NATURAL SCIENCES TRIPOS Part II

Saturday 29 May 2010 9.00 am to 12.00 noon

EXPERIMENTAL AND THEORETICAL PHYSICS (2)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (2)

Candidates offering the **whole** *of this paper should attempt a total of* **six** *questions, three from Section A* **and** *three from Section B. The questions to be attempted are* **A1**, **A2** *and* **one** *other question from Section A and* **B1**, **B2** *and* **one** *other question from Section B.*

Candidates offering **half** *of this paper should attempt a total of* **three** *questions,* **either** *three from Section A* **or** *three from Section B. The questions to be attempted are* **A1**, **A2** *and* **one** *other question from Section A or* **B1**, **B2** *and* **one** *other question from Section B. These candidates will leave after* **90 minutes.**

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains **7** *sides, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

A separate Answer Book should be used for each section.

STATIONERY REQUIREMENTS
2×20 Page Answer Book
Metric graph paper
Rough workpad
Yellow master coversheet

SPECIAL REQUIREMENTS
Mathematical Formulae handbook
Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

ADVANCED QUANTUM PHYSICS

A1 *Attempt* **all** *parts of this question. Answers should be concise, and relevant formulae may be assumed without proof.*

(a) Write down the operator corresponding to the spin component in the direction $\widehat{\boldsymbol{n}} = (1/\sqrt{2})(1, 0, 1)$ for a spin-half particle, and calculate its eigenvalues and non-normalised eigenvectors. [4]

$\left[\textit{The Pauli spin matrices } \sigma_x, \sigma_y, \sigma_z \textit{ are } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \right]$

(b) For a wavefunction of the form $\psi = Axf(r)$, where $r^2 = x^2 + y^2 + z^2$, show that the uncertainty in the angular momentum component L_x is zero. [4]

(c) A two-dimensional harmonic oscillator is described by a potential of the form

$$V = \frac{1}{2}m\omega^2\left[(x^2 + y^2) + \alpha(x - y)^2\right],$$

where α is a positive constant. Find the ground-state energy of the oscillator. [4]

A2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) the addition of angular momenta in quantum mechanics;

(b) Fermi's golden rule;

(c) space and spin wavefunctions for identical particles.

A3 *Attempt* **either** *this question* **or** *question A4.*

A system is described by a Hamiltonian $\widehat{H}_0$ possessing a complete set of non-degenerate eigenstates $|n^0\rangle$ of energy E_n^0. Show that, when a small perturbing potential $\widehat{H}_1$ is applied to the system, the first-order contribution to the change in energy of the n^{th} eigenstate is $E_n^1 = \langle n^0|\widehat{H}_1|n^0\rangle$. (Note that the superscripts are used to label the order of each term.) [6]

Show also that the amplitude of the state $|m^0\rangle$ in the first-order perturbation expansion of the wave function for the nth state ($m \neq n$) is

$$\langle m^0|n^1\rangle = \frac{\langle m^0|\widehat{H}_1|n^0\rangle}{E_n^0 - E_m^0}.$$ [2]

Hence show that the second-order contribution to the change in energy of the n^{th} state is

$$E_n^2 = \sum_{m\neq n} \frac{\left|\langle m^0|\widehat{H}_1|n^0\rangle\right|^2}{E_n^0 - E_m^0}.$$ [5]

A particle moving in a simple-harmonic potential of frequency ω is described by the Hamiltonian

$$\widehat{H}_0 = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right),$$

where

$$[a, a^\dagger] = 1.$$

A small perturbation is applied to the system, giving the total Hamiltonian $\widehat{H} = \widehat{H}_0 + \widehat{H}_1$, where

$$\widehat{H}_1 = \beta\left[(a^\dagger)^3 + 3(a^\dagger)^2 a + 3a^\dagger a^2 + a^3\right].$$

Find the energy of the perturbed ground state up to and including terms of second order in β. [8]

Describe briefly circumstances in which perturbation theory in the form described above may break down or need modification. [4]

[*The m^{th} excited state of a simple-harmonic oscillator is given by*

$$|m\rangle = \frac{(a^\dagger)^m}{\sqrt{m!}}|0\rangle,$$

where $|0\rangle$ is the ground state.]

(TURN OVER

A4 *Attempt* **either** *this question* **or** *question A3.*

State Hund's rules and explain the underlying physical principles upon which they are based. [7]

The neutral zirconium atom has electronic configuration

$$[\mathrm{Kr}](5\mathrm{s})^2(4\mathrm{d})^2.$$

Determine the spectroscopic terms for the possible multiplets and predict which is the ground state, assuming that LS coupling holds approximately in zirconium. [5]

A sample of zirconium is placed in a magnetic field of 1 T. Show that it can absorb microwaves with a wavelength of approximately 32 mm. [6]

State the selection rules that apply to the total angular momentum quantum numbers J and m_J. Determine the spectrum for transitions in zirconium between a ${}^1\mathrm{D}_2$ state and a ${}^3\mathrm{F}_2$ state in the presence of a weak magnetic field B, expressing the line separations in terms of $\mu_B B$, where μ_B is the Bohr magneton. [7]

[*You may assume the formula for the Landé g-factor:*

$$g = \frac{3J(J+1) + S(S+1) - L(L+1)}{2J(J+1)}.$$
]

SECTION B

OPTICS AND ELECTRODYNAMICS

B1 *Attempt* **all** *parts of this question. Answers should be concise, and relevant formulae may be assumed without proof.*

(a) The power gain of a Hertzian electric dipole is given by $G = (3/2)\sin^2\theta$. What is the maximum effective area in μm^2 of such a dipole when illuminated by light of wavelength 500 nm? [4]

(b) A 1D photonic crystal is fabricated from equal volumes of two materials of refractive index $n_1 = 1.4$ and $n_2 = 1.6$, with a periodicity 200 nm. Sketch the dispersion relation for the crystal and find the approximate wavelength at the bandgap for normal incidence light. [4]

(c) For an electron, the phase difference between two paths enclosing a magnetic flux Φ is $\Delta = e\Phi/\hbar$. Indicate the origin of this equation and estimate the magnetic field strength B required to convert constructive to destructive electron interference around a circular conducting loop of radius 1 μm. [4]

B2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) what spatial and spectral coherence reveal about a light source;

(b) how antennas can be made directional;

(c) the properties of synchrotron radiation.

(TURN OVER

B3 *Attempt* **either** *this question* **or** *question B4.*

Define the *four-vector potential* and *four-vector current* and explain how they can be used to write Maxwell's equations in Lorentz-covariant form. [5]

Using the four-vector transformation given by

$$\begin{pmatrix} a_0' \\ a_1' \\ a_2' \\ a_3' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix},$$

derive the following transformation relations for electric and magnetic fields from an inertial frame S to an inertial frame S' moving relative to S with velocity u in the x-direction:

$$\begin{aligned} E_x' &= E_x, & B_x' &= B_x \\ E_y' &= \gamma(E_y - uB_z), & B_y' &= \gamma(B_y + uE_z/c^2) \\ E_z' &= \gamma(E_z + uB_y), & B_z' &= \gamma(B_z - uE_y/c^2). \end{aligned}$$ [6]

A large thin conducting plate lying in the x–y plane carries a charge σ per unit area and a current density in the y-direction of magnitude J_y (per unit width), both measured in the rest frame of the plate. Find the electric field outside the plate in its rest frame, and show that the electrostatic potential in the region $z > 0$ is given by

$$\phi = -\frac{\sigma z}{2\epsilon_0}.$$ [2]

Given that the magnetic vector potential above the plate is $A = (0, -\mu_0 J_y z/2, 0)$, deduce the magnetic field measured in the rest frame of the plate. [2]

By explicitly transforming the fields from the rest frame, calculate the electric and magnetic fields measured in a frame moving relative to the plate with relativistic velocity u in the x-direction. [4]

Hence find expressions for the charge and current densities measured in the moving frame. Account qualitatively for how the charge and current densities in the moving frame differ from those measured in the rest frame. [6]

B4 *Attempt* **either** *this question* **or** *question B3.*

Explain the use of Jones vectors and Jones matrices to describe the propagation of polarised light through optical components. [4]

Show that the Jones matrix for a linear polariser mounted with its transmitting axis at an angle θ to the transverse x-axis of a propagating light beam is

$$\underline{\underline{J}}(\theta) = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix},$$

and use this result to prove that no light can pass through crossed polarisers. [5]

An optical system consists of N linear polarisers mounted sequentially, with the transmitting axis of the n^{th} polariser oriented at an angle $n\theta$ ($n = 1, 2, \ldots, N$) to the x-axis. By evaluating the matrix $\underline{\underline{R}}(\theta)\cdot\underline{\underline{J}}(\theta)$, or otherwise, show that

$$\begin{pmatrix} j_{x'} \\ j_{y'} \end{pmatrix} = \cos^N\theta \begin{pmatrix} 1 & \tan\theta \\ 0 & 0 \end{pmatrix} \begin{pmatrix} j_x \\ j_y \end{pmatrix},$$

where (j_x, j_y) is a Jones vector describing the incident light beam in a coordinate basis defined by the x-axis, and $(j_{x'}, j_{y'})$ is the Jones vector for the transmitted light in a coordinate basis defined such that the x'-axis is oriented along the transmitting axis of the final polariser. [6]

For the case that the incident light beam is linearly polarised along the x-direction and emerges with its plane of polarisation rotated through 90°, find the fractional transmitted intensity for a system containing $N = 20$ polarisers and show that the number of polarisers needed to reduce the intensity loss below 1% is given approximately by

$$N \approx \frac{(10\pi)^2}{4}.$$ [8]

Explain why it is impracticable to construct such a low-loss polarisation rotation element from conventional linear polarisers. [2]

[*The matrix which rotates the orientation of axes through an angle θ is*

$$\underline{\underline{R}}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$
]

END OF PAPER

A1a NST II ETP (2) MAY 2010 (p.1)

Find the Pauli spin matrix in the direction $\hat{n} = (1,0,1)/\sqrt{2}$.

Spin matrix $\parallel \hat{n} = \underline{\sigma}\cdot\hat{n}$ where $\underline{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$.

$$\sigma_n = \underline{\sigma}\cdot\hat{n} = \left(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right)\frac{1}{\sqrt{2}} = \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\frac{1}{\sqrt{2}}. \qquad \boxed{1}$$

$$\sigma_n\begin{pmatrix}a\\ b\end{pmatrix} = \alpha\begin{pmatrix}a\\ b\end{pmatrix} \Rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}a+b\\ a-b\end{pmatrix} = \alpha\begin{pmatrix}a\\ b\end{pmatrix} \qquad \boxed{1}$$

$\Rightarrow \quad a+b = \sqrt{2}\alpha a \Rightarrow b = (\sqrt{2}\alpha - 1)\,a$ (1)

and $a-b = \sqrt{2}\alpha b \Rightarrow a = b(1+\alpha\sqrt{2})$ (2)

$= ((\alpha\sqrt{2})^2 - 1)\,a$ from (1) $\boxed{1}$

$\therefore \quad 1 = 2\alpha^2 - 1 \quad (a \neq 0)$

$\therefore 2\alpha^2 = 2$

$\therefore \alpha^2 = \pm 1.$ (e'values ±1 as expected)

$\boxed{1}$

$\therefore$ non normalised e'vectors are $\begin{pmatrix}1\\ \pm\sqrt{2}-1\end{pmatrix}$.

[Normalisation: $1 = 1 + (\pm\sqrt{2}-1)^2$
$= 2 + 2 \mp 2\sqrt{2}$
$= 4 \mp 2\sqrt{2}$.]
not required.

PAPER 2, SECTION A

ADVANCED QUANTUM PHYS.

A i b

$$\psi = A x f(r) \qquad (r^2 = x^2 + y^2 + z^2)$$

$$\hat{\underline{L}} = \underline{r} \wedge \hat{\underline{p}} \qquad \hbar \qquad \boxed{1}$$

$$L_x = \frac{\hbar}{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \boxed{1/2} \quad \therefore \; [L_x, x] = 0.$$

NB $\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$ $\boxed{\tfrac{1}{2}}$

$$\therefore L_x \psi = \frac{\hbar}{i} A x \left(y \frac{\partial r}{\partial z}\frac{df}{dr} - z\frac{\partial r}{\partial y}\frac{df}{dr}\right)$$

$$= \frac{\hbar}{i} A x \left(yz f' - zy f'\right) = 0. \qquad \boxed{1}$$

$\therefore L_x^2 \psi = 0$.

$\therefore$ Uncertainty $\Delta L_x = \left(\langle \hat{L}_x^2 \rangle - \langle \hat{L}_x \rangle^2\right)^{1/2} = 0.$ $\boxed{1}$

A1c

$V = \frac{1}{2} m\omega^2 [(x^2+y^2) + \alpha(x-y)^2]$.

Try to write in new coords $x' = \frac{x+y}{\sqrt{2}}$, $y' = \frac{x-y}{\sqrt{2}}$ $\boxed{1}$

($\sqrt{2}$ preserves length).

$\therefore \frac{2V}{m\omega^2} = \frac{2x'^2 + 2y'^2}{2} + 2\alpha y'^2 = x'^2 + y'^2(1+2\alpha)$. $\boxed{1}$

$\therefore$ This looks like 2 1D ~~Sch eqns~~ potentials added together, indep of each other. $\boxed{1}$

So energies are $(n+\frac{1}{2})\hbar\omega$ and $(m+\frac{1}{2})\hbar(\omega\sqrt{1+2\alpha})$

So total energy of g.s ($m=n=0$) is $\frac{1}{2}\hbar\omega(1+\sqrt{1+2\alpha})$. $\boxed{1}$

or

Could instead say:

let $x^2+y^2+\alpha(x-y)^2 = \beta(x+y)^2 + \gamma(x-y)^2$

$= (\beta+\gamma)x^2 + (\beta+\gamma)y^2 + xy(2\beta-2\gamma)$ $\boxed{1}$

(x^2) $\therefore$ $\beta+\gamma = 1+\alpha$ ①

(xy) $\cancel{2}(\beta-\gamma) = -\cancel{2}\alpha$ ②

(y^2) $\beta+\gamma = 1+\alpha$ (as ①)

① + ② $\Rightarrow 2\beta = 1$ $\therefore \beta = \frac{1}{2}$

① − ② $\Rightarrow 2\gamma = 1+2\alpha$ $\therefore \gamma = \frac{1}{2}(1+2\alpha)$.

$\therefore V = \frac{1}{2} m\omega^2 \left(\left(\frac{x+y}{\sqrt{2}}\right)^2 + \left(\frac{x-y}{\sqrt{2}}\right)^2 (1+2\alpha) \right)$ $\boxed{1}$

NB $\frac{1}{\sqrt{2}}$ in new coords to preserve length, so we can use the standard results $\boxed{1}$

$\therefore$ Like 1D ptls in $\frac{x+y}{\sqrt{2}}$ and $\frac{x-y}{\sqrt{2}}$, indep. of each other, so energy is sum of these 1D energies, so for ground state

$E = \frac{1}{2}\hbar\omega + \frac{1}{2}\hbar\omega\sqrt{1+2\alpha}$ $\boxed{1}$

[gives expected result if $\alpha = 0$] ✓

Brief notes, ~~6~~ 7 marks each

2) Addition of angular momenta: p 58 –

Often need to add angular momenta, eg $\underline{L}+\underline{S}$, $\underline{J}_1+\underline{J}_2$, two spins, etc. ~~$\underline{S}_1+\underline{S}_2$~~ ~~etc~~.

(1

Consider 2 indep. ang. mom. $\underline{J}_1, \underline{J}_2$, with $\underline{J}=\underline{J}_1+\underline{J}_2$, $[\underline{J}_1,\underline{J}_2]=0$.

Usually require a basis in which $\underline{J}^2$ is diagonal, together with J_z, $\underline{J}_1^2$, $\underline{J}_2^2$.

(quantum no.s J, m_J, j_1, j_2)

Instead of the basis $\underline{J}_1, J_{1z}, \underline{J}_2, J_{2z}$

with q-no.s $\{j_1, m_1, j_2, m_2$.

Use matrix elements called Clebsch–Gordon coeffs to convert between these bases.

Strategy to generate basis states: (1) Find basis state with maximal $\{\underline{J}^2\} = J_{max}$ and $M_J = J_{max}$.

– easy since take all the highest components.
(eg $M_1 = J_{1max}$, $M_2 = J_{2max}$).

(2) Use lowering op $\hat{J}_-$ to find all states with $J=J_{max}$, and $M_J = J_{max}-1, \dots, -J_{max}$

(3) From state with $J=J_{max}$ & $M_J = J_{max}-1$, obtain state with $J = J_{max}-1$ and $M_J = J_{max}-1$ by orthogonality. Repeat step 2, etc.

May give Examples

(plenty of marks)

~~Time dep pert theory~~

p143-4 p164

Fermi's golden rule ~~(and its application to the Born approximation)~~

Do time-dependent pert.-theory for harmonic p^tl $V(t) = V e^{-i\omega t}$, which is turned on abruptly at time $t=0$, when system is state $|i\rangle$ (eg atom in oscillating electric field) $\boxed{1}$

FGR gives probability that system lies in state $|f\rangle$ at time t $\boxed{1}$

$$c_f^{(1)}(t) \propto \int_0^t dt' \langle f|V|i\rangle e^{i(\omega_{fi}-\omega)t'}$$ $\boxed{1}$

$$P_{i\to f}(t) = |c_f^{(1)}(t)|^2 \propto |\langle f|V|i\rangle|^2 \left(\frac{\sin((\omega_{fi}-\omega)\frac{t}{2})}{(\omega_{fi}-\omega)/2}\right)^2$$

As $t \to \infty$, this ( ) tends to a δ function $\times t$. $\boxed{1}$

∴ divide by t to get transition rate.

$$R_{i\to f}(t) = \lim_{t\to\infty} \frac{P_{i\to f}(t)}{t} = \frac{2\pi}{\hbar^2} |\langle f|V|i\rangle|^2 \delta(\omega_{fi}-\omega)$$ $\boxed{1}$

$$(\hbar\omega_{fi} = E_f - E_i).$$

– FGR $\boxed{1}$

Long-time limit reached when $t \gg \dfrac{1}{\omega_{fi}-\omega}$ (short still compared with mean transition time)

~~The~~ Notice that : (a) final states must exist over a continuous energy range to match $\Delta E = \hbar\omega$ for fixed ω, or $\boxed{1}$

(b) pert. must cover a sufficiently wide spectrum so that a discrete transition with fixed $\Delta E = \hbar\omega$ is possible. ($\boxed{1}$)

Space and spin wave functions for identical particles in QM or Particle

p78 – 80

Lots to say – too much for 6 marks?

Identical particles are indistinguishable in QM.
(– i.e. particles having any internal q. no (eg spin) the same.)

Two-particle wavefns $\psi(x_1, x_2)$ give probabilities $|\psi(x_1,x_2)|^2 dx_1 dx_2$ of finding simultaneously one particle in x_1 to $x_1 + dx_1$ & the other in x_2 to $x_2 + dx_2$

– but this prob'y must be symmetric since we can't tell which particle is which.

So wavefns must be symm. or antisymm under exchange of particles.

i.e. $\psi(x_1, x_2) = \pm\, \psi(x_2, x_1)$

Particles always have one sign or other (+ for bosons, – for fermions)

For multiple non-interacting identical particles, construct wavefn using a Slater determinant. if $\psi_a(1)$, $\psi_b(2)$, $\psi_c(3)$ — are wavefns for particles 1, 2, 3 etc, need to combine terms of the form $\psi_a(1)\psi_b(2)\psi_c(3)$ etc, S. Det is

$$\frac{1}{\sqrt{N!}}\begin{vmatrix} \psi_a(1) & \psi_b(1) & \psi_c(1) & \cdots \\ \psi_a(2) & & & \\ \psi_a(3) & & & \\ \vdots & & & \end{vmatrix}$$

Clearly antisymmetric. (or + sign for bosons.)

Space & spin wavefns are independent & so are multiplied together. Each must be symm or a'symm, & product must satisfy requirement for fermion or boson. (fermion asymm = sym × as. or as. × sym)

eg total wavefn $\psi(x_1, x_2)\chi(s_1, s_2)$ for 2 particles, spins s_1, s_2.

For fermions, if spatial state sym., then spin state asymm $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ & vice versa (3 spin states, triplet, $|\uparrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$, $\frac{1}{\sqrt{2}}(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle)$.)

Pauli exclusion ppl ...

H.S.

$$H = H_0 + H_1 \quad ①$$

(p63)

$$H_0 |n^{(0)}\rangle = E_n^{(0)} |n^{(0)}\rangle \quad ②$$

Use factor λ with H_1 to show order of approx.

Seek $|n\rangle$ s.t. $(H_0 + \lambda H_1)|n\rangle = E_n |n\rangle$. $\boxed{1}$

Expand as $|n\rangle = |n^{(0)}\rangle + \lambda |n^{(1)}\rangle + \lambda^2 |n^{(2)}\rangle + O(\lambda^3)$

$E_n = E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + O(\lambda^3)$. $\boxed{1}$

$$\therefore (H_0 + \lambda H_1)\left(|n^{(0)}\rangle + \lambda |n^{(1)}\rangle + \lambda^2 |n^{(2)}\rangle + O(\lambda^3)\right)$$

$$= \left(E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + O(\lambda^3)\right)\left(|n^{(0)}\rangle + \lambda |n^{(1)}\rangle + \lambda^2 |n^{(2)}\rangle + O(\lambda^3)\right) \quad \boxed{1}$$

$$\therefore \cancel{H_0 |n^{(0)}\rangle} + \lambda\left(H_0 |n^1\rangle + H_1 |n^0\rangle\right) + \lambda^2\left(H_0 |n^{(2)}\rangle + H_1 |n^1\rangle\right) + O(\lambda^3)$$

$$= \cancel{E_n^0 |n^0\rangle} + \lambda\left(E_n^{(0)} |n^1\rangle + E_n^{(1)} |n^0\rangle\right) + \lambda^2\left(|n^1\rangle E_n^{(1)} + E_n^0 |n^{(2)}\rangle + E_n^2 |n^0\rangle\right) \quad ③ \quad \boxed{1}$$

1st order in λ:

$$H_0 |n^1\rangle + H_1 |n^0\rangle = E_n^0 |n^1\rangle + E_n^1 |n^0\rangle \quad ④$$

$\langle n^0| \times ④$:

$$\underbrace{\langle n^0| H_0 |n^1\rangle}_{= E_n^0 \langle n^0|} + \langle n^0| H_1 |n^0\rangle = E_n^0 \langle n^0|n^1\rangle + E_n^1 \underbrace{\langle n^0|n^0\rangle}_{=1} \quad \boxed{1}$$

so cancels on RHS.

$$\therefore \underline{\underline{E_n^{(1)} = \langle n^0| H_1 |n^0\rangle}} \quad ⑤$$

1st order shift $\boxed{1}$

$(\boxed{6})$

2nd part – see printed answer.

Find $|n^1\rangle$: take $m \neq n$.

$\langle m^0| \times (4)$:
$$\underbrace{\langle m^0|H_0}_{E_m^0\langle m^0|}|n^1\rangle + \langle m^0|H_1|n^0\rangle = E_n^0\langle m^0|n^1\rangle + 0 .$$ (since m, n orthog.

$$\therefore \quad \langle m^0|n^1\rangle = \frac{\langle m^0|H_1|n^0\rangle}{E_n^0 - E_m^0} \qquad (6)$$

$\boxed{2}$ ($/2$)

2nd order: λ^2 terms in (3): $\boxed{1}$

$$H_0|n^2\rangle + H_1|n^1\rangle = |n^1\rangle E_n^1 + E_n^0|n^2\rangle + E_n^2|n^0\rangle \qquad (7)$$

$\langle n^0| \times (7)$

$$\cancel{\underbrace{\langle n^0|H_0}_{E_n^0\langle n^0|}|n^2\rangle} + \langle n^0|H_1|n^1\rangle = E_n^1\langle n^0|n^1\rangle + \cancel{E_n^0\langle n^0|n^2\rangle} + E_n^2\underbrace{\langle n^0|n^0\rangle}_{=1} \qquad (8)$$

$\boxed{1}$

Choose Normalisation cond^n to be $\langle n^0|n\rangle = 1 = \langle n^0|n^0\rangle + \lambda\langle n^0|n^1\rangle + \lambda^2\langle n^0|n^2\rangle + \ldots$

$\therefore \langle n^0|n^1\rangle = 0 = \langle n^0|n^2\rangle$ ~~etc~~ (equating coeffs of λ) (9)

$\boxed{1}$

in (8), $E_n^2 = \langle n^0|H_1|n^1\rangle$ $\boxed{1}$

But ~~from (6)~~, $|n^1\rangle = \sum_{m \neq n} |m^0\rangle\langle m^0|n^1\rangle$ (no cmpt of $|n^0\rangle$ by (9)).

$$\therefore E_n^2 = \sum_{m \neq n} \langle n^0|H_1|m^0\rangle\langle m^0|n^1\rangle = \frac{\langle n^0|H_1|m^0\rangle\langle m^0|H_1|n^0\rangle}{E_n^0 - E_m^0}$$

from (6)

$$= \frac{|\langle n^0|H_1|m^0\rangle|^2}{E_n^0 - E_m^0}$$

$\boxed{1}$

($\boxed{5}$)

A3 *Attempt* ***either*** *this question or question A4.*

Pert. theory - see written answer.

Since H_1 has all as to the right and all $a^\dagger$s to the left, and since $a|0>=0$, we have that

$$\Delta E^{(1)} = 0$$

[2]

For the second-order correction, we have that

$$\Delta E^{(2)} = \sum_{m\neq 0} < 0|H_1|m > \frac{1}{E_0 - E_m} < m|H_1|0 >$$

where $|m>= \frac{(a^\dagger)^m}{\sqrt{m!}}|0>$. Note also that

$$E_0 - E_m = -m\hbar\omega.$$

[1]

Clearly, the only non-zero matrix element in the sum is [2]

$$< 3|H_1|0 >= \beta < 3|(a^\dagger)^3|0 >= \sqrt{3!} < 3|3 > \beta = \sqrt{6}\beta.$$

Hence [2]

$$\Delta E^{(2)} = -2\frac{\beta^2}{\hbar\omega}.$$

[1]

Ways in which perturbation theory may break down:

- If a state is degenerate with some other(s), a perturbation may have new eigenstates that are linear combinations of the degenerate eigenfunctions, ie there may be large contributions from the other degenerate eigenstates. [2]
- If the perturbing potential is such that the form of the wavefunction is totally different when the perturbation parameter λ changes sign then there is no way that the energy shift can be represented as a power series in λ. eg in 1D a localised perturbation o strength λ can either cause bound states or a continuum, the lowest state of which does not depend on λ. [2]

A4 *Attempt* ***either*** *this question or question A3.*

Hund's rules apply for LS coupling: [1]

(a) Combine the spins of the electrons to obtain possible values of total spin S. The largest permitted value of S lies lowest in energy. (Maximising S makes the spin wavefunction as symmetric as possible. This tends to make the spatial wavefunction antisymmetric, and hence reduces the Coulomb repulsion.) [2]

(b) For each value of S, find the possible values of total angular momentum L. The largest value of L lies lowest in energy. (Maximising L also tends to keep the electrons apart. This is less obvious, though a simple classical picture of electrons rotating round the nucleus in the same or different senses makes it at least plausible.) [2]

(c) Couple the values of L and S to obtain the values of J. If the subshell is less than half full, the smallest value of J lies lowest; otherwise, the largest value of J lies lowest. (The separation of energies for states of different J arises from treating the spin-orbit term as a perturbation (fine structure).) [2]

[Kr]$(5s)^2(4d)^2$ is like a noble gas and 4 extra electrons. The 5s pair fill the 5s level and so can be ignored. This leaves 2 electrons in the 4d states, each with $\ell = 2$. [1]

For LS coupling, individual orbital angular momenta ℓ_1 and ℓ_2, and combined orbital ang. mom. L, spin S and total ang. mom. J (and components M_L, M_J) are good quantum numbers. [1]

These are fermions, so if spin wavefn is symmetric ($S = 1$), spatial wavefn is antisymmetric, so L is odd, and vice versa. If $S = 1$, L is odd:

m_{ℓ_1}	m_{ℓ_2}	M_L
2	1	3
2	0	2
2	-1	1
2	-2	0
1	0	1
1	-1	0
1	-2	-1
0	-1	-1
0	-2	-2
-1	-2	-3

(Table not necessary.) This consists of the $L = 3$ and $L = 1$ states. J takes values between $L \pm S$. Thus, using the notation $^{2S+1}L_J$, the states are $^3\mathrm{F}_{2,3,4}$ and $^3\mathrm{P}_{0,1,2}$. [2]

If $S = 0$, L is even $(= 4, 2, 0)$ and $J = L$. Thus the possibilities are $^1\mathrm{G}_4$, $^2\mathrm{D}_2$ and $^1\mathrm{S}_0$. [1]

By Hund's rules, max S so $S = 1$, max L so $L = 3$, band $<$ half full so min J gives $J = 2$, i.e. $^3\mathrm{F}_2$ is the ground state. [2]

Here. for $S = 1, L = 3, J = 2$,
$g = \frac{3J(J+1)+S(S+1)-L(L+1)}{2J(J+1)} = (18 + 2 - 12)/12 = 2/3$. [1]

In a magnetic field of $B = 1\mathrm{T}$, the states will split to energies $M_J g \mu_\mathrm{B} B (= M_J \times 6.18 \times 10^{-24}\mathrm{J})$. [1]

There are $2J + 1 = 5$ levels. Microwaves will excite transitions between adjacent levels, at energy $E = g\mu_\mathrm{B}B$. [1]

$E = h\nu = hc/\lambda$, so wavelength $\lambda = hc/g\mu_\mathrm{B}B = \frac{6.63\times10^{-34}\times3\times10^8}{(2/3)\times9.27\times10^{-24}\times1} = 0.0321\mathrm{m}$

(TURN OVER

$= 32.1$ cm. [2]

Selection rules: $\Delta J = \pm 1, 0$ (but $0 \to 0$ is not allowed) and $\Delta M_J = \pm 1, 0$. (Other selection rules are for ideal LS coupling, so may not apply here.) [2]

Excited $^1D_2$ state has $S = 0, L = 2, J = 2$, so $g = (18 - 6)/12 = 1$. $2J + 1 = 5$ levels again. [1]

For transitions from $^1D_2$ (initial) to $^3F_2$ (ground state, final), $\Delta J = 0$ (OK).

The following are allowed transitions (with energy ΔE, and taking E_0 to be the energy difference between the states at $B = 0$) since $\Delta M_J = \pm 1, 0$:

m_{J_i}	m_{J_f}	$(\Delta E - E_0)/\mu_B B$
2	2	2-4/3=2/3
	1	2-2/3=4/3
1	2	1-4/3=-1/3
	1	1-2/3=1/3
	0	1
0	1	-2/3
	0	0
	-1	2/3
-1	0	-1
	-1	-1+2/3=-1/3
	-2	-1+4/2=2/3
-2	-1	-2+2/3=-4/3
	-2	-2+4/3=-2/3

[2]

Combining these, to get the degeneracies:

$(\Delta E - E_0)/\mu_B B$	degeneracy
4/3	1
1	1
2/3	2
1/3	2
0	1
-1/3	2
-2/3	2
-1	1
-4/3	1

[1]

This is 9 distinct transitions, with those at $\pm 2/3$ and $\pm 1/3 \mu_B B$ twice as strong as the others. [1]

END OF PAPER

NST II ETP(2) MAY 2010 / PAPER 4, SEC - B

answers 09-10

B1

(a) $A_{eff} = \frac{\lambda^2}{4\pi} G(\theta)$ [1]

$A_{max} = \frac{\lambda^2}{4\pi} \frac{3}{2}$ [1]

$= 0.5^2 \frac{3}{8\pi}$ μm^2 [1]

$= 0.03$ μm^2 [1]

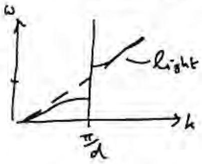
[bookwork]

(b) a periodic arrangement of different n, on λ-scale.

[bookwork]

$n_1 = 1.4$, $n_2 = 1.6$

equal volumes, so $\bar{n} = 1.5$

[2] 

light line $\frac{\omega}{k} = \frac{c}{\bar{n}}$

so $\omega = \frac{c}{\bar{n}} \frac{\pi}{d}$

so $\lambda = \frac{2\pi c}{\omega} = 2\bar{n}d = 600\,nm$ [2]

(c) from $\phi = \frac{q}{\hbar}\int A.dr$, from $\hat{p} = -i\hbar\nabla - qA$ [2]

$\psi \sim \exp\left\{ i(\underline{k}.\underline{r} + \frac{q\underline{A}.\underline{r}}{\hbar})\right\}$

$\Delta_1 - \Delta_2 = \frac{q}{\hbar}\Phi_s$ ← flux enclosed

need $\pi = \frac{e}{\hbar} BA$

so $B = \frac{\hbar}{er^2} \approx 7 \times 10^{-4}\ T$ [2]

[bookwork]

B2

(a) - sensing of wavefronts; relative phase

- mutual coherence function

$\Gamma(\tau) = \langle f(t) f^*(t-\tau) \rangle$ temporal coherence

- fringe visibility

$V(\tau) = \frac{|\Gamma(\tau)|}{I_0}$ $V(0) = 1$, $0 < V < 1$

- power spectrum

$P(\omega) = FT[\gamma(\tau)]$

- Michelson interferometer

- Spatial coherence lateral coherence

$V = |\gamma(u = kd)|$

$\gamma(u) = \frac{FT[I(\theta)]}{I_0}$

- coherence volume

- stellar interferometer

– stellar interferometer

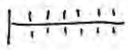
(b) – Hertzian dipoles, quadrupoles, ... have specific θ, ϕ pattern
– antenna emit a combination of ED, MD, EQ, ...

– Power gain $G(\theta,\phi)$ = $\dfrac{\text{flux in specific dir}^n}{\text{total flux}}$
↳ emission pattern

– flux $N = E \wedge H$

– from ED, $N = E_\theta B_\phi / \mu_0$ in far field

diagram of emission pattern
$G \propto \sin^4\theta \sin^2\phi$

– cancellation / interference of EM rad^n in some dir^{n}s
– phased array steering
– half-wave antenna
– stub antenna
–  explained: Yagi-Uda antenna

(c) – circular motion
$\omega_B = \dfrac{qB}{\gamma m}$

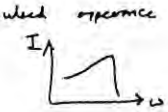
– power radiated
$$P = \frac{\mu_0 q^4 \gamma^2 B^2 u^2}{6\pi c m^2}$$

– radius $R = \dfrac{u}{\omega_B}$ velocity u

– $\lambda \sim \dfrac{2\pi c}{\omega_B} < R$ so not ED

–  in rest frame ⇒  lab frame
$\Delta\theta \sim \dfrac{1}{\gamma}$

– pulsed appearance to observer

–  spectrum
cut off $\nu_c \sim \gamma^3 \omega_B$

– uses

B3

(a) $A = \left(\dfrac{\phi}{c}, \underline{A}\right)$ [1]
$J = (c\rho, \underline{J})$ [1]

$\Box^2 A = \mu_0 J$ all Maxwell eq^{n}s manifestly indep. of frame. [1]

$\Box . J$ is a Lorentz invariant, ⇒ cons. of charge (in all frames) [1]

$\Box . A$ [again scalar product of 4-vectors, so invariant] [1]
↖ gauge condition. [bookwork]

(b) $\phi' = \gamma(\phi - vA_x)$. $A_x' = \gamma(A_x - v\phi/c^2)$. $A_y' = A_y$. $A_z' = A_z$ [2]

(b) $\phi' = \gamma(\phi - vA_x)$, $A_x' = \gamma(A_x - v\phi/c^2)$, $A_y' = A_y$, $A_z' = A_z$ [2]

$B' = \nabla\wedge A'$, $E' = -\nabla'\phi' - \frac{\partial A'}{\partial t'}$ with $\frac{1}{c}\frac{\partial}{\partial t'} = \frac{1}{c}\left(\gamma\frac{\partial}{\partial t} + \gamma v\frac{\partial}{\partial x}\right)$ [2]

From chain rule $\frac{\partial}{\partial x'} = \gamma\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right)$

since $\Box$ is 4 vector

$$B_x' = \frac{\partial A_z'}{\partial y'} - \frac{\partial A_y'}{\partial z'} = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = B_x \quad [1]$$

$$B_y' = \frac{\partial A_x'}{\partial z'} - \frac{\partial A_z'}{\partial x'} = \gamma\left(\frac{\partial A_x}{\partial z} - \frac{v}{c^2}\frac{\partial \phi}{\partial z}\right) - \gamma\left(\frac{\partial A_z}{\partial x} + \frac{v}{c^2}\frac{\partial A_z}{\partial t}\right)$$

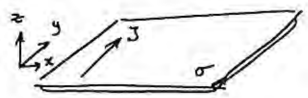
$$= \gamma\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \gamma\frac{v}{c^2}\left(-\frac{\partial \phi}{\partial z} - \frac{\partial A_z}{\partial t}\right) = \gamma\left(B_y + \frac{v}{c^2}E_z\right)$$

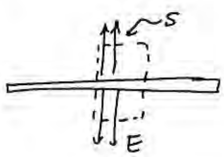
[1]

$$E_x' = E_x \qquad B_x' = B_x$$
$$E_y' = \gamma(E_y - vB_z) \qquad B_y' = \gamma\left(B_y + \frac{v}{c^2}E_z\right)$$
$$E_z' = \gamma(E_z + vB_y) \qquad B_z' = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

[bookwork]

(c)





$$\int E.dS = \frac{Q}{\epsilon_0} \quad [1]$$

$$2ES = \frac{\sigma S}{\epsilon_0} \quad \text{outwards}$$

$$E = \frac{\sigma}{2\epsilon_0}$$

$$E = -\nabla\phi$$

so $\phi = -\frac{\sigma z}{2\epsilon_0}$ [1]

[bookwork]

(d)

$\nabla^2 A = -\mu_0 J$ $\nabla . A = 0$ Coulomb gauge

$B = \nabla\wedge A$ [1]

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & -\frac{\mu_0 J z}{2} & 0 \end{vmatrix} \quad [1]$$

$$= \left(\frac{\mu_0 J}{2},\ 0,\ 0\right)$$ [seen before]

indep. of height (field lines don't spread out)

(e) $E_{/\!/} = 0$, $E_\perp = \frac{\sigma}{2\epsilon_0}$, $B_\perp = 0$, $B_{/\!/} = \frac{\mu_0 J}{2}$

$E_{/\!/}' = E_{/\!/} = 0$ [1]

$E_\perp' = \gamma(E_\perp + v\wedge B_\perp) = \gamma\frac{\sigma}{2\epsilon_0}$ [1]

$B_{/\!/}' = B_{/\!/} = \mu_0 J/2$ [1]

$B_\perp' = \gamma\left(B_\perp - \frac{v}{c^2}\wedge E_\perp\right) = -\frac{v}{c^2}\frac{\sigma}{2\epsilon_0}\gamma$ [1]

[similar to problems]

(f) Lorentz contraction of plate by γ $\rightarrow\boxed{}\leftarrow$ [1]

squeezes up charge so charge density $\rightarrow \gamma\sigma$ [1]

$B_{/\!/}$ unchanged as current $\perp$ to velocity [1]

but now $B_\perp$ as charge on plate moves.

$$J_{eff} = \frac{2B_\perp}{\mu_0} = \frac{v\sigma}{c^2}\frac{\gamma}{\epsilon_0\mu_0} = v\sigma\gamma \qquad [2]$$

Lorentz contracted charge $\gamma\sigma$ moves at v : ✓ [1]

[seen in different form before]

B4

(a) Jones vector gives ratio of components of optical field along orthogonal axes. [1] [1]
State of light at a point [1]

Jones matrix gives transformation of Jones vector when passing through a component. [1]

[Goodwork]

(b)

field strength along polariser axes is
$(\cos\theta, \sin\theta)$ [1]

fraction transmitted is then $\cos\theta\,(\cos\theta, \sin\theta)$ [1]
$\sin\theta\,(\cos\theta, \sin\theta)$ [1]

giving $\underline{\underline{J}}_\theta$

(c) assume first polariser vertical so
$$\underline{v} = \begin{pmatrix}1\\0\end{pmatrix}$$

then $$\underline{\underline{J}}_\theta \underline{v} = \begin{pmatrix}\cos^2\theta\\ \cos\theta\sin\theta\end{pmatrix} \qquad [1]$$

if $\theta = \pi/2$, $\underline{\underline{J}}_{\pi/2}\underline{v} = 0$ [1]

no light transmitted [seen in notes]

(d)
$$\underline{\underline{R}}(\theta)\,\underline{\underline{J}}(\theta) = \begin{pmatrix}c & s\\ -s & c\end{pmatrix}\begin{pmatrix}c^2 & cs\\ cs & s^2\end{pmatrix} \qquad [1]$$

$$= \begin{pmatrix}c^3+cs^2 & c^2s+s^3\\ -sc^2+c^2s & -cs^2+cs^2\end{pmatrix} \qquad [1]$$

$$= \begin{pmatrix}c & s\\ 0 & 0\end{pmatrix} \qquad [1]$$

where $c = \cos\theta$, $s = \sin\theta$

Many polarisers gives

$$(RJ)^N = \begin{pmatrix}c & s\\ 0 & 0\end{pmatrix}^N \qquad [1]$$

$$= \begin{pmatrix}c & s\\ 0 & 0\end{pmatrix}\begin{pmatrix}c & s\\ 0 & 0\end{pmatrix}\begin{pmatrix}c & s\\ 0 & 0\end{pmatrix}^{N-2}$$

$$= \begin{pmatrix}c^2 & cs\\ 0 & 0\end{pmatrix}\begin{pmatrix}c & s\\ 0 & 0\end{pmatrix}\begin{pmatrix}c & s\\ 0 & 0\end{pmatrix}^{N-3}$$

$$= \begin{pmatrix}c^3 & c^2s\\ 0 & 0\end{pmatrix}\begin{pmatrix}c & s\\ 0 & 0\end{pmatrix}^{N-3} \qquad [1]$$

$$= \begin{pmatrix} c^N & c^{N-1}s \\ 0 & 0 \end{pmatrix}$$

$$= \cos^N\theta \begin{pmatrix} 1 & \tan\theta \\ 0 & 0 \end{pmatrix} \qquad [1]$$

[new problem]

(e) for N polarisers  $\frac{\pi}{2} = (N+1)\theta$ [1]

so $$(RJ)^N = \cos^N\left(\frac{\pi/2}{N+1}\right)\begin{pmatrix} 1 & \tan\frac{\pi/2}{N+1} \\ 0 & 0 \end{pmatrix}$$

For large N, small θ, $\cos\theta \to 1 - \frac{\theta^2}{2} + \dots$ [1]

$$\text{Transmission} = (RJ)^N \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \cos^N\left(\frac{\pi/2}{N+1}\right) \qquad [1]$$

$$\simeq \left[1 - \left(\frac{\pi/2}{N+1}\right)^2\right]^N$$

$$\simeq 1 - \frac{N\pi^2/4}{(N+1)^2}$$

$$\simeq 1 - \frac{\pi^2}{4N} \qquad [1]$$

$$\text{Intensity} = t^2 = \left(1 - \frac{\pi^2}{4N}\right)^2 \qquad [1]$$

$$\simeq 1 - \frac{\pi^2}{2N}$$

$N = 20 \Rightarrow I_t = 75.3\%$ [1]

to get $I_t = 99\%$

$\frac{\pi^2}{2N} = \frac{1}{100}$ so $N = \frac{(10\pi)^2}{2} = 493$ [2]

[new problem]

(f) Problem is loss

- Fresnel loss because polarisers mean that $n_i \neq n_{i+1}$ [1]
- intrinsic loss of polarisers [1]

[new problem]