UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Friday 24 May 2019 1.30 pm to 3.30 pm

PHYSICS (2)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (2)

RELATIVITY

Candidates offering this paper should attempt a total of ***five*** *questions:* ***three*** *questions from Section A and* ***two*** *questions from Section B.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains ***five*** *sides, including this coversheet, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
2×20 Page Answer Book	Mathematical Formulae handbook
Metric graph paper	Approved calculator allowed
Rough workpad	
Yellow master coversheet	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

*Attempt **all** questions in this Section. Answers should be concise and relevant formulae may be assumed without proof.*

1 Show that for an isolated free massive particle moving at any speed u in the laboratory frame decay to a single photon is impossible. [4]

2 In Minkowski spacetime, a spaceship is moving along the x-axis of some inertial frame S. An astronaut on board measures the spaceship to have a constant acceleration g. Show that the components u^μ of the spaceship's four-velocity in S satisfy

$$\frac{\mathrm{d}^2 u^\mu}{\mathrm{d}\tau^2} = \frac{g^2 u^\mu}{c^2}.$$

[4]

3 The metric for a 2-dimensional space is

$$\mathrm{d}s^2 = y^2 \mathrm{d}x^2 + x^2 \mathrm{d}y^2.$$

Using the Lagrangian method, find the geodesic equations in this space. Show that $y(\lambda) = mx(\lambda)$, where m is a constant and λ is the affine parameter along the geodesic, is a possible solution. [4]

[*It is not necessary to find an explicit form for $x(\lambda)$*]

SECTION B
*Attempt **two** questions from this section*

B4 The energy-momentum tensor of a perfect fluid is

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu - p g^{\mu\nu},$$

where ρ is the proper density, p is the isotropic pressure in the instantaneous rest frame and u^μ are the components of the fluids' four-velocity. By considering the components of this energy-momentum tensor in an inertial cartesian coordinate system, explain the physical interpretation of the components of this tensor. Explain also the physical interpretation of $\nabla_\mu T^{\mu\nu} = 0$. [4]

If the fluid is a dust, then the pressure $p = 0$. Starting from the condition $\nabla_\mu T^{\mu\nu} = 0$, show by contracting with u_ν that $\nabla_\mu(\rho u^\mu) = 0$. Hence deduce that the particles of the dust each travel along a geodesic of the spacetime in which the fluid moves. [5]

Starting from the assumption that the four-force and four-velocity f_μ for electromagnetism are related by $f_\mu = qF_{\mu\nu}u^\nu$, where $F_{\mu\nu}$ is the Maxwell field-strength tensor, discuss how electromagnetism can be represented as a relativistic field theory. Complete mathematical details are not required. You should consider the symmetry and transformation properties of $F_{\mu\nu}$, the relationship to the electric and magnetic field components, and how Maxwell's equations are represented in a covariant form. [7]

If each dust particle now carries a charge q, show that the individual particles of the fluid obey the equation

$$u^\mu \nabla_\mu u^\nu = \frac{q}{m} F^{\nu\alpha} u_\alpha .$$

[3]

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B5 The Schwarzschild metric is

$$\mathrm{d}s^2 = c^2\left(1-\frac{2\mu}{r}\right)\mathrm{d}t^2 - \left(1-\frac{2\mu}{r}\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\theta^2 - r^2\sin^2\theta\mathrm{d}\phi^2.$$

Show that the geodesic equations for the worldline $x^\mu(\tau)$ of a massive particle in the equatorial plane $\theta = \pi/2$ are

$$\begin{aligned}\left(1-\frac{2\mu}{r}\right)\dot{t} &= k\\ r^2\dot{\phi} &= h\\ \left(1-\frac{2\mu}{r}\right)^{-1}\ddot{r} + \frac{\mu c^2}{r^2}\dot{t}^2 - \left(1-\frac{2\mu}{r}\right)^{-2}\frac{\mu}{r^2}\dot{r}^2 - r\dot{\phi}^2 &= 0,\end{aligned}$$

where k and h are constants along the geodesic and dots denote $\frac{\mathrm{d}}{\mathrm{d}\tau}$. Give a physical interpretation of k and h. [4]

Hence show that for such a particle

$$\dot{r}^2 + \frac{h^2}{r^2}\left(1-\frac{2\mu}{r}\right) - \frac{2\mu c^2}{r} = c^2(k^2-1),$$

and give a brief physical interpretation of this equation. [3]

For geodesic motion of a massive particle in the circle $r = R$, show that

$$k^2 = \frac{(R-2\mu)^2}{R(R-3\mu)} \qquad \text{and} \qquad h^2 = \frac{\mu c^2 R^2}{R-3\mu}.$$

[6]

A test of general relativity is to compare the times as measured by a clock stationary on the surface of the Earth and by one in a satellite in orbit around the Earth. In one such experiment a satellite is in a circular polar orbit of radius R_s and a standard clock on the satellite measures the proper time to perform one orbit $\Delta\tau_s$. A second clock is at rest at the North Pole on Earth and records the time $\Delta\tau_E$ of successive observations of the satellite being directly overhead. Show that

$$\frac{\Delta\tau_s}{\Delta\tau_E} = \left(1-\frac{3\mu}{R_s}\right)^{1/2}\left(1-\frac{2\mu}{R_E}\right)^{-1/2},$$

where R_E is the radius of the Earth. Estimate this ratio for realistic experimental conditions taking the orbital period to be of order 3 hours. [6]

[*Take R_E = 6370* km *and you may assume that the radius of a geo-stationary orbit around the earth is 42100* km.]

B6 What do we mean by the term *fundamental observer* in cosmology? By considering appropriate symmetries for our standard cosmological model, what properties must fundamental observers have? Explain also how this leads to the adoption of synchronous coordinates. [7]

The Robertson-Walker metric describing such a universe is

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - a^2(t)\left[\mathrm{d}\chi^2 + S_K^2(\chi)\mathrm{d}\Omega^2\right],$$

where

$$S_K(\chi) = \begin{cases} \sin(\sqrt{K}\chi)/\sqrt{K} & \text{for} \quad K > 0 \\ \chi & \text{for} \quad K = 0 \\ \sinh(\sqrt{|K|}\chi)/\sqrt{|K|} & \text{for} \quad K < 0 \end{cases}.$$

Explain the physical interpretation of the coordinates t and χ. By considering the proper distance between two fundamental observers with coordinates $\chi = 0$ and $\chi = \Delta\chi$ show that this metric represents an expanding universe and derive an expression for the Hubble parameter $H(t)$. [3]

A photon is emitted by a fundamental observer at coordinates $(t_E, 0, 0, 0)$ and received later by a fundamental observer at $(t_R, \chi_R, 0, 0)$. The four-momentum of the photon is $p^\mu = (p^0, p^1, 0, 0)$. By considering the geodesic equation in the form

$$\frac{\mathrm{d}p_\mu}{\mathrm{d}\lambda} = \frac{1}{2}\frac{\partial g_{\nu\rho}}{\partial x^\mu}p^\nu p^\rho,$$

where λ is an affine parameter, show that p_1 is constant. [3]

Using the null condition, find an expression for p_0 and hence show that the photon is redshifted by z, where

$$1 + z \equiv \frac{\lambda_R}{\lambda_E} = \frac{\nu_E}{\nu_R} = \frac{a(t_R)}{a(t_E)}.$$

[3]

Two photons are received by an observer at a proper time t_R with an angular separation of $\Delta\theta$, having been emitted at time t_E from the ends of a rod of proper length l, each end being at a fixed coordinate χ. Show that

$$l = \frac{a(t_0)S(\chi)}{1 + z}\Delta\theta.$$

[3]

END OF PAPER