NST Part II Physics

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ADVANCED QUANTUM PHYSICS

Examples Sheet 1 - Revision

1. **Operator methods and measurement**

   The Hamiltonian $\hat{H}$ has two normalized eigenstates $|\psi_1\rangle$ and $|\psi_2\rangle$, corresponding to different eigenvalues E_1 and E_2.

   (a) Show that $|\psi_1\rangle$ and $|\psi_2\rangle$ are orthogonal.

   (b) An observable $\hat{A}$ has the properties $\hat{A}|\psi_1\rangle = |\psi_2\rangle$ and $\hat{A}|\psi_2\rangle = |\psi_1\rangle$; calculate its eigenvalues and eigenvectors (as combinations of $|\psi_1\rangle$ and $|\psi_2\rangle$).

   (c) At time $t = 0$, a measurement of $\hat{A}$ results in the measured value -1. Find the state of the system $|\psi(t)\rangle$ at times $t > 0$, and show that the probability that a measurement of $\hat{A}$ again gives the value -1 is given by $P(t) = \cos^2[(E_1 - E_2)t/2\hbar]$.

2. **Probability Flux**

   Consider the state $\psi(x) = Ae^{ikx} + Be^{-ikx}$. Show that the probability flux is $j = (|A|^2 - |B|^2)(\hbar k/m)$. (The cross terms vanish, so we can associate the two parts of j with the corresponding parts of ψ.)

3. **Ladder operators**

   The potential energy of a one-dimensional harmonic oscillator of mass m and angular frequency ω is given by $V(\hat{x}) = m\omega^2\hat{x}^2/2$. Using the raising and lowering operators,

   $$\hat{a}^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}(-i\hat{p} + m\omega\hat{x}), \qquad \hat{a} = \frac{1}{\sqrt{2m\hbar\omega}}(i\hat{p} + m\omega\hat{x}),$$

   show that:

   (a) The expectation values of the position and momentum are zero for an energy eigenstate $|\psi_n\rangle$.

   (b) The expectation values of the potential and kinetic energies are each equal to $(n + 1/2)(\hbar\omega/2)$ where n is the quantum number of the state $|\psi_n\rangle$.

   (c) The uncertainties Δx and Δp in position and momentum are related by $\Delta x\, \Delta p = (n + 1/2)\hbar$.

   [The ladder operators have the properties $\hat{a}^\dagger|\psi_n\rangle = \sqrt{n+1}|\psi_{n+1}\rangle$ and $\hat{a}|\psi_n\rangle = \sqrt{n}|\psi_{n-1}\rangle$.]

4. **Matrix methods**

Show that for a system with orbital angular momentum $\ell = 1$, for the basis of states $|\phi_1\rangle = |Y_{11}\rangle$, $|\phi_2\rangle = |Y_{10}\rangle$, $|\phi_3\rangle = |Y_{1-1}\rangle$, the angular momentum operators may be represented by the matrices

$$\hat{L}_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{L}_y = \frac{i\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{L}_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

A rotating body has the Hamiltonian

$$\hat{H} = \frac{\hat{L}_x^2}{2I_x} + \frac{\hat{L}_y^2}{2I_y} + \frac{\hat{L}_z^2}{2I_z}.$$

Find the energy levels and corresponding eigenstates when $\ell = 1$.

5. **Spin**

In terms of the Pauli matrices $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$, the operator corresponding to the component of spin along the axis (θ, ϕ) in spherical polar coordinates for a spin-half particle is $(\hbar/2)\boldsymbol{\sigma} \cdot \boldsymbol{n}$, where $\boldsymbol{n}$ is the unit vector $\boldsymbol{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Show that the eigenvalues of spin in this direction are $\pm\hbar/2$ (as expected), and deduce the corresponding wavefunctions. Hence, infer the wavefunctions for particles whose spins are aligned along the $+x$, $-x$, $+y$ and $-y$ directions.

6. **Identical particles**

Two non-interacting, indistinguishable particles of mass m move in the one-dimensional potential $V(x)$ given by

$$V(x) = \begin{cases} 0 & 0 < x < L \\ \infty & \text{otherwise} \end{cases}.$$

Show that the energy of the system is of the form $E = (n_1^2 + n_2^2)\varepsilon$, where n_1 and n_2 are integers, and find an expression for ε.

Consider the state with $E = 5\varepsilon$ for each of the following three cases:

(a) spin-zero particles;

(b) spin-1/2 particles in a spin-singlet state;

(c) spin-1/2 particles in a spin-triplet state.

In each case, state the symmetries of the spin and spatial components of the two-particle wavefunction. Write down the spatial wavefunction $\psi(x_1, x_2)$, and sketch the probability density $|\psi(x_1, x_2)|^2$ in the (x_1, x_2) plane.

Describe qualitatively how the energies of these states would change if the particles carried electric charge and hence interacted with each other.

7. **Heisenberg picture**

In the Heisenberg picture, time dependent operators $\hat{A}(t) \equiv e^{i\hat{H}t/\hbar}\hat{A}e^{-i\hat{H}t/\hbar}$ are introduced. For the one-dimensional harmonic oscillator of question 3, show that the ladder operators $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ in the Heisenberg representation satisfy

$$\begin{aligned}\hat{a}(t) &\equiv e^{i\hat{H}t/\hbar}\hat{a}(0)e^{-i\hat{H}t/\hbar} = e^{-i\omega t}\hat{a}(0)\,,\\ \hat{a}^\dagger(t) &\equiv e^{i\hat{H}t/\hbar}\hat{a}^\dagger(0)e^{-i\hat{H}t/\hbar} = e^{i\omega t}\hat{a}^\dagger(0)\,.\end{aligned}$$

Use this result to demonstrate that the position operator in the Heisenberg representation obeys the equation of motion

$$\frac{\mathrm{d}\hat{x}(t)}{\mathrm{d}t} = \frac{\hat{p}(t)}{m}\,.$$

Show that this last result holds also for the more general case $\hat{H} = \hat{p}^2/2m + V(\hat{x})$.

ANSWERS

1. (b) $\pm 1\,;\ \ \big(|\psi_1\rangle \pm |\psi_2\rangle\big)/\sqrt{2}$;

4. (b) $(\hbar^2/2)(I_x^{-1} + I_y^{-1})$, $(\hbar^2/2)(I_x^{-1} + I_z^{-1})$, $(\hbar^2/2)(I_y^{-1} + I_z^{-1})$,
   $|Y_{10}\rangle$, $(|Y_{11}\rangle + |Y_{1,-1}\rangle)/\sqrt{2}$, $(|Y_{11}\rangle - |Y_{1,-1}\rangle)/\sqrt{2}$.

5. $+\hbar/2$: $|\psi\rangle = \cos(\theta/2)|\uparrow\rangle + e^{i\phi}\sin(\theta/2)|\downarrow\rangle$;
   $-\hbar/2$: $|\psi\rangle = \sin(\theta/2)|\uparrow\rangle - e^{i\phi}\cos(\theta/2)|\downarrow\rangle$