UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Tuesday 26 May 2015 1.30 pm to 3.30 pm

PHYSICS (1)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (1)

THERMAL AND STATISTICAL PHYSICS

Candidates offering this paper should attempt a total of **three** *questions. The questions to be attempted are* **1**, **2** *and* **one** *other question.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains **four** *sides, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
2×20 Page Answer Book	Mathematical Formulae handbook
Rough workpad	Approved calculator allowed
Yellow master coversheet	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

THERMAL AND STATISTICAL PHYSICS

1 *Attempt* **all** *parts of this question. Answers should be concise and relevant formulae may be assumed without proof.*

(a) Obtain the equation of state of a non-ideal gas, from the van der Waals partition function

$$Z = \frac{1}{N!}\left(\frac{mk_{\mathrm{B}}T}{2\pi\hbar^2}\right)^{3N/2}(V - Nb)^N \exp\left[aN^2/k_{\mathrm{B}}TV\right],$$

where a and b are constants and other symbols have their usual meanings. [4]

(b) Using the relation $U = TS - pV + \mu N$, obtain the Gibbs-Duhem equation $\mathrm{d}\mu = -s\,\mathrm{d}T + v\,\mathrm{d}p$, where s is the entropy per particle, and v is the volume per particle. [4]

(c) Show that fluctuations in the internal energy U of an ideal gas in equilibrium with a reservoir of temperature T have variance $\langle(\Delta U)^2\rangle = k_{\mathrm{B}}T^2C_V$. Estimate the fractional fluctuation $\sqrt{\langle(\Delta U)^2\rangle}/U$ in the internal energy of a sample of monatomic ideal gas of volume 1 cm^3 at room temperature and pressure, and comment on your answer.

[*You may assume that* $(\partial^2F/\partial U^2)_V = 1/(C_VT)$ *for an ideal gas in thermal equilibrium.*] [4]

2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) non-ideal gases and the virial expansion;

(b) Bose-Einstein condensation;

(c) the Fermi-Dirac distribution and the specific heat of metals at low temperature.

3 *Attempt* **either** *this question* **or** *question 4.*

Explain what is meant by the concept of the *grand canonical ensemble*, and discuss its application to thermodynamic systems. [5]

Show that the grand partition function Ξ_k for a level of energy ϵ_k of an ideal gas in a box at temperature T is given in the classical limit by

$$\Xi_k \approx 1 + \exp\left[-\frac{(\epsilon_k - \mu)}{k_B T}\right] ,$$

where μ is the chemical potential. [3]

Given that the density of states for a classical gas composed of atoms of mass m in a box of volume V is $g(\epsilon) = (Vm^{3/2}/\sqrt{2}\pi^2\hbar^3)\epsilon^{1/2}$, show that the grand potential is given by

$$\Phi = -k_B T \frac{V}{\lambda^3} \exp\left(\frac{\mu}{k_B T}\right) ,$$

where $\lambda = \sqrt{2\pi\hbar^2/mk_B T}$. Hence show that the chemical potential is given by

$$\mu = k_B T \ln\left(\frac{N\lambda^3}{V}\right) ,$$

where N is the number of atoms in the box. [8]

$\left[\right.$*You may wish to use the results* $\mathrm{d}\Phi = -S\,\mathrm{d}T - p\,\mathrm{d}V - N\,\mathrm{d}\mu$, $\int_0^\infty x^2 e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}/4$. $\left.\right]$

A vapour at pressure p and temperature T is in thermal equilibrium with a surface. A molecule of the vapour may attach itself to the surface at one of a fixed number of surface sites, lowering the energy of the molecule by an amount ϵ. Using the grand partition function for an adsorbtion site, or otherwise, show that the chemical potential of the adsorbed species is

$$\mu_s = -\epsilon + k_B T \ln\left[\frac{\theta}{(1-\theta)z_s}\right] ,$$

where θ is the fraction of surface sites occupied by adsorbed molecules, and the partition function $z_s(T)$ accounts for molecular vibration with respect to the surface. [5]

Treating the vapour as an ideal classical gas, and neglecting internal degrees of freedom in the vapour phase, obtain an expression for θ in terms of p and T, and sketch the dependence of θ on p at fixed T. [4]

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4 *Attempt* **either** *this question* **or** *question 3.*

Give a brief outline of the Landau theory of phase transitions. [5]

The free energy of a system at temperature T can be written as

$$F(\phi, T) = \frac{1}{2}c(T^2 - T_0^2)\phi^2 - \frac{aT}{12\pi}\phi^3 + \frac{\lambda}{4}\phi^4, \qquad (*)$$

where $\phi \geq 0$ is an order parameter, and c, a, λ and T_0 are positive constants.

Determine the range of temperatures for which the free energy $F(\phi)$ is a local minimum at $\phi = 0$, and the range for which it is a local maximum at $\phi = 0$. [2]

The quantity γ is defined as

$$\gamma \equiv \frac{a^2}{72\pi^2\lambda c} .$$

By considering the zeros of $F(\phi, T)$ with $\phi > 0$, or otherwise, show that there exists a globally stable phase with $\phi > 0$ and free energy $F(\phi, T) < 0$ in two cases: a) for all temperatures T if $\gamma > 1$; b) for temperatures $T < T^* \equiv T_0/\sqrt{1-\gamma}$ if $\gamma < 1$. For temperature $T = T^*$, show that there is a stable solution with $\phi = \phi^*$, where $\phi^* \equiv aT^*/6\pi\lambda$. Sketch the form of $F(\phi)$ for the two cases $\gamma > 1$ and $\gamma < 1$, for a range of different temperatures T. [9]

As the system cools from very high temperature, explain why a phase transition is inevitable if $\gamma < 1$, and possible if $\gamma > 1$. State the temperature at which this transition occurs, and explain why the transition is first order in nature. [4]

In a simplified model of the electroweak phase transition in the early Universe, the free energy is given by the form (*) above, with $\lambda/a = m_{\mathrm{H}}^2/m_0^2$ and $c/a = A + m_{\mathrm{H}}^2/m_0^2$, where $m_0 = 223.9\,\mathrm{u}$, $A = 0.95$, m_{H} is the mass of the Higgs boson, and u is the unified atomic mass unit.

The Universe is thought to undergo a first order transition from a high temperature phase corresponding to the overall minimum of $F(\phi)$, with $\gamma < 1$ and $\phi^* > T^*$. Show that this is possible only for a finite range of m_{H}, and determine whether $m_{\mathrm{H}} = 134\,\mathrm{u}$ lies within this range. [5]

END OF PAPER