Pt II Electrodynamics and Optics

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Answer 1

The incoming field has the form

$$\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{1}$$

and so the emergent polarisation is given by

$$\begin{aligned} &\begin{pmatrix} a & 0 \\ 0 & ib \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \\ = &\begin{pmatrix} a\cos\theta \\ ib\sin\theta \end{pmatrix}. \end{aligned} \tag{2}$$

LCP requires

$$\begin{aligned} b\sin\theta &= a\cos\theta \\ \tan\theta &= \frac{a}{b} \\ \tan(\theta-\pi) &= \frac{a}{b} \\ \theta &= \tan^{-1}\frac{a}{b} \\ \theta &= \pi + \tan^{-1}\frac{a}{b}. \end{aligned} \tag{3}$$

RCP requires

$$\begin{aligned} b\sin\theta &= -a\cos\theta \\ \tan\theta &= -\frac{a}{b} \\ \tan(\theta-\pi) &= -\frac{a}{b} \\ \theta &= -\tan^{-1}\frac{a}{b} \\ \theta &= \pi - \tan^{-1}\frac{a}{b}. \end{aligned} \tag{4}$$

For $a = 0.8$ and $b = 0.9$, $\theta = 41.6°$.

Answer 2

We must have

$$\begin{aligned}
P &= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta N(r,\theta,\phi) r^2 \sin\theta \qquad (5)\\
C^{-1} &= \int_0^{2\pi} d\phi \sin^2\phi \int_0^{\pi} d\theta \sin^3\theta \\
C^{-1} &= \pi \int_0^{\pi} d\theta \sin\theta \left(1-\cos^2\theta\right) \\
&= \pi \int_{-1}^{1} du \left(1-u^2\right) \\
&= \pi \left[u - \frac{1}{3}u^3\right]_{-1}^{+1} \\
C &= \frac{3}{4\pi}.
\end{aligned}$$

Giving

$$N(r,\theta,\phi) = \frac{3}{4\pi} \frac{\sin^2\phi \sin^2\theta}{r^2} P, \qquad (6)$$

The maximum gain is given by

$$\begin{aligned}
N(r,\theta,\phi) &= \frac{\frac{3}{4\pi}\frac{1}{r^2}P}{\frac{P}{4\pi r^2}} \qquad (7)\\
&= 3 = 4.77\,\mathrm{dB}.
\end{aligned}$$

Answer 3

The detected power has the form

$$P(d) \propto 1 + \exp(-2d/c\tau_c)\cos(2d\omega_0/c). \tag{8}$$

The oscillation distance is $d_0 = \pi c/\omega_0 = \lambda_0/2$, and the exponential decay distance is $d_d = c\tau_c/2$.

Δf is the full width half maximum of the resonance, and so $1/\tau_c = \Delta f/2$. $Q = f_0/\Delta f = f_0\tau_c/2$, but $\tau = 2d/c$, and say we require $\tau = 2\tau_c$, then $Q = f_0\tau/4 = f_0 d/2c = d/2\lambda_0$. Basically we require $d > 2\lambda_0 Q$. Hence $d_{min} = 60$ m, which would be a challenge. This needs to be sampled at $\Delta d = \lambda_0/4$, $N_{min} = 8Q$, which would be about 8000.

Answer 4

The governing equation has the form

$$m\ddot{\mathbf{r}} = -e\left(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B}_0\right). \tag{9}$$

giving

$$\begin{aligned} x &= \frac{e}{m\omega^2}E_x - i\frac{\omega_c}{\omega}y \\ y &= \frac{e}{m\omega^2}E_y + i\frac{\omega_c}{\omega}x. \end{aligned} \tag{10}$$

The plasma is neutral, and so opposite charges are present. However, if the charges have very different mass, it is sufficient to consider the lightest mass particle only. In the above we have only take into account electrons. Also, time-harmonic behaviour is assumed.

The dipole moment per unit volume is

$$\mathbf{P} = -2ne\mathbf{r}, \tag{11}$$

and the susceptibility is defined through

$$\mathbf{P} = \epsilon_0 \underline{\underline{\chi}} \cdot \mathbf{E} \tag{12}$$

Considering only the transverse components:

$$\epsilon = \left[\begin{array}{cc} 1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2} & i\frac{\omega_c\omega_p^2}{\omega(\omega^2-\omega_c^2)} \\ -i\frac{\omega_c\omega_p^2}{\omega(\omega^2-\omega_c^2)} & 1 - \frac{\omega_p^2}{\omega^2-\omega_c^2} \end{array}\right]. \tag{13}$$

The matrix elements become

$$\epsilon = \left[\begin{array}{cc} 1 - \frac{\omega_p^2}{\omega^2 - \omega\omega_c} & 0 \\ 0 & 1 - \frac{\omega_p^2}{\omega^2+\omega\omega_c} \end{array}\right] \tag{14}$$

The permittivity tensor is diagonal in this basis, and the two circular polarisations propagate with different phase velocities.

For the LCP, cut off occurs when

$$\begin{aligned} 1 - \frac{\omega_p^2}{\omega^2 - \omega\omega_c} &= 0 \\ \omega^2 - \omega\omega_c - \omega_p^2 &= 0 \end{aligned} \tag{15}$$

and for RCP

$$\begin{aligned} 1 - \frac{\omega_p^2}{\omega^2 + \omega\omega_c} &= 0 \\ \omega^2 + \omega\omega_c - \omega_p^2 &= 0. \end{aligned} \tag{16}$$

These give

$$\omega = \pm\frac{\omega_c}{2} + \left[\left(\frac{\omega_c}{2}\right)^2 + \omega_p^2\right]^{1/2}. \tag{17}$$

The two cut-off frequencies are LCP = 10.5 MHz and RCP = 9.5 MHz, and so the cut-off frequency splits, one becoming higher and the other lower.

At 10 Mhz, therefore, RCP would emerge because the other is cut off.

Actually, there is also a pole at 1 MHz, below which the LCP will start propagating again.

Answer 5

Retarded time has the functional form $F(r, r', t - |r - r'|/c)$. The argument $t - |r - r'|/c$ is the difference between the observation time and the time that the disturbance would have taken to reach the observation point.

Taking $r' = 0$,

$$\begin{aligned} \frac{\partial [F]}{\partial r} &= \frac{\partial F}{\partial r} - \frac{1}{c}\frac{\partial F}{\partial t} \\ \frac{\partial [F]}{\partial t} &= \frac{\partial F}{\partial t}. \end{aligned} \tag{18}$$

In spherical coordinates:

$$\begin{aligned} A_r &= \frac{\mu_0}{4\pi r}[\dot{p}]\cos\theta \\ A_\theta &= -\frac{\mu_0}{4\pi r}[\dot{p}]\sin\theta \\ A_\phi &= 0. \end{aligned} \tag{19}$$

Using $\mathbf{B} = \nabla \times \mathbf{A}$ in spherical coordinates gives

$$B_\phi = \frac{\mu_0}{4\pi}\sin\theta\left\{\frac{[\dot{p}]}{r^2} + \frac{[\ddot{p}]}{rc}\right\}. \tag{20}$$

Now $\mathbf{E} = -\dot{\mathbf{A}} - \nabla\phi$

$$\begin{aligned} E_r &= -\frac{\mu_0\cos\theta}{4\pi r}[\ddot{p}] - \frac{\cos\theta}{4\pi\epsilon_0}\left\{-2\frac{[p]}{r^3} - \frac{[\dot{p}]}{r^2 c}\right\} + \frac{\cos\theta}{4\pi\epsilon_0 c}\left\{\frac{[\dot{p}]}{r^2} + \frac{[\ddot{p}]}{rc}\right\} = \frac{2\cos\theta}{4\pi\epsilon_0}\left\{\frac{[p]}{r^3} + \frac{[\dot{p}]}{r^2 c}\right\} \\ E_\theta &= \frac{\mu_0\sin\theta}{4\pi r}[\ddot{p}] + \frac{\sin\theta}{4\pi\epsilon_0}\left\{\frac{[p]}{r^3} + \frac{[\dot{p}]}{r^2 c}\right\} = \frac{\sin\theta}{4\pi\epsilon_0}\left\{\frac{[p]}{r^3} + \frac{[\dot{p}]}{r^2 c} + \frac{[\ddot{p}]}{rc^2}\right\} \\ E_\phi &= 0. \end{aligned} \tag{21}$$

We have

$$\begin{aligned} [p_0] &= e^{-(t-r/c)/\tau} \\ [\dot{p}_0] &= \frac{-1}{\tau}e^{-(t-r/c)/\tau} \\ [\ddot{p}_0] &= \frac{1}{\tau^2}e^{-(t-r/c)/\tau}. \end{aligned} \tag{22}$$

In the far field

$$\begin{aligned} B_\phi &= \frac{\mu_0}{4\pi}\sin\theta\left\{\frac{[\ddot{p}_0]}{rc}\right\} \\ E_\theta &= \frac{\sin\theta}{4\pi\epsilon_0}\left\{\frac{[\ddot{p}_0]}{rc^2}\right\}, \end{aligned} \tag{23}$$

and the Poynting flux becomes

$$\begin{aligned} N &= \frac{\mu_0}{16\pi^2 c}\sin^2\theta\frac{[\ddot{p}_0]^2}{r^2} \\ &= \frac{\mu_0}{16\pi^2 c}\sin^2\theta\frac{1}{r^2}\frac{1}{\tau^4}e^{-2(t-r/c)/\tau}. \end{aligned} \tag{24}$$

There is no power prior to the start of the decay at $t = r/c$. The signal turns on abruptly, and then decays with a time constant that is one half of the radiating dipole moment. The peak power scales as $1/\tau^4$ and so can be appreciable for short pulses even at large distances.