UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Friday 31 May 2013 13.30 to 15.30

EXPERIMENTAL AND THEORETICAL PHYSICS (3)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (3)

Candidates offering this paper should attempt a total of **three** *questions. The questions to be attempted are* **1**, **2** *and* **one** *other question.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains **four** *sides, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS
2×20 Page Answer Book
Rough workpad
Yellow master coversheet

SPECIAL REQUIREMENTS
Mathematical Formulae handbook
Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

ADVANCED QUANTUM PHYSICS

1 *Attempt* **all** *parts of this question. Answers should be concise and relevant formulae may be assumed without proof.*

(a) For two fermions with spin $S_1 = S_2 = 3/2$, by considering the operator $(\boldsymbol{S}_1 + \boldsymbol{S}_2)^2$, or otherwise, find all eigenvalues of the operator $\boldsymbol{S}_1 \cdot \boldsymbol{S}_2$. [4]

(b) A non-relativistic particle of mass m is confined in a smoothly varying one-dimensional potential. The wavefunction of the particle has an overall extent $\sim a$ and has n approximately equally spaced nodes. Estimate the particle's kinetic energy. [4]

(c) A particle is in the ground state of an infinitely deep one-dimensional rectangular potential well of width L. Find the unnormalised probability distribution, $p(k)$, for the particle's wavevector k. [4]

2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) spatial and spin wavefunctions for identical particles;

(b) coherent and non-classical states of light;

(c) the Stark effect.

3 *Attempt* **either** *this question* **or** *question 4.*

A particle of mass m is trapped in a one-dimensional harmonic potential with a trapping frequency ω,

$$V = \frac{1}{2}m\omega^2(x - x_0)^2,$$

where x_0 is the position of the trap centre. Sketch the wavefunctions of the particle, $\psi_n(x)$, in the ground state, $|n = 0\rangle$, and the first excited state, $|n = 1\rangle$. State the energies of the two states, $\hbar\omega_0$ and $\hbar\omega_1$. By considering the potential energy, or otherwise, show that, up to a dimensionless factor of order unity, the characteristic spatial extent of both states is given by $a_0 = \sqrt{\hbar/(m\omega)}$. [5]

Initially, $x_0 = 0$ and the particle is in the $|n = 0\rangle$ state. We want to excite it to the $|n = 1\rangle$ state by shaking the trap. Specifically, in the time interval $0 \le t \le T$, the position of the trap centre oscillates according to

$$x_0 = \varepsilon\cos(\Omega t),$$

where $\varepsilon \ll a_0$, $T \gg 1/\omega$ and $|\Omega - \omega| \ll \omega$. Writing the general time-dependent wavefunction in the form

$$|\Psi(t)\rangle = \sum_n c_n(t)e^{-i\omega_n t}|\psi_n\rangle,$$

the coefficients c_n, for $n \neq 0$, are, within first-order perturbation theory, approximately

$$c_n(t) = \frac{1}{i\hbar}\int_{-\infty}^{t} e^{i(\omega_n-\omega_0)t'}\langle\psi_n|\widehat{V}(t')|\psi_0\rangle \mathrm{d}t',$$

where $\widehat{V}(t')$ is the time-dependent perturbing potential.

Identify $\widehat{V}(t')$ corresponding to the shaking of the trap in this problem and hence show that

$$c_1(T) = -\varepsilon\frac{m\omega^2}{i\hbar}\int_0^T e^{i\omega t}\langle\psi_1|x|\psi_0\rangle\cos(\Omega t)\mathrm{d}t,$$

clearly stating all the steps in your derivation. [4]

Explain why $c_n(T)$ is zero for any even n, why $|c_n(T)| \ll |c_1(T)|$ for all odd $n > 1$, and why, for evaluating $c_1(T)$, it is a good approximation to replace $\cos(\Omega t)$ with $(1/2)e^{-i\Omega t}$. [6]

Approximating $\langle\psi_1|x|\psi_0\rangle \approx a_0$, show that the probability, p_1, that the particle is found in the state $|n = 1\rangle$ at time T is

$$p_1 = \left(\frac{\varepsilon\omega}{a_0}\right)^2 \frac{\sin^2((\omega - \Omega)T/2)}{(\omega - \Omega)^2}\,.$$ [6]

For a fixed T, sketch the function $p_1(\Omega)$ and comment on its main features. [4]

4 *Attempt* **either** *this question* **or** *question 3.*

With help of a sketch, explain what is meant by the Einstein coefficients, B_{12}, B_{21}, and A_{21}, for radiative transitions in a two-level system with non-degenerate energy levels E_1 and E_2, where $E_2 > E_1$. [5]

A two-level system is in equilibrium with black-body radiation in free space. Given that the energy density per unit angular frequency ω of black-body radiation is

$$u(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{\exp(\hbar\omega/k_B T) - 1},$$

show that

$$B_{12} = B_{21} \quad \text{and} \quad A_{21} = B_{21} \frac{\hbar\omega_0^3}{\pi^2 c^3},$$

where $\hbar\omega_0 = E_2 - E_1$. [6]

Now consider two parallel conducting plates lying in the $x - y$ plane and separated along z by $d \ll c/\omega_0$. Between the plates, an electric field oscillating at angular frequency ω_0 must be linearly polarised along z and radiation can propagate only in the $x - y$ plane.

By considering only the modes propagating in the $x - y$ plane, show that, between the plates, for all frequencies $\omega \ll c/d$,

$$u(\omega) = \frac{\hbar\omega^2}{2\pi c^2 d} \frac{1}{\exp(\hbar\omega/k_B T) - 1},$$

and comment on how this affects the ratio of the A and B coefficients. [8]

Now suppose that $\hbar\omega_0$ corresponds to the energy splitting between the ground state ($n = 1$) and the first excited state ($n = 2$) of the hydrogen atom. A hydrogen atom in the $2p$ state is placed between the plates. Neglecting spin-orbit coupling and numerical factors of order unity, but clearly stating any assumptions or approximations you make, discuss how placing the atom between the plates affects the spontaneous lifetime of the $2p$ state. By considering the relevant selection rules for dipolar transitions, distinguish between the cases of $2p$ states with quantum numbers $m_\ell = 1, 0$ and -1 (labelled with respect to the z-axis). [6]

END OF PAPER