UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Tuesday 24 May 2016 9:00 am to 11:00 am

PHYSICS (2)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (2)

RELATIVITY

Candidates offering this paper should attempt a total of **three** *questions. The questions to be attempted are* **1**, **2** *and* **one** *other question.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains **four** *sides, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS
2×20 Page Answer Book
Rough workpad
Yellow master coversheet

SPECIAL REQUIREMENTS
Mathematical Formulae handbook
Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

1 *Attempt* **all** *parts of this question. Answers should be concise, and relevant formulae may be assumed without proof.*

(a) A two-dimensional Riemannian space has the metric

$$ds^2 = dr^2 + \alpha^2 r^4 d\phi^2$$

where α is a constant, $0 < r < \infty$ and $0 \leq \phi < 2\pi$. Using the Lagrangian method, find the geodesic equations for this metric. State what conserved quantity corresponds to the lack of dependence of the metric coefficients on ϕ, and show explicitly from your equations that this quantity is indeed constant. Also show explicitly that the Lagrangian itself is constant along a geodesic. [4]

(b) For a given observer, the *kinetic energy* (KE) of a particle of rest mass m is defined as $(\gamma - 1)\,mc^2$, where γ is the Lorentz boost factor of the particle in the frame of the observer. A large particle disintegrates into a number of smaller particles which fly apart. Show that the total KE liberated in the explosion, i.e. the total KE after the explosion minus the total KE before the explosion, is a Lorentz invariant. [4]

(c) A small radio transmitter radiates isotropically at a frequency ν_0 as measured in its rest frame. It is attached to the rim of a circular disc of radius r which is rotating at a constant angular velocity ω as measured in the laboratory frame. What are the maximum and minimum frequencies as measured by a stationary receiver located in the same plane as the disc and at a distance d ($> r$) from its centre?

A second receiver is attached to the rim of the disc at an angle α around the circumference with respect to the position of the transmitter, and rotates rigidly with the disc and the transmitter. What frequency will be measured at this receiver? [4]

[*Hint: involved calculations are not needed in either case.*]

2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) the principle of equivalence;

(b) the formation of a black hole as seen by a distant observer;

(c) the bending of light as a test of general relativity.

3 *Attempt* **either** *this question* **or** *question 4.*

Define the Ricci tensor $R_{\mu\nu}$ in terms of the Riemann tensor $R^{\mu}{}_{\nu\alpha\beta}$, and the Ricci scalar R in terms of $R_{\mu\nu}$. [2]

The Bianchi identity in General Relativity can be used to show that

$$\nabla_\mu \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) = 0 \, .$$

Discuss briefly the relationship between this identity and the conservation condition $\nabla_\mu T^{\mu\nu} = 0$, where $T^{\mu\nu}$ is the stress–energy tensor for a field or matter. [3]

The stress–energy tensor for a perfect fluid composed of dust (i.e. zero pressure) has the form

$$T^{\mu\nu} = \rho u^\mu u^\nu .$$

Briefly describe the quantities ρ and u^μ, and by considering a local cartesian inertial frame, give a physical interpretation for the quantities T^{0i}, for $i = 1, 2, 3$. [3]

For such a fluid, show that the conservation condition $\nabla_\mu T^{\mu\nu} = 0$ leads to the requirement that each of the dust particles individually travels along a geodesic of the spacetime in which the fluid moves. [7]

Briefly describe the Faraday tensor $F^{\mu\nu}$, including its relationship with the electric and magnetic field vectors $\boldsymbol{E}$ and $\boldsymbol{B}$, and the Lorentz invariant quantities that can be constructed from $F^{\mu\nu}$. [4]

If each of the dust particles in a fluid of density ρ carries charge q and has rest mass m, then it may be shown that the stress–energy tensor of the fluid now obeys the relation

$$\nabla_\mu T^{\mu\nu} = F^{\nu\alpha} j_\alpha$$

where $j^\alpha = (\rho/m) q u^\alpha$ is the 4-current of the fluid. Demonstrate from this that individual particles of the fluid now obey the equation of motion

$$u^\mu \nabla_\mu u^\nu = \frac{q}{m} F^{\nu\alpha} u_\alpha$$

and interpret this relation physically. [6]

4 *Attempt* **either** *this question* **or** *question 3.*

The Schwarzschild metric for the vacuum around a spherically symmetric body of mass M is

$$ds^2 = \left(1 - \frac{2\mu}{r}\right) c^2\,dt^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)$$

where $\mu = GM/c^2$. Using the Lagrangian method, obtain the geodesic equations for a particle of non-zero mass moving in the $\theta = \pi/2$ plane in this metric, and in particular demonstrate

$$\left(1 - \frac{2\mu}{r}\right)\dot{t} = k, \quad c^2\left(1 - \frac{2\mu}{r}\right)\dot{t}^2 - \left(1 - \frac{2\mu}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = c^2 \quad \text{and} \quad r^2\dot{\phi} = h \quad (*)$$

where k and h are constants. [3]

Using the alternative form of the geodesic equation:

$$\dot{p}_\mu = \tfrac{1}{2}\left(\partial_\mu g_{\nu\sigma}\right) p^\nu p^\sigma$$

give a physical interpretation for the constants k and h. [2]

By using the equations $(*)$, construct the 'energy equation' for particle motion, in the form

$$\dot{r}^2 + \frac{h^2}{r^2}\left(1 - \frac{2\mu}{r}\right) - \frac{2c^2\mu}{r} = c^2(k^2 - 1)\,.$$

[2]

Now consider the case of a particle projected radially outwards from a radius $r_0 > 2\mu$, which just escapes to infinity, and has zero velocity there. Using the energy equation, demonstrate that

$$r(\tau) = \left(r_0^{3/2} + a\tau\right)^{2/3}$$

gives the radius $r(\tau)$ reached by the particle after an elapse of proper time τ, and find the constant a. [5]

Calculate the velocity of the particle described in the previous paragraph relative to a stationary observer at a general radius r at the moment the particle passes by the observer. [6]

[*You may find it helpful to define relative velocity in terms of rate of change of proper distance with respect to proper time, both as measured by a given observer. Alternative methods will be accepted.*]

The particle is now projected from r_0 with an outward velocity such that it has a non-zero velocity u at infinity. Find the velocity of the particle as measured by a stationary observer at radius r at the moment the particle passes by them, in this new case, expressing your answer in terms of M, r and u. [7]

[*Note that an expression for the radial position of the particle as a function of proper time is not needed.*]

END OF PAPER