E&O paper 2015

1(a). $\begin{bmatrix} a \\ b \end{bmatrix} \longrightarrow$

$J_1 = J_4 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$J_2 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \end{bmatrix}\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \quad ; \theta = \omega t.$$

$$J_3 = \begin{bmatrix} \cos^2 2\theta & -\cos 2\theta \sin 2\theta \\ -\cos 2\theta \sin 2\theta & \sin^2\theta \end{bmatrix} \quad ; \quad \theta = \omega t.$$

light from left :- $J = J_4 J_3 J_2 J_1$

$$\begin{bmatrix} \cos^2 2\theta & -\cos 2\theta \sin 2\theta \\ -\cos 2\theta \sin 2\theta & \sin^2 2\theta \end{bmatrix}$$

$$J_4 J_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \cos^2 2\theta & -\cos 2\theta \sin 2\theta \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$J_2 J_1 = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}\begin{bmatrix} \cos^2\theta & 0 \\ \cos\theta\sin\theta & 0 \end{bmatrix}.$$

$$\begin{bmatrix} \cos^2\theta & 0 \\ \cos\theta\sin\theta & 0 \end{bmatrix}$$

$$J_4 J_3 J_2 J_1 = \begin{bmatrix} \cos^2 2\theta & -\cos 2\theta \sin 2\theta \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta . \cos^2 2\theta - \cos\theta\sin\theta\cos 2\theta \sin 2\theta . & 0 \\ 0 & 0 \end{bmatrix}$$

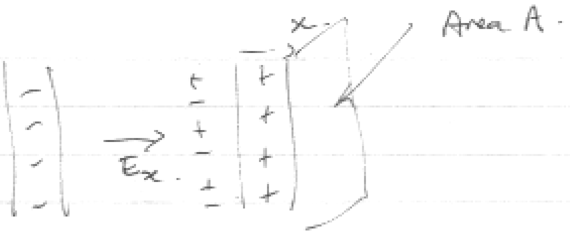
$$= \begin{bmatrix} \cos\theta\cos 2\theta(\cos\theta\cos 2\theta - \sin\theta\sin 2\theta) & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta \cos 2\theta \cos 3\theta . & 0 \\ 0 & 0 \end{bmatrix}$$

$$I_{in} = |a|^2 + |b|^2 = 2|a|^2.$$

$$I_{out} = \cos^2\theta \cos^2 2\theta \cos^2 3\theta |a|^2 = \frac{1}{2} I_{in} \cos^2\theta \cos^2 2\theta \cos^2 3\theta$$

1(b).



Gauss :- $E_x \epsilon_0 A = -n\frac{q}{2}Ax + n\frac{(-q)}{2}Ax$

$$\therefore E_x = -\frac{nqx}{\epsilon_0}$$

Equation of motion for each mass

$$m\ddot{x} = F_x = qE_x = -\frac{nq^2 x}{\epsilon_0}$$

S.H.M at plasma angular frequency

$$\omega_p = \sqrt{\frac{nq^2}{m\epsilon_0}}$$

On switching on B field induced currents screen interior* at all frequencies (angular) $\omega < \omega_p$ ie over a range $l > \lambda_L = c/\omega_p$. *Lenz's law.

1(c). Evidence that phase $\Delta\phi = \omega\Delta t - \underline{k}.\Delta\underline{r}$ is a Lorentz invariant is empirical. viz. Michelson-Morley experiment.

However, $(c\Delta t ; \Delta\underline{r})$ is prototype 4-vector so, $(\frac{\omega}{c}, \underline{k})$ is also a 4-vector.

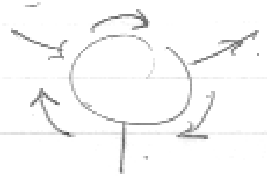
2 (a) Faraday effect :-

Rotation of plane of polarization of light as it propagates in the direction of an applied magnetic field.

$$\Delta\phi = \underbrace{V}_{\text{Verdet constant}}\ \underbrace{B}_{\text{magnetic field}}\ \underbrace{L}_{\text{path length}}$$

The rotation is non-reciprocal arising as it does from circular birefringence reversing with field direction.

This makes possible non-reciprocal two-ports (isolators) and three-ports (circulators)



(b) Homogeneous (i) and inhomogeneous (ii) broadening are distinguished.

eg (i) Intrinsic line broadening :-

$$\Delta\nu = \frac{h}{\tau_s} \qquad \tau_s = \text{spontaneous radiative lifetime}$$

~~de~~

Typically the lineshape is Lorentzian. For dipole allowed visible radiation $\tau_s \sim 10^{-8}$ s.

(ii) Doppler broadening due to thermal motion of atoms. eg in gas lasers

2(b) (Cont.) Collisional broadening in high pressure gas discharges. Source can approximate thermal spectrum - when electron and ion ~~tem~~ distributions reach thermal equilibrium -

2(c) Rayleigh scattering :-

eg. Sky light scattering. — Hertzian dipole (elastic) scattering with amplitude $\propto \frac{1}{\lambda^4}$
Hence, blue scattering predominant.

Thomson scattering - free electron accelerates under the influence of the em wave $\underline{E}$-field. Scattering amplitude independent of ~~θ~~ λ
Elastic scattering (like Rayleigh) unless in the quantum regime. cf. Sun's coronal scattering - during eclipse.
↳ Compton scattering.

3 Radiation resistance : resistance in the equivalent electrical circuit of an antenna that represents the radiation load on the circuit.

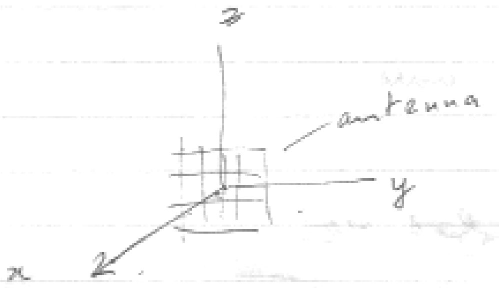
Power gain : Poynting vector of the radiation field of an antenna in the θ, ϕ direction divided by the Poynting vector averaged over 4π solid angle. It is highly anisotropic for a highly directional antenna.

Effective area : ratio of power intercepted by an antenna to the Poynting vector of the incident field.

Thermodynamic argument for $A_{eff} = \frac{\lambda^2}{4\pi} G(\theta, \phi)$.

Equate power absorbed in a matched load from black body radiation (in Rayleigh-Jeans approximation) (This depends on $A_{eff}(\theta, \phi)$). with the power radiated due to Johnson noise voltage fluctuations in load resistor. (This depends on power gain).

3(cont)



$$(\theta,\phi) = (\tfrac{\pi}{2}, 0)$$

$$G(\theta,\phi) = \frac{\cancel{C}\, N(\theta,\phi)}{\frac{1}{4\pi}\iint C \exp\left[-\frac{(\theta-\pi/2)^2}{2\alpha^2}\right]\exp\left[\frac{-\phi^2}{2\alpha^2}\right]\sin\theta\, d\theta\, d\phi}.$$

$$\alpha \ll \pi/2.$$

Substitute $\frac{(\theta-\pi/2)}{\sqrt{2}\alpha} = x$. $\frac{\phi}{\sqrt{2}\alpha} = y$.

$$G(\tfrac{\pi}{2},0) = \frac{4\pi\cancel{C}}{\iint \cancel{C}\exp(-x^2)\exp(-y^2)\sqrt{2}\alpha\, dx \sqrt{2}\alpha\, dy}$$

$$\doteq \frac{4\pi}{2\alpha^2\sqrt{\pi}\sqrt{\pi}} = \frac{2}{\alpha^2}.$$

$$\underline{N}(D,\pi/2,0) = \frac{2}{\alpha^2}\frac{P}{4\pi D^2}\hat{\underline{n}} = \underline{E}\times\underline{H} = \frac{\underline{E}\times\underline{B}}{\mu_0} = \frac{E^2}{c\mu_0}\hat{\underline{n}}. \quad (1)$$

Target dipole: $p = 4\pi\epsilon_0 a^3 E.$

$$p^2 = (4\pi\epsilon_0 a^3)^2 E^2. \quad (2)$$

$$N(\text{at detector}) = \frac{3}{2}\cdot\frac{8\pi^3\nu^4}{3\epsilon_0 c^3}p^2$$

reflected power intercepted $= \frac{3}{2}\cdot\frac{8\pi^3\nu^4}{3\epsilon_0 c^3}p^2\frac{A_{\text{eff}}}{4\pi D^2}.$ (3)

3 (cont)
$$\frac{V_m^2}{2R} = \frac{3}{2} \cdot \frac{8\pi^3 \nu^4 p^2}{3\epsilon_0 c^3} \frac{A_{eff}}{4\pi D^2} . \qquad (4)$$

substitute $A_{eff} = \frac{\lambda^2}{4\pi} G(\pi/2, 0) = \frac{\lambda^2}{4\pi} \frac{2}{\alpha^2}$.

and p^2 from (2).

and E^2 from (1)

to get implicit relation for D in terms of, ν, P, R, V_m, a

$$(4) \quad \frac{V_m^2}{2R} = \frac{4\pi^{\cancel{3}} \nu^4 p^2}{\epsilon_0 c^3} \frac{1}{4\cancel{\pi} D^2} \cdot \frac{\lambda^2}{4\cancel{\pi}} \frac{2}{\alpha^2}.$$

$$= \frac{1}{2} \frac{\pi \nu^4 \lambda^2}{\epsilon_0 c^3 D^2 \alpha^2} (4\pi\epsilon_0 a^3)^2 c\mu_0 \cdot \frac{2}{\alpha^2} \cdot \frac{P}{4\pi D^2}.$$

$$= \cancel{\frac{1}{2}} \frac{\pi \nu^2 \cancel{c^2}}{\epsilon_0 \cancel{c^3} D^2 \alpha^2} (4\pi\epsilon_0 a^3)^2 \frac{\cancel{c}\mu_0 \cancel{2} P}{\alpha^2 4\pi D^2}.$$

$$= \frac{\cancel{\pi} \nu^2 (\overset{2}{\cancel{4}}\pi a^3)^2 \epsilon_0 \mu_0 P}{\cancel{4}\cancel{\pi} \alpha^4 D^4}.$$

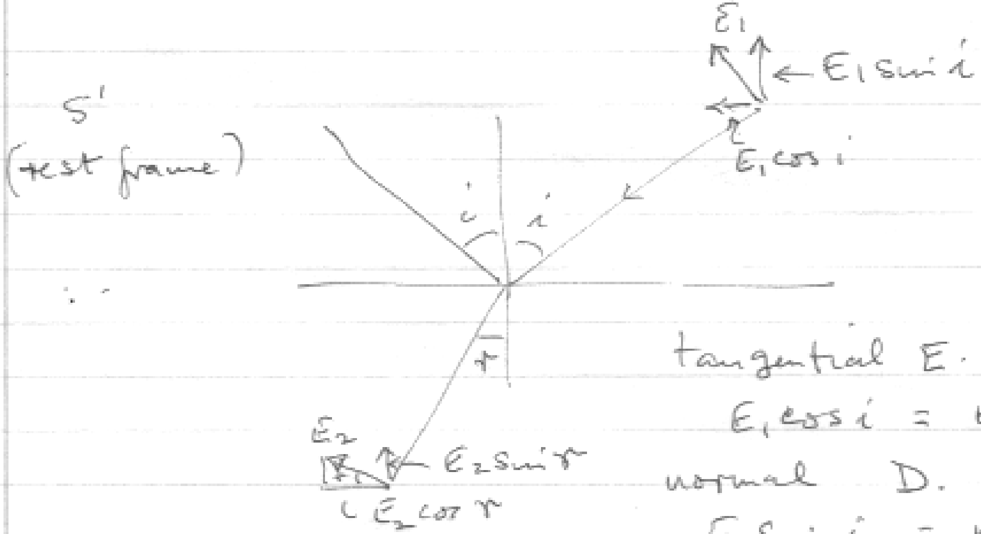
∴

$$\frac{V_m^2}{2R} = 4\pi^2 \frac{\nu^2 a^6}{c^2} \frac{P}{\alpha^4 D^4}.$$

Plenty of scope for errors here but at least it is dimensionally O.K.

4. First statement is true only for an elliptically polarized wave travelling in the x-direction. In this case E_x and E_y transform similarly under the particular Lorentz boost given and, phase being Lorentz invariant, the ellipticity is preserved.

The second part of the question provides a counter-example to the statement in the general case.



∴

tangential E ∴

$$\left.\begin{aligned} &E_1 \cos i = E_2 \cos r \\ &\text{normal D.} \\ &E_1 \sin i = n^2 E_2 \sin r \end{aligned}\right\} *$$

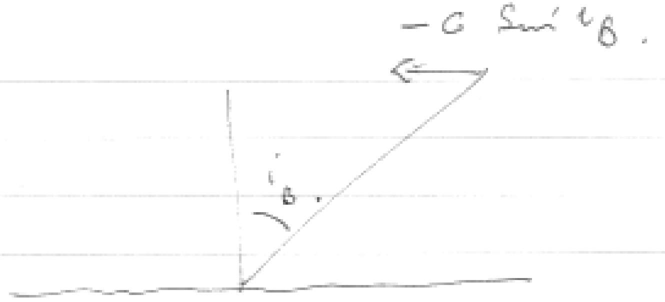
At Brewster angle ∴ $i = \frac{\pi}{2} - r$.

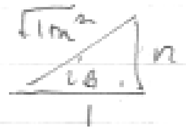
$$\therefore \sin i = \sin\left(\frac{\pi}{2} - r\right) = \cos r$$

Snell: $n \sin r = \sin i$

$$* \left\{\begin{aligned} &\therefore E_1 \cos i = E_2 \sin i \\ &E_1 \sin i = n E_2 \sin r \end{aligned}\right. \qquad \therefore \tan i_B = n$$

4(cont).





$\tan i_B = n$.

Lorentz velocity transform:-

$$u_x = \frac{u_x' + v}{1 + \frac{u_x' v}{c^2}} = 0.$$

$$u_x' = -c \sin i_B. = \frac{-c \cdot n}{\sqrt{1+n^2}}$$

$$\therefore \quad -c \sin i_B + v = 0.$$

$$v = \frac{c \cdot n}{\sqrt{1+n^2}}.$$

This speed of slab yields s-polarized reflection in S'.

∴