UNIVERSITY OF
CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Wednesday 25 May 2016 1.30 pm to 3.30 pm

PHYSICS (4)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (4)
OPTICS AND ELECTRODYNAMICS

Candidates offering this paper should attempt a total of **three** *questions. The questions to be attempted are* **1**, **2** *and* **one** *other question.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains **four** *sides, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
2×20 Page Answer Book	Mathematical Formulae handbook
Rough workpad	Approved calculator allowed
Yellow master coversheet	

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OPTICS AND ELECTRODYNAMICS

1 *Attempt* **all** *parts of this question. Answers should be concise and relevant formulae may be assumed without proof.*

(a) A charged particle with kinetic energy T passes through a slab of material with refractive index n. Electromagnetic radiation is observed in a direction at angle θ to the particle's direction of motion. What is the particle's mass? [4]

(b) Compare the coherence length and coherence width of sunlight at the surface of the Earth. [4]

(c) At time $t = 0$ an electron is in a circular orbit of radius a_0 around a proton. Show that, according to classical physical arguments alone and assuming that the fraction loss of energy per orbit is small, the radius r of the orbit will decrease with t according to:

$$r^3 = a_0^3 - 4r_0^2 ct$$

where $r_0 = \dfrac{e^2}{4\pi\epsilon_0 mc^2}$ is the classical radius of the electron. [4]

[*The instantaneous power P radiated by an electron with acceleration $\boldsymbol{\alpha}$ is given by the Larmor formula* $P = \dfrac{\mu_0 e^2}{6\pi c}|\boldsymbol{\alpha}|^2$.]

2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) birefringence;

(b) synchrotron radiation;

(c) photonic crystals.

3 *Attempt* **either** *this question* **or** *question 4.*

Define the *magnetic vector potential* $\boldsymbol{A}$ and explain what is meant by a *gauge transformation.* [3]

Show that in free space in the presence of a steady free current density $\boldsymbol{J}$ and with suitable choice of gauge:

$$\nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J}, \qquad [4]$$

and hence show that:

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int_{\text{all space}} \frac{\boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \, \mathrm{d}^3\boldsymbol{r}'. \qquad [3]$$

If $\boldsymbol{J}$ varies explicitly with time, explain how this expression should be modified to take into account the effects of *retardation.* [2]

A long straight wire lies along the z-axis. At $t = 0$ a current I is suddenly established uniformly along the wire, which remains always electrically uncharged. Show that at time t at a distance r ($< ct$) from the wire:

$$\boldsymbol{A}(r, t) = \widehat{z} \frac{\mu_0 I}{2\pi} \ln\left(\frac{ct}{r} + \sqrt{\left(\frac{ct}{r}\right)^2 - 1} \right). \qquad [7]$$

Hence calculate the electric and magnetic fields $\boldsymbol{E}(r, t)$ and $\boldsymbol{H}(r, t)$, and the Poynting vector $\boldsymbol{N}(r, t)$ for some fixed value of r. [4]

Sketch graphs of, and comment on, your results. [2]

[*You may use the result* $\displaystyle\int \frac{\mathrm{d}z}{\sqrt{z^2 + a^2}} = \ln\left(z + \sqrt{z^2 + a^2}\right) + \text{constant}$.

In cylindrical co-ordinates (r, θ, z), $\nabla \times \boldsymbol{A} = \begin{vmatrix} \widehat{\boldsymbol{r}}/r & \widehat{\boldsymbol{\theta}} & \widehat{z}/r \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix}$.]

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4 *Attempt* **either** *this question* **or** *question 3.*
The power $\langle P \rangle$ radiated by an electric dipole p is given by

$$\langle P \rangle = \frac{\mu_0 \langle \ddot{p}^2 \rangle}{6\pi c}.$$

Show that Rayleigh scattering from spherical dielectric particles of size much smaller than the wavelength of the incident radiation is strongly dependent on the frequency of the radiation. [4]

If the incident radiation is propagating along Ox and is linearly polarized along Oz, describe, without detailed calculation, the angular profile of the scattered radiation intensity and its polarization. [2]

Radiation propagating along Ox is scattered by a needle-shaped particle at the origin O. The particle has length much less than the radiation wavelength, and diameter much smaller still, and is formed from a material with a very large dielectric constant. The long axis of the particle is oriented along the direction defined by (θ, ϕ) in standard spherical polar co-ordinates. For a given incident intensity I_0, what are the polarization states and relative intensities of the radiation scattered along the Oy-direction when the incident radiation is:

(a) linearly polarized along Oz;

(b) linearly polarized along Oy. [6]

Use your results for cases (a) and (b) to deduce the polarization states of the radiation scattered along the Oy-direction when the incident radiation is

(c) right hand circularly polarized;

(d) unpolarized? [4]

In case (a) above, the particle is now replaced by a diffuse cloud of similar particles with random orientation. The inter-particle spacings are large compared with the coherence length of the incoming radiation so that the particles scatter mutually incoherently. Neglecting multiple scattering, calculate the relative intensities of radiation measured by a distant observer along Oy through a linear polarizer with its axis first parallel to Ox, and then parallel to Oz, and comment on your results. [7]

How would this be modified if the randomly oriented particles were densely packed to form a continuous dielectric medium? [2]

END OF PAPER

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OPTICS AND ELECTRODYNAMICS

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(a) A charged particle with kinetic energy T passes through a slab of material with refractive index n. Electromagnetic radiation is observed in a direction at angle θ to the particle's direction of motion. What is the particle's rest mass? [4]

UNSEEN but straightforward

Cerenkov: $\cos\theta = \dfrac{c}{nv} \quad \rightarrow \quad v = \dfrac{c}{n\cos\theta} \quad \rightarrow \quad \gamma = \dfrac{1}{\sqrt{1 - \frac{1}{n^2\cos^2\theta}}}$ [2]

$T = (\gamma - 1)mc^2 \quad \rightarrow \quad m = \dfrac{T}{(\gamma-1)c^2}$ with γ as above. [2]

(b) Compare the coherence length and coherence width of sunlight at the surface of the Earth. [4]

BOOKWORK recollection

$l_{\mathrm{w}} \sim \dfrac{\lambda}{\alpha}$ [1]

$l_{\mathrm{l}} \sim \dfrac{\lambda^2}{2\Delta\lambda}$ [1]

UNSEEN but straightforward

with $\alpha \sim 0.5^\circ \sim 0.01$ rad, $\lambda \sim 600$ nm, $\Delta\lambda \sim 300$ nm so [2]

$l_{\mathrm{w}} \sim 6 \times 10^{-5}$ m $\qquad l_{\mathrm{l}} \sim 36 \times 10^{-14}/(2 \times 3 \times 10^{-7}) \sim 6 \times 10^{-7}$ m.

(c) At time $t = 0$ an electron is in a circular orbit of radius a_0 around a proton. Show that, according to classical physical arguments alone and assuming that the fraction loss of energy per orbit is small, the radius r of the orbit will decrease with t according to:

$$r^3 = a_0^3 - 4r_0^2 ct$$

where $r_0 = \dfrac{e^2}{4\pi\epsilon_0 mc^2}$ is the classical radius of the electron. [4]

[*The instantaneous power P radiated by an electron with acceleration $\boldsymbol{\alpha}$ is given by the Larmor formula $P = \dfrac{\mu_0 e^2}{6\pi c}|\boldsymbol{\alpha}|^2$.*]

ALL UNSEEN

The acceleration is almost entirely radial, given by $m\alpha = \dfrac{e^2}{4\pi\epsilon_0 r^2}$. So

$$\frac{\mathrm{d}U}{\mathrm{d}t} = -\frac{\mu_0 e^2}{6\pi c}\left(\frac{e^2}{4\pi\epsilon_0 m r^2}\right)^2 = -\frac{\mu_0}{6\pi}\frac{r_0^2 c^3}{r^4} \quad \text{since} \quad r_0 = \frac{e^2}{4\pi\epsilon_0 m c^2}$$

Now

$$U = -\frac{e^2}{4\pi\epsilon_0 r} + \frac{mv^2}{2} = -\frac{e^2}{8\pi\epsilon_0 r}$$

So

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \frac{e^2}{8\pi\epsilon_0 r^2}\dot{r} = -\frac{\mu_0}{6\pi}\frac{r_0^2 c^3}{r^4} \qquad \text{from above}$$

So

$$3\dot{r}r^2 = -4r_0^2 c \qquad\qquad r^3 = -4r_0^2 ct + \text{const.} = a_0^3 - 4r_0^2 ct$$

assuming motion always non-relativistic. [4]

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2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) birefringence;

Birefringence

1.Optical anisotropy (birefringence) occurs in materials which are structurally anisotropic — e.g. calcite.
Their relative permittivity (and therefore also their refractive index) is a Hermitian matrix :

$$\boldsymbol{D} = \epsilon_0\, \underline{\underline{\epsilon}} \cdot \boldsymbol{E} \qquad D_i = \epsilon_0 \sum_j \epsilon_{ij} E_j \qquad \begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = \epsilon_0 \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

So in general $\boldsymbol{D} \not\parallel \boldsymbol{E}$.
For lossless media the ϵ_{ij} are real, and then $\underline{\underline{\epsilon}}$ is symmetric and can therefore be diagonalized by a rotation of the Cartesian axes.
So there is a set of orthogonal axes $\{\widehat{\boldsymbol{e}}_1,\ \widehat{\boldsymbol{e}}_2,\ \widehat{\boldsymbol{e}}_3\}$ – the ***principal axes*** – such that:

$$\underline{\underline{\epsilon}} = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix} = \begin{pmatrix} n_1^2 & 0 & 0 \\ 0 & n_2^2 & 0 \\ 0 & 0 & n_3^2 \end{pmatrix}$$

where n_1, n_2 and n_3 are the *principal refractive indices*.
If $n_1 \neq n_2 \neq n_3$, the material is ***biaxial***.

Materials (such as calcite) which have two of the principal refractive indices equal are ***uniaxial***.

$$\underline{\underline{\epsilon}} = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_1 \end{pmatrix} \qquad \underline{\underline{\epsilon}} = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix} \qquad \underline{\underline{\epsilon}} = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix}$$

For uniaxial systems it is conventional to take $n_1 = n_2 \neq n_3$ then:

$$\underline{\underline{\epsilon}} = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix} = \begin{pmatrix} n_o^2 & 0 & 0 \\ 0 & n_o^2 & 0 \\ 0 & 0 & n_e^2 \end{pmatrix}$$

where the "o" denotes the "ordinary" directions, and "e" the "extraordinary" direction – the ***optic axis***.
The ***birefringence*** for a uniaxial material is defined as $\Delta n = n_e - n_o$, and can be positive or negative.

2. Linearly Polarized EM Waves in Anisotropic Materials
If $\boldsymbol{D}$ lies along a principal axis (say $\widehat{e}_1$), $\boldsymbol{E} \parallel \boldsymbol{D}$. Then $\boldsymbol{E} \times \boldsymbol{H} = \boldsymbol{N} \parallel \boldsymbol{k}$, and the wave equation is identical to that for an isotropic medium with an ϵ corresponding to that for the axis along which $\boldsymbol{D}$ and $\boldsymbol{E}$ are directed.

3. Uniaxial Materials
For a *uniaxial* material:

$$\boldsymbol{D} = \epsilon_0 \underline{\underline{\epsilon}} \cdot \boldsymbol{E} = \epsilon_0 \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \tag{1}$$

So even if $\boldsymbol{D} \nparallel \widehat{e}_1$ or $\widehat{e}_2$, but lies in the $\widehat{e}_1$-$\widehat{e}_2$ plane for which $\epsilon_1 = \epsilon_2 = n_o^2$, $\boldsymbol{D} \parallel \boldsymbol{E}$.
So for $\boldsymbol{D} \perp Oz$, the optic axis, $\boldsymbol{E} \parallel \boldsymbol{D}$ whatever its direction in this plane, and the wave velocity is c/n_o – the "*ordinary*" ray: $\boldsymbol{E} \times \boldsymbol{H} = \boldsymbol{N}$ is parallel to $\boldsymbol{k}$.
But if $\boldsymbol{D}$ is ***not*** in the $\widehat{e}_1$-$\widehat{e}_2$ plane, it cannot be assumed that $\boldsymbol{E} \parallel \boldsymbol{D}$. For this "*extraordinary*" ray: $\boldsymbol{E} \times \boldsymbol{H} = \boldsymbol{N}$ is **NOT** parallel $\boldsymbol{k}$ – the *phase and the energy may propagate in different directions.*

4. Double Refraction
The complex effect of anisotropic refractive index is to alter the angle of refraction for the two possible linear polarizations – the ordinary and extraordinary rays. For normal incidence the phase propagates into a slab of anisotropic material along the surface normal. For the ordinary ray the energy propagates along the normal too. For the extraordinary ray the phase still propagates along the surface normal, but the energy propagates at an angle to the normal: the ray is therefore *laterally shifted* when it emerges from the crystal. So an object can give rise to two distinct images when viewed through an anisotropic material. Each image has different polarization properties.

5. Optical Elements: Waveplates (or Retarders)

A plane-polarized EM wave $e^{i(kz-\omega t)}$ travels along Oz at different speeds c/n_f or c/n_s depending on whether $\boldsymbol{E} \parallel Ox$ or Oy. So the plate applies phase terms depending on the different *optical thicknesses* for different polarizations. Quarter- and half-wave plates can, with linear polarizers, be used to manipulate and analyze the polarization state of light in detail. e.g.: start with linear polarization and create circular.

Induced birefringence:

6. Photoelasticity: is the birefringence induced when an otherwise isotropic material is subjected to *stress.* The corresponding distortion of the material, at molecular level, changes the dielectric response, producing an anisotropic permittivity tensor. A transparent *isotropic* object placed between crossed linear polarizers would not change the initial polarization, so no light should be transmitted. But if stressed to induce birefringence, linearly polarized light passing through the material has its polarization state affected in complex ways depending on the stress field, and some light is transmitted, allowing patterns of stress in transparent mechanical structures placed between crossed polarizers to be visualized.

7. The Kerr Effect: In an applied electric field $\boldsymbol{E}_0$ an otherwise isotropic material can become uniaxially birefringent, with the optic axis along $\boldsymbol{E}_0$. Suitable materials can therefore be used to make *voltage-controlled wave-plates.*

(b) synchrotron radiation;

SynchrotronRadiation

1. Occurs for circular motion of charged particle with speed $u \longrightarrow c$, $\gamma \gg 1$.
As for non-relativistic cyclotron motion, the acceleration $\boldsymbol{a}$ is perpendicular to the velocity $\boldsymbol{u}$, and in the IRF it is $\boldsymbol{\alpha}$, a *constant*, not oscillatory.
But here it turns out (see below) that, in contrast with the cyclotron case, the characteristic wavelength emitted

$$\lambda_s \ll R = \frac{u \to c}{\omega_B}$$

so the rotating/oscillating dipole picture in lab frame S does not apply.
So the radiated fields are *not harmonically time-varying* as for the cyclotron case. There is a continuous shear of the lines of force. There is no "natural" frequency, but (see below) a *continuum* of frequencies centred on $\nu_s = \dfrac{c}{\lambda_s}$.

2. But *in the particle's IRF the motion is non-relativistic* – and the instantaneous radiated power is still given by the Larmor formula E:

$$P = \frac{\mu_0 |\ddot{\boldsymbol{p}}|^2}{6\pi c} = \frac{\mu_0 q^2 |\boldsymbol{\alpha}|^2}{6\pi c}$$

The angular distribution of the radiated power in the IRF S′ is the usual doughnut dipole shape, but in the *laboratory frame S*, as β increases, the

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radiated *power becomes more closely confined to the forward direction*, the direction of motion, and the rearward lobe becomes much weaker. At highly relativistic speeds $\gamma \gg 1$, the angular half-width of the forward lobe $\to 1/\gamma$.

3. So a stationary observer O receives radiation only when the angle between the observer's line of sight to the particle and the particle's velocity is $< 1/\gamma$. i.e. when the observer's line of sight is approximately *tangential* to the particle's circular path. The field at O therefore is therefore *pulsed.*

4. The pulse width can be estimated as follows...
O first receives radiation from the point A (emitted at time t_{A}) at a time

$$t_{\mathrm{A}} + \frac{d + R/\gamma}{c}$$

The end of the pulse (emitted at time $t_{\mathrm{B}} = t_{\mathrm{A}} + 2R/\gamma u$) is detected at time

$$t_{\mathrm{B}} + \frac{d - R/\gamma}{c}$$

The *duration Δt of the pulse* is therefore:

$$\begin{aligned}\Delta t &\sim \frac{1}{c}\left(d - \frac{R}{\gamma} - d - \frac{R}{\gamma}\right) + \frac{2R}{\gamma u} \\ &= \frac{2R}{\gamma}\left(\frac{1}{u} - \frac{1}{c}\right) \\ &= \frac{2}{\gamma\omega_B}\left(1 - \frac{u}{c}\right) \\ &\sim \frac{1}{\gamma^3\omega_B}\end{aligned} \tag{2}$$

using $(1 - u/c) \approx 1/2\gamma^2$.

5. The radiation is *pulsed in time*, so is *spectrally broad* with some characteristic frequency and wavelength:

$$\nu_{\mathrm{s}} \sim \frac{1}{\Delta t} \sim \gamma^3\omega_B \qquad \lambda_{\mathrm{s}} \sim \frac{c}{\gamma^3\omega_B} \sim \frac{R}{\gamma^3} \ll R \tag{3}$$

6. This process produces significant energy loss in electron synchrotrons, and *limits the energy* that can be reached in circular *accelerators.* Bad for particle physicists.

7. For high γ, $\nu_{\mathrm{s}} \gg \omega_B = \dfrac{u(\approx c)}{R}$ producing *hard X-rays and a broad continuum at lower frequencies.*
E.S.R.F.: $\gamma \sim 12000$ $\quad 2\pi R = 844$ m $\quad \lambda_{\mathrm{s}} \sim 5$ Å
So synchrotrons now used as a powerful *source of highly collimated X-rays* for spectroscopy and X-ray diffraction studies. Good for everyone except particle physicists.

8. Synchrotron radiation is also very important astronomically, as it also occurs naturally in astronomical objects such as supernovae and radio galaxies.

(c) photonic crystals.

Photonic Structures

1.Photonics concerns the manipulation of light using *artificial microstructures.* These necessarily involve structured media, with *interfaces* between media of different properties. For simplicity, take isotropic, non-chiral media, and $\mu = 1$.
Consider a *periodic structure* – $\epsilon(\boldsymbol{r})$ varies with position $\boldsymbol{r}$ in a periodic way described by an underlying *lattice* with lattice vectors $\boldsymbol{R}$ and a corresponding *reciprocal lattice* with vectors $\boldsymbol{G}$.

$$\boldsymbol{G} \cdot \boldsymbol{R} = 2\pi \times \text{an integer}$$

Because it is periodic in the lattice $\{\boldsymbol{R}\}$, $\epsilon(\boldsymbol{r})$ can be expanded as a Fourier sum over the reciprocal lattice vectors $\boldsymbol{G}$:

$$\epsilon(\boldsymbol{r}) = \sum_{\boldsymbol{G}} \epsilon_{\boldsymbol{G}}\, e^{i\boldsymbol{G}\cdot\boldsymbol{r}} \tag{4}$$

The electric field can also be written as a *3D Fourier transform*:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{(2\pi)^3} \int \tilde{\boldsymbol{E}}(\boldsymbol{q})\, e^{i\boldsymbol{q}\cdot\boldsymbol{r}} d\boldsymbol{q} \tag{5}$$

Substituting Eqns 4 and 5 into the wave equation for transverse EM waves of frequency ω:

$$\nabla^2 \boldsymbol{E} - \frac{\omega^2}{c^2}\, \epsilon \boldsymbol{E} = 0 \tag{6}$$

$$-q^2 \int \tilde{\boldsymbol{E}}(\boldsymbol{q})\, e^{i\boldsymbol{q}\cdot\boldsymbol{r}} d\boldsymbol{q} - \frac{\omega^2}{c^2} \sum_{\boldsymbol{G}} \epsilon_{\boldsymbol{G}} \int \tilde{\boldsymbol{E}}(\boldsymbol{q})\, e^{i(\boldsymbol{G}+\boldsymbol{q})\cdot\boldsymbol{r}} d\boldsymbol{q} = 0 \tag{7}$$

$$-q^2 \int \tilde{\boldsymbol{E}}(\boldsymbol{q})\, e^{i\boldsymbol{q}\cdot\boldsymbol{r}} d\boldsymbol{q} - \frac{\omega^2}{c^2} \sum_{\boldsymbol{G}\epsilon_{\boldsymbol{G}}} \int \tilde{\boldsymbol{E}}(\boldsymbol{q} - \boldsymbol{G})\, e^{i\boldsymbol{q}\cdot\boldsymbol{r}} d\boldsymbol{q} = 0 \tag{8}$$

... the last step being a simple shift of the origin of the integral. Eqn 8 requires the coefficient of $e^{i\boldsymbol{q}\cdot\boldsymbol{r}}$ to be zero:

$$-q^2 \tilde{\boldsymbol{E}}(\boldsymbol{q}) - \frac{\omega^2}{c^2} \sum_{\boldsymbol{G}} \epsilon_{\boldsymbol{G}} \tilde{\boldsymbol{E}}(\boldsymbol{q} - \boldsymbol{G}) = 0 \tag{9}$$

where the sum is over all reciprocal lattice vectors $\boldsymbol{G}$.
Clearly, the coefficients $\tilde{\boldsymbol{E}}(\boldsymbol{q} - \boldsymbol{G})$ in the Fourier expansion (Eqn 5) are linked. If $\epsilon(\boldsymbol{r}) = \text{const.}$, the solutions are simple plane waves. For the modulated $\epsilon(\boldsymbol{r})$, Eqn. 9 suggests solutions, labeled by wavevector $\boldsymbol{q}$, in the form of a sum of the linked plane waves:

$$\boldsymbol{E}_{\boldsymbol{q}}(\boldsymbol{r}) = \sum_{\boldsymbol{G}} \tilde{\boldsymbol{E}}(\boldsymbol{q} + \boldsymbol{G})\, e^{i(\boldsymbol{q}+\boldsymbol{G})\cdot\boldsymbol{r}} \tag{10}$$

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Hence, with $\boldsymbol{r} \rightarrow \boldsymbol{r} + \boldsymbol{R}$:

$$\boldsymbol{E}_{\boldsymbol{q}}(\boldsymbol{r} + \boldsymbol{R}) = \sum_{\boldsymbol{G}} \tilde{E}(\boldsymbol{q} + \boldsymbol{G})\ \mathrm{e}^{i(\boldsymbol{q}+\boldsymbol{G})\cdot(\boldsymbol{r}+\boldsymbol{R})} = \boldsymbol{E}_{\boldsymbol{q}}(\boldsymbol{r}) \times \mathrm{e}^{i(\boldsymbol{q}+\boldsymbol{G})\cdot \boldsymbol{R}}$$

$\boldsymbol{G} \cdot \boldsymbol{R} = 2\pi \times$ integer, so: ....

$$\boxed{\boldsymbol{E}_{\boldsymbol{q}}(\boldsymbol{r} + \boldsymbol{R}) = \boldsymbol{E}_{\boldsymbol{q}}(\boldsymbol{r})\ \mathrm{e}^{i\boldsymbol{q}\cdot \boldsymbol{R}}} \tag{11}$$

So it must be that

$$\boxed{\boldsymbol{E}_{\boldsymbol{q}}(\boldsymbol{r}) = U_{\boldsymbol{q}}(\boldsymbol{r})\ \mathrm{e}^{i\boldsymbol{q}\cdot\boldsymbol{r}} \qquad \text{where} \qquad U_{\boldsymbol{q}}(\boldsymbol{r} + \boldsymbol{R}) = U_{\boldsymbol{q}}(\boldsymbol{r})} \tag{12}$$

Eqns 11 and 12 are equivalent statements of ***Bloch's Theorem*** (or Floquet's Theorem) – a general result for solutions of wave equations in periodic systems.

2. Now simplify enormously and take as an illustrative model the *1-D periodic dielectric multilayer* with EM waves polarized along Ox and travelling along Oz:
   By analogy with electronic band structures, the wave equation has bands of propagating solutions, and band gaps, frequency ranges in which q becomes *complex*, with $q_{\mathrm{r}} = n\pi/d$. Then the EM wave is attenuated and does not propagate along Oz.
   There is no absorption since ϵ_{a} and ϵ_{b} are real – the wave is evanescent and there is a "*photonic band gap*". Under these conditions the multilayer is a perfect mirror – a "*Bragg reflector*".
3. Artificial photonic materials allow the control of photons and their interactions *via* the *photonic band structure*.
4. Defects can produce localized states.... photonic nanocavities.
5. And in Nature: butterfly wings, beetles carapaces etc.

3 *Attempt* **either** *this question* **or** *question 4.*

Define the *magnetic vector potential* $\boldsymbol{A}$ and explain what is meant by a *gauge transformation.* [3]

Show that in free space in the presence of a steady free current density $\boldsymbol{J}$ and with suitable choice of gauge:

$$\nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J},$$ [4]

and hence show that:

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int_{\text{all space}} \frac{\boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \, \mathrm{d}^3\boldsymbol{r}'.$$ [3]

If $\boldsymbol{J}$ varies explicitly with time, explain how this expression should be modified to take into account the effects of *retardation.* [2]

A long straight wire lies along the z-axis. At $t = 0$ a current I is suddenly established uniformly along the wire, which remains always electrically uncharged. Show that at time t at a distance r ($< ct$) from the wire:

$$\boldsymbol{A}(r, t) = \widehat{\boldsymbol{z}} \, \frac{\mu_0 I}{2\pi} \ln \left(\frac{ct}{r} + \sqrt{\left(\frac{ct}{r}\right)^2 - 1} \right).$$ [7]

Hence calculate the electric and magnetic fields $\boldsymbol{E}(r,t)$ and $\boldsymbol{H}(r,t)$, and the Poynting vector $\boldsymbol{N}(r,t)$ for some fixed value of r. [4]

Sketch graphs of, and comment on, your results. [2]

[*You may use the result* $\displaystyle\int \frac{\mathrm{d}z}{\sqrt{z^2 + a^2}} = \ln \left(z + \sqrt{z^2 + a^2} \right) + \text{const.}$

In cylindrical co-ordinates (r, θ, z), $\nabla \times \boldsymbol{A} = \begin{vmatrix} \widehat{\boldsymbol{r}}/r & \widehat{\boldsymbol{\theta}} & \widehat{\boldsymbol{z}}/r \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix}$.]

BOOKWORK

Since $\nabla \cdot \boldsymbol{B} = 0$ and $\nabla \cdot (\nabla \times\ ) \equiv 0$, $\boldsymbol{B}$ can be written as $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ with $\boldsymbol{A}$ as a vector potential.

$\boldsymbol{B} = \nabla \times \boldsymbol{A}$ is unaffected by the addition of a term $\nabla\psi$ to $\boldsymbol{A}$ since $\nabla \times \nabla\psi \equiv 0$. So the gauge transformation $\boldsymbol{A} \to \boldsymbol{A}' + \nabla\psi$, for any scalar function ψ, leaves $\boldsymbol{B}$ unchanged.

Since $\nabla \cdot \boldsymbol{A} = \nabla \cdot \boldsymbol{A}' + \nabla^2\psi$ and ψ is arbitrary, $\nabla \cdot \boldsymbol{A}$ can be chosen arbitrarily. $\nabla \cdot \boldsymbol{A} = 0$ is the Coulomb gauge, appropriate for the following example.

For steady currents in vacuum, $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$ and since $\boldsymbol{B} = \nabla \times \boldsymbol{A}$:

$$\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} = \mu_0 \boldsymbol{J}$$

Choosing the Coulomb gauge $\nabla \cdot \boldsymbol{A} = 0$:

$$\nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J}$$

This is of exactly the same form as *Poisson's equation* for the scalar electrostatic potential:

$$\nabla^2\phi = -\rho/\epsilon_0$$

for which the solution is:

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int_{\text{all space}} \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}^3\boldsymbol{r}'$$

Each component of $\boldsymbol{A}$ must obey a similar equation and therefore in general

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int_{\text{all space}} \frac{\boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}^3\boldsymbol{r}' \longrightarrow \frac{\mu_0}{4\pi}\int_{\text{all space}} \frac{I}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}\boldsymbol{l}'$$

where, for a line current I, $\boldsymbol{J}(\boldsymbol{r}')\mathrm{d}^3\boldsymbol{r}' \longrightarrow I\mathrm{d}\boldsymbol{l}'$.

Taking into account the finite speed c at which EM influences travel in vacuum, $\boldsymbol{A}$ at $(\boldsymbol{r},t)$ will depend on $\boldsymbol{J}$ at $\boldsymbol{r}'$ at time $t-\frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}$, so:

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\int_{\text{all space}} \frac{\boldsymbol{J}\left(\boldsymbol{r}', t-\frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right)}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}^3\boldsymbol{r}'$$

UNSEEN

Take r in the $z=0$ plane.

Since the wire is uncharged the electrostatic potential $\phi = 0$ everywhere, at all times.

For $r > ct$, $\boldsymbol{A} = 0$.

For $r < ct$, the wire in the interval $z = \pm\sqrt{c^2t^2-r^2}$ contributes to $\boldsymbol{A}$, and since I is constant for $t>0$:

$$\begin{aligned}
\boldsymbol{A}(r,t) &= \frac{\mu_0}{4\pi}\int_{-\sqrt{c^2t^2-r^2}}^{\sqrt{c^2t^2-r^2}} \frac{I}{\sqrt{r^2+z^2}}\,\mathrm{d}\boldsymbol{z} \\
&= \frac{\mu_0}{2\pi}\int_0^{\sqrt{c^2t^2-r^2}} \frac{I}{\sqrt{r^2+z^2}}\,\mathrm{d}\boldsymbol{z} \qquad \text{by symmetry} \\
&= \widehat{\boldsymbol{z}}\,\frac{\mu_0 I}{42\pi}\left[\ln(z+\sqrt{z^2+r^2})\right]_0^{\sqrt{c^2t^2-r^2}} \\
&= \widehat{\boldsymbol{z}}\,\frac{\mu_0 I}{2\pi}\ln\left(\frac{\sqrt{c^2t^2-r^2}+ct}{r}\right) \\
&= \widehat{\boldsymbol{z}}\,\frac{\mu_0 I}{2\pi}\ln\left(\frac{ct}{r}+\sqrt{\left(\frac{ct}{r}\right)^2-1}\right)
\end{aligned}$$

For $\boldsymbol{E}$ and $\boldsymbol{H}$ (with $\phi=0$):

$$\boldsymbol{E} = -\frac{\partial\boldsymbol{A}}{\partial t} = -\widehat{\boldsymbol{z}}\,\frac{\mu_0 I}{2\pi}\,\frac{1}{\left(\frac{ct}{r}+\sqrt{\left(\frac{ct}{r}\right)^2-1}\right)}\left(\frac{c}{r}+\frac{c^2t/r^2}{\sqrt{\left(\frac{ct}{r}\right)^2-1}}\right)$$

$$\mu_0 \boldsymbol{H} = \nabla \times \boldsymbol{A} = -\widehat{\boldsymbol{\theta}}\,\frac{\partial A_z}{\partial r} = -\widehat{\boldsymbol{\theta}}\,\frac{\mu_0 I}{2\pi}\,\frac{1}{\left(\frac{ct}{r} + \sqrt{\left(\frac{ct}{r}\right)^2 - 1}\right)}\,\left(-\frac{ct}{r^2} - \frac{c^2t^2/r^3}{\sqrt{\left(\frac{ct}{r}\right)^2 - 1}}\right)$$

With $\alpha = ct/r$ these results become, for $ct/r > 1$:

$$\boldsymbol{E} = -\widehat{\boldsymbol{z}}\,\frac{\mu_0 I}{2\pi}\,\frac{1}{t}\,\frac{1}{\left(\alpha + \sqrt{\alpha^2 - 1}\right)}\,\left(\alpha + \frac{\alpha^2}{\sqrt{\alpha^2 - 1}}\right) = -\widehat{\boldsymbol{z}}\,\frac{\mu_0 I}{2\pi}\,\frac{1}{t}\,\frac{\alpha}{\sqrt{\alpha^2 - 1}}$$

$$\boldsymbol{H} = \widehat{\boldsymbol{\theta}}\,\frac{I}{2\pi}\,\frac{1}{\alpha + \sqrt{\alpha^2 - 1}}\,\left(\frac{\alpha}{r} + \frac{\alpha^2/r}{\sqrt{\alpha^2 - 1}}\right) = \widehat{\boldsymbol{\theta}}\,\frac{I}{2\pi}\,\frac{1}{r}\,\frac{\alpha}{\sqrt{\alpha^2 - 1}}$$

GRAPHS

As $t \to \infty$, $\boldsymbol{H} \to \widehat{\boldsymbol{\theta}}\,\dfrac{I}{2\pi r}$ as expected from Ampere's law.

There is an electric field in the z-direction, so there is an energy flux in the radial direction given by the Poynting vector:

$$\boldsymbol{N} = \widehat{\boldsymbol{r}}\,\frac{\mu_0 I^2}{4\pi^2}\,\frac{1}{rt}\,\frac{\alpha^2}{\alpha^2 - 1} = \widehat{\boldsymbol{r}}\,\frac{\mu_0 I^2}{4\pi^2}\,\frac{1}{r}\,\frac{c^2 t}{c^2t^2 - r^2}.$$

As $t \to \infty$, $\boldsymbol{E} \to 0$ and $\boldsymbol{N} \to 0$ as we approach the steady state. There must be an outward energy flow to provide the field energy density corresponding to the ultimate steady state magnetic field given by Ampere's Law.

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4 *Attempt* **either** *this question* **or** *question 3.*

The power $\langle P\rangle$ radiated by an electric dipole p is given by

$$\langle P\rangle = \frac{\mu_0\langle \ddot{p}^2\rangle}{6\pi c}.$$

Show that Rayleigh scattering from spherical dielectric particles of size much smaller than the wavelength of the incident radiation is strongly dependent on the frequency of the radiation. [4]

If the incident radiation is propagating along Ox and is linearly polarized along Oz, describe, without detailed calculation, the angular profile of the scattered radiation intensity and its polarization. [2]

Radiation propagating along Ox is scattered by a needle-shaped particle at the origin O. The particle has length much less than the radiation wavelength, and diameter much smaller still, and is formed from a material with a very large dielectric constant. The long axis of the particle is oriented along the direction defined by (θ, ϕ) in standard spherical polar co-ordinates. For a given incident intensity I_0, what are the polarization states and relative intensities of the radiation scattered along the Oy-direction when the incident radiation is:

(a) linearly polarized along Oz;

(b) linearly polarized along Oy. [6]

Use your results for cases (a) and (b) to deduce the polarization states of the radiation scattered along the Oy-direction when the incident radiation is

(c) right hand circularly polarized;

(d) unpolarized? [4]

In case (a) above, the particle is now replaced by a diffuse cloud of similar particles with random orientation. The inter-particle spacings are large compared with the coherence length of the incoming radiation so that the particles scatter mutually incoherently. Neglecting multiple scattering, calculate the relative intensities of radiation measured by a distant observer along Oy through a linear polarizer with its axis first parallel to Ox, and then parallel to Oz, and comment on your results. [7]

How would this be modified if the randomly oriented particles were densely packed to form a continuous dielectric medium? [2]

BOOKWORK

For Rayleigh scattering, $p = \alpha E$, where α is the particle's *isotropic* polarizability. This requires particle size $\ll \lambda$ so that E can be taken to be spatially uniform.

$E = E_0 \exp(-\mathrm{i}\omega t)$ so $\ddot{p} = -\omega^2 E_0 \alpha$.

So $\langle P\rangle = \dfrac{\mu_0\langle \ddot{p}^2\rangle}{6\pi c} = \dfrac{\mu_0\omega^4 E_0^2\alpha^2}{6\pi c} \propto \omega^4.$

BOOKWORK

The induced dipole is oscillating in the z-direction, so the radiation pattern follows that of a hertzian dipole oriented along Oz. The directional intensity is proportional to the gain function $G(\theta, \phi) = \frac{3}{2}\cos^2\theta$, and the radiation is linearly polarized in the θ-direction; i.e. vertically for propagation in the xy-plane.

UNSEEN

Because the particle has length much less than the radiation wavelength, and diameter much smaller still, the problem relates to Rayleigh scattering, i.e. to the induced dipole moment set up by the incident fields.

For incident fields along the axis of the particle, the boundary condition that $E_{||}$ across the interface is conserved requires that the internal field is comparable to the external, while for fields perpendicular to the axis of the particle the boundary condition that $D_{\perp}$ across the interface is conserved requires the internal field to be much less than the external. Whatever the orientation of the incident field E_0, the induced dipole $\boldsymbol{p}$ will therefore be (very largely) along the direction of the particle's axis; the particle is effectively a linear polarizer in 3D.

For the given scattering geometry: only p_x and p_z give rise to scattering in the Oy direction. Take $I_0 = E_0^2$ to calculate relative intensities.

(a) the component of the incident field along the axis of the particle is $E_0\cos\theta$, so $p_x \propto \cos\theta \times \sin\theta\cos\phi$ and $p_z \propto \cos\theta \times \cos\theta$. These are in phase and combine to produce linearly polarized radiation propagating in the Oy direction with components $E_x \propto \sin\theta\cos\theta\cos\phi$ and $E_z \propto \cos^2\theta$, i.e. linearly polarized at an angle $\tan^{-1}\left(\dfrac{\sin\theta\cos\phi}{\cos\theta}\right)$ to Oz. The overall intensity $I \propto \cos^2\theta(\cos^2\theta + \sin^2\theta\cos^2\phi)$.

(b) the component of the incident field along the axis of the particle is $E_0\sin\theta\sin\phi$, so $p_x \propto \sin\theta\sin\phi \times \sin\theta\cos\phi$ and $p_z \propto \sin\theta\sin\phi \times \cos\theta$. These are in phase and combine to produce radiation in the Oy direction with components $E_x \propto \sin^2\theta\sin\phi\cos\phi$ and $E_z \propto \sin\theta\cos\theta\sin\phi$, i.e. linearly polarized at an angle $\tan^{-1}\left(\dfrac{\sin\theta\cos\phi}{\cos\theta}\right)$ to Oz. The overall intensity $I \propto \sin^2\theta\sin^2\phi(\cos^2\theta + \sin^2\theta\cos^2\phi)$.

(c) right hand circularly polarized radiation can be written as $(E_0/\sqrt{2})(\widehat{\boldsymbol{y}} + i\widehat{\boldsymbol{z}})$, so the situation is described by a superposition of (a) and (b) with a $\pi/2$ phase shift. So $p_x \propto (\cos\theta + i\sin\theta\sin\phi) \times \sin\theta\cos\phi$ and $p_z \propto (\cos\theta + i\sin\theta\sin\phi) \times \cos\theta$. This again produces radiation in the Oy direction with components $E_x \propto p_x$ and $E_z \propto p_z$, i.e. linearly polarized at an angle $\tan^{-1}\left(\dfrac{\sin\theta\cos\phi}{\cos\theta}\right)$ to Oz.

(d) unpolarized radiation can be written as an incoherent superposition of $(E_0/\sqrt{2})\widehat{\boldsymbol{y}}$ and $(E_0/\sqrt{2})\widehat{\boldsymbol{z}}$, so the situation is described by an incoherent superposition of (a) and (b). So $p_x \propto (\cos^2\theta + \sin^2\theta\sin^2\phi)^{1/2} \times \sin\theta\cos\phi$ and $p_z \propto (\cos^2\theta + \sin^2\theta\sin^2\phi)^{1/2} \times \cos\theta$. This again produces linearly polarized radiation in

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the Oy direction with components $E_x \propto p_x$ and $E_z \propto p_z$, i.e. at an angle $\tan^{-1}\left(\frac{\sin\theta\cos\phi}{\cos\theta}\right)$ to Oz.

i.e. in each case the result is linearly polarized at $\tan^{-1}\left(\frac{\sin\theta\cos\phi}{\cos\theta}\right)$ to Oz, simply because however it is induced, the dipole always has components $p_z \propto \cos\theta$ and $p_x \propto \sin\theta\cos\phi$.

For the diffuse cloud of randomly oriented particles:

A needle along (θ, ϕ) (from (a)) produces linearly polarized radiation propagating in the Oy direction with components $E_x \propto \sin\theta\cos\theta\cos\phi$ and $E_z \propto \cos^2\theta$.

So a z-linear polarizer would transmit an intensity $\propto \cos^4\theta$. Integrating intensity (since the scatterers are incoherent) over all directions for the needles:

$$I_z \propto \int_0^{2\pi}\int_0^{\pi} \cos^4\theta \ \sin\theta \ \mathrm{d}\theta \ \mathrm{d}\phi = 2\pi\left[-\frac{1}{5}\cos^5\theta\right]_0^{\pi} = \frac{4\pi}{5}$$

Likewise a x-linear polarizer would transmit an intensity $\propto \sin^2\theta\cos^2\theta\cos^2\phi$, and integrating:

$$\begin{aligned} I_x \propto & \int_0^{2\pi}\int_0^{\pi} \sin^2\theta\cos^2\theta\cos^2\phi \ \sin\theta \ \mathrm{d}\theta \ \mathrm{d}\phi = \pi\int_0^{\pi} \sin^2\theta\cos^2\theta \ \sin\theta \ \mathrm{d}\theta \\ &= \pi\int_0^{\pi}(1-\cos^2\theta)\cos^2\theta \ \sin\theta \ \mathrm{d}\theta \\ &= \pi\left[-\frac{\cos^3\theta}{3}+\frac{\cos^5\theta}{5}\right]_0^{\pi} = \pi\left(\frac{1}{3}-\frac{1}{5}+\frac{1}{3}-\frac{1}{5}\right) = \frac{4\pi}{15} \end{aligned}$$

So the scattered beam along Oy must be *elliptically polarized*, or *partially polarized* with either an elliptical or linear polarized beam added to an unpolarized beam. Any elliptical component is not possible since this would require phase registration of $\pi/2$ between x and z components which is not possible since each particle scatters with x and z components in phase, and with mutual incoherence. So the beam is a superposition of unpolarized and z-polarized radiation. That there is such a linear polarized element is unsurprising since particles aligned along Oz are much more strongly excited by the incoming z-polarized radiation.

When densely packed, the particles form a dielectric medium with an isotropic polarizability, so the overall induced dipole moment lies along Oz, just as for a small spherical particle. The radiation scattered along Oy is therefore linearly polarized along Oz.

END OF PAPER

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