NATURAL SCIENCES TRIPOS Part II

Saturday 31st May 2008 9.00 to 12.00

EXPERIMENTAL AND THEORETICAL PHYSICS (4)

Candidates offering the **whole** *of this paper should attempt a total of* **six** *questions, three from Section A* **and** *three from Section B. The questions to be attempted are* **A1, A2** *and* **one** *other question from Section A and* **B1, B2** *and* **one** *other question from Section B.*

Candidates offering **half** *of this paper should attempt a total of* **three** *questions,* **either** *three from Section A* **or** *three from Section B. The questions to be attempted are* **A1, A2** *and* **one** *other question from Section A or* **B1, B2** *and* **one** *other question from Section B.*

Answers to **each** *question should be tied up separately, with the number of the question written clearly on the cover sheet.*

The **approximate** *number of marks allocated to each part of a question is indicated in the right margin. This paper contains 7 sides, and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS
Script paper
Metric graph paper
Rough work paper
Blue coversheets
Tags

SPECIAL REQUIREMENTS
Mathematical formulae handbook
Approved calculators allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

SOFT CONDENSED MATTER AND BIOPHYSICS

A1 *Attempt this question.*

Give concise answers to all three parts of the question. Relevant formulae may be assumed without proof.

(a) The fluid sap in a leaf at the top of a tree is held in thin capillaries of diameter 1μm. Assuming complete wetting, and that Laplace pressure is responsible for maintaining the column of sap from the ground up to the leaves, estimate the maximum possible height of the tree. [*Take the surface tension of sap to be* $72 \times 10^{-3}\mathrm{Nm}^{-1}$.] [4]

(b) The phase behaviour of a certain liquid mixture at temperature T can be described by the regular solution model with interaction parameter $\chi = 600\mathrm{K}/T$. Calculate the temperature at the critical point. [4]

(c) Ignoring any effect from gravity, the amplitude spectrum of thermal fluctuations of the interface between two fluids, as a function of the wavevector $\mathbf{q}$, is approximately

$$\langle h^2(\mathbf{q})\rangle = \frac{k_\mathrm{B}T}{A(\gamma q^2)},$$

where γ is the interfacial tension. This interface is in a square container with area $A = 1\mathrm{cm}^2$.

At room temperature, what is the average undulation amplitude due to thermal fluctuations for the interface between water and hexane, which has an interface tension of $38 \times 10^{-3}\mathrm{Nm}^{-1}$? [*Assume a small wavelength cutoff at the molecular scale, taking the molecular size to be* 0.2nm.] [4]

A2 *Attempt this question.*

Write brief notes on **two** of the following: [13]

(a) the liquid-crystal phase;

(b) factors that determine the entropic elasticity of rubber;

(c) interactions between colloidal particles;

(d) biological swimmers at low Reynolds number.

A3 *Attempt* **either** *this question* **or** *question A4.*

(a) Define what is meant by linear viscoelasticity. [4]

(b) Describe the Maxwell model, and show that its relaxation modulus is of the form

$$G(t) = G_0 \exp(-t/\tau).$$

[3]

(c) From the relaxation modulus given above, calculate the real and imaginary parts of the complex modulus, $G'(\omega)$ and $G''(\omega)$, for the Maxwell model. Identify the frequency above which the material would appear solid-like. [4]

(d) Define the compliance $J(t)$ of a fluid. Show that for a Newtonian fluid with viscosity η, $J(t) = t/\eta$. [2]

(e) The stochastic thermal motion of a colloidal sphere of radius a suspended in a material is recorded, measuring the mean-square displacement as a function of time. This microrheological experiment can be understood in analogy to a creep experiment, by considering the stress to be $\sigma_0 = \sqrt{\langle \sigma^2(t) \rangle} = k_B T/(\pi a^3)$ and the strain to be $\langle \boldsymbol{r}^2(t) \rangle / a^2$, where $\boldsymbol{r}(t)$ is the particle's position at time t.

Justify this treatment. [2]

Using the compliance given above, consider the result you expect from a microrheological creep experiment on a Newtonian fluid, and check that it is consistent with the Stokes-Einstein diffusion law. [3]

(f) Using the assumptions of the previous part, obtain the equation describing the mean-square displacement as a function of time for the case of a colloidal particle in a viscoelastic fluid described by the Maxwell model. [2]

(g) Sketch the average mean-square displacement as a function of time of 1μm radius beads at room temperature embedded in a tenuous 3d network with mesh size $\xi = 500$nm. Indicate significant features and relevant time and length scales. Comment on the feasibility of measuring the elastic modulus of this material by using video particle tracking. [*Assume the material is a Maxwell fluid, and take the viscosity to be* $\eta = 10^{-3}$Pa s.] [5]

A4 *Attempt* **either** *this question* **or** *question A3.*

(a) Consider the following random walks, each with an equal and large number of steps:

(i) 2d random walk;
(ii) 2d self-avoiding random walk;
(iii) projection on 2d of a 3d random walk;
(iv) projection on 2d of a 3d self-avoiding random walk.

Order these walks in order of increasing average end-to-end distance, providing justification for your choice. [5]

(b) Define the Kuhn length l_K of a polymer chain. What is the maximum possible value of l_K? What is l_K for a random walk with step size b? Discuss qualitatively the effect on the Kuhn length of introducing a fixed angle constraint between neighboring bonds in the polymer. [4]

(c) Flory's estimate of the scaling exponent of a polymer chain can be applied to chains in two dimensions. Show that the Flory exponent ν_{2d} for a self-avoiding polymer is equal to 3/4. [*The entropic elasticity has the same scaling form as for polymers in the bulk, but the excluded volume term can be estimated as*

$$E_{rep} = N k_B T b \frac{N}{A},$$

where A is the area occupied by the polymer of N monomers.] [3]

(d) (i) How is the polymer overlap concentration defined?

(ii) The polymer Poly-(Vinyl-Acetate) (PVAc) spreads on a water-air interface, with each of its monomers confined to the interface, forming a two-dimensional monolayer. The water-air interface acts as a good solvent for this polymer. For PVAc of polymerisation number $N = 10^4$, calculate the area fraction covered by polymer at the overlap concentration. [5]

(e) Calculate the scaling form of the osmotic pressure as function of the surface area fraction, for the PVAc monolayer described above, in the dilute and semidilute regimes. [8]

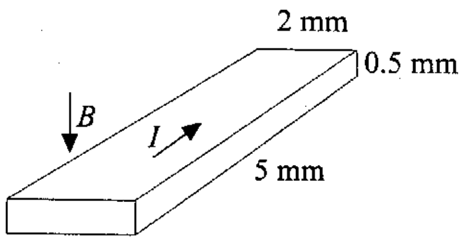
SECTION B

QUANTUM CONDENSED MATTER PHYSICS

B1 *Attempt this question.*

Give concise answers to all three parts of the question. Relevant formulae may be assumed without proof.

(a) The resistance of a uniform slab of semiconductor, of width 2 mm, thickness 0.5 mm and length 5 mm, is measured to be 1000 Ω along its length. A current $I = 1$ mA flows along the slab and a magnetic field $B = 1$ T is applied perpendicular to the widest face of the slab, as shown below. The Hall voltage, measured across the slab, is 10 μV. Find the carrier density and mobility. [4]



(b) For a monatomic crystal with a simple-cubic lattice, show that the number of electronic states per atom in the first Brillouin zone is $2N$ per band, where N is the number of atoms in the crystal. How does this help to understand which materials are metals or insulators? [4]

(c) What are plasmons? Estimate the plasma frequency $\omega_{\mathrm{p}} = (ne^2/\epsilon_0 m)^{1/2}$ in a typical metal (interatomic spacing ~ 3 Å). [4]

B2 *Attempt this question.*

Write brief notes on **two** of the following: [13]

(a) a comparison of the band structures derived using the tight-binding and nearly-free-electron approximations;

(b) the Thomas-Fermi approximation, and screening in semiconductors;

(c) the Stoner-Hubbard model for magnetism in metals;

(d) the integer quantum Hall effect.

(TURN OVER

B3 *Attempt* **either** *this question* **or** *question B4.*

Discuss, with appropriate sketches, how conservation laws affect the scattering of electrons in semiconductors and metals by

(a) photons;

(b) acoustic phonons;

(c) optical phonons. [11]

An electron at energy E_1 above the bottom of the conduction band of a direct-gap semiconductor excites another electron from the valence band into the conduction band leaving a hole in the $\mathbf{k} = \mathbf{0}$ state. Establish the minimum value of E_1 for the process to occur, in terms of the bandgap E_g. Assume that the electron effective mass in the conduction band is independent of energy and that the electron energy varies as the square of the wavevector. [9]

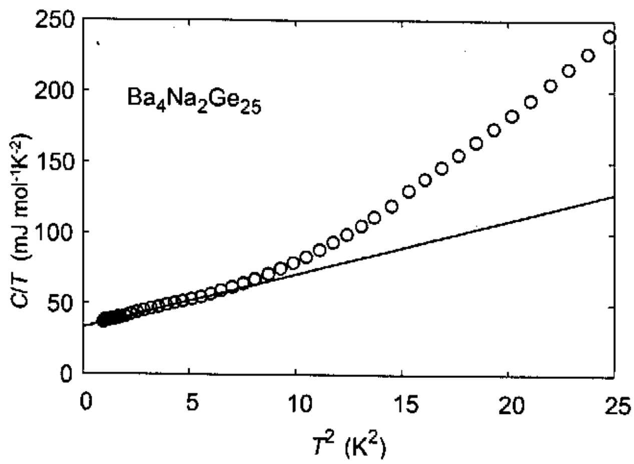
Describe the subsequent energy-loss processes. [5]

B4 *Attempt* **either** *this question* **or** *question B3.*

Show that the molar heat capacity C of metals at low temperature, T, takes the form

$$C = \gamma T + \beta T^3 \quad ,$$

where $\gamma = \frac{\pi^2}{3} k_B^2 g(E_F)$ and $\beta = \frac{12\pi^4}{5} N_A k_B \theta_D^{-3}$ are material-dependent constants. $g(E_F)$ is the molar density of states at the Fermi energy, E_F, θ_D is the Debye temperature, and N_A is Avogadro's number. [*It is not necessary to deduce the precise form of the prefactors.*] [7]



The graph shows the heat capacity per mole of the metallic compound $Ba_4Na_2Ge_{25}$, measured up to 5 K, and plotted as C/T vs. T^2. The line is a linear fit to the low-temperature region. The molar volume of $Ba_4Na_2Ge_{25}$ is $V_m = 4.6 \times 10^{-4}$ m^3.

From the measured data, extract the parameter γ and find $g(E_F)$. Assuming a free-electron model with a single, parabolic band and two conduction electrons per formula unit, estimate the Fermi energy, the effective mass, and the Fermi wave vector, k_F. [10]

From the measured data, extract the parameter β and find θ_D. Estimate the Debye wave vector k_D and the speed of sound in this material. [6]

Why does the measured heat capacity at temperatures $T^2 > 10$ K^2 deviate significantly from the low-temperature form discussed above? What form do you expect the molar heat capacity to take at even higher temperatures $T \gg \theta_D$? [2]

END OF PAPER