UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Tuesday 28 May 2013 9.00 am to 11.00 am

EXPERIMENTAL AND THEORETICAL PHYSICS (2)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (2)

Relativity — ANSWERS

1 (a) [*Unseen, although similar to an example sheet question.*]

Suppose observers are at rest at the origins of frames S and S', respectively, which are in standard configuration. For incoming photon, the standard aberration formulae are

$$\cos\theta' = \frac{\beta + \cos\theta}{1 + \beta\cos\theta}, \qquad \sin\theta' = \frac{\sin\theta}{\gamma(1 + \beta\cos\theta)}.$$

If the planet is small and distant, then θ is small and so $\sin\theta \approx \theta$ and $\cos\theta \approx 1$, and similar for θ'. In this approximation, the first aberration formula is satisfied identically and the second gives

$$\theta' \approx \frac{\theta}{\gamma(1 + \beta)} = \left(\frac{1 - \beta}{1 + \beta}\right)^{1/2} \theta = \left(\frac{c - v}{c + v}\right)^{1/2} \theta.$$

If the observers are momentarily coincident, for the observed angular size of the planet to differ by a factor of 2, one requires

$$\left(\frac{c + v}{c - v}\right)^{1/2} = 2,$$

which implies $v = 3c/5$.

(b) [*Bookwork, but setting the fluid pressure to zero.*]

The energy-momentum tensor for dust is $T^{\mu\nu} = \rho u^\mu u^\nu$ and the fluid equation of motion is $\nabla_\mu T^{\mu\nu} = 0$. Hence,

$$\nabla_\mu(\rho u^\mu u^\nu) = \nabla_\mu(\rho u^\mu)u^\nu + \rho u^\mu \nabla_\mu u^\nu = 0. \qquad (*)$$

Contracting this expression with u_ν gives

$$c^2\nabla_\mu(\rho u^\mu) + \rho u^\mu u_\nu \nabla_\mu u^\nu = 0,$$

But $u_\nu u^\nu = c^2$, which implies that $u_\nu \nabla_\mu u^\nu = 0$ and so the second term on the LHS vanishes to yield $\nabla_\mu(\rho u^\mu) = 0$. Substituting this result back into $(*)$ gives the required answer $u^\mu \nabla_\mu u^\nu = 0$.

Using the standard expression for the covariant derivative,

$$u^\mu \nabla_\mu u^\nu = u^\mu(\partial_\mu u^\nu + \Gamma^\nu{}_{\sigma\mu} u^\sigma) = \frac{du^\nu}{d\tau} + \Gamma^\nu{}_{\sigma\mu} u^\sigma u^\mu.$$

For each dust particle $u^\mu = \dot{x}^\mu$, so

$$\ddot{x}^\nu + \Gamma^\nu{}_{\sigma\mu} \dot{x}^\sigma \dot{x}^\mu = 0,$$

which is the equation of a geodesic.

(c) [*Unseen, although related to a calculation in the lectures.*]

$$ds^2 = c^2\, dt^2 - a^2(t)(dx^2 + dy^2 + dz^2) \quad \Rightarrow \quad \mathcal{L} = c^2 \dot{t}^2 - a^2(t)(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

The Euler–Lagrange equation for t gives

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{t}}\right) = \frac{\partial \mathcal{L}}{\partial t} \ \Rightarrow\ \frac{d}{d\tau}(2c^2 \dot{t}) = -2aa'(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \ \Rightarrow\ \ddot{t} + \frac{2aa'}{c^2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = 0,$$

and that for x gives (and similarly for y and z by symmetry)

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x} \ \Rightarrow\ \frac{d}{d\tau}(-2a^2 \dot{x}) = 0 \ \Rightarrow\ \ddot{x} + \frac{2a'}{a}\dot{x}\dot{t} = 0$$

These are the geodesic equations, which have the generic form $\ddot{x}^\nu + \Gamma^\nu{}_{\sigma\mu} \dot{x}^\sigma \dot{x}^\mu = 0$, so one can read off the connection coefficients (taking care with factors of 2)

$$\Gamma^0{}_{11} = \Gamma^0{}_{22} = \Gamma^0{}_{33} = \frac{aa'}{c^2} \qquad \text{and} \qquad \Gamma^1{}_{10} = \Gamma^2{}_{20} = \Gamma^3{}_{30} = \frac{a'}{a}.$$

One can satisfy $g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = c^2$ and the geodesic equations for $\dot{x} = \dot{y} = \dot{z} = 0$ and $\dot{t} = 1$, which corresponds to a stationary observer, for whom proper time equals the coordinate time t.

2 (a) *The equivalence principle and local inertial coordinates*

Newtonian equation of motion of a particle of inertial mass m_I is

$$\frac{d^2\boldsymbol{x}}{dt^2} = -\frac{m_G}{m_I}\nabla\phi.$$

But experimental fact: m_G/m_I is the same for all particles $\Rightarrow$ can choose units so that ratio is unity. Thus, trajectory of particle under gravity is independent of the nature of the particle. Equivalence of m_G and m_I is a remarkable coincidence in the Newtonian theory, verified one part in 10^{11} (by Dicke and co-workers).

The equivalence of gravitational and inertial mass led Einstein to his classic 'elevator' thought experiment. Consider an observer in a freely falling elevator: particles released from rest remain floating 'weightless'; particles shot from one side to other appear to move in a straight line at constant velocity, rather than along a curved path.

These observations would hold exactly if the gravitational field of the Earth were truly uniform, but force acts radially inwards towards CoM with strength $\propto 1/r^2$ So, if elevator free falls for a long time, and/or 'elevator' is very large, particles released from rest in elevator would: draw in horizontally (falling along radial lines); and draw out vertically (varying field strength with radius). These are the tidal forces from inhomogeneity in gravitational field, once main acceleration subracted; always present in general for a finite elevator (laboratory)

But, if we make the provisos that: the time interval of observation is short; and the elevator cabin is spatially small, then the freely-falling elevator (which may have x, y, z coordinates chalked on its wall and an elevator clock measuring time t) resembles a Cartesian inertial frame of reference. The one arrives at the (strong) equivalence principle (EP): in a freely-falling (nonrotating) laboratory occupying a small region of spacetime, the laws of physics are those of special relativity.

EP $\Rightarrow$ at any event P in spacetime, must be able to define a coordinate system X^μ so that, in the local neighbourhood of P, the line element takes the form

$$ds^2 \approx \eta_{\mu\nu}\, dX^\mu\, dX^\nu,$$

where the equality holds exactly at the event P. Coordinates X^μ define a local Cartesian inertial frame in vicinity of P (in which laws of SR hold). Mathematically, corresponds to constructing about event P a coordinate system X^μ such that

$$g_{\mu\nu}(P) = \eta_{\mu\nu} \qquad \text{and} \qquad (\partial_\sigma g_{\mu\nu})_P = 0.$$

Thus, near point P, the metric in a local inertial coordinate system X^μ is

$$g_{\mu\nu} = \eta_{\mu\nu} + \tfrac{1}{2}(\partial_\sigma\partial_\rho g_{\mu\nu})_P X^\sigma X^\rho + \cdots,$$

where the sizes of the second derivatives (related to the spacetime curvature) determine the region over which approximation remains valid.

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(b) *The use of accretion discs around compact objects as a test of general relativity*

Orbits of particles and photons are probes of the geometry of spacetime. Information about the geometry produced by compact massive objects or black holes can be obtained from observations of the orbits of particles in the accretion disc that often surrounds them.

Small trace of iron found in the accreting matter. Incident radiation from X-ray flares above and below the disc ⇒ fluorescence from highly ionised atoms in the disc ⇒ electron in the atom is de-excited ⇒ 6.4 keV spectral line in X-ray band.

Frequency of the photons as measured by some observer at infinity is (i) gravitationally redshifted by an amount that depends on the radius from which they were emitted; (ii) Doppler shifted by an amount that depends on the speed and direction (relative to the distant observer) of the emitting material. Angular size of discs in such systems typically far smaller than the width of the observing beam of any telescope ⇒ observed spectral line is integrated over disc. Use line-profile to probe the strong-field regime of gravity. Detailed shape of the line depends on the mass and rotation of the central object, the inclination of the disc to the line of sight and relativistic beaming effects.

Mathematically, the ratio of a photon's frequency at reception to that at emission is given by

$$\frac{\nu_R}{\nu_E} = \frac{\boldsymbol{p}(R)\cdot\boldsymbol{u}_R}{\boldsymbol{p}(E)\cdot\boldsymbol{u}_E} = \frac{p_\mu(R)u_R^\mu}{p_\mu(E)u_E^\mu},$$

where $\boldsymbol{p}(E)$ and $\boldsymbol{p}(R)$ are the photon 4-momenta at emission and reception respectively, $\boldsymbol{u}_E$ is the 4-velocity of the material at emission and $\boldsymbol{u}_R$ is the 4-velocity of the observer at reception.

Consider a non-rotating central object (Schwarzchild geometry in t, r, θ, ϕ coordinates), with disc oriented edge-on to the observer, so all orbits in $\theta = \pi/2$ plane. Assuming an observer fixed at infinity, so $[u_R^\mu] = (1, 0, 0, 0)$. For emitting material of disc $[u_E^\mu] = u_E^0(1, 0, 0, \Omega)$, where $\Omega \equiv d\phi/dt = (GM/r^3)^{1/2}$. Can fix u_E^0 using $g_{\mu\nu}u^\mu u^\nu = c^2$ to yield $u_E^0 = (1 - 3\mu/r)^{-1/2}$, so

$$\frac{\nu_R}{\nu_E} = \frac{p_0(R)}{p_0(E)u_E^0 + p_3(E)u_E^3} = \left(1 - \frac{3\mu}{r}\right)^{1/2} \frac{p_0(R)}{p_0(E)} \left[1 \pm \frac{p_3(E)}{p_0(E)}\Omega\right]^{-1},$$

with $+/-$ for emitting matter moving towards/away from the observer. Since the Schwarzschild metric is stationary, $p_0(E) = p_0(R)$ and $p_3(E)/p_0(E)$ can, in general, be fixed using $g^{\mu\nu}p_\mu p_\nu = 0$ and the photon geodesic equations.

There are, however, two simple special cases: (i) when $\phi = 0$ or $\phi = \pi$, then $p_3(E) = 0$; and (ii) when $\phi = -\pi/2$ or $\phi = \pi/2$, then $p_1(E) = 0$. Thus, respectively one has

$$\frac{\nu_R}{\nu_E} = \left(1 - \frac{3\mu}{r}\right)^{1/2} \quad \text{and} \quad \frac{\nu_R}{\nu_E} = \frac{(1 - 3\mu/r)^{1/2}}{1 \pm (r/\mu - 2)^{-1/2}}.$$

The first result also holds if the disc is viewed face-on, since the motion of the emitting matter is always transverse to the observer.

(c) *Advanced Eddington–Finkelstein coordinates in the Schwarzchild spacetime*

In standard Schwarzschild coordinates (t, r, θ, ϕ), the line-element has a coordinate singular at $r = 2\mu$. Also, the worldlines both of radially moving photons and massive particles cross $r = 2\mu$ only at $t = \pm\infty$. This suggests the t coordinate is inappropriate to describe the Schwarzschild spacetime at $r \leq 2\mu$.

Circumvent problem of unsatisfactory coordinates by 'probing' spacetime with geodesics, which are coordinate-independent and hence unaffected by the boundaries of coordinate validity. One of many possibilities is to use as probes the null worldlines of radially moving photons.

Consider radially infalling photon worldlines. Main idea is to use the integration constant as the new coordinate, which we denote by p. Thus, we make the coordinate transformation

$$p = ct + r + 2\mu \ln \left| \frac{r}{2\mu} - 1 \right| ,$$

where the 'advanced time parameter' p is a null coordinate. Since p is constant along the entire worldline of the radially ingoing photon, it will be a 'good' coordinate wherever that worldline penetrates. Differentiating this expression and substituting for dt in the Schwarzschild line element gives

$$ds^2 = \left(1 - \frac{2\mu}{r}\right) dp^2 - 2\, dp\, dr - r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right),$$

which is regular for $0 < r < \infty$, which is the range of r-values probed by an infalling photon geodesic. Common practice to work instead with the related timelike coordinate t', defined by $ct' \equiv p - r$, yielding the line-element

$$ds^2 = c^2 \left(1 - \frac{2\mu}{r}\right) dt'^2 - \frac{4\mu c}{r}\, dt' dr - \left(1 + \frac{2\mu}{r}\right) dr^2 - r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right),$$

which is again regular for $0 < r < \infty$. The coordinates (t', r, θ, ϕ) are called *advanced Eddington–Finkelstein* coordinates. Incoming and outgoing photon worldlines are given by

$$\begin{aligned} ct' &= -r + \text{constant}, \\ ct' &= r + 4\mu \ln \left| \frac{r}{2\mu} - 1 \right| + \text{constant}. \end{aligned}$$

Thus ingoing photon worldlines are continuous straight lines making an angle of 45° with the r-axis and is valid for $0 < r < \infty$. Similarly, infalling massive particle worldlines must lie inside the forward lightcone at each point, and are hence continuous across $r = 2\mu$. The lightcone structure changes at $r = 2\mu$: once you have crossed the boundary $r = 2\mu$ your future is directed towards the singularity. Similarly, a photon (or particle) starting at $r < 2\mu$ cannot escape to the region $r > 2\mu$. The Schwarzschild radius $r = 2\mu$ thus defines an *event horizon* (see sketch below).

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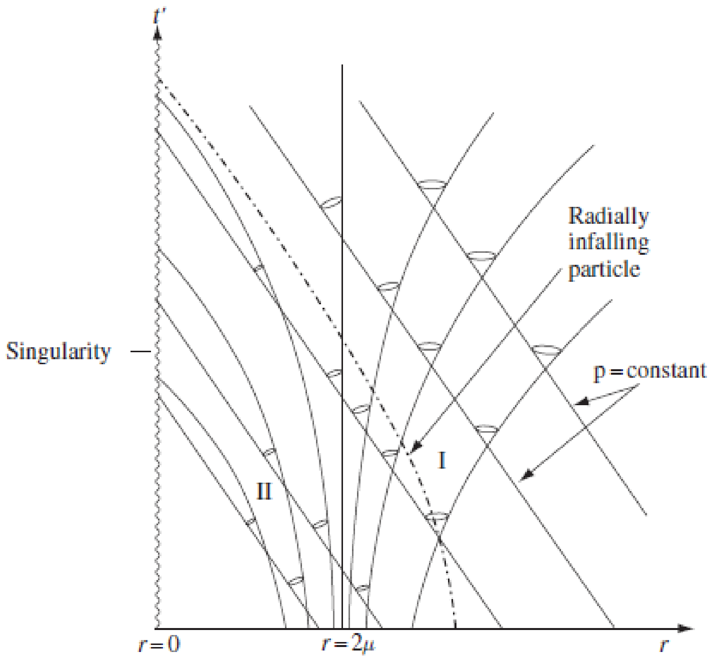

t'
Singularity
Radially
infalling
particle
p = constant
I
II
r = 0
r = 2μ
r

3 [*All unseen, but based on the discussion of a uniformly accelerated observer in the lectures and on an example sheet question.*]

Let $p = t - x$, $q = t + x$, so inverse is $t = \frac{1}{2}(p + q)$, $x = \frac{1}{2}(p - q)$. Therefore,

$$ds^2 = dt^2 - dx^2 = \tfrac{1}{4}(dp + dq)^2 - \tfrac{1}{4}(dp - dq)^2 = dp\,dq.$$

In general $ds^2 = g_{\mu\nu}\,dx^\mu\,dx^\nu$, so in lightcone coordinates $g_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$.

The frames S and S' are in standard configuration, so use standard Lorentz transformation formulae (with $c = 1$)

$$t' = \frac{t - vx}{\sqrt{1 - v^2}}, \qquad x' = \frac{x - vt}{\sqrt{1 - v^2}}.$$

Therefore

$$p' = t' - x' = \frac{t - vx - x + vt}{\sqrt{1 - v^2}} = \frac{(1 + v)(t - x)}{\sqrt{1 - v^2}} = \left(\frac{1 + v}{1 - v}\right)^{1/2} p$$
$$q' = t' + x' = \frac{t - vx + x - vt}{\sqrt{1 - v^2}} = \frac{(1 - v)(t + x)}{\sqrt{1 - v^2}} = \left(\frac{1 - v}{1 + v}\right)^{1/2} q.$$

Hence $p' = \alpha p$ and $q' = q/\alpha$, where $\alpha = \left(\frac{1+v}{1-v}\right)^{1/2}$.

If $\boldsymbol{u}(\tau)$ and $\boldsymbol{a}(\tau)$ are the spaceship 4-velocity and 4-acceleration, respectively, for a spaceship with uniform proper acceleration a, one requires

$$\boldsymbol{u} \cdot \boldsymbol{u} = 1 \quad \Rightarrow \quad g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 1 \quad \Rightarrow \quad \dot{p}\dot{q} = 1$$
$$\boldsymbol{a} \cdot \boldsymbol{a} = -a^2 \quad \Rightarrow \quad g_{\mu\nu}\ddot{x}^\mu \ddot{x}^\nu = -a^2 \quad \Rightarrow \quad \ddot{p}\ddot{q} = -a^2$$

Now $\dot{p} = 1/\dot{q} \Rightarrow \ddot{p} = -\ddot{q}/\dot{q}^2$, which, on substituting into $\ddot{p}\ddot{q} = -a^2$, gives $(\ddot{q}/\dot{q})^2 = a^2$. Since $q = t + x$, we must take the postive square root, and on integrating we obtain the first-order linear ordinary differential equation $\dot{q} - aq = k$, where k is an integration constant. Multiplying through by the integrating factor $e^{-a\tau}$ and integrating again gives $q(\tau) = Ae^{a\tau} + B$, where A and B are constants. But $\dot{p} = 1/\dot{q} = e^{-at}/(Aa)$, so $p(\tau) = -e^{-at}/(Aa^2) + C$, where C is a constant. From the boundary conditions given, $p(0) = -1/a$ and $q(0) = 1/a$, so the constants are given $A = 1/a$, $B = C = 0$. Thus, the required solution is

$$p(\tau) = -\frac{1}{a}e^{-a\tau}, \qquad q(\tau) = \frac{1}{a}e^{a\tau}.$$

From the previous result, one immediately finds

$$t = \tfrac{1}{2}(p + q) = \frac{1}{a}\sinh a\tau, \qquad x = \tfrac{1}{2}(p - q) = \frac{1}{a}\cosh a\tau,$$

so the spaceship worldine is the right-hand branch of the hyperbola $x^2 - t^2 = a^{-2}$, which crosses the x-axis at $x = 1/a$ and has asymptotes $x = \pm t$ (to be sketched). The line $x = t$

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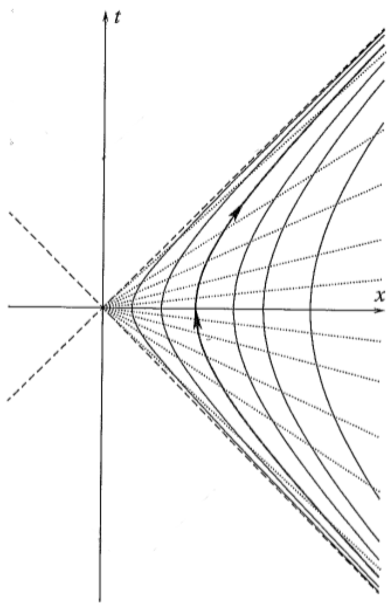
is an event horizon for the astronaut, namely no signal emitted in the region $t > x$ can ever be received by the astronaut.

A photon is emitted with frequency ν_0 by a stationary star at $x = L$ $(0 < L < 1/a)$. In (t, x) coordinates, the photon 4-momentum is $[p^\mu] = h\nu_0(1, 1)$. On reception at proper time τ, the astronaut, with 4-velocity $\boldsymbol{u}(\tau)$, measures the photon to have energy $E = \boldsymbol{p} \cdot \boldsymbol{u} = p_\mu u^\mu$. But $[u^\mu] = (\dot{t}, \dot{x}) = (\cosh a\tau, \sinh a\tau)$, so

$$E = p_\mu u^\mu = h\nu_0(\cosh a\tau - \sinh a\tau) = h\nu_0 e^{-a\tau}.$$

Thus, the measured frequency of the photon is $\nu = \nu_0 e^{-a\tau}$. In order for the photon to reach the astronaut, it must be emitted before $t = L$, since $x = t$ is the astronaut's event horizon.

Rindler coordinates (ξ, ζ) are given by $t = \frac{1}{a}e^{a\zeta}\sinh a\xi$ and $x = \frac{1}{a}e^{a\zeta}\cosh a\xi$. By inspection, the spaceship worldline corresponds to $\xi(\tau) = \tau$ and $\zeta(\tau) = 0$, so (ξ, ζ) are the coordinates of the comoving frame of the astronaut. The curves ζ = constant are hyperbolae intersecting the x-axis at $x = \frac{1}{a}e^{a\zeta}$ with asymptotes $x = \pm t$, and the curves ξ = constant are straight lines through the origin with slope $\tanh a\xi$. Note that Rindler coordinates are incomplete, since they only cover the domain $x > |t|$.



4 The FRW line-element is

$$ds^2 = c^2 dt^2 - R^2(t)\left[d\chi^2 + S^2(\chi)(d\theta^2 + \sin^2\theta\, d\phi^2)\right],$$

where $S(\chi) = \sin\chi, \chi$ or $\sinh\chi$ depending on whether the universe is spatially closed, flat or open, respectively.

[*Bookwork*] For a radial photon path $ds = d\theta = d\phi = 0$, so for two successive photon wave crests

$$\int_{t_e}^{t_0} \frac{c\,dt}{R(t)} = \int_0^{\chi} d\chi = \int_{t_e+\delta t_e}^{t_0+\delta t_0} \frac{c\,dt}{R(t)} \quad \Rightarrow \quad \int_{t_e}^{t_e+\delta t_e} \frac{c\,dt}{R(t)} = \int_{t_0}^{t_0+\delta t_0} \frac{c\,dt}{R(t)}.$$

But δt_e and δt_0 are very small compared with the characteristic expansion time, so can take $R(t)$ to be constant in both integrands, hence

$$\frac{\delta t_e}{R(t_e)} = \frac{\delta t_0}{R_0} \quad \Rightarrow \quad 1 + z = \frac{\nu_e}{\nu_0} = \frac{\delta t_e}{\delta t_0} = \frac{R_0}{R(t_e)}.$$

The physical interpretation is that the photon wavelength is stretched in proportion to the increase in scale factor.

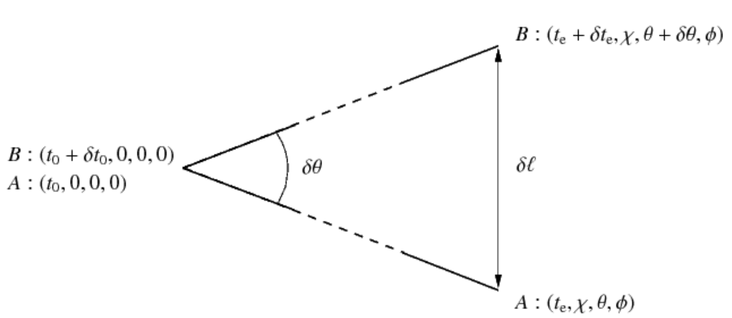
[*Bookwook*] From above $\chi = \int_{t_e}^{t_0} \frac{c\,dt}{R(t)}$, but the cosmic time interval dt is related to a redshift interval dz by

$$dz = d(1 + z) = d\left(\frac{R_0}{R}\right) = -\frac{R_0}{R^2}\dot{R}\,dt = -(1 + z)H(z)\,dt,$$

where $H(t) \equiv \dot{R}/R$ is the Hubble parameter. Hence, expressions for $\chi(z)$ and the lookback time $t_0 - t_e$ are given by

$$\chi(z) = \frac{c}{R_0}\int_0^z \frac{d\bar{z}}{H(\bar{z})}, \qquad t_0 - t_e = \int_0^z \frac{d\bar{z}}{(1 + \bar{z})H(\bar{z})}.$$

[*Unseen, but closely related to the derivation of angular-diameter distance given in the lectures*] Consider the emission and reception of two photons A and B, with coordinates as shown below:



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From the figure, $v = \delta\ell/\delta t_{\rm e}$ and $\dot\theta = \delta\theta/\delta t_0$, but from the FRW metric $\delta\ell = R(t_{\rm e})S(\chi)\,\delta\theta$. Hence the proper-motion distance is

$$d_M \equiv \frac{v}{\dot\theta} = \frac{R(t_e)S(\chi)\,\delta\theta}{\delta t_{\rm e}}\,\frac{\delta t_0}{\delta\theta} = (1+z)R(t_{\rm e})S(\chi) = R_0 S(\chi).$$

[*Unseen*] From the FRW metric

$$dV_0 = (R_0\,d\chi)(R_0^2 S^2(\chi)\,d\Omega) = R_0^3 S^2(\chi)\,d\chi\,d\Omega = R_0^3 S^2(\chi)\,\frac{d\chi}{dz}\,dz\,d\Omega.$$

But, using the previously derived expression for $\chi(z)$, one has $d\chi/dz = c/[R_0 H(z)]$, hence

$$dV_0 = \frac{cR_0^2 S^2(\chi)}{H(z)}\,dz\,d\Omega.$$

[*Unseen*] In general, $dN = n(z)\,dV(z) = n(z)dV_0(1+z)^{-3}$. If the galaxies are neither destroyed nor further ones created after formation at $z = z_{\rm f}$, then $n(z) = n_0(1+z)^3$. Hence, using the above expression for dV_0, the number N per unit solid angle is

$$N = cn_0R_0^2 \int_0^{z_{\rm f}} \frac{S^2(\chi(z))}{H(z)}\,dz.$$

In the spatially-flat EdS universe, $S(\chi) = \chi$ and $H(z) = H_0(1+z)^{3/2}$. Thus,

$$\chi(z) = \frac{c}{R_0H_0}\int_0^z \frac{dz}{(1+z)^{3/2}} = \frac{c}{R_0H_0}\left[-2(1+z)^{-1/2}\right]_0^z = \frac{2c}{R_0H_0}\left[1-(1+z)^{-1/2}\right].$$

Substituting these expressions into the above result of N gives

$$N = \frac{4c^3 n_0}{H_0^3}\int_0^{z_{\rm f}} \frac{[1-(1+z)^{-1/2}]^2}{(1+z)^{3/2}}\,dz.$$

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