UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

May–June 2020 **1 hour 15 minutes**

PHYSICS (4)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (4)

OPTICS AND ELECTRODYNAMICS

Candidates offering this paper should attempt a total of ***four*** *questions:* ***three*** *questions from Section A and* ***one*** *question from Section B.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains ***four*** *sides, including this coversheet. You may use the formula handbook for values of constants and mathematical formulae, which you may quote without proof.*

You have 75 minutes (plus any pre-agreed individual adjustment) to answer this paper. Do not start to read the questions on the subsequent pages of this question paper until the start of the time period.

Please treat this as a closed-book exam and write your answers within the time period. Downloading and uploading times should not be included in the allocated exam time. If you wish to print out the paper, do so in advance. You can pause your work on the exam in case of an external distraction, or delay uploading your work in case of technical problems.

Section A and the chosen section B question should be uploaded as separate pdfs. Please name the files 1234X_Qi.pdf, where 1234X is your examination code and i is the number of the question/section (A or 4 or 5).

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Master coversheet	Mathematical Formulae handbook
	Approved calculator allowed

SECTION A

*Attempt **all** questions in this Section. Answers should be concise and relevant formulae may be assumed without proof.*

1 A plane wave travelling in the $\widehat{\boldsymbol{z}}$-direction of a Cartesian coordinate system is incident on an imperfect quarter-wave plate having the following Jones matrix:

$$\begin{pmatrix} a & 0 \\ 0 & \mathrm{i}b \end{pmatrix},$$

where a and b are real positive constants. If the incident wave is linearly polarised in the direction of θ with respect to $\widehat{\boldsymbol{x}}$, derive expressions for the angles θ for which the emergent wave is (i) left circularly polarised and (ii) right circularly polarised. If for a particular plate $a = 0.8$ and $b = 0.9$, calculate the smallest angle θ for which the emergent beam is circularly polarised. [4]

2 A certain array of antennas produces the Poynting flux

$$N(r, \theta, \phi) = C \frac{\sin^2 \phi \sin^2 \theta}{r^2} P,$$

when a time-averaged power P is fed into its terminals from a matched source. C is a constant of proportionality, θ the polar angle, ϕ the azimuthal angle, and r the distance to the observation point. What is the value of C, and what is the maximum power gain of the antenna in dB? [4]

3 Electromagnetic radiation leaking from a warm microwave cavity has the following autocorrelation function:

$$\Gamma(\tau) = \exp(-|\tau|/\tau_{\mathrm{c}}) \exp(\mathrm{i}\omega_0 \tau).$$

The emitted radiation is passed through a Michelson interferometer having a total differential path length $2d$, and then detected using a power meter. If the cavity is resonant at $f_0 = 10$ GHz, and has a quality factor of $Q = f_0/\Delta f = 1000$, where Δf is the FWHM, what minimum differential path length, and minimum number of samples, would be needed to establish the primary form of the autocorrelation function in an experiment? Justify your reasoning. [4]

SECTION B

Attempt one question from this section

B4 A linearly-polarised time-harmonic plane wave propagating in the $\widehat{\boldsymbol{z}}$ direction is incident on a neutrally charged plasma. A static magnetic field $\boldsymbol{B}_0 = B_0\widehat{\boldsymbol{z}}$ is also present.

(a) Write down an equation of motion for the plasma, stating any assumptions made. [2]

(b) Derive an expression, in matrix form, for the Cartesian permittivity tensor $\underline{\underline{\varepsilon}}$, taking the number density of electrons to be n. Express your answer in terms of the plasma and cyclotron frequencies,

$$\omega_\mathrm{p} = \sqrt{\frac{ne^2}{\varepsilon_0 m}} \qquad \omega_\mathrm{c} = \frac{eB_0}{m},$$

respectively. [6]

(c) Derive the permittivity tensor for left and right circular polarisation by calculating the matrix elements of $\underline{\underline{\varepsilon}}$ with respect to the basis vectors $(1,\mathrm{i})^\mathrm{T}/\sqrt{2}$ and $(1,-\mathrm{i})^\mathrm{T}/\sqrt{2}$. [4]

(d) Describe the nature of $\underline{\underline{\varepsilon}}$ in the new basis. [2]

(e) Suppose that, for some physical system, $f_\mathrm{p} = 10\,\mathrm{MHz}$ and $f_\mathrm{c} = 1\,\mathrm{MHz}$. Calculate the cut-off frequencies for the two circular states of polarisation. [4]

(f) If a linearly polarised wave is applied having $f = 10\,\mathrm{MHz}$, what emerges? [1]

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B5 (a) Explain what is meant by 'retarded time', and why it is useful. [2]

(b) $[F]$ denotes the retarded form of the general function $F(r,t)$ for a source at the origin; derive expressions for $\partial[F]/\partial r$ and $\partial[F]/\partial t$. [2]

(c) For a Hertzian dipole located at the origin, the scalar and vector potentials are given by

$$\boldsymbol{A}(r,t) = \frac{\mu_0}{4\pi r}[\dot{\boldsymbol{p}}]$$
$$\phi(r,\theta,t) = \frac{\cos\theta}{4\pi\varepsilon_0}\left\{\frac{[p]}{r^2} + \frac{[\dot{p}]}{rc}\right\},$$

respectively, where $[\boldsymbol{p}]$ is the retarded dipole moment. Using these potentials derive expressions, using spherical-polar coordinates, for all non-zero components of the electric and magnetic fields associated with the $\hat{\boldsymbol{z}}$-directed dipole moment $\boldsymbol{p} = p_0\hat{\boldsymbol{z}}$. [8]

A $\hat{\boldsymbol{z}}$-directed static dipole moment p_0, located at $\boldsymbol{r}' = 0$, spontaneously decays exponentially to zero according to the following form:

$$\boldsymbol{p}(t) = p_0\hat{\boldsymbol{z}} \quad \text{where} \quad p_0 = \begin{cases} 1 & (t<0) \\ \mathrm{e}^{-t/\tau} & (t>0) \end{cases}.$$

(d) At very great distances, the electric and magnetic fields fall as $1/r$. Derive expressions for the electric and magnetic fields in the far field, and thereby derive an expression for the Poynting flux as a function of time. [5]

(e) A distant observer measures the radiated power density with a receiver. Plot the form of the detected signal as a function of time, drawing attention to any notable features. [2]

[*In spherical-polar coordinates* (r,θ,φ),

$$\nabla f = \left(\frac{\partial f}{\partial r},\ \frac{1}{r}\frac{\partial f}{\partial \theta},\ \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\right)$$

and

$$\nabla\times\boldsymbol{v} =$$
$$\left(\frac{1}{r\sin\theta}\left(\frac{\partial(v_\varphi\sin\theta)}{\partial\theta} - \frac{\partial v_\theta}{\partial\varphi}\right),\ \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\varphi} - \frac{\partial(rv_\varphi)}{\partial r}\right),\ \frac{1}{r}\left(\frac{\partial(rv_\theta)}{\partial r} - \frac{\partial v_r}{\partial\theta}\right)\right).$$
]

END OF PAPER