UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Friday 4 June 2021 11.00-13.00

PHYSICS (5)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (5)

ASTROPHYSICAL FLUID DYNAMICS

Candidates offering this paper should attempt a total of ***five*** *questions:* ***three*** *questions from Section A and* ***two*** *questions from Section B.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains ***five*** *sides, including this coversheet, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
2 × 20 Page Answer Book	Mathematical Formulae handbook
Metric graph paper	Approved calculator allowed
Rough workpad	
Yellow master coversheet	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

*Attempt **all** questions in this Section. Answers should be concise and relevant formulae may be assumed without proof.*

1 Someone explains the momentum equation as follows: "The momentum of a volume element δV of fluid with velocity $\mathbf{u}$ and density ρ is $\rho\mathbf{u}\,\delta V$. The force that acts (ignoring gravity) is $-\boldsymbol{\nabla} p\,\delta V$, where p is the pressure, so Newton's law tells us

$$\frac{D}{Dt}\left(\rho\mathbf{u}\,\delta V\right) = -\boldsymbol{\nabla} p\,\delta V, \text{ or}$$
$$\frac{D}{Dt}\left(\rho\mathbf{u}\right) = -\boldsymbol{\nabla} p,$$

which is the momentum equation.". What has gone wrong, and what is the correct argument?

An incompressible inviscid fluid flows without the application of any external forces except gravity. Based on measurements of the velocity $\mathbf{u}(\mathbf{r},t)$ alone, would it be possible to tell if the fluid was replaced with another of different density? Explain.

[4]

2 If the magnetic field due to the Sun's dipole moment decayed as $1/r^3$ it would be roughly 10^{-11} T at the Earth's orbit. Instead, it is roughly 100 times greater. Suggest a possible explanation in terms of the solar wind, which may be treated as a highly conducting plasma. The Sun's radius is 0.5% of the radius of the Earth's orbit. [4]

3 The dispersion relation of waves on the surface of a fluid of density ρ with surface tension σ in a gravitational field g is

$$\omega(k) = \sqrt{|k|\left(g + \sigma k^2/\rho\right)}.$$

A stone dropped into water produces circular ripples with the most pronounced ripple corresponding to the minimum phase velocity. Find the speed of this ripple if $\sigma = 7.2\times10^{-2}$ N m^{-1} and $\rho = 1000$ kg m^{-3}.

[4]

SECTION B

Attempt two questions from this section

4 (a) Describe briefly the circumstances under which shocks are likely to form in astrophysical systems, and explain the physical origin of the Rankine-Hugoniot equations for a normal shock

$$\begin{aligned} \rho_2 u_2 &= \rho_1 u_1 \\ p_2 + \rho_2 u_2^2 &= p_1 + \rho_1 u_1^2 \\ \frac{\gamma}{\gamma-1}\frac{p_2}{\rho_2} + \frac{1}{2}u_2^2 &= \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1} + \frac{1}{2}u_1^2. \end{aligned}$$

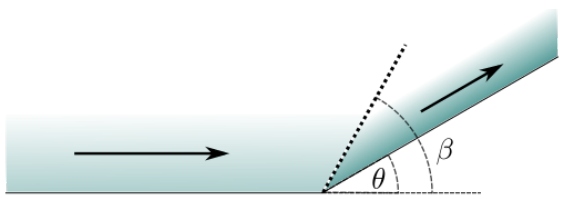
The subscripts 1 and 2 refer to upstream and downstream, respectively. $u_{1,2}$ are the normal velocities, $p_{1,2}$ the pressures and $\rho_{1,2}$ the densities. γ is the adiabatic index of the gas. [5]

(b) Take the limit in which the upstream pressure is negligible. Show that in this case the downstream velocity is related to the upstream velocity by

$$u_2 = \frac{\gamma-1}{\gamma+1}u_1.$$ [3]

(c) Gas travelling supersonically with a Mach number $M \gg 1$ is flowing parallel to a surface (see figure below). The surface has a concave corner of external angle θ. An oblique shock forms making an angle β with respect to the surface before the corner, and flow after the corner is again parallel to the surface. Show that β and θ are related by

$$\tan\beta = \frac{\gamma+1}{\gamma-1}\tan(\beta-\theta).$$ [5]



(d) Show that for $\theta < \theta_{\max} \equiv \arcsin(1/\gamma)$ there are two possible solutions for the shock angle β. Find the behaviour of these solutions as $\theta \to 0$, and suggest which is likely to be observed. [5]

(e) What do you think happens for $\theta > \theta_{\max}$? [1]

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5 This question concerns an isothermal fluid obeying $p = K\rho$, where ρ is the density, p is the pressure, and K is a constant.

(a) Find the equation describing $\rho(r)$ for a self-gravitating spherically symmetric fluid body in hydrostatic equilibrium, where r is the radial distance from the centre. [4]

(b) Show that the substitution $\rho = \rho_c e^{-\Psi}$, $r = a\xi$ with $a = \sqrt{\frac{K}{4\pi G\rho_c}}$ puts this equation in the form

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\Psi}{d\xi}\right) = e^{-\Psi}. \qquad (\star)$$

State and explain the boundary conditions on Ψ if ρ_c is the central density. [3]

(c) Show that the number of particles inside a sphere of radius $a\xi$ is

$$N = \frac{4\pi}{m}\rho_c\left(\frac{K}{4\pi G\rho_c}\right)^{3/2}\xi^2\frac{d\Psi(\xi)}{d\xi},$$

where m is the mass of the particles. [3]

The stability of the body can be analyzed by considering

$$\left(\frac{\partial p}{\partial V}\right)_{N,T} = \frac{K}{4\pi r^2}\left(\frac{\partial \rho}{\partial r}\right)_{N,T},$$

which is negative in a stable system.

(d) Find $\delta\rho$ when $\rho_c \to \rho_c + \delta\rho_c$ and $\xi \to \xi + \delta\xi$. Find the relation between $\delta\rho_c$ and $\delta\xi$ if N is fixed. Express the criterion $\delta\rho|_N = 0$ (i.e. at fixed N) in terms of the function $\Psi(\xi)$ and its first derivative. [5]

(e) The numerical solution of $(\star)$ shows that $\xi = 6.5$ is the smallest value where the criterion of part (d) is satisfied. Find the critical radius for stability. Compare this with the Jeans length $\lambda_{\mathrm{J}} = \sqrt{\frac{\pi c_s^2}{G\rho_0}}$ for a body of uniform density ρ_0 with speed of sound c_s, and comment on the merits of each model for gravitational collapse. [4]

6 This question concerns the stability of rotating *incompressible* fluids.

(a) An incompressible inviscid (zero viscosity) fluid may rotate with an angular velocity $\Omega(r)$, where the dependence on the radial coordinate r is arbitrary. Explain why this is, and give an expression for the radial pressure gradient. [2]

(b) When the viscosity is finite, the possible form of $\Omega(r)$ is restricted. Find the general behaviour in this case. [5]

The remainder of the question concerns the inviscid case. The stability of the flow may be assessed by expanding the velocity in small axisymmetric deviations. In cylindrical coordinates (r, ϕ, z) the velocity $\mathbf{u}(r, z, t)$ is written

$$\mathbf{u}(r,z,t) = u_r(r,z,t)\hat{\mathbf{r}} + [\Omega(r)r + u_\phi(r,z,t)]\,\hat{\boldsymbol{\phi}} + u_z(r,z,t)\hat{\mathbf{z}}.$$

The deviations u_r, u_ϕ, and u_z satisfy the equations

$$\begin{aligned}
\frac{\partial u_r}{\partial t} - 2\Omega(r)u_\phi &= -\frac{\partial \pi}{\partial r} \\
\frac{\partial u_\phi}{\partial t} + \left(\frac{d}{dr}(r^2\Omega)\right)\frac{u_r}{r} &= 0 \\
\frac{\partial u_z}{\partial t} &= -\frac{\partial \pi}{\partial z} \\
\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} &= 0
\end{aligned} \qquad (\star)$$

(c) Briefly explain the origin of these equations, and the meaning of $\pi(r, z, t)$. [3]

Now consider flow in a thin region between two concentric cylinders with radii $R_<$ and $R_>$ where $R_> - R_< \ll R_<$. In this case it is possible to ignore the second term of $(\star)$, which becomes

$$\frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z} = 0.$$

(d) Assume $u_i(r, z, t) = u_i(r)e^{i(kz-\omega t)}$ for $i = r, \phi, z$ and $\pi(r, z, t) = \pi(r)e^{i(kz-\omega t)}$. Show that $u_r(r)$ obeys the ordinary differential equation

$$-\frac{d^2u_r}{dr^2} + \frac{k^2}{\omega^2}\Phi(r)u_r = -k^2u_r, \qquad (\dagger)$$

where the function $\Phi(r)$ depends only on $\Omega(r)$. State and justify the boundary conditions. [5]

(e) What condition on $\Phi(r)$ guarantees that the flow is stable? Explain your answer. $\big[$*Hint:* $(\dagger)$ *is a time-independent Schrödinger equation*$\big]$ [4]

END OF PAPER