TSP 2010

1) a) N identical, distinguishable particles

2 level system, $\epsilon = 0$ - non degenerate

$\epsilon > 0$ - degeneracy g

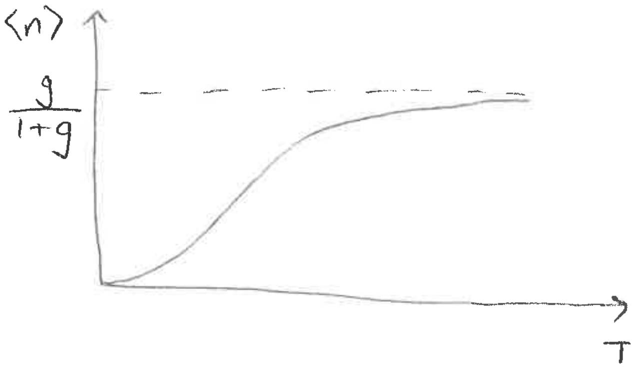
$$\Xi = 1 + g e^{-\beta(\epsilon - \mu)}$$

$$\phi = -k_B T \ln \Xi = -k_B T \ln\left(1 + g e^{-\beta(\epsilon-\mu)}\right)$$

$$\langle n \rangle = -\frac{\partial \phi}{\partial \mu} = \frac{g e^{-\beta(\epsilon-\mu)}}{1 + g e^{-\beta(\epsilon-\mu)}}.$$

low T - $\langle n \rangle \to 0$

high T - $\langle n \rangle \sim \frac{g}{1+g}$



b) $S = A(NVU)^{1/3}$

$U(S,V,N)$: $U = \left(\frac{S}{A}\right)^3 \frac{1}{NV}$

$$T = \frac{\partial U}{\partial S} = \frac{3S^2}{A^3} \frac{1}{NV}$$

$$P = -\frac{\partial U}{\partial V} = \left(\frac{S}{A}\right)^3 \frac{1}{NV^2}$$

$$P(V,N,T) = \frac{1}{A^3 N V^2}\left(\frac{A^3 N V T}{3}\right)^{3/2} = \sqrt{\frac{N}{V}}\left(\frac{AT}{3}\right)^{3/2}$$

c) Spin-1/2 fermions in 2D

$$g(E) = \frac{Am}{\pi\hbar^2}$$

Find chemical potential of N fermions in T=0 limit

$$N = \int \cancel{g}\, g(E)\, n(E)\, dE = \frac{Am}{\pi\hbar^2}\int_0^{\infty} \frac{dE}{e^{\beta(E-\mu)}+1}$$

$n(E)$ becomes step function in zero T limit

$$n(E) = \begin{cases} 0 & E > \cancel{E_F}\ \mu \\ 1 & E < \cancel{E_F}\ \mu \end{cases}$$

$$N = \frac{Am}{\pi\hbar^2}\int_0^{E_F} dE = \frac{AmE_F}{\pi\hbar^2}$$

$$\mu(T=0) = E_F \Rightarrow \mu = \frac{\pi\hbar^2 N}{Am}$$

3) Fermi-Dirac distribution

Show that probability that state with energy $\mu+\delta$ is occupied is equal to the probability that a state with energy $\mu-\delta$ is not occupied

$$P(E=\mu+\delta) = \frac{1}{e^{\beta\delta}+1}$$

$$P(E=\mu-\delta) = \frac{1}{e^{-\beta\delta}+1}$$ - probability that state with energy $\mu-\delta$ is occupied

probability that state with energy $\mu-\delta$ is not occupied is

$$1-P(E=\mu-\delta) = \frac{e^{-\beta\delta}}{e^{-\beta\delta}+1} = \frac{1}{e^{\beta\delta}+1} = P(E=\mu+\delta)$$

$E_k = \pm\hbar v_F|k|$, massless fermions with $g=2$, $\mu=0$ at all T

Show that $$E(T)-E(0) = 4\int \frac{d^2k}{(2\pi)^2}\frac{|E_k|}{\exp(\beta|E_k|)+1}$$

$$g(E) = \frac{d^2k}{(2\pi/L)^2}\, g = \frac{2A}{(2\pi)^2}\, d^2k = \frac{2d^2k}{(2\pi)^2} \quad \text{per unit area}$$

$$E(T) = \int_0^\infty E\, g(E)\, n(E)\, dE$$

fermions – $n(E) = \dfrac{1}{e^{\beta(E-\mu)}+1} = \dfrac{1}{e^{\beta E}+1}$ for $\mu = 0$

$$E(T) = E(0) + 2\int \frac{2d^2k}{(2\pi)^2}\,\frac{|E_k|}{e^{\beta|E_k|}+1}$$

– extra factor of 2 for 2 symmetric branches?

$$E(T) - E(0) = 4\int \frac{d^2k}{(2\pi)^2}\,\frac{|E_k|}{e^{\beta|E_k|}+1}$$

calculate contribution to heat capacity

$$E(T) - E(0) = 4\int \frac{2\pi k\, dk}{(2\pi)^2}\,\frac{\hbar v_F k}{e^{\beta\hbar v_F k}+1}$$

$x = \beta\hbar v_F k$, $dx = \beta\hbar v_F\, dk$

$$E(T) - E(0) = \frac{2}{\pi}\hbar v_F \int_0^\infty \frac{x^2 dx \,.\, (\beta\hbar v_F)^{-3}}{e^x+1}$$

$$= \frac{2}{\pi}\frac{(k_BT)^3}{(\hbar v_F)^2}\underbrace{\int_0^\infty \frac{x^2 dx}{e^x+1}}_{1.8}$$

$$C = \frac{\partial E}{\partial T} = \frac{6}{\pi}\cdot 1.8\, \frac{k_B^3 T^2}{(\hbar v_F)^2} = 8.2\times 10^{-13}\, T^2$$

b) 2D phonons, low T limit

$$g(\omega)\, d\omega = 2 \cdot \frac{2\pi k\, dk}{(2\pi/L)^2} = \frac{A\omega\, d\omega}{\pi v_p^2}$$

at low T, occupation of higher energy modes is negligible

$$U = \int_0^\infty \frac{A\omega\, d\omega}{\pi v_p^2}\,\frac{\hbar\omega}{e^{\beta\hbar\omega}-1} = \frac{\hbar}{\pi v_p^2}\int_0^\infty \frac{\omega^2 d\omega}{e^{\beta\hbar\omega}-1}$$

$x = \beta\hbar\omega$, $dx = \beta\hbar\, d\omega$

$$U = \frac{\hbar}{\pi v_p^2}\int_0^\infty \frac{1/(\beta\hbar)^3\, x^2 dx}{e^x-1} = \frac{(k_BT)^3}{\pi(\hbar v_p)^2}\underbrace{\int_0^\infty \frac{x^2 dx}{e^x-1}}_{2.4}$$

$$C = \frac{3}{\pi}\frac{k_B^3 T^2}{(\hbar v_p)^2}\cdot 2.4 = 1.4\times 10^{-9}\, T^2$$

– dominant contribution from phonons

4) Canonical ensemble - partition function of dilute classical system of N particles interacting through a pair potential $\varphi(r)$

Hamiltonian $\mathcal{H} = \sum_i p_i^2/2m + \sum_{j>i} \varphi(r_{ij})$

$$Z = \sum_{\text{microstates}} e^{-\beta \mathcal{H}(p,r)} = \frac{1}{N!} \int e^{-\beta \mathcal{H}} d^3r_1 \dots d^3r_N \frac{d^3p_1 \dots d^3p_N}{(2\pi\hbar)^{3N}}$$

- sum over all particles

momentum integrals give $\left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3N/2}$

- same as for a single free particle, to Nth power (separable integrals)

$$Z = \frac{1}{N!}\left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3N/2} \int e^{-\beta \sum_{j>i}\varphi(r_{ij})} d^3r_1 \dots d^3r_N$$

$$= \frac{1}{N!}\left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3N/2} Z_\varphi$$

difference in internal energy (compared with ideal gas)

$$U = -\frac{\partial \ln Z}{\partial \beta}$$

$$\ln Z = N - N\ln N + \frac{3}{2}N \ln\left(\frac{mk_BT}{2\pi\hbar^2}\right) + \ln Z_\varphi$$

$$-\frac{\partial \ln Z}{\partial \beta} = \frac{3}{2}Nk_BT - \frac{\partial \ln Z_\varphi}{\partial \beta} = U_{ideal} - \frac{\partial \ln Z_\varphi}{\partial \beta}$$

$$\text{difference} = \frac{\partial \ln Z_\varphi}{\partial \beta}$$

$$-\frac{\partial \ln Z_\varphi}{\partial \beta} = \frac{1}{Z_\varphi}\int \sum_{j>i}\varphi(r_{ij}) \exp\left[-\beta \sum_{j>i}\varphi(r_{ij})\right] d^3r_1 \dots d^3r_N$$

$$= \frac{N(N-1)}{2Z_\varphi}\int \varphi(r_{12}) e^{-\sum_{j>i}\beta\varphi(r_{ij})} d^3r_1 \dots d^3r_N$$

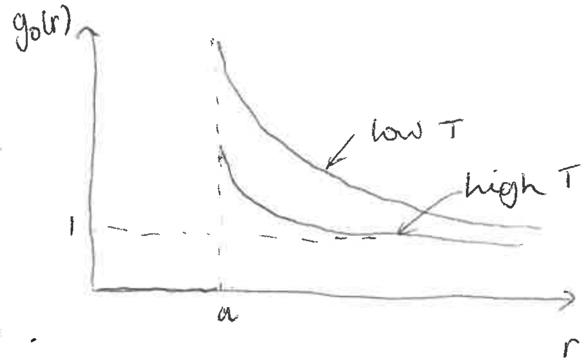
$$= \frac{N^2}{2V}\int_0^\infty \varphi(r)\, g(r)\, 4\pi r^2 dr$$

$g_0(r) = e^{-\beta\varphi(r)}$, φ converges to zero rapidly for $r \to \infty$

$\hookrightarrow g(r) \sim 1$

$$\text{difference} \sim \frac{N^2}{2V}\int_a^\infty 4\pi r^2 \varphi(r)\, dr$$

$$\varphi(r) = \begin{cases} \infty & r<a \\ \psi(r) & r>a \end{cases}$$



$\psi(r) = -J/r^6$

$g_0(r) = 0 \quad r<a \quad$ for any T

for $r>a$, $g_0(r) = \exp(\beta J/r^6)$

$$u \sim \frac{3}{2}Nk_BT - \frac{N^2}{2V}\int_a^\infty \frac{4\pi J}{r^4}\,dr = \frac{3}{2}Nk_BT - \frac{8\pi JN^2}{Va^3}$$

2nd virial coefficient

$$B_2(T) = \frac{1}{2}\int d^3r\left[1-\exp(-\beta\varphi(r))\right]$$

$$= 2\pi\int_0^a r^2dr + 2\pi\int_a^\infty r^2dr\left[1-\exp(\beta J/r^6)\right]$$

$$= \frac{2}{3}\pi a^3 + 2\pi\int_{a/(\beta J)^{1/6}}^\infty (\beta J)^{1/3}x^2\left[1-\exp(1/x^6)\right](\beta J)^{1/6}\,dx$$

$$= \frac{2}{3}\pi a^3 + 2\pi(\beta J)^{1/2}\underbrace{\int_{a/(\beta J)^{1/6}}^\infty x^2\,dx\left[1-\exp(1/x^6)\right]}_{= -\frac{1}{3}\frac{(\beta J)^{1/2}}{a^3}}$$

$$B_2(T) = \frac{2}{3}\pi a^3 - \frac{2}{3}\pi\frac{\beta J}{a^3}$$

Boyle temp $-$ when $B_2(T) = 0$

$a^6 = \beta J$

$$T_B = \frac{J}{k_Ba^6}$$