Algorithmic Challenges: Knuth-Morris-Pratt Algorithm

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String Processing and Pattern Matching Algorithms

Algorithms and Data Structures

Outline

1. Exact Pattern Matching
2. Safe Shift
3. Prefix Function
4. Computing Prefix Function
5. Implementation
6. Analysis
7. Knuth-Morris-Pratt Algorithm

Exact Pattern Matching

Input: Strings T (Text) and P (Pattern).

Output: All such positions in T (Text) where P (Pattern) appears as a substring.

(For all strings in this module we use 0-based indices)

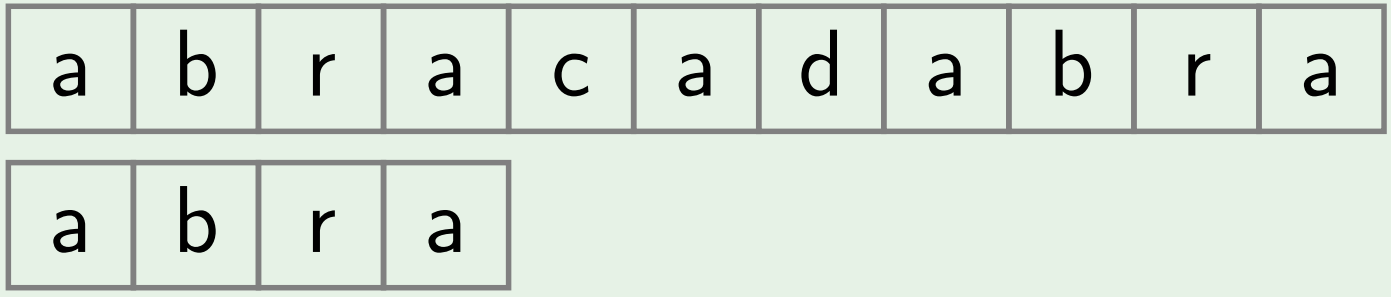
Brute Force Algorithm

- Slide the Pattern down Text

Brute Force Algorithm

- Slide the Pattern down Text
- Running time $\Theta(|T||P|)$

Brute Force Algorithm



Brute Force Algorithm

0	1	2	3	4	5	6	7	8	9	10
a	b	r	a	c	a	d	a	b	r	a

a	b	r	a

Output: []

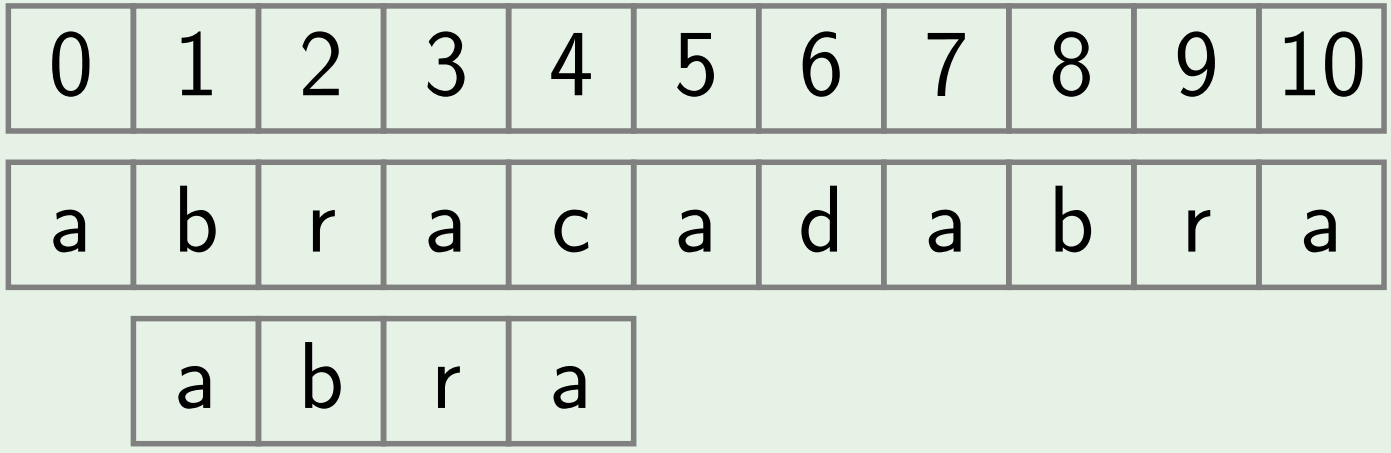
Brute Force Algorithm

0	1	2	3	4	5	6	7	8	9	10
a	b	r	a	c	a	d	a	b	r	a

a	b	r	a

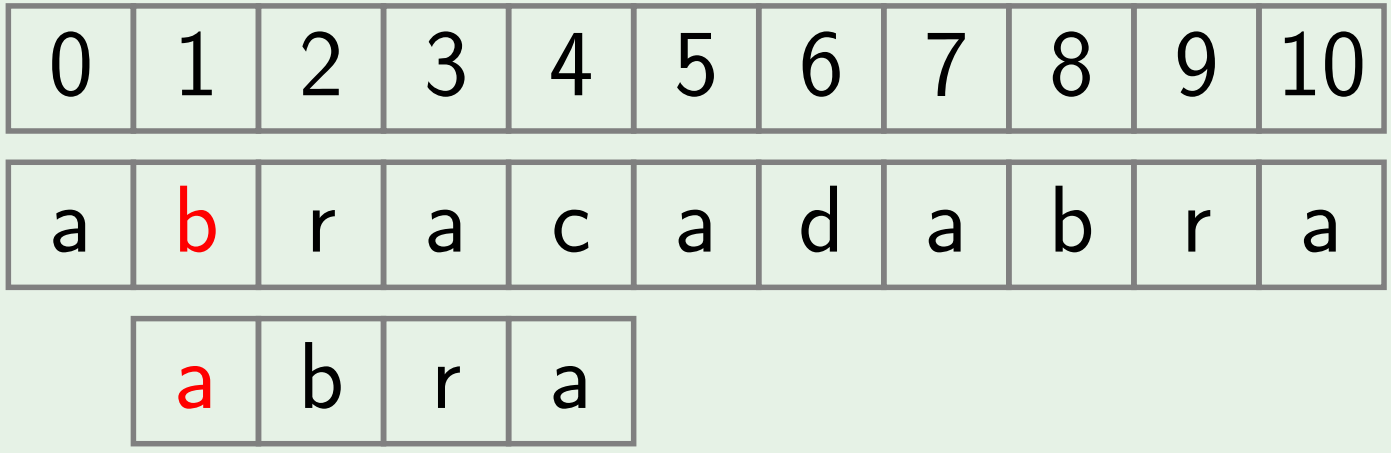
Output: [0]

Brute Force Algorithm



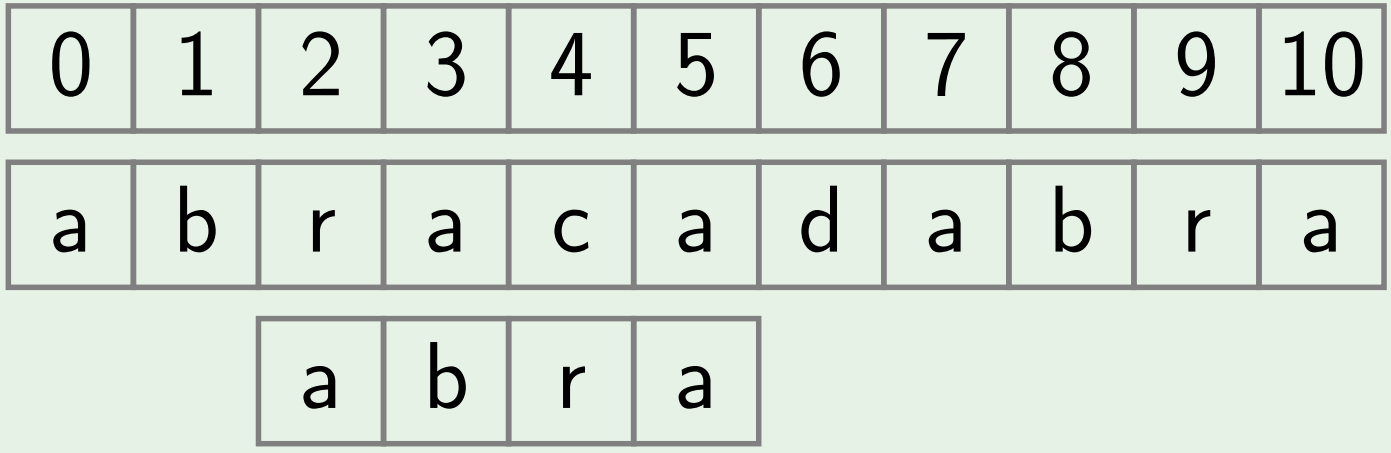
Output: [0]

Brute Force Algorithm



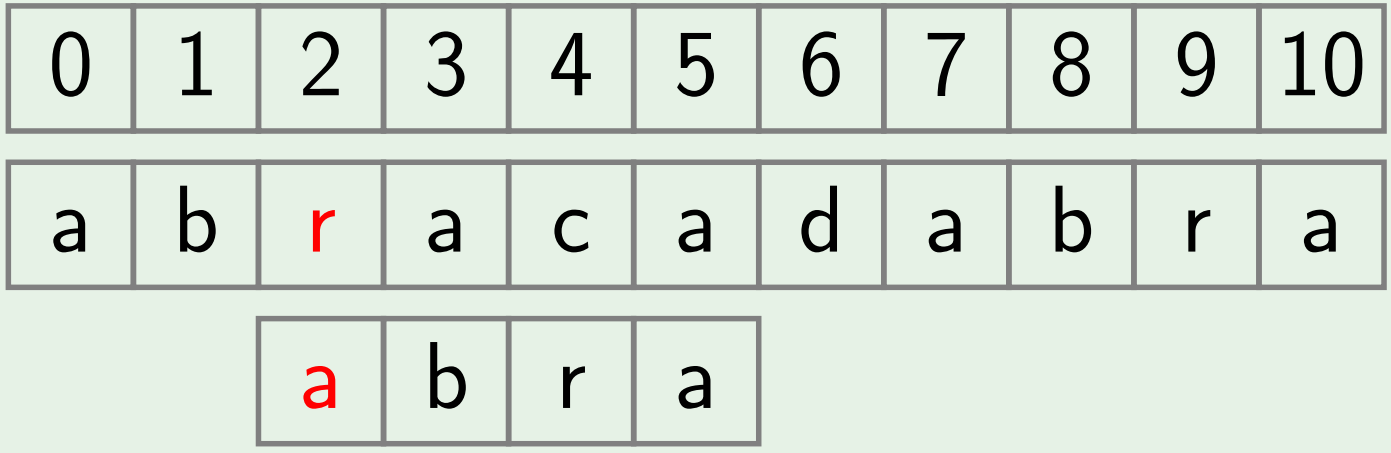
Output: [0]

Brute Force Algorithm



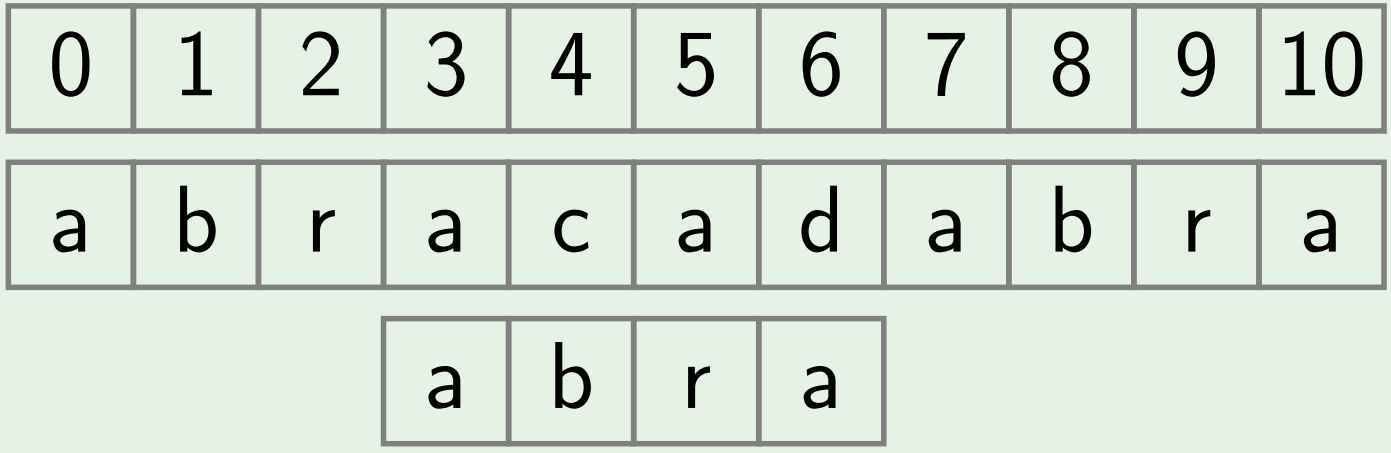
Output: [0]

Brute Force Algorithm



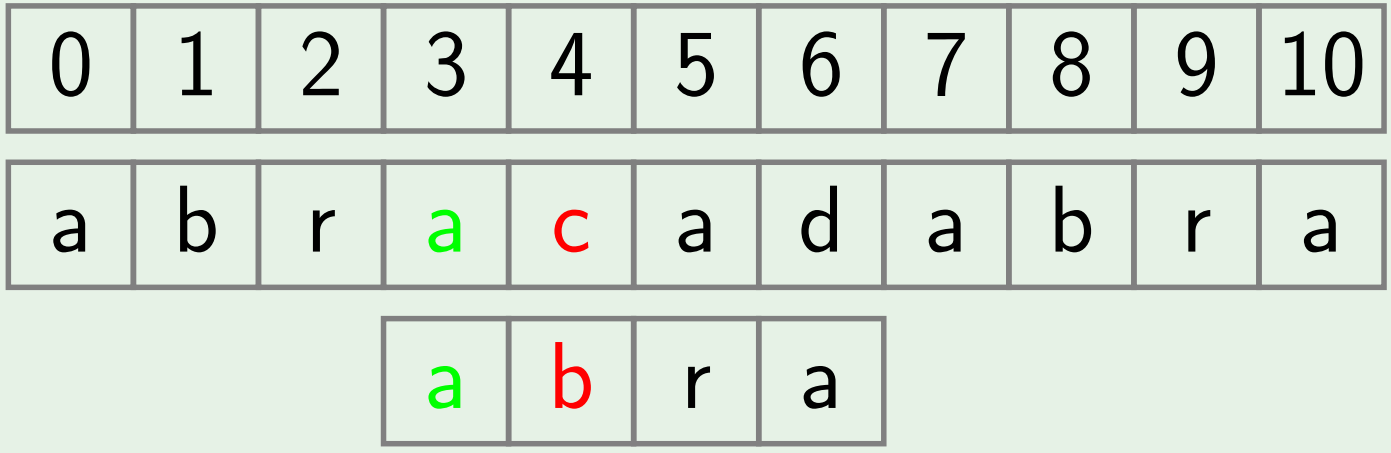
Output: [0]

Brute Force Algorithm



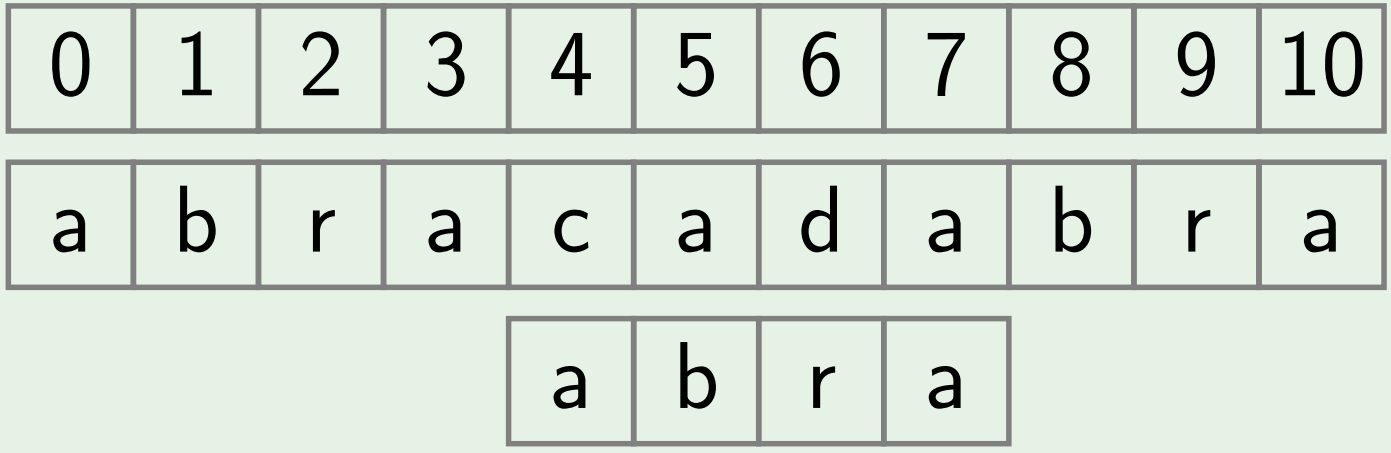
Output: [0]

Brute Force Algorithm



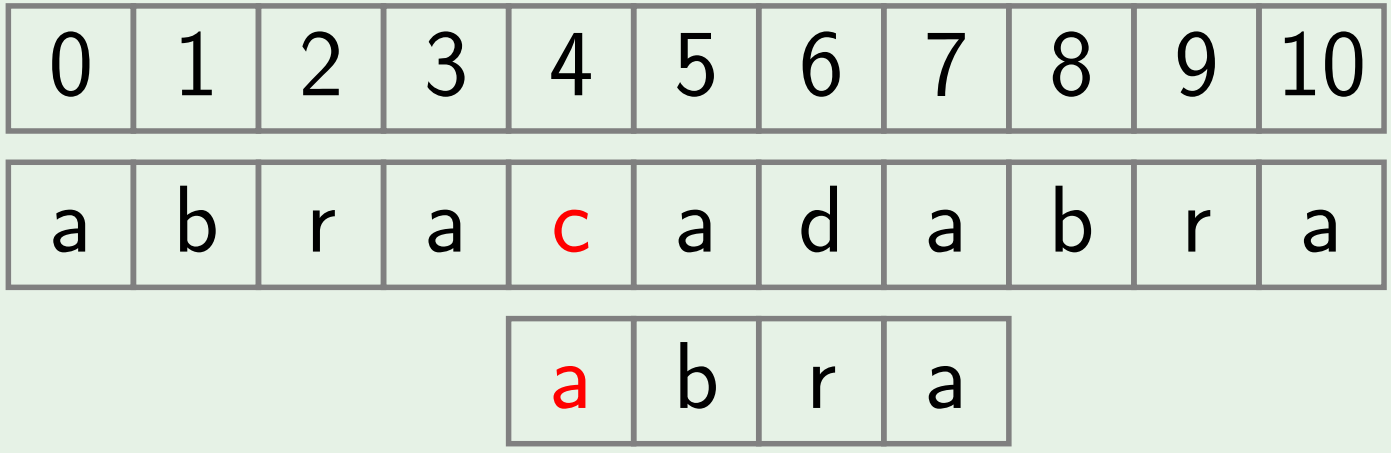
Output: [0]

Brute Force Algorithm



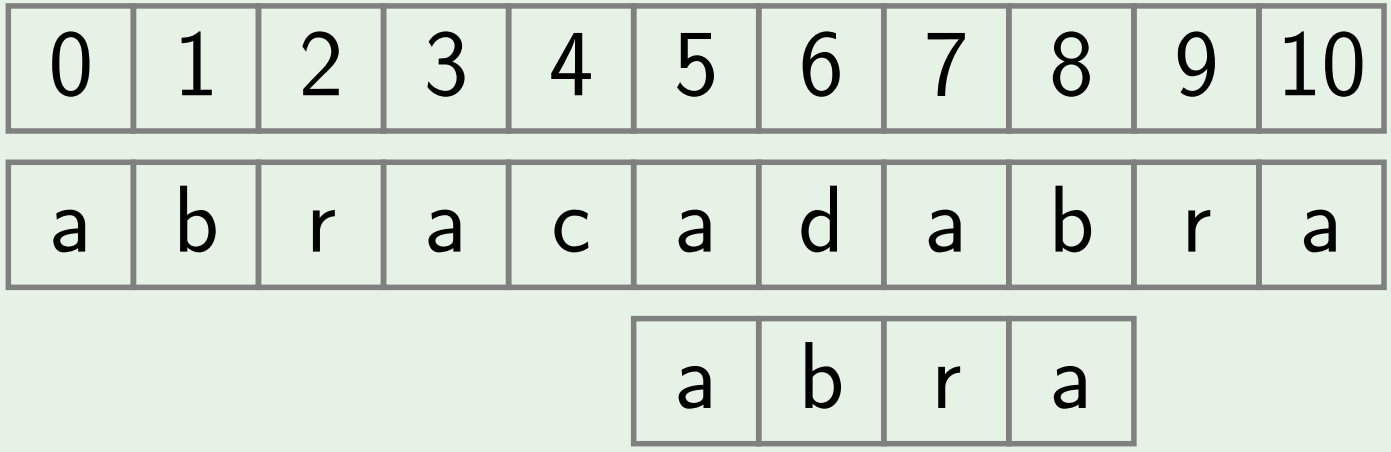
Output: [0]

Brute Force Algorithm



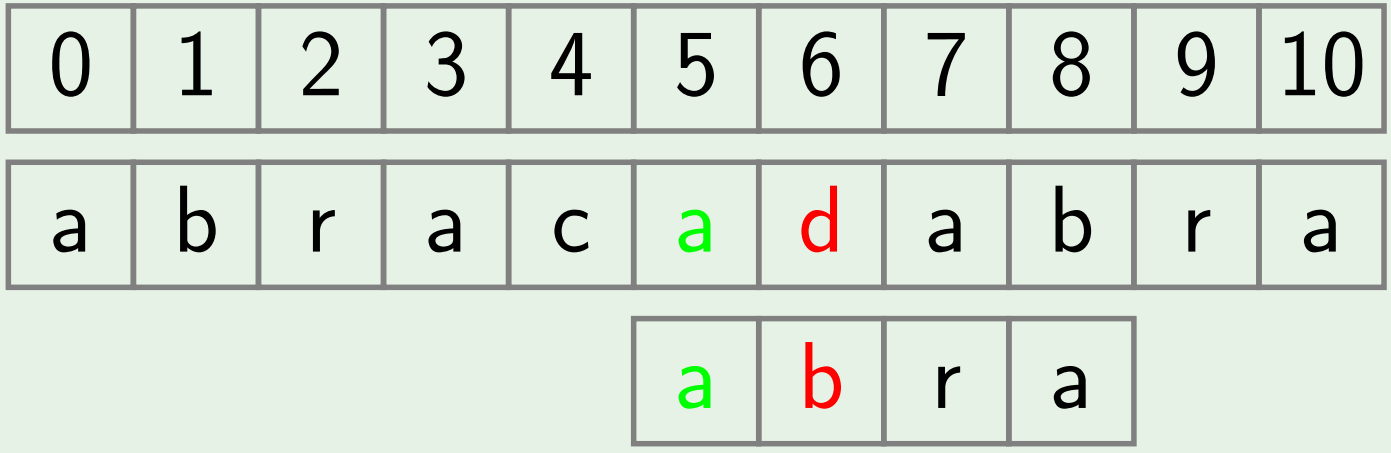
Output: [0]

Brute Force Algorithm



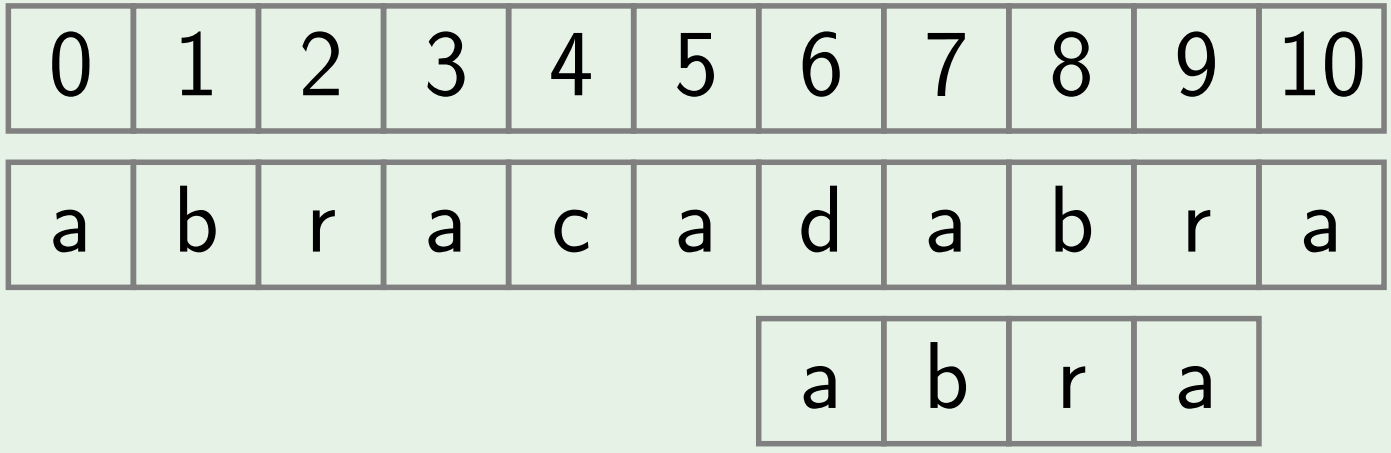
Output: [0]

Brute Force Algorithm



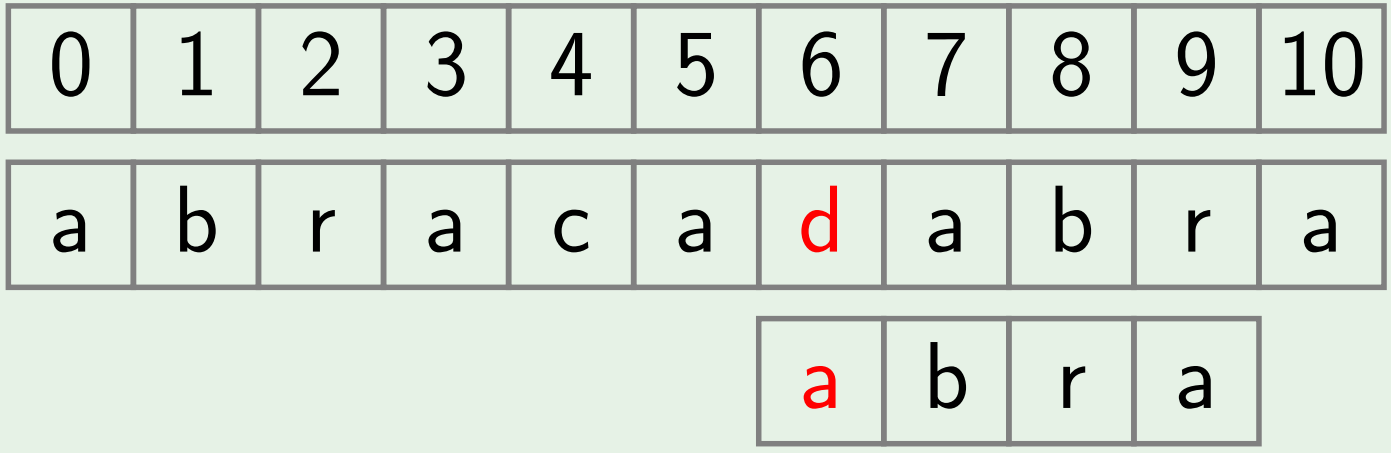
Output: [0]

Brute Force Algorithm



Output: [0]

Brute Force Algorithm



Output: [0]

Brute Force Algorithm

0	1	2	3	4	5	6	7	8	9	10
a	b	r	a	c	a	d	a	b	r	a
							a	b	r	a

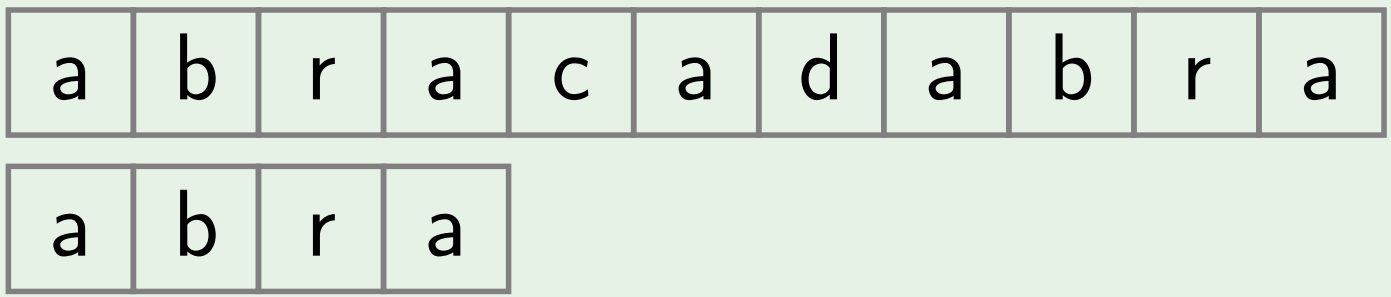
Output: [0]

Brute Force Algorithm

0	1	2	3	4	5	6	7	8	9	10
a	b	r	a	c	a	d	a	b	r	a
							a	b	r	a

Output: [0,7]

Skipping Positions



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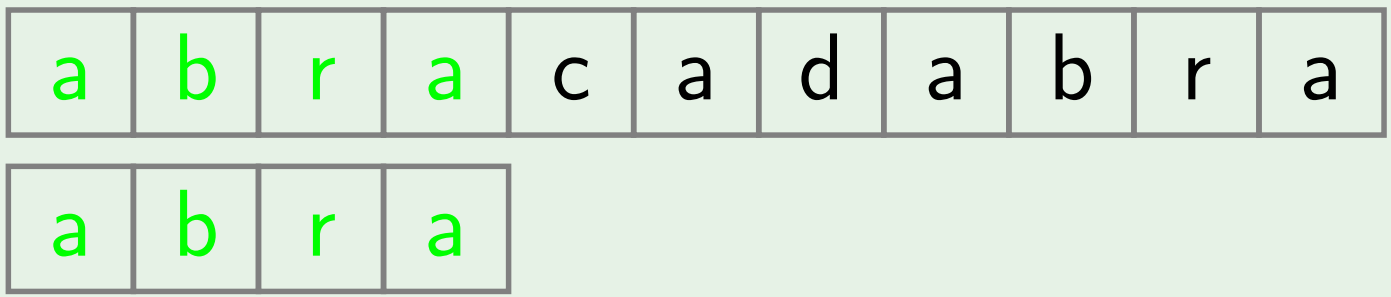

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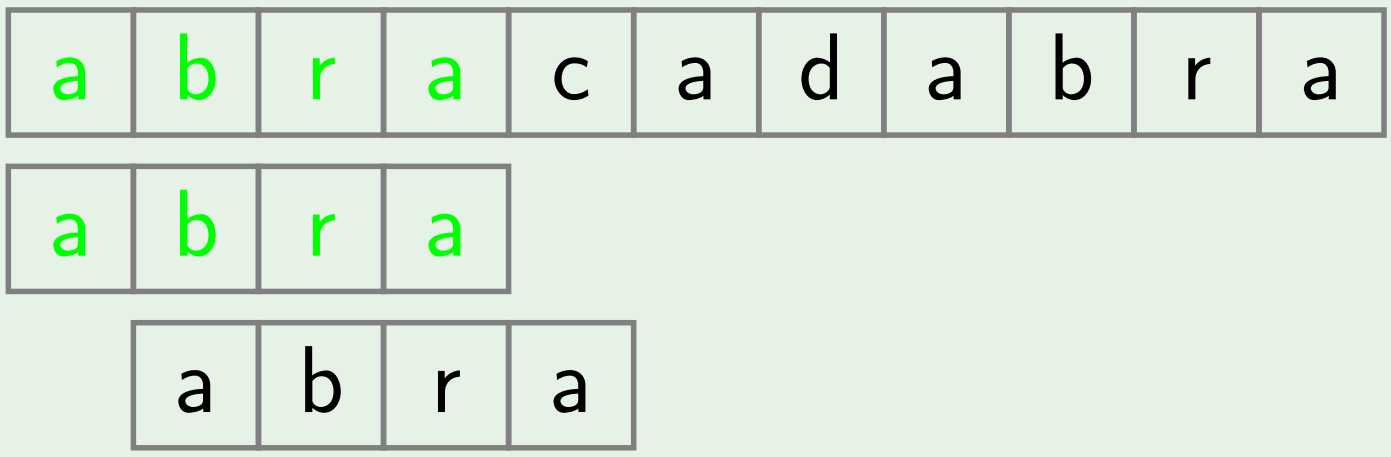

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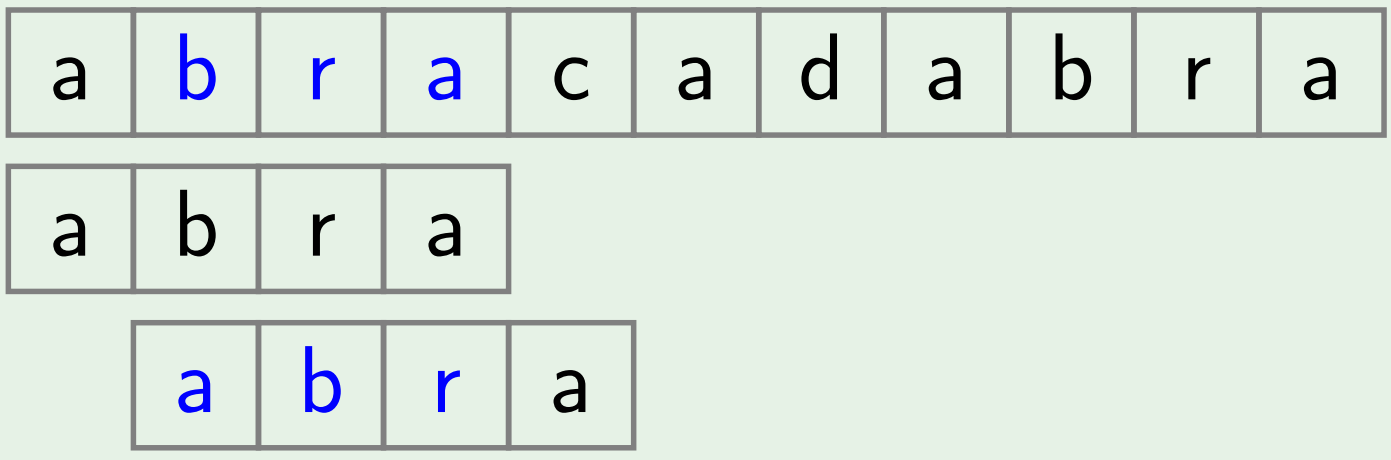

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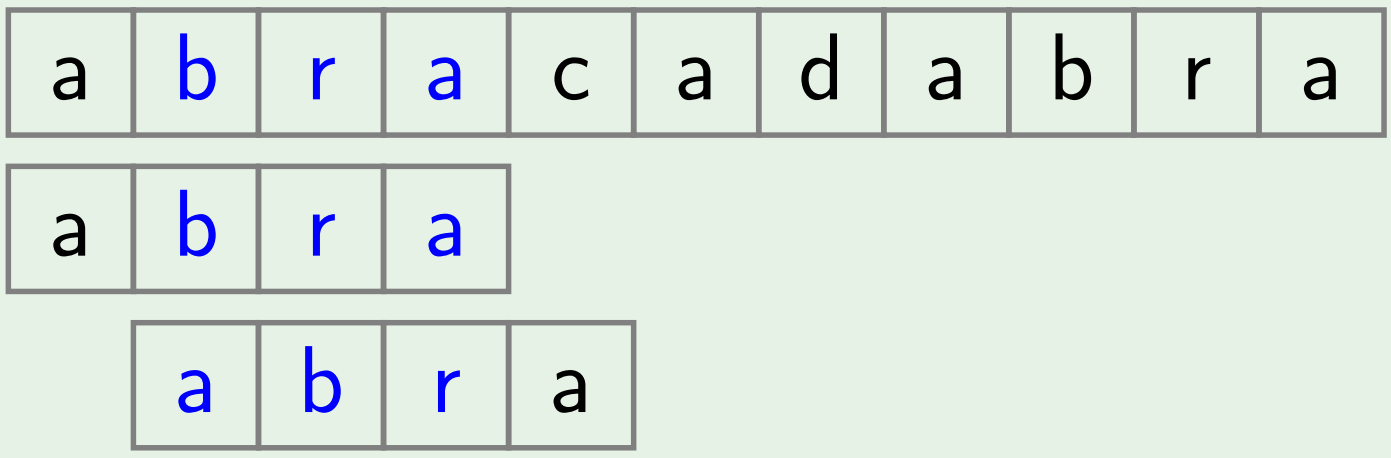

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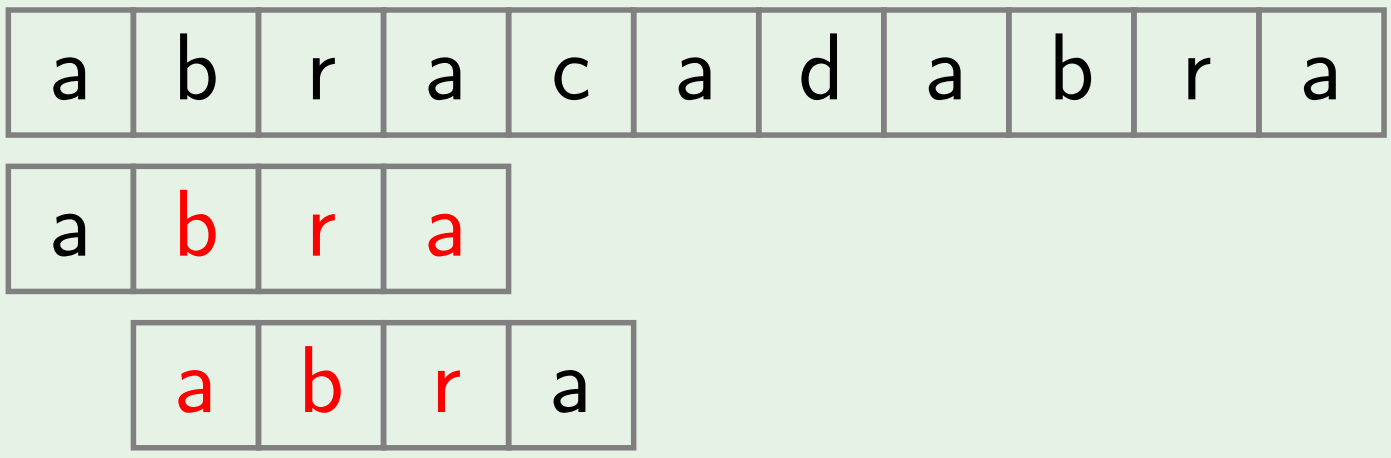

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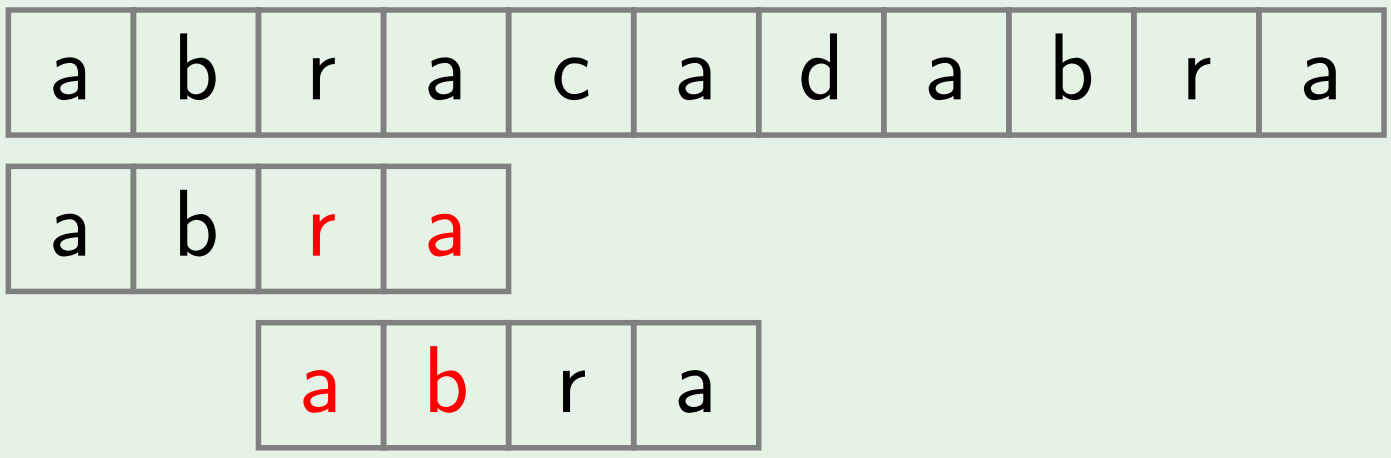

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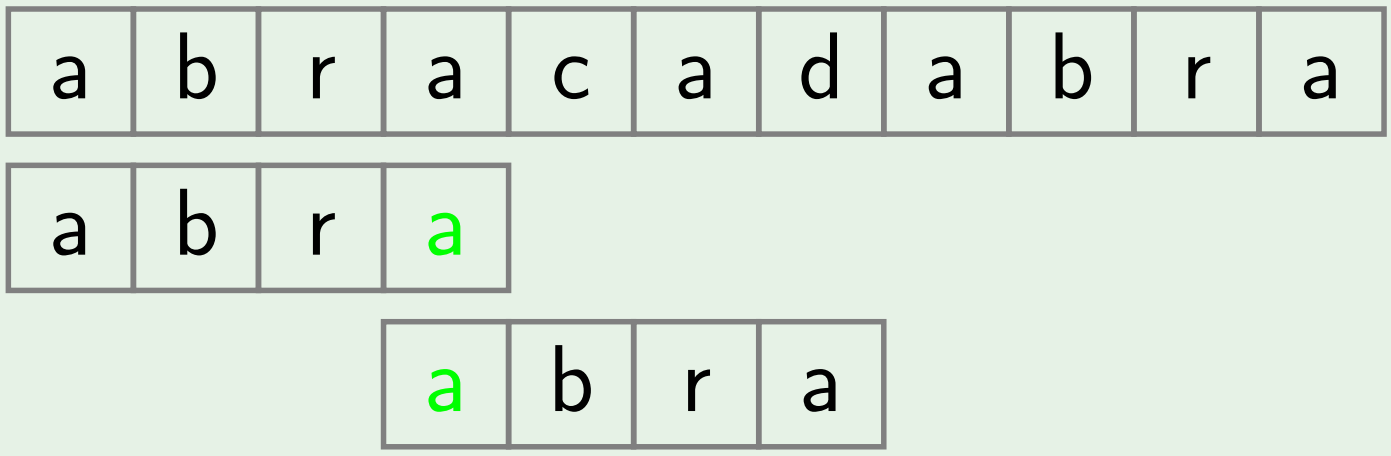

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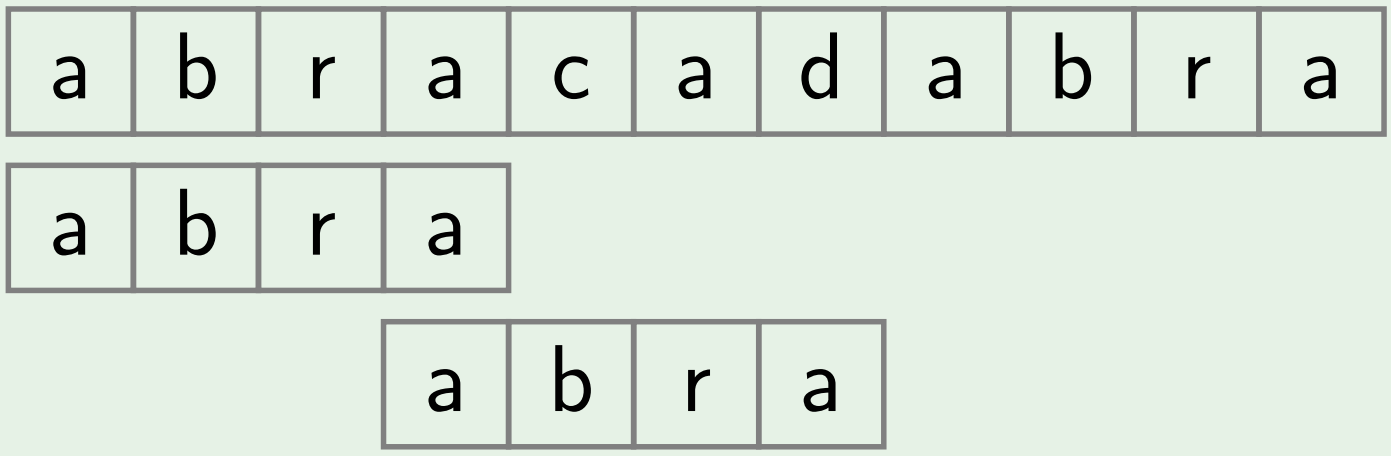

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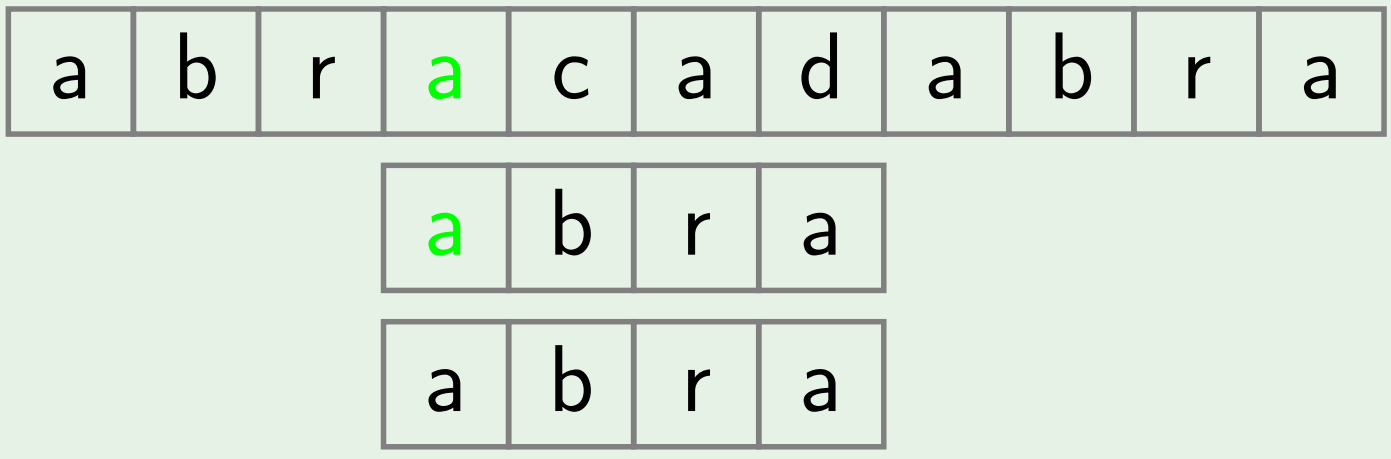

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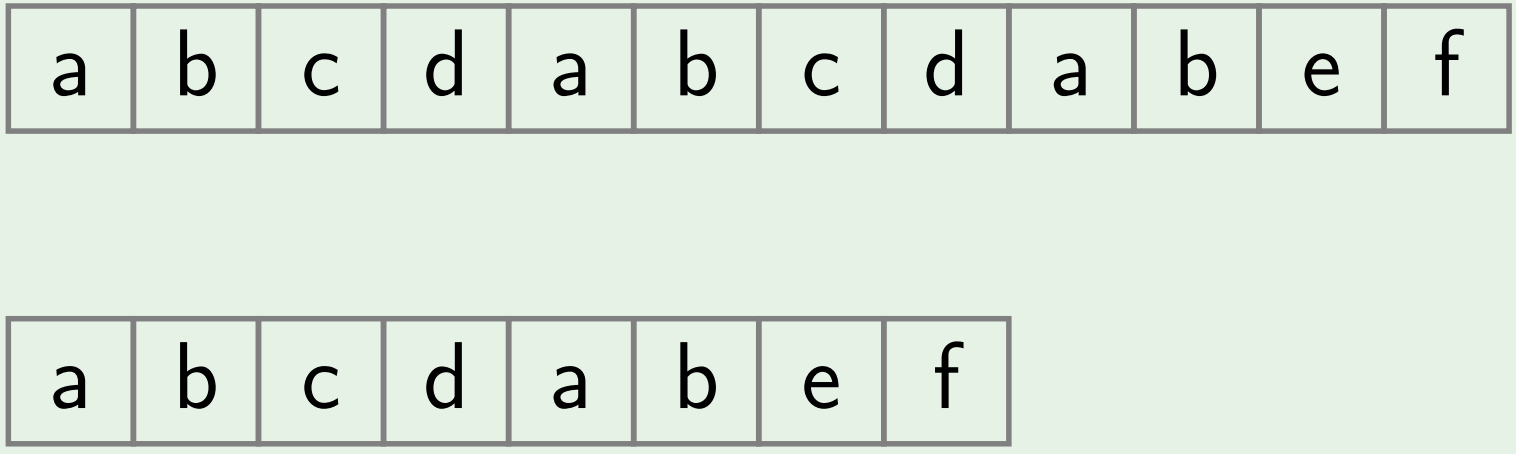

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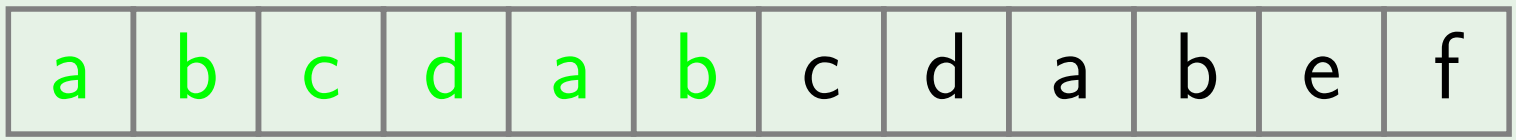

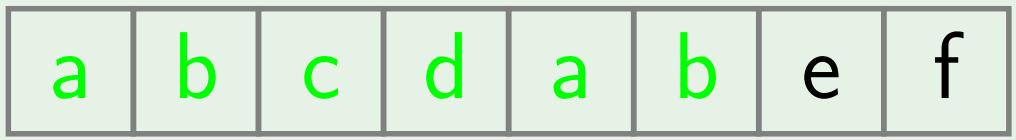

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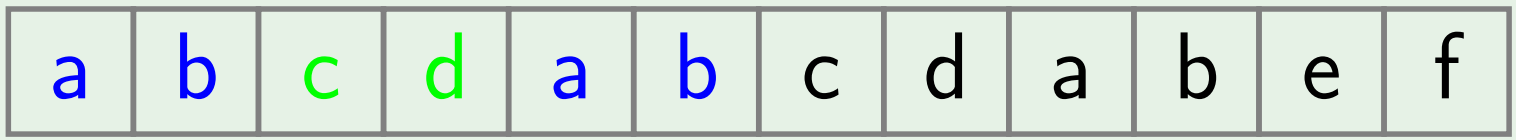

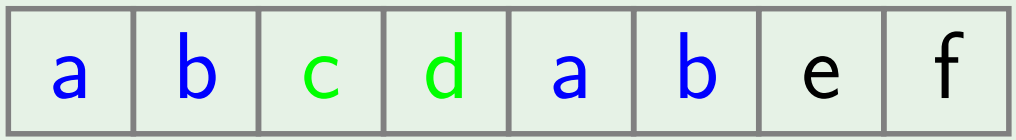

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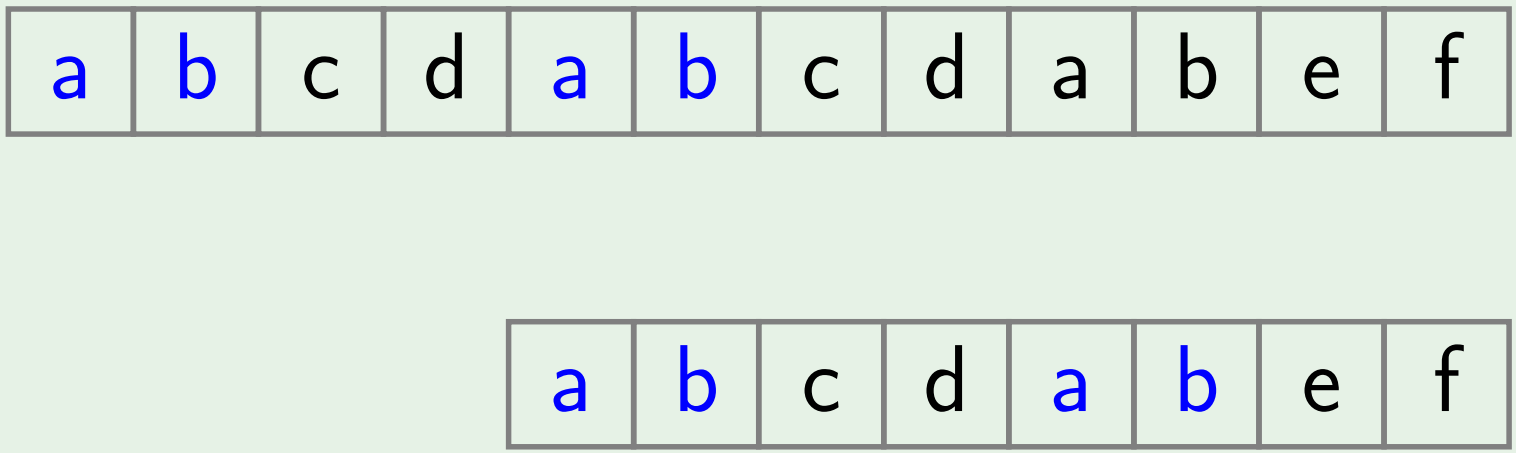

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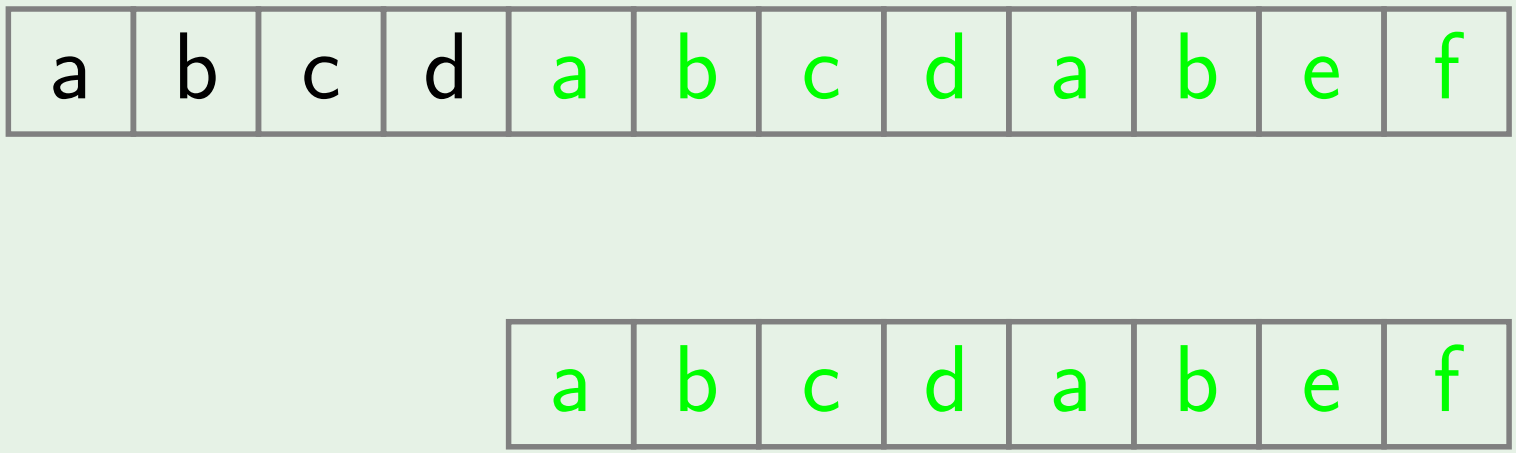

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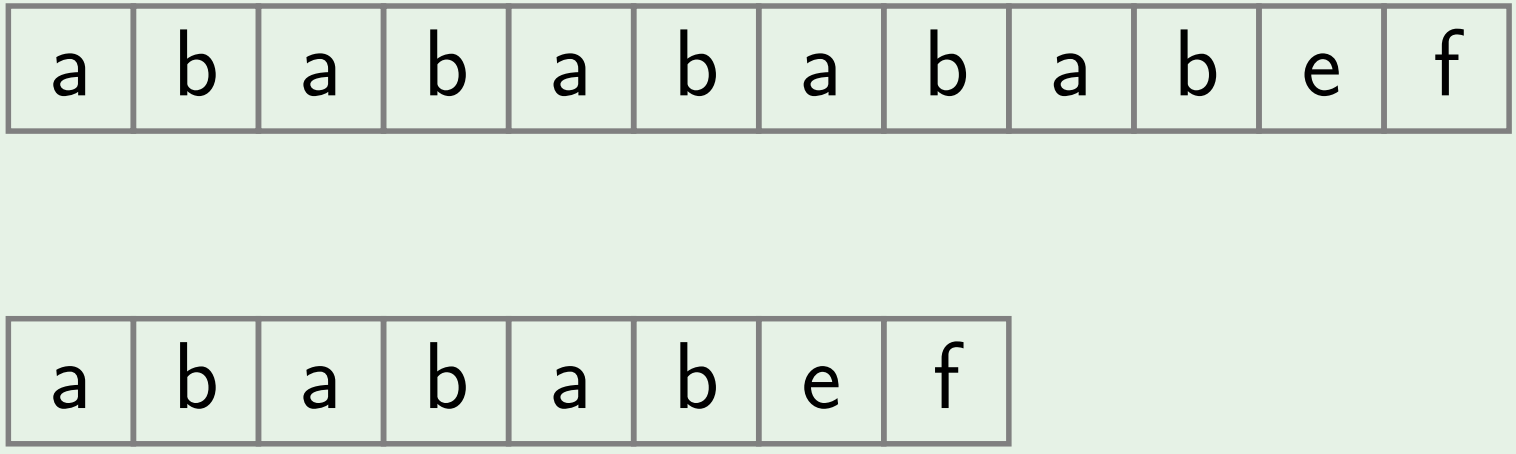

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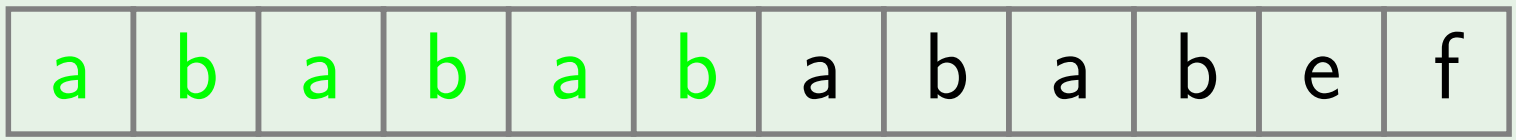

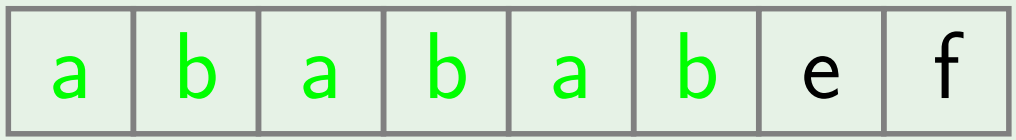

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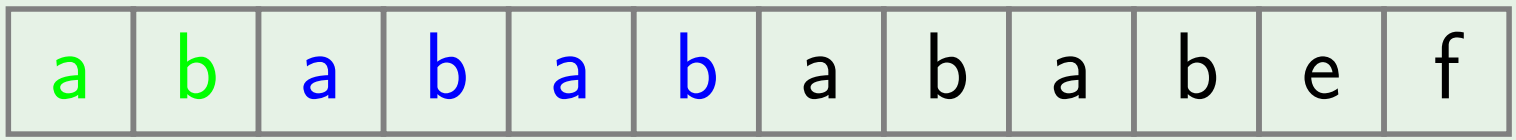

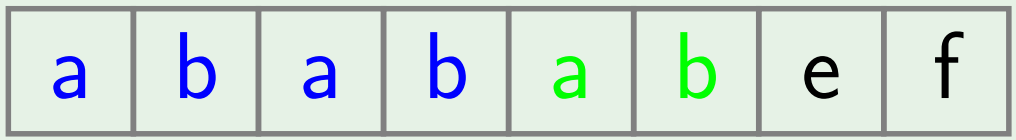

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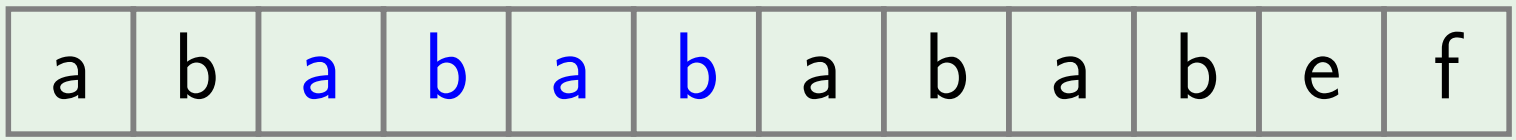

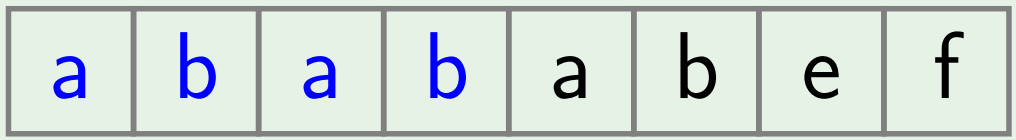

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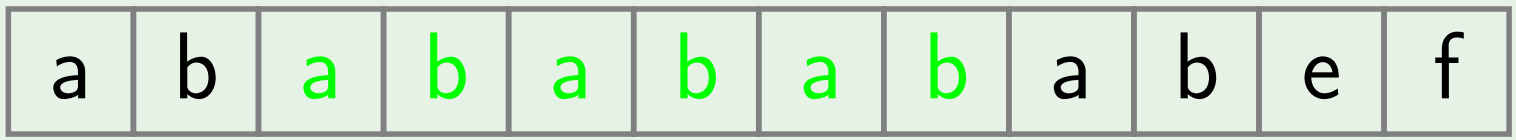

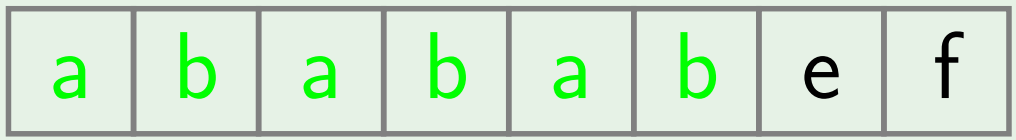

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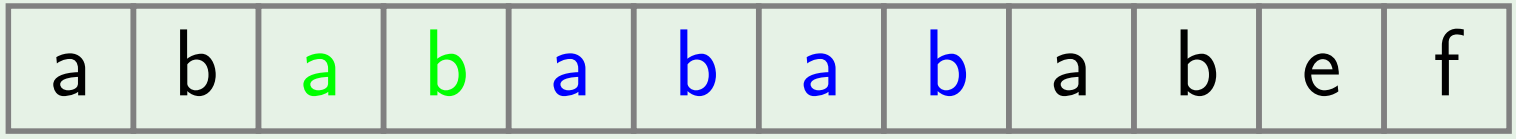

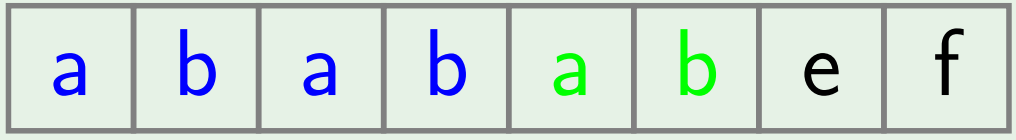

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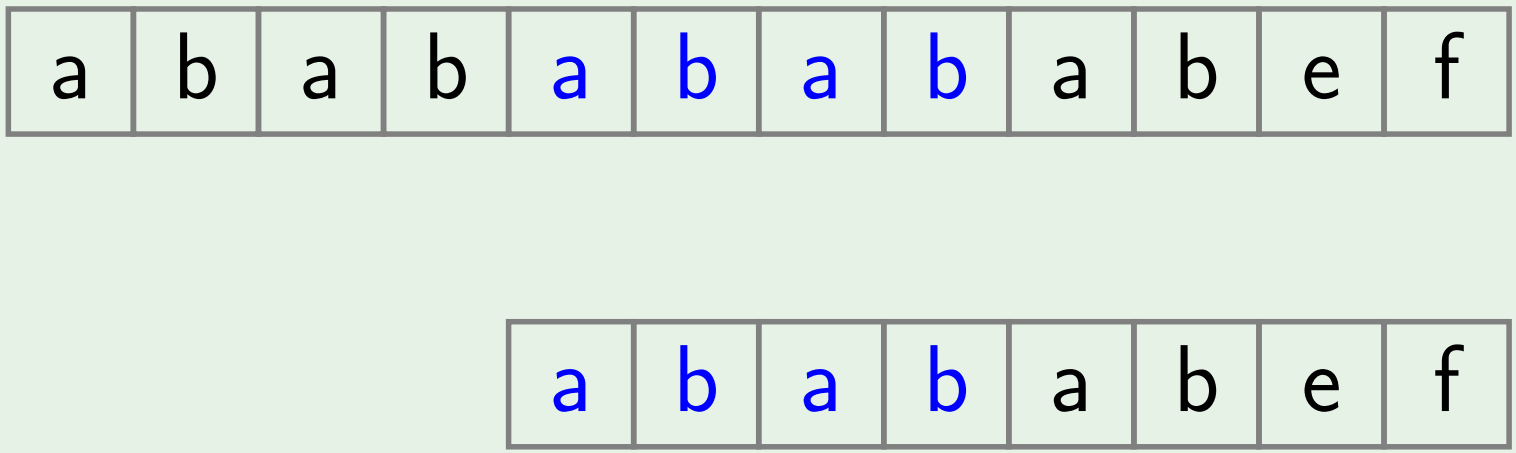
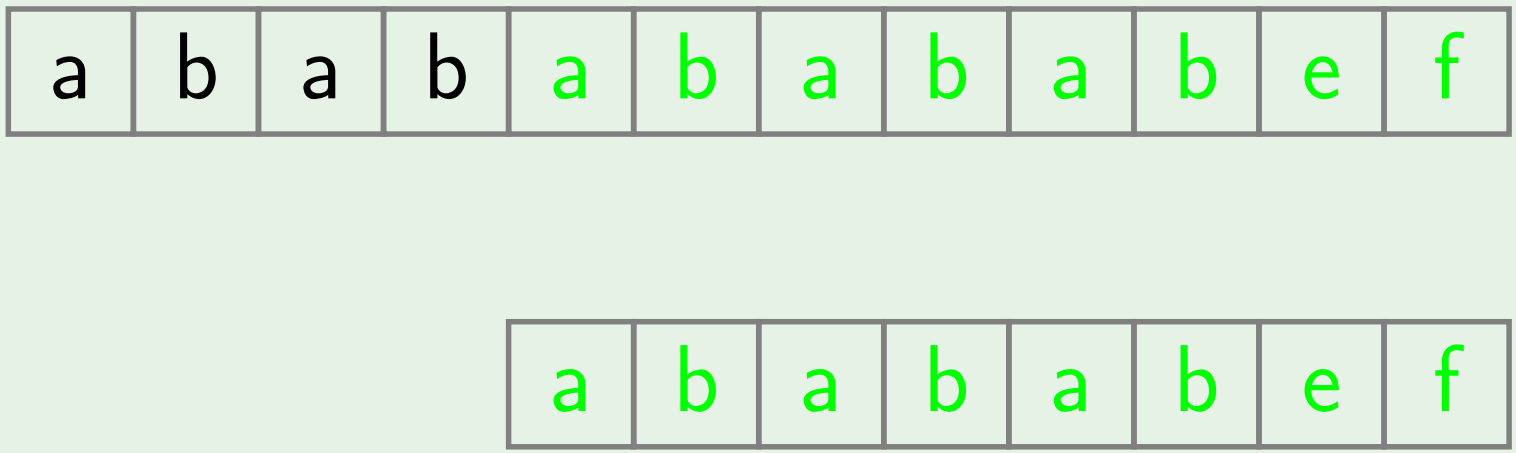


Skipping Positions



Definitions

Definition

Border of string S is a prefix of S which is equal to a suffix of S, but not equal to the whole S.

Example

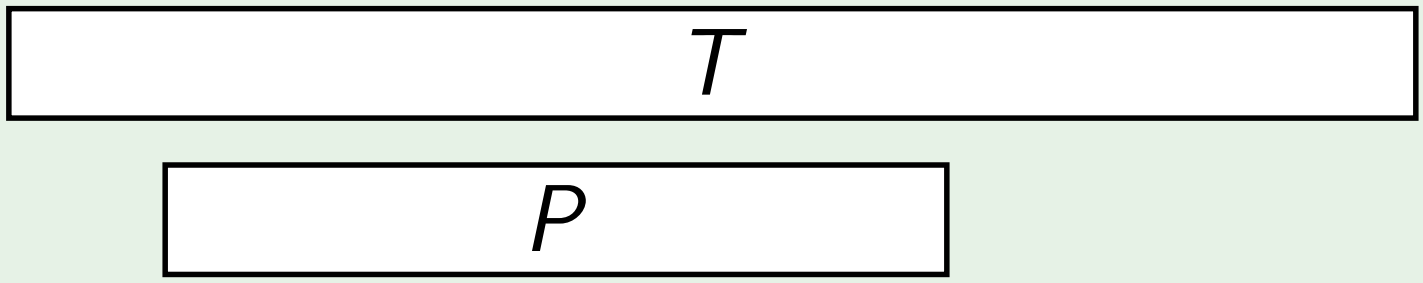
"a" is a border of "arba"

"ab" is a border of "abcdab"

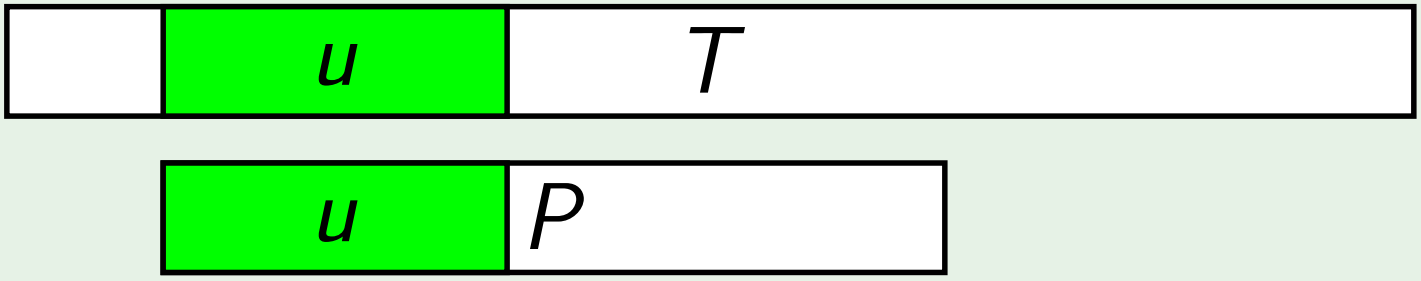
"abab" is a border of "ababab"

"ab" is not a border of "ab"

Shifting Pattern

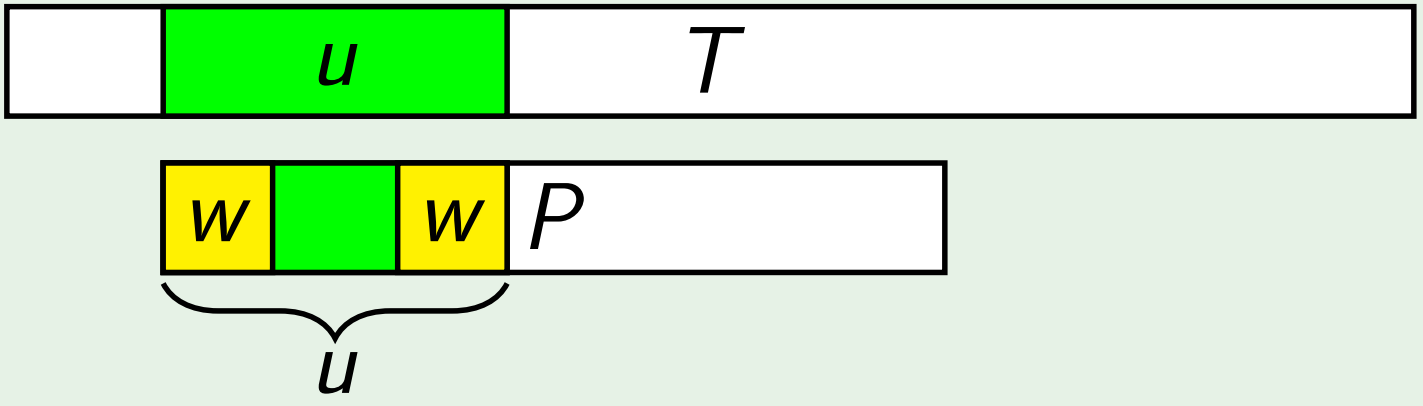


Shifting Pattern



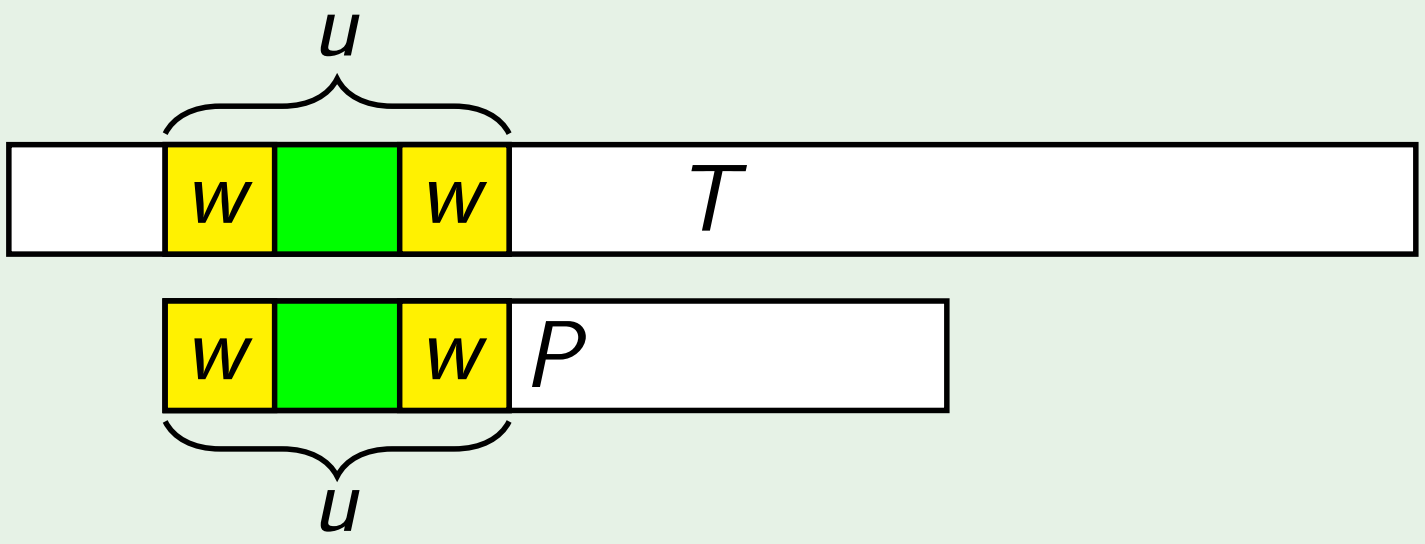
- Find longest common prefix u

Shifting Pattern



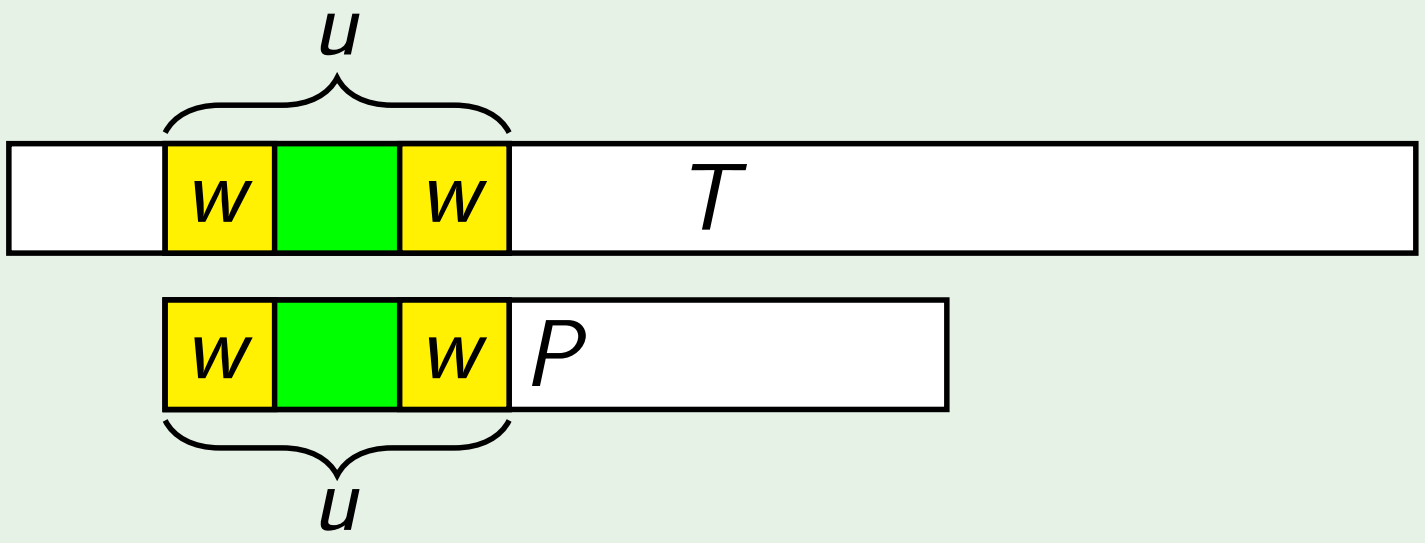
- Find longest common prefix u
- Find w — the longest border of u

Shifting Pattern



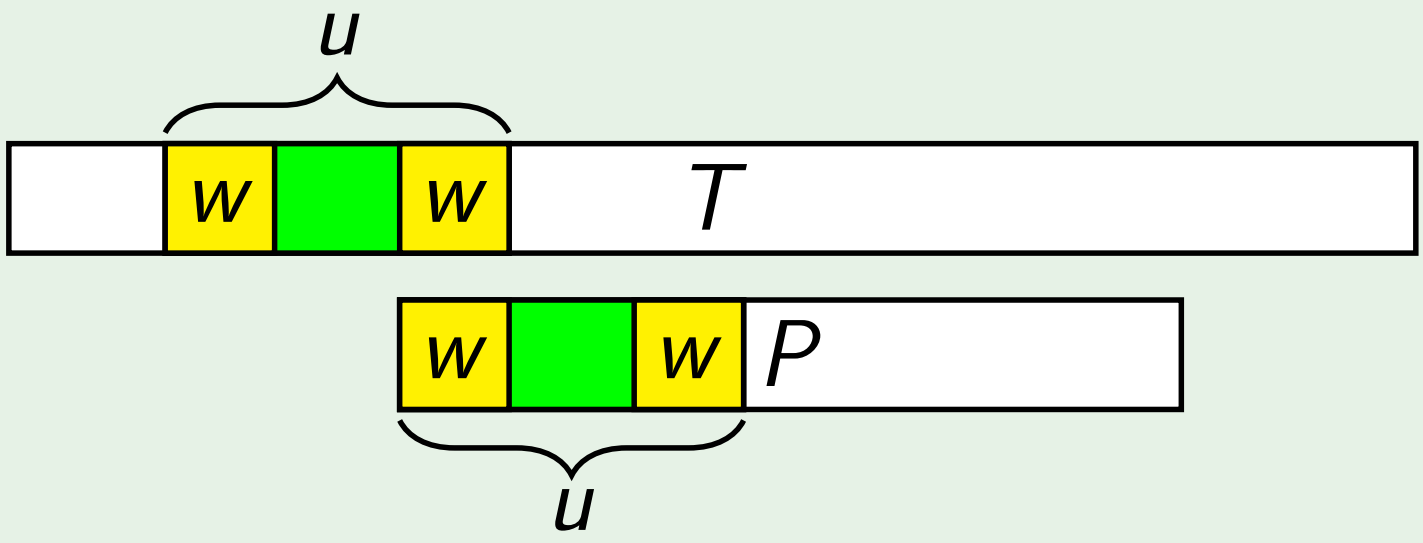
- Find longest common prefix u
- Find w — the longest border of u

Shifting Pattern



- Find longest common prefix u
- Find w — the longest border of u
- Move P such that prefix w in P aligns with suffix w of u in T

Shifting Pattern



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- But we shouldn't miss any of the pattern occurrences in the text

- Now you know we can skip some of the comparisons
- But we shouldn't miss any of the pattern occurrences in the text
- Is it safe to shift the pattern this way?

Outline

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2. Safe Shift
3. Prefix Function
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7. Knuth-Morris-Pratt Algorithm

Suffix notation

Definition

Denote by S_k suffix of string S starting at position k.

Examples

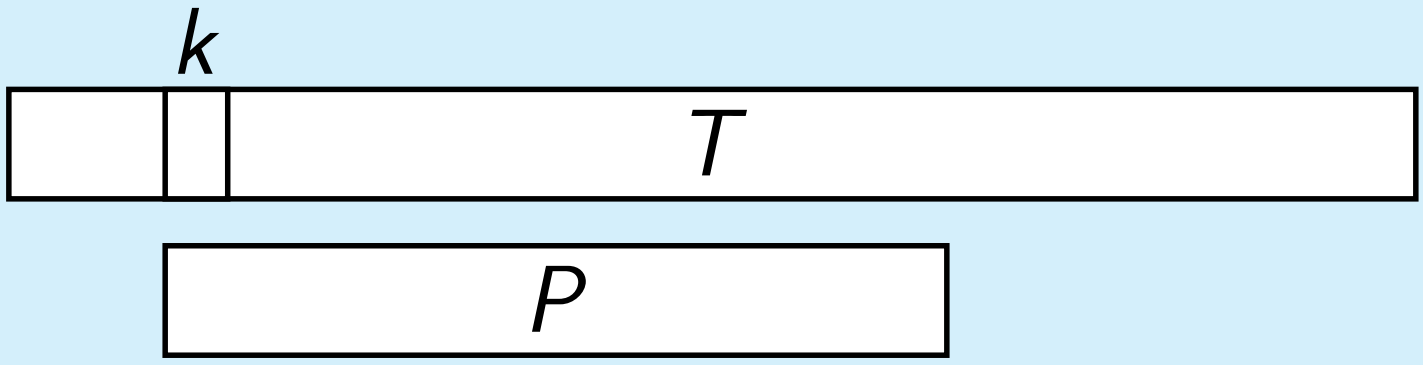
$S =$ "abcd" $\Rightarrow S_2 =$ "cd"

$T =$ "abc" $\Rightarrow T_0 =$ "abc"

$P =$ "aa" $\Rightarrow P_1 =$ "a"

Safe shift

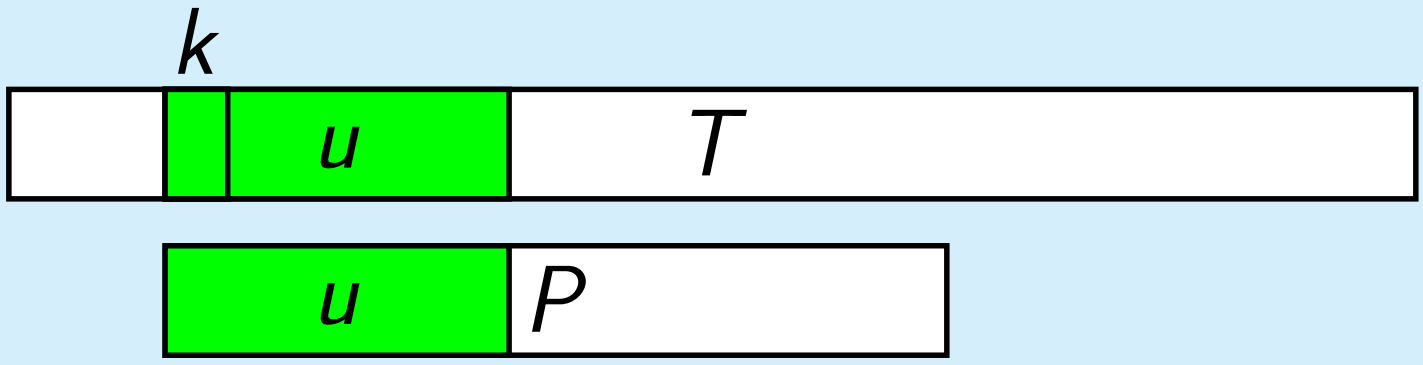
Lemma



Let u be the longest common prefix of P and T_k. Let w be the longest border of u. Then there are no occurrences of P in T starting between positions k and $(k + |u| - |w|)$ — the start of suffix w in the prefix u of T_k.

Safe shift

Lemma



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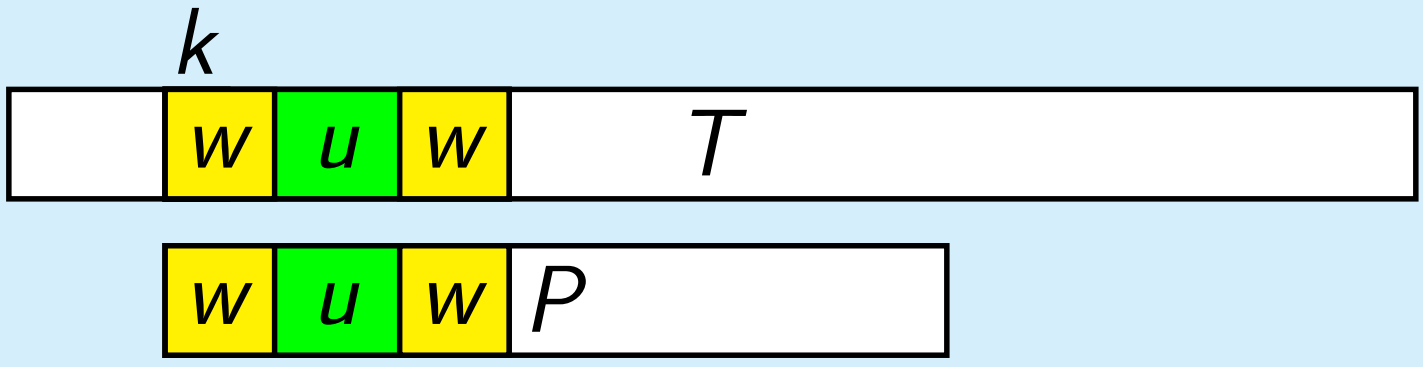
Safe shift

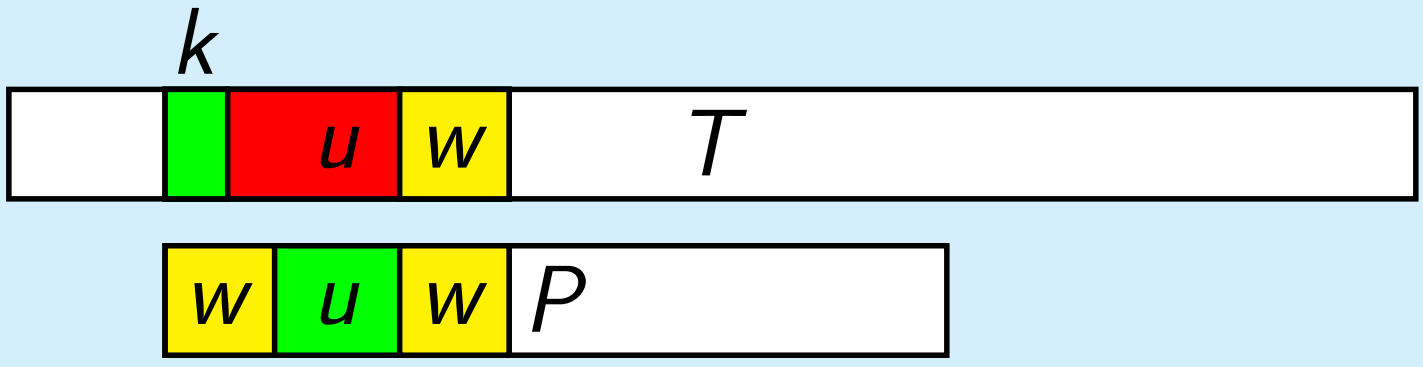
Lemma



Let u be the longest common prefix of P and T_k. Let w be the longest border of u. Then there are no occurrences of P in T starting between positions k and $(k + |u| - |w|)$ — the start of suffix w in the prefix u of T_k.

Safe shift

Lemma



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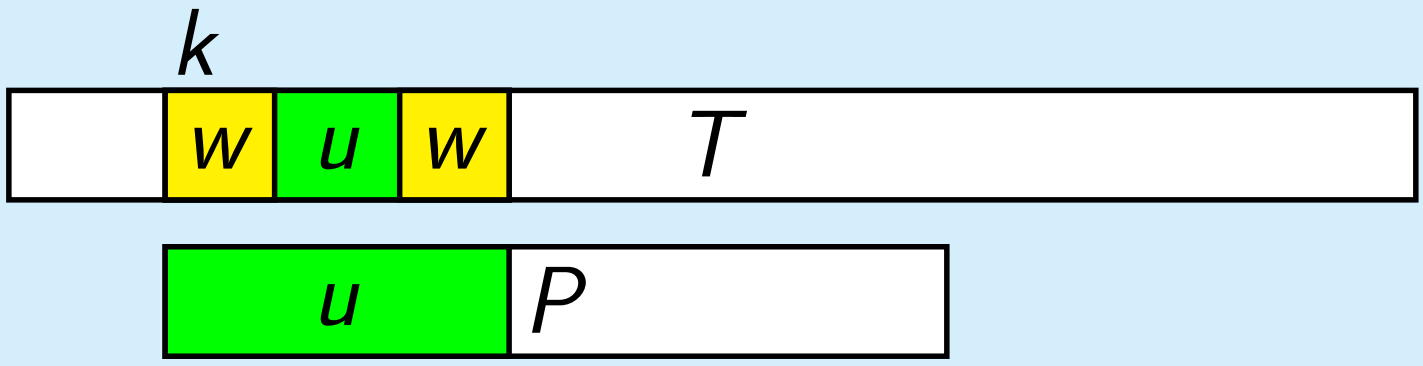
Proof



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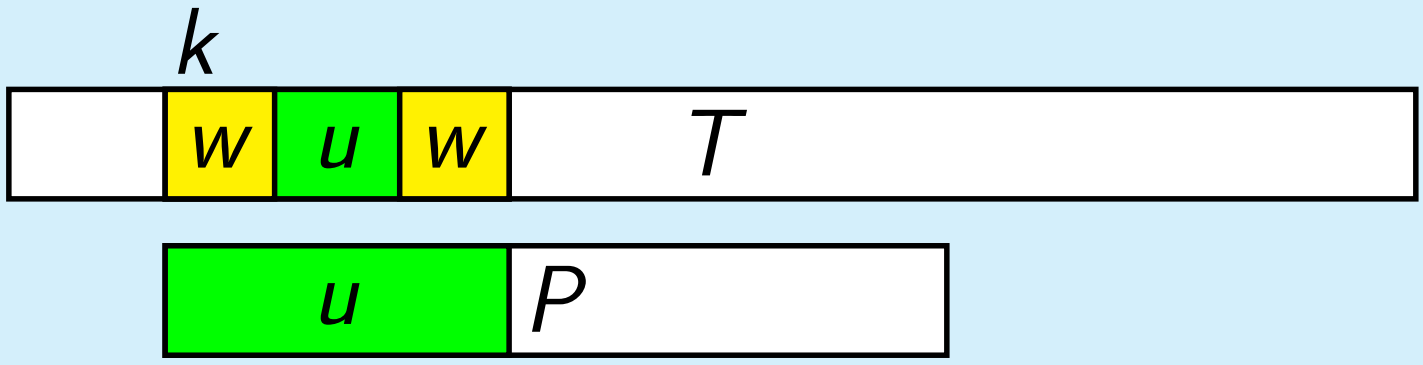
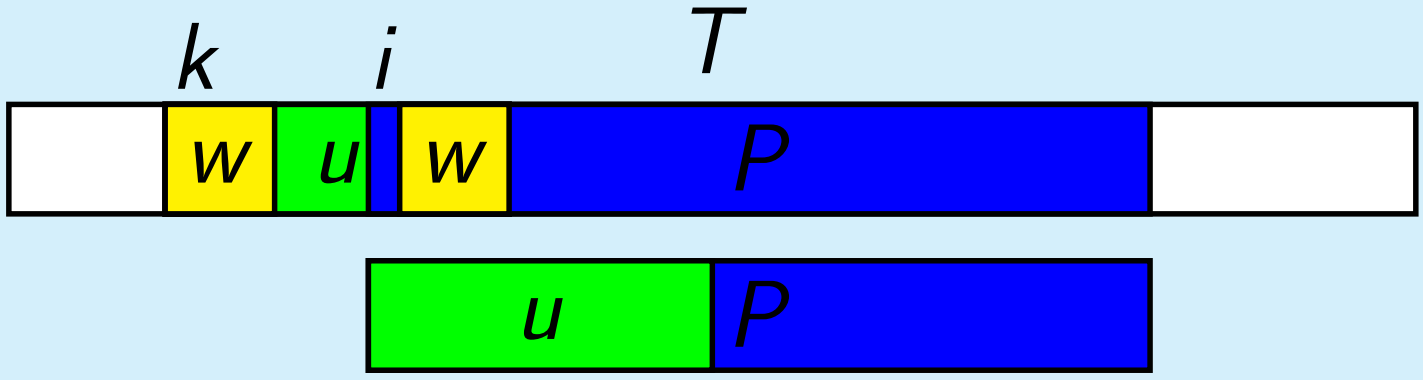


- Suppose P occurs in T in position i between k and start of suffix w

Proof



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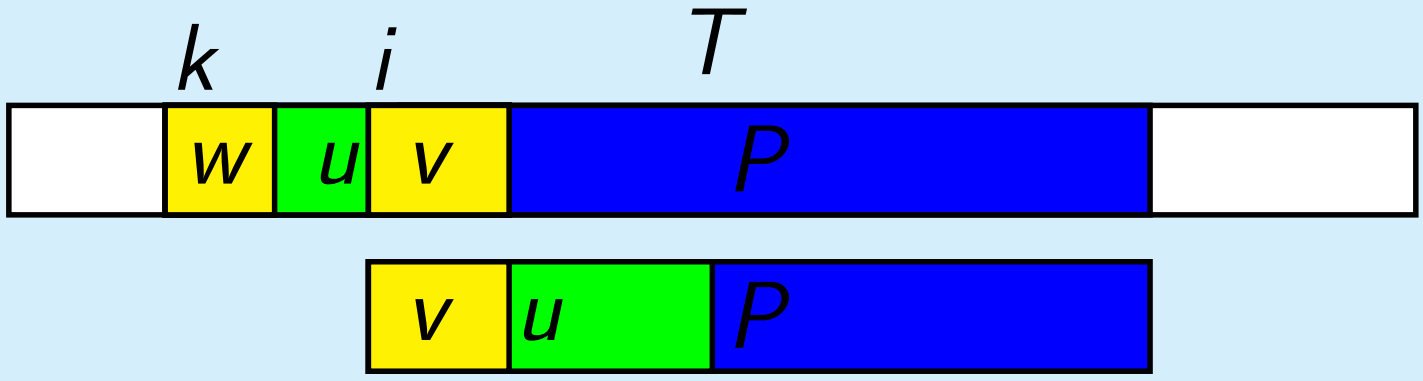

- Suppose P occurs in T in position i between k and start of suffix w
- Then there is prefix v of P equal to suffix in u, and v is longer than w □

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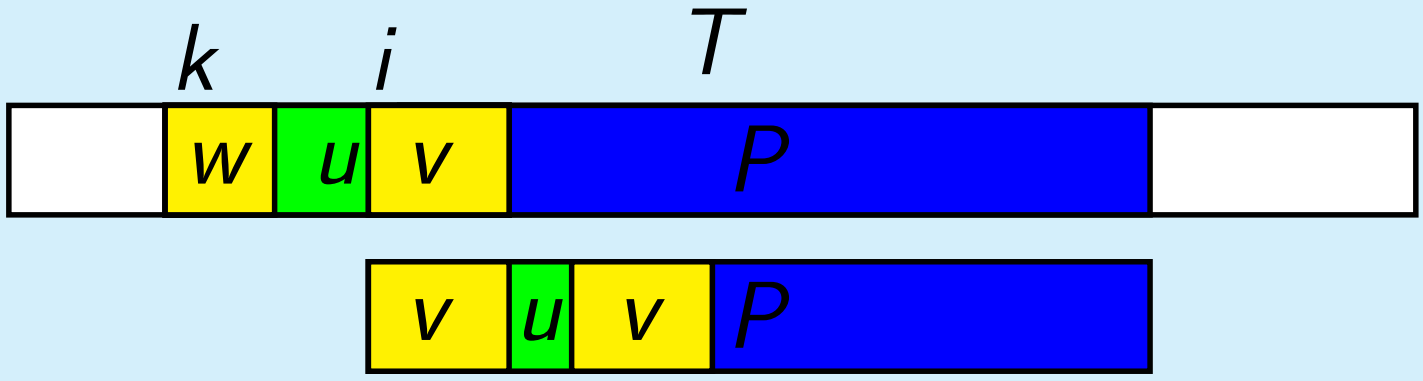

- Then there is prefix v of P equal to suffix in u, and v is longer than w
- v is a border longer than w, but w is longest border of u — contradiction □

- Now you know it is possible to avoid many of the comparisons which Brute Force algorithm does

- Now you know it is possible to avoid many of the comparisons which Brute Force algorithm does
- But how to determine the best pattern shifts?

Outline

1. Exact Pattern Matching
2. Safe Shift
3. Prefix Function
4. Computing Prefix Function
5. Implementation
6. Analysis
7. Knuth-Morris-Pratt Algorithm

Prefix function

Definition

Prefix function of a string P is a function $s(i)$ that for each i returns the length of the longest border of the prefix $P[0..i]$.

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1	2	3	4	0	1	1	2

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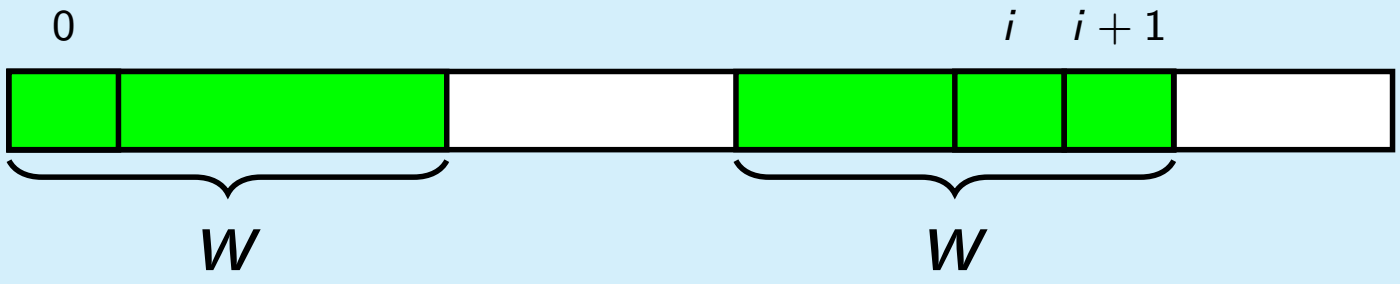

- Take the longest border w of $P[0..i+1]$

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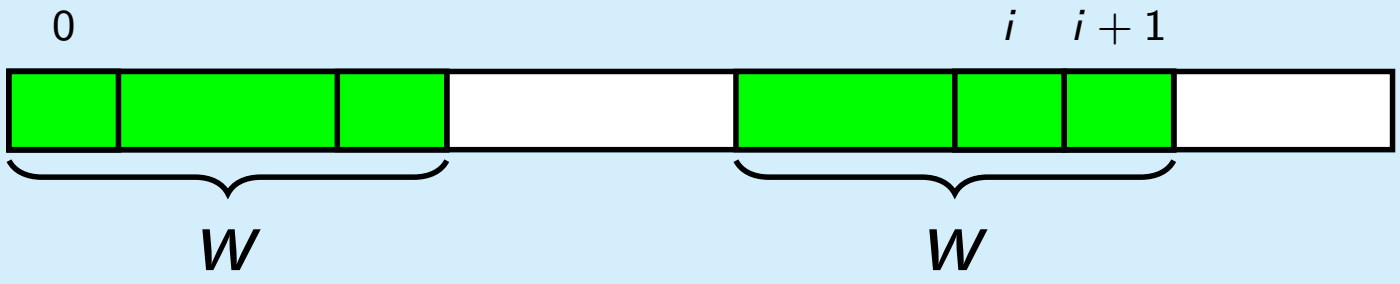
Proof



- Take the longest border w of $P[0..i+1]$
- Cut the last character from w — it's a border of $P[0..i]$ now

□

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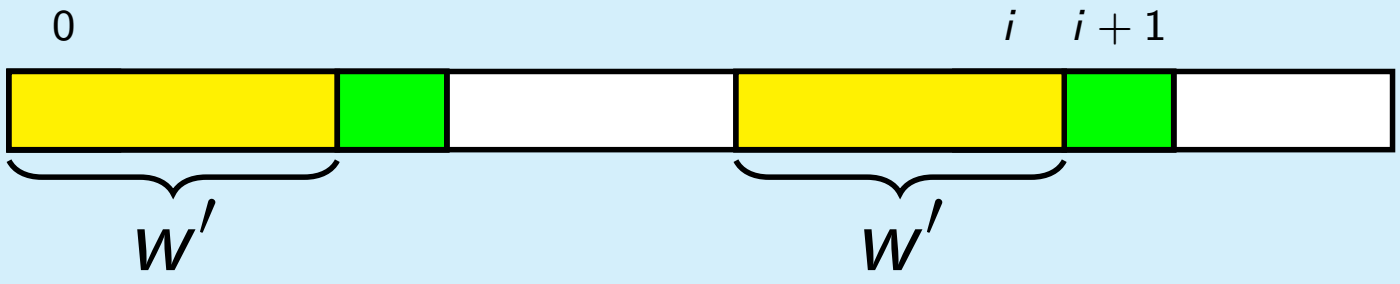


- Take the longest border w of $P[0..i+1]$
- Cut the last character from w — it's a border of $P[0..i]$ now

□

Corollary

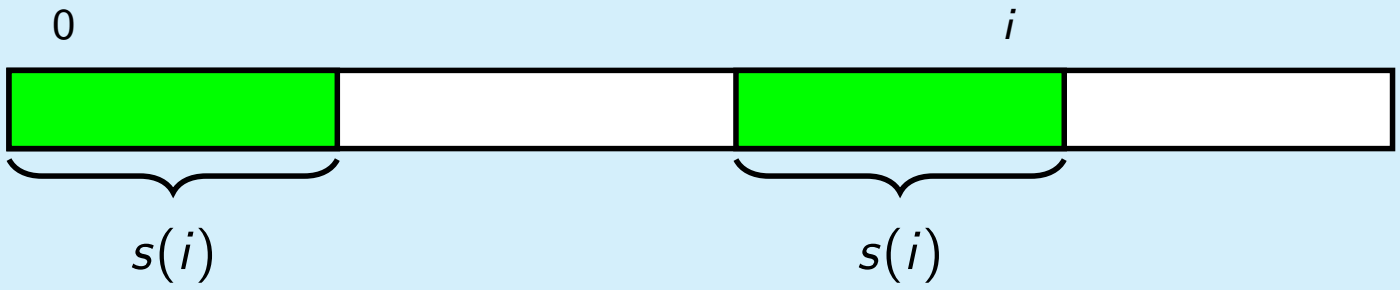
$s(i+1) \leq s(i) + 1$

Enumerating borders

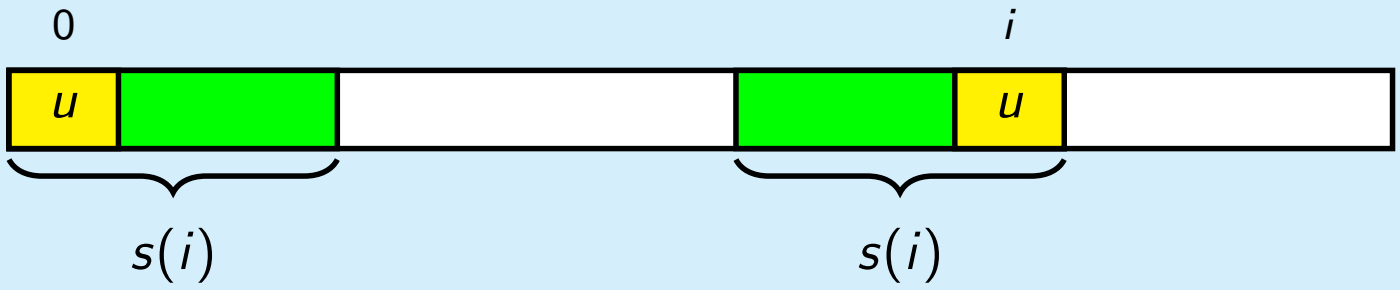
Lemma

If $s(i) > 0$, then all borders of $P[0..i]$ but for the longest one are also borders of $P[0..s(i) - 1]$.

Proof

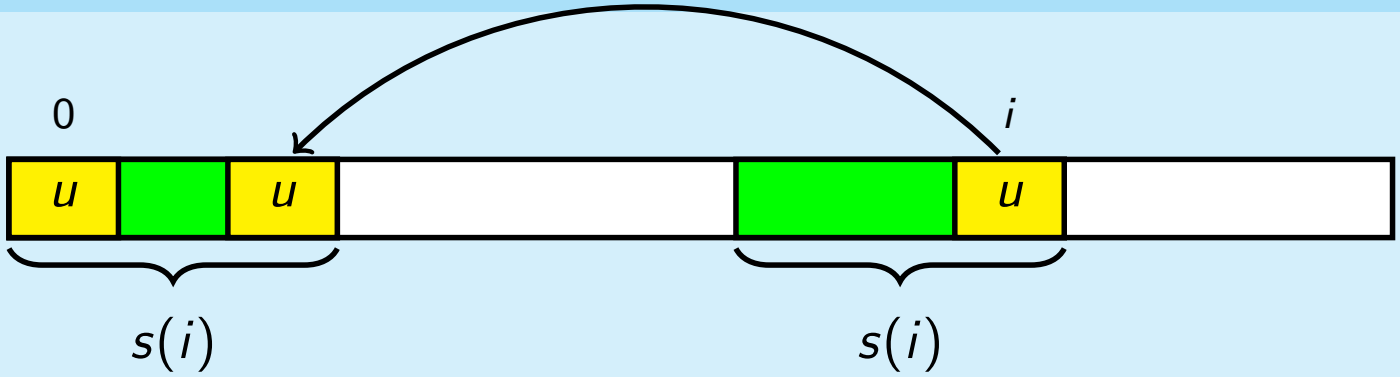


Proof



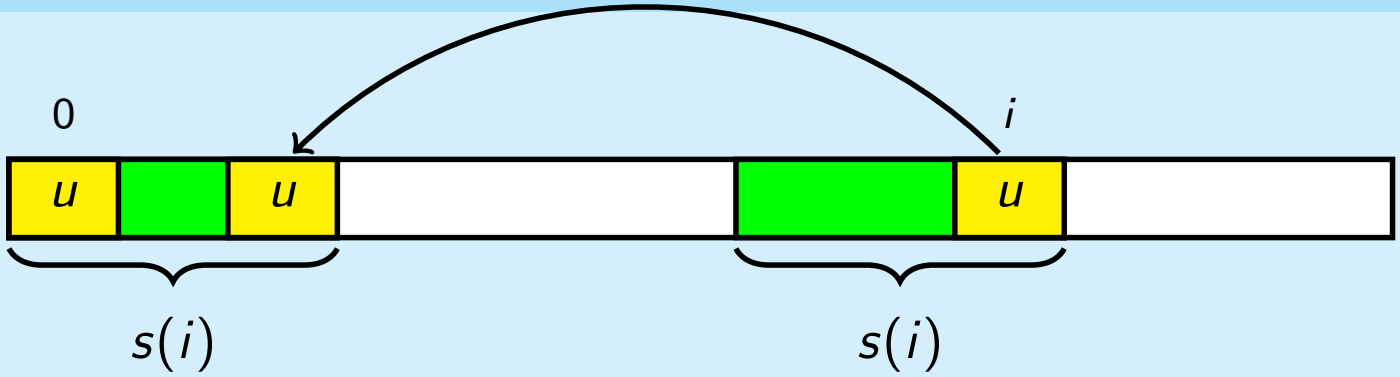
- Let u be a border of $P[0..i]$ such that $|u| < s(i)$

Proof



- Let u be a border of $P[0..i]$ such that $|u| < s(i)$
- Then u is both a prefix and a suffix of $P[0..s(i) - 1]$

Proof



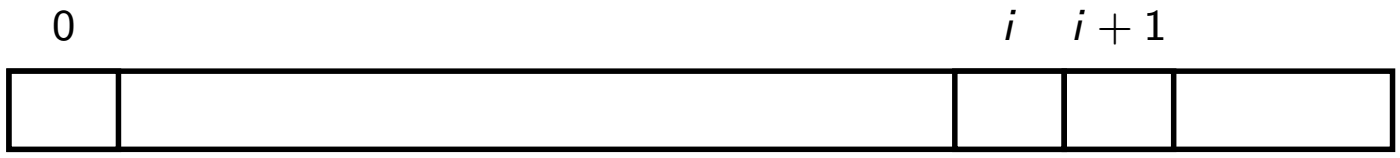
- Let u be a border of $P[0..i]$ such that $|u| < s(i)$
- Then u is both a prefix and a suffix of $P[0..s(i)-1]$
- $u \neq P[0..s(i)-1]$, so u is a border of $P[0..s(i)-1]$

Enumerating borders

Corollary

All borders of $P[0..i]$ can be enumerated by taking the longest border b_1 of $P[0..i]$, then the longest border b_2 of b_1, then the longest border b_3 of b_2, . . . , and so on.

Computing $s(i+1)$



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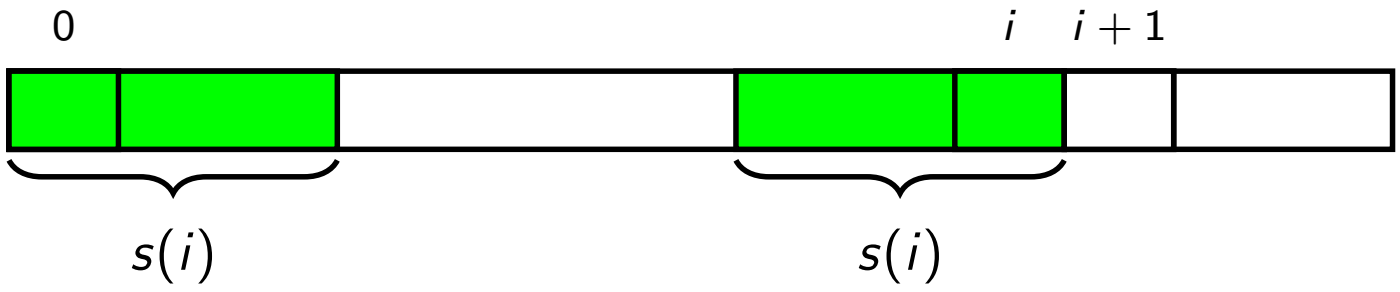

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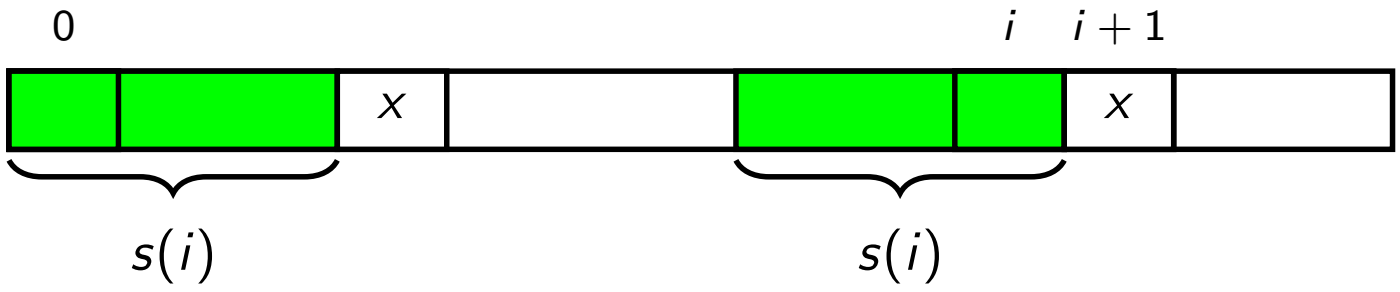

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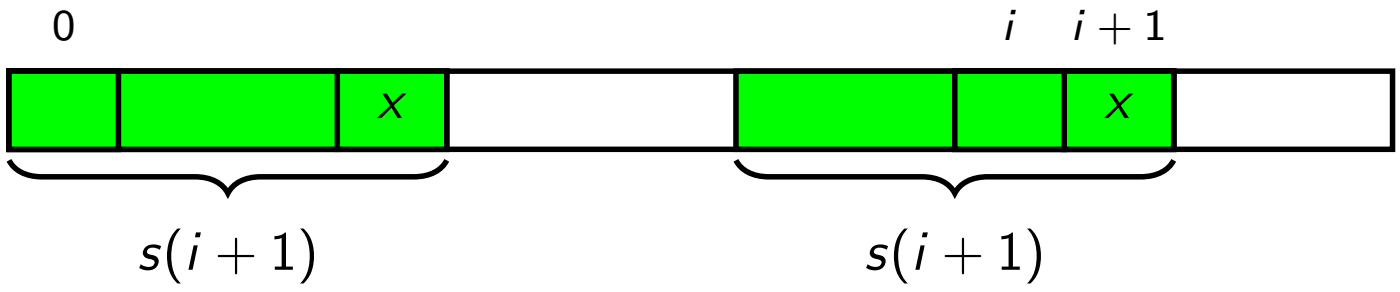
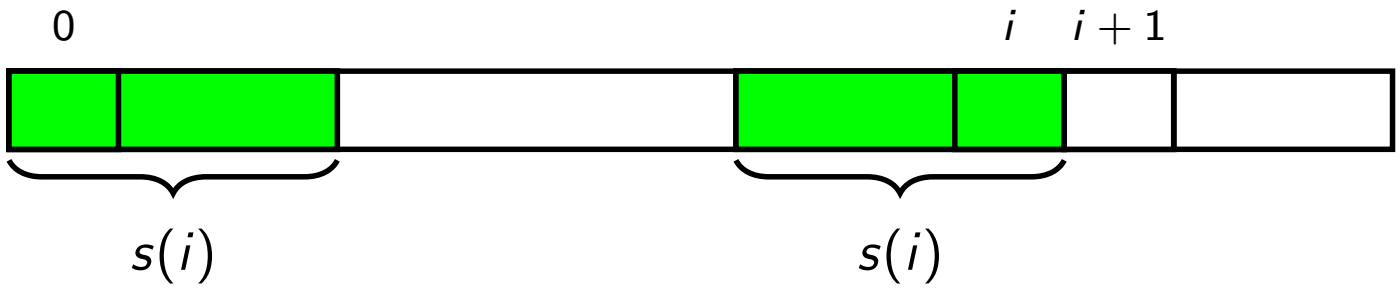


$$s(i+1) = s(i) + 1$$

Computing $s(i+1)$



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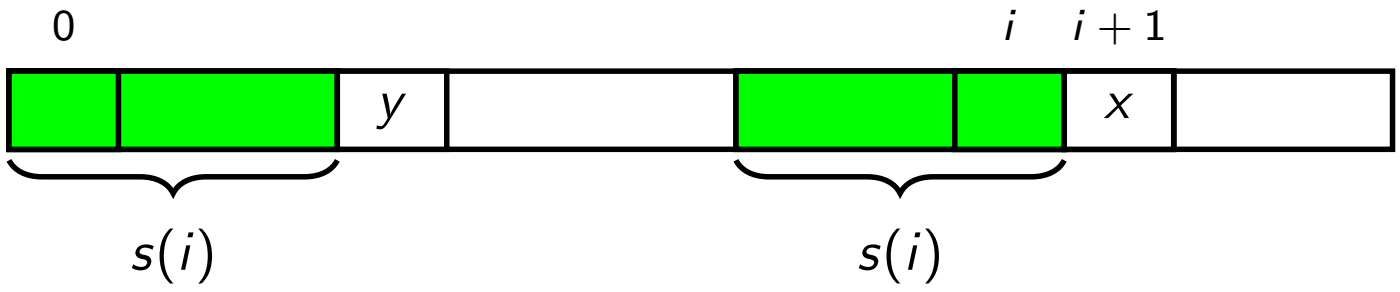

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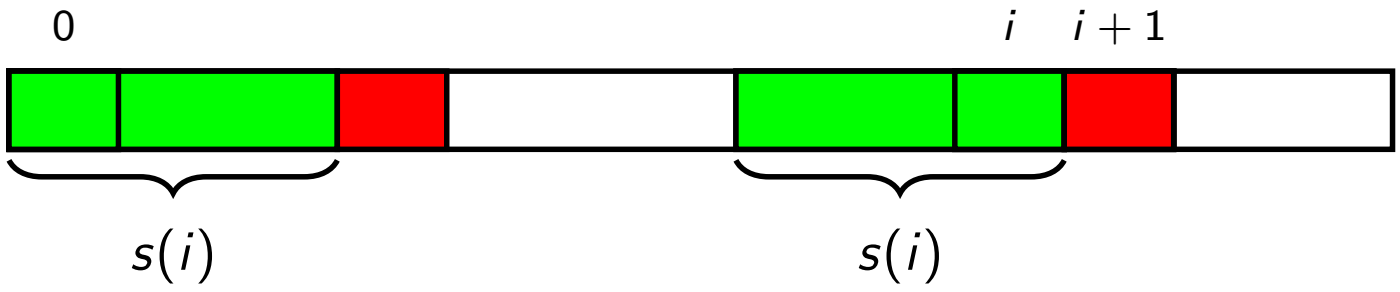

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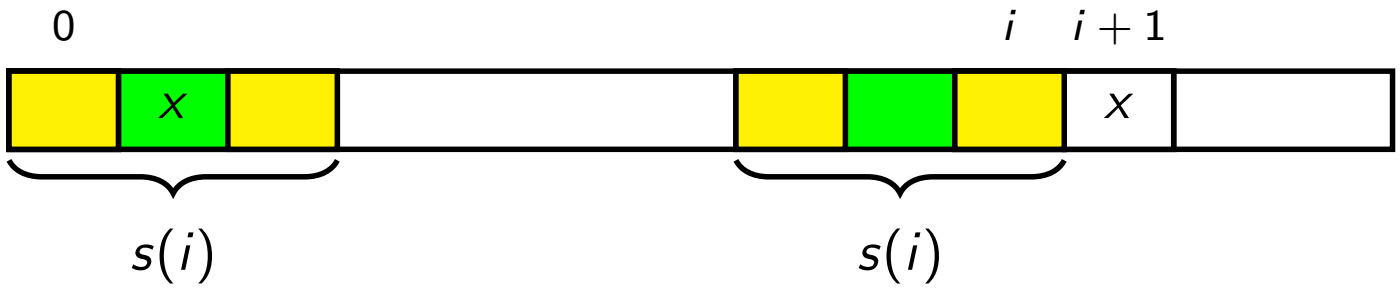

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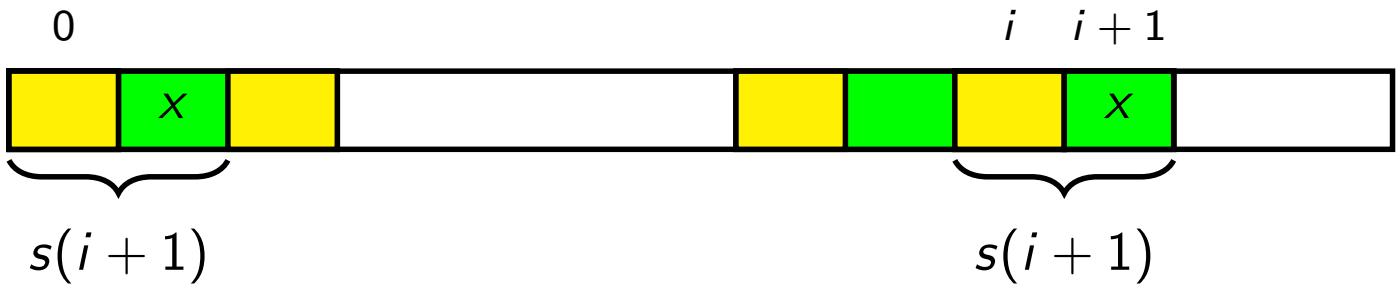

$$s(i+1) = |\textit{some border of } P[0..s(i)-1]| + 1$$

- Now you know lots of properties of prefix function

- Now you know lots of properties of prefix function
- But how to compute all of its values??

Outline

1. Exact Pattern Matching
2. Safe Shift
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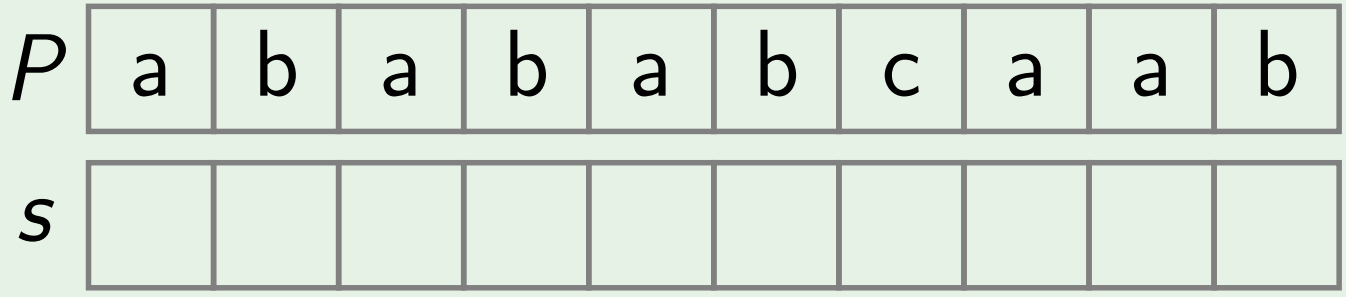
7. Knuth-Morris-Pratt Algorithm

Example



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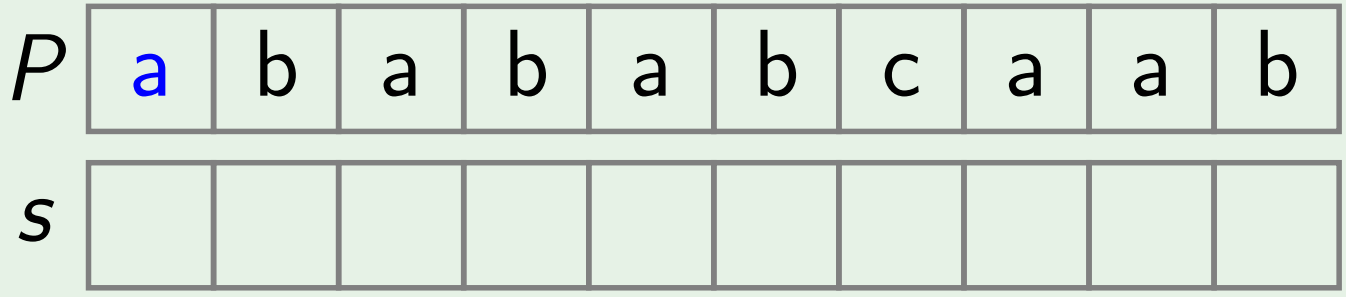

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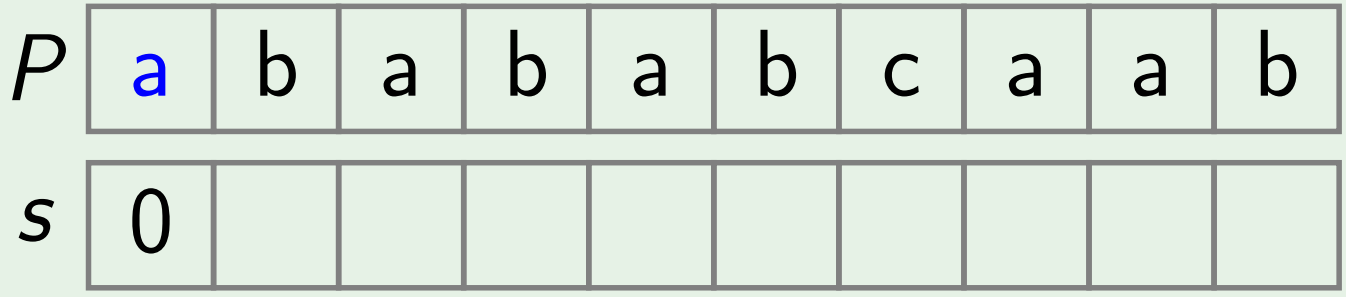

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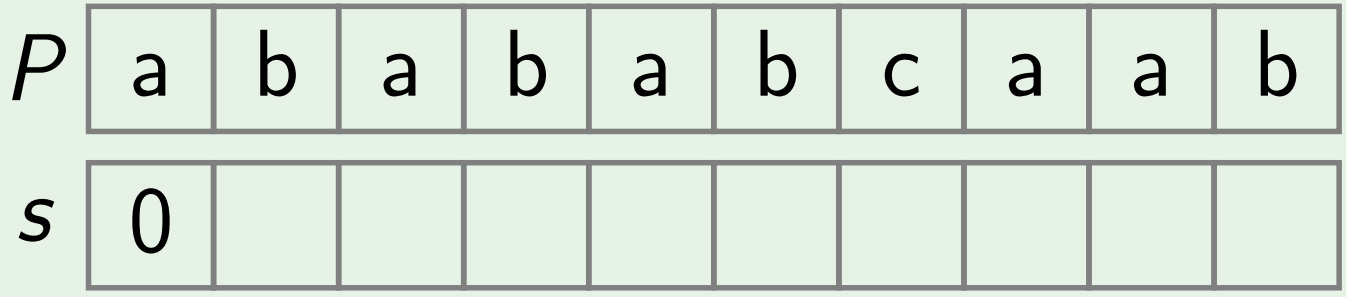

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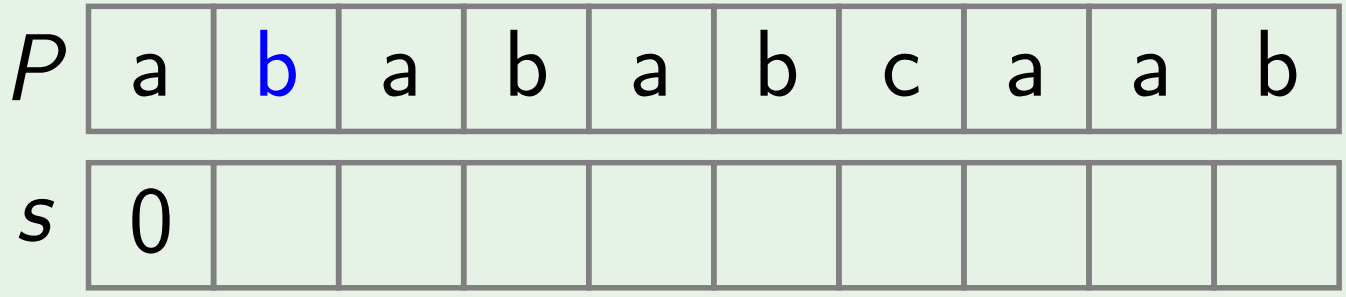

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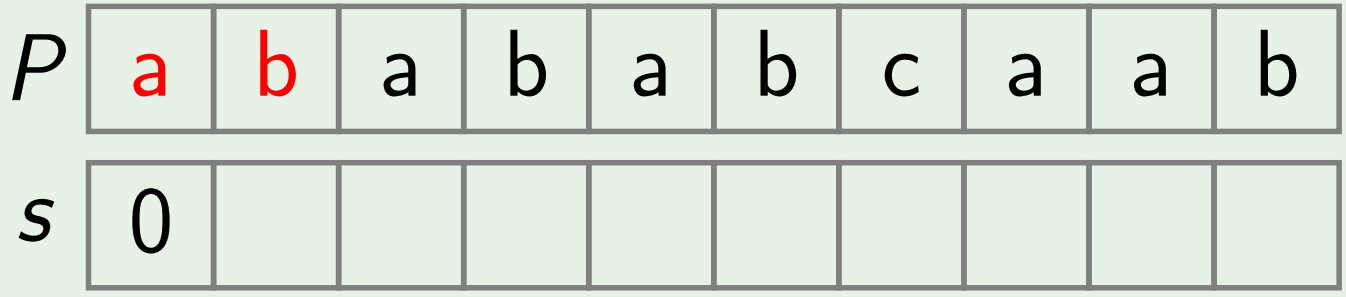

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0								

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P	a	b	a	b	a	b	c	a	a	b
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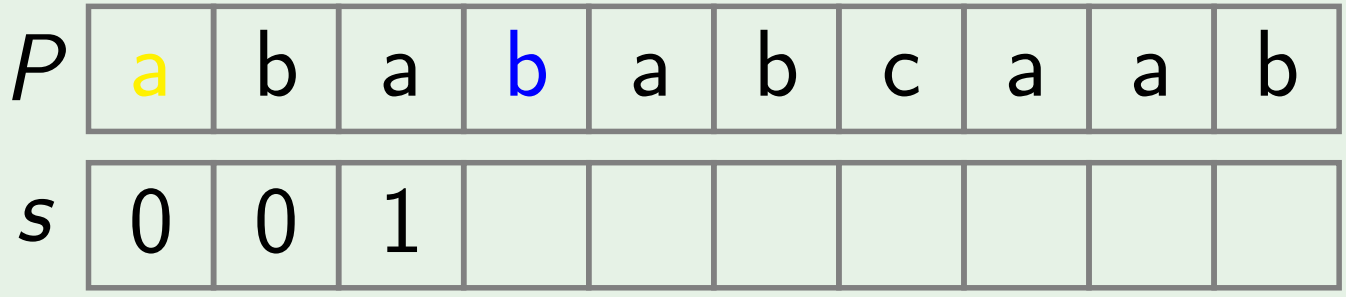
Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1							

Example



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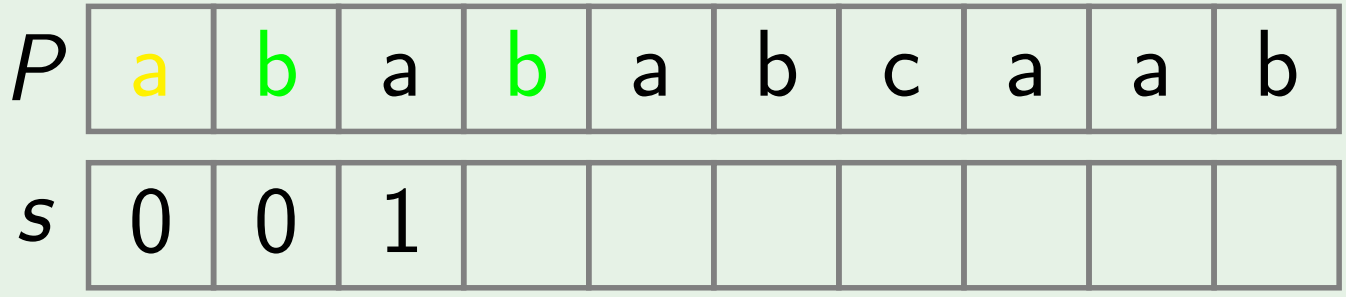

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4. Computing Prefix Function
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7. Knuth-Morris-Pratt Algorithm

ComputePrefixFunction(P)

```
s ← array of integers of length |P|
s[0] ← 0, border ← 0
for i from 1 to |P| − 1:
    while (border > 0) and (P[i] ≠ P[border]):
        border ← s[border − 1]
    if P[i] == P[border]:
        border ← border + 1
    else:
        border ← 0
    s[i] ← border
return s
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Outline

1. Exact Pattern Matching
2. Safe Shift
3. Prefix Function
4. Computing Prefix Function
5. Implementation
6. Analysis
7. Knuth-Morris-Pratt Algorithm

Lemma

The running time of `ComputePrefixFunction` is $O(|P|)$.

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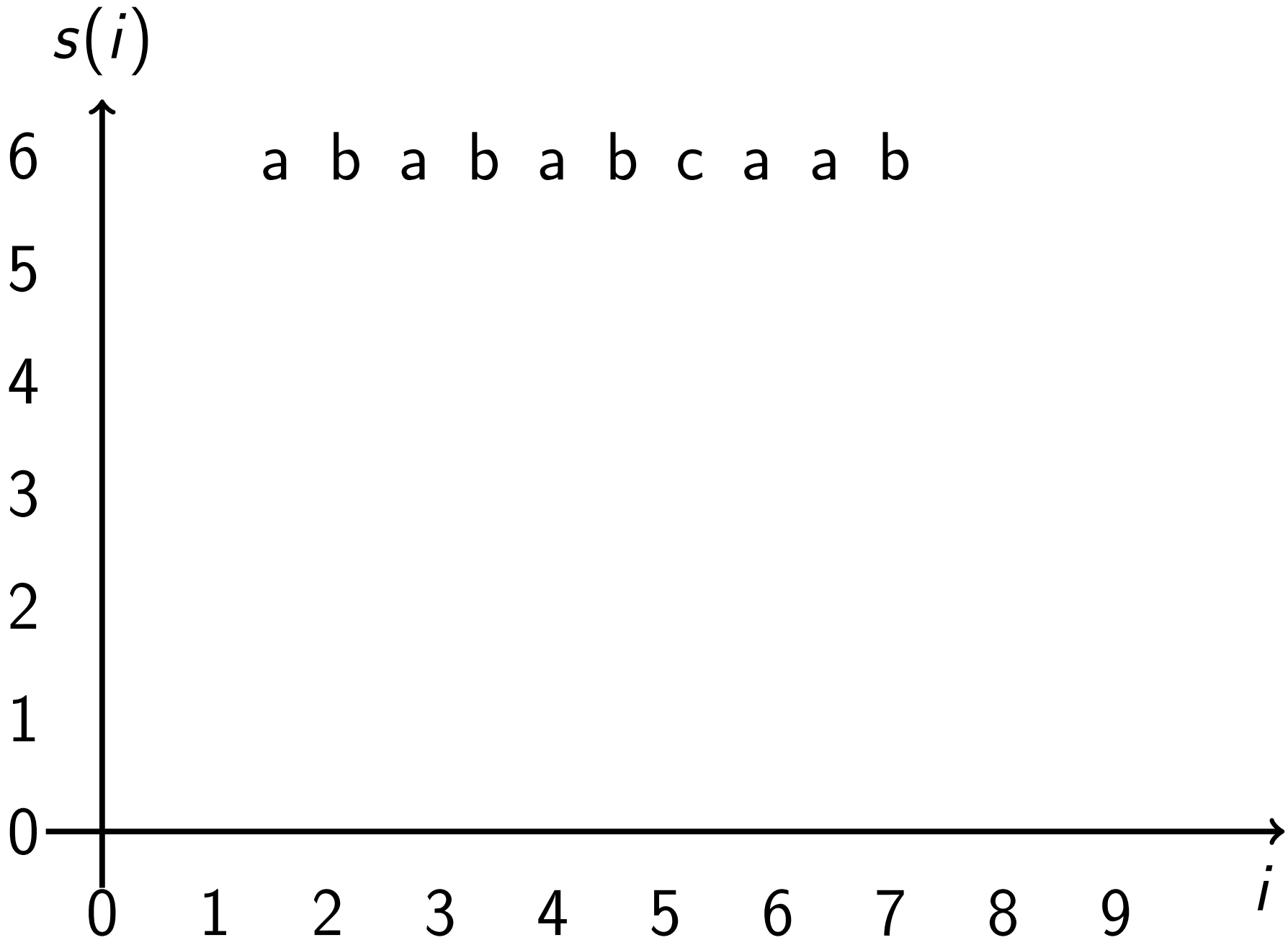
- Now we will bound the number of the while loop iterations by $O(|P|)$


$s(i)$
6
5
4
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2
1
0
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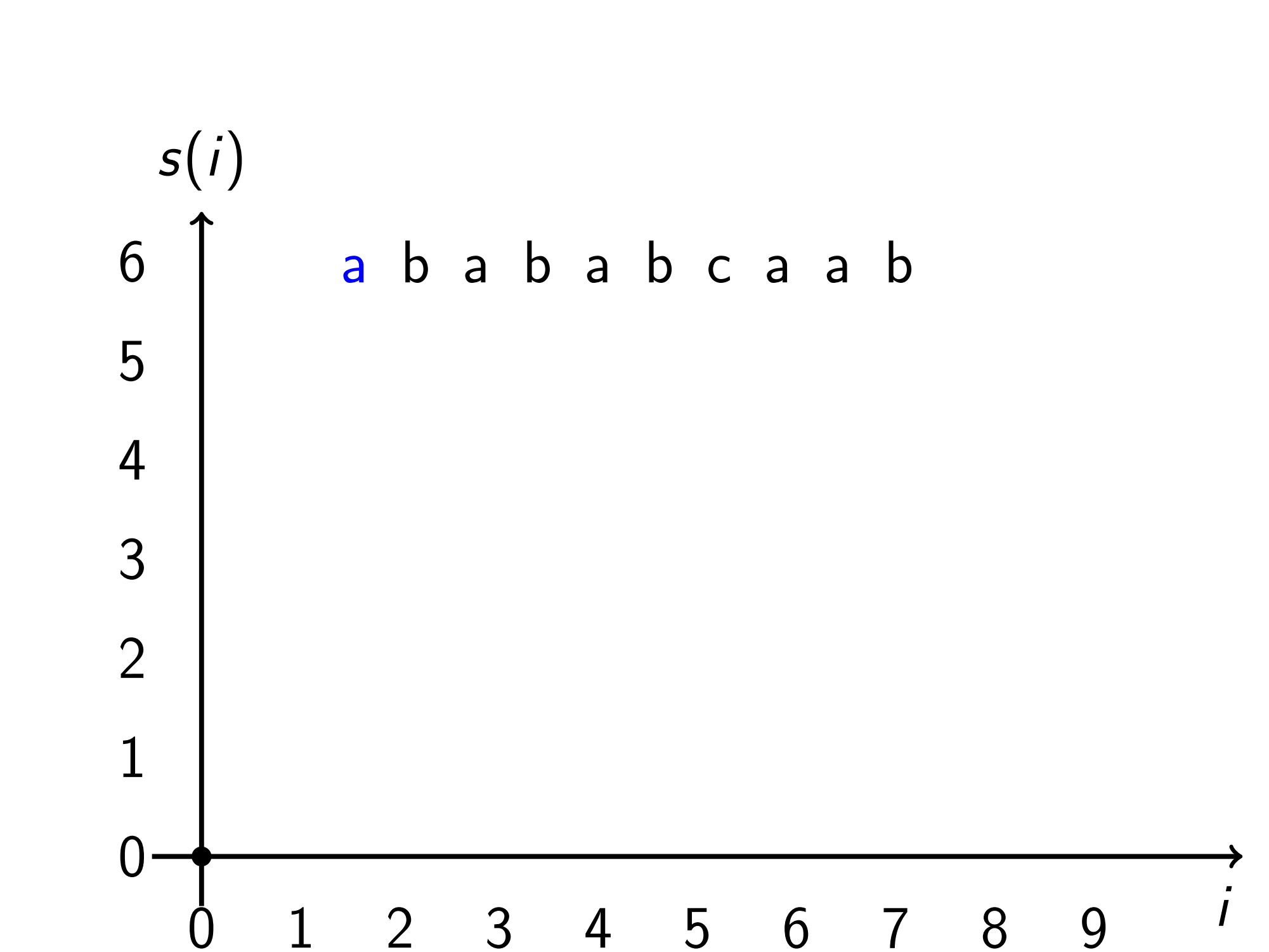
0 1 2 3 4 5 6 7 8 9
i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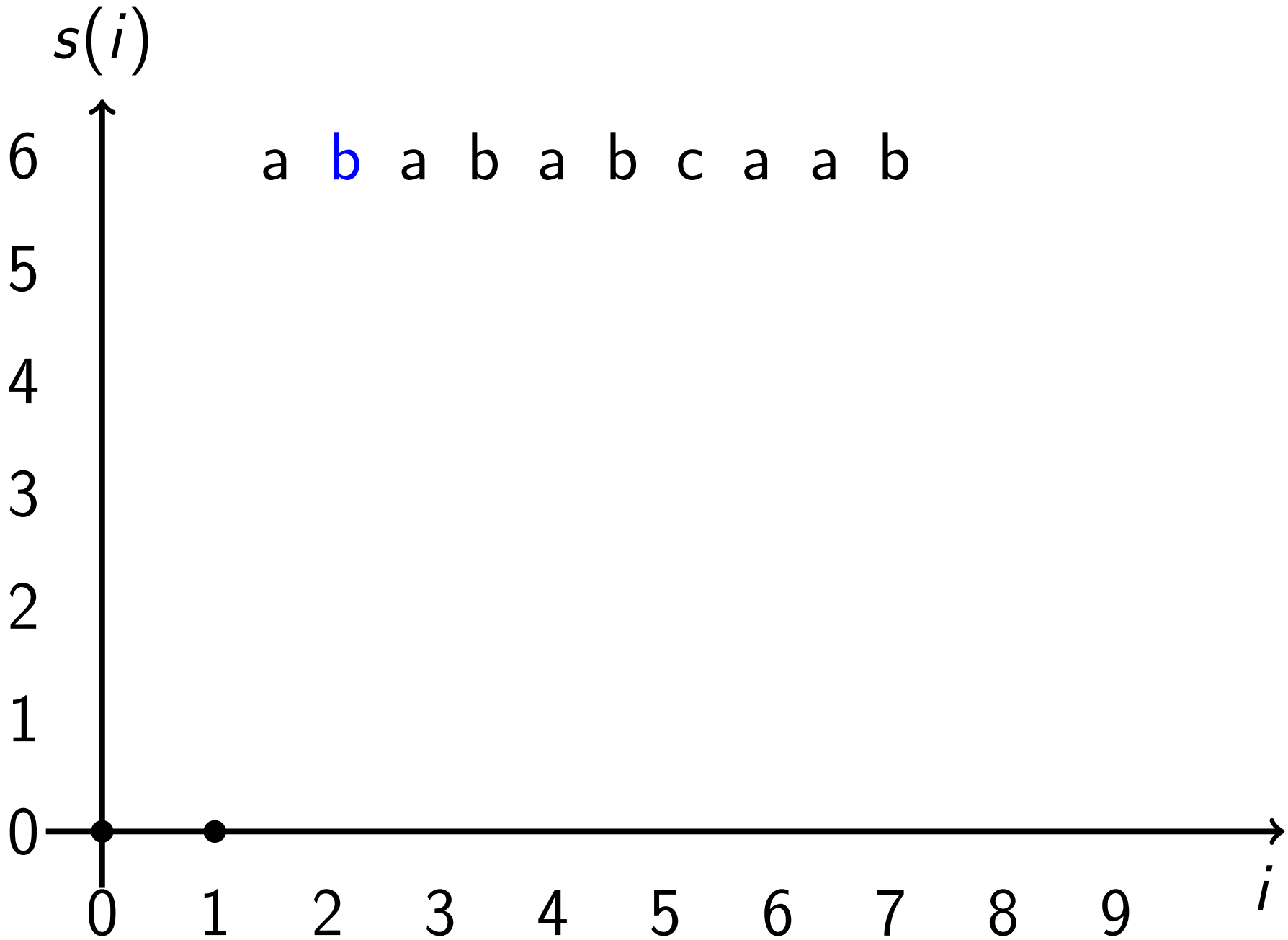
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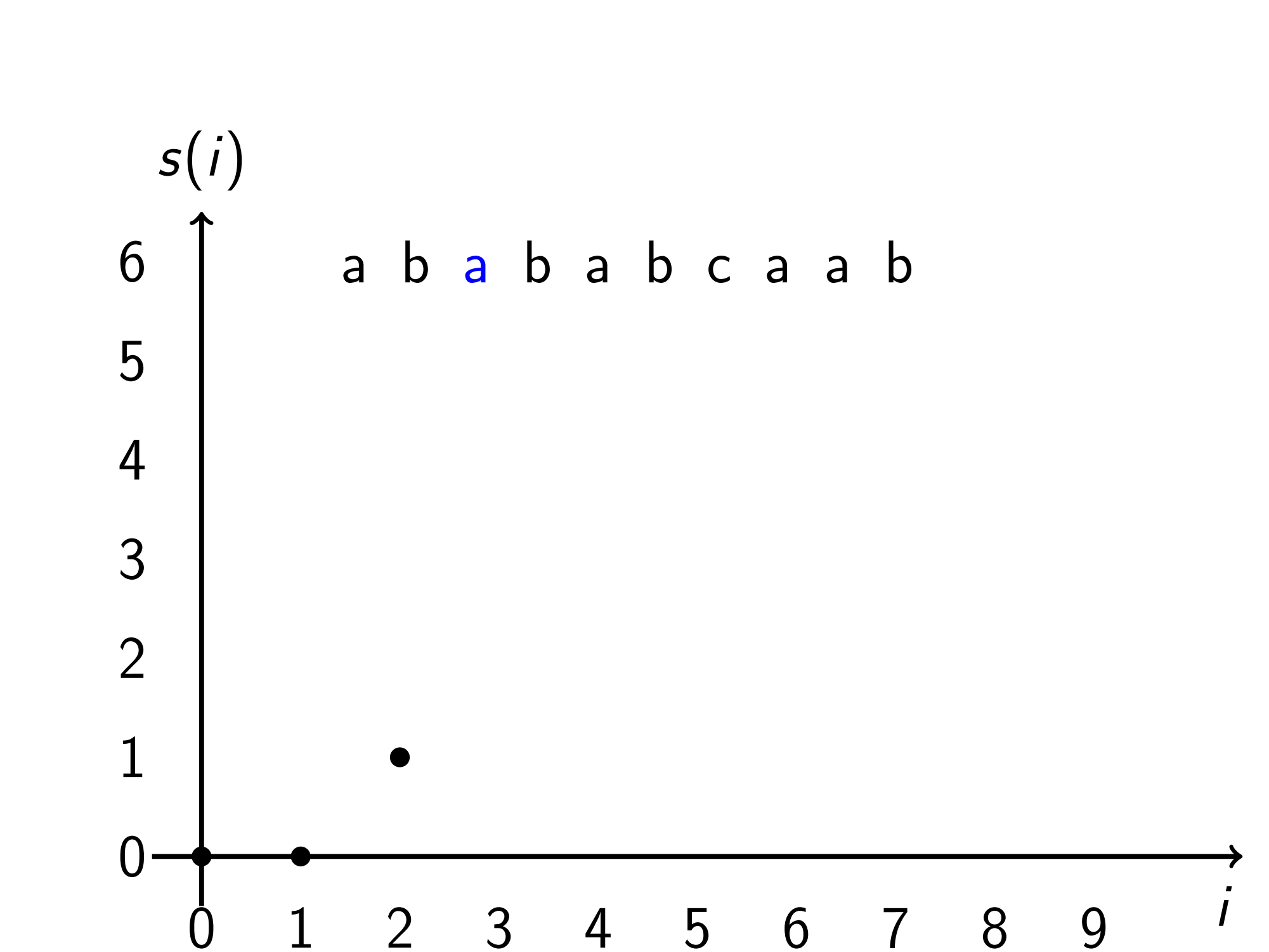
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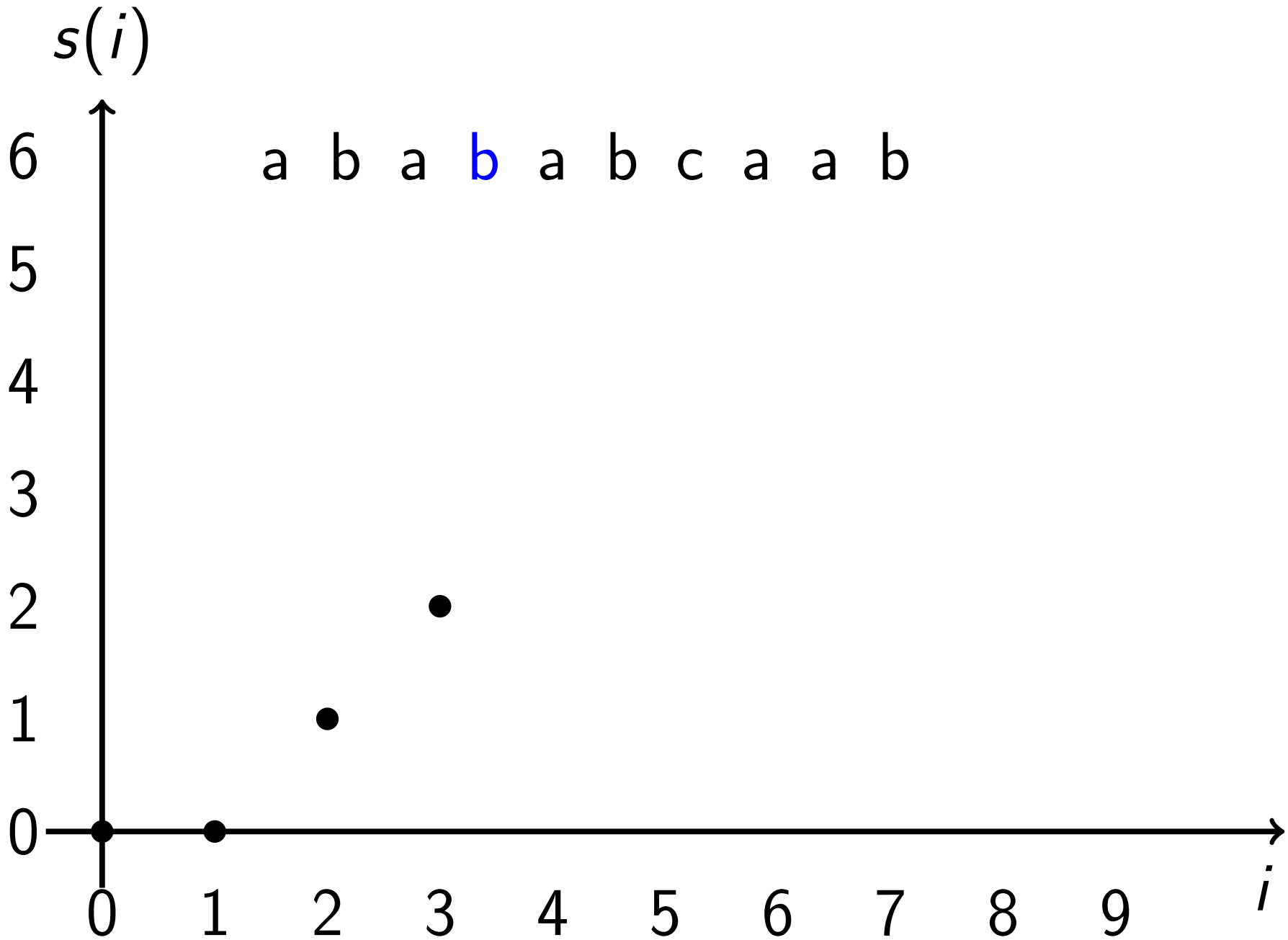
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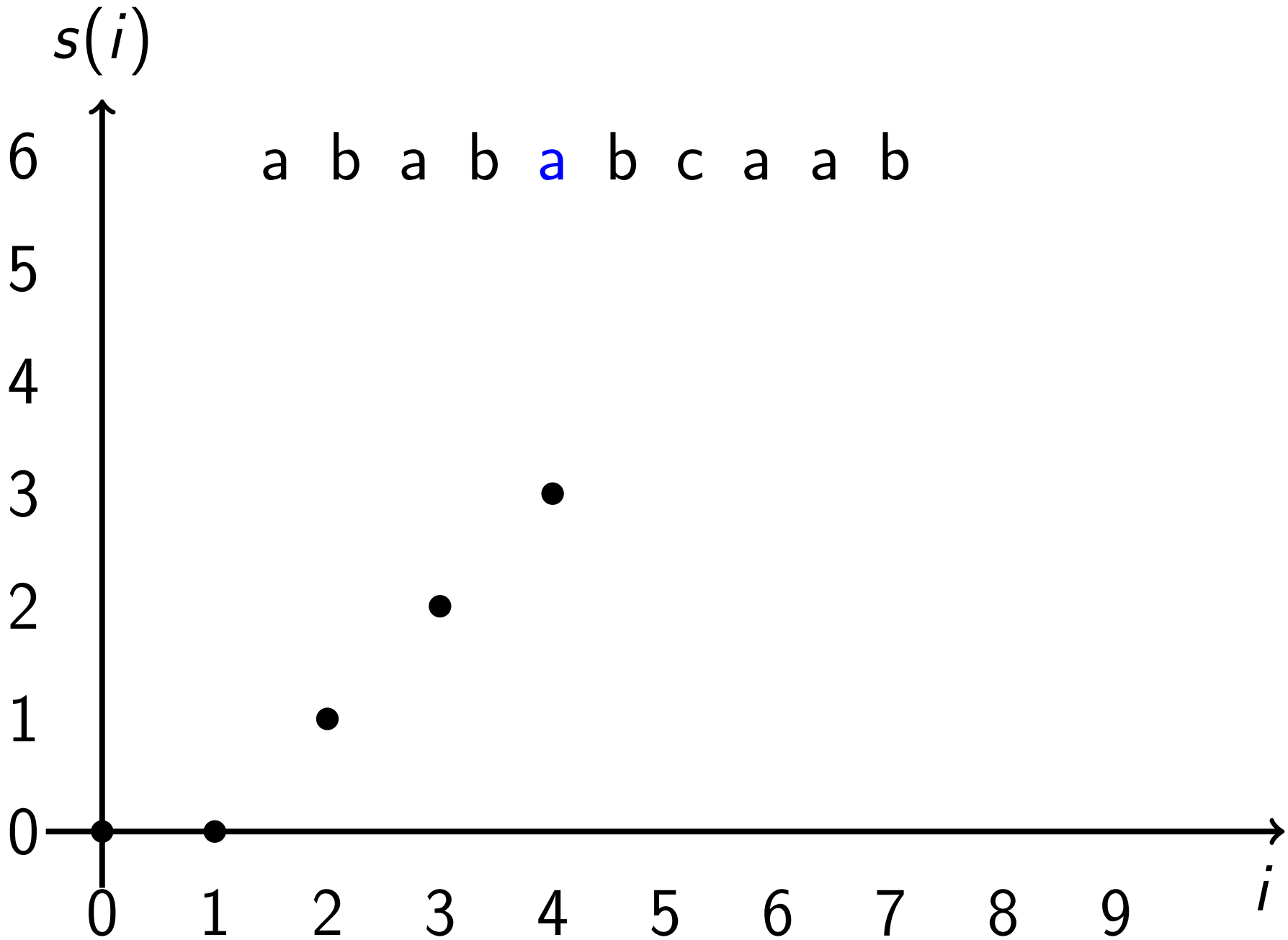
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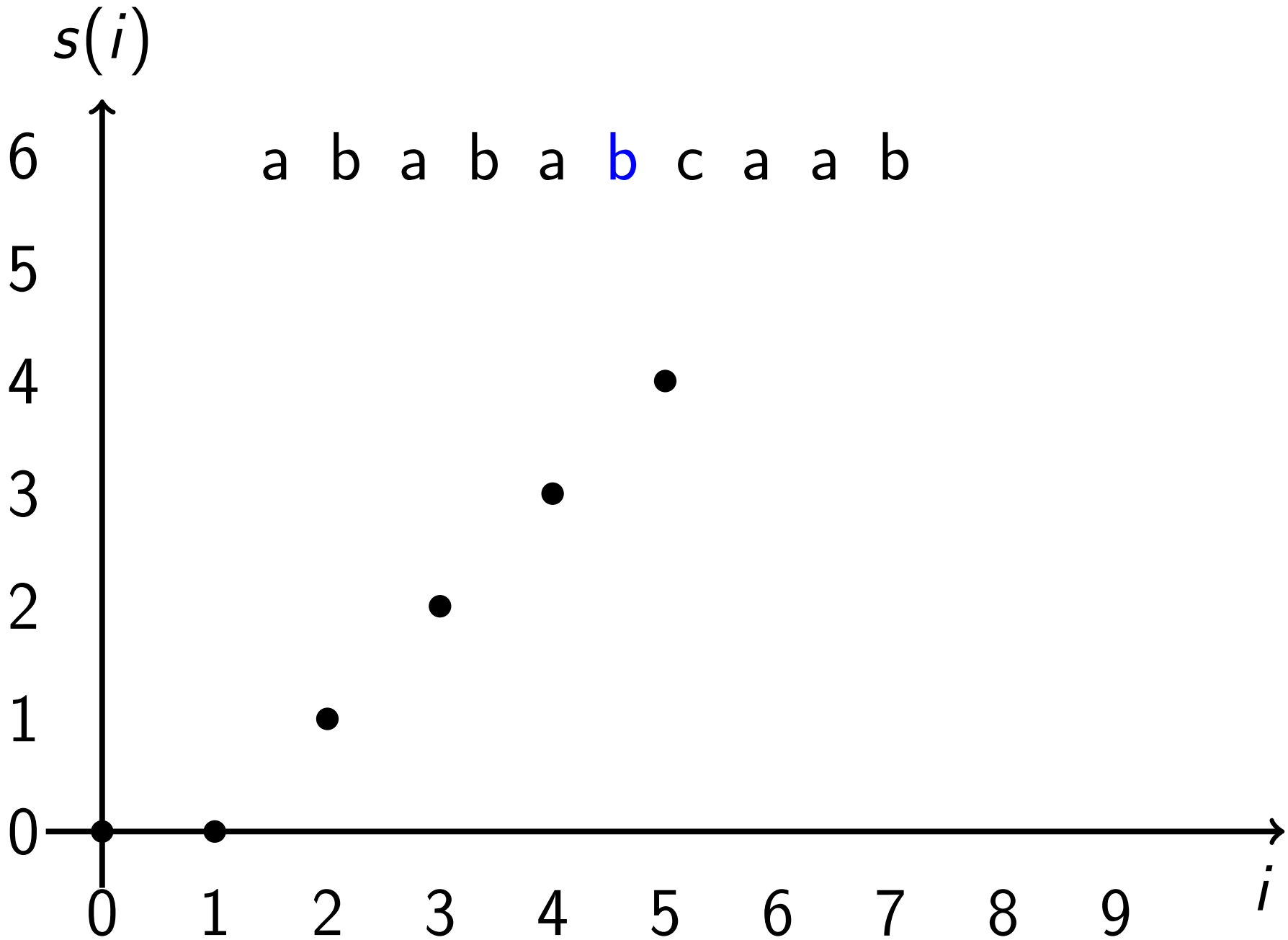
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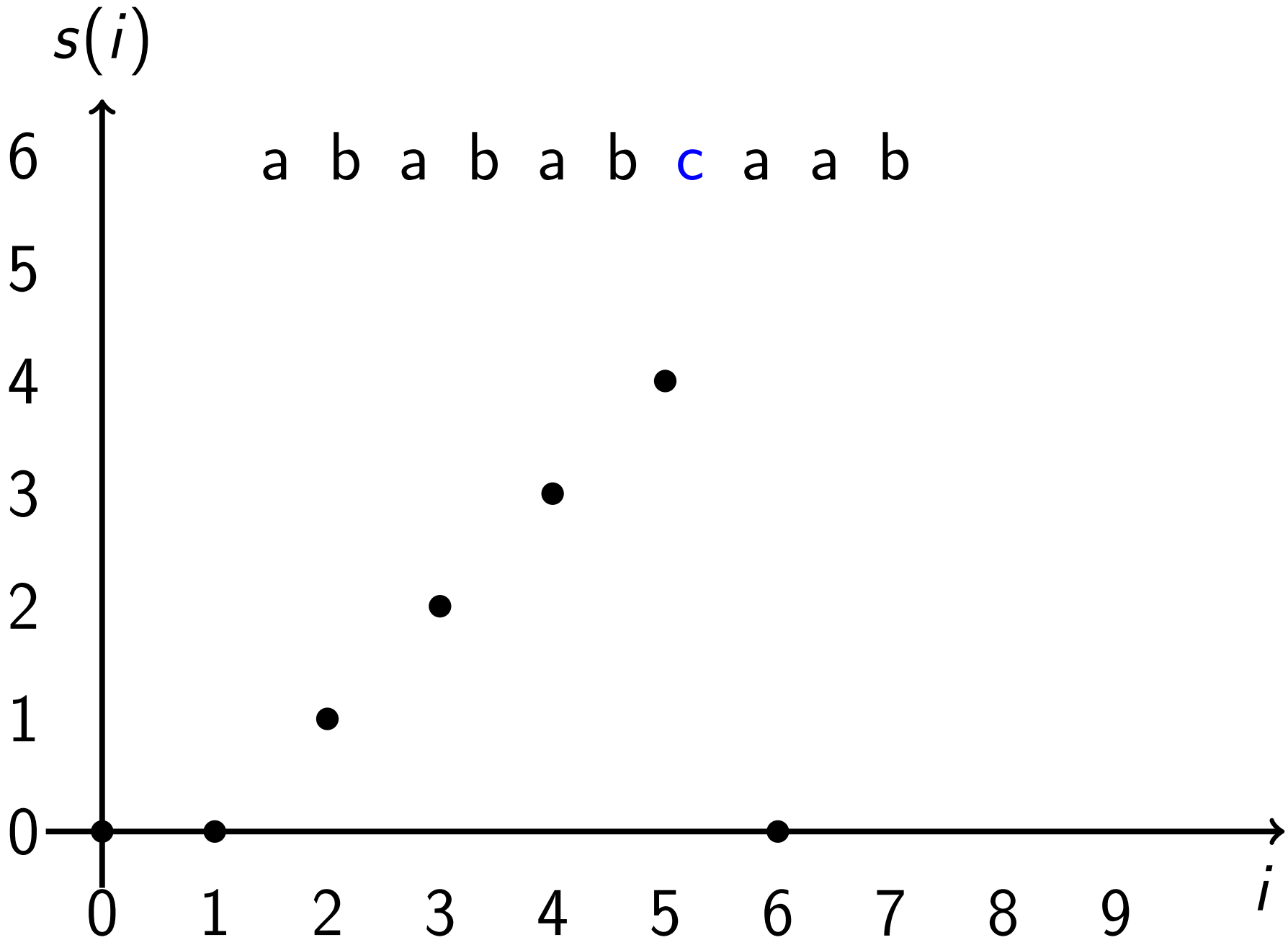
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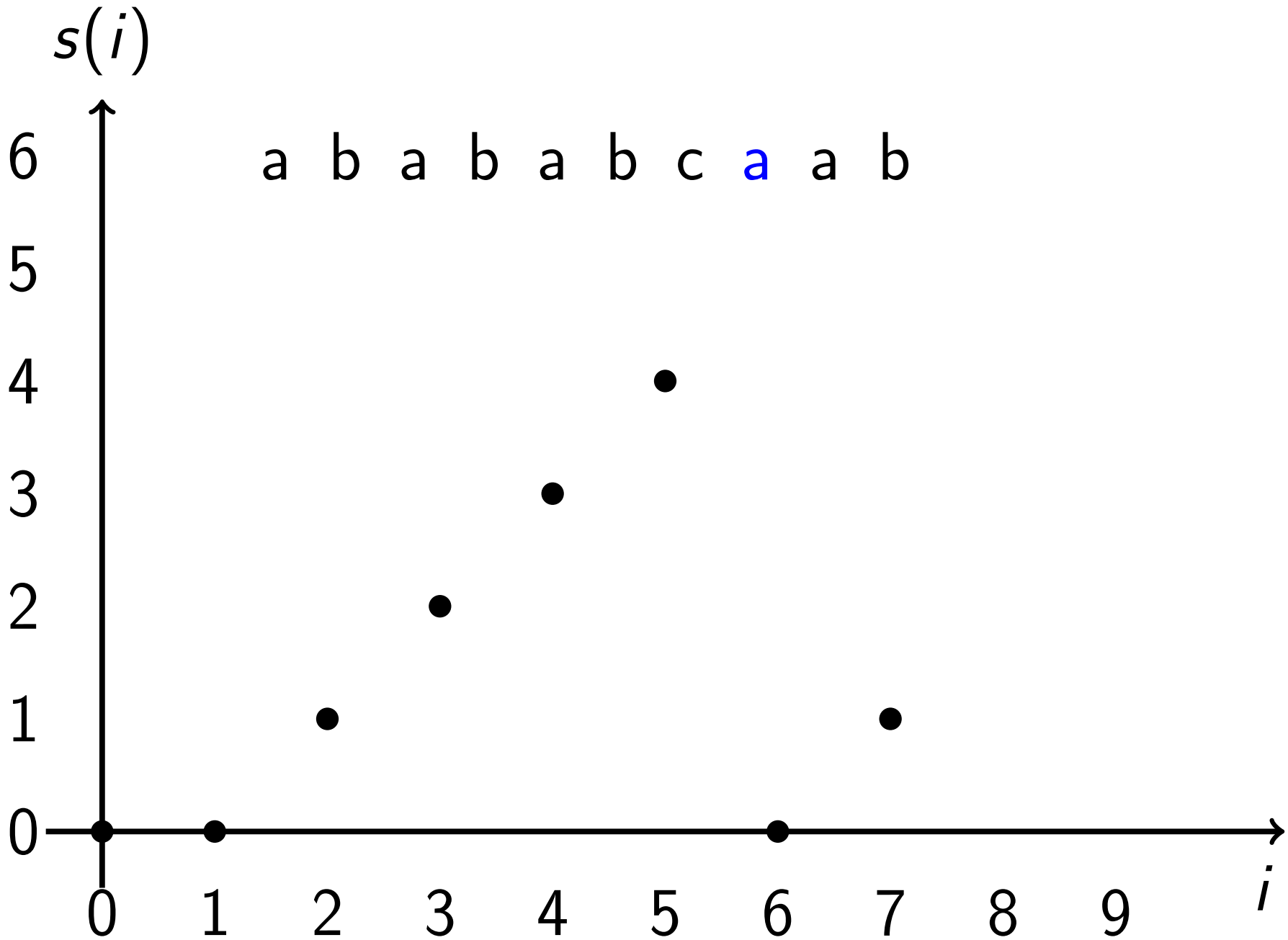
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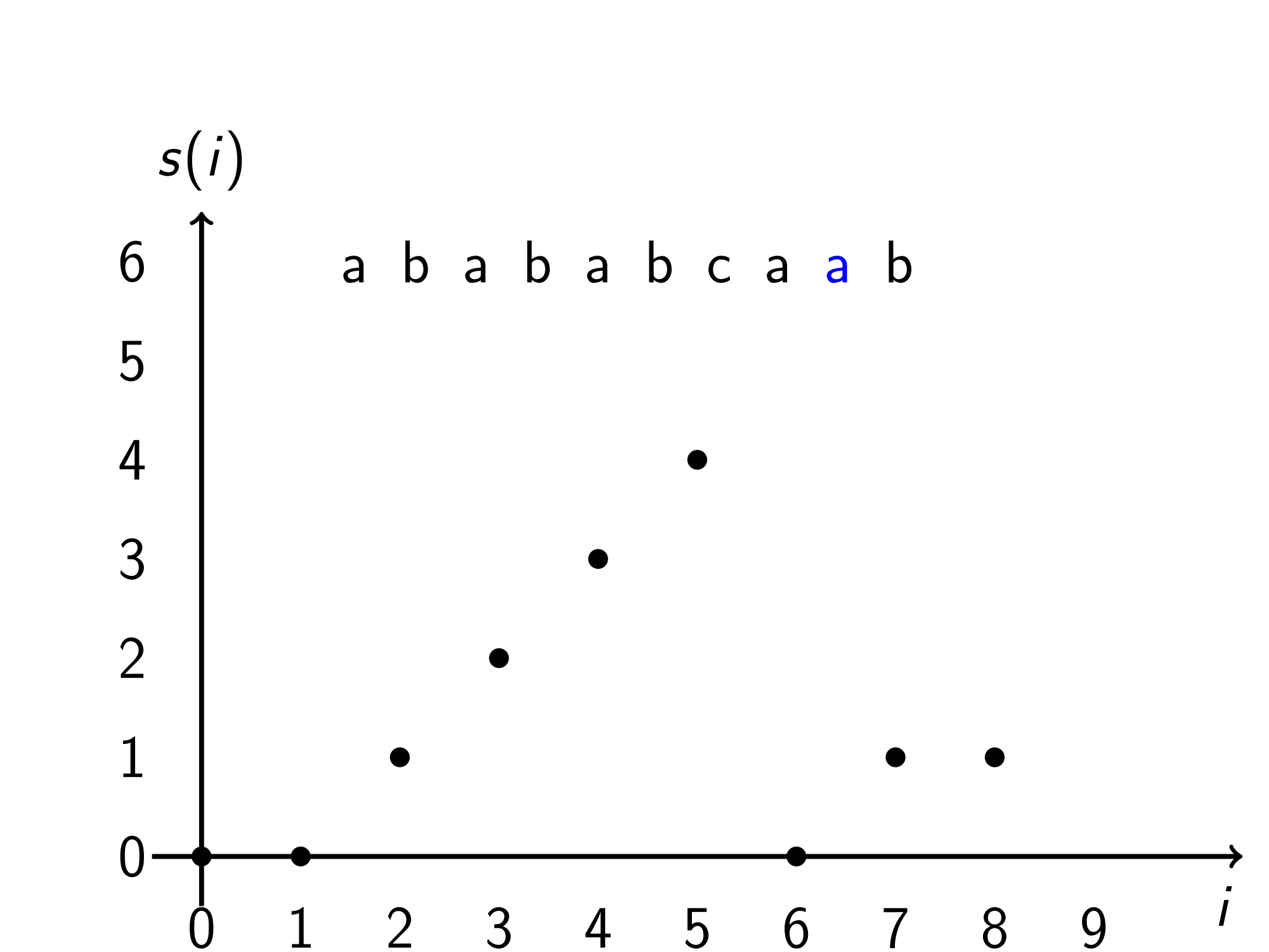
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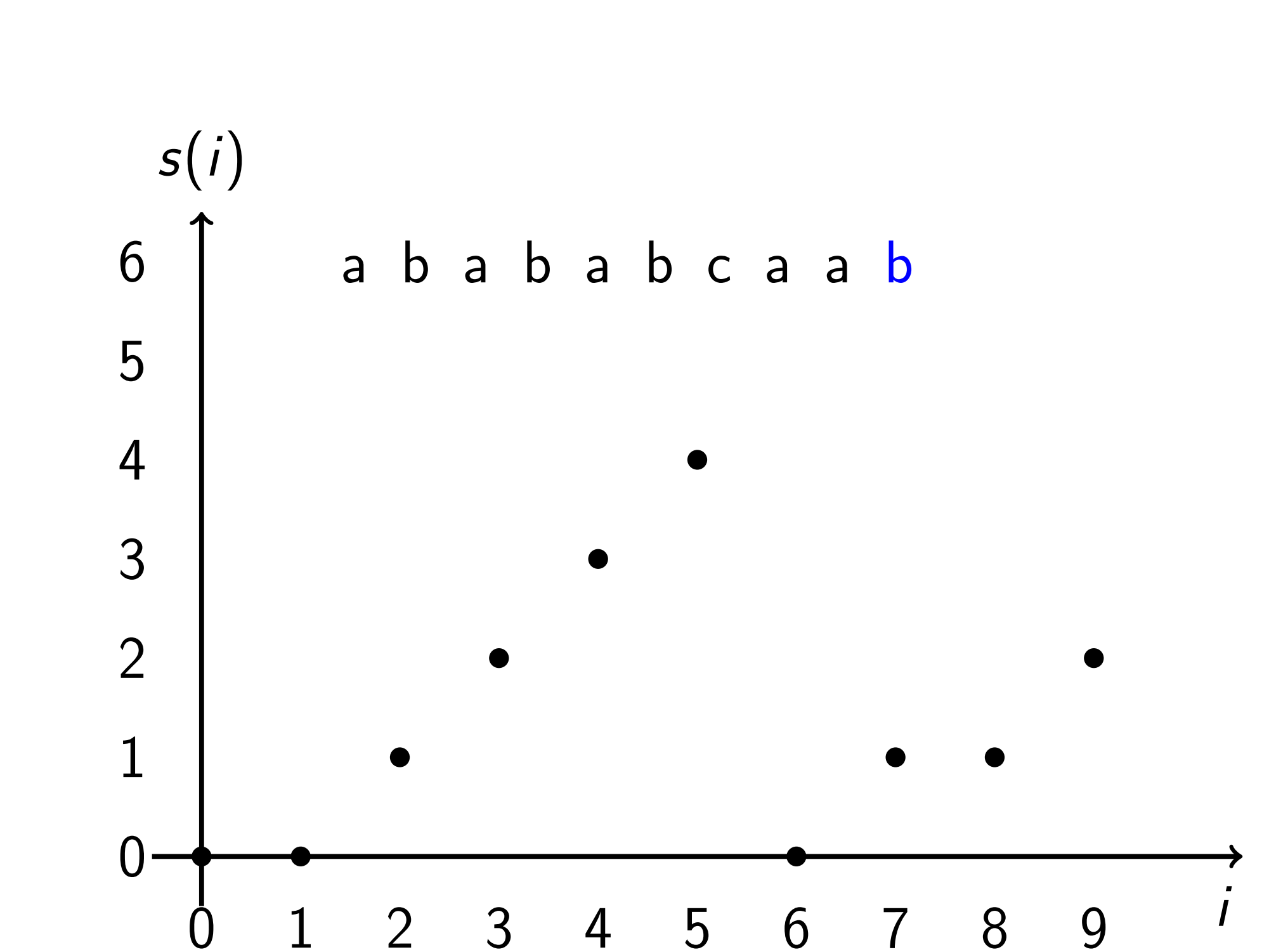
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(continued)

- *border* can increase at most by 1 on each iteration of the for loop

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- *border* is decreased at least by 1 on each iteration of the while loop
- $border \geq 0$
- So there are $O(|P|)$ iterations of the while loop

- Now you know how to compute prefix function in linear time

- Now you know how to compute prefix function in linear time
- But how to find pattern in text??

Outline

1. Exact Pattern Matching
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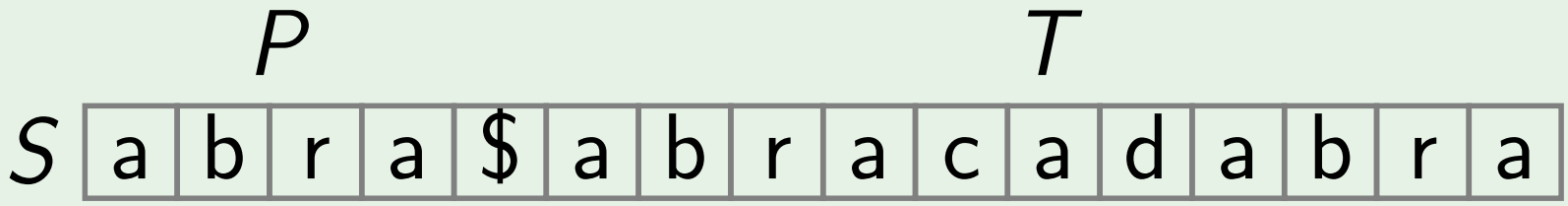
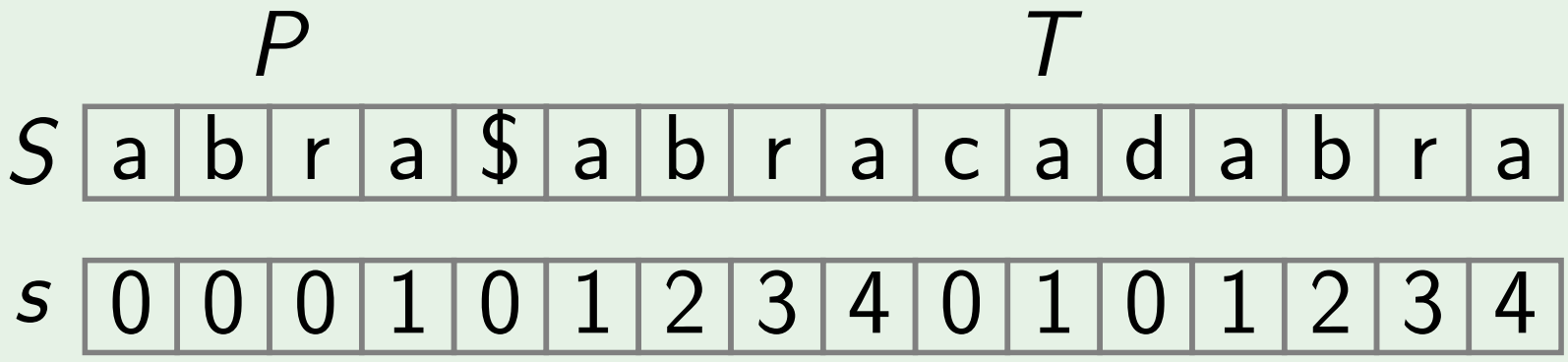
Algorithm



To search for pattern P in text T:

- Create new string $S = P + \text{'\$'} + T$, where '\$' is a special character absent from both P and T

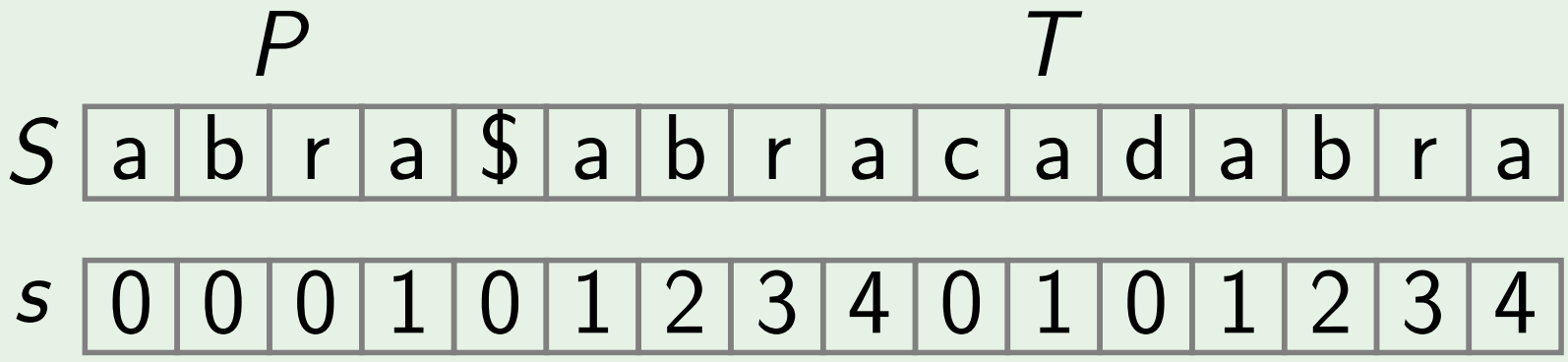
Algorithm



To search for pattern P in text T:

- Compute prefix function s for string S

Algorithm



To search for pattern P in text T:

- Compute prefix function s for string S
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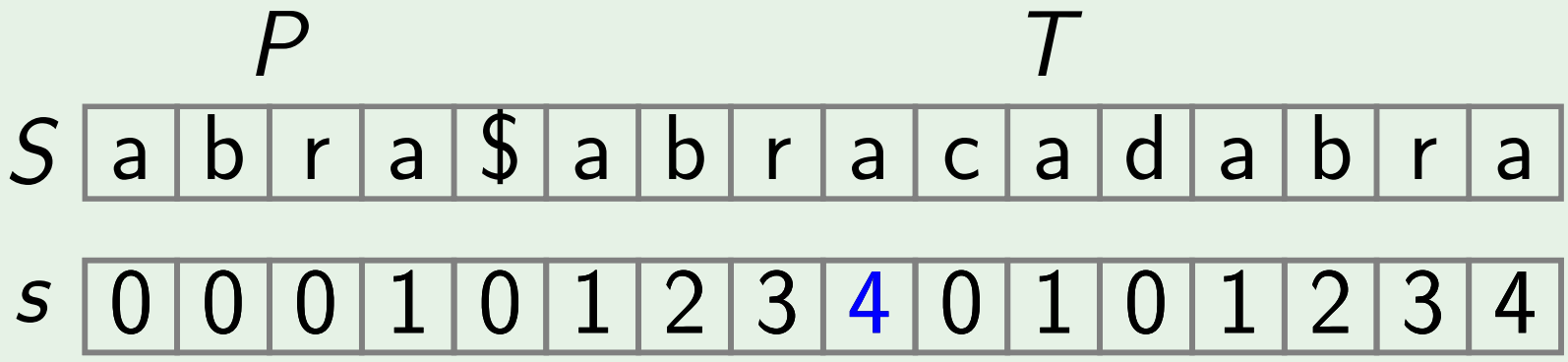
Algorithm



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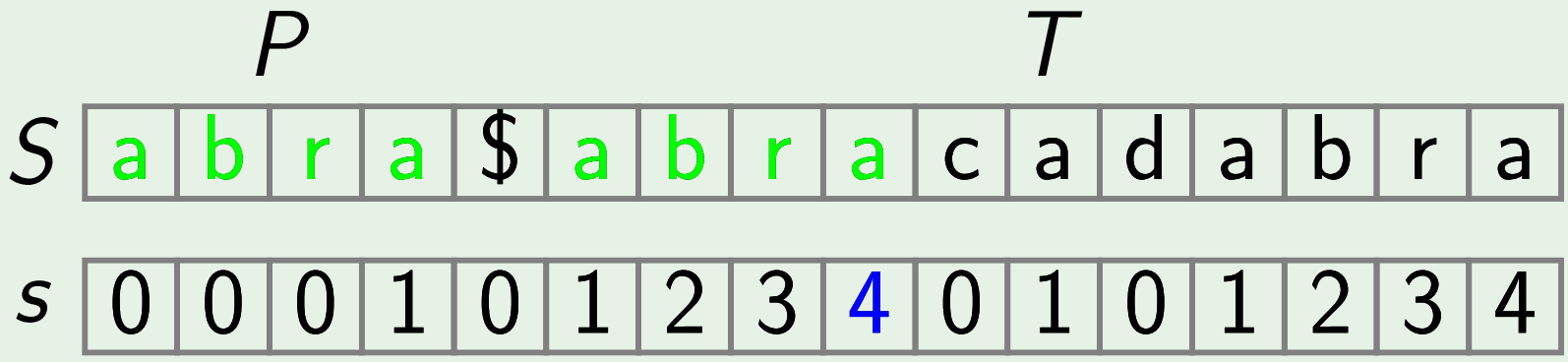

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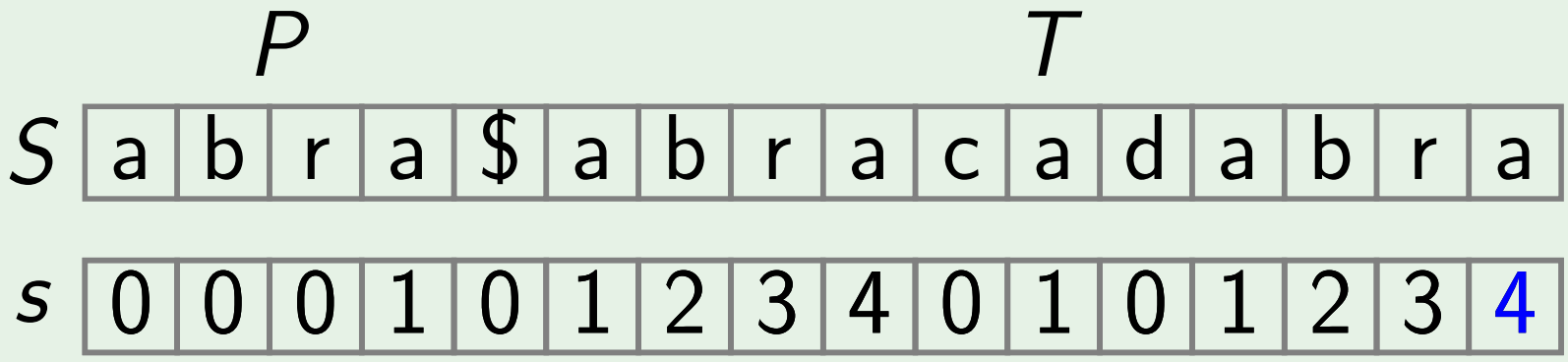

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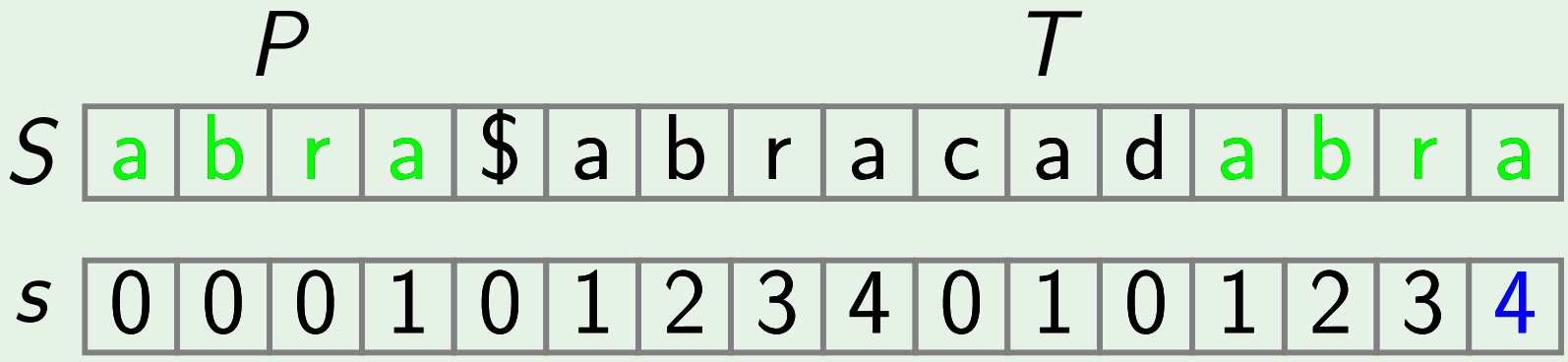

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Explanation

- For all i, $s(i) \leq |P|$ because of the special character '\$'
- If $i > |P|$ and $s(i) = |P|$, then $P = S[0..|P| - 1] = S[i - |P| + 1..i] = T[i - 2|P|..i - |P| - 1]$
- If $s(i) < |P|$, no full occurrence of $|P|$ ends in position i

FindAllOccurrences(P, T)

```
S ← P + '$' + T
s ← ComputePrefixFunction(S)
result ← empty list
for i from |P| + 1 to |S| − 1:
    if s[i] == |P|:
        result.Append(i − 2|P|)
return result
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FindAllOccurrences(P, T)

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Lemma

The running time of Knuth-Morris-Pratt algorithm is $O(|P| + |T|)$.

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Proof

- Building string S is $O(|P| + |T|)$
- Computing prefix function is $O(|S|) = O(|P| + |T|)$
- The for loop runs $O(|S|) = O(|P| + |T|)$ iterations

Conclusion

- Can search pattern in text in linear time
- Can compute prefix function of a string in linear time
- Can enumerate all borders of a string