A1

Hunds Rules applied to $2p^2$

$S = 1$

$L = 1$

Shell less than half filled $\Rightarrow$ $J = |L - S| = 0$.

$\boxed{{}^3P_0}$.

Lowest energy excited states come from

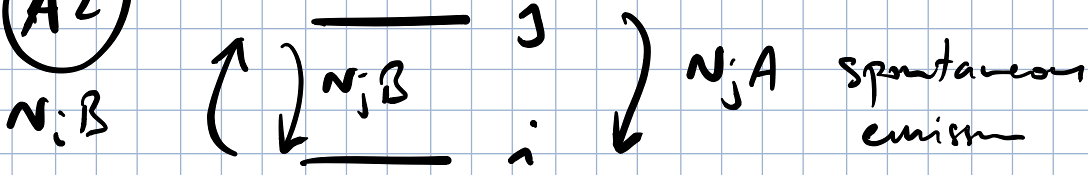
$L=1$, $S=1$ added in different ways

(spin-orbit energy scale)

$\Rightarrow$ $J = 1, 2$.

$\boxed{{}^3P_1 \,,\, {}^3P_2}$

A2



Steady state requires:

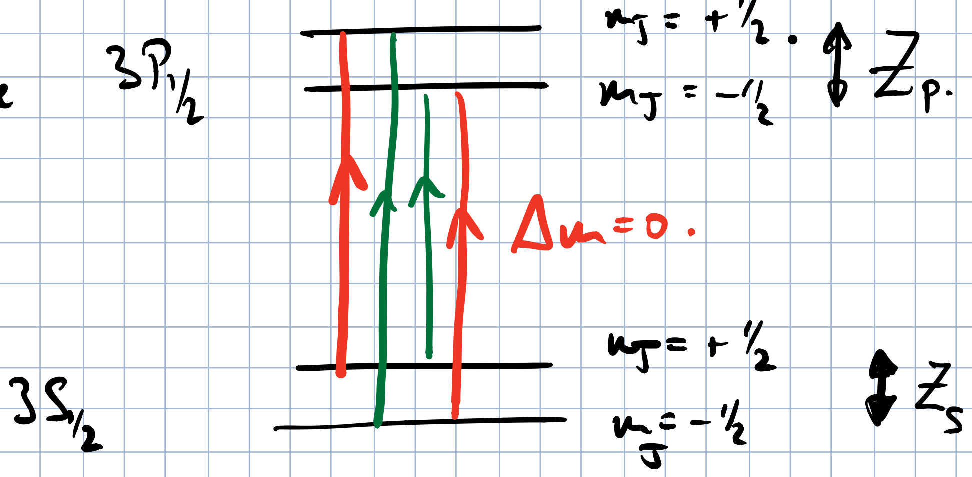
$$N_i B = N_j (A+B)$$

A/B = (N_i-N_j)/N_j

Ratio of Spontaneous/Stimulated = (N_j A)/(N_j B) = A/B = (N_i/N_j-1)

A3

D1 line



(i) Light polarized parallel to B induces $\Delta m = 0$ transitions

$\Rightarrow$ Two lines, split by $|Z_S - Z_p|$

(ii) Light polarized perpendicular to B induces $\Delta m = \pm 1$ transitions.

$\Rightarrow$ Two lines split by $Z_S + Z_p$

B4 (a)

- By taking a trial wavefunction $\psi(x)$ which depends on some parameter(s), one can find an upper bound on the groundstate energy by minimizing $\frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$.

- The associated wavefunction is an approximation to the exact groundstate

- The accuracy improves with increasing variational freedom.

Proof: Write $|\psi\rangle = \sum_\lambda \psi_\lambda |\lambda\rangle$

where $|\lambda\rangle$ are exact eigenstates of $\hat{H}$.

$$\frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\sum_\lambda |\psi_\lambda|^2 E_\lambda}{\sum_\lambda |\psi_\lambda|^2} \geq \frac{\sum_\lambda |\psi_\lambda|^2 E_0}{\sum_\lambda |\psi_\lambda|^2}$$

$E_\lambda \geq E_0$.

groundstate

$$\Rightarrow \quad \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} \geq \varepsilon_0.$$

(b) $$\frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\int \varphi_0^*(\lambda x)\, H\, \varphi_0(\lambda x)\, dx}{\int |\varphi_0(\lambda x)|^2\, dx}$$

$$= \frac{\int \varphi_0^*(\lambda x) \frac{-\hbar^2}{2m} \frac{\partial^2}{dx^2} \varphi_0(\lambda x)\, dx}{\int |\varphi_0(\lambda x)|^2 dx}$$

$$+ \frac{\int \varphi_0^*(\lambda x)\, a x^2\, \varphi_0(\lambda x)\, dx}{\int |\varphi_0(\lambda x)|^2 dx}$$

Rescale to $x' = \lambda x$

$$\frac{-\frac{\hbar^2}{2m} \lambda^2 \int \varphi_0'(x') \frac{d^2}{dx'^2} \varphi_0(x')\, dx' \frac{1}{\lambda}.}{\int |\varphi_0(x')|^2 dx'\, {}^1\!/\!_\lambda}$$

$$+ \frac{1}{\lambda^2} \frac{\int \varphi_0(x')\, a x'^2\, \varphi_0(x')\, dx'\ 1/\lambda}{\int |\varphi_0(x')|^2 dx'\ 1/\lambda}$$

$$+ \lambda^2 \langle T \rangle_0 + \frac{1}{\lambda^2} \langle V \rangle_0 .$$

Minimize : $2\lambda \langle T \rangle_0 - \frac{2}{\lambda} \langle V \rangle_0 = 0.$

But minimum is at $\lambda = 1$.

$$\Rightarrow \boxed{\langle T \rangle_0 = \langle V \rangle_0 .}$$

Following the same logic:

(c) $$\frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle} = \lambda^2 \langle T \rangle_0 + \lambda^p \langle \tilde{V} \rangle_0$$

$$\Rightarrow \quad 2\lambda \langle T \rangle_0 + p \lambda^{p-1} \langle \tilde{V} \rangle_0 = 0.$$

i.e. $$\langle T \rangle_0 = -\frac{p}{2} \langle \tilde{V} \rangle_0 .$$

Total energy is $\langle T\rangle_0 + \langle U\rangle_0$

$$= \langle T\rangle_0 - \left(\frac{2}{p}\right)\langle T\rangle_0$$

$$= \langle T\rangle_0 \left(1 - 2/p\right)$$

$\langle T\rangle_0 \geq 0$

$\Rightarrow$ for $1 - 2/p > 0$

i.e. $p > 2$ $\quad E_0 > 0$

$\Rightarrow$ No bound state.

B5

$$i\hbar \frac{\partial|\psi\rangle}{\partial t} = H_0|\psi\rangle + \Delta V(t)|\psi\rangle$$

Write $|\psi(t)\rangle = e^{\frac{-iE_i t}{\hbar}}|i\rangle + \sum_{j\neq i} \gamma_{ji}(t)|j\rangle e^{\frac{-iE_j t}{\hbar}} + \ldots$

$$\Rightarrow \cancel{\hbar E_i e^{\frac{-iE_i t}{\hbar}}}|i\rangle + \sum_{j\neq i}\gamma_{ji}\cancel{\hbar E_j |j\rangle e^{\frac{-iE_j t}{\hbar}}}$$

$$+ i\hbar \sum_j \dot{\gamma}_{ji}|j\rangle e^{\frac{-iE_j t}{\hbar}}$$

$$\simeq \cancel{\hbar E_i|i\rangle e^{\frac{-iE_i t}{\hbar}}} + \Delta V(t) e^{\frac{-iE_i t}{\hbar}}|i\rangle$$

$$+ \sum_{j\neq i} \gamma_{ji}(t) \cancel{e^{\frac{-iE_j t}{\hbar}}} E_j|j\rangle.$$

$\langle j|$

$$\Rightarrow i\hbar\dot{\gamma}_{ji} = e^{i\omega_{ji}t}\langle j|\Delta\hat{V}|i\rangle$$

$$\dot{\gamma}_{ji} = \frac{1}{i\hbar} e^{i\omega_{ji}t}\langle j|\Delta\hat{V}|i\rangle.$$

$$\Rightarrow \boxed{\gamma_{ji}(\infty) = \frac{1}{i\hbar}\int_{-\infty}^{\infty} e^{i\omega_{ji}t'}\langle j|\Delta\hat{V}(t')|i\rangle}$$

(b) $$\gamma_{ji} = \frac{1}{i\hbar}\int_{-\infty}^{\infty} e^{i\omega_{ji}t} \langle j| b\hat{x} + vt\hat{y} |i\rangle \times \left(-\frac{Ze^2}{(b^2+v^2t^2)^{3/2}}\right)$$

$$= -2\frac{Ze^2}{i\hbar}\int_0^{\infty} \frac{\cos(\omega_{ji}t)}{(b^2+v^2t^2)^{3/2}} \langle j|b\hat{x}|i\rangle dt$$

$$- \frac{Ze^2}{i\hbar}\int_0^{\infty} \frac{i\sin(\omega_{ji}t)}{(b^2+v^2t^2)^{3/2}} \langle j| vt\hat{y} |i\rangle dt$$

Need.

$$b\int_0^{\infty} \frac{\cos(\omega t)}{(b^2+v^2t^2)^{3/2}} dt \qquad \int_0^{\infty} \frac{\sin \omega t \ (vt)}{(b^2+v^2t^2)^{3/2}} dt$$

Work: $z = \omega t$.

$$b\int_0^{\infty} \frac{\cos z}{(b^2+ v^2/\omega^2 z^2)^{3/2}} \frac{dz}{\omega} \qquad \int_0^{\infty} \frac{\sin z}{(b^2+\frac{v^2}{\omega^2}z^2)^{3/2}} \left(\frac{vz}{\omega}\right)\frac{dz}{\omega}$$

$$= b\left(\frac{\omega^2}{v^2}\right)^{3/2}\int_0^{\infty} \frac{\cos z \ dz}{\left(\frac{\omega^2b^2}{v^2}+z^2\right)^{3/2}} \frac{dz}{\omega} \quad \Bigg| \quad \frac{v}{\omega^2}\frac{\omega^3}{v^3}\int_0^{\infty} \frac{\sin z \ z \ dz}{\left(\frac{\omega^2b^2}{v^2}+z^2\right)^{3/2}}$$

$$\beta = \frac{\omega b}{v} \ll 1$$

$$\int_0^\infty \frac{\sin t \; z}{(\beta^2+z^2)^{3/2}}\, dz$$

$$= K_0(\beta)$$

$$\simeq \frac{1}{\beta^2}.$$

$$\simeq \Gamma + \log(2/\beta)$$

$$\sim b \frac{\cancel{\omega^2}}{v^3} \frac{v^2}{\cancel{\omega^2} b^2}$$

$$\sim \frac{\omega}{v^2}\left(\Gamma \ldots\right)$$

$$\sim \frac{1}{vb}$$

$$\simeq \frac{1}{vb} \underbrace{\frac{b\omega}{v}}_{\ll 1.}$$

For $\omega b/v \ll 1$ need only keep $\langle x^2 \rangle$ term

$$\Rightarrow \quad \gamma_{ji} \simeq -\frac{2Ze^2}{i\hbar} \frac{1}{vb} \langle j|\hat{x}|i\rangle$$

$$\Rightarrow \boxed{P_{ji} \equiv |\gamma_{ji}|^2 = \left(\frac{2Ze^2}{\hbar vb}\right)^2 |\langle j|\hat{x}|i\rangle|^2}$$

(c) Assuming that the energy loss is due to the above mechanism, we expect a total rate of loss of energy as

$$\delta E = \sum_{b} \sum_{j} P_{ji} (E_j - E_i)$$

$\uparrow$ sum over all charges in the material.

There is an overall prefactor $\propto \left(\frac{Z}{v}\right)^2$

For a <u>non-relativistic</u> ion $E = mv^2$

we expect $\delta E \propto \frac{1}{E}$.

Indeed this seems to apply quite well, e.g.

${}^{12}_{6}C$ has $\delta E = 30$ MeV at $E = 50$ MeV.

$\Rightarrow$ expect $\delta E = \frac{30}{2} = 15$ MeV at 100 MeV

which is approximately correct.

The dependence on atomic number + isotope follows by noting that

$$v^2 = E/M.$$

Overall prefactor is

$$\left(\frac{Z}{v}\right)^2 = \frac{MZ^2}{E} = AZ^2\left(\frac{m_p}{E}\right)$$

(i) For a given Z, expect δE to increase with A. (As seen in data)

(ii) <u>If fully ionized</u>, expect δE to increase with atomic number Z.

(Also borne out in the data)

[Qualitative comments sufficient here].

B6 a) Groundstate: $\uparrow$ and $\downarrow$

in $(n_x, n_y, n_z) = (0,0,0)$

$\Rightarrow$ degeneracy 1

First excited state:

one particle in $(0,0,0)$ $\uparrow$ or $\downarrow$

one particle in any one of
$(1,0,0)$, $(0,1,0)$, $(0,0,1)$ $\uparrow$ or $\downarrow$

$\Rightarrow$ $2 \times 6 = 12$ choices.

b) The total angular momentum, $\hat{\underline{L}}^2$, commutes with the Hamiltonian because the Hamiltonian has rotational symmetry.

The total spin $\hat{\underline{S}}^2 = \hbar^2 (S)(S+1)$

Commutes with the Hamiltonian, since it does not depend on the spin operators at all.

c)

$$\Phi_x \propto r \sin\theta \cos\varphi \, e^{-r^2/2a_0^2}$$

$$\Phi_y \propto r \sin\theta \sin\varphi \, e^{-r^2/2a_0^2}$$

$$\Phi_z \propto r \cos\theta \, e^{-r^2/2a_0^2}$$

$\Rightarrow$ Combine $\Phi_x \pm i\Phi_y \propto Y_{1,\pm 1}$

$\Phi_z \propto Y_{1,0}$.

(Looking only at angular part)

$\Rightarrow$ $l = 1$ and $m_l = 0, \pm 1$.

d) <u>In ground state</u>: 2 particles are in $l = 0$.

$\Rightarrow L = 0$.

$S = 0$ required by antisymmetry

$\Rightarrow (2L+1)(2S+1) = 1$

In the 1st excited state:

Two particles are in $\ell = 0$ and $\ell = 1$ orbitals.

$\Rightarrow \quad L = 1.$

$S = 0$ or $S = 1$ are both possible

$\Rightarrow \quad (2L+1)(2S+1) = 3 \cdot 1 + 3 \cdot 3 = 12$ ✓.

(e) $$\Delta H = A\, \underline{\hat{S}} \cdot \underline{\hat{L}}$$

$$= \frac{A}{2} \left(\hat{J}^2 - \hat{L}^2 - \hat{S}^2 \right)$$

$\underline{\underline{S=0}}$: $L = 1$, $J = 1$. $\Rightarrow \Delta H = 0.$

$\Rightarrow$ 3-fold degenerate level

$\underline{\underline{S=1}}$: $L = 1$ $J = 0, 1, 2$

$$\Delta H_{J=0} = \frac{A}{2}\left(-1(2)-(1)(2)\right)\hbar^2$$

$$= -2A\hbar^2$$

$$\Delta H_{J=1} = \frac{A}{2}\hbar^2\left(1.2 - 1.2 - 1.2\right)$$

$$= -A\hbar^2$$

$$\Delta H_{J=2} = \frac{A\hbar^2}{2}\left(2.3 - 4\right)$$

$$= +A\hbar^2.$$

$\Rightarrow$