UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Wednesday 01 June, 2022 13.30 to 15.30

PHYSICS (5)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (5)

ASTROPHYSICAL FLUID DYNAMICS

*Candidates offering this paper should attempt a total of* ***five*** *questions: all* ***three*** *questions from Section A and* ***two*** *questions from Section B.*

*The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains* ***seven*** *sides, including this coversheet, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

| STATIONERY REQUIREMENTS | SPECIAL REQUIREMENTS |
| --- | --- |
| $2 \times 20$ Page Answer Book | Mathematical Formulae handbook |
| Rough workpad | Approved calculator allowed |
| Yellow master coversheet | |

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

*Attempt **all** questions in this Section. Answers should be concise and relevant formulae may be assumed without proof.*

1 A wave moving through a one dimensional fluid is described by the velocity profile $u(x,t) = u_0 \sin(kx - \omega t)$. If a particle starts at position $X_0$ at time $t = 0$, find its position $x(t)$ to second order in the velocity amplitude $u_0$. [4]

2 A large comet impacts Earth in the Pacific Ocean off the coast of Chile. If $10^{22}$ J is deposited into the atmosphere, will the atmospheric blast wave reach Lansing, Michigan, 8,500 km away? The height of the atmosphere is roughly 10 km. [4]

3 The induction equation describing the magnetic field in a highly conducting fluid is

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) \, .$$

Show that

$$\frac{D}{Dt} \int_S \mathbf{B} \cdot d\mathbf{S} = 0,$$

where $S$ is a surface that moves with the fluid. [4]



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SECTION B

*Attempt two questions from this section*

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(a) What is the physical meaning of the Brunt–Väisälä frequency? Find an expression for this frequency for adiabatic perturbations of a gas with adiabatic index $\gamma$ in a temperature and pressure gradient. [5]

(b) A fluid in a vertical gravitational field of strength $g$ has a density when at rest that depends linearly on the vertical coordinate $z$: $\rho = \rho_0 + \rho_0' z$. Assuming that the flow is incompressible ($\nabla \cdot \mathbf{u} = 0$), show that when $\rho_0' z \ll \rho_0$ small perturbations with velocity $(u_x, 0, u_z)$ obey the system of equations:

$$
\begin{aligned}
\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} &= 0 \\
\rho_0 \frac{\partial u_x}{\partial t} &= -\frac{\partial \pi}{\partial x} \\
\rho_0 \frac{\partial u_z}{\partial t} &= -\frac{\partial \pi}{\partial z} - \varrho g \\
\frac{\partial \varrho}{\partial t} &= -u_z \rho_0',
\end{aligned}
$$

where $\varrho$ is the perturbation to the density and you should identify $\pi$. [5]

(c) Show that the dispersion relation $\omega(\mathbf{k})$ of waves with wavevector $\mathbf{k} = (k_x, 0, k_z)$ is given by

$$\omega^2 = N^2 \frac{k_x^2}{k_x^2 + k_z^2}.$$

Give an expression for $N$. When is the motion stable? [5]

(d) Find the group velocity associated with this dispersion relation and comment on its relation to the wavevector $\mathbf{k}$. [4]

5 (a) Using the momentum equation as an example, explain how the Rankine-Hugoniot equations for a normal shock are derived. What effect does a finite viscosity have on the form of a shock? [5]

When pressure may be neglected, shock formation in one dimensional flow can be described by the equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0.$$

(b) Consider a shock profile moving with velocity $V$: $u(x,t) = u(x - Vt)$. Find the relationship between $V$ and the downstream ($u_1$) and upstream ($u_2$) velocities. [4]

When viscosity $\nu$ is included the equation becomes

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}.$$

(c) $u(x,t)$ can be written in terms of another function $\Psi(x,t)$ as

$$u(x,t) = -2\nu\frac{\partial}{\partial x}\log\Psi(x,t).$$

Find the equation satisfied by $\Psi(x,t)$ and show that

$$\Psi(x,t) = \exp\left(\nu\kappa_1^2 t + \kappa_1 x\right) + \exp\left(\nu\kappa_2^2 t + \kappa_2 x\right)$$

is a solution, where $\kappa_{1,2}$ are arbitrary real constants. [5]

(d) Show that the above solution for $\Psi(x,t)$ describes a velocity $u(x,t)$ that tends to two different values $u_{1,2}$ as $x \to \pm\infty$, and find the relationship betwen $u_{1,2}$ and $\kappa_{1,2}$. Find the velocity $V$ with which the profile moves and the width of the profile. [5]



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6 This question concerns the gravitational stability of an incompressible fluid cylinder.

(a) Show that the pressure at a distance $r$ from the axis of an incompressible cylinder of radius $R$ and constant density $\rho$ is [4]

$$p(r) = \pi G \rho^2 \left(R^2 - r^2\right), \qquad \text{for } r < R.$$

To investigate the stability of the cylinder to axisymmetric perturbations a small deformation is introduced, described by a sinusoidal variation of the radius with axial coordinate $z$

$$r(z,t) = R + \epsilon \mathrm{Re}\left[e^{i(kz-\omega t)}\right]. \qquad (\star)$$

(b) Show that the linear equations describing small deviations from equilibrium are

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} &= -\frac{\nabla \pi}{\rho} - \nabla \psi \\ \nabla \cdot \mathbf{u} &= 0 \\ \nabla^2 \psi &= 0\,, \end{aligned}$$

where you should explain the physical meaning of the quantities $\pi$ and $\psi$. [4]

Inside the boundary $(\star)$ $\psi$ is given by

$$\psi(r,z,t) = -4\pi\epsilon G \rho R K_0(\kappa) I_0(kr) \mathrm{Re}\left[e^{i(kz-\omega t)}\right],$$

where $\kappa = kR$ and $I_0(x)$ and $K_0(x)$ are modified Bessel functions. In addition, incompressibility implies

$$\psi + \frac{\pi}{\rho} = \epsilon I_0(kr) \mathrm{Re}\left[\Pi_0 e^{i(kz-\omega t)}\right]$$

for some constant $\Pi_0$.

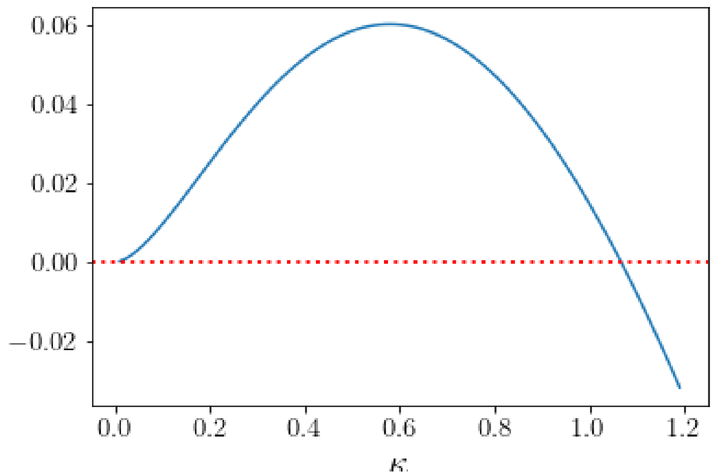
(c) Show that [3]

$$\Pi_0 = \frac{\omega^2}{k I_0'(\kappa)}.$$

(d) The pressure must vanish on the boundary $(\star)$. Use this to show [4]

$$-\frac{\omega^2}{4\pi G\rho} = \kappa \frac{I_0'(\kappa)}{I_0(\kappa)} \left[K_0(\kappa) I_0(\kappa) - \frac{1}{2}\right]. \qquad (\dagger)$$

(e) The right hand side of $(\dagger)$ is shown in the figure overleaf. What can you conclude about the stability of perturbations of wavevector $k$? [4]

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