Part II Experimental and Theoretical Physics
Astrophysical Fluids — Answers — 2018

Q1.1. Streamlines have $\dot{R} = b; R\dot{\phi} = a$ so $\mathrm{d}R/R = (b/a)\mathrm{d}\phi \Rightarrow \log r = (b/a)\phi + \log r_0$, $r = r_0 \exp((b/a)\phi)$. The streamlines are spirals.

The flow is steady so $\partial_t \boldsymbol{u} = 0$. The continuity equation then can be written as $\boldsymbol{u}\cdot\boldsymbol{\nabla}\rho = -\rho\boldsymbol{\nabla}\cdot\boldsymbol{u}$. In cylindrical polars $\boldsymbol{\nabla}\cdot\boldsymbol{u} = R^{-1}\partial_R(Ru_r) + R^{-1}\partial_\phi u_\phi$, so here $\boldsymbol{\nabla}\cdot\boldsymbol{u} = b/R$ and $b\partial_r\rho = -\rho b/R \Rightarrow \log\rho = -\log R +$ constant. Hence $\rho \propto R^{-1}$.

The other flow has $u_R = bR^2, U_\phi = aR^2$, so the equation for the streamlines is the same as for the first flow and the streamlines are spirals again. However, $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 3bR$ and this time we get $\rho \propto R^{-3}$.

Q1.2. The continuity equation $\partial_t\rho + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}) = 0$ can be written as $\dfrac{D\rho}{Dt} + \rho\boldsymbol{\nabla}\cdot\boldsymbol{u}$ so, if $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$ the density doesn't change along the streamline.

This doesn't mean that the density is constant, since it can vary across the streamlines.

Q1.3. Rewriting the flow in polars we get $\boldsymbol{u} = U(1 - a^2/R^2)\cos\phi\ \hat{\boldsymbol{e}}_R - U(1 + a^2/R^2)\sin\phi)\ \hat{\boldsymbol{e}}_\phi$. Using $\dot{R} = u_R, R\dot{\phi} = u_\phi$ we get

$$\frac{\mathrm{d}R(1 + a^2/R^2)}{R(1 - a^2/R^2)} = -\frac{\cos\phi\ \mathrm{d}\phi}{\sin\phi} = -\mathrm{d}\log\sin\phi \tag{1}$$

Integrating we find $\log(R - a^2/R) = -\log\sin\phi +$ constant, so $(R - a^2/R)\sin\phi =$ constant and represents the flow around a cylinder.

Q1.4. From $u_x = 2/x, u_y = 1$ we get $\frac{1}{2}x\mathrm{d}x = \mathrm{d}y$ so $y = \frac{1}{4}(x^2 - x_0^2) + y_0$. Proceeding as in Q1.1 we find $\boldsymbol{\nabla}\cdot\boldsymbol{u} = -2/x^2$ and

$$\frac{2}{x}\frac{\partial\rho}{\partial x} + \frac{\partial\rho}{\partial y} = \rho\frac{2}{x^2} \tag{2}$$

Separating variables so that $\rho(x, y) = X(x)Y(y)$ we get

$$\frac{Y'}{Y} = -\frac{2}{x}\frac{X'}{X} + \frac{2}{x^2} = C \tag{3}$$

where C is a constant. We find $Y \propto e^{Cy}$ and $\log X = -\frac{1}{4}Cx^2 + \log x +$ constant. Bearing in mind that both signs of x are possible, we get

$$\rho(x, y) \propto |x|e^{-\frac{1}{4}Cx^2}e^{Cy} \tag{4}$$

and substituting for y we write the solution along the streamline $\rho(x, y) = \dfrac{\rho_0}{x_0}|x|$ where the x^2 terms have cancelled in the exponential and the constant ones absorbed into ρ_0.

Write the concentration $C = Q\rho = \dfrac{Q_0\rho_0}{x_0}|x|e^{-t}$. We need to find t, but we can do that from $\dot{x} = 2/x \Rightarrow t = \frac{1}{4}(x^2 - x_0^2)$. We get

$$\rho(x, y) = \frac{\rho_0}{x_0}|x| \exp(\tfrac{1}{4}(x_0^2 - x^2)) \propto |x| \exp(-\tfrac{1}{4}x^2) \tag{5}$$

This function has a maximum at $x = \pm\sqrt{2}$ so, if the flow is started from $|x_0| < \sqrt{2}$ the concentration will reach a maximum before decaying again.

Q1.5. This is too easy, but I will say that; (a) needs brackets; (b) doesn't need brackets, so might consider lending (a) some and (c) is just Leibnitz. None of it needs suffix notation. Real men don't use suffix notation...

The second part is more instructive. From $\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} = \frac{1}{2}\boldsymbol{\nabla}\boldsymbol{u}^2 - \boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{u})$ we find

$$\partial_t \boldsymbol{u} = \boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{u}) - \boldsymbol{\nabla}(\tfrac{1}{2}u^2) - \boldsymbol{\nabla}\Phi - \frac{1}{\rho}\boldsymbol{\nabla}P \tag{6}$$

Taking the curl, and assuming $\boldsymbol{\nabla}\times\boldsymbol{u} = 0$ at $t = 0$ it nearly all vanishes immediately: $\partial_t(\boldsymbol{\nabla}\times\boldsymbol{u}) = 0 + 0 + 0 + 0 - \boldsymbol{\nabla}(1/\rho)\times\boldsymbol{\nabla}P$. However, if $P = P(\rho)$ then $\boldsymbol{\nabla}P \parallel \boldsymbol{\nabla}\rho$ and the vorticity remains zero.

Q1.6. Using Gauss' theorem we find $\boldsymbol{g} = -4\pi G\rho z\,\hat{\boldsymbol{e}}_z$ for $-a < z < a$ and $|\boldsymbol{g}| = -4\pi G\rho a$ otherwise. Orbits are SHM with period $\sqrt{\dfrac{\pi}{G\rho}} = 6.88 \times 10^6$ yr, and we find a velocity of 29 km s^{-1} at the mid point.

Using $\boldsymbol{\nabla}P = \rho\boldsymbol{g}$, the pressure distribution is $P(z) = 2\pi G\rho^2(a^2 - z^2)$.

Q1.7. Bernoulli says: $\frac{1}{2}v^2 + \dfrac{u + P}{\rho} + \Phi =$ constant along a streamline. Here $u = P/(\gamma - 1)$ is the internal energy per unit volume and I have reverted to using $\boldsymbol{v}$ for velocity. If the flow is dominated by the gravity then we set $u + P = 0$ and get

$$v^2 = v_0^2 + 2GM\left(\frac{1}{r} - \frac{1}{r_0}\right) \tag{7}$$

Continuity gives $\rho v r^2 = \rho_0 v_0 r_0^2$ so

$$\rho = \frac{\rho_0 r_0^2 v_0}{r^2\sqrt{v_0^2 + 2GM(1/r - 1/r_0)}} \tag{8}$$

The temperature is related to the density by $T = T_0(\rho/\rho_0)^{\gamma-1}$.

If the initial velocity is the escape velocity from r_0, the results simplify and give

$$\rho = \frac{\rho_0 r_0^2 v_0}{\sqrt{2GM}r^{3/2}} \tag{9}$$

and $T \propto r^{-1}$.

If the gas is very hot at r_0 then thermal effects are important at first, but the rapid cooling as the gas expands means that the flow becomes very supersonic. For the case given we find $P \propto R^{-5/2}$ and $\nabla P \propto R^{-7/2}$.

This question provides an instructive and astrophysically accurate model of the solar wind.

Q1.8. Case (i) is an extreme Kepler orbit with $\omega^2 = GM/a^3$ with $a = r_0/2$. Time to the centre is a half period so $T = \sqrt{\pi^2 r_0^3/8GM}$.

Alternatively, you can solve the equation $\mathrm{d}r/\mathrm{d}t = \sqrt{2GM(1/r - 1/r_0)}$. I'd suggest the substitution $r = r_0 \sin^2 u$, yielding $2\sin^2 u \mathrm{d}u = \sqrt{2GM}\mathrm{d}t$. The integral of the LHS is $\pi/2$, hence result.

Case (ii) is SHM with $\omega^2 = GM/r_0^3$. Time to the centre is a quarter period so $T = \sqrt{\pi^2 r_0^3/4GM}$.

The mass inside any shell doesn't change as the infall proceeds, so the answer is the same as case (i) above. Since $M(r) \propto r^3$ the time for all shells is the same and they all arrive at once. In terms of the density the answer is $T = \sqrt{3\pi/32G\rho}$.

Q1.9. (i) The scale height of the atmosphere is about $kT/\mu g = 8.7$ km for a mean molecular weight of 29. The lectures give a definition of a "fluid" having $nl^3 > 1$, where l is the scale of interest (here 9 km). Using a exponential density model, we get about 500 km. However, the fluid is a collisionless one by that stage. The mean free path is $\lambda = 1/(\sigma n)$, and taking $4\pi r_0^2$ with $r_0 = 10^{-10}$ m we find that the mean free path becomes about equal to the scale height at a density of 9×10^{14} m^{-3}, which is about 24 scale heights or 200 km. The constant gravity assumption breaks down at about 1500 km.

(ii) The worst case scenario is when the cloud has enough ram pressure ρv^2 to strip our atmosphere within 7 minutes (being the time that the Earth takes to immerse at our orbital velocity). Using $\overline{m} = 2m_{\mathrm{p}}$ we get $n = 7 \times 10^{22}$ m^{-3}. Clearly any appreciable fraction of that density would be disastrous, and would strip the atmosphere of the necessary oxygen. If the density was 10^{18} m^{-3} the atmosphere would last a year.... Lots of interesting discussion points here (neutrals, magnetospheres, stand-off shocks, mechanism of ablation.)

To affect the ozone layer (where the pressure is 0.1% atmospheric) scale down the numbers accordingly.

At the other end of the scale, the ram pressure of the solar wind is rather variable, and solar storms regularly dump the contents of our radiation belts (causing aurorae and magnetic storms). This gives $n = 10^8$ m^{-3}, which is on the low side for a molecular cloud. A proper molecular cloud is around $n = 10^{11}$ m^{-3}, which would be more interesting.

Q1.10. A very good estimate of the temperature is (Bernoulli, or method of dimensions) $\dfrac{kT}{\mu} = \dfrac{GM}{R} = 1.15 \times 10^7$ K using solar parameters ($R_S = 7 \times 10^8$ m) and

$\mu = 0.5m_{\rm p}$ (allowance for electrons in the average mass).

A reasonably good estimate of the pressure (method of dimensions) is $P = GM^2/R^4 = 1.1 \times 10^{15}$ Pa. setting $P = \frac{1}{3}aT^4$ yields $T = 4.6 \times 10^7$ K. The first question on the next sheet suggests that the pressure is $P = \dfrac{3GM^2}{8\pi R^4}$.

Thank goodness the Sun isn't actually supported by radiation pressure, because such stars are very unstable.

Q2.1. The gravity inside the planet is $\frac{4}{3}\pi G\rho r$, so the pressure is $P = \frac{2}{3}\pi G\rho^2(a^2 - r^2)$ (see the last question on the first sheet...). This fixes the radius as $a = \sqrt{\dfrac{3P}{2\pi G\rho^2}}$, and the mass as $M = \frac{4}{3}\pi a^3 = \dfrac{2}{3\sqrt{2\pi}\rho^2}\left(\dfrac{3P}{G}\right)^{3/2}$.

Q2.2. The radial equilibrium is $\Omega^2 = GM/R^3$. The tidal acceleration out of the plane is $g(z) = GMz/R^3$ and, using $\dfrac{\mathrm{d}P}{\mathrm{d}z} = \dfrac{kT}{\mu}\dfrac{\mathrm{d}\rho}{\mathrm{d}z} = -g\rho$ we get $\rho \propto \exp\left(-\frac{\mu GM}{2kTR^3}z^2\right)$, a Gaussian. The scale height H is $\sqrt{\frac{kT}{\mu\Omega^2}} \ll R$. For $T = 10^5$ K we get 1.4×10^{11} m, which is very nearly the radius of the Earth's orbit (1.5×10^{11} m). For $T = 10^3$ K it'll be 10% of the radius. As for what "very much less than" means, that's up to you.... This is a good problem — relevant to accretion discs.

Q2.3. The potential given is $\Phi = \dfrac{GM}{(r^2 + b^2)^{1/2}}$. We find $4\pi G\rho = \boldsymbol{\nabla}^2\Phi = \dfrac{3GMb^2}{(r^2 + b^2)^{5/2}}$, so that $\rho \propto \Phi^5$ as required. The gradient of the pressure is $-\dfrac{3GM^2b^2r}{4\pi(r^2 + b^2)^4}$ and the pressure itself is $P = \dfrac{GM^2b^2}{8\pi(r^2 + b^2)^3} \propto \rho^{6/5}$. This is isobaric with index $n = 5$. The internal energy is

$$U = \frac{1}{\gamma - 1}\int \mathrm{d}r\ 4\pi r^2 P = \frac{1}{\gamma - 1}\frac{GM^2b^2}{2}\int_0^\infty \frac{\mathrm{d}r\ r^2}{(r^2 + b^2)^3}. \tag{10}$$

The final integral is $\dfrac{\pi}{16b^3}$. One of my students did this as a contour integral, others used substitutions. I didn't do it those old-fashioned ways...If you want to, you can express this answer as

$$U = \frac{1}{\gamma - 1}\left(\frac{\pi}{4}\right)\left(\frac{3}{4\pi}\right)^{6/5} KM^{6/5}b^{-3/5} \tag{11}$$

but I really can't see any advantage (other than that's what the question asks for).

Cathie Clark (previous lecturer) pointed out that the index $n = 5$ is no indication of the internal energy content $u = P/(\gamma - 1)$. Indeed, if $\gamma = 6/5$, a polytrope is gravitationally unstable, the critical value being $4/3$. As a model for a radiative star, $n = 5$ is quite acceptable, and convectively stable (see later).

Several students tried to use the virial theorem to get the internal energy via $3(\gamma-1)U+\Omega=0$. This works fine, but there is an error in the course book that has $2U+\Omega=0$. Actually, that's right of course, since $\gamma=5/3$ for nearly all the matter in the Universe, and certainly all stable stars!

Q2.4. The density distribution for the isothermal slab is $\rho \propto \mathrm{sech}^2 az$. The potential is parabolic in the middle, linear towards infinity.

The potential is $\Phi = 2\frac{kT}{\mu}\log\cosh az$, with $a=\sqrt{\dfrac{2\pi G\rho_0\mu}{kT}}$. By conservation of energy we get $v^2 = 4\dfrac{kT}{\mu}\log\left(\dfrac{\cosh az_0}{\cosh az}\right)$.

I was asked to provide a quick derivation of the isothermal slab profile. This is straight out of Pringle & King's book.

We start by writing the equation of hydrostatic equilibrium:

$$\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}z} = -\frac{\mathrm{d}\Phi}{\mathrm{d}z} \tag{12}$$

Now set $\mathrm{d}m=\rho\mathrm{d}z$ ($m=0$ at $z=0$) and use $P=v_\mathrm{s}^2\rho$ to get

$$v_\mathrm{s}^2\frac{\mathrm{d}\rho}{\mathrm{d}m} = -\frac{\mathrm{d}\Phi}{\mathrm{d}z} \Rightarrow v_\mathrm{s}^2\frac{\mathrm{d}^2\rho}{\mathrm{d}m^2} = -\frac{\mathrm{d}^2\Phi}{\mathrm{d}z^2}\frac{\mathrm{d}z}{\mathrm{d}m} = -4\pi G \tag{13}$$

This allows us to integrate to find the form of $\rho(m)$. Applying the boundary condition at $z=0$ we get

$$\rho = \frac{\mathrm{d}m}{\mathrm{d}z} = \frac{2\pi G}{v_\mathrm{s}^2}(M^2-m^2) \tag{14}$$

where M is the total mass of the (half) slab. This has solution $m = M\tanh(2\pi GMz/v_\mathrm{s}^2)$, and we're there, since $M=\sqrt{\dfrac{v_\mathrm{s}^2\rho_0}{2\pi G}}$.

Q2.5. The work done per unit mass is the integral $-\int P\mathrm{d}V = \int_0^\rho \mathrm{d}\rho'\ \frac{P}{\rho^2}$. For $P=K\rho^{1+1/n}$ it evaluates to $nK\rho^{1/n}=nP/\rho$. The internal energy per unit volume is $P/(\gamma-1)$, so the internal energy per unit mass is $P/\rho(\gamma-1)$. The two are only equal if $\gamma=1+1/n$.

The internal energy is $U\propto\int \mathrm{d}r\ r^2\ K\rho^{1+1/n}$. I put the bookwork into the next question. The radius scales as $r_\mathrm{max}\propto\rho_c^{1/2(1/n-1)}K^{1/2}$. The scaling is $U\propto K\rho_c^{1+1/n}(K\rho_c^{-1+1/n})^{3/2}=K^{5/2}\rho_c^{(-1+5/n)/2}$. The mass scales as $\rho_c r_\mathrm{max}^3\propto\rho_c^{(-1+3/n)/2}K^{3/2}$ (see next question). You get the answer $U\propto M^{(5-n)/(3-n)}$ if you ignore the variation of K. If you use $K\propto\rho_c^{-1/n}$ you get $U\propto M$ of course, as this is the assumption that the central temperature is constant. Perhaps the question needs some rewording here to make the students consider both cases.

Q2.6. The polytrope has $P = K\rho^{1+1/n}$, so $\mathrm{d}P = -\rho \mathrm{d}\Phi$ becomes $\mathrm{d}\Phi = -K(1+1/n)\rho^{-1+1/n}\mathrm{d}\rho$. This integrates to give $\Phi_t - \Phi = K(n+1)\rho^{1/n}$ and rearranges to give

$$\rho = \rho_c \left(\frac{\Phi_t - \Phi}{\Phi_t - \Phi_c}\right)^n \equiv \rho_c \theta^n \tag{15}$$

where $\Phi_{c,t}$ are the potentials at the centre and the surface of the star ($\rho_t = 0$). To get the dimensionless Lane-Emden equation we use the scaled radius $\xi = \sqrt{\frac{4\pi G\rho_c}{\Phi_t - \Phi_c}} r$ to get

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2 \frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n \tag{16}$$

We can rearrange the expression $\Phi_t - \Phi_c = K(n+1)\rho_c^{1/n}$ and the definition of ξ to

$$\xi = \sqrt{\frac{4\pi G\rho_c^{1-1/n}}{K(n+1)}} r \tag{17}$$

The mass evaluates to $M = 4\pi\rho_c \left(\frac{4\pi G\rho_c^{1-1/n}}{K(n+1)}\right)^{-3/2} \int_0^{\xi_{\max}} \mathrm{d}\xi\ \xi^2\theta^n$. This predicts $M \propto \rho_c^{1/2(3/n-1)} K^{3/2}$, and

$$R \equiv r_{\max} = \xi_{\max} \left(\frac{4\pi G\rho_c^{1-1/n}}{K(n+1)}\right)^{-1/2} \propto \rho_c^{1/2(1/n-1)} K^{1/2} \tag{18}$$

For a monatomic gas we expect $n = 3/2$, but we have to guess how K scales, and let's set that so that $T_c \propto P/\rho_c$ is constant, i.e. $K \propto \rho_c^{-1/n}$.

We find $R \propto \rho_c^{-1/2}$ and $M \propto \rho_c^{-1/2}$, i.e. $M \propto R$. Heavy stars are less dense in the middle, though they are in fact slightly hotter, and the scaling law for main-sequence stars is more like $R \propto M^{0.8}$.

The next question from 2014 and earlier has been deleted, presumably because I pointed out that it was unphysical...

Q2.7. The acoustic impedance is $Z = \sqrt{\gamma P_0 \rho}$ (Part IB Waves...) and relates the amplitude of the pressure to the velocity: $\delta P = Z\delta v$. STP is a can of worms, and not very unique! Lets take 10^5 Pa and $\rho = 1.3$ kg m^{-3}. The sound speed is 328 m s^{-1} and impedance $Z = 427$. If $\delta P = 0.001 P_0$ then $\delta v = 23.4$ cm s^{-1}. This sound level is 130 db — the threshold of pain. That's quite loud!

Most students get 0.33 ms^{-1}. They are out by a factor of $\gamma = 7/5$, so let's derive it correctly. We only need the first-order terms in a Fourier expansion of the variables: $\{v_k, \rho_k, P_k\} \propto e^{i(kx-\omega t)}$. The zero-order terms are $\{v_0 = 0, \rho_0, P_0\}$. The continuity equation is $-i\omega\rho_k = -ik\rho_0 v_k$. The force equation is $-i\omega\rho_0 v_k = -ikP_k$. We also need $P_k/P_0 = \gamma\rho_k/\rho_0$. Putting them together we get $(\omega^2 - k^2 v_\mathrm{s}^2)v_k = 0$

(and for the other variables), with $v_s^2 = \gamma P_0/\rho_0$ as expected. Rearranging the original equations shows that the amplitudes are related by

$$\frac{P_k}{P_0} = \gamma \frac{v_k}{v_s} \tag{19}$$

and hence the acoustic impedance is as given above.

Q2.8. When thinking about shocks **always use the frame in which the shock is stationary and the flow is normal**. From the Rankine-Hugoniot conditions we have $v_2/v_1 = ((\gamma-1) + 2/M_*^2)/(\gamma+1)$. For this case we have the transverse velocity: $v_\perp = v_1 \tan\theta = v_2 \tan(\theta+\chi)$ and we have to remember that the relevant Mach number is $M_*^2 = M^2\cos^2\theta$. Use the fact that the transverse velocity is the same on both sides to set $\tan(\theta+\chi) = (\tan\theta + \tan\chi)/(1-\tan\theta\tan\chi)$ and get

$$\cot\chi = \cot\theta \frac{1+\alpha\tan^2\theta}{\alpha - 1}$$

where $\alpha \equiv v_1/v_2$. We have $\alpha - 1 = \dfrac{2(M^2\cos^2\theta - 1)}{(\gamma-1)M^2\cos^2\theta + 2}$ and $\alpha\tan^2\theta = \dfrac{(\gamma+1)M^2\sin^2\theta}{(\gamma-1)M^2\cos^2\theta+2}$. Finally

$$\cot\chi = \cot\theta \frac{(\gamma-1)M^2\cos^2\theta + 2 + (\gamma+1)M^2\sin^2\theta}{2(M^2\cos^2\theta - 1)} = \cot\theta\left(\frac{(\gamma+1)M^2}{2(M^2\cos^2\theta-1)} - 1\right)$$

Q2.9. We assume initially that the compression is large (isothermal shock) and the post-shock flow is stationary, so that the collision velocity is $v = 2$ km s^{-1}. The collision time is $2r/v = 3\times10^{13}$ s or 9.3×10^5 yr. Almost all the kinetic energy is lost so the cooling time is $\frac{1}{2}v^2/\dot{Q} = 2\times10^{10}$ s, which is much shorter than the collision time, so the shock is isothermal The shocked layer will actually have spread out to a size more than $2r$ in the other dimensions.

The shock is isothermal, and we can get the the density after the shock from the condition $P_1 + \rho_1 v_1^2 = P_2 + \rho_2 v_2^2$, which we approximate to $\rho_2 kT/\mu = P_2 = \rho_1 v_1^2$. This is about $\frac{\rho_2}{\rho_1} \approx \frac{\mu v^2}{kT} = 48$ for neutral hydrogen or 97 for H_2 (the shock speed is insufficient to dissociate molecular hydrogen). We then suppose it relaxes back to an isothermal slab with parameter $H^2 = \dfrac{kT}{2\pi G\rho_0\mu}$. The column density is given as $\Sigma = 2\rho_0 H = 0.1$ kg m^{-2} , so $H = \dfrac{kT}{\pi G\Sigma\mu} = 2\times10^{15}$ m for molecular hydrogen. The starting size was $x = 4\times3\times10^{16}/97 = 1.2\times10^{15}$ m, so the fraction $x/H \approx 0.6$.

The column density is astrophysically reasonable and corresponds to a H_2 density of about 5×10^8 m^{-3}.

Q3.1. The downstream pressure is given by the shock conditions $P_2 = P_1 + \frac{2\rho_1 v_1^2}{\gamma+1}\left(1 - \frac{1}{M_1^2}\right)$. For a strong shock $\frac{v_1}{v_2} = \frac{\gamma+1}{\gamma-1}$ and using $\rho_1 v_1 = \rho_2 v_2$ this gives $\frac{\gamma P_2}{\rho_2} = \frac{2\gamma}{\gamma-1} v_2^2$ as required. The downstream Mach number is $M_2^2 = (\gamma-1)/2\gamma$ for a strong shock. The downstream γ might not be the same as the upstream one though! This requires more thought...

In the frame where the shock is stationary $v_1 = 3 \times 10^6$ m s^{-1}. We can rearrange the above to the more useful form $\frac{kT}{\mu} = \frac{2(\gamma-1)}{(\gamma+1)^2} v_1^2$. This gives $T = 1.02 \times 10^8$ K, using $\gamma = 5/3$ and $\mu = 0.5 m_\mathrm{u}$ The worked answer gets 2.8×10^8 K by a very round-about argument, and has clearly missed the factor of 2 for the electrons.

The question asks for the ratio of the sound speeds. I can't think this is useful, but I get

$$\frac{c_{\mathrm{s}2}}{c_{\mathrm{s}1}} = M_1 \frac{\sqrt{2\gamma(\gamma-1)}}{\gamma+1} \tag{20}$$

Q3.2. For spherical outflow or accretion we get a relation between v and r. In my lecture course I write

$$\tfrac{1}{2}\left(1 - \frac{c_\mathrm{s}^2}{v^2}\right)\frac{\mathrm{d}v^2}{\mathrm{d}r} = -\frac{GM}{r^2}\left(1 - \frac{2c_\mathrm{s}^2 r}{GM}\right)$$

but the form in the book is clearly a rearrangement of that

$$\tfrac{1}{2}\left(v^2 - c_\mathrm{s}^2\right)\frac{\mathrm{d}\log v}{\mathrm{d}r} = \frac{2c_\mathrm{s}^2}{r}\left(1 - \frac{GM}{2c_\mathrm{s}^2 r}\right) \tag{21}$$

At the sonic point $r = \frac{GM}{2c_\mathrm{s}^2} = 2 \times 10^9$ m (The gas will be ionised at that temperature so we have to allow for the electrons ($\mu = 0.5 m_\mathrm{u}$.)

Q3.3. The sonic point is again $r = \frac{GM}{2c_\mathrm{s}^2}$ Bondi (1952) showed that

$$\dot{M} = \pi G^2 M^2 \frac{\rho(\infty)}{c_\mathrm{s}^3(\infty)}\left(\frac{2}{5-3\gamma}\right)^{\frac{5-3\gamma}{2(\gamma-1)}} \tag{22}$$

The funny factor involving γ varies smoothly from $e^{3/2}$ at $\gamma = 1$ to 1 at $\gamma = 5/3$. If $c_\mathrm{s} = 1$ km s^{-1} and it's isothermal molecular hydrogen, we will use $\gamma = 1$ and get $r = 6.7 \times 10^{13}$ m and $\dot{M} = 8.4 \times 10^{14}$ kg s^{-1}.

Looking at the worked answer it's clear that the solution of $\dot{M} = AM^2$ is required, i.e. $1/M_0 - 1/M = At$, so, for doubling the mass we have $t = m_0/2\dot{M} = 1.2 \times 10^{15}$ s, or 3.8×10^7 yr. The doubling time is inversely proportional to the initial mass.

Q3.4. This is a variant on the blast-wave formula and is simply $r \propto \left(\frac{L}{\rho}\right)^{1/5} t^{3/5}$.

They stall when $\frac{\mathrm{d}r}{\mathrm{d}t} \propto \left(\frac{L}{\rho}\right)^{1/5} t^{-2/5} = c_\mathrm{s}$ (i.e. constant). That means $t_\mathrm{stall} \propto L^{1/2}\rho^{-1/2}$ and indeed $r_\mathrm{stall} \propto L^{1/2}\rho^{-1/2}$. Why we are interested in the area of bubbles viewed from above I really can't imagine, but clearly it's $\propto L$ independent of the arrangement of the star cluster. If the cluster is big enough, however, the bubbles can break out of the Galactic plane.

Q3.5. If $\frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}z} < \frac{\gamma}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}z}$, the atmosphere is unstable to adiabatic mixing and it will convect. (Be very careful with this formula, because it presumes that gravity acts downwards, so that $\frac{\mathrm{d}P}{\mathrm{d}z}$ is negative… A useful thing to remember is the the isothermal case ($n = \infty$) is always stable, as the temperature doesn't fall with height. If the barotropic index n satisfies $1 + 1/n > \gamma$, convection will set in and the star will return to adiabatic $P \propto \rho^\gamma$.

Q3.6. (a) The condition for thermal instability is $\left(\frac{\partial \dot{Q}}{\partial T}\right)_P < 0$, as then a reduction in temperature increases the rate of energy loss and vice versa Here $\dot{Q} = A\rho T^{1/2} - C$. This means $\dot{Q} = \frac{A\mu P}{kT^{1/2}} - C$. We have $\left(\frac{\partial \dot{Q}}{\partial T}\right)_P = -\frac{A\mu P}{2kT^{3/2}} < 0$. The system is thermally unstable.

(b) The heating cooling balance is $C = A\rho T^{1/2}$. Now $P = \frac{kT\rho}{\mu} = \frac{kC^2}{\mu\rho A^2}$. That means the density decreases with pressure, which is very bad indeed, since the pressure is highest in the middle of the slab. It's highly Rayleigh-Taylor unstable. Or convectively unstable, if you prefer.

Use of the previous formula ($1 + 1/n > \gamma$) doesn't work, as $\mathrm{d}P/\mathrm{d}\rho < 0$ and again reverses the sign of the inequality. **Be careful!**.

Q3.7. This is a silly question that depends on the exact definition of Jeans' mass and sound crossing time. The worked answer given to us says the sound crossing time for a cloud of radius R is R/c_s. The collapse time is definitely $t_\mathrm{ff} = \sqrt{\frac{3\pi}{32G\rho_0}}$. What the Jeans mass is I'm not sure (other than its general form, of course), but they were told it's $M_\mathrm{J} = \frac{\pi^{3/2}c_\mathrm{s}^3}{G^{3/2}\rho_0^{1/2}}$. We get a cloud size of $R = \left(\frac{3}{4\pi}\right)^{1/3}\left(\frac{\pi c_\mathrm{s}^2}{G\rho_0}\right)^{1/2}$. We get $\frac{t_\mathrm{ff}}{t_\mathrm{c}} = \left(\frac{4\pi}{3}\right)^{1/3}\left(\frac{3}{32}\right)^{1/2} = 0.493569$. I say it's 1 by definition…

If the sphere collapses homogeneously $\rho R^3 =$ constant, and we suppose it's isothermal. The Jeans mass $\propto \rho^{-1/2} \propto R^{3/2}$. So the number of Jeans masses in a cloud increases and the cloud fragments further.

Q3.8. We have $g = GMz/R^3$ as in Q2.1. Here $\rho = \rho_0(z^2 - z_m^2)^2$, which isn't even dimensionally correct... I assume that it is zero past z_m and the question really ought to specify that! Nice question, though. We can get the barotropic index from $\frac{\mathrm{d}P}{\mathrm{d}\rho}\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}z} = -\frac{GMz}{R^3}$. We find $\frac{\mathrm{d}\rho}{\mathrm{d}z} = -4\rho_0(z_m^2 - z^2)z = -4\rho_0^{1/2}\rho^{1/2}z$. Hence $\frac{\mathrm{d}P}{\mathrm{d}\rho} = \frac{GM}{4R^3}\frac{\rho^{1/2}}{\rho_0^{1/2}}$. Integrating we get $P \propto \rho^{3/2}$.

This is unstable for $\gamma = 1.4$ (molecular hydrogen) and stable for $\gamma = 5/3$ (HI).

Q3.9. The jet is isothermal and in pressure equilibrium with an isothermal slab so $\rho_\mathrm{j}(z) = \frac{\rho_0 T_\mathrm{s}}{T_\mathrm{j}}\mathrm{sech}^2(z/z_0)$. For a mass flow of $\dot{M}$ we have $\dot{M} = \rho_\mathrm{j} v_\mathrm{j} A_\mathrm{j} =$ constant. The area of the jet will be a minimum at the sonic point so $A_\mathrm{min} = \frac{\dot{M}}{c_\mathrm{j}\rho_{0\mathrm{j}}}\cosh^2(z_\mathrm{min}/z_0)$.

Bernoulli is $\frac{1}{2}v^2 + \int \frac{\mathrm{d}P}{\rho} + \phi_\mathrm{g} =$ constant, but there's a variant for isothermal flow. The $\int \frac{\mathrm{d}P}{\rho}$ term is $\frac{kT_\mathrm{j}}{\mu}\log\rho_\mathrm{j}$ and $0 = -\boldsymbol{\nabla}P - \rho\boldsymbol{\nabla}\phi_\mathrm{g}$ for the slab gives the gravitational potential as $\phi = \frac{kT_\mathrm{s}}{\mu}\log\rho_\mathrm{s}$. The worked answer ignores this term, which is OK for $T_\mathrm{j} \gg T_\mathrm{s}$.

From the base of the jet to the sonic point we get

$$\tfrac{1}{2}c_\mathrm{s}^2 + \frac{kT_\mathrm{j}}{\mu}\log\rho_\mathrm{j}|_{A_\mathrm{min}} = \tfrac{1}{2}\left(\frac{\dot{M}}{\rho_{0\mathrm{j}}A_0}\right)^2 + \frac{kT_\mathrm{j}}{\mu}\log\rho_{0\mathrm{j}} \tag{23}$$

where $v_0 = \left(\frac{\dot{M}}{\rho_{0\mathrm{j}}A_0}\right)$ is the initial velocity of the jet That means

$$\tfrac{1}{2}\left(c_\mathrm{s}^2 - v_0^2\right) = \frac{kT_\mathrm{j}}{\mu}\log\left[\cosh^2(z_\mathrm{min}/z_0)\right] \tag{24}$$

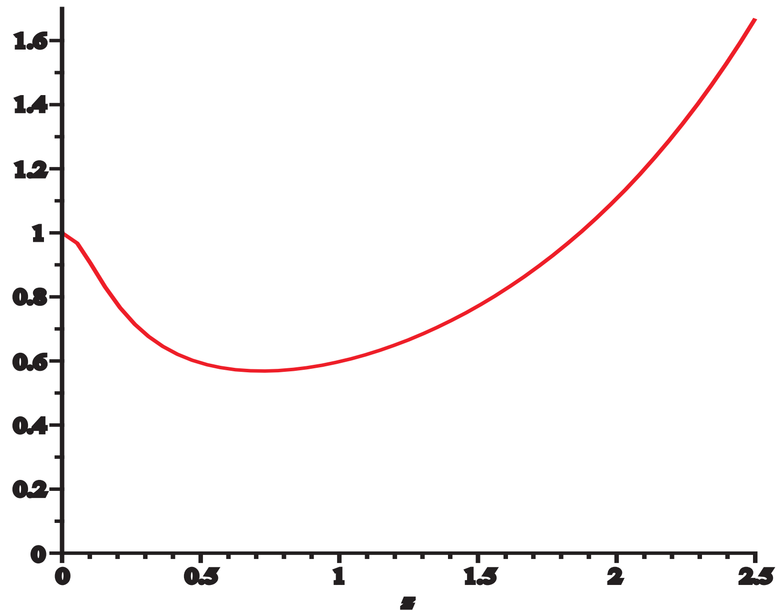
so that

$$\cosh^2(z_\mathrm{min}/z_0) = \exp\left[\frac{\mu}{2kT_\mathrm{j}}\left(c_\mathrm{s}^2 - v_0^2\right)\right] \tag{25}$$

and hence

$$A_\mathrm{min} = \frac{\dot{M}}{c_\mathrm{j}\rho_{0\mathrm{j}}}\exp\left[\frac{\mu}{2kT_\mathrm{j}}\left(c_\mathrm{s}^2 - v_0^2\right)\right] \tag{26}$$

The velocity at the minimum is of course c_j and the height z_min can be found from (25).



Now applying Bernoulli to a general point of the jet

$$\tfrac{1}{2}\left(\frac{\dot{M}}{\rho_{0\mathrm{j}}A_{\mathrm{j}}(z)}\right)^2 + \frac{kT_{\mathrm{j}}}{\mu}\log\rho_{\mathrm{j}}(z) = \tfrac{1}{2}\left(\frac{\dot{M}}{\rho_{0\mathrm{j}}A_0}\right)^2 + \frac{kT_{\mathrm{j}}}{\mu}\log\rho_{0\mathrm{j}} \tag{27}$$

Multiplying through by $2\rho_{0\mathrm{j}}^2/\dot{M}^2$ and using the formula for $\rho(z)$ we get

$$\left(\frac{\cosh^2(z/z_{\mathrm{s}})}{A_{\mathrm{j}}(z)}\right)^2 = \frac{1}{A_{0\mathrm{j}}^2} + \frac{2kT_{\mathrm{j}}\rho_{0\mathrm{j}}^2}{\mu\dot{M}^2}\log\left(\cosh^2(z/z_{\mathrm{s}})\right) \tag{28}$$

and hence

$$A_{\mathrm{j}}(z) = \frac{A_{0\mathrm{j}}\cosh^2(z/z_{\mathrm{s}})}{\left(1 + \frac{2kT_{\mathrm{j}}\rho_{0\mathrm{j}}^2}{\mu\dot{M}^2}\log\left(\cosh^2(z/z_{\mathrm{s}})\right)\right)^{1/2}} \tag{29}$$

The jet is thinner in the middle that at the start, and flares out exponentially at the end. The figure shows the radius versus z/z_0 for a typical case.

4.1 The shock condition has $P_1 + \rho_1 v_1^2 = P_2 + \rho_2 v_2^2$. If P_1 can be neglected, $v_2/v_1 = (\gamma-1)/(\gamma+1) = 1/4$ for $\gamma = 5/3$. After the shock we have $P_2 = 3/4\rho_1 v_1^2$, rising by one third to $P_2 = \rho_1 v_1^2$ when the cooling takes over.

For a general γ, the post-shock pressure is greater than $2\rho_1 v_1^2/(\gamma+1)$, and the ram pressure is less than $\rho_1 v_1^2(\gamma-1)/(\gamma+1)$, so $\rho_2 v_2^2/P_2 < \frac{1}{2}(\gamma-1)$ as required.

On the assumption the that pressure $P = \rho kT/\mu \approx \rho_1 v_1^2$ and using $\rho v = \rho_1 v_1$ we get $v = kT/\mu v_1$, which we can re-express as $v = v_2 T/T_2$. Using the cooling law $C_P\dot{T} = -KT^2$ and $\mathrm{d}T/\mathrm{d}t = v\mathrm{d}T/\mathrm{d}x$ we get $T = T_2\exp(-\frac{KT_2}{C_P v_2}x)$ as required.

4.2 **Method 1**: balance viscous forces on element between r and $r + \mathrm{d}r$ with total force on unit length of pipe $2\pi r\mathrm{d}r\Delta P/d$. The stress is $\tau_{rz} = \eta\mathrm{d}v_z/\mathrm{d}r$ and the force

$2\pi r\eta \mathrm{d}v_x/\mathrm{d}r$. The net force is $\mathrm{d}(\eta \mathrm{d}v_z/\mathrm{d}r)$. Dividing by $\mathrm{d}r$ thus get

$$\eta \frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}v_z}{\mathrm{d}r}\right) = -\frac{r\Delta P}{d} \tag{30}$$

Method 2: Use the Navier-Stokes equation $0 = -\nabla P + \eta\nabla^2 \boldsymbol{v}$. This is easy for this case because the flow is in the same direction everywhere. We just have to remember the expression for ∇^2 in cylindrical polars. Clearly, we get the same equation.

So, in either case, we integrate

$$r\frac{\mathrm{d}v_z}{\mathrm{d}r} = -\frac{r^2\Delta P}{2\eta d} + A \tag{31}$$

Dividing by r and integrating again we get the Poiseuille profile

$$v_z = -\frac{r^2\Delta P}{4\eta d} + A\log(r) + B \tag{32}$$

The velocity has to vanish at the walls, i.e. at $r = R_1$ and $r = R_2$

$$0 = -\beta R_1^2 + A\log R_1 + B = -\beta R_2^2 + A\log R_2 + B \tag{33}$$

where $\beta = \Delta P/4d\eta$. We find $\beta(R_2^2 - R_1^2) = A\log(R_2/R_1)$ and

$$B = \beta R_1^2 - A\log(R_1) = \beta R_1^2 - \beta(R_2^2 - R_1^2)\frac{\log R_1}{\log(R_2/R_1))} \tag{34}$$

The mass flow rate is $Q = 2\pi\rho\int_{R_1}^{R_2} \mathrm{d}r\ rv$, which is perfectly horrible, but not quite as bad as given in the worked answer, because they've made an error. Maple assures me that the answer is

$$Q = \frac{\pi\beta\rho}{2}\left((R_2^4 - R_1^4) - \frac{(R_2^2 - R_1^2)^2}{\log(R_2/R_1)}\right) \tag{35}$$

This approaches the usual answer with no central wall rather slowly. The bit in the middle really slows it down...

4.3 We now have $\rho g\sin\alpha = -\eta \mathrm{d}^2 v/\mathrm{d}y^2$, so that $\eta \mathrm{d}v/\mathrm{d}y = \rho g\sin\alpha(d - y)$, so there is no stress at the free surface. Integrating again $v = \rho g\sin\alpha(dy - \frac{1}{2}y^2)/\eta$ and we find the total flow rate $Q = \rho g\sin\alpha d^3/3\eta$.

4.4 I think you are just expected to check that the answer is a solution of the Navier-Stokes equation, but I'll derive the Green function for the diffusion equation

$$\frac{\partial G}{\partial t} - \nu\frac{\partial^2 G}{\partial y^2} = \delta(t)\delta(y) \tag{36}$$

Take the Fourier transform: $(i\omega + \nu k^2)\tilde{G}(\omega, k) = 1$, and back transform

$$G(t,x) = \frac{1}{(2\pi)^2}\int \mathrm{d}\omega \mathrm{d}k \, \frac{e^{i(ky-\omega t)}}{i\omega + \nu k^2} = \frac{1}{2\pi}\int \mathrm{d}k \;\exp(iky - \nu k^2 t)\;\Theta(t) \tag{37}$$

where $\Theta(t)$ is a Heaviside function. Complete the square of the Gaussian integral and get the result

$$G(x,t) = \sqrt{\frac{1}{4\pi\nu t}}\exp\left(\frac{-y^2}{4\nu t}\right)\;\Theta(t) \tag{38}$$

The shear layer is just a line of these Green functions of strength $u_x(y)G(t, y - y')$.

4.5 The velocity is $u_R(t,R)\hat{\boldsymbol{e}}_R + R\Omega(t,R)\hat{\boldsymbol{e}}_\phi$, so the continuity equation is

$$\int \mathrm{d}z\,(\dot{\rho} + \nabla\cdot(\rho\boldsymbol{u})) \Rightarrow \frac{\partial\Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\Sigma u_R) = 0 \tag{39}$$

The force equation $\rho\partial_t\boldsymbol{u} + \rho\boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla P + \rho\nu\nabla^2\boldsymbol{u}$ separates into a transverse part and a radial part.

Method 1: It is entirely possible to work with the differential equation provided that you remember that $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\phi$ are functions of ϕ. The convective term is

$$\boldsymbol{u}\cdot\nabla\boldsymbol{u} = \left(-\Omega^2 R + u_R\frac{\partial u_R}{\partial R}\right)\hat{\boldsymbol{e}}_R + \left(2\Omega + R\frac{\partial\Omega}{\partial R}\right)u_R\hat{\boldsymbol{e}}_\phi \tag{40}$$

The more tricky term is a nasty Leibniz — best of luck…

$$\nabla^2(u_R\hat{\boldsymbol{e}}_R + R\Omega\hat{\boldsymbol{e}}_\phi) = \left(\frac{\partial^2 u_R}{\partial R^2} + \frac{1}{R}\frac{\partial u_R}{\partial R} - \frac{u_R}{R^2}\right)\hat{\boldsymbol{e}}_R + \left(3\frac{\partial\Omega}{\partial R} + R\frac{\partial^2\Omega}{\partial R^2}\right)\hat{\boldsymbol{e}}_\phi \tag{41}$$

It was pointed out to me that I ignored a term $\frac{1}{3}\eta\nabla\nabla\cdot\boldsymbol{u}$. There is, however, no transverse component from this term…

The transverse part is thus

$$\rho\partial_t(\Omega R) + \rho u_R\left(2\Omega + R\frac{\partial\Omega}{\partial R}\right) = \eta\left(3\frac{\partial\Omega}{\partial R} + R\frac{\partial^2\Omega}{\partial R^2}\right) \tag{42}$$

The form given as the answer is this multiplied by R and has a $\partial_t(\rho R) + \partial_R(R\rho u_R) = 0$ (from the continuity equation) added to it to give

$$\partial_t(R^2\Sigma\Omega) + \frac{1}{R}\partial_R(\Sigma R^3\Omega u_R) = \frac{\nu\Sigma}{R}\partial_R(R^3\partial_R\Omega) \tag{43}$$

Method 2: Alternatively (and more sensibly), we note that there is no viscous stress unless Ω varies with radius, so that the stress is $\eta R\mathrm{d}\Omega/\mathrm{d}R$, and we multiply by $2\pi R$ to get the total stress on an element and by R to get the torque, so that the couple on an element is $2\pi\eta\partial_R(R^3\partial\Omega/\partial R)$. The terms on the LHS are the rate of change of the angular momentum $\partial_t(2\pi R\Sigma(\Omega R)R)$ and the net amount of angular momentum convected in the radial direction $R^{-1}\partial_R(u_R(2\pi R)R\Omega R)$.

There is, of course, another equation

$$\partial_t u_R - \Omega^2 R + u_R \partial_R u_R = -\rho^{-1}\partial_R P + \tfrac{4}{3}\nu\left(\frac{\partial^2 u_R}{\partial R^2} + \frac{1}{R}\frac{\partial u_R}{\partial R} - \frac{u_R}{R^2}\right) \tag{44}$$

(I have added the term $\frac{1}{3}\eta\nabla\nabla\cdot\boldsymbol{u}$).

4.6 This is a section taken from my plasma physics course, which I hope is helpful.

A plasma is a highly conducting fluid. In a stationary medium we have the constitutive relation (Ohm's law) $\boldsymbol{j} = \sigma\boldsymbol{E}$. In a (non-relativistic) medium this becomes $\boldsymbol{j} = \sigma(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})$. If the medium is highly conducting and we let $\sigma \to \infty$, so $\boldsymbol{E} = -\boldsymbol{v}\times\boldsymbol{B}$. Taken together with the induction equation $\boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}$, this gives

$$\frac{\partial\boldsymbol{B}}{\partial t} = \boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{B}) \tag{45}$$

One can show that this means that the magnetic field lines are "frozen in" and move with the fluid velocity.

In the force equation of fluid dynamics we have to add a term $\boldsymbol{j}\times\boldsymbol{B}$. For fluid motions we ignore the displacement current and set $\nabla\times\boldsymbol{B} = \mu_0\boldsymbol{J}$ to get

$$\rho\frac{D\boldsymbol{v}}{Dt} = -\boldsymbol{\nabla}P + \rho\boldsymbol{g} - \frac{1}{\mu_0}(\boldsymbol{B}\times(\boldsymbol{\nabla}\times\boldsymbol{B})) \tag{46}$$

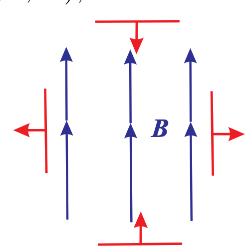
We can write $\left[\frac{1}{\mu_0}(\boldsymbol{B}\times(\boldsymbol{\nabla}\times\boldsymbol{B}))\right]_i = \boldsymbol{\nabla}_j P^{\text{mag}}_{ij}$ where $P^{\text{mag}}_{ij} = \left(\frac{B^2\delta_{ij}}{2\mu_0} - \frac{B_iB_j}{\mu_0}\right)$ (we have used $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$).

There is an additional anisotropic magnetic pressure, which has a magnitude $\frac{\boldsymbol{B}^2}{2\mu_0}$. Magnetic field provides a pressure force perpendicular to the field, but a tension in the direction of the field lines.

The tension leads to a new form of magnetohydrodynamic wave — Alfven waves, which are like waves on magnetic strings.

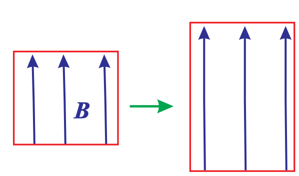
Magnetic pressure: $P^{\text{mag}}_{ij} = \left(\frac{B^2\delta_{ij}}{2\mu_0} - \frac{B_iB_j}{\mu_0}\right)$. Take $\boldsymbol{B} = (0, 0, B)$, so that

$$P_{ij} = \begin{pmatrix} \frac{B^2}{2\mu_0} & 0 & 0 \\ 0 & \frac{B^2}{2\mu_0} & 0 \\ 0 & 0 & -\frac{B^2}{2\mu_0} \end{pmatrix} \tag{47}$$



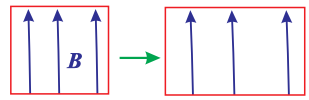
Magnetic pressure is positive (i.e. pushes) perpendicular to $\boldsymbol{B}$ but has tension parallel to $\boldsymbol{B}$. Magnetic pressure produces waves, just like sound waves: $\frac{\omega}{k} = \sqrt{\frac{\gamma P}{\rho}}$ (γ is adiabatic index).

Parallel to $\boldsymbol{B}$: (vertical stretch)
$\boldsymbol{B}$ stays the same.



$P \propto B^2 \propto V^0$
$\Rightarrow \gamma_{\parallel} = 0.$

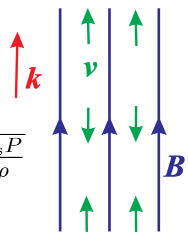
Perpendicular to $\boldsymbol{B}$: (horizontal stretch)
$\boldsymbol{B} \propto V^{-1}$.
$P \propto B^2 \propto V^{-2}$
$\Rightarrow \gamma_{\perp} = 2.$



(1) $\boldsymbol{k}$ parallel to $\boldsymbol{B}$

(A) $\boldsymbol{v}$ parallel to $\boldsymbol{k}$

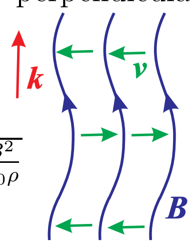
$\frac{\omega}{k} = v_{\rm s} = \sqrt{\frac{\gamma_{\rm s} P}{\rho}}$



Sound wave (x1)

(B) $\boldsymbol{v}$ perpendicular to $\boldsymbol{k}$

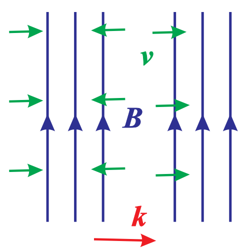
$v_{\rm a} = \sqrt{\frac{B^2}{\mu_0 \rho}}$



Alfven waves (x2)

(2) $\boldsymbol{k}$ perpendicular to $\boldsymbol{B}$

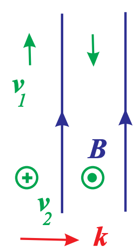
(A) $\boldsymbol{v}$ parallel to $\boldsymbol{k}$



Magnetosound (x2)

$v_{\rm m} = \sqrt{v_{\rm s}^2 + v_{\rm a}^2}$

(B) $\boldsymbol{v}$ perpendicular to $\boldsymbol{k}$



No restoring force (x4)

For arbitrary directions of $\boldsymbol{k}$ and $\boldsymbol{v}$ the magnetohydrodynamic modes are mixtures of the above.

This question actually asks about "fast magnetosound", for which $v_{\rm m} = \sqrt{v_{\rm s}^2 + v_{\rm a}^2}$.

4.7 The free-fall time (see earlier) $\approx (G\rho)^{-1/2}$ and the sound-crossing time is $\approx R\sqrt{\rho\mu_0}/B$. hence $M_{\rm J} \approx \rho R^3 = B^3 \rho^{-2} (G\mu_0)^{-3/2}$ as required.

If it collapses, $B \propto R^{-2}$ and $\rho \propto R^{-3}$, hence result.