Part II Physics : 2021

ELECTRODYNAMICS AND OPTICS

EXAMPLES

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Polarization

1. Without using any special device, how can the direction of the polarizing axis of a polaroid sheet be determined?

2. What are the time- and space-dependences of the electric and magnetic fields for light with the complex representations $\boldsymbol{E}_+ = \boldsymbol{E}_1 + i\boldsymbol{E}_2$ and $\boldsymbol{E}_- = \boldsymbol{E}_1 - i\boldsymbol{E}_2$, where $\boldsymbol{E}_{1,2}$ represent the electric fields of $\pm z$-going waves of angular frequency ω, linearly-polarized along the x- and y-directions respectively, and $|\boldsymbol{E}_1| = |\boldsymbol{E}_2|$.

   Classify $\boldsymbol{E}_\pm$ as describing LCP or RCP light, distinguishing clearly between the cases of positive and negative wavevector k_z.

3. Show that the Jones matrix for a polaroid sheet with its transmitting axis at an angle θ to the x-axis is
$$\underline{\underline{\boldsymbol{J}}}_\theta = \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}$$

4. Explain how a uniaxial birefringent material can be used to make a quarter-wave plate. If the principal refractive indices are $n_\mathrm{o}, n_\mathrm{o}$ and n_e, what is the minimum thickness the plate can be for light of free-space wavelength λ?

5. A light beam is elliptically polarized with an axial ratio of 3 to 1 and the major axis vertical. What are the possible orientations of linearly polarized light which can be obtained by passing the beam through a quarter-wave plate?

6. When a light beam is passed through a linear polarizing filter, maximum and minimum intensities (5 and 2 units respectively) are found for vertical and horizontal planes of polarization. When the beam is passed through a quarter-wave plate with the fast axis vertical and then through the polarizing filter the maximum intensity is found at an angle of 26.6° to the vertical. What intensity is transmitted in this case? What is the degree of polarization before and after passing through the quarter-wave plate?

7. Young's double slit arrangement is illuminated with plane polarized, monochromatic light. The slits are covered by quarter-wave plates oriented to produce circular polarizations of opposite handedness. The "fringe" system is observed through a plane polarizing filter. What is observed as this filter is rotated?

   [Hint: Consider the polarization state you get when superposing RCP and LCP waves with a phase difference ϕ between them]

8. (a) Using Jones matrices, show that ideal crossed linear polarizers extinguish light of any polarization.

   (b) A quarter-wave plate is inserted between crossed polarizers, with its fast axis at an angle θ to the transmission axis Ox of the first polarizer. What is the resulting Jones matrix?

   (c) If unpolarized light of intensity I is incident on this system, how does the transmitted intensity depend on θ? What is the maximum transmitted intensity, and at what values of θ does this occur?

9. A non-magnetic uniaxial crystal has principal refractive indices $n_{\mathrm{o}}, n_{\mathrm{o}}$ and n_{e} in a Cartesian co-ordinate system with the z-axis aligned with the optic axis. A plane-wave with wavevector $\boldsymbol{k} = k(\sin\theta, 0, \cos\theta)$ has fields

$$\boldsymbol{D}(\boldsymbol{r}, t) = D(-\cos\theta, 0, \sin\theta)e^{i\boldsymbol{k}\cdot\boldsymbol{r}-i\omega t}$$
$$\boldsymbol{H}(\boldsymbol{r}, t) = H(0, 1, 0)e^{i\boldsymbol{k}\cdot\boldsymbol{r}-i\omega t}$$

   (a) Find the corresponding $\boldsymbol{E}$ and $\boldsymbol{B}$ fields, and show that these fields satisfy all of Maxwell's Equations provided the speed of the wave is c/n_{b}, with

$$\frac{n_{\mathrm{b}}^2 \sin^2\theta}{n_{\mathrm{e}}^2} + \frac{n_{\mathrm{b}}^2 \cos^2\theta}{n_{\mathrm{o}}^2} = 1$$

   (b) Find the direction of the Poynting vector and discuss your answer.

10. Born has demonstrated that a "molecule" composed of four identical polarizable spheres, interacting by Coulomb fields, can produce optical rotation. Would this be

true if the spheres were at the corners of a regular tetrahedron? Might one expect optical rotation from a tri-atomic molecule?

11. An EM cavity of length L in the z-direction is formed between two perfect plane mirrors. What are the frequencies of the cavity modes (*i.e.* the standing waves of light with wavevector parallel to the z-axis) in the cases where the cavity contains:

    (a) a non-magnetic uniaxial crystal with principal refractive indices n_{o} and n_{e} with the optic axis aligned (i) parallel; (ii) perpendicular to the z-axis;

    (b) a Faraday medium with average refractive index n and Verdet constant $\mathcal{V}$ with a magnetic field $\boldsymbol{B}$ along the z-axis.

    [Hint: Using a Jones vector to specify each wave, write down a general solution for the electric field in the cavity in terms of $+z$ and $-z$-going waves of each polarized wave, remembering that each may have a different wavevector k. Use the boundary conditions at $z = 0$ and $z = L$ to establish necessary relations between the ks for each wave, and then substitute $k = n_? \omega/c$, where $n_?$ is the refractive index appropriate for each wave. $n_?$ in each case requires some thought.]

12. Show that for a plasma in a magnetic field $\boldsymbol{B}$ at low frequencies ($\omega \ll \omega_{\mathrm{c}}, \omega \ll \omega_{\mathrm{p}}^2/\omega_{\mathrm{c}}$, where ω_{p} and ω_{c} is the plasma and cyclotron frequencies) there exist propagating RCP modes with wavevector $\boldsymbol{k} \parallel \boldsymbol{B}$, but no such propagating LCP modes. Derive the dispersion relation of the propagating modes at low frequency, and obtain an expression for their group velocity.

    [The propagating modes are "helicon" modes, and are responsible for "whistlers", a characteristic type of audio-frequency radio interference most commonly encountered at high latitudes. They are brief, intermittent pulses at audible frequencies, rapidly descending in pitch.]

Coherence

13. A star ejects atomic hydrogen in the form of a thin luminous gaseous shell. A Michelson interferometer is used as a Fourier transform spectrometer to examine radiation from an area of sky near the star so as to include contributions from the front and back of the shell but not from the star itself. The visibility curve obtained for the H spectral line of wavelength 656 nm is sketched below. Explain its general form and *estimate* the velocity of expansion of the shell, and the apparent linewidth. If the linewidth arises from thermal broadening, what temperature is implied?

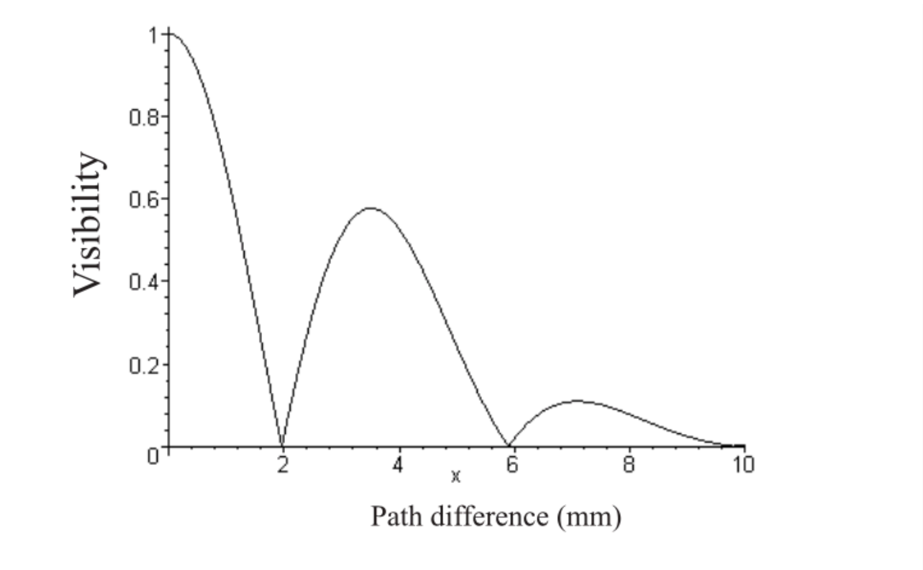

Figure 1: Fringe visibility as a function of optical path difference.

14. A Michelson interferometer forms fringes with cadmium red light of wavelength 643.847 nm and linewidth 0.0013 nm.

(a) Estimate the coherence length for the light assuming a gaussian line-shape.

(b) Estimate the visibility of the fringes when one mirror is moved by distances d = 10 mm and 50 mm from the position of zero path difference between the arms.

(c) If the line-shape were a top-hat function, at what mirror position d would the visibility first fall to zero?

15. A long hot wire of width $w = 0.1$ mm is placed in the focal plane of a lens of focal length $f = 100$ mm. Light from the glowing wire passes through the lens and a filter which transmits only a very small range of wavelengths near $\lambda = 600$ nm, and falls onto a screen placed normal to the axis behind the lens. Show that the degree of lateral coherence γ for light arriving at two points on the screen separated by d in a direction perpendicular to the wire is:

$$\gamma(d) = \operatorname{sinc}\left(\frac{\pi w d}{f\lambda}\right)$$

What is the smallest separation d for which the degree of coherence is zero?

Electrodynamics

16. Starting from the relevant expressions for the magnetic flux density components

$$B_r = \frac{\mu_0 m \cos\theta}{2\pi r^3} \qquad B_\theta = \frac{\mu_0 m \sin\theta}{4\pi r^3}$$

find a magnetic vector potential $\boldsymbol{A}(\boldsymbol{r})$ suitable for describing the fields due to a static magnetic dipole moment $\boldsymbol{m}$ aligned along Oz.

17. Show that (b) below represents the components of a real magnetic flux density, whereas (a) does not. For the case (b) find the current density distribution $\boldsymbol{J}$ required to produce the field, and the corresponding vector potential $\boldsymbol{A}$.

(a) $\dfrac{B_0 b}{r^3}\left((x-y)\,z,\ (x-y)\,z,\ x^2-y^2\right)$ in Cartesian co-ordinates.

(b) $B_0 b^2\left(\dfrac{zr}{(b^2+z^2)^2},\ 0,\ \dfrac{1}{b^2+z^2}\right)$ in *cylindrical* polar coordinates.

Radiation

18. A dipole antenna is enclosed in a sealed plastic box and radiates at a wavelength of 10 cm. Suggest (non-destructive) experiments to determine its orientation and whether it is an electric or a magnetic dipole.

How could this information be deduced if the dipole antenna is disconnected from any power supply and its terminals shorted with a matched load?

19. A magnetic dipole at the origin lies in the x-y-plane and is rotating at constant angular frequency about the z-axis.

(a) Show that in the x-y-plane the radiation pattern is circular and the emitted radiation is polarized parallel to Oz.

(b) What is the polarization of the radiation emitted at an angle θ to the rotation axis Oz?

20. A pulsar is can be represented as a constant magnetic moment $\boldsymbol{M}$ rotating with an angular velocity ω in vacuum about an axis perpendicular to $\boldsymbol{M}$. By considering the rotating magnet as equivalent to two orthogonal magnets varying in phase quadrature, show that the energy loss due to radiation causes ω to obey the equation $\omega\ddot{\omega} = 3\dot{\omega}^2$.

[Assume the formula for radiation loss for a magnetic dipole, and that $\dot{\omega} \ll \omega^2$.]

For the pulsar in the Crab Nebula, the period $T = 33$ ms and $\dot{T} = 36$ ns/day. Assuming that it is a sphere (of uniform density) with radius 7 km and a mass equal to that of the Sun (2×10^{30} kg), estimate $\boldsymbol{M}$ and hence the magnetic field at its equator.

21. What are meant by the *radiation resistance* and *power gain* of an antenna? For an antenna which consists of a plane loop of wire of area a^2 operating at a frequency $\omega \ll c/a$, calculate the radiation resistance and power gain.

Radiation, linearly-polarized with its magnetic field perpendicular to the plane of the loop, is incident from a direction in that plane. Find the cross-section for combined scattering and absorption when the antenna is connected to a matched load.

[Assume without proof that such a loop radiates mean power $\mu_0 I_0^2 \omega^4 a^4 / (12\pi c^3)$ when a current $I_0 \cos \omega t$ flows in it.]

22. Estimate the number of molecules above each m^2 of the Earth's surface. Hence, on the assumption that a molecule can be represented as a perfectly conducting sphere of radius 0.1 nm, estimate the reduction in the (ultra-violet) radiation with wavelength $\lambda = 320$ nm arriving from the Sun at noon on the Equator due to scattering by the atmosphere.

[Assume that half of the radiation scattered still reaches the Earth's surface and ignore multiple scattering.]

23. Estimate the degree of polarization of the overhead sky at noon on a clear spring day (say March 21) in Cambridge.

Relativistic Electrodynamics and Radiation

24. In the standard arrangement of co-ordinate frames, a slab of non-dispersive medium with refractive index n is at rest in S$'$ and has thickness d in the Oz' direction. In the laboratory frame S the slab moves along the x-axis with speed $v = \beta c$.

    (i) Show that, for light moving through the slab parallel to its motion, the refractive index as determined in S is:

    $$n_v = \frac{n+\beta}{n\beta+1} \approx n - (n^2-1)\beta \quad \text{for small } \beta$$

    (ii) Show that a ray of light falling on the slab at normal incidence in S passes through the slab at an angle $\alpha = \sin^{-1}(\beta/n)$ to the Oz' axis as measured in S$'$. Hence show that in frame S the ray is laterally displaced by a distance:

    $$a = \gamma d \beta \frac{n^2-1}{\sqrt{n^2-\beta^2}}$$

    after passing through the slab. [$\gamma = 1/\sqrt{1-\beta^2}$.]

25. A particle with charge q and rest mass m moves in an electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$. Show that its 4-velocity is $(\gamma c, \tilde{\boldsymbol{u}} c)$, where $\dfrac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}\tau} = c\tilde{\boldsymbol{u}}$ and τ is the proper time measured by an observer moving with the particle. [Note that $\tilde{\boldsymbol{u}}$ is not the 3-velocity $\boldsymbol{u}$.]

    Show also that the particle's equation of motion can be written in the form:

    $$\frac{\mathrm{d}\tilde{\boldsymbol{u}}}{\mathrm{d}\tau} = \frac{q}{mc}\left(\gamma \boldsymbol{E} + c\tilde{\boldsymbol{u}} \times \boldsymbol{B}\right)$$

    and that

    $$\frac{\mathrm{d}\gamma}{\mathrm{d}\tau} = \frac{q}{mc}\boldsymbol{E} \cdot \tilde{\boldsymbol{u}}$$

26. (i) A parallel plate capacitor is at rest in S$'$ with its plates perpendicular to Oy'. One plate at $y' = 0$ has zero potential and the other at $y' = s$ is at potential $-V_0$. Write down the 4-potential between the plates in S$'$. Transform the 4-potential to the frame S in which the capacitor is moving at speed $c\beta$ along the positive x-direction. Hence find $\boldsymbol{E}$ and $\boldsymbol{B}$ between the plates in S, and interpret your results.

    (ii) Repeat (i) when the plates are perpendicular to Ox', with the plate at $x' = 0$ at zero potential and the other at $x' = s$ at potential $-V_0$.

    Are the results for (i) and (ii) consistent when two such perpendicularly oriented capacitors are considered together in S$'$, electrically connected in parallel?

27. Electric and magnetic fields, $\boldsymbol{E}$ and $\boldsymbol{B}$, as viewed in frame S are such that when they are examined in frame S$'$, which is moving relative to S with constant velocity v along the common x-axis, the electric field is unaltered but the magnetic field is reversed. Show that the two fields are perpendicular in both frames and that, if $\boldsymbol{E} = (E_x, E_y, E_z)$, then

$$\boldsymbol{B} = \frac{\gamma - 1}{\gamma v}\,(0, -E_z, E_y)$$

28. A plane conducting non-magnetic sheet of uniform thickness is oriented normal to the y and y' axes in frames S and S$'$. S$'$ moves at a constant velocity v in the positive x-direction in S, and the sheet is at rest in S$'$. In S, there is a uniform magnetic field having components $(0, 0, B)$, and the electric field is zero. Transform these fields to S$'$ and hence deduce:

    (a) the fields $\boldsymbol{E}'_0$ and $\boldsymbol{B}'_0$ outside the sheet;
    (b) the fields $\boldsymbol{E}'$ and $\boldsymbol{B}'$ inside the sheet;
    (c) the surface charge density in S$'$ on either side of the sheet.

    Transform the results for (b) back to S, and hence find $\boldsymbol{E}$ and $\boldsymbol{B}$ inside the sheet. Interpret these results from the point of view of an observer at rest in S.

29. Derive expressions for the fields due to a uniformly moving charge.

    Two particles, each with charge q, are constrained to move with the same constant velocity v parallel to the x-axis. Find the force on each due to the presence of the other when, instantaneously, one is at the origin and the other is at:

    (a) $(r, 0, 0)$;
    (b) $(0, r, 0)$;
    (c) $\left(r/\sqrt{2}, r/\sqrt{2}, 0\right)$.

30. (i) By direct substitution, show that the quantities $|\boldsymbol{E}|^2 - c^2|\boldsymbol{B}|^2$ and $\boldsymbol{E}\cdot\boldsymbol{B}$ are invariant under the Lorentz Transformations.

    (ii) A region of space contains isotropic electromagnetic radiation of mean energy density $\langle U\rangle$. Show that an observer moving in the x-direction with speed v through the region will measure a mean energy density $\langle U'\rangle$ and mean Poynting vector $\langle \boldsymbol{N'}\rangle$ given by:

$$\langle U'\rangle = \langle U\rangle\left(\frac{4\gamma^2 - 1}{3}\right) \qquad\qquad \langle \boldsymbol{N'}\rangle = \left(-\frac{4v\langle U\rangle\gamma^2}{3}\ ,0\ ,0\right)$$

    where $\langle\ldots\rangle$ denotes time averaging.

31. A 1 GeV electron passes into a region in which there is a transverse magnetic field of 1 T. Estimate the typical frequency of the synchrotron radiation emitted, and the fractional loss of energy per orbit.

32. (For enthusiasts only.) A particle of charge q and rest mass m is placed in an electromagnetic plane wave described by:

$$\begin{aligned} E_y &= cB_0 \sin\left[\omega\left(t - x/c\right)\right], E_x = E_z = 0 \\ B_z &= B_0 \sin\left[\omega\left(t - x/c\right)\right], B_x = B_y = 0 \end{aligned}$$

If the wave is "strong"(i.e. $qB_0/m\omega > 1$) it attains relativistic speeds and the force on it arising from the wave's magnetic field cannot be ignored. Applying the results of Q.25, write down the equations of motion.

If the particle is introduced at rest at $x = t = \tau = 0$, show that:

(a) $\tilde{u}_z = 0$;

(b) $\tilde{u}_x = \gamma - 1$;

(c) $(t - x/c) = \tau$, where x refers to the particle's position at time τ.

Find the maximum energy E_{max} attained by the particle and sketch its motion.

Answers

A4. $l_{\rm min} = \dfrac{\pi\ c}{2\omega|n_{\rm e} - n_{\rm o}|} = \dfrac{\lambda}{4|n_{\rm e} - n_{\rm o}|}$.

A5. $\pm 18.4^\circ$ to the vertical.

A6. 6 units; 5/7, 5/7.

A8. (b) $\underline{\underline{\boldsymbol{J}}}_{\rm tot} = \begin{pmatrix} 0 & 0 \\ (1-i)\sin\theta\cos\theta & 0 \end{pmatrix}$.

(c) $I_{\rm out} = I\sin^2(2\theta)/4$. $I_{\rm out}^{\rm max} = I/4$, at $\theta = \pi/4 + n\pi/2$.

A9. (a) $\boldsymbol{E}(\boldsymbol{r},t) = D\left(-\dfrac{\cos\theta}{\epsilon_0 n_{\rm o}^2}, 0, \dfrac{\sin\theta}{\epsilon_0 n_{\rm e}^2}\right)\ e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$: $\boldsymbol{B}(\boldsymbol{r},t) = \mu_0 H(0,1,0)\ e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$.

(b) $\boldsymbol{N} = +\dfrac{cD^2}{\epsilon_0 n_{\rm b}}\cos^2(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)\left(\dfrac{\sin\theta}{n_{\rm e}^2}, 0, \dfrac{\cos\theta}{n_{\rm o}^2}\right) \nparallel \boldsymbol{k}$ unless $\theta = \dfrac{n\pi}{2}$.

A11. (a) (i) $\omega = \dfrac{\pi c}{n_{\rm o} L}\ell$; (ii) $\omega = \dfrac{\pi c}{n_{\rm o} L}\ell$ and $\dfrac{\pi c}{n_{\rm e} L}\ell$, where ℓ is an integer.

(b) $\omega = \dfrac{\pi c}{nL}\ell \pm \dfrac{\mathcal{V}Bc}{n}$.

A12. $\omega = c^2 k^2 \dfrac{\omega_{\rm c}}{\omega_{\rm p}^2}$; $v_{\rm g} = 2c\sqrt{\dfrac{\omega\omega_{\rm c}}{\omega_{\rm p}^2}}$, increasing with ω.

A13. $v \sim 2.4 \times 10^4$ m s^{-1}; $\delta\lambda \sim 4 \times 10^{-2}$ nm $\longrightarrow T \sim 10,000$ K.

A14. (a) $l_{\rm c} \sim 160$ mm; (b) $V \sim 0.98$, $V \sim 0.69$; (c) ~ 160 mm.

A15. 0.6 mm.

A16. $\boldsymbol{A} = \left(0,\ 0,\ \dfrac{\mu_0 m\sin\theta}{4\pi r^2}\right)$.

A17. $\boldsymbol{J} = \left(0,\ \dfrac{B_0 b^2 r}{\mu_0}\ \dfrac{b^2 - 3z^2}{\left(b^2+z^2\right)^3}\ ,\ 0\right)\ ;\quad \boldsymbol{A} = \left(0,\ \dfrac{B_0 b^2 r}{2\left(b^2+z^2\right)}\ ,\ 0\right).$

A20. 10^{27} A m^2; 10^9 T.

A21. $R_\mathrm{r} = \dfrac{\mu_0 \omega^4 a^4}{6\pi c^3}\ ;\ G(\theta,\phi) = \frac{3}{2}\sin^2\theta\ ;\ A_\mathrm{tot} = 3\pi c^2/\omega^2\ .$

A22. $\sim 2\times 10^{29}$ m^{-2}; $\sim 13\%$.

A23. $\sim 45\%$.

A26. (i) $\boldsymbol{E} = (0, \gamma V_0/s, 0)$; $\boldsymbol{B} = (0, 0, \beta\gamma V_0/cs)$.

(ii) $\boldsymbol{E} = (V_0/s, 0, 0)$; $\boldsymbol{B} = (0,0,0)$.

A28. (a) $\boldsymbol{E}_0' = (0, -\gamma v B, 0)$; $\boldsymbol{B}_0' = (0,0,\gamma B)$;

(b) $\boldsymbol{E}' = (0,0,0)$; $\boldsymbol{B}' = \boldsymbol{B}_0' = (0,0,\gamma B)$;

(c) $\sigma'_\mathrm{top} = +\epsilon_0 E'_{0y'} = -\epsilon_0\gamma v B$, $\sigma'_\mathrm{bot} = -\epsilon_0 E_{0y'} = +\epsilon_0\gamma v B$;

(d) $\boldsymbol{E} = (0, \gamma^2 v B, 0)$; $\boldsymbol{B} = (0,0,\gamma^2 B)$.

A29. (a) F/γ^2 along the x-axis, where $F = q^2/4\pi\epsilon_0 r^2$;

(b) F/γ along the y-axis;

(c) $2\gamma^{-1}F\left[\left(\gamma^4+1\right)/\left(\gamma^2+1\right)^3\right]^{1/2}$ at $\tan^{-1}\left(1/\gamma^2\right)$ to the x-axis.

A31. $\sim 10^{17}$ Hz (most emission is rather below ν_s) $\equiv 3$ nm, soft X-rays; fractional loss per orbit $\sim 3\times 10^{-5}$.

A32. $E_\mathrm{max} = mc^2[1 + 2\left(qB_0/m\omega\right)^2]$.