Department of Physics
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Part II Experimental and
Theoretical Physics

Electrodynamics and Optics

Answers to Problems

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[Modifications by NRC.]

1

2 In general,

$$\tilde{\boldsymbol{E}}_\pm = E_0(1, \pm i, 0)e^{ikz-i\omega t}$$

Thus the time dependence of the electric field at a fixed position (choose $z = 0$ for convenience) for the two cases is

$$\begin{aligned} \boldsymbol{E}_\pm &= \Re\left[E_0(1, \pm i, 0)e^{-i\omega t}\right] \\ \boldsymbol{E}_+ &= E_0(\cos\omega t, \sin\omega t, 0) \\ \boldsymbol{E}_- &= E_0(\cos\omega t, -\sin\omega t, 0) \end{aligned}$$

The electric field vector rotates in the $x - y$ plane with fixed magnitude.

If k is positive, such that the wave is moving in the $+\hat{\boldsymbol{z}}$-direction, then for an observer looking back into the source of the light the electric field corresponding to $\tilde{\boldsymbol{E}}_+$ rotates in an anticlockwise sense, and that corresponding to $\tilde{\boldsymbol{E}}_-$ rotates in a clockwise sense. For k positive, $\tilde{\boldsymbol{E}}_+$ describes LCP light; $\tilde{\boldsymbol{E}}_-$ describes RCP light.

If k is negative, such that the wave is moving in the $+\hat{\boldsymbol{z}}$-direction, then for an observer looking back into the source of the light the electric field corresponding to $\tilde{\boldsymbol{E}}_+$ rotates in a clockwise sense, and that corresponding to $\tilde{\boldsymbol{E}}_-$ rotates in an anticlockwise sense. For k negative, $\tilde{\boldsymbol{E}}_+$ describes RCP light; $\tilde{\boldsymbol{E}}_-$ describes LCP light.

The sign of k is important in determining the *direction* of the magnetic field, since $\boldsymbol{E} \times \boldsymbol{B}$ must be in the same direction as $\boldsymbol{k}$.

k positive

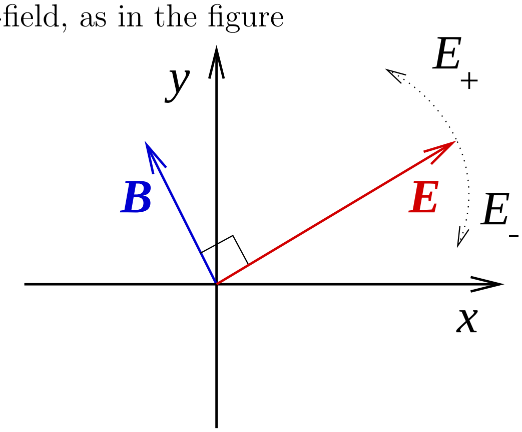
The complex amplitudes for the magnetic fields of the linearly-polarised modes are

$$\begin{aligned} \tilde{\boldsymbol{B}}_1 &= \frac{E_0}{c}(0, 1, 0)e^{ikz-i\omega t} \\ \tilde{\boldsymbol{B}}_2 &= \frac{E_0}{c}(-1, 0, 0)e^{ikz-i\omega t} \end{aligned}$$

So, at $z = 0$, we have

$$\begin{aligned} \boldsymbol{B}_\pm &= \Re\left[\frac{E_0}{c}(\mp i, 1, 0)e^{-i\omega t}\right] \\ \boldsymbol{B}_+ &= \frac{E_0}{c}(-\sin\omega t, \cos\omega t, 0) \\ \boldsymbol{B}_- &= \frac{E_0}{c}(\sin\omega t, \cos\omega t, 0) \end{aligned}$$

The magnetic field rotates in the same sense as the $\boldsymbol{E}$-field. staying at 90° to the $\boldsymbol{E}$-field, as in the figure

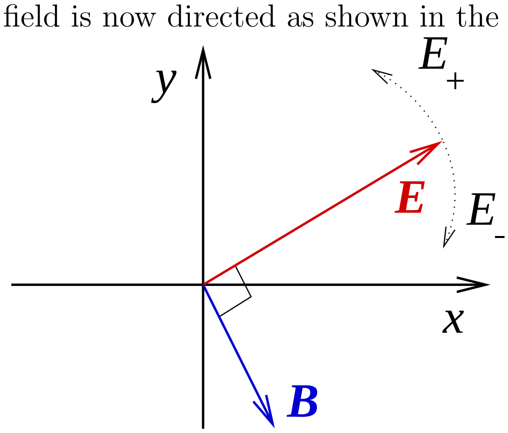


k negative

For the same electric field vector, the magnetic field has the opposite sign for modes propagating in the opposite direction. At $z = 0$, we have

$$\begin{aligned}
\boldsymbol{B}_\pm &= \Re\left[\frac{E_0}{c}(\pm i, -1, 0)e^{-i\omega t}\right] \\
\boldsymbol{B}_+ &= \frac{E_0}{c}(\sin\omega t, -\cos\omega t, 0) \\
\boldsymbol{B}_- &= \frac{E_0}{c}(-\sin\omega t, -\cos\omega t, 0)
\end{aligned}$$

The magnetic field is now directed as shown in the figure



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4 In a birefringent material the two linear polarisations have different wave speeds: c/n_a and c/n_b where $n_{a,b}$ are the refractive indices of the two polarisations. For a plate of thickness l waves of the two polarisations that enter at the same time will emerge with a temporal separation

$\Delta t = l(n_a/c) - l(n_b/c) = l(n_a - n_b)/c$. The slab will behave as a quarter-wave plate provided

$$\omega \Delta t = (\pi/2) + 2\pi n$$

where n is any integer, such that a phase difference of $\pi/2$ is introduced between the two polarisations.

The maximal difference in speed is achieved by orienting the crystal such that the optic axis is parallel to the surface of the slab. Then, one polarisation has the electric field along the optic axis (with refractive index n_e), and the other polarisation has electric field perpendicular to the optic axis (refractive index n_o). The minimal thickness is therefore

$$l_{\min} = \frac{\pi\ c}{2\omega |n_\mathrm{e} - n_\mathrm{o}|} = \frac{\lambda}{4|n_\mathrm{e} - n_\mathrm{o}|}$$

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8 (a) Linear polarisers aligned with the x and y axes are described by

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Crossed polarisers have Jones matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

so light of any polarisation is extinguished.

(b) The Jones matrix for a quarter wave plate at an angle θ to the x axis (towards the positive y axis) can be obtained by considering the rotation matrix:

$$\underline{\underline{R}} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

which has the property that it relates the components of a Jones vector $\begin{pmatrix} a \\ b \end{pmatrix}_{x,y}$ with respect to the linear polarisation basis along the x and y axes, to the components of the Jones vector $\begin{pmatrix} a' \\ b' \end{pmatrix}_{x',y'}$ with respect to axes x' and y' which are at angle θ to the original x, y axes, *i.e.*

$$\begin{pmatrix} a' \\ b' \end{pmatrix}_{x',y'} = \underline{\underline{R}} \begin{pmatrix} a \\ b \end{pmatrix}_{x,y}$$

In the x', y' system the Jones matrix for the $\lambda/4$-plate with fast axis along x' is simply

$$\underline{\underline{M'}} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}_{x',y'}$$

and so the outgoing Jones vector, in the x', y' basis is

$$\begin{pmatrix} c' \\ d' \end{pmatrix}_{x',y'} = \underline{\underline{M'}} \begin{pmatrix} a' \\ b' \end{pmatrix}_{x',y'} = \underline{\underline{M'}}\,\underline{\underline{R}} \begin{pmatrix} a \\ b \end{pmatrix}_{x,y}$$

Converting to the original basis, via

$$\begin{pmatrix} c \\ d \end{pmatrix}_{x,y} = \underline{\underline{R}}^{-1} \begin{pmatrix} c' \\ d' \end{pmatrix}_{x',y'}$$

we find that

$$\begin{pmatrix} c \\ d \end{pmatrix}_{x,y} = \underline{\underline{R}}^{-1}\underline{\underline{M}}\;\underline{\underline{R}} \begin{pmatrix} a \\ b \end{pmatrix}_{x,y}$$

and so, in the x, y basis, the overall Jones Matrix for the inclined $\lambda/4$-plate is

$$\begin{aligned} \underline{\underline{R}}^{-1}\underline{\underline{M'}}\,\underline{\underline{R}} &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\theta + i\sin^2\theta & (1-i)\sin\theta\cos\theta \\ (1-i)\sin\theta\cos\theta & i\cos^2\theta + \sin^2\theta \end{pmatrix} \end{aligned}$$

Between crossed polarised, the total Jones matrix is (light passes the x-polariser first)

$$\begin{aligned} M_{\text{tot}} &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos^2\theta + i\sin^2\theta & (1-i)\sin\theta\cos\theta \\ (1-i)\sin\theta\cos\theta & i\cos^2\theta + \sin^2\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ (1-i)\sin\theta\cos\theta & 0 \end{pmatrix} \end{aligned}$$

The incident unpolarised light can be viewed as an incoherent mixture of x- and y- linearly-polarised, each of intensity $I/2$. From the Jones matrix, we see: the y-polarised light is not transmitted at all; the x-polarised light emerges as y-polarised light, and with its electric field strength reduced from the incident value by a factor of $|(1-i)\sin\theta\cos\theta| = \sqrt{2}|\sin\theta\cos\theta|$. Therefore the final intensity of the emerging light is

$$I_{\text{out}} = \frac{I}{2}(\sqrt{2}\sin\theta\cos\theta)^2 = \frac{I}{4}\sin^2(2\theta)$$

The maximium value, for $\theta = \pi/4 + n\pi/2$, is $I_{\text{out}}^{\text{max}} = I/4$.

9

10 A regular tetrahedron of four identical spheres is non-chiral: it can be superposed on its mirror image. Therefore a "molecule" of four identical spheres in this geometry could not induce optical rotation. [If the four spheres were all different in some way, then this could be optically-active.]

A molecule formed from 3 atoms cannot be chiral (even if the atoms are not identical): the molecule is invariant under reflection in the plane in which the 3 atoms lie. Therefore, a molecule formed from three atoms cannot be optically active. [Quite generally, for the same reason, no planar molecule can be optically-active.]

11 Let us choose the geometry in which waves propagate along the z-axis, with the mirrors at $z = 0, L$. The general optical field for standing waves with frequency ω, may be written

$$\begin{aligned} \tilde{\boldsymbol{E}} &= A_1(1,i,0)e^{ik_1z-i\omega t} + A_2(1,i,0)e^{-ik_2z-i\omega t} \\ &\quad +A_3(1,-i,0)e^{ik_3z-i\omega t} + A_4(1,-i,0)e^{-ik_4z-i\omega t} \end{aligned}$$

We have used basis vectors that correspond to RCP and LCP radiation, since these have simple propagation properties in media showing optical activity and the Faraday effect. Note that we choose a convention with $k_i > 0$, so waves 1 and 3 are in the $+z$ direction, and waves 2 and 4 are in the $-z$ direction.

At the mirror at $z = 0$ we have the boundary condition $E_x = E_y = 0$. This requires $A_1 + A_2 = 0$ and $A_3 + A_4 = 0$, and we can write

$$\begin{aligned} \tilde{\boldsymbol{E}} &= A_1(1,i,0)\left[e^{ik_1z} - e^{-ik_2z}\right]e^{-i\omega t} \\ &\quad +A_3(1,-i,0)\left[e^{ik_3z} - e^{-ik_4z}\right]e^{-i\omega t} \end{aligned}$$

Applying the boundary condition at the other mirror, $E_x = E_y = 0$ at $z = L$, we find that for the mode with amplitude A_1 we require

$$\begin{aligned} e^{ik_1L} - e^{-ik_2L} &= 0 \\ e^{-ik_2L}\left[e^{i(k_1+k_2)L} - 1\right] &= 0 \\ \Rightarrow (k_1 + k_2) &= \frac{2\pi}{L}p \end{aligned} \tag{1}$$

where p is an integer. Similarly, for the mode with amplitude A_3, we require

$$(k_3 + k_4) = \frac{2\pi}{L}q \tag{2}$$

where q is an integer.

Conditions (1,2) determine the frequencies of the two modes, once we relate the wavevectors k_p to the freqnencies ω. We need to know the appropriate refractive index, n_p, for each wave, from which $k_i = (\omega/c)n_i$.

The medium has average refractive index, n, *i.e.*

$$n = \frac{1}{2}(n_L + n_R)$$

where n_L and n_R are the refractive indices for LCP and RCP light. We may write the specfic rotatory power as (see lecture 3)

$$\alpha = \frac{\omega}{2c}(n_L - n_R)$$

Thus,

$$\begin{aligned} n_L &= n + \frac{c\alpha}{\omega} \\ n_R &= n - \frac{c\alpha}{\omega} \end{aligned}$$

We consider in turn the cases of (b) an optically active medium and (c) a Faraday medium.

(a) ...

(b) Optically-active medium

Wave 1: This is an LCP wave, travelling in the $+z$ direction. The refractive index is

$$n_1 = n + \frac{c\alpha}{\omega}$$

Thus,

$$k_1 = \frac{\omega}{c}n_1 = \frac{\omega}{c}\left(n + \frac{c\alpha}{\omega}\right)$$

Wave 2: This is *an RCP wave*, since it is travelling in the $-z$ direction. (See problem 2 for the importance of the sign of k in assigning RCP or LCP.)

The refractive index is therefore

$$n_2 = n - \frac{c\alpha}{\omega}$$

and

$$k_2 = \frac{\omega}{c}n_2 = \frac{\omega}{c}\left(n - \frac{c\alpha}{\omega}\right)$$

Therefore, the condition for the mode with amplitude A_1, Eqn (1), is

$$\begin{aligned}(k_1 + k_2) &= \frac{2\pi}{L}p \\ \frac{2\omega}{c}n &= \frac{2\pi}{L}p \\ \Rightarrow \omega &= \frac{\pi c}{nL}p\end{aligned}$$

Thus the resonant frequencies for this mode are *independent of the specific rotatory power of the optically active medium.*

By similar reasoning, it follows that waves 3 and 4 are RCP and LCP respectively. Thus

$$\begin{aligned}k_3 &= \frac{\omega}{c}n_3 = \frac{\omega}{c}\left(n - \frac{c\alpha}{\omega}\right) \\ k_4 &= \frac{\omega}{c}n_4 = \frac{\omega}{c}\left(n + \frac{c\alpha}{\omega}\right)\end{aligned}$$

and the condition for the mode with amplitude A_3, Eqn (2), is

$$\omega = \frac{\pi c}{nL}q$$

i.e. also at the same set of frequencies.

(c) Faraday Medium

For definiteness, we assume the magnetic field to point in the $+z$ direction, $\boldsymbol{B} = B\hat{\boldsymbol{z}}$. (The final results would be the same for the other choice of sign, $\boldsymbol{B} = -B\hat{\boldsymbol{z}}$.)

Wave 1: This is an LCP wave, travelling in the $+z$ direction, so its direction of propagation is parallel to the magnetic field. The specific rotatory power for this wave is therefore (from the definition of the Verdet constant)

$$\alpha_1 = -\mathcal{V}B$$

The refractive index is

$$n_1 = n + \frac{c\alpha_1}{\omega} = n - \frac{c\mathcal{V}B}{\omega}$$

Thus,

$$k_1 = \frac{\omega}{c}n_1 = \frac{\omega}{c}\left(n - \frac{c\mathcal{V}B}{\omega}\right)$$

Wave 2: As before, we note that this is an RCP wave since it is travelling in the $-z$ direction. However, we must also now note that its direction of propagation is opposite to the magnetic field. Therefore the specific rotatory power for this wave is

$$\alpha_2 = -\mathcal{V}(-B) = +\mathcal{V}B$$

The refractive index is

$$n_2 = n - \frac{c\alpha_2}{\omega} = n - \frac{c\mathcal{V}B}{\omega}$$

Thus,

$$k_2 = \frac{\omega}{c}n_2 = \frac{\omega}{c}\left(n - \frac{c\mathcal{V}B}{\omega}\right)$$

The condition for the mode with amplitude A_1, Eqn (1), is

$$\begin{aligned} (k_1 + k_2) &= \frac{2\pi}{L}p \\ 2\frac{\omega}{c}\left(n - \frac{c\mathcal{V}B}{\omega}\right) &= \frac{2\pi}{L}p \\ \Rightarrow \omega &= \frac{\pi c}{nL}p + \frac{\mathcal{V}Bc}{n} \end{aligned}$$

Thus the resonant frequencies for this mode are *shifted by a constant amount that depends on the Verdet constant.*

Applying the same reasoning, it follows that waves 3 and 4 are RCP and LCP respectively with $\alpha_3 = -\mathcal{V}B$ and $\alpha_4 = +\mathcal{V}B$. This leads to the conclusion that

$$k_3 = \frac{\omega}{c}n_3 = \frac{\omega}{c}\left(n + \frac{c\mathcal{V}B}{\omega}\right)$$

$$k_4 = \frac{\omega}{c}n_4 = \frac{\omega}{c}\left(n + \frac{c\mathcal{V}B}{\omega}\right)$$

and the condition for the mode with amplitude A_3, Eqn (2), is

$$\omega = \frac{\pi c}{nL}q - \frac{\mathcal{V}Bc}{n}$$

i.e. the resonant frequencies for this mode are shifted by a constant amount in the *opposite* direction to the other mode.

The answer to this question illustrates that, despite many similarities, there *is* a difference between an optically-active medium and a material showing the Faraday effect. In a geometry where light is passed back and forth through the material upon reflection:

(i) For the optically active material the phase shifts on forward and backward passes *cancel*. As a result there is no net rotation of the plane of polarisation of a linearly-polarised beam.

(ii) For the Faraday material the phase shifts on forward and backward passes *add*. As a result there is a continued rotation of the plane of polarisation of a linearly-polarised beam.

12 From lectures, the wavevectors for LCP and RCP are obtained from the wave-equation

$$k_{L,R}^2 = \omega^2 \mu_0 \epsilon_{L,R}(\omega)$$

where

$$\begin{aligned} \epsilon_R(\omega) &= \epsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2(1+\omega_c/\omega)}\right) \\ \epsilon_L(\omega) &= \epsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2(1-\omega_c/\omega)}\right) \end{aligned}$$

At low frequencies, $\omega \ll \omega_c$, we have

$$\begin{aligned} \epsilon_R(\omega) &\simeq \epsilon_0 \left(1 - \frac{\omega_p^2}{\omega\omega_c}\right) \\ \epsilon_L(\omega) &\simeq \epsilon_0 \left(1 + \frac{\omega_p^2}{\omega\omega_c}\right) \end{aligned}$$

and, furthermore, for $\omega \ll \omega_p^2/\omega_c$,

$$\begin{aligned} \epsilon_R(\omega) &\simeq -\frac{\epsilon_0\omega_p^2}{\omega\omega_c} \\ \epsilon_L(\omega) &\simeq \frac{\epsilon_0\omega_p^2}{\omega\omega_c} \end{aligned}$$

Thus, the wavevector for RCP waves $k_R = \omega\sqrt{\mu_0\epsilon_L(\omega)}$ is imaginary: these waves are evanescent.

For LCP waves, the wavevector is real. The dispersion relation is

$$\begin{aligned} k_L^2 &= \omega^2\mu_0\epsilon_R(\omega) = \omega^2\mu_0\epsilon_0\frac{\omega_p^2}{\omega\omega_c} \\ \omega &= \frac{k_L^2}{\mu_0\epsilon_0}\frac{\omega_c}{\omega_p^2} = c^2k_L^2\frac{\omega_c}{\omega_p^2} \end{aligned}$$

The group velocity is

$$\frac{d\omega}{dk_L} = 2c^2 k_L \frac{\omega_c}{\omega_p^2}$$

which increases as $k_L \propto \sqrt{\omega}$ increases. Therefore higher frequency waves travel with higher group velocity.

[The generation of these "helicon" modes in the ionosphere accounts for the "whistlers" which can be heard as radio interference in northern latitudes. Helicon pulses generated from electromagnetic activity travel along the magnetic field lines of the earth towards the pole (these waves can travel the whole way around the earth from one pole to the other). Owing to the dispersion the higher frequencies arrive earlier, leading to the characteristic falling tone of the whistler.]

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Electrodynamics

17 Using spherical polar co-ordinates, the magnetic field components of a magnetic dipole m are

$$\begin{aligned} B_r &= \frac{\mu_0 m \cos\theta}{2\pi r^3} \\ B_\theta &= \frac{\mu_0 m \sin\theta}{4\pi r^3} \\ B_\varphi &= 0 \end{aligned}$$

We find $\boldsymbol{A}$ using

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \boldsymbol{e}_r & r\boldsymbol{e}_\theta & r\sin\theta \boldsymbol{e}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ A_r & rA_\theta & r\sin\theta A_\varphi \end{vmatrix}$$

We can simplify the analysis by noting that, from the symmetry of the current source (azimuthal current of the magnetic dipole), only A_φ will be non-zero. We thus obtain

$$\begin{aligned} B_r &= \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(r\sin\theta A_\varphi) \\ B_\theta &= -\frac{1}{r\sin\theta}\frac{\partial}{\partial r}(r\sin\theta A_\varphi) \\ B_\varphi &= 0 \end{aligned}$$

Equating these with the known magnetic field components leads to

$$\begin{aligned} \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(r\sin\theta A_\varphi) &= \frac{\mu_0 m\cos\theta}{2\pi r^3} \\ \Rightarrow r\sin\theta A_\varphi &= \frac{\mu_0 m\sin^2\theta}{4\pi r} + f(r,\varphi) \end{aligned}$$

and

$$\begin{aligned} -\frac{1}{r\sin\theta}\frac{\partial}{\partial r}(r\sin\theta A_\varphi) &= \frac{\mu_0 m\sin\theta}{4\pi r^3} \\ \Rightarrow r\sin\theta A_\varphi &= \frac{\mu_0 m\sin^2\theta}{4\pi r} + g(\theta,\varphi) \end{aligned}$$

For consistency of the two results, we require $f = g$, and we can choose the gauge such that $f = g = 0$ and

$$A_\varphi = \frac{\mu_0 m\sin\theta}{4\pi r^2}$$

[An alternative method is to note that by symmetry the only non-zero component of $\boldsymbol{A}$ is $A_\varphi(r,\theta)$. Then one can use Stokes' theorem to relate the line integral of $\boldsymbol{A}$ around a circuit of constant r and θ, $\int \boldsymbol{A}\cdot d\boldsymbol{l} = A_\varphi 2\pi r\sin\theta$, to the magnetic flux through the surface bounded by that loop, $\int \boldsymbol{B}\cdot d\boldsymbol{S} = \int B_r 2\pi r^2\sin\theta d\theta = \frac{\mu_0 m}{2r}\sin^2\theta$. Hence $A_\varphi 2\pi r\sin\theta = \frac{\mu_0 m}{2r}\sin^2\theta$, yielding A_φ as before.]

18 (a) A real magnetic field must satisfy $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$. Let us calculate this for $\boldsymbol{B} = \frac{B_0 b}{r^3}\left((x-y)\,z,\,(x-y)\,z, x^2-y^2\right)$. Noting that

$$\frac{\partial}{\partial x_i}\frac{1}{r^3} = \frac{-3x_i}{r^5}$$

we find

$$\begin{aligned}
\nabla\cdot\boldsymbol{B} &= \frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}\\
&= B_0 b\left[\left(\frac{z}{r^3}-\frac{3x(x-y)z}{r^5}\right)+\left(\frac{-z}{r^3}-\frac{3y(x-y)z}{r^5}\right)+\left(-\frac{3(x^2-y^2)z}{r^5}\right)\right]\\
&= \frac{B_0 b}{r^5}(-3)\left[z(x^2-y^2)+z(x-y)(x+y)\right]\\
&= \frac{-6B_0 b}{r^5}z(x^2-y^2)\neq 0
\end{aligned}$$

Thus, $\boldsymbol{B}$ is not a real magnetic field. [NB it would be if $\boldsymbol{B}=\frac{B_0 b}{r^3}\left((x-y)\,z,\,(x-y)\,z,\text{-}(x^2-y^2)\right)$.]

(b) Now $\boldsymbol{B}=B_0 b^2\left(\frac{zr}{(b^2+z^2)^2},0,\frac{1}{b^2+z^2}\right)$ in cylindrical polar coordinates.

$$\begin{aligned}
\nabla\cdot\boldsymbol{B} &= \frac{1}{r}\frac{\partial}{\partial r}(rB_r)+\frac{1}{r}\frac{\partial B_\varphi}{\partial\varphi}+\frac{\partial B_z}{\partial z}\\
&= B_0 b^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{zr^2}{(b^2+z^2)^2}\right)\frac{\partial}{\partial z}\left(\frac{1}{b^2+z^2}\right)\right]\\
&= B_0 b^2\left[\frac{2z}{(b^2+z^2)^2}-\left(\frac{2z}{b^2+z^2}\right)\right]\\
&= 0
\end{aligned}$$

It follows that this represents a real magnetic field.

We calculate the current density $\boldsymbol{j}$ from

$$\begin{aligned}
\mu_0\boldsymbol{j} &= \nabla\times\boldsymbol{B}\\
&= \frac{1}{r}\begin{vmatrix}\boldsymbol{e}_r & r\boldsymbol{e}_\varphi & \boldsymbol{e}_z\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z}\\ B_r & rB_\varphi & B_z\end{vmatrix}\\
&= \frac{B_0 b^2}{r}\begin{vmatrix}\boldsymbol{e}_r & r\boldsymbol{e}_\varphi & \boldsymbol{e}_z\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z}\\ \frac{zr}{(b^2+z^2)^2} & 0 & \frac{1}{b^2+z^2}\end{vmatrix}\\
&= \frac{B_0 b^2}{r}\left[\boldsymbol{e}_r(0)+r\boldsymbol{e}_\varphi\frac{\partial}{\partial z}\left(\frac{zr}{(b^2+z^2)^2}\right)+\boldsymbol{e}_z(0)\right]\\
&= B_0 b^2\left[\frac{r(b^2+z^2)^2-2zr(2z)(b^2+z^2)}{(b^2+z^2)^4}\right]\boldsymbol{e}_\varphi\\
&= B_0 b^2\left[\frac{r(b^2+z^2)-4rz^2}{(b^2+z^2)^3}\right]\boldsymbol{e}_\varphi\\
\Rightarrow j_\varphi &= \frac{B_0 b^2 r}{\mu_0}\frac{b^2-3z^2}{(b^2+z^2)^3}
\end{aligned}$$

To find the vector potential $\boldsymbol{A}$ we can invert the equation for $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$. We simplify this process by noting that, because $\boldsymbol{j} = j_\varphi \boldsymbol{e}_\varphi$, the only non-zero component of $\boldsymbol{A}$ is A_φ. Thus.

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} = \frac{1}{r} \begin{vmatrix} \boldsymbol{e}_r & r\boldsymbol{e}_\varphi & \boldsymbol{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ 0 & rA_\varphi & 0 \end{vmatrix}$$

from which we find

$$\begin{aligned} B_r &= -\frac{1}{r}\frac{\partial(rA_\varphi)}{\partial z} = B_0 b^2 \frac{zr}{(b^2+z^2)^2} \\ \Rightarrow A_\varphi &= \frac{B_0 b^2 r}{2(b^2+z^2)} + \frac{f(r,\varphi)}{r} \\ B_z &= \frac{1}{r}\frac{\partial(rA_\varphi)}{\partial r} = B_0 b^2 \frac{1}{b^2+z^2} \\ \Rightarrow A_\varphi &= \frac{B_0 b^2 r}{2(b^2+z^2)} + \frac{g(\varphi,z)}{r} \end{aligned}$$

We require $f = g$, and can choose the gauge such that $f = g = 0$ and

$$A_\varphi = \frac{B_0 b^2 r}{2(b^2+z^2)}$$

Radiation

19 Investigate the fields at a distance $r \gg 10$cm (say 2m) from the emitting antenna, in order to be certain to probe the far-field (radiation) region.

The power distribution of both electric and magnetic dipoles has the angular distribution $\sin^2\theta$, where θ is the polar angle relative to the axis of symmetry. Using a well-characterised receiving antenna (let us say a Hertzian dipole), identify the nodal line ($\theta = 0, \pi$) along which there is no radiated power. (Be certain to probe for both polarisations of radiation; using a Hertzian dipole as a receiver this can be done by rotating the axis of the receiving dipole about the line directed towards the emitting antenna.)

Having determined the axis of the emitting antenna, probe the radiation fields in the plane $\theta = \pi/2$. By rotating the axis of the receiving dipole, determine whether the emitted radiation is polarised along the axis of the emitting dipole (the emitting antenna is then an *electric dipole*), or perpendicular to the axis of the emitting dipole (the emitting antenna is then a *magnetic dipole*).

[If the dipole just has its terminals shorted, and is not transmitting, it's orientation and nature could still be probed by performing a (polarisation resolved) scattering measurement with incident radiation.]

20 (a) By the rotational symmetry of the situation, any two directions in the plane perpendicular to the axis of rotation are physically equivalent, up to an offset in time related to the time it takes the dipole to rotate through the angular difference between these two directions. From the point of view of the radiation emitted, this offset in time can only introduce a *phase difference* in the radiation along the two directions; the radiated power must be the same in the two directions. Hence the emitted power must be the same in all directions in this plane, so the radiation pattern is a circle.

(b) Consider still the case where we view the radiation in the plane perpendicular to the axis of rotation. Since we know the nature of the radiation fields produced an oscillating magnetic dipole, it is very helpful to view the rotating dipole as composed of two such oscillating magnetic dipoles at right angles to each other: one oscillating along the line of sight, and one oscillating perpendicular to the line of sight (and also perpendicular to the axis of rotation). Since Maxwell's equations are linear, we can deduce the radiation fields of the rotating dipole by superposing the radiation fields produced by each of these two oscillating dipoles. The magnetic dipole oscillating along the line of sight produces *no radiation fields* in the direction of observation. The dipole oscillating perpendicular to the line of sight produces linearly polarised radiation, with the magnetic field in the plane perpendicular the axis of rotation and the electric field parallel to the axis of rotation. Thus, the net radiation in the plane perpendicular to the axis of rotation is *linearly polarised, with polarisation vector parallel to the axis of rotation*.

Consider now the radiation emitted along the axis of rotation of the dipole. Viewing, still, the net radiation fields as the sums of the radiation fields of the two orthognal oscillating dipoles, we see that in this direction we get equal contributions from *both* these dipoles. To be definite, let's focus on the net electric field (identical reasoning holds for the magnetic field). The total electric field consists of the (vector) sum of the two orthogonal electric fields from the two oscillating dipoles: these contributions have equal magnitude but are $\pi/2$ out of phase with each other (since the two oscillating dipoles must be $\pi/2$ out of phase to describe the rotating dipole). Thus, the total electric field of the radiation emitted along the axis of rotation is rotating in the same sense as the rotating dipole (so too is the total magnetic field). The radiation along the axis of rotation is therefore *circularly polarised*.

In a general direction, somewhere between the axis of rotation and the plane perpendicular to this axis, the radiation is *elliptically polarised*.

21 We first determine the rate of energy loss from the rotating dipole, by viewing it as the sum of two dipoles

$$\boldsymbol{m}(t) = \boldsymbol{e}_x m_0 \cos\omega t + \boldsymbol{e}_y m_0 \sin\omega t \equiv \boldsymbol{e}_x m_1(t) + \boldsymbol{e}_y m_2(t)$$

The fields are then found by superposing the fields of the two dipoles.

$$\boldsymbol{B}(t) = \boldsymbol{B}_1(t) + \boldsymbol{B}_2(t) \qquad \boldsymbol{E}(t) = \boldsymbol{E}_1(t) + \boldsymbol{E}_2(t)$$

The Poynting flux is

$$\begin{aligned} \boldsymbol{N}(t) &= \frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B} \\ &= \frac{1}{\mu_0}(\boldsymbol{E}_1+\boldsymbol{E}_2)\times(\boldsymbol{B}_1+\boldsymbol{B}_2) \\ &= \frac{1}{\mu_0}\left[\boldsymbol{E}_1\times\boldsymbol{B}_1 + \boldsymbol{E}_2\times\boldsymbol{B}_2 + \boldsymbol{E}_1\times\boldsymbol{B}_2 + \boldsymbol{E}_2\times\boldsymbol{B}_1\right] \end{aligned}$$

and the average flux is

$$\langle\boldsymbol{N}\rangle = \frac{1}{\mu_0}\left[\langle\boldsymbol{E}_1\times\boldsymbol{B}_1\rangle + \langle\boldsymbol{E}_2\times\boldsymbol{B}_2\rangle + \langle\boldsymbol{E}_1\times\boldsymbol{B}_2\rangle + \langle\boldsymbol{E}_2\times\boldsymbol{B}_1\rangle\right]$$

But $\boldsymbol{E}_1$ and $\boldsymbol{B}_1$ vary as $\cos[\omega(t-r/c)]$, whereas $\boldsymbol{E}_2$ and $\boldsymbol{B}_2$ vary as $\sin[\omega(t-r/c)]$. Therefore the cross-terms average to zero and we are left with

$$\langle\boldsymbol{N}\rangle = \langle\boldsymbol{N}_1\rangle + \langle\boldsymbol{N}_2\rangle$$

i.e. since the two dipoles are $\pi/2$ out of phase, the time-averaged Poynting fluxes add. In particular, the average power loss is twice the average power loss of a single magnetic dipole

$$\langle P\rangle = 2\times\frac{\mu_0\omega^4 m_0^2}{12\pi c^3}$$

For the dynamical part, we make use of the conservation of energy. The total energy stored in the rotating body is $E = \frac{1}{2}I\omega^2$, where I is the moment of inertia. This energy is lost through radiated power. We make use of the assumption that $\dot{\omega} \ll \omega^2$ (which ensures that the change of energy in one period of rotation is small compared to the energy stored), to equate the rate of loss of energy to the *average* power loss

$$\frac{dE}{dt} = -\langle P \rangle = -\kappa\omega^4$$

where $\kappa = (\mu_0 \omega^4 m_0^2)/(6\pi c^3)$ was derived above. Thus

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}I\omega^2\right) &= -\kappa w^4 \\ I\omega\dot{\omega} &= -\kappa\omega^4 \\ I\dot{\omega} &= -\kappa\omega^3 \\ I\ddot{\omega} &= -3\kappa\omega^2\dot{\omega} \end{aligned}$$

Dividing the last line by the preceding line, we find

$$\begin{aligned} \frac{\ddot{\omega}}{\dot{\omega}} &= \frac{3\dot{\omega}}{\omega} \\ \Rightarrow \omega\ddot{\omega} &= 3\dot{\omega}^2 \end{aligned}$$

In the calculation for the pulsar, we note that the period p is

$$\begin{aligned} p &= \frac{2\pi}{\omega} \\ \dot{p} &= -\frac{2\pi\dot{\omega}}{\omega^2} \\ \frac{\dot{\omega}}{\omega^3} &= -\frac{\dot{p}p}{(2\pi)^2} \\ \Rightarrow -\frac{\kappa}{I} = -\frac{\mu_0 {m_0}^2}{6\pi c^3 I} &= -\frac{\dot{p}p}{(2\pi)^2} \\ \Rightarrow m_0^2 &= \frac{3I\dot{p}pc^3}{2\pi\mu_0} \end{aligned}$$

Taking the moment of inertia of a uniform sphere $I = \frac{2}{5}MR^2 = 3.9 \times 10^{37}\mathrm{kgm}^2$, we find $m_0 = 2.4 \times 10^{27}\mathrm{Am}^2$.

At a distance $R = 7\mathrm{km}$, which satisfies $R \ll \lambda = \frac{2\pi c}{\omega}$ and is therefore in the near-field region (where the fields are those of a static dipole), we get

$$B_\theta = \frac{\mu_0 m_0}{4\pi R^3}\sin\theta \simeq 7 \times 10^8 \sin\theta \mathrm{T}$$

22 The radiation resistance of an antenna is the ratio of the mean power *radiated* to the mean square current at the contact to the aerial

$$R_r \equiv \langle P\rangle/\langle I^2\rangle \, .$$

The power gain is defined in terms of the time-averaged radial Poynting flux, N, as

$$G(\theta,\varphi) = \frac{N(\theta,\varphi)}{\frac{1}{4\pi}\int N(\theta,\varphi)\,\mathrm{d}\Omega}$$

where the integral is over the surface of a sphere (total solid angle 4π).

The dimensions of the antenna are small compared to the wavelength of the radiation, so it will behave as a magnetic dipole. The radiation fields are

$$\begin{aligned} B_\theta &= \frac{\mu_0 \sin\theta[\ddot{m}]}{4\pi r c} \\ E_\varphi &= -cB_\theta \end{aligned}$$

The radial Poynting flux is

$$N_r = \frac{1}{\mu_0}(\boldsymbol{E}\times\boldsymbol{B})_r = \frac{\mu_0 \sin^2\theta[\ddot{m}]^2}{16\pi^2 r^2 c^3}$$

and the power loss is

$$\begin{aligned} P &= \frac{\mu_0[\ddot{m}]^2}{16\pi^2 r^2 c^3}\int_0^\pi 2\pi r^2 \sin\theta \sin^2\theta \mathrm{d}\theta \\ &= \frac{\mu_0[\ddot{m}]^2}{6\pi c^3} \end{aligned}$$

Using $m = I_0 a^2 \cos(\omega t)$, such that $\langle[\ddot{m}]^2\rangle = (1/2)\omega^4 I_0^2 a^4$, we can determine the average power loss and find

$$\begin{aligned} R_r &\equiv \langle P\rangle/\langle I^2\rangle = \frac{(1/2)\omega^4 I_0^2 a^4(\mu_0/6\pi c^3)}{(1/2)I_0^2} \\ &= \frac{\mu_0\omega^4 a^4}{6\pi c^3} \end{aligned}$$

The power gain is

$$G(\theta,\varphi) = \frac{\sin^2\theta}{\frac{1}{4\pi}\int \sin^2\theta \mathrm{d}\Omega} = \frac{3}{2}\sin^2\theta$$

Two methods may be used to determine the cross-section

(i) *Direct calculation*

By Faraday's law, the induced voltage in the loop (whose normal we align with the z-axis) is

$$V = -\dot{\Phi} = -a^2 \dot{B}_z \quad \Rightarrow \quad \langle V^2 \rangle = \omega^2 a^4 \langle B_z^2 \rangle$$

The incident power is

$$N_i = \frac{1}{\mu_0} \langle \boldsymbol{E} \times \boldsymbol{B} \rangle = \frac{c}{\mu_0} \langle B_z^2 \rangle$$

(we assume the radiation is polarised for maximum flux linkage, $\boldsymbol{B} = B_z \boldsymbol{e}_z$). We therefore have

$$\langle V^2 \rangle = \frac{\omega^2 a^4 \mu_0}{c} N_i$$

The equivalent circuit for the system contains a voltage source connected in series with the radiation resistance (R_r) and the load resistor (R_r too since it is matched). The total power scattered *and* absorbed is given by the total power dissipated in these two resistors

$$P_{tot} = \langle \left(\frac{V}{2R_r} \right)^2 (2R_r) \rangle = \frac{\omega^2 a^4 \mu_0}{2cR_r} N_i$$

Thus the total cross-section is

$$A_{tot} \equiv \frac{P_{tot}}{N_i} = \frac{3\pi c^2}{\omega^2}$$

(ii) *Using* $A_{\text{eff}} = \frac{\lambda^2}{4\pi} G$

We have already calculated the power gain, so know that the effective area is

$$A_{\text{eff}} = \frac{\lambda^2}{4\pi} G = \frac{3\lambda^2}{8\pi} \sin^2\theta$$

The effective area is the cross-section for power dissipated in the load. For a matched load, an equal amount of power is scattered (dissipated in the radiation resistance), so the total cross section for scattering and absorption is

$$A_{tot} = 2A_{\text{eff}} = \frac{3c^2}{4\pi\nu^2} \sin^2\theta = \frac{3\pi c^2}{\omega^2} \sin^2\theta$$

In the question the radiation is incident with $\theta = \pi/2$, so

$$A_{tot} = \frac{3\pi c^2}{\omega^2} .$$

23 The pressure at the earth's surface is equal to the total weight of the air above unit area,

$$P = 1\,\text{atm} = 10^5\text{Nm}^{-2} = N\bar{m}g$$

where we have written the weight of the atmosphere per unit area in terms of the number of molecules per unit area, N, the acceleration due to gravity, $g = 9.81\text{ms}^{-1}$ (assumed constant), and the mean molecular mass $\bar{m}$. We consider air to be made up of 80% nitrogen (28 amu) and 20% oxygen (32 amu),

$$\bar{m} = (0.8 \times 28 + 0.2 \times 32) \times 1.6 \times 10^{-27}\text{kg} = 4.61 \times 10^{-26}\text{kg}$$

Hence we obtain an estimate of N

$$N = \frac{10^5}{4.61 \times 10^{-26} \times g} \approx 2.2 \times 10^{29}\text{m}^{-2}$$

The scattering cross-section of a single molecule is determined by the Rayleigh scattering formula ($\lambda \gg$ molecular size a)

$$\sigma = \frac{\mu_0^2\omega^4\alpha^2}{6\pi} = \frac{8\pi\omega^4 a^6}{3c^4} \simeq 1.24 \times 10^{-30}\text{m}^2$$

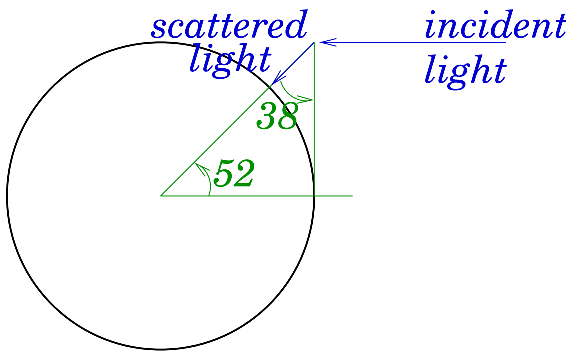
where we have used the polarisability $\alpha = 4\pi\epsilon_0 a^3$ for a metallic sphere of radius $a = 0.1$nm, and wavelength $\lambda = 320$nm.

The probability of scattering for radiation passing through the column of air above the earth is given by the cross-section for an individual molecule, σ, multiplied by the number of molecules per unit area, N. The scattering probability is

$$p = N\sigma = 0.27$$

(We are told to ignore multiple scattering; this is accurate provided $p \ll 1$.) Because only half the radiation is scattered back to Earth, the probability that the radiation is lost is $\frac{1}{2}0.27 = 0.135$. The reduction in UV radiation incident from the Sun due to scattering is about 13.5%.

24 The latitude of Cambridge is 52°, so in March (close to the Spring equinox) the geometry for scattering is as in the diagram



Applying the formula for the polarisation, P of scattered radiation at a scattering angle $\theta = 38°$ (derived in lectures, and in the handout), we find

$$\begin{aligned} P &= \frac{\sin^2\left(\frac{\pi}{2}\right) - \sin^2\theta}{\sin^2\left(\frac{\pi}{2}\right) + \sin^2\theta} \\ &= \frac{1 - \sin^2(38°)}{1 + \sin^2(38°)} \\ &= 0.45 \end{aligned}$$

The polarisation of the sky is about 45%.

Relativistic Electrodynamics

25

26

27 The 4-velocity is defined to be $\mathbf{U} = (\gamma c, \tilde{\boldsymbol{u}} c)$, so the 3-velocity is

$$\boldsymbol{u} = \tilde{\boldsymbol{u}} c/\gamma$$

The equation of motion is

$$\boldsymbol{f} = \frac{\mathrm{d}(\gamma m \boldsymbol{u})}{\mathrm{d}t} = q\left(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}\right)$$

Rewriting in terms $\tilde{\boldsymbol{u}}$, and using $\mathrm{d}t = \gamma \mathrm{d}\tau$, we find

$$\begin{aligned} \frac{1}{\gamma}\frac{\mathrm{d}(mc\tilde{\boldsymbol{u}})}{\mathrm{d}\tau} &= q\left(\boldsymbol{E} + \frac{c\tilde{\boldsymbol{u}}}{\gamma} \times \boldsymbol{B}\right) \\ \frac{\mathrm{d}\tilde{\boldsymbol{u}}}{\mathrm{d}\tau} &= \frac{q}{mc}\left(\gamma \boldsymbol{E} + c\tilde{\boldsymbol{u}} \times \boldsymbol{B}\right) \end{aligned}$$

The 4-force is

$$\begin{aligned}\mathbf{F} &\equiv \frac{\mathrm{d}}{\mathrm{d}\tau}(cm\gamma, \boldsymbol{p}) = \left(cm\frac{\mathrm{d}\gamma}{\mathrm{d}\tau}, \gamma\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}\right) \\ &= \left(cm\frac{\mathrm{d}\gamma}{\mathrm{d}\tau}, \gamma\boldsymbol{f}\right)\end{aligned}$$

Taking the 4-vector inner product with $\mathbf{U}$, we find

$$\begin{aligned}\mathbf{F}\cdot\mathbf{U} &= \gamma c^2 m\frac{\mathrm{d}\gamma}{\mathrm{d}\tau} - \gamma c\boldsymbol{f}\cdot\tilde{\boldsymbol{u}} \\ &= \gamma c\left[mc\frac{\mathrm{d}\gamma}{\mathrm{d}\tau} - q\boldsymbol{E}\cdot\tilde{\boldsymbol{u}}\right]\end{aligned}$$

However, working in the IRF we know $\mathbf{F}\cdot\mathbf{U} = 0$, so

$$\begin{aligned}\gamma c\left[mc\frac{\mathrm{d}\gamma}{\mathrm{d}\tau} - q\boldsymbol{E}\cdot\tilde{\boldsymbol{u}}\right] &= 0 \\ \frac{\mathrm{d}\gamma}{\mathrm{d}\tau} &= \frac{q}{mc}\boldsymbol{E}\cdot\tilde{\boldsymbol{u}}\end{aligned}$$

28 In the frame S', the electrostatic potential is that of a parallel plate capacitor (NB the electric field between the plates is uniform)

$$\phi' = \begin{cases} (-V_0)(y'/s) & 0 \le y' \le s \\ 0 & y' < 0 \\ -V_0 & s < y' \end{cases}$$

There are no currents, so $\boldsymbol{A}' = 0$, and the potential 4-vector is

$$A'^{\mu} = \begin{cases} (-V_0 y'/(sc), 0, 0, 0) & 0 \le y' \le s \\ (0,0,0,0) & y' < 0 \\ (-V_0/c, 0, 0, 0) & s < y' \end{cases}$$

Using the (inverse) Lorentz transformation, we transform the potential 4-vector from the frame S$'$ to the frame S (in which the capacitor is moving at velocity βc in the positive x-direction). Making use of the fact that $y' = y$, we find that in S

$$A^{\mu} = \begin{cases} (\gamma(-V_0 y/(sc)), \gamma\beta(-V_0 y/(sc)), 0, 0) & 0 \le y \le s \\ (0,0,0,0) & y < 0 \\ (-\gamma V_0/c, -\gamma\beta V_0/c, 0, 0) & s < y \end{cases}$$

Then we proceed to compute the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields. In the regions outside the plates ($y < 0$, $y > s$) the scalar and vector potentials are constants in space and time, so the fields vanish $\boldsymbol{E} = \boldsymbol{B} = \boldsymbol{0}$. The fields are non-zero only in the region $0 \le y \le s$. Here we find

$$\begin{aligned} \boldsymbol{E} &= -\boldsymbol{\nabla}\phi - \dot{\boldsymbol{A}} \\ &= -\boldsymbol{\nabla}\left(-\frac{\gamma V_0 y}{s}\right) - 0 \\ &= \left(\frac{\gamma V_0}{s}\right)\boldsymbol{e}_y \\ \Rightarrow \boldsymbol{E} &= \left(0, \frac{\gamma V_0}{s}, 0\right) \end{aligned}$$

For $\boldsymbol{B}$, we have

$$\begin{aligned} \boldsymbol{B} &= \boldsymbol{\nabla} \times \boldsymbol{A} \\ &= \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{\gamma\beta V_0 y}{sc} & 0 & 0 \end{vmatrix} \\ &= \frac{\gamma\beta V_0}{sc}\boldsymbol{e}_z \\ \Rightarrow \boldsymbol{B} &= \left(0, 0, \frac{\gamma\beta V_0}{sc}\right) \end{aligned}$$

29 Let the fields in S and S' be

$$\boldsymbol{E} = (E_x, E_y, E_z) \qquad \boldsymbol{B} = (B_x, B_y, B_z)$$

and

$$\boldsymbol{E}' = (E'_x, E'_y, E'_z) \qquad \boldsymbol{B}' = (B'_x, B'_y, B'_z)$$

respectively. These components are related by the Lorentz transformations for fields

$$\begin{aligned} E'_x &= E_x & B'_x &= B_x \\ E'_y &= \gamma(E_y - vB_z) & B'_y &= \gamma\left(B_y + \tfrac{vE_z}{c^2}\right) \\ E'_z &= \gamma(E_z + vB_y) & B'_z &= \gamma\left(B_z - \tfrac{vE_y}{c^2}\right) \end{aligned}$$

We are told that in the moving frame $\boldsymbol{E}' = \boldsymbol{E}$ and $\boldsymbol{B}' = -\boldsymbol{B}$. Using this to substitute for the components of $\boldsymbol{E}'$ and $\boldsymbol{B}'$ in the above formulae leads to

$$\begin{aligned} E_x &= E_x & B_x &= 0 \\ (1-\gamma)E_y &= -\gamma v B_z & -(1+\gamma)B_y &= \tfrac{\gamma v}{c^2}E_z \\ (1-\gamma)E_z &= \gamma v B_y & -(1+\gamma)B_z &= -\tfrac{\gamma v}{c^2}E_y \end{aligned}$$

Solving these equations for (B_x, B_y, B_z) in terms of (E_x, E_y, E_z), we find [NB the six equations are overcomplete; it is important to check that the answer is consistent with *all* equations]

$$\begin{aligned} B_x &= 0 \\ B_y &= \frac{1-\gamma}{\gamma v} E_z \\ B_z &= \frac{\gamma - 1}{\gamma v} E_y \\ \Rightarrow \boldsymbol{B} &= \frac{\gamma-1}{\gamma v}(0, -E_z, E_y) \end{aligned}$$

We check $\boldsymbol{B}\cdot\boldsymbol{E}$

$$\boldsymbol{B}\cdot\boldsymbol{E} = \frac{\gamma-1}{\gamma v}\begin{pmatrix} 0 \\ -E_z \\ E_y \end{pmatrix} \cdot \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

so the 2 fields are perpendicular in S. Since $\boldsymbol{B}'\cdot\boldsymbol{E}' = (-\boldsymbol{B})\cdot(\boldsymbol{E}) = 0$, the fields are also perpendicular in S$'$.

[Note that, in fact, $\boldsymbol{B}\cdot\boldsymbol{E}$ is a Lorentz invariant (see problem 32). We can therefore use $\boldsymbol{B}\cdot\boldsymbol{E} = \boldsymbol{B}'\cdot\boldsymbol{E}' = (-\boldsymbol{B})\cdot(\boldsymbol{E}) = -\boldsymbol{B}\cdot\boldsymbol{E} = 0$ to state directly that the electric and magnetic fields are orthogonal in all frames if $\boldsymbol{B}' = -\boldsymbol{B}$ and $\boldsymbol{E}' = \boldsymbol{E}$.]

30 In S, we know that the E and B (uniform magnetic field) fields are

$$\boldsymbol{E} = (0,0,0) \qquad \boldsymbol{B} = (0,0,B)$$

(a) We make use of the Lorentz transformation of the fields,

$$\begin{aligned} E'_x &= E_x & B'_x &= B_x \\ E'_y &= \gamma(E_y - vB_z) & B'_y &= \gamma\left(B_y + \tfrac{v}{c^2}E_z\right) \\ E'_z &= \gamma(E_z + vB_y) & B'_z &= \gamma\left(B_z - \tfrac{v}{c^2}E_y\right) \end{aligned}$$

to find the fields in S$'$ outside the plane conducting sheet:

$$\boldsymbol{E}'_0 = (0, -\gamma v B, 0) \qquad \boldsymbol{B}'_0 = (0,0,\gamma B)\,.$$

(b) We now deduce the fields inside the conducting sheet. Since the sheet is a conductor, we must have $\boldsymbol{E}' = (0,0,0)$. For simplicity, we take $\mu_r = 1$ such that $\boldsymbol{B}_\parallel$ is conserved at the interface, $\boldsymbol{B}' = \boldsymbol{B}'_0$.

(c) We determine the charge density on the surfaces of the conductor in S$'$, using Gauss' theorem to relate the surface charge density σ' to the discontinuity in normal electric field. On the top and bottom surfaces, we find,

$$\sigma'_{top} = +\epsilon_0 E_0'^z = -\gamma\epsilon_0 VB \qquad \sigma'_{bottom} = -\epsilon_0 E_0'^z = +\gamma\epsilon_0 VB \qquad .$$

Finally, we transform the fields inside the conductor back to S, to find

$$\boldsymbol{E}_i = (0, \gamma^2 VB, 0) \qquad \boldsymbol{B}_i = (0, 0, \gamma^2 B)$$

Interpretation in S: Outside the conductor, there is a field $\boldsymbol{B}_0 = (0, 0, B)$ and no electric field; inside we have found that $\boldsymbol{B}_i = (0, 0, \gamma^2 B)$ is larger, and the electric field is non-zero, $\boldsymbol{E}_i = (0, \gamma^2 VB, 0)$.

As the conductor is moving at speed V through a magnetic field, the charge carriers in the conductor experience a Lorentz force $\boldsymbol{f} = -qVB_i^z\boldsymbol{e}_y$, where $\boldsymbol{B}_i = (0, 0, B_i^z)$ is the field inside the conductor (we will determine B_i^z presently). In order to prevent charge motion, in the steady-state situation an electric field will build up in the y-direction to oppose the Lorentz force, $E_i^y = +VB_i^z$. To generate this electric field there must be a surface charge density (equal and opposite charge densities on the two surfaces for charge neutrality): $\sigma_{top} = -\sigma_{bottom} = -E_i^y\epsilon_0$. These charge densities are moving at speed V, so there are *surface currents*: the total surface current per unit width of the slab is $j^x_{top} = -j^x_{bottom} = -VE_i^y\epsilon_0$. The surface currents cause a discontinuity in $B_\parallel$ across the top and bottom surfaces of the conductor (apply the integral form of Ampère's theorem). This leads to a formula for B_i^z:

$$\begin{aligned}
B_0^z - B_i^z &= \mu_0(-VE_y\epsilon_0) = \frac{1}{c^2}(-V^2 B_i^z) \\
B_0^z - B_i^z &= -\beta^2 B_i^z \\
B_i^z &= \frac{1}{1-\beta^2} B_0^z = \gamma^2 B_0^z \\
\Rightarrow B_i^z &= \gamma^2 B \quad \text{(hence: } E_i^y = V\gamma^2 B\text{)},
\end{aligned}$$

as was found previously.

31 One can follow the derivation in the handout. An alternative derivation is given here.

We take the charge to be at the space-time origin of both S and S′. At a general point (x', y', z') in the rest frame of the charge S′, the fields are

$$\begin{aligned}
\boldsymbol{E}' &= \frac{q}{4\pi\epsilon r'^3}(x', y', z') \\
\boldsymbol{B}' &= (0, 0, 0)
\end{aligned}$$

We obtain the fields in the frame S by the (inverse) Lorentz Transformation

$$\begin{aligned}
\boldsymbol{E} &= \left(E'_x, \gamma\left(E'_y + vB'_z\right), \gamma\left(E'_z - vB'_y\right)\right) = \frac{q}{4\pi\epsilon r'^3}\left(x', \gamma y', \gamma z'\right) \\
\boldsymbol{B} &= \left(B'_x, \gamma\left(B'_y - \frac{v}{c^2}E'_z\right), \gamma\left(B'_z + \frac{v}{c^2}E'_y\right)\right) = \frac{q}{4\pi\epsilon r'^3}(0, -\frac{\gamma v z'}{c^2}, \frac{\gamma v y'}{c^2})
\end{aligned}$$

We need to relate (x', y', z') to (x, y, z) by the Lorentz transformation for position. To do so, we note that $t = 0$ (the time when the charge is at the origin of S), and

$$\begin{aligned}
x' &= \gamma x \\
y' &= y \\
z' &= z \\
r' &= \sqrt{x'^2 + y'^2 + z'^2} = \sqrt{r^2 + (\gamma^2 - 1)x^2}
\end{aligned}$$

Therefore

$$\begin{aligned}
\boldsymbol{E} &= \frac{q}{4\pi\epsilon}\frac{\gamma}{\left[r^2 + (\gamma^2 - 1)x^2\right]^{3/2}}(x, y, z) \\
\boldsymbol{B} &= \frac{q}{4\pi\epsilon}\frac{\gamma v/c^2}{\left[r^2 + (\gamma^2 - 1)x^2\right]^{3/2}}(0, -z, y)
\end{aligned}$$

(a) Suppose $(x, y, z) = (r, 0, 0)$,

$$\begin{aligned}
\boldsymbol{E} &= \left(\frac{q}{4\pi\epsilon_0}\frac{1}{\gamma^2 r^2}, 0, 0\right) \\
\boldsymbol{B} &= (0, 0, 0)
\end{aligned}$$

Hence the force on the particle at this point is along the x-axis, and of size

$$f_x = qE_x = \frac{q^2}{4\pi\epsilon_0}\frac{1}{\gamma^2 r^2} = \frac{F}{\gamma^2}$$

where $F = q^2/(4\pi\epsilon r^2)$.

(b) Suppose $(x, y, z) = (0, r, 0)$,

$$\begin{aligned}
\boldsymbol{E} &= \left(0, \frac{q}{4\pi\epsilon_0}\frac{\gamma}{r^2}, 0\right) \\
\boldsymbol{B} &= \left(0, 0, \frac{q}{4\pi\epsilon_0}\frac{\gamma v/c^2}{r^2}\right)
\end{aligned}$$

By the Lorentz force law, the force $\boldsymbol{f} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$ is along the y-axis of size

$$\begin{aligned} f_y &= qE_y - qvB_z \\ &= \frac{\gamma q^2}{4\pi\epsilon r^2} - \frac{\gamma q^2 v^2}{4\pi\epsilon_0 r^2 c^2} \\ &= \frac{\gamma q^2}{4\pi\epsilon r^2}\left(1 - \frac{v^2}{c^2}\right) \\ &= \frac{\gamma q^2}{4\pi\epsilon r^2}\frac{1}{\gamma^2} = \frac{F}{\gamma}\,. \end{aligned}$$

(c) Suppose $(x, y, z)(r/\sqrt{2}, r/\sqrt{2}, 0)$, then $r^2 + (\gamma^2 - 1)x^2 = (\gamma^2 + 1)r^2/2$, and

$$\begin{aligned} \boldsymbol{E} &= \frac{q}{4\pi\epsilon_0}\frac{\gamma}{r^3[(\gamma^2+1)/2]^{3/2}}\left(r/\sqrt{2}, r/\sqrt{2}, 0\right) \\ &= \frac{q}{4\pi\epsilon_0 r^2}\frac{2\gamma}{(\gamma^2+1)^{3/2}}\,(1,1,0) \\ \boldsymbol{B} &= \frac{q}{4\pi\epsilon}\frac{\gamma v/c^2}{r^3[(\gamma^2+1)/2]^{3/2}}\left(0, 0, r/\sqrt{2}\right) \\ &= \left(0, 0, \frac{q}{4\pi\epsilon r^2}\frac{\gamma v/c^2}{(\gamma^2+1)^{3/2}}\right) \end{aligned}$$

Using the Lorentz force, we find

$$\begin{aligned} f_x &= qE_x = \frac{q^2}{4\pi\epsilon r^2}\frac{2\gamma}{(\gamma^2+1)^{3/2}} \\ &= F\frac{2\gamma}{(\gamma^2+1)^{3/2}} \\ f_y &= qE_y - evB_z = eE_y - \frac{v^2}{c^2}eE_y \\ &= \frac{1}{\gamma^2}E_y = \frac{q^2}{4\pi\epsilon r^2}\frac{2}{\gamma(\gamma^2+1)^{3/2}} \\ &= F\frac{2}{\gamma(\gamma^2+1)^{3/2}} \\ f_z &= 0 \end{aligned}$$

Therefore the total force is

$$
\begin{aligned}
|\boldsymbol{f}| &= \sqrt{f_x{}^2 + f_y{}^2 + f_z^2} \\
&= \sqrt{\frac{4\gamma^2 F^2}{(\gamma^2+1)^3} + \frac{4F^2}{\gamma^2(\gamma^2+1)^3}} \\
&= \frac{2F}{\gamma}\sqrt{\frac{\gamma^4+1}{(\gamma^2+1)^3}}
\end{aligned}
$$

and it is directed at an angle $\tan^{-1}(f_y/f_x) = \tan^{-1}(1/\gamma^2)$ to the x-axis.

32 (a) The fields $\boldsymbol{E}'$ and $\boldsymbol{B}'$ are related to the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ by Lorentz transformation :

$$
\boldsymbol{E} = \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}
$$

$$
\boldsymbol{E}' = \begin{pmatrix} E_x \\ \gamma(E_y - vB_z) \\ \gamma(E_z + vB_y) \end{pmatrix} \qquad \boldsymbol{B}' = \begin{pmatrix} B_x \\ \gamma\left(B_y + \frac{v}{c^2}E_z\right) \\ \gamma\left(B_z - \frac{v}{c^2}E_y\right) \end{pmatrix}
$$

To show that the quantities $\boldsymbol{E}^2 - c^2\boldsymbol{B}^2$, and $\boldsymbol{E}\cdot\boldsymbol{B}$ are invariant under Lorentz transformations, let us calculate $\boldsymbol{E}'^2 - c^2\boldsymbol{B}'^2$ and $\boldsymbol{E}'\cdot\boldsymbol{B}'$

$$
\begin{aligned}
\boldsymbol{E}'^2 - c^2\boldsymbol{B}'^2 &= E_x{}^2 + \gamma^2(E_y - vB_z)^2 + \gamma^2(E_z + vB_y)^2 \\
&\quad -c^2B_x^2 - \gamma^2c^2\left(B_y + \frac{v}{c^2}E_z\right)^2 - \gamma^2c^2\left(B_z - \frac{v}{c^2}E_y\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&= E_x{}^2 + \gamma^2(E_y{}^2 - 2vE_yB_z + v^2B_z{}^2) + \gamma^2(E_z{}^2 + 2vE_zB_y + v^2B_y{}^2) \\
&\quad -c^2B_x{}^2 - c^2\gamma^2\left(B_y{}^2 + \frac{2v}{c^2}B_yE_z + \frac{v^2}{c^4}E_z{}^2\right) \\
&\quad -c^2\gamma^2\left(B_z{}^2 + \frac{2v}{c^2}B_zE_y + \frac{v^2}{c^4}E_y{}^2\right) \\
&= E_x{}^2 + \gamma^2\left(1 - \frac{v^2}{c^2}\right)E_y{}^2 + \gamma^2\left(1 - \frac{v^2}{c^2}\right)E_z{}^2 \\
&\quad -c^2B_x{}^2 - c^2\gamma^2\left(1 - \frac{v^2}{c^2}\right)B_y{}^2 - c^2\gamma^2\left(1 - \frac{v^2}{c^2}\right)B_z{}^2 \\
&= \boldsymbol{E}^2 - c^2\boldsymbol{B}^2
\end{aligned}
$$

using $\frac{1}{\gamma^2} = \left(1 - \frac{v^2}{c^2}\right)$.

Similarly, we find

$$\begin{aligned}
\boldsymbol{E}' \cdot \boldsymbol{B}' &= \begin{pmatrix} E_x \\ \gamma(E_y - vB_z) \\ \gamma(E_z + vB_y) \end{pmatrix} \cdot \begin{pmatrix} B_x \\ \gamma\left(B_y + \frac{v}{c^2}E_z\right) \\ \gamma\left(B_z - \frac{v}{c^2}E_y\right) \end{pmatrix} \\
&= E_xB_x + \gamma^2(E_y - vB_z)\left(B_y + \frac{v}{c^2}E_z\right) + \gamma^2(E_z + vB_y)\left(B_z - \frac{v}{c^2}E_y\right) \\
&= E_xB_x + \gamma^2\left(E_yB_y - vB_zB_y + \frac{v}{c^2}E_yE_z - \frac{v^2}{c^2}E_zB_z\right) \\
&\quad +\gamma^2\left(E_zB_z + vB_zB_y - \frac{v}{c^2}E_yE_z - \frac{v^2}{c^2}E_zB_z\right) \\
&= E_xB_x + \gamma^2\left(1 - \frac{v^2}{c^2}\right)E_yB_y + \gamma^2\left(1 - \frac{v^2}{c^2}\right)E_zB_z \\
&= \boldsymbol{E} \cdot \boldsymbol{B}
\end{aligned}$$

Hence we have shown that both $\boldsymbol{E}^2 - c^2\boldsymbol{B}^2$ and $\boldsymbol{E} \cdot \boldsymbol{B}$ are invariant under Lorentz transformations.

(b) First we note that we can write the mean energy

$$\langle U \rangle = \frac{\langle \boldsymbol{B}^2 \rangle}{2\mu_0} + \frac{\epsilon_0 \langle \boldsymbol{E}^2 \rangle}{2} = \frac{\langle \boldsymbol{B}^2 \rangle}{2\mu_0} + \frac{\langle \boldsymbol{E}^2 \rangle}{2\mu_0 c^2}$$

where the brackets $\langle\rangle$ denote the time average. For isotropic electromagnetic radiation, we have $\langle \boldsymbol{E}^2 \rangle = c^2 \langle \boldsymbol{B}^2 \rangle$, and

$$\frac{1}{2\mu_0}\langle {B_x}^2 \rangle = \frac{1}{2\mu_0}\langle {B_y}^2 \rangle = \frac{1}{2\mu_0}\langle {B_z}^2 \rangle = \frac{\epsilon_0}{2}\langle {E_x}^2 \rangle = \frac{\epsilon_0}{2}\langle {E_y}^2 \rangle = \frac{\epsilon_0}{2}\langle {E_z}^2 \rangle = \frac{\langle U \rangle}{6}$$

Furthermore, since the mean power flux (mean Poynting vector $\boldsymbol{N}$) is zero for isotropic electromagnetic radiation, we have

$$\langle E_yB_z - B_zE_y \rangle \propto N_x = 0$$

Now we calculate $\langle U' \rangle$ by direct substitution, and make use of the above results to find

$$
\begin{aligned}
\langle U'\rangle &= \frac{1}{2\mu_0 c^2}\langle {E_x}^2+\gamma^2(E_y-vB_z)^2+\gamma^2(E_z+vB_y)^2\rangle \\
&\quad+\frac{1}{2\mu_0}\langle {B_x}^2+\left(B_y+\frac{v}{c^2}E_z\right)^2+\gamma^2\left(B_z-\frac{v}{c^2}E_y\right)^2\rangle \\
&= \frac{1}{2\mu_0}\langle\frac{{E_x}^2}{c^2}+\frac{\gamma^2}{c^2}({E_y}^2-2vE_yB_z+v^2{B_z}^2)+\frac{\gamma^2}{c^2}({E_z}^2+2vE_zB_y+v^2{B_y}^2)\rangle \\
&\quad+\frac{1}{2\mu_0}\langle {B_x}^2+\gamma^2\left({B_y}^2+\frac{2v}{c^2}B_yE_z+\frac{v^2}{c^4}{E_z}^2\right)+\gamma^2\left({B_z}^2+\frac{2v}{c^2}B_zE_y+\frac{v^2}{c^4}{E_y}^2\right)\rangle \\
&= \frac{1}{2\mu_0}\langle\frac{{E_x}^2}{c^2}+\frac{\gamma^2}{c^2}{E_y}^2+\frac{\gamma^2}{c^2}{E_z}^2+\frac{v^2\gamma^2}{c^2}{B_y}^2+\frac{v^2\gamma^2}{c^2}{B_z}^2+\frac{2\gamma^2v}{c^2}(E_zB_y-E_yB_z)\rangle \\
&\quad+\frac{1}{2\mu_0}\langle\frac{v^2\gamma^2}{c^4}{E_y}^2+\frac{v^2\gamma^2}{c^4}{E_z}^2+{B_x}^2+\gamma^2{B_y}^2+\gamma^2{B_z}^2+\frac{2\gamma^2v}{c^2}(E_zB_y-E_yB_z)\rangle \\
&= \frac{1}{2\mu_0}\langle\frac{{E_x}^2}{c^2}+\frac{\gamma^2}{c^2}\left(1+\frac{v^2}{c^2}\right)({E_y}^2+{E_z}^2)\rangle \\
&\quad+\frac{1}{2\mu_0}\langle {B_x}^2+\gamma^2\left(1+\frac{v^2}{c^2}\right)\left(B_y^2+B_z^2\right)\rangle \\
&= \frac{\langle U\rangle}{6}+\gamma^2\left(1+\frac{v^2}{c^2}\right)\left(\frac{2\langle U\rangle}{6}\right)+\frac{\langle U\rangle}{6}+\gamma^2\left(1+\frac{v^2}{c^2}\right)\left(\frac{2\langle U\rangle}{6}\right) \\
&= \frac{\langle U\rangle}{3}+\frac{2\gamma^2\langle U\rangle}{3}\left(1+\frac{v^2}{c^2}\right) \\
&= \frac{\langle U\rangle}{3}+\frac{2\gamma^2\langle U\rangle}{3}\left(2-\frac{1}{\gamma^2}\right) \\
&= \langle U\rangle\frac{4\gamma^2-1}{3}
\end{aligned}
$$

Since the observer is moving in the x-direction with velocity v, the mean Poynting vector can only be in the x-direction, $\boldsymbol{N}'=N'_x\hat{\boldsymbol{e}}_x$ and

$$
\begin{aligned}
\langle N'_x\rangle &= \frac{1}{\mu_0}\langle E'_yB'_z-E'_zB'_y\rangle \\
&= \frac{\gamma^2}{\mu_0}\langle(E_y-vB_z)\left(B_z-\frac{v}{c^2}E_y\right)-(E_z+vB_y)\left(B_y+\frac{v}{c^2}E_z\right)\rangle \\
&= -\frac{\gamma^2}{\mu_0}\langle\frac{v}{c^2}({E_y}^2+{E_z}^2)+v({B_y}^2+{B_z}^2)+(E_zB_y-E_yB_z)+\frac{v^2}{c^2}(B_yE_z-B_zE_y)\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{\gamma^2 v}{\mu_0}\langle\frac{{E_y}^2+{E_z}^2}{c^2}+{B_y}^2+{B_z}^2\rangle \\
&= -\frac{4v\langle U\rangle\gamma^2}{3} \\
\Rightarrow \langle \boldsymbol{N}\rangle &= \left(-\frac{4v\langle U\rangle\gamma^2}{3},0,0\right)
\end{aligned}
$$

Again, we have used $\langle E_zB_y - E_yB_z\rangle = 0$.

[An alternative method is to use the transformation of the average stress-energy tensor. In a frame containing isotropic radiation this takes the form

$$
\langle M^{\mu\nu}\rangle = \begin{pmatrix} \langle U\rangle & 0 & 0 & 0 \\ 0 & \langle U\rangle/3 & 0 & 0 \\ 0 & 0 & \langle U\rangle/3 & 0 \\ 0 & 0 & 0 & \langle U\rangle/3 \end{pmatrix}
$$

(using $\langle \boldsymbol{N}\rangle = 0$, $\langle E_x^2\rangle = \langle E_y^2\rangle = \langle E_z^2\rangle$, $\langle E_xE_y\rangle = \langle E_yE_z\rangle = \langle E_zE_x\rangle = 0$ and similarly for $\boldsymbol{B}$). The mean energy density and Poynting flux in the frame S$'$ can then be found from the transformation $\langle {M'}^{\alpha\beta}\rangle = \Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}{}_{\nu}\langle M^{\mu\nu}\rangle$.]

Radiation and relativistic electrodynamics

33 For a magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{e}}_z$, the equation of motion of the electron, of charge $-e$, is

$$
\begin{aligned}
\frac{\mathrm{d}(\gamma m\boldsymbol{u})}{\mathrm{d}t} &= -e\boldsymbol{u}\times\boldsymbol{B} \\
\dot\gamma m\boldsymbol{u} + \gamma m\dot{\boldsymbol{u}} &= -eB\boldsymbol{u}\times\hat{\boldsymbol{e}}_z
\end{aligned}
$$

Take the dot product of this equation with $\boldsymbol{u}$

$$
\dot\gamma m u^2 + \gamma m\boldsymbol{u}\cdot\dot{\boldsymbol{u}} = 0
$$

By direct differentiation, $\dot\gamma = (\gamma^3/c^2)(\dot u u) \Rightarrow \boldsymbol{u}\cdot\dot{\boldsymbol{u}} = (\dot\gamma c^2/\gamma^3)$, so inserting in the above equation we find

$$
\begin{aligned}
\dot\gamma m u^2 + \gamma m(\dot\gamma c^2/\gamma^3) &= 0 \\
\dot\gamma m c^2(\beta^2 + 1/\gamma^2) &= 0 \\
\dot\gamma m c^2 &= 0
\end{aligned}
$$

Thus, the particle has $\dot\gamma = 0$, and

$$
\gamma m\dot{\boldsymbol{u}} = -eB\boldsymbol{u}\times\hat{\boldsymbol{e}}_z
$$

which describes circular motion at angular frequency ω

$$\omega = \frac{eB}{\gamma m}$$

Using $\gamma mc^2 = 1\text{GeV}$, and $m = 9.1 \times 10^{-31}\text{kg}$, we find that $\gamma = 1954$. Now, taking $e = 1.6 \times 10^{-19}\text{C}$ and $B = 1\text{T}$ we get $\omega = 9.0 \times 10^{7}\text{rads}^{-1}$.

In a synchrotron, the radiation is strongly beamed, causing the radiation to arrive as a series of pulses, each with characteristic duration $\delta t \simeq \frac{1}{\gamma^3\omega}$. The characteristic frequency is set by the inverse of this time

$$\nu_s = \gamma^3\omega$$

which for the above parameters is approximately $\nu_s \sim 7 \times 10^{17}\text{Hz}$. Since the peak intensity occurs at slightly less than ν_s, a reasonable assessment of the typical frequency is of order 10^{17}Hz.

The power radiated is

$$P = \frac{\mu_0 e^4 \gamma^2 B^2 u^2}{6\pi c m^2}$$

so the fractional loss in energy in one period is

$$\begin{aligned}
\frac{\Delta E}{E} &= P\left(\frac{2\pi}{\omega}\right)\frac{1}{\gamma mc^2} \\
&= \frac{\mu_0 e^4 \gamma^2 B^2 u^2}{6\pi c m^2}\left(\frac{2\pi}{\omega}\right)\frac{1}{\gamma mc^2} \\
&= \frac{\mu_0 e^4 \gamma B^2 (u^2/c^2)}{3\omega c m^3} \\
&= \frac{\mu_0 e^4 \gamma B^2 (1 - 1/\gamma^2)}{3\omega c m^3} \\
&\sim 2.6 \times 10^{-5}
\end{aligned}$$

34 From problem 27, we know that

$$\begin{aligned}
E_x &= 0 \qquad E_y = cB_0 \sin\left[\omega\left(t - \frac{x}{c}\right)\right] \qquad E_z = 0 \\
B_x &= 0 \qquad B_y = 0 \qquad B_z = B_0 \sin\left[\omega\left(t - \frac{x}{c}\right)\right]
\end{aligned}$$

From the equation for $\mathrm{d}\gamma/\mathrm{d}\tau$ derived above, we have

$$\begin{aligned}
\frac{\mathrm{d}\gamma}{\mathrm{d}\tau} &= \frac{q}{mc} E_y \tilde{u}_y \\
&= \frac{q}{mc}(cB_0\tilde{u}_y)\sin\left[\omega\left(t - \frac{x}{c}\right)\right] \\
&= \frac{qB_0\tilde{u}_y}{m}\sin\left[\omega\left(t - \frac{x}{c}\right)\right] \qquad (3)
\end{aligned}$$

Expanding the equation for $\mathrm{d}\tilde{\boldsymbol{u}}/\mathrm{d}t$ in components leads to

$$\frac{\mathrm{d}\tilde{u}_x}{\mathrm{d}\tau} = \frac{qB_0\tilde{u}_y}{m}\sin\left[\omega\left(t-\frac{x}{c}\right)\right] \tag{4}$$

$$\frac{\mathrm{d}\tilde{u}_y}{\mathrm{d}\tau} = \frac{qB_0(\gamma-\tilde{u}_x)}{m}\sin\left[\omega\left(t-\frac{x}{c}\right)\right] \tag{5}$$

$$\frac{\mathrm{d}\tilde{u}_z}{\mathrm{d}\tau} = 0 \tag{6}$$

(a) By equation (6), it is clear that $\tilde{u}_z =$ constant. Since $\tilde{u}_z = 0$ at $t = 0$, we get

$$\tilde{u}_z = 0$$

(b) Divide equation (4) by equation (3) to find

$$\frac{\mathrm{d}\tilde{u}_x}{\mathrm{d}\gamma} = 1$$

Integrate this, for $\tilde{u}_x$ from 0 to $\tilde{u}_x$ and for γ from 1 to γ, to find

$$\tilde{u}_x = \gamma - 1$$

(c) The x-component of the 3-velocity is

$$\begin{aligned} u_x \equiv \frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{c}{\gamma}\tilde{u}_x \\ &= \frac{c}{\gamma}(\gamma-1) \end{aligned}$$

$$\begin{aligned} \int_0^x \mathrm{d}x &= \int_0^t \left[c - \frac{c}{\gamma}\right]\mathrm{d}t \\ x &= ct - c\int_0^t \frac{1}{\gamma}\mathrm{d}t \\ &= ct - c\int_0^\tau \mathrm{d}\tau \\ &= c(t-\tau) \\ \Rightarrow (t - x/c) &= \tau \end{aligned}$$

Where x is the position at the time t (measured in S) and τ is the particle's proper time.

Using (b) and (c), the equations to solve become

$$\begin{aligned}\frac{\mathrm{d}\gamma}{\mathrm{d}\tau} &= \frac{qB_0\tilde{u}_y}{m}\sin\omega\tau \\ \frac{\mathrm{d}\tilde{u}_y}{\mathrm{d}\tau} &= \frac{qB_0}{m}\sin\omega\tau\end{aligned}$$

Integrating the second equation gives

$$\tilde{u}_y = \frac{qB_0}{m\omega}\left(1-\cos\omega\tau\right)$$

which, inserted in the other equation, leads to

$$\begin{aligned}\frac{\mathrm{d}\gamma}{\mathrm{d}\tau} &= \frac{q^2B_0^2}{m^2\omega}\left(1-\cos\omega\tau\right)\sin\omega\tau \\ \int_1^\gamma \mathrm{d}\gamma &= \int_0^\tau \frac{q^2B_0^2}{m^2\omega}\left(1-\cos\omega\tau\right)\sin\omega\tau\mathrm{d}\tau \\ \gamma - 1 &= \frac{q^2B_0^2}{2m^2\omega^2}\left(1-\cos\omega\tau\right)^2 \\ \Rightarrow \gamma &= 1+\frac{q^2B_0^2}{2m^2\omega^2}\left(1-\cos\omega\tau\right)^2\end{aligned}$$

The greatest energy attained by the particle is

$$E_{max} = \gamma_{max}mc^2 \quad \text{where} \quad \gamma_{max} = 1+\frac{2q^2B_0^2}{m^2\omega^2}$$

We can determine the trajectory of the particle by noting that the position obeys

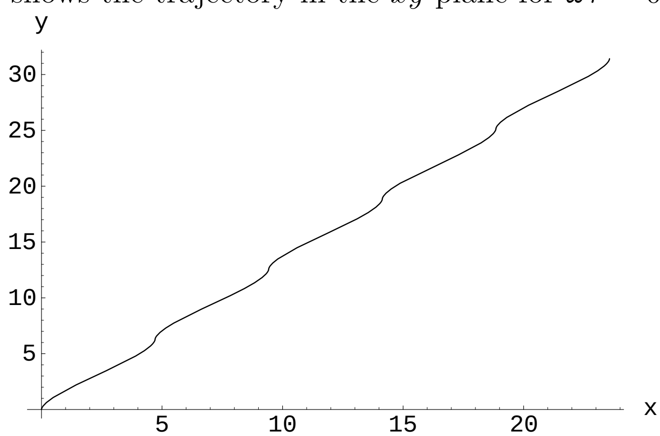
$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} &= \boldsymbol{u} \\ \frac{1}{\gamma}\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}\tau} &= \boldsymbol{u} \\ \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}\tau} &= \gamma\boldsymbol{u} = c\tilde{\boldsymbol{u}} \\ \Rightarrow \boldsymbol{r}(\tau) &= c\int_0^\tau \tilde{\boldsymbol{u}}\mathrm{d}\tau\end{aligned}$$

Applying this to the x and y components of position (the z component is zero since $\tilde{u}_z = 0$), we find

$$
\begin{aligned}
x(\tau) &= c\int_0^{\tau}\left[\frac{q^2B_0^2}{2m^2\omega^2}\left(1-\cos\omega\tau\right)^2\right]\mathrm{d}\tau \\
&= \frac{cq^2B_0^2}{2m^2\omega^3}\left[\frac{3}{2}\omega\tau - 2\sin\omega\tau + \frac{1}{4}\sin(2\omega\tau)\right] \\
y(\tau) &= c\int_0^{\tau}\left[\frac{qB_0}{m\omega}\left(1-\cos\omega\tau\right)\right]\mathrm{d}\tau \\
&= \frac{cqB_0}{m\omega^2}\left[\omega\tau - \sin\omega\tau\right]
\end{aligned}
$$

These results specify the trajectory $(x(\tau), y(\tau), 0)$ as a function of τ.

The plot shows the trajectory in the xy-plane for $\omega\tau = 0 \to 10\pi$.



[Lengths in the y-direction are in units of $(c/\omega)[qB_0/(m\omega)]$; lengths in the x-direction are in units of $(c/\omega)[qB_0/(m\omega)]^2$.]