UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Friday 1 June 2018 9.00 am to 11.00 am

PHYSICS (5)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (5)

ASTROPHYSICAL FLUID DYNAMICS

Candidates offering this paper should attempt a total of **three** *questions. The questions to be attempted are* **1** *and* **two** *questions from Section B.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains **five** *sides, including this coversheet, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS
2 × 20 Page Answer Book
Metric graph paper
Rough workpad
Yellow master coversheet

SPECIAL REQUIREMENTS
Mathematical Formulae handbook
Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

draft: 17th May 2018

SECTION A

Answers should be concise and relevant formulae may be assumed without proof.

A1 *Attempt* **all** *parts of this question.*

(a) The net cooling rate per unit mass in a gas heated by cosmic rays can be written as

$$\dot{Q} = A\rho T^{\alpha} - H\,,$$

where A and H are constants, ρ is the density and T the temperature of the gas. Assuming the gas is at constant pressure, determine whether it is thermally stable when cooling is due to optically thin Bremsstrahlung with a spectral index $\alpha = 0.5$ [4]

(b) A fluid of conductivity σ moves at a non-relativistic velocity $\boldsymbol{u}$. In this limit the current density is given by $\boldsymbol{j} = \sigma(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B})$ where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic fields in the fluid. Show that the magnetic field satisfies the following differential equation

$$\frac{\partial \boldsymbol{B}}{\partial t} \approx \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) + \frac{1}{\mu_0 \sigma}\nabla^2 \boldsymbol{B}\,.$$

[4]

(c) Show that the vertical density structure for a self-gravitating slab of gas, with a polytropic equation of state $p = K\rho^2$ and in hydrostatic equilibrium, satisfies the differential equation

$$\frac{\mathrm{d}^2\rho}{\mathrm{d}z^2} + \frac{2\pi G}{K}\rho = 0\,.$$

[4]

SECTION B
Attempt **two** *questions from this section*

B2 Define what is meant by the 'Mach cone' and, by considering how disturbances propagate in supersonic flow, explain why shocks are likely to form. [3]

Derive the Rankine–Hugoniot conditions at a normal adiabatic shock front:

$$\begin{aligned} \rho_2 u_2 &= \rho_1 u_1 , \\ p_2 + \rho_2 u_2^2 &= p_1 + \rho_1 u_1^2 , \\ \frac{\gamma}{\gamma-1}\frac{p_2}{\rho_2} + \frac{1}{2}u_2^2 &= \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1} + \frac{1}{2}u_1^2 , \end{aligned}$$

where ρ is the density, p the pressure, γ the adiabatic index and u the normal velocity. The subscripts 1 and 2 refer to the upstream and downstream conditions respectively. [5]

For a strong shock the upstream pressure can be neglected. Show that, in the limit of a strong shock,

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{\gamma+1}{\gamma-1}.$$ [5]

A large, planar, stationary slab of monatomic gas has initial thickness L, density ρ and pressure p. A strong shock incident on the front of the slab, and parallel to the plane of the slab, propagates through the slab. When the shock reaches the far side of the slab, what is the thickness of the slab and with what speed is the gas moving? [2]

The slab of gas lies close to a star. The effect of the shock is to cause the slab of gas to escape the gravitational influence of the star. Show that the square of the Mach number of the shock must be around five times the gravitational potential energy of the slab in the field of the star divided by the thermal energy of the cloud. [4]

B3 Show that the Lagrangian derivative of some quantity Q can be written as

$$\frac{DQ}{Dt} = \frac{\partial Q}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} Q,$$

where $\boldsymbol{u}$ is the velocity field. [2]

Without a detailed derivation, discuss the origin of Euler's equation for the motion of fluid in a gravitational potential ψ

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = -\frac{1}{\rho} \boldsymbol{\nabla} p - \boldsymbol{\nabla} \psi,$$

where p and ρ are the pressure and density fields. [2]

Show that the quantity

$$H = \frac{1}{2} u^2 + \int \frac{\mathrm{dp}}{\rho} + \psi$$

is constant along a streamline. [3]

Spherical accretion of gas occurs onto a star of mass M. If the accretion is steady show that at a radius r

$$(u^2 - c_\mathrm{s}^2) \frac{\mathrm{d} \ln u}{\mathrm{d} r} = \frac{2 c_\mathrm{s}^2}{r} \left(1 - \frac{GM}{2 c_\mathrm{s}^2 r}\right),$$

where $c_\mathrm{s} = \sqrt{\mathrm{d}p/\mathrm{d}\rho}$ is the sound-speed in the gas, and find an expression for the sonic radius r_s. [4]

If the accretion is also isothermal, show that

$$u^2 = 2 c_\mathrm{s}^2 \ln\left(\frac{\rho_\infty}{\rho}\right) + \frac{2GM}{r},$$

where ρ_∞ is the density at infinity. [3]

Find an expression for the accretion rate onto the star in terms of M, ρ_∞ and c_s. [5]

B4 The equations governing the behaviour of a self-gravitating fluid are

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) &= 0, \\
\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} &= -\frac{1}{\rho} \boldsymbol{\nabla} p - \boldsymbol{\nabla} \psi, \\
\nabla^2 \psi &= 4\pi G \rho,
\end{aligned}$$

where $\boldsymbol{u}$, p and ρ are respectively the velocity, pressure and density fields, and ψ is the gravitational potential.

Write down the equations for a stationary fluid, of density ρ_0, pressure p_0 and gravitational potential ψ_0, which is in hydrostatic equilibrium. Briefly discuss why there is a problem when we consider an infinite static uniform medium, and note the approach adopted by Jeans to circumvent this problem. [3]

By following this approach and considering small changes of the form $p = p_0 + \Delta p$, $\rho = \rho_0 + \Delta \rho$, $\psi = \psi_0 + \Delta \psi$ and $u = \Delta u$, find the linearised forms of the above equations which are valid to first order in the perturbed quantities – assume the sound speed c_s is constant. [5]

Assuming a wave-like solution of the form $\Delta p = p_1 \mathrm{e}^{\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)}$ and similarly for the other perturbed quantities, show that the dispersion relation has the form

$$\omega^2 = c_\mathrm{s}^2 (k^2 - k_\mathrm{J}^2),$$

and find a expression for the Jeans wavenumber, k_J and Jeans length $\ell_\mathrm{J} = 2\pi / k_\mathrm{J}$. [4]

What is the criterion for a growing unstable mode which leads to gravitational collapse? Which modes, and hence which physical scales, grow fastest? [2]

Show that the Jeans length is approximately the length scale over which the sound-crossing time is the same as the free-fall time under self gravity. [5]

END OF PAPER