = \mathbf{W}\boldsymbol{\Lambda}\mathbf{W}^T$
- Compute the DMD modes: $\boldsymbol{\Phi} = \mathbf{X}_{0,2}\mathbf{V}\boldsymbol{\Sigma}^{-1}\mathbf{W}$
- Compute the amplitudes: $\mathbf{b} = \boldsymbol{\Phi}^+\mathbf{x}_{0,1}$.

A single snapshot is represented as: $\mathbf{x}_i = \sum_{j=1}^{q} \boldsymbol{\phi}_j \lambda_j^{i-1} b_j$

# Exercise 2: DMD's properties

The properties of the DMD modes are quite different from the POD modes:

- As the POD, the DMD decouples the spatial information in $\mathbf{\Phi}$ from the temporal information in $\mathbf{\Lambda}$.

- The DMD **modes** are **not orthogonal**.

- The identification of the **most important modes** is **not straightforward**.

- The **interpretation** of modes with a **decaying** or **exploding growth factors is challenging**.

# Resources

The tasks of the two exercises can be found in this **Github repo**:

https://github.com/burn-research/ercoftac2023-workshop/tree/main/modal_analysis

The repo contains the **two exercises** as two Jupyter notebooks, as well as **the data** and some resources for beginners in Python.

To write and execute the code you have 4 options:

- Download the data and solve the exercise on your local machine.
- Use Spyder (Python IDE) on Marenostrum (requires display software).
- Use Jupyter notebooks on Marenostrum (no display software needed).
- Use Google Colab notebooks on your browser.