UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Saturday 4 June 2011 9.00 am to 12.00 noon

EXPERIMENTAL AND THEORETICAL PHYSICS (2)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (2H)

Candidates offering the **whole** *of this paper should attempt a total of* **six** *questions, three from Section A* **and** *three from Section B. The questions to be attempted are* **A1**, **A2** *and* **one** *other question from Section A and* **B1**, **B2** *and* **one** *other question from Section B.*

Candidates offering **half** *of this paper should attempt a total of* **three** *questions,* **either** *three from Section A* **or** *three from Section B. The questions to be attempted are* **A1**, **A2** *and* **one** *other question from Section A or* **B1**, **B2** *and* **one** *other question from Section B. These candidates will leave after* **90 minutes**.

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains **7** *sides, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

A separate Answer Book should be used for each section.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
2 × 20 Page Answer Book	Mathematical Formulae handbook
Metric graph paper	Approved calculator allowed
Rough workpad	
Yellow master coversheet	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

ADVANCED QUANTUM PHYSICS

A1 *Attempt* **all** *parts of this question. Answers should be concise, and relevant formulae may be assumed without proof.*

(a) Among the levels of the hydrogen atom with $n = 2$, two of the levels show a first-order Stark shift in opposite senses and the other two show almost no shift at low applied electric field. Explain these observations in general terms (without detailed calculation). [4]

(b) What is the ${}^{2S+1}L_J$ value of the $(3p)^3$ electronic ground state of P ($Z = 15$), assuming LS coupling? [4]

(c) Sketch the wavefunction of the 10^{th} excited state in a simple harmonic potential. Indicate how the main features can be explained using the WKB approximation, or otherwise. [4]

A2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) uniform magnetic fields in quantum mechanics, including a discussion of the Aharonov-Bohm effect and Landau levels;

(b) the Hartree method in the central-field approximation;

(c) coupling of matter to electromagnetic radiation, including a discussion of emission and absorption of photons by atoms.

A3 *Attempt* **either** *this question* **or** *question A4.*

Explain how the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are employed in relation to spin-$\frac{1}{2}$ particles. Find the normalised wavefunctions $|X_+\rangle$ and $|X_-\rangle$ for neutrons polarised respectively along the positive and negative x-directions, in terms of the eigenfunctions $|\alpha\rangle$ and $|\beta\rangle$ for neutrons polarised respectively along the positive and negative z-directions. [6]

If the neutron, initially in state $|\alpha\rangle$ at $t = 0$, is subjected to a constant magnetic field B in the x-direction, show that its wavefunction becomes

$$|\psi(t)\rangle = |\alpha\rangle \cos\omega t + \mathrm{i}\,|\beta\rangle \sin\omega t,$$

where $\omega \equiv \mu B/\hbar$ and μ is the magnetic moment of the neutron. Obtain also the expectation values of each of the components of the magnetic moment as a function of time, and interpret the results. [8]

In an interferometer, a beam of monoenergetic neutrons initially in state $|\alpha\rangle$ is split into two separate beams that propagate in different directions, one of which is subjected to a magnetic field as above. The beams travel equal effective path lengths and are then recombined coherently, i.e. their wavefunctions are added. Show that the number of neutrons detected in the combined beam is proportional to $(1 + \cos\omega t)$. [5]

Sketch graphs of the numbers of neutrons detected in the combined beam, and of the z-component of $\langle\boldsymbol{\mu}\rangle$ for each of the two beams before recombination, against ωt. Give a physical interpretation of the curves in terms of the components of the combined wavefunction. [6]

[*You may find useful the relation*

$$\mathrm{e}^{\mathrm{i}\theta(\boldsymbol{n}\cdot\boldsymbol{\sigma})} = \boldsymbol{I}\cos\theta + \mathrm{i}(\boldsymbol{n}\cdot\boldsymbol{\sigma})\sin\theta\,,$$

in the usual notation; $\boldsymbol{I}$ *is the identity matrix.*]

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A4 *Attempt* **either** *this question* **or** *question A3.*

Outline, without a detailed derivation, how perturbation theory leads to Fermi's Golden Rule,

$$R_{\mathrm{i}\to\mathrm{f}}(t) = \frac{2\pi}{\hbar^2}|\langle \mathrm{f}|\widehat{V}|\mathrm{i}\rangle|^2\delta(\omega_{\mathrm{fi}} - \omega),$$

defining the symbols. [7]

Indicate how Fermi's Golden Rule can be applied to the case of a continuous density of final states. [2]

A beam of spinless particles, each of mass m and momentum $\hbar\boldsymbol{k}$, is incident on a fixed, weak, localised, scattering potential $V(\boldsymbol{r})$. Define the differential scattering cross-section $\mathrm{d}\sigma/\mathrm{d}\Omega$ and use Fermi's Golden Rule to derive the Born approximation for scattering to momentum $\hbar\boldsymbol{k}'$,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(\boldsymbol{q}) = \left|\frac{m}{2\pi\hbar^2}\int V(\boldsymbol{r})\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}}\mathrm{d}^3\boldsymbol{r}\right|^2,$$

where $\boldsymbol{q} = \boldsymbol{k}' - \boldsymbol{k}$. (You may assume that the particle flux of a plane wave of unit amplitude is $\hbar k/m$.) [5]

For a potential of the form $V = V_0\exp(-r^2/a^2)$, where a is a constant, show that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left|\frac{mV_0a^3\sqrt{\pi}}{2\hbar^2}\,\mathrm{e}^{-q^2a^2/4}\right|^2.$$

(You may find it helpful to work in Cartesian coordinates.) [5]

Show that there is no angular dependence in $\mathrm{d}\sigma/\mathrm{d}\Omega$ if the energy of the particles is sufficiently small. Also find an approximate expression for the total scattering cross-section to second order in $|\boldsymbol{k}|a$. [6]

[*You may use the results*

$$\int_{-\infty}^{\infty}\mathrm{e}^{-z^2/a^2}\cos qz\,\mathrm{d}z = \mathrm{e}^{-q^2a^2/4}a\sqrt{\pi}, \quad \int_{-\infty}^{\infty}\mathrm{e}^{-z^2/a^2}\,\mathrm{d}z = a\sqrt{\pi}.$$
]

SECTION B

OPTICS AND ELECTRODYNAMICS

B1 *Attempt* **all** *parts of this question. Answers should be concise, and relevant formulae may be assumed without proof.*

(a) An 11.5 GeV alpha particle of rest mass 3.73 GeV is fired through a dielectric block and produces Cerenkov light at 47° to its trajectory. What is the refractive index of the block? [4]

(b) Light with wavelength 500 nm from a star of diameter 4×10^8 km which is 2×10^{15} km away is used to form interference fringes from a double slit. Estimate the smallest double slit separation needed to reduce the fringe visibility to zero. [4]

(c) What is the Jones matrix of a birefringent plate producing an optical phase difference of π, which is oriented with its fast axis at 45° to the x-axis? [4]

B2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) the polarisation dependence of light scattering;

(b) the power gain and effective area of aerials;

(c) how magnetic fields produced by currents can be accounted for by the relativistic transformation of charge densities.

B3 *Attempt* **either** *this question* **or** *question B4.*

The fields radiated by a Hertzian dipole aligned along the z-axis are given by

$$B_\phi = \frac{\mu_0 \sin\theta}{4\pi}\left\{\frac{[\dot{p}]}{r^2} + \frac{[\ddot{p}]}{rc}\right\},$$

$$E_\theta = \frac{\sin\theta}{4\pi\varepsilon_0}\left\{\frac{[p]}{r^3} + \frac{[\dot{p}]}{r^2 c} + \frac{[\ddot{p}]}{rc^2}\right\},$$

$$E_r = \frac{2\cos\theta}{4\pi\varepsilon_0}\left\{\frac{[p]}{r^3} + \frac{[\dot{p}]}{r^2 c}\right\},$$

where r, θ and ϕ are the conventional spherical-polar coordinates and $[p]$ is the retarded dipole moment.

Explain with reasoning which of these terms you would expect to see dominate the radiation pattern of this emitter in the near- and far-field regions. [3]

Draw a diagram of the far-field emission pattern, together with labelled field orientations and polarisations. [4]

Using these expressions, calculate the Poynting flux in the far field, averaged over a cycle. Hence calculate the instantaneous power radiated by the dipole in terms of its frequency, its charges and their maximum separation. [6]

One model of a vibrating linear triatomic molecule, aligned along the z-axis, has a central stationary charge of $+2q$ surrounded by two charges of $-q$ moving equally in opposite directions on either side (along the z-axis, each spaced on average a from the centre). By considering the radiating B-field arising from the sum of two oppositely directed currents with slightly different retarded times, calculate the radiated magnetic field. [6]

Sketch the emission pattern of this quadrupole, and calculate the ratio of its power radiated to that from an equivalent Hertzian dipole if the emission wavelength is $6\,\mu$m and $a = 1$ nm. [6]

B4 *Attempt* **either** *this question* **or** *question B3.*

What are 4-vectors in relativistic electrodynamics, and how are they used to form Lorentz-invariant quantities? [5]

Write down the frequency 4-vector of a plane wave, and by applying the Lorentz transformation between two frames S and S' in standard configuration, show that a plane wave travelling in the xy-plane at angle θ to the x-axis in S transforms to the S' frame in the following way:

$$\begin{aligned} \frac{\omega'}{c} &= \gamma\left(\frac{\omega}{c} - \beta k \cos\theta\right), \\ k' \cos\theta' &= \gamma\left(k\cos\theta - \beta\frac{\omega}{c}\right), \\ k' \sin\theta' &= k\sin\theta, \end{aligned}$$

where $\beta = v/c$, $\gamma = 1/\sqrt{1-\beta^2}$ and v is the velocity, directed along the $+x$ axis, of the origin of the S' frame as measured in S. [4]

A transparent isotropic dielectic medium moving in the negative x' direction at speed v in frame S' is stationary in frame S, where it has refractive index n. Using the equations above, calculate the refractive index n' experienced by light moving in the $+x'$ direction in the frame S'. [4]

Explain what happens to the wavelength of the light in frame S'. [2]

Show that light travelling inside the dielectric along the y-axis in frame S experiences in S' a refractive index:

$$n' = \frac{n}{\gamma}\sqrt{1 + \left(\frac{\gamma\beta}{n}\right)^2}.$$

[4]

A Michelson interferometer is embedded in the medium. By considering the transformations you have calculated, or otherwise, discuss how the interference pattern at the output will change in the moving frame S'. [3]

Explain the concept of birefringence in optics, and discuss briefly if this moving medium is birefringent (when at rest it is isotropic). [3]

END OF PAPER