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Lent/Easter Terms 2015

Part II Particle and Nuclear Physics
Outline solutions

1. **Natural Units; practice in using natural units, and converting between natural and SI units**

   (a)

   $$\lambda = \frac{1}{m} = 7.15 \text{ GeV}^{-1} = 1.41 \text{ fm}$$

   using $\hbar c = 0.197$ GeV fm to change units.

   (b) We have $\alpha \approx 1/137$ and $\sqrt{s}$ =91.2 GeV.

   $$\Rightarrow \sigma = 2.68 \cdot 10^{-8} \text{ GeV}^{-2} = 2.68 \cdot 10^{-8} \times (0.197 \text{ fm})^2 = 1.04 \cdot 10^{-11} \text{ barns} = 0.0104 \text{ nb}$$

   (c) Using standard dimensional analysis to insert the missing factors of $\hbar$ and c we find:

   $$\lambda = \frac{\hbar}{mc} \quad \text{and} \quad \sigma = \frac{4\pi\alpha^2\hbar^2c^2}{3s}$$

   (noting that α is dimensionless, so equal to 1/137 in any system of units).

2. **Semi-Empirical Mass Formula**

   (a) Bookwork.

   (b) Treat nucleus as uniformly charged sphere of radius $R = R_0A^{\frac{1}{3}}$, charge Ze. Find self-energy by bringing spheres of charge from infinity. Charge of sphere of radius r is $Ze(r/R)^3$, and charge of spherical shell between r and $r + \mathrm{d}r$ is $Ze(4\pi r^2\mathrm{d}r)/(\frac{4}{3}\pi R^3)$. So, the self-energy is

   $$\int_0^R Ze\frac{4\pi r^2\mathrm{d}r}{\frac{4}{3}\pi R^3}\frac{Ze(r/R)^3}{4\pi\epsilon_0 r} = \frac{3Z^2e^2}{20\pi\epsilon_0 R}$$

   Hence,

   $$a_\mathrm{C} = \frac{3e^2}{20\pi\epsilon_0 R_0} = 0.72 \text{ MeV}$$

   (c) The B/A vs A curve peaks around $A \sim 56$, so for light nuclei fusion can be energetically favourable, while for very heavy nuclei fission is possible.

(d) At fixed A the most stable nucleus will be that with lowest mass (not necessarily the greatest binding energy). So we require $\left.\frac{\partial M}{\partial Z}\right)_A = 0$

$$\Rightarrow (m_p - m_n) + \frac{2a_C Z}{A^{\frac{1}{3}}} - \frac{4a_A(A-2Z)}{A} = 0 \quad \Rightarrow \quad Z = \frac{(m_n - m_p) + 4a_A}{2a_C A^{-\frac{1}{3}} + 8a_A/A}$$

Inserting numerical values for $A = 300$ gives $Z = 114$ etc.

(e) By analogy with (b), and neglecting the proton-neutron mass difference, we have the gravitational p.e. given by:

$$-\frac{3GM^2}{5R} = -\frac{3Gm_n^2 A^2}{5R_0 A^{\frac{1}{3}}} = -a_G A^{\frac{5}{3}} \quad \text{with} \quad a_G = \frac{3Gm_n^2}{5R_0} = 5.8 \cdot 10^{-37} \text{ MeV}$$

If neutrons only, $N = A$ and $Z = 0$, so the binding energy reduces to

$$B = a_V N - a_S N^{\frac{2}{3}} - a_A N + a_G N^{\frac{5}{3}}$$

which must be > 0 if the "nucleus" is bound. The a_S term is negligible, so we find $N^{\frac{2}{3}} > (a_A - a_V)/a_G = 5 \cdot 10^{55}$. So $M > 8 \cdot 10^{28}$ kg which is 0.07 solar masses.

3. **The Fermi Gas model**

Density of states:

$$g(\epsilon) = BA\epsilon^{\frac{1}{2}} \quad \text{where} \quad B = \frac{4\sqrt{2}m^{\frac{3}{2}}R_0^3}{3\pi\hbar^3}$$

Fermi energy is defined by

$$N = \int_0^{\epsilon_F} g(\epsilon)\mathrm{d}\epsilon = \frac{2}{3}BA\epsilon_F^{\frac{3}{2}} \quad \Rightarrow \quad \epsilon_F = \left(\frac{3N}{2BA}\right)^{\frac{2}{3}}$$

If $N = Z = \frac{1}{2}A$, neutrons and protons have the same Fermi energy

$$\overline{\epsilon}_F = \left(\frac{3A}{4BA}\right)^{\frac{2}{3}} = \left(\frac{3}{4B}\right)^{\frac{2}{3}} = 33.2 \text{ MeV}$$

(n.b. non-relativistic) and the corresponding Fermi momentum $= \sqrt{2m\overline{\epsilon}_F} = 250$ MeV$/c$.

Total KE carried by the neutrons is

$$\int_0^{\epsilon_F} \epsilon g(\epsilon)\mathrm{d}\epsilon = \frac{2}{5}BA\epsilon_F^{\frac{5}{2}} = \frac{2}{5}BA\left(\frac{3N}{2BA}\right)^{\frac{5}{3}} = \frac{3}{5}\left(\frac{3}{2BA}\right)^{\frac{2}{3}} N^{\frac{5}{3}}$$

Adding in the protons similarly, total KE is

$$\frac{3}{5}\left(\frac{3}{2BA}\right)^{\frac{2}{3}}(N^{\frac{5}{3}} + Z^{\frac{5}{3}})$$

Writing $N = \frac{1}{2}A(1+\alpha)$ and $Z = \frac{1}{2}A(1-\alpha)$ where $\alpha = (N-Z)/A$, we can expand to second order in α

$$(N^{\frac{5}{3}} + Z^{\frac{5}{3}}) \approx \left(\frac{A}{2}\right)^{\frac{5}{3}}\left[2 + \frac{10}{9}\frac{(N-Z)^2}{A^2}\right]$$

So the asymmetry term in the energy is

$$\frac{3}{5}\left(\frac{3}{2BA}\right)^{\frac{2}{3}}\left(\frac{A}{2}\right)^{\frac{5}{3}}\frac{10}{9}\frac{(N-Z)^2}{A^2} \equiv a_A\frac{(N-Z)^2}{A}$$

from which we find, after a little algebra, $a_A = \frac{1}{3}\overline{\epsilon}_F = 11$ MeV. This is about half the fitted value for a_A.

Pairing energy $\sim 1/g(\epsilon_F)$ in this model.

$$g(\overline{\epsilon}_F) = BA\overline{\epsilon}_F^{\frac{1}{2}} = \frac{3A}{4\overline{\epsilon}_F}$$

so pairing energy $\sim 4\overline{\epsilon}_F/3A \sim 0.44$ MeV, taking $A = 100$ and $\overline{\epsilon}_F = 33$ MeV. For comparison, SEMF fit gives 33.5 MeV$/A^{\frac{3}{4}} \sim 1$ MeV. Again the Fermi gas model is smaller by about a factor of two, because it only takes account of KE, and not of any PE contribution, i.e. the inter-nucleon attraction.

4. **Practice in using cross-sections**

   (a) Number of scattering centres per unit area $= \rho/M_A$ where $\rho = 0.1$ kg m^{-2}, and M_A is the mass per nucleus. Rate of scattering $= 10^5(\rho/M_A)\sigma = 688$ s^{-1} (since the total cross-section $\sigma = 270.01$ b); transmitted intensity $= 99312$ s^{-1}.

   (b) Rate of fissions $= 688 \times (200/270.01) = 510$ s^{-1}.

   (c) Total rate of elastic scattering $= 688 \times (0.01/270.01) = 0.0255$ s^{-1}. Area of sphere at radius 10 m is 400π m^2, so flux of neutrons $= 0.0255/400\pi = 2\ 10^{-5}$ s^{-1}m^{-2}.

5. **Nuclear sizes and Form Factors**

Start from

$$F(q^2) = \int \mathrm{d}^3\underline{r}\ e^{i\underline{q}\cdot\underline{r}}\rho(r)$$

For the purpose of integration, take $\underline{q}$ as the polar axis, hence:

$$\begin{aligned} F(q^2) &= \int 2\pi \mathrm{d}(\cos\theta')\ r^2\mathrm{d}r\ e^{iqr\cos\theta'}\rho(r) \\ &= \int 2\pi r^2\mathrm{d}r\rho(r)\left[\frac{e^{iqr\cos\theta'}}{iqr}\right]_0^\pi = \frac{4\pi}{q}\int_0^\infty r\sin qr\rho(r)\mathrm{d}r \end{aligned}$$

Then, for $qr \ll 1$, expand the sin keeping the first two terms:

$$F(q^2) = \frac{4\pi}{q}\int_0^\infty r\left[qr - \frac{(qr)^3}{3!}\cdots\right]\rho(r)\mathrm{d}r = 1 - \frac{1}{6}q^2\overline{R^2}\cdots$$

For a 200 MeV electron (ultrarelativistic) $E = p = k$ in natural units, so $k = 1.02 \times 10^{15}$ m^{-1}. $q = 2k\sin\frac{1}{2}\theta = 1.95 \times 10^{14}$ m^{-1}. So, given $|F(q^2)|^2 = 0.7$ we obtain $\sqrt{\overline{R^2}} = 5 \times 10^{-15}$ m.

Oscillatory structure in $F(q^2)$ suggests non-smoothness in $\rho(r)$, i.e. more like a top-hat than a Gaussian form (c.f. Fourier transforms of a Gaussian and a top-hat in 1D.) Best fit is a top-hat with rounded corners (e.g. a Fermi function).

6. **Mirror Nuclei**
The pair of mirror nuclei have $(N, Z) = (\frac{1}{2}(A \pm 1), \frac{1}{2}(A \mp 1))$. They have different Coulomb energies;

$$\Delta E_{\mathrm{C}} = \frac{a_{\mathrm{C}}}{A^{\frac{1}{3}}}\left[\left(\frac{A+1}{2}\right)^2 - \left(\frac{A-1}{2}\right)^2\right] = \frac{a_{\mathrm{C}} A}{A^{\frac{1}{3}}} = \frac{3\alpha A}{5R}$$

using $a_{\mathrm{C}} = 3e^2/(20\pi\epsilon_0 R_0) = 3\alpha/5R_0$ and $R = R_0 A^{\frac{1}{3}}$. The only other terms which are differ in the SEMF between the mirror nuclei are $Z(m_p+m_e)+(A-Z)m_n$, so $M(A, Z+1)-M(A, Z) = 2m_e + E_{max} = \Delta E_{\mathrm{C}} + m_p + m_e - m_n \;\; \Rightarrow \;\; \Delta E_{\mathrm{C}} = E_{max} + 1.8$ MeV. Insert numbers to get the answers on the problem sheet. Find $R/A^{\frac{1}{3}}$ is reasonably constant, but higher than the usual value of R_0.

7. **Scattering in Nuclear Physics**
Born approximation is based on first order perturbation theory, so it is suitable when the potential is weak. In contrast, the partial wave method is general, but involves the calculation of an infinite number of partial waves. However, if the projectile's energy is small enough, and/or the potential is sufficiently short range, so that $kR \ll 1$, then only the $\ell = 0$ partial wave is significant in the neighbourhood of the potential, and the method is quite easy to apply. Of course, if the potential is also weak, then either approach is viable.

The question doesn't imply that $V(r)$ is weak, but we are asked about the low energy limit, so the partial wave approach is the obvious choice. Wave functions are:

$$r < b \quad \psi = A\sinh qr \qquad \text{where} \qquad q = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}} \approx \sqrt{\frac{2mV_0}{\hbar^2}}$$

$$r > b \quad \psi = B\sin(kr + \delta_0) \qquad \text{where} \qquad k = \sqrt{\frac{2mE}{\hbar^2}}$$

Matching boundary conditions at $r = b$ we have:

$$A\sinh qb = B\sin(kb + \delta_0) \quad ; \quad qA\cosh qb = kB\cos(kb + \delta_0)$$

Dividing:

$$\frac{\tanh qb}{q} = \frac{\tan(kb + \delta_0)}{k} \approx \frac{kb + \delta_0}{k} \quad \text{as k, } \delta_0 \to 0$$

Hence:

$$-f(\theta) \equiv \frac{\delta_0}{k} = \frac{\tanh qb}{q} - b \quad \Rightarrow \quad \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left[\frac{\tanh qb}{q} - b\right]^2$$

8. **Breit-Wigner Formula in Nuclear Scattering**
We have $\Gamma_n = 0.0104$ eV and $\Gamma_\gamma = 0.105$ eV, so that $\Gamma = \Gamma_n + \Gamma_\gamma = 0.1154$ eV, assuming that there are no other decay modes. Hence, $\sigma_n/\sigma_\gamma = \Gamma_n/\Gamma_\gamma$, and given $\sigma_\gamma = 75$ nb, we obtain $\sigma_n = 7.4$ kb.

The c.m. energy of the neutron is approximately equal to the lab. energy of 2.2 eV, since the target is much more massive. We thus infer $\lambda = \sqrt{h^2/2mE} = 1.92 \times 10^{-11}$ m. We then know everything in the cross-section formula apart from g (note that $E = E_0$ at the resonance peak). We find g=0.77$\approx \frac{3}{4}$, which corresponds to $J = 1$ for the spin of the compound nucleus.

9. **Nuclear Shell Model**
Sequence of energy levels is given in lecture notes. Each has degeneracy $(2j+1)$. Shell model predicts that the spin and parity is that of the unpaired nucleon. Hence:

	Z	N	Unpaired nucleon	predicted J^P	c.f. experiment?
${}^{3}_{2}\mathrm{He}$	2	1	$1\mathrm{s}_{1/2}$	$\frac{1}{2}^{+}$	yes
${}^{9}_{4}\mathrm{Be}$	4	5	$1\mathrm{p}_{3/2}$	$\frac{3}{2}^{-}$	yes
${}^{7}_{3}\mathrm{Li}$	3	4	$1p_{3/2}$	$\frac{3}{2}^{-}$	yes
${}^{12}_{6}\mathrm{C}$	6	6	None	0^{+}	yes
${}^{13}_{7}\mathrm{C}$	6	7	$1p_{1/2}$	$\frac{1}{2}^{-}$	yes
${}^{15}_{7}\mathrm{N}$	7	8	$1p_{1/2}$	$\frac{1}{2}^{-}$	yes
${}^{17}_{8}\mathrm{O}$	8	9	$1d_{5/2}$	$\frac{5}{2}^{+}$	yes
${}^{23}_{11}\mathrm{Na}$	11	12	$1d_{5/2}$	$\frac{5}{2}^{+}$	no
${}^{131}_{54}\mathrm{Xe}$	77	54	$1h_{11/2}$	$\frac{11}{2}^{-}$	no
${}^{207}_{82}\mathrm{Pb}$	125	82	$2f_{5/2}$	$\frac{5}{2}^{-}$	no

The last three disagree with experiment. ${}^{23}_{11}\mathrm{Na}$ has a significant electric quadrupole moment, which tells us that the nucleus is not spherical, unlike the assumption in the simple shell model. In the case of ${}^{131}_{54}\mathrm{Xe}$ and ${}^{207}_{82}\mathrm{Pb}$, levels $2d_{3/2}$ and $3p_{1/2}$ respectively are predicted to lie very close to the naively predicted level, which would explain the observed J^P values. It is often favourable to fill a higher-j level at the expense of a lower-lying lower-j level, because the pairing energy is greater for high j.

10. **Magnetic moments of nuclei**
Method – find the state of the unpaired nucleon, and use the Schmidt limit formulae from notes: $\mu = g_J \mu_N m_J$ for $m_J = J$ where

$$g_J = g_\ell \pm \frac{g_s - g_\ell}{2\ell + 1} \quad \text{for the cases} \quad j = \ell \pm \frac{1}{2}$$

and $g_s = 5.58$ $g_\ell = 1$ for an unpaired proton and $g_s = -3.83$ $g_\ell = 0$ for an unpaired neutron. Hence we have:

${}^{3}_{2}\mathrm{He}$ unpaired n in $1s_{1/2}$ gives $\mu = -1.91\mu_N$

${}^{9}_{4}\mathrm{Be}$ unpaired n in $1p_{3/2}$ gives $\mu = -1.91\mu_N$

${}^{7}_{3}\mathrm{Li}$ unpaired p in $1p_{3/2}$ gives $\mu = 3.79\mu_N$

${}^{13}_{6}\mathrm{C}$ unpaired n in $1p_{1/2}$ gives $\mu = +0.64\mu_N$

${}^{15}_{7}\mathrm{N}$ unpaired p in $1p_{1/2}$ gives $\mu = -0.26\mu_N$

${}^{17}_{8}\mathrm{O}$ unpaired n in $1d_{5/2}$ gives $\mu = -1.91\mu_N$

Some correlation with experiment, but by no means perfect agreement. Least successful are ${}^{9}_{4}\mathrm{Be}$ and ${}^{7}_{3}\mathrm{Li}$ which have a partially filled $1p_{3/2}$ — in this case the simple shell model usually predicts correctly the largest component of the wavefunction, but there are other ways of coupling the angular momenta of the nucleons, which presumably contribute. The other four nuclides are all closed shell ± 1 nucleon, and the model is closer to reality.

Harder part: The $d + p$ model of ${}^3_2$He is an example of an additional contribution to the wavefunction, which could still have the same J^P as the shell model predicts. We can work out the magnetic moment for this case by noting that there are still only two contributions to the total angular momentum, the spin of the proton ($= \frac{1}{2}$) and the spin of the deuteron (=1). We can therefore adapt the regular formula by replacing ℓ by $S_d = 1$ and g_ℓ by $g_d = \frac{1}{2}(g_p + g_n) = 0.875$. This gives $\mu = -0.35\mu_N$, much worse than the shell model. The shell model, assuming like nucleons pair whenever possible, may not give exactly the right wave-function, but it generally gives the largest term in the wavefunction correctly.

11. **Spin-parity of Energy Levels**

(a) Need to form an antisymmetric wave function for three $j = \frac{5}{2}$ neutrons, so they must have different m_j values. enumerate the possibilities and the corresponding values of total M_J:

m_j values	M_J
$\left(\frac{5}{2}, \frac{3}{2}, \frac{1}{2}\right)$	$\frac{9}{2}$
$\left(\frac{5}{2}, \frac{3}{2}, -\frac{1}{2}\right)$	$\frac{7}{2}$
$\left(\frac{5}{2}, \frac{3}{2}, -\frac{3}{2}\right)$, $\left(\frac{5}{2}, \frac{1}{2}, -\frac{1}{2}\right)$	$\frac{5}{2}$
$\left(\frac{5}{2}, \frac{3}{2}, -\frac{5}{2}\right)$, $\left(\frac{5}{2}, \frac{1}{2}, -\frac{3}{2}\right)$, $\left(\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}\right)$	$\frac{3}{2}$
$\left(\frac{5}{2}, \frac{1}{2}, -\frac{5}{2}\right)$, $\left(\frac{5}{2}, -\frac{1}{2}, -\frac{3}{2}\right)$, $\left(\frac{3}{2}, \frac{1}{2}, -\frac{3}{2}\right)$	$\frac{1}{2}$

plus similar negative values of M_J. There's only one way to form $M_J = \frac{9}{2}, \frac{7}{2}$ but two ways to make $M_J = \frac{5}{2}$ and three ways for $M_J = \frac{3}{2}, \frac{1}{2}$. Need $J = \frac{9}{2}, \frac{5}{2}, \frac{3}{2}$ (only) to account for this set of M_J values.

(b) Phonons are bosons, so all combinations of m_j are possible, since any can yield a symmetric wavefunction. Tabulate these for two phonons:

m_j values	M_J
(2,2)	4
(2,1)	3
(2,0), (1,1)	2
(2,-1), (1,0)	1
(2,-2), (1,-1), (0,0)	0

plus similar negative values of M_J. We can see that these form $J = 4, 2, 0$ only. Likewise for three phonons:

m_j values	M_J
(2,2,2)	6
(2,2,1)	5
(2,2,0), (2,1,1)	4
(2,2,-1), (2,1,0), (1,1,1)	3
(2,2,-2), (2,1,-1), (2,0,0), (1,1,0)	2
(2,1,-2), (2,0,-1), (1,1,-1), (1,0,0)	1
(2,0,-2), (2,-1,-1), (1,1,-2), (1,0,-1), (0,0,0)	0

plus similar negative values of M_J. This time the number of possible combinations changes at $M_J = 6, 4, 3, 2, 0$ indicating as before that these are the permitted values of J.

12. **Energy Levels; excited states of nuclei**

    Shell model predicts that the ground state of even-even nuclei should have $J^P = 0^+$. Only $^{18}_{9}\mathrm{F}$ is not even-even, so it must be (d).

    $^{18}_{8}\mathrm{O}$ and $^{18}_{10}\mathrm{Ne}$ are mirror nuclei, so one expects similar level schemes. They must be (b) and (e). The difference between them is the Coulomb energy, which will be greater for $^{18}_{10}\mathrm{Ne}$. To first order this has been subtracted out by measuring the energies w.r.t. the ground state. However, the Coulomb repulsion is slightly less for excited states than for the ground state, because their wavefunctions will be spatially larger, and we have therefore overcompensated for their Coulomb energy by aligning the ground states. The upshot is that the excited states for $^{18}_{10}\mathrm{Ne}$ will lie lower w.r.t. the ground state. Hence, $^{18}_{10}\mathrm{Ne}$ is (b) and $^{18}_{8}\mathrm{O}$ is (e). This subtle effect is called the Thomas-Ehrman shift.

    $^{208}_{82}\mathrm{Pb}$ is doubly magic, so we expect it to be very stable with a large first excitation energy — it must be (a).

    By elimination, $^{166}_{68}\mathrm{Er}$ must be (c). Its low-lying excitations clearly form a series of rotational states with energy $\propto J(J+1)$. This is as expected for a heavy nuclide far from closed shells, which is quite likely to be non-spherical.

13. **Rate equations**

    (a) Denote the number of $^{198}\mathrm{Au}$ nuclei by N_1 and the number of $^{198}\mathrm{Hg}$ nuclei by N_2. We then have:

    $$\begin{aligned}\dot{N}_1 &= R - \lambda N_1 \\ \dot{N}_2 &= \lambda N_1\end{aligned}$$

    where the decay rate is the reciprocal of the mean lifetime, $\lambda = 2.89 \times 10^{-6}\ \ \mathrm{s}^{-1}$, and $R = 10^{10}\ \mathrm{s}^{-1}$. Integrating, we obtain

    $$N_1 = \frac{R}{\lambda}\left(1 - e^{-\lambda t}\right) = 2.69 \times 10^{15} \ \text{ for } \ t = 6 \text{ days}$$

    Further integration gives

    $$N_2 = R\left[t + \frac{e^{-\lambda t}}{\lambda} - \frac{1}{\lambda}\right] = 2.50 \times 10^{15}$$

    The equilibrium number of $^{197}\mathrm{Au}$ nuclei is given by the limit of N_1 for large t, which is given by $N_1 = R/\lambda = 3.46 \times 10^{15}$.

    (b) The rate equations are:

    $$\begin{aligned}\dot{N}_{\mathrm{Cs}} &= -\lambda_{\mathrm{Cs}} N_{\mathrm{Cs}} \quad \Rightarrow \quad N_{\mathrm{Cs}}(t) = N_{\mathrm{Cs}}(0) e^{-\lambda_{\mathrm{Cs}} t} \\ \dot{N}_{\mathrm{Ba}} &= \lambda_{\mathrm{Cs}} N_{\mathrm{Cs}} - \lambda_{\mathrm{Ba}} N_{\mathrm{Ba}} = \lambda_{\mathrm{Cs}} N_{\mathrm{Cs}}(0) e^{-\lambda_{\mathrm{Cs}} t} - \lambda_{\mathrm{Ba}} N_{\mathrm{Ba}}\end{aligned}$$

This can be solved by various standard methods (e.g. integrating factor; complementary function + particular integral), noting the initial condition $N_{\text{Ba}}(0) = 0$ to yield

$$N_{\text{Ba}}(t) = \frac{\lambda_{\text{Cs}} N_{\text{Cs}}(0)}{\lambda_{\text{Ba}} - \lambda_{\text{Cs}}} \left[e^{-\lambda_{\text{Cs}} t} - e^{-\lambda_{\text{Ba}} t} \right]$$

The Ba activity (i.e. its rate of decay, $\lambda_{\text{Ba}} N_{\text{Ba}}(t)$) is maximised when $N_{\text{Ba}}(t)$ is maximised. i.e.

$$\lambda_{\text{Cs}} e^{-\lambda_{\text{Cs}} t} = \lambda_{\text{Ba}} e^{-\lambda_{\text{Ba}} t} \quad \Rightarrow \quad t = \frac{\ln(\lambda_{\text{Cs}}/\lambda_{\text{Ba}})}{(\lambda_{\text{Cs}} - \lambda_{\text{Ba}})} = 33.5 \text{ min}$$

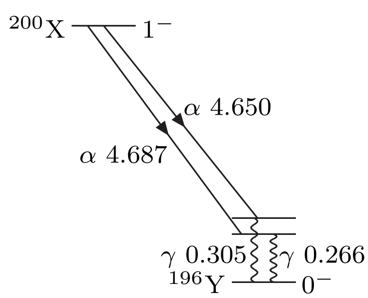
Activity of Ba relative to initial activity of Cs is

$$\frac{\lambda_{\text{Ba}} N_{\text{Ba}}(t)}{\lambda_{\text{Cs}} N_{\text{Cs}}(0)} = \frac{\lambda_{\text{Ba}}}{\lambda_{\text{Cs}}} \frac{\lambda_{\text{Cs}}}{\lambda_{\text{Ba}} - \lambda_{\text{Cs}}} \left[e^{-\lambda_{\text{Cs}} t} - e^{-\lambda_{\text{Ba}} t} \right] = 0.087$$

so the maximum activity is 87 μCi.

14. **Alpha decay**

(a) The likely decay scheme is this (not to scale):



Note the energies of the α and γ almost add up to the same total, but not quite. This is because of the small recoil kinetic energy carried by the daughter nucleus, which is slightly different in the two decay chains.

The α-particle has $J^P = 0^+$ (even-even nucleus). The decay $1^- \to 0^-$ would require orbital angular momentum $\ell = 1$ in order to conserve total angular momentum, but then parity wouldn't be conserved. The excited states of ^{196}Y could be 1^-, with $\ell = 0$ in the α-decay, followed by M1 photon transition to the ground state, or 1^+, with $\ell = 1$ in the α-decay, and E1 transition to the ground state, or 2^+, with $\ell = 1$ in the α-decay, and E2 transition to the ground state.

(b) Geiger-Nuttall tells us that (for fixed Z as in this case):

$$\ln \tau_{\frac{1}{2}} = a + bQ^{-\frac{1}{2}}$$

For the Thorium isotopes given, a plot of $\ln \tau_{\frac{1}{2}}$ against $Q^{-\frac{1}{2}}$ shows good linear behaviour. One can thus use the graph (or a fit to the graph) to interpolate and determine $\tau_{\frac{1}{2}}$ for $^{224}_{90}$Th using the given value of Q.

15. **Nuclear β-decay**
Main ingredients: Fermi transitions have $S(e\nu) = 0$ and Gamow-Teller have $S(e\nu) = 1$. Allowed have $\ell = 0$ and "$n^{\rm th}$-forbidden" have $\ell = n$. Then apply usual QM rules for combination of angular momenta. Odd ℓ leans to a parity change, even ℓ does not. The lowest permitted value of ℓ will dominate.

    (i) $\frac{1}{2}^+ \to \frac{1}{2}^+$ is consistent with either Fermi or G-T transitions — can couple either $S(e\nu) = 0$ or $S(e\nu) = 1$ to $\frac{1}{2}$ and get $\frac{1}{2}$. Superallowed, because overlap between nucleon wavefunctions must be maximal.

    (ii) $0^+ \to 1^+$ is consistent with $\ell = 0$ and $S(e\nu) = 1$ (but not $S(e\nu) = 0$). So pure G-T and superallowed (based on the ft value).

    (iii) $0^+ \to 0^+$ is consistent with $\ell = 0$ and $S(e\nu) = 0$ (but not $S(e\nu) = 1$). So pure Fermi and superallowed (based on the ft value).

    (iv) $\frac{3}{2}^+ \to \frac{3}{2}^+$ is consistent with either Fermi or G-T transitions — can couple either $S(e\nu) = 0$ or $S(e\nu) = 1$ to $\frac{3}{2}$ and get $\frac{3}{2}$. Allowed (based on ft).

    (v) $2^- \to 0^+$ has a parity change, so ℓ is odd. Consistent with $\ell = 1$ and $S(e\nu) = 1$ (but not $S(e\nu) = 0$). So pure G-T and first forbidden.

    (vi) $1^- \to 0^+$ has a parity change, so ℓ is odd. Consistent with $\ell = 1$ and $S(e\nu) = 0$ or $S(e\nu) = 1$, so mixed Fermi and G-T, first forbidden.

    (vii) $\frac{7}{2}^+ \to \frac{3}{2}^+$ has no parity change, but $\ell = 0$ does not work. Hence $\ell = 2$ with either $S(e\nu) = 0$ or $S(e\nu) = 1$, so mixed Fermi and G-T, second forbidden.

16. **Nuclear β-decay**
Here, as well as considering selection rules, as in the preceding question, we also have to consider which decays are energetically possible. In terms of atomic masses, the relevant constraints are:

$$\begin{array}{rcl} \beta^- & : & M(Z) - M(Z+1) > 0 \\ \beta^+ & : & M(Z) - M(Z-1) - 2m_e > 0 \\ \text{ElectronCapture} & : & M(Z) - M(Z-1) > 0 \end{array}$$

Based on this, the Ce and Nd nuclides should be stable. The possible decays are:

- La$\to$Ce by β^- decay. $2^- \to 0^+$ implies first forbidden G-T (c.f. Qu. 15(v)).
- Pr$\to$Ce by EC. $2^- \to 0^+$ implies first forbidden G-T.
- Pr$\to$Nd by β^- decay. $2^- \to 0^+$ implies first forbidden G-T.
- Pm$\to$Nd by EC or β^+ decay. $1^+ \to 0^+$ implies allowed G-T (c.f. Qu. 15(ii)).
- Sm$\to$Pm by EC or β^+ decay. $0^+ \to 1^+$ implies allowed G-T.

The shortest lifetime will correspond to the allowed decay with the greatest energy release, i.e. Pm. The longest lifetime will correspond to the most forbidden decay with the smallest energy release, i.e. Pr. Agrees with experiment.

17. **Gamma decay; selection rules**

We are told the ground state is $J^P = \frac{3}{2}^+$. γ_3 is an E2 transition to the g.s., so $\underline{\Delta J} = 2$ and no parity change. This is consistent with the 2.51 MeV state having $J^P = \frac{1}{2}^+, \frac{3}{2}^+, \frac{5}{2}^+$ or $\frac{7}{2}^+$. But the first three of these could also decay to the g.s. by M1 transition, not seen. So the 2.51 MeV state has $J^P = \frac{7}{2}^+$.

γ_1 is M1, so $\underline{\Delta J} = 1$ and no parity change. This is consistent with the 2.10 MeV state having $J^P = \frac{5}{2}^+, \frac{7}{2}^+$ or $\frac{9}{2}^+$. But if it was $\frac{5}{2}^+$ or $\frac{7}{2}^+$ it could decay to the g.s. by E2 (or M1 in the latter case), not seen. So the 2.10 MeV state has $J^P = \frac{9}{2}^+$.

γ_2 is E1, so $\underline{\Delta J} = 1$ and a parity change. This is consistent with the 1.20 MeV state having $J^P = \frac{5}{2}^-, \frac{7}{2}^-$ or $\frac{9}{2}^-$. But if it was $\frac{5}{2}^-$ it could decay to the g.s. by E1, which is not seen. So the 1.20 MeV state has $J^P = \frac{9}{2}^-$ or $\frac{7}{2}^-$ (which can decay to the g.s. by the higher order processes E3 or M2/E3 respectively).

18. **Nuclear Fission**

(a) Suppose initial neutron (momentum p_0, mass m) hits nucleus (mass Am where $A = 12$ in this case) head on. Final particles have momenta p_1 and p_2 in the same direction. We then have:

$$p_0 = p_1 + p_2 \quad \text{(conservation of momentum)}$$
$$\frac{p_0^2}{2m} = \frac{p_1^2}{2m} + \frac{p_2^2}{2Am} \quad \text{(conservation of energy)}$$

Eliminate p_2 and solve for p_1/p_0 we find

$$\frac{p_1}{p_0} = \frac{1-A}{1+A} = -\frac{11}{13}$$

Hence the maximal fractional energy loss for the neutron in one collision is $1 - (11/13)^2 = \frac{48}{169}$. The smallest number of collisions, n, to reduce the energy to 0.025 eV is therefore given by

$$\left(\frac{11}{13}\right)^{2n} = \frac{0.025}{2.5 \times 10^6} \quad \Rightarrow \quad n = 55$$

This requires n consecutive head-on collisions, which is highly unlikely, so this is a considerable underestimate.

(b) Call the numbers of each nuclide present N_U, N_C and N_B in obvious notation. The fraction of ^{235}U by weight is

$$f = \frac{235 N_U}{235 N_U + 12 N_C}$$

where the contribution of B is negligible in f. We therefore have

$$\frac{N_U}{N_C} = \frac{12f}{235(1-f)} \quad ; \quad \frac{N_B}{N_C} = \frac{12}{10} \times 10^{-6} = 1.2 \times 10^{-6}$$

The probability that a neutron is absorbed by a ^{235}U nucleus and generates fission is therefore

$$P = \frac{N_U \sigma_{\text{U;fiss.}}}{N_C \sigma_C + N_U \sigma_U + N_B \sigma_B} = \frac{\frac{N_U}{N_C} \times 580}{0.04 + \frac{N_U}{N_C} \times 700 + 1.2 \times 10^{-6} \times 3800}$$

The multiplication factor is given by $k = 2.5P < 1$, which leads to $\frac{N_U}{N_C} < 5.94 \times 10^{-5}$ which in turn implies $f < 1.16 \times 10^{-3}$.

19. **Nuclear Fusion** The two oxygen nuclei each have radius $R \sim 1.2\ \mathrm{fm} \times \mathrm{A}^{\frac{1}{3}} = 3.0$ fm. The Coulomb barrier to fusion is therefore the potential energy when the nuclei are $2R$ apart, i.e.

$$\frac{Z^2 e^2}{4\pi\epsilon_0 \times 2R} = \frac{Z^2 \alpha}{2R} = 0.08\ \mathrm{fm}^{-1} = 15\ \mathrm{MeV}.$$

Setting this equal to $k_{\mathrm{B}}T$ gives $T = 1.7 \times 10^8$ K.

20. **Relativistic Kinematics**

(a) Conservation of energy:

$$m_X = E_a + E_b$$

Conservation of momentum:

$$\sqrt{E_a^2 - m_a^2} = \sqrt{E_b^2 - m_b^2}$$

Eliminating E_b we have

$$E_b^2 = m_X^2 + E_a^2 - 2m_X E_a = E_a^2 - m_a^2 + m_b^2 \quad \Rightarrow E_a = \frac{m_X^2 + m_a^2 - m_b^2}{2m_X}$$

and the result for E_b follows by interchanging a and b. If $m_a = m_b$ both reduce to $E_a = E_b = \frac{1}{2} m_X$.

(b)

$$p_a = \sqrt{E_a^2 - m_a^2} = \frac{\sqrt{m_X^4 + m_a^4 + m_b^4 - 2m_X^2 m_a^2 - 2m_X^2 m_b^2 - 2m_a^2 m_b^2}}{2m_X} = p_b$$

If $m_a = m_b$ this reduces to $p_a = p_b = \frac{1}{2}\sqrt{m_X^2 - 4m_a^2}$

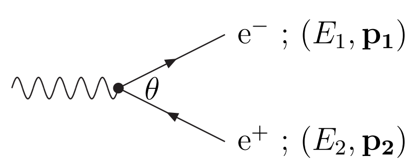
(c) c.m. energy $\sqrt{s}$ is given by

$$s = (E_e + E_p)^2 - (\mathbf{p}_e + \mathbf{p}_p)^2 \approx (920 + 27.5)^2 - (920 - 27.5)^2 \quad \Rightarrow \quad \sqrt{s} = 318.1\,\mathrm{GeV}$$

This is sufficient energy to produce a Higgs boson. The case of fixed-target collision can be obtained from the same formula, setting $E_p = m_p$ and $p_p = 0$, giving $s \approx 2E_e m_p$, from which we obtain $E_e = 53.9$ TeV.

21. **Relativistic Kinematics**

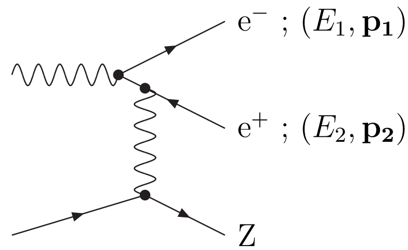
(a) In vacuo we have:



The initial state has $E^2 - p^2$=0, so equating this to the invariant for the final state:

$$0 = (E_1 + E_2)^2 - (\mathbf{p}_1 + \mathbf{p}_2)^2 = 2m_e^2 + 2E_1E_2 - 2p_1p_2\cos\theta$$

and the right hand side is manifestly greater than zero. Hence the decay is forbidden. In matter, we can have the following:



The internal electron line is virtual, so no problem to conserve (E, p) at each vertex. The spectator (labelled "Z") could be an electron, or more likely a heavy nucleus which would have a greater coupling.

(b) Energies of π^- and Ξ^0 are 313.8 and 2316 MeV respectively. So,

$$m_\Omega^2 = m_\pi^2 + m_\Xi^2 + 2E_\pi E_\Xi - 2p_\pi p_\Xi \cos\theta \;\;\Rightarrow\;\; m_\Omega = 1689\,\mathrm{MeV}$$

Hence $E_\Omega = 2316 + 313.8 = 2630$ MeV and $p_\Omega = \sqrt{E_\Omega^2 - p_\Omega^2} = 2016$ MeV.

(c) In the lab, $t = L/\beta$, and the proper lifetime $\tau = t/\gamma$. Noting that $\beta = p/E$ and $\gamma = E/m$ we have $\tau = mL/p = 7\,10^{-11}$ s (all formulae in natural units).

22. **Partial width, decay rate and branching fraction**

(a) Total width of decay $\Gamma(K^+) = \hbar/\tau = 5.47\,10^{-8}$ eV. Hence branching fraction for the decay $K^+ \to \pi^+\pi^0$ is given by $\Gamma(K^+ \to \pi^+\pi^0)/\Gamma(K^+) = 1.2/5.47 = 0.219$.

(b) Beam of K^+ have proper lifetime $\tau = 1.2 \times 10^8$ s; their laboratory lifetime will be $\gamma\tau$ and the mean decay length $\lambda = \beta c\gamma\tau$. We have $\beta\gamma = \frac{p}{E} \cdot \frac{E}{m} = \frac{p}{m} = 20.26$ and thus obtain the undecayed fraction as $\exp[-L/\beta\gamma c\tau] = 0.254$.

The c.m. momentum of the daughter π^+ is $p^* = 0.205$ GeV, using the result from Qu. 20(b) and its energy $E^* = 0.248$ GeV. The minimum and maximum laboratory energy can be obtained from a Lorentz transform: $E_{min} = \gamma(E^* - \beta p^*) = 0.88$ GeV and $E_{max} = \gamma(E^* + \beta p^*) = 9.18$ GeV.

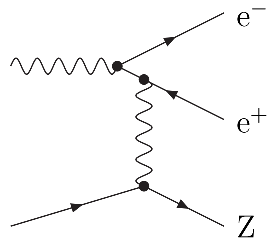
23. **Breit-Wigner Formula**

In this case, we have $\Gamma = 2.5$ GeV; $\Gamma_i = \Gamma_f = 0.0337\Gamma = 0.08425$ GeV; $E = 90$ GeV, $E_0 = 91.2$ GeV. Since the Z^0 has $J = 1$, the spin degeneracy factor $g = \frac{3}{4}$ and $\lambda = 2\pi/(45$ GeV$)$=0.0275 fm (remembering that this is the wavelength of the colliding particles in the c.m. frame). Inserting all this into the Breit-Wigner formula we obtain $\sigma = 1.07$ nb.

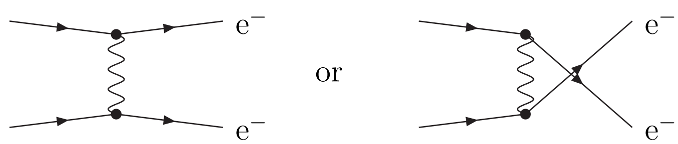
24. **Feynman Diagrams and QED**

Draw the lowest order Feynman diagram(s) for each of the following processes:

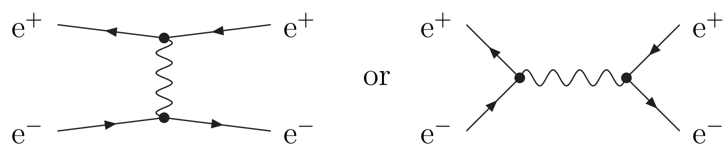
(a) $\gamma \to e^+e^-$ (in matter)



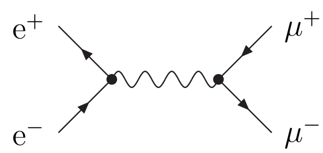
(b) $e^- + e^- \to e^- + e^-$



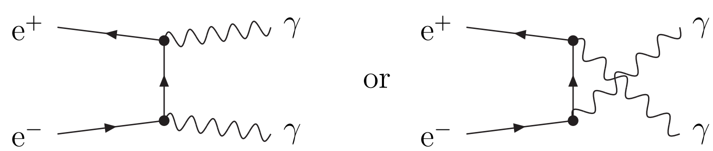
(c) $e^+ + e^- \to e^+ + e^-$



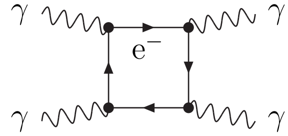
(d) $e^+ + e^- \to \mu^+ + \mu^-$



(e) $e^+ + e^- \to \gamma + \gamma$



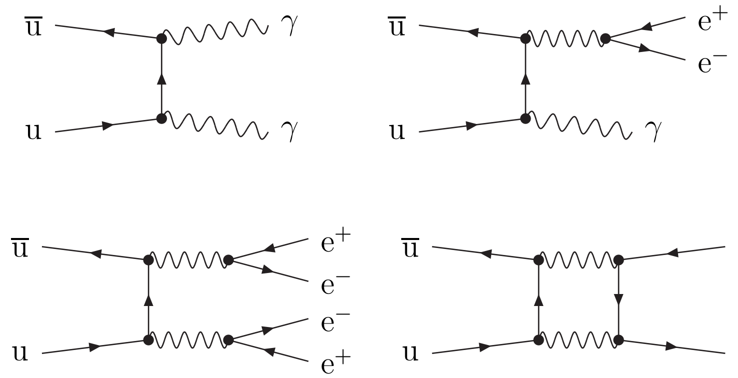
(f) $\gamma + \gamma \to \gamma + \gamma$



(other charged particles could appear in the loop, and the final photons could be crossed).

25. **Feynman Diagrams and QED**

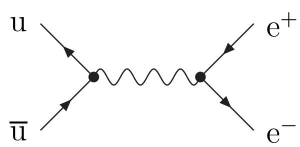
(a) The relevant Feynman diagrams are:



Also can have $\mathrm{d}\overline{\mathrm{d}}$ initial state, but less likely owing to charge. Compare each decay with the predominant $\gamma\gamma$ mode, counting powers of α:

- $\mathrm{e}^+\mathrm{e}^-\gamma$ — one extra vertex means matrix element is multiplied by a factor $2\sqrt{\alpha}$, where the factor 2 asises because either photon could convert to $\mathrm{e}^+\mathrm{e}^-$, and hence the rate $\propto 4\alpha \sim 2.9\%$;
- $\mathrm{e}^+\mathrm{e}^-\mathrm{e}^+\mathrm{e}^-$ — two extra vertices, so rate $\propto \alpha^2 \sim 5\,10^{-5}$;
- $\mathrm{e}^+\mathrm{e}^-$ — again two extra vertices, so rate $\propto \alpha^2 \sim 5\,10^{-5}$ again. Presumably the extra electron propagator accounts for the additional suppression compared to $\mathrm{e}^+\mathrm{e}^-\mathrm{e}^+\mathrm{e}^-$. Note that the quarks can't just decay into a single virtual photon, since this would not conserve angular momentum (the photon has $J = 1$).

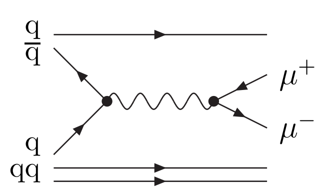
(b) Unlike the π^0, the ρ^0 has the same J^P as the photon, so this Feynman diagram is allowed:



The partial width is given by $\Gamma_{\mathrm{e}^+\mathrm{e}^-} = B(\mathrm{e}^+\mathrm{e}^-)\hbar/\tau$. Inserting numbers, we get $\Gamma_{\mathrm{e}^+\mathrm{e}^-} = 6.0$ keV for the ρ^0 and $\Gamma_{\mathrm{e}^+\mathrm{e}^-} = 1.6\,\mu$eV for the π^0; a ratio of 4×10^9. This is because of the extra vertex factors and propagators in the π^0 case, and also the greater phase space in ρ^0 decay.

26. **Drell-Yan process**

Typical Feynman diagram:



The cross section will be proportional to $\sum_i e_i^2$, where the sum runs over all $\mathrm{q}\overline{\mathrm{q}}$ pairs which can annihilate, and e_i is the charge of the i^{th} quark. Hence, in π^+ ($\mathrm{u}\overline{\mathrm{d}}$) interactions with a proton (uud) the cross-section $\propto \frac{1}{3}^2$, in π^+ ($\mathrm{u}\overline{\mathrm{d}}$) interactions with a neutron (udd) the cross-section $\propto 2 \times \frac{1}{3}^2$, in π^- ($\mathrm{d}\overline{\mathrm{u}}$) interactions with a proton (uud) the cross-section $\propto 2 \times \frac{2}{3}^2$ and in π^- ($\mathrm{d}\overline{\mathrm{u}}$) interactions with a neutron (udd) the cross-section $\propto \frac{2}{3}^2$. Hence the ratios are 1 : 2 : 8 : 4.

In pp scattering, there are no antiquarks, so the cross-section is zero to leading order. (However, in higher orders in QCD, the gluons in the proton can create virtual q$\bar{\text{q}}$ pairs, so the rate is non-zero.) In $\bar{\text{p}}$p collisions the rate should be $\propto \left(4 \times \frac{2}{3}^2 + \frac{1}{3}^2\right) = \frac{17}{9}$, i.e. 17 times that for π^+p.

27. **Colour factors in QCD** Follow the approach in lectures for $q\bar{q}$ in, for example, a $r\bar{b}$ colour combination, which is manifestly not colourless. In this case, the possible gluon exchanges are $(r\bar{r} - b\bar{b})/\sqrt{2}$ or $(r\bar{r} + b\bar{b} - 2g\bar{g})/\sqrt{6}$. The first of these gives coupling $\beta/\sqrt{2}$ at the quark vertex and $-(-\beta/\sqrt{2})$ at the antiquark vertex (where the first $-$ arises because it's an antiquark and the second from the sign in the gluon colour state). Likewise exchange of the second gluon yields $\beta/\sqrt{6}$ at the quark vertex and $-(\beta/\sqrt{6})$ at the antiquark vertex. Multiplying the colour factors and adding the two exchange diagrams we get

$$V = \frac{\beta^2}{2r} - \frac{\beta^2}{6r} = +\frac{\beta^2}{3r} = \frac{\alpha_s}{6r} \,.$$

The case of qq in a symmetric colour state $(rg + gr)/\sqrt{2}$ is just the same as in lectures except that we don't get the $-$ sign associated with the $r\bar{g}$ exchange symmetry. Hence

$$V = +\frac{\beta^2}{r} - \frac{\beta^2}{3r} = +\frac{2\beta^2}{3r} = \frac{\alpha_s}{3r} \,.$$

Both of these cases give a repulsive potential. Of course, the forces in hadrons involve more than single gluon exchanges, but this does help to make plausible the idea that hadrons can only be formed in colour singlet states.

28. **Spin and Parity**
The reaction is this:

$$\underbrace{\pi^- d}_{\ell=0} \longrightarrow \underbrace{nn}_{L}$$

We are given that the deuteron has spin-parity $J^P = 1^+$ and the pion has spin 0. Hence, the initial state has total $J = 1$ and parity equal to that of the pion, P_π. J^P must both be conserved. Combining the angular momenta in the final state, we see that there are four ways of achieving $J = 1$, namely ${}^3S_1$, ${}^3P_1$, ${}^1P_1$ and ${}^3D_1$ in spectroscopic notation. However, the two neutrons are identical fermions, and their overall wavefunction must be antisymmetric under particle exchange. The only state which satisfies this is ${}^3P_1$. Hence $L = 1$ and the parity of the final state is $(-1)^L = -1$, and consequently $P_\pi = -1$.

29. **Hadron masses in the quark model**

 (a) The quark model prediction for the mass of a meson with spin J and $\ell = 0$ is

$$M = m_1 + m_2 + \frac{A}{2m_1m_2}\left(J(J+1) - \frac{3}{2}\right)$$

i.e. $m_1 + m_2 + A/4m_1m_2$ if $J = 1$ and $m_1 + m_2 - 3A/4m_1m_2$ if $J = 0$.
Most of the mesons should be straightforward, given their quark compositions and spins. Only two cases may cause some difficulty:

The η has quark wavefunction $\frac{1}{6}(\mathrm{u\overline{u}}+\mathrm{d\overline{d}}-2\mathrm{s\overline{s}})$, i.e. a probability $\frac{2}{3}$ of being in the $s\overline{s}$ configuration etc. This gives the mass as $\frac{2}{3}\times 768+\frac{1}{3}\times 140$=559 MeV.
The η' has quark wavefunction $\frac{1}{3}(\mathrm{u\overline{u}}+\mathrm{d\overline{d}}+\mathrm{s\overline{s}})$, i.e. a probablity $\frac{1}{3}$ of being in the $s\overline{s}$ configuration etc. This gives the mass as $\frac{1}{3}\times 768+\frac{2}{3}\times 140$=349 MeV – not a successful prediction!
The left side of the identity:

$$\frac{\rho-\pi}{K^*-K}=\frac{A/m_u^2}{A/m_u m_s}=\frac{m_s}{m_u}=1.61$$

and represents the ratio of the s and u quark masses appearing in the colour magnetic moments. The right side gives

$$2\left(\frac{3K^*+K}{3\rho+\pi}\right)-1=2\frac{4(m_u+m_s)}{4(m_u+m_u)}-1=\frac{m_s}{m_u}=1.57$$

and therefore gives the ratio of the s and u quark masses appearing in the constituent mass terms.

(b) Each pair of quarks in a $\mathrm{J}^P=\frac{3}{2}^+$ baryon must have $S=1$. Hence the mass formula gives:

$$M=\sum_i m_i+\frac{A}{4}\sum_{i<j}\frac{1}{m_i m_j} \quad .$$

Predictions are therefore:

$$\begin{aligned}
\Delta \quad &: \quad M = \; 3m_u+\frac{3A}{4}\frac{1}{m_u^2}=1230\text{ MeV}\\
\Sigma \quad &: \quad M = \; 2m_u+m_s+\frac{A}{4}\left(\frac{1}{m_u^2}+\frac{2}{m_s m_u}\right)=1377\text{ MeV}\\
\Xi \quad &: \quad M = \; m_u+2m_s+\frac{A}{4}\left(\frac{1}{m_s^2}+\frac{2}{m_s m_u}\right)=1529\text{ MeV}\\
\Omega \quad &: \quad M = \; 3m_s+\frac{3A}{4}\frac{1}{m_s^2}=1687\text{ MeV}
\end{aligned}$$

30. **Magnetic Moments in the quark model**

(a) The two u quarks must have an overall antisymmetric wavefunction. They have symmetric spatial wavefunction (ℓ=0), symmetric flavour wavefunction (trivially), and the colour singlet wavefunction is antisymmetric. Hence their spin wavefunction is symmetric, and so $S=1$.

(b) From derivation in lecture notes:

$$|p\uparrow\rangle=\frac{1}{\sqrt{6}}\left(2\,|u\uparrow u\uparrow d\downarrow\rangle-\;|u\uparrow u\downarrow d\uparrow\rangle-\;|u\downarrow u\uparrow d\uparrow\rangle\right)$$

Expression for neutron arises by exchanging u and d:

$$|n\uparrow\rangle=\frac{1}{\sqrt{6}}\left(2\,|d\uparrow d\uparrow u\downarrow\rangle-\;|d\uparrow d\downarrow u\uparrow\rangle-\;|d\downarrow d\uparrow u\uparrow\rangle\right)$$

(c)

$$\mu_u = \frac{\frac{2}{3}e\hbar}{2m_u} \quad ; \quad \mu_d = \frac{-\frac{1}{3}e\hbar}{2m_u} = -\tfrac{1}{2}\mu_u$$

(d) Using the wavefunctions just derived, p has probability $\frac{2}{3}$ of being in the $|u\!\uparrow\! u\!\uparrow\! d\!\downarrow\rangle$ state etc., so:

$$\mu_p = \tfrac{2}{3}(2\mu_u - \mu_d) + \tfrac{1}{3}\mu_d = \tfrac{4}{3}\mu_u - \tfrac{1}{3}\mu_d = \tfrac{3}{2}\mu_u$$
$$\mu_n = \tfrac{2}{3}(2\mu_d - \mu_u) + \tfrac{1}{3}\mu_u = \tfrac{4}{3}\mu_d - \tfrac{1}{3}\mu_u = -\mu_u$$

and the ratio $\mu_n/\mu_p = \frac{2}{3}$, compared to data: 0.684. Not too bad. Requires $m_u \approx 0.33$ GeV, which is around one third of the proton's mass – not stupid.

(e) By comparison with the cases just considered,

$$\mu_{\Sigma^+} = \tfrac{4}{3}\mu_u - \tfrac{1}{3}\mu_s$$
$$\mu_{\Sigma^-} = \tfrac{4}{3}\mu_d - \tfrac{1}{3}\mu_s$$

Hence,

$$\mu_{\Sigma^+} - \mu_{\Sigma^-} = \tfrac{4}{3}[\mu_u - \mu_d]$$

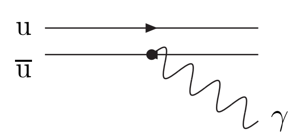
and from (d)

$$\mu_p - \mu_n = \tfrac{5}{3}[\mu_u - \mu_d]$$

which leads to the required result.

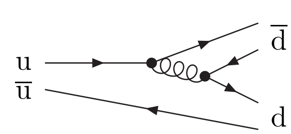
31. **Feynman diagrams in QCD; conservation laws**

(a) $\rho^0 \to \pi^0\gamma$



Electromagnetic decay - needs to conserve J^P, which is fine, so long as the photon is emitted in a 1^+ state, i.e. it is a magnetic dipole (M1) transition.

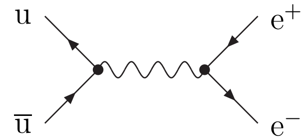
(b) $\rho^0 \to \pi^+\pi^-$



OK, by the strong interaction. J^P is conserved, so long as the two pions are formed in an $\ell = 1$ orbital state.

(c) $\rho^0 \to \pi^0\pi^0$. Same Feynman diagram as for $\pi^+\pi^-$ can be drawn, just replacing d by u. But in this case the two π^0s are identical bosons, so they are forbidden to be in an $\ell = 1$ (anitsymmetric) state. Hence angular momentum cannot be conserved, and the decay is absolutely forbidden.

(d) The ρ^0 has the same J^P as the photon, so this Feynman diagram is allowed:



Expect $\pi^+\pi^-$ to have the highest rate (strong) followed by $\pi^0\gamma$ and then e^+e^- (more vertices than $\pi^0\gamma$) while $\pi^0\pi^0$ is simply forbidden.

In comparing decays of ρ^0 and ω^0, we need to remember that the decay can proceed from either the $u\overline{u}$ or the $d\overline{d}$ part of the wavefunction, and that the two contributions will interfere. Considering the decays to $\pi^0\gamma$, schematically we can write the matrix elements as

$$M(\rho^0 \to \pi^0\gamma) \sim \left\langle u\bar{u} - d\bar{d}\right| q \left|u\bar{u} - d\bar{d}\right\rangle \sim \tfrac{2}{3} + (-\tfrac{1}{3}) = \tfrac{1}{3}$$

$$M(\omega^0 \to \pi^0\gamma) \sim \left\langle u\bar{u} + d\bar{d}\right| q \left|u\bar{u} - d\bar{d}\right\rangle \sim \tfrac{2}{3} - (-\tfrac{1}{3}) = 1$$

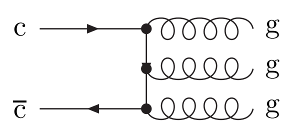
Hence the ratio of rates should naïvely be $\sim$1/9. In the e^+e^- case, we can write

$$M(\rho^0 \to e^+e^-) \sim \left\langle u\bar{u} - d\bar{d}\right| q \left|\gamma\right\rangle \sim \tfrac{2}{3} - (-\tfrac{1}{3}) = 1$$

$$M(\omega^0 \to e^+e^-) \sim \left\langle u\bar{u} + d\bar{d}\right| q \left|\gamma\right\rangle \sim \tfrac{2}{3} + (-\tfrac{1}{3}) = \tfrac{1}{3}$$

so in this case the ratio of rates should be $\sim$9/1.

32. **α_s in charmonium decay**



By analogy with QED we expect the rate for charmonium decay to be

$$\Gamma(ggg) = \frac{2(\pi^2-9)}{9\pi}\left(\tfrac{4}{3}\alpha_s\right)^6 m_c \quad .$$

Taking the charm quark mass to be $m_c \approx \frac{1}{2}M_{J//\psi} \approx 1.55$ GeV, and assuming $\Gamma(J/\psi \to \text{hadrons}) = \Gamma(\text{ggg})$, we obtain $\alpha_s = 0.23$.

Similarly for the Υ, replacing the charm quark mass by $m_b \approx \frac{1}{2}M_\Upsilon \approx 4.73$ GeV, we obtain $\alpha_s = 0.18$. We note that α_s runs (decreases) with energy scale as expected.

33. **Breit-Wigner for J/ψ production; Zweig's rule**

(a) In this case, $J = 1$, and $\Gamma_i = \Gamma_f = B\Gamma$. Integrate the Breit-Wigner formula using the substitution $E - E_0 = \frac{1}{2}\Gamma \tan\theta$:

$$\begin{aligned}
\sigma' &= \frac{3\lambda^2}{16\pi} B^2\Gamma^2 \int_0^\infty \frac{1}{[(E-E_0)^2 + \frac{1}{4}\Gamma^2]} \mathrm{d}E \\
&= \frac{3\lambda^2}{16\pi} B^2\Gamma^2 \int_{-\pi/2}^{\pi/2} \frac{\frac{1}{2}\Gamma \sec^2\theta \,\mathrm{d}\theta}{\frac{1}{4}\Gamma^2 \sec^2\theta} \\
&= \frac{3\lambda^2}{16\pi} B^2\Gamma^2 \frac{2\pi}{\Gamma} = \frac{3}{8}\lambda^2 B^2 \Gamma
\end{aligned}$$

Note we have implicitly assumed that $\Gamma \ll E_0$, allowing us to set the lower limit of integration to $-\frac{1}{2}\pi$.

(b) The measured cross-section is given by a convolution:

$$\sigma_{meas}(E) = \int_0^\infty \sigma_{true}(E') f(E'-E) \mathrm{d}E' \quad .$$

Now, the spectrum f is a probability distribution, so

$$\int_0^\infty f(E'-E)\mathrm{d}E = \int_0^\infty f(E'-E)\mathrm{d}E' = 1 \quad .$$

Hence,

$$\int_0^\infty \sigma_{meas}(E)\mathrm{d}E = \int_0^\infty \int_0^\infty \sigma_{true}(E') f(E'-E)\mathrm{d}E\mathrm{d}E' = \int_0^\infty \sigma_{true}(E)\mathrm{d}E \quad ,$$

by performing the integration over E'.

(c) The fraction is given by

$$\frac{\int_{-0.6}^{0.6}(1+\cos^2\theta)\mathrm{d}\cos\theta}{\int_{-1}^{1}(1+\cos^2\theta)\mathrm{d}\cos\theta} = \frac{\left[\cos\theta + \frac{1}{3}\cos^3\theta\right]_{-0.6}^{0.6}}{\left[\cos\theta + \frac{1}{3}\cos^3\theta\right]_{-1}^{1}} = 0.504$$

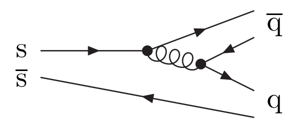
(since $\mathrm{d}\Omega \propto \mathrm{d}\cos\theta$).

(d) Only possible to make rough estimates — these are mine.... Taking $\mathrm{e}^+\mathrm{e}^-$ for example, peak cross-section above background ~ 80 nb, FWHM=$\Gamma \sim 0.003$ GeV, so that

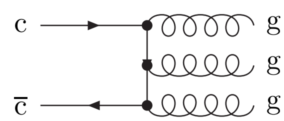
$$\sigma' \sim \frac{80 \times 3}{0.504} = 500 \text{ nb.MeV}$$

(with a large uncertainty). Peak cross-section above background for $\mu^+\mu^- \sim 90$ nb and for hadrons ~ 2200 nb. So $B \sim \frac{80/0.504}{(80+90)/0.504+2200} \sim 0.065$. The wavelength $\lambda = (2\pi/1550)\ \mathrm{MeV}^{-1} = 0.825$ fm. Hence $\Gamma \sim 70$ keV and $\Gamma_{\mathrm{e^+e^-}} \sim 5$ keV.

(e) The J/ψ and ϕ mesons have similar leptonic widths Γ_{ee} because both decays involve the same Feynman diagram — annihilation into a single virtual photon. The total widths Γ are dominated by the hadronic decay modes. The ϕ can decay into a pair of strange K-mesons



whilst the J/ψ is below the threshold for decay into charmed mesons, so the quarks must annihilate into three gluons



This more complicated decay is "Zweig-suppressed" by the extra coupling constants and propagators needed.

34. **Fermi theory of β-decay**
Bookwork gives:

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}p} = \frac{G_{\mathrm{F}}^2}{2\pi^3}\,(E_0 - E)^2\,p^2$$

(omitting the suffices "e" for brevity). This is the momentum spectrum of the electrons, which can be regarded as the probability density function for electron momentum, once normalised.

If the electron is highly relativistic we approximate $p = E$, so the mean energy is.

$$\frac{\displaystyle\int_0^{E_0} E\cdot(E_0 - E)^2 E^2 \mathrm{d}E}{\displaystyle\int_0^{E_0} (E_0 - E)^2 E^2 \mathrm{d}E} = \tfrac{1}{2}E_0$$

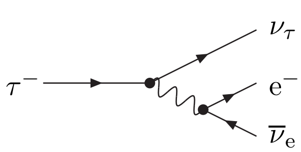
If the electron is non-relativistic, we have $E = p^2/2m$, and the upper limit of the integral is $\sqrt{2mE_0}$, so the mean kinetic energy is

$$\frac{\displaystyle\int_0^{\sqrt{2mE_0}} \frac{p^2}{2m}\cdot\left(E_0 - \frac{p^2}{2m}\right)^2 p^2\mathrm{d}p}{\displaystyle\int_0^{\sqrt{2mE_0}} \left(E_0 - \frac{p^2}{2m}\right)^2 p^2\mathrm{d}p} = \tfrac{1}{3}E_0$$

In the relativistic case, the total rate is

$$\Gamma = \frac{G_{\mathrm{F}}^2}{2\pi^3}\int_0^{E_0}(E_0 - E)^2 E^2 \mathrm{d}E = \frac{G_{\mathrm{F}}^2 E_0^5}{60\pi^3}\quad .$$

35. **Weak decays of the τ-lepton**
The relevant Feynman diagrams will be:



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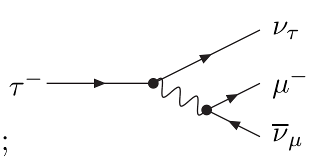

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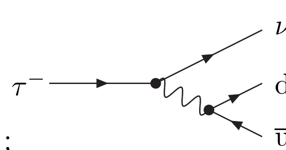
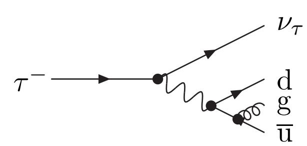


The matrix elements for $\tau^- \to e^-\nu_\tau\bar{\nu}_e$ and $\tau^- \to \mu^-\nu_\tau\bar{\nu}_\mu$ should be equal, and the small difference in the rates simply reflects phase space. The decay $\tau^- \to d\bar{u}$ should be multiplied by a factor 3 for colour, and by $\cos^2\theta_c$ where θ_c is the Cabbibo angle. However, there is also the possibility of the decay $\tau^- \to s\bar{u}$ with rate $\propto 3\sin^2\theta_c$, so the total rate for hadronic final states is enhanced simply by the colour factor of 3. Energy conservation does not permit the τ to decay to heavier quarks sich as charm. The observed ratio of 3.5 is accounted for by higher order QCD corrections, such as:



We have $\Gamma(\tau \to e) = B(\tau \to e)/\tau_\tau$ and $\Gamma(\mu \to e) = 1/\tau_\mu$. Then , from Sargent's rule, since $E_0 \propto m_{\ell^-}$,

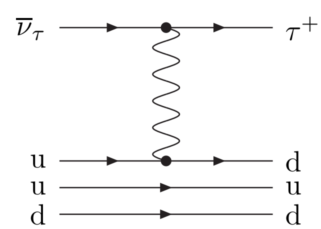
$$\frac{\Gamma(\tau \to e)}{\Gamma(\mu \to e)} = \left(\frac{m_\tau}{m_\mu}\right)^5$$

and hence

$$\tau_\tau = B(\tau \to e) \cdot \tau_\mu \cdot \left(\frac{m_\mu}{m_\tau}\right)^5 = 0.298 \text{ ps}$$

36. **Weak interactions of the ν_τ; kinematics**
In a beam of antineutrinos, it is proposed to search for $\bar{\nu}_\tau$ via their interactions on nucleons in a stationary target to produce τ-leptons.

(a) Simplest process is $\overline{\nu}_\tau p \longrightarrow \tau^+ n$:



(b) Equating the $E^2 - p^2$ invariant before and after the collision, we have at threshold for this reaction:

$$s = (E_\nu + m_p)^2 - E_\nu^2 = 2E_\nu m_p + m_p^2 = (m_n + m_\tau)^2$$

in obvious notation, and hence, rearranging,

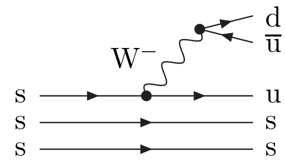
$$E_\nu = \frac{(m_n + m_\tau)^2 - m_p^2}{2m_p} = 3.47 \text{ GeV}$$

(c) The γ-factor of the c.m. system is $(E_\nu + m_p)/(m_n + m_\tau) = 1.62$, and hence in the lab, $E_\tau = \gamma m_\tau = 2.88$ GeV. (Many other ways to do this)

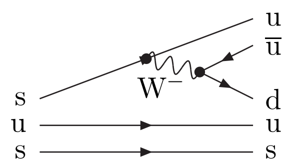
(d) Velocity of c.m. system is $\beta = 0.787$; lab lifetime $t_\tau = \gamma\tau_\tau$, and so average lab distance travelled is $\beta\gamma\tau_\tau$ in natural units, $= 1.1 \times 10^{-4}$ m.

37. **Decays of the Ω^-**

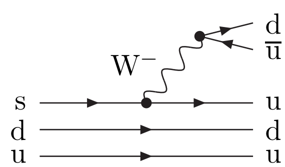
$\Omega^- \to \Xi^0\pi^-$:



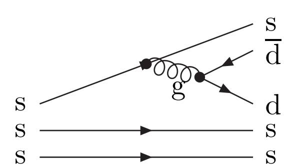
$\Xi^0 \to \Lambda^0\pi^0$:



$\Lambda^0 \to \mathrm{p}\pi^-$:

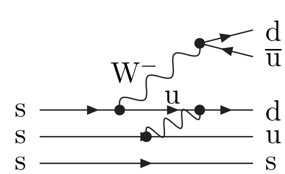


Strong decay $\Omega^- \to \Xi^-\overline{\mathrm{K}^0}$



Looks plausible, but the decay is not observed because energy is not conserved — the daughters are more massive than the parent, so absolutely forbidden.

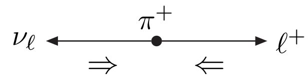
Several possibilities for $\Omega^- \to \Lambda^0\pi^-$, for example:



However, all possibilities involve two W-bosons, because we have to turn two strange quarks into non-strange. So this is second-order weak, and highly suppressed relative to the permitted first-order decays to $\Xi\pi$ or $\Lambda^0 K^-$.

38. **Helicity in the weak interaction**

(a) In the centre-of-mass frame of the pion:



The double arrows indicate the helicities of the emitted particles. The neutrino always has negative helicity, so angular momentum conservation demands that the ℓ^+ also have negative helicity (the disfavoured state).

(b) Calling the c.m. momentum p^*, energy conservation gives

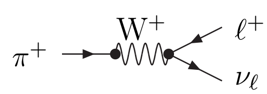
$$m_\pi = p^* + \sqrt{p^{*2} + m_\ell^2}$$

which, after rearrangement yields:

$$p^* = \frac{m_\pi^2 - m_\ell^2}{2m_\pi} \quad .$$

The velocity is given by $v/c = p^*/E^*$, so plugging in numbers gives $v/c = 0.99997$ for the decay to electron, and 0.271 for the decay to muon.

(c) This is the Feynman diagram:



(d) The probability for producing the ℓ^+ in the "wrong" negative helicity state is proportional to $(1 - \frac{v}{c})$, $=3\times10^{-5}$ for the electron case and 0.73 for the muon: a ratio of 4×10^{-5}. This accounts for the large part of the observation, with the difference in phase space also contributing.

(e) From (b) we have (changing notation for the masses):

$$p^* = \frac{M^2 - m^2}{2M} \quad \Rightarrow \quad E^* = \frac{M^2 + m^2}{2M} \quad .$$

Hence the helicity factor is

$$\frac{1}{2}\left(1 - \frac{v}{c}\right) = \frac{1}{2}\left(1 - \frac{p^*}{E^*}\right) = \frac{m^2}{M^2 + m^2} \quad .$$

Now for the phase space factor:

$$E_0 = p^* + \sqrt{p^{*2} + m^2} \quad \Rightarrow \quad \frac{dE_0}{dp^*} = 1 + \frac{p^*}{E^*} = \frac{2M^2}{M^2 + m^2}$$

so that

$$\rho(E_0) \propto p^{*2}\frac{dE_0}{dp^*} = \left(\frac{M^2 - m^2}{2M}\right)^2 \left(\frac{M^2 + m^2}{2M^2}\right) \quad .$$

Multiplying by the helicity factor we obtain the decay rate as

$$\Gamma \propto \frac{m^2(M^2 - m^2)^2}{8M^4}$$

as required. The numerator is the bit which determines the ratio between electron and muon final states; putting numbers in we get 1.28×10^{-4}, in pretty good agreement with observation. The residual discrepancy arises from photon radiation from the final state lepton, which is enhanced for the electron because of its smaller mass.

39. **Decays of the W boson**
Plugging in numbers,

$$\Gamma(\mathrm{W}^- \to \mathrm{e}^- \bar{\nu}_\mathrm{e}) = \frac{\mathrm{G_F}}{\sqrt{2}} \frac{\mathrm{M_W^3}}{6\pi} = 0.23\,\mathrm{GeV}\ .$$

The total width is the sum of the leptonic and hadronic contributions:

$$\Gamma_\mathrm{W} = n_\nu \Gamma(\mathrm{W}^- \to \mathrm{e}^- \bar{\nu}_\mathrm{e}) + 2 \times 3 \times \Gamma(\mathrm{W}^- \to \mathrm{e}^- \bar{\nu}_\mathrm{e}) = (6 + n_\nu)\Gamma(\mathrm{W}^- \to \mathrm{e}^- \bar{\nu}_\mathrm{e})$$

where the first term reflects the number of lepton generations accessible, n_ν, and the second term the hadronic decays. Hadronic decays to the third generation involving the top quark are kinematically forbidden, so we have a factor 2 for generations and 3 for colour. Given $\Gamma_\mathrm{W} = 2.1$ GeV, we deduce $n_\nu = 3.1$, i.e. 3 since it must be an integer. More correctly there should be a QCD correction factor (of ∼1.04) applied to the hadronic term, which gives a closer estimate.

40. **The Z boson**

(a) Considering the first generation fermions, we can construct the following table

Fermion	Q	I_3 l.h.	I_3 r.h.	c_L	c_R	$c_L^2 + c_R^2$	Branching fraction
e^-	-1	$-\frac{1}{2}$	0	-0.270	0.230	0.126	0.0344
ν_e	0	$+\frac{1}{2}$	0	0.5	0	0.250	0.0684
u	$\frac{2}{3}$	$+\frac{1}{2}$	0	0.347	-0.153	0.144	0.1179
d'	$-\frac{1}{3}$	$-\frac{1}{2}$	0	-0.423	0.077	0.185	0.1519

To obtain the total rate in order to normalise the branching fractions, we need to include a colour factor of 3 for the quark final states, and recollect than only two generations of u-like quarks can be formed in Z decays (the t-quark being too massive), but three generations of the other fermions. An improved treatment would be to include a factor $(1 + \alpha_s/\pi)$ for gluon emission for the quark final states, roughly a 4% correction.

(b) From a graph we can make rough estimates: M_{Z^0} ∼91.5 GeV and Γ_Z ∼2.8 GeV. The peak cross-section (setting $E = M_{\mathrm{Z}^0}$ and $\Gamma_\mathrm{e} = \Gamma_\tau$) is

$$\sigma_0(\mathrm{Z}^0 \longrightarrow \tau^+\tau^-) = \frac{3\pi}{4p^2} \frac{4\Gamma_\mathrm{e}\Gamma_\tau}{\Gamma_\mathrm{Z}^2}$$

where p is the c.m. momentum of the beams at the peak, i.e. $\frac{1}{2}M_{\mathrm{Z}^0} = 45.75$ GeV. From the graph σ_0 is $\sim$ 1.5 nb, so inserting our estimates we get Γ_τ/Γ=0.0146 and hence Γ_τ ≈82 MeV.

The main reason why the measured resonance curve is asymmetric is initial-state radiation, i.e. emission of one or more photons from the electron and positron *before* they form the Z^0. This reduces the collision energy at the point when the resonance is formed. This will tend to decrease the cross-section at or below the peak, and increase the cross-section when we are significantly above the peak. The distorting effect needs to be calculated carefully from QED before extracting a reliable value for the Z mass and width. There are also small additional effects, like tides and trains to be considered, but initial-state radiation is much the largest effect.

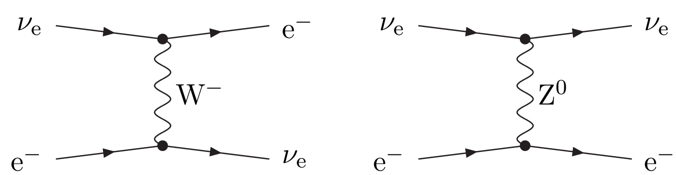
(c) The total width is obtained by summing all contributions:

$$\Gamma_Z = \Gamma_{\mu^+\mu^-}(3 + R) + \Gamma_{\nu\overline{\nu}} \cdot 3$$

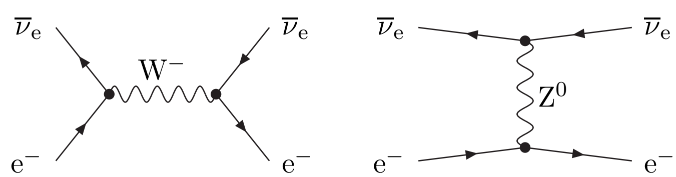
where R is measured as 20.7, and the factors of 3 reflect lepton universality, assuming that only three generations of leptons contribute. This gives $\Gamma = 2473.7$ MeV, and $\tau_Z = 1/\Gamma = 2.65 \times 10^{-25}$ s.

41. **Neutrino scattering**

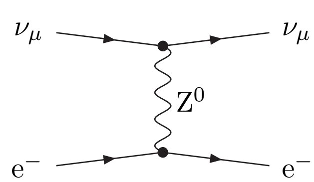
(a) $\nu_e\, e^- \to \nu_e\, e^-$



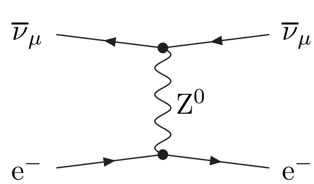
(b) $\bar{\nu}_e\, e^- \to \bar{\nu}_e\, e^-$



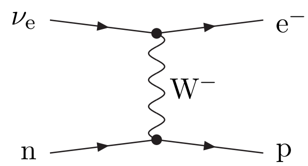
(c) $\nu_\mu\, e^- \to \nu_\mu\, e^-$



(d) $\bar{\nu}_\mu\, e^- \to \bar{\nu}_\mu\, e^-$

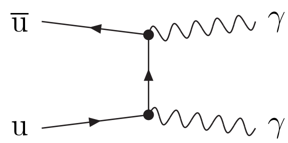


(e) $\nu_e\, n \to e^-\, p$

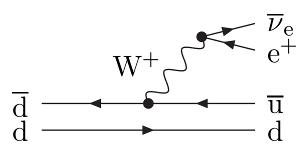


42. **Weak decays and conservation laws**

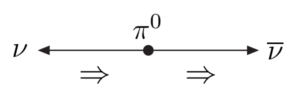
(a) $\pi^0 \to \gamma\gamma$ OK, electromagnetic decay. This is the Feynman diagram:



$\pi^0 \to \pi^- e^+ \nu_e$ – Feynman diagram is OK (see below), but kinematically impossible (π^- is heavier than the π^0.

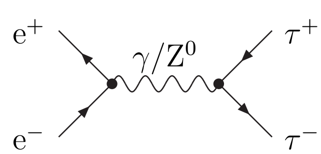


In the centre-of-mass frame of the pion the decay looks like this:

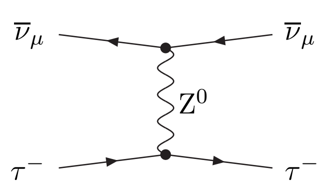


We see that $\pi^0 \to \nu\bar{\nu}$ is absolutely forbidden by angular momentum conservation since neutrinos (antineutrinos) *always* have the helicity states shown, and the π^0 has $J = 0$.

(b) $e^+e^- \to \tau^+\tau^-$ is an allowed electroweak process.

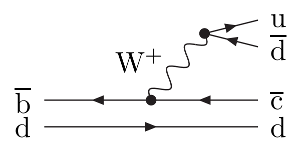


$\bar{\nu}_\mu + \tau^- \to \tau^- + \bar{\nu}_\mu$ is an allowed weak process.

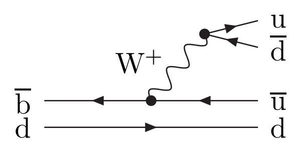


$\nu_\tau + p \to \tau^+ + n$ does not conserve tau-lepton number. There is no valid Feynman diagram.

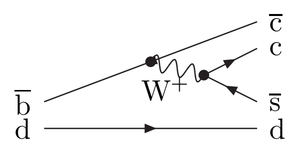
(c) $B^0(\overline{b}d) \to D^-(\overline{c}d)\pi^+$ is a weak (flavour changing) decay. Rate $\propto V_{bc} \cos\theta_C$.



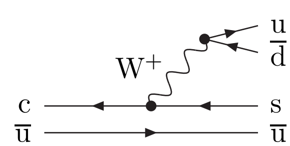
$B^0 \to \pi^+\pi^-$ is a weak (flavour changing) decay. Rate $\propto V_{bu} \cos\theta_C$. Much suppressed compared to $D\pi^+$ because of the small CKM matrix element V_{bu}.



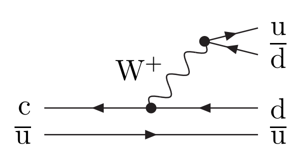
$B^0 \to J/\psi K^0$ is a weak (flavour changing) decay. Rate $\propto V_{bc} \cos\theta_C$. Same vertices as $D\pi^+$ but suppressed a bit by phase space.



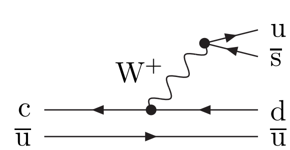
(d) $D^0(c\overline{u}) \to K^-\pi^+$ is an allowed weak decay; rate $\propto \cos^2\theta_C$.



$D^0 \to \pi^+\pi^-$ is an allowed weak decay; rate $\propto \cos\theta_C \sin\theta_C$.



$D^0 \to K^+\pi^-$ is an allowed weak decay; rate $\propto \sin^2\theta_C$.



43. **Neutrino Oscillations**

At $t = 0$ the state is

$$\psi(0) = |\nu_\mu\rangle = \nu_2 \cos\theta + \nu_3 \sin\theta \ ,$$

so inserting the usual time dependence for the mass eigenstates:

$$\psi(t) = \nu_2 e^{-iE_2 t/\hbar} \cos\theta + \nu_3 e^{-iE_3 t/\hbar} \sin\theta \ .$$

Hence the probability of being in the ν_μ state at time t is given by

$$\begin{aligned} |\langle\nu_\mu|\psi(t)\rangle|^2 &= \left| e^{-iE_2 t/\hbar} \cos^2\theta + e^{-iE_3 t/\hbar} \sin^2\theta \right|^2 \\ &= \cos^4\theta + \sin^4\theta + 2\sin^2\theta\cos^2\theta\cos(E_3 - E_2)t/\hbar \\ &= 1 - \sin^2 2\theta \sin^2 \tfrac{1}{2}(E_3 - E_2)t/\hbar \end{aligned}$$

which gives the required answer, noting that $t \approx L/c$ for neutrinos.

If m_2 and m_3 are very much less than the neutrino momentum, p,

$$E_2 = (p^2 + m_2^2)^{\frac{1}{2}} \approx p\left(1 + \frac{m_2^2}{2p^2}\right)$$

and hence

$$(E_3 - E_2) = (m_3^2 - m_2^2)/2p$$

which yields the required answer; now in natural units.

Inserting the given numerical values into the formula, we have

$$\left|\frac{\nu_\mu(L)}{\nu_\mu(0)}\right|^2 = 1 - 0.9\sin^2\left(1.27 \times \frac{2.5 \times 10^{-3} \times 730}{p}\right)$$

Evaluate at some typical momenta and sketch:

p /GeV	$\left\|\frac{\nu_\mu(L)}{\nu_\mu(0)}\right\|^2$
1	0.51
1.5	0.10
2	0.24
3	0.56
4	0.73
5	0.80

From the result of qu.36, the threshold for production of a τ^- is 3.5 GeV, so observation is just about possible, though the oscillation probabllity is not high at and above this energy.