TSP 2017

1) a) Near Fermi energy ϵ_F, $g(\epsilon) = g(\epsilon_F)$

Electronic states which had energy ϵ in non-superconducting phase correspond to excitations from ground state of superconductor with energy

$$E(\epsilon) = \sqrt{(\epsilon - \epsilon_F)^2 + |\Delta|^2}$$

Show that $D(E)$ = density of excitations = $\dfrac{g(\epsilon_F) E}{\sqrt{E^2 - |\Delta|^2}}$

$$dE = \frac{(\epsilon - \epsilon_F)\, d\epsilon}{\sqrt{(\epsilon - \epsilon_F)^2 + |\Delta|^2}}$$

$$D(E)\, dE = g(\epsilon)\, d\epsilon = g(\epsilon_F)\, d\epsilon \quad \text{near Fermi energy}$$

$$D(E) = g(\epsilon_F) \frac{d\epsilon}{dE} = \frac{\sqrt{(\epsilon - \epsilon_F)^2 + |\Delta|^2}}{(\epsilon - \epsilon_F)} g(\epsilon_F)$$

$$= \frac{E g(\epsilon_F)}{\sqrt{E^2 - |\Delta|^2}}$$

b) permutations of word quagga

2 repeated letters

$6! = 720$ possible combinations, $\dfrac{6!}{2!2!}$ distinct combinations $= 180$

M cells, N particles, each cell is empty or filled

${}^{M}C_{N}$ ways of arranging particles

$$= \frac{M!}{N!(M-N)!}$$

c) Gibbs probability for Grand Canonical Ensemble

$$p_i = \frac{e^{-(E_i - \mu N_i)\beta}}{\sum_j e^{-(E_j - \mu N_j)\beta}}$$ - probability of system being in a state with energy E_i, occupation N_i

normalised by $\Xi = \sum_j e^{-(E_j - \mu N_j)\beta}$

- total number of possible states

Fermions: $\sum_j e^{-(E_j - \mu N_j)\beta} = 1 + e^{-(\epsilon - \mu)\beta}$

- any state can only be occupied by 0 or 1 fermions

for state i, $p_i = \frac{e^{-(\epsilon-\mu)\beta}}{1 + e^{-(\epsilon-\mu)\beta}} = \frac{1}{e^{\beta(\epsilon-\mu)}+1} = \langle n \rangle$

Bosons: denominator $= \sum_0^\infty \left(e^{-(\epsilon-\mu)\beta}\right)^n = \frac{1}{1 - e^{-\beta(\epsilon-\mu)}}$

$$p_i = \frac{e^{-(\epsilon-\mu)\beta}}{1 - e^{-\beta(\epsilon-\mu)}} = \frac{1}{e^{\beta(\epsilon-\mu)} - 1} = \langle n \rangle$$

3) a) classical ideal gas of spin-1/2 particles in B field

2 energy levels, $E = \pm \mu_B B(r)$

consider spin up and down particles as separate subsystems

Spin up: $E = -\mu_B B(r)$

$Z_1 = \frac{\sigma V}{\lambda^3} Z_{int}$, $\sigma = 1$ for spin-1/2 particles and Z_{int} represents internal degrees of freedom

$Z_1 = \frac{V}{\lambda^3} e^{\beta \mu_B B}$, $Z_N = \frac{1}{N_\uparrow!} Z_1^{N_\uparrow} = \frac{1}{N_\uparrow!}\left(\frac{V}{\lambda^3} e^{\beta\mu_B B}\right)^{N_\uparrow}$

$F = -k_B T \ln Z_{N_\uparrow} = k_B T(N_\uparrow \ln N_\uparrow - N_\uparrow) - N_\uparrow k_B T \ln \frac{V}{\lambda^3} - N_\uparrow \mu_B B$

$\mu_\uparrow = \frac{\partial F}{\partial N_\uparrow} = k_B T \ln N_\uparrow - k_B T \ln \frac{V}{\lambda^3} - \mu_B B$

$n_Q(T) = \frac{1}{\lambda^3}$ $\Rightarrow$ $\mu_\uparrow = k_B T \ln\left(\frac{N_\uparrow}{V n_Q}\right) - \mu_B B$

$= k_B T \ln\left(\frac{n_\uparrow}{n_Q}\right) - \mu_B B$

For spin down particles, $E = \mu_B B(r)$

- replace $\mu_B B$ with $-\mu_B B$ and $n_\uparrow$ with $n_\downarrow$

$$\mu_\downarrow = k_B T \ln\left(\frac{n_\downarrow}{n_Q}\right) + \mu_B B(r)$$

Magnetisation per unit volume

$$Z_1 = (e^{\beta\mu_B B} + e^{-\beta\mu_B B}) V n_Q = 2 V n_Q \cosh\beta\mu_B B$$

$$F_1 = -k_B T \ln Z_1 = -k_B T \ln[2 n_Q \cosh\beta\mu_B B]$$

$$M_1 = \frac{-\partial F_1}{\partial B} = \mu \tanh\beta\mu_B B$$

$M = \frac{\mu N}{V} \tanh\beta\mu_B B$ per unit volume for N atoms

$$n_\uparrow = n_Q \exp(\beta\mu + \beta\mu_B B), \quad n_\downarrow = n_Q \exp(\beta\mu - \beta\mu_B B)$$

$$n = n_\uparrow + n_\downarrow = n_Q e^{\beta\mu}(e^{\beta\mu_B B} + e^{-\beta\mu_B B}) = 2 n_Q e^{\beta\mu} \cosh\beta\mu_B B$$

$$M = 2\mu_B n_Q e^{\beta\mu} \sinh\beta\mu_B B$$

high T $k_BT \gg \mu_B B$

$\sinh x \sim x$ for small x

$$M(r) \sim 2\mu_B n_Q \frac{\mu_B B(r)}{k_BT} e^{\beta\mu}$$

$$\sim \frac{2\mu_B^2 n_Q B(r)}{k_BT} \quad \text{for } k_BT \gg \mu$$

Curie law $M(r) \propto \frac{1}{T}$ - applies to this system at high T

particle density $n(r)$ in equilibrated gas

$n(r) = 2n_Q e^{\beta\mu}\cosh(\beta\mu_B B)$

b) N atoms, n vacancies, N+n sites in total

vacancy formation energy ϵ

min Gibbs free energy

Show that average no of vacancies is

$$n = \frac{N}{e^{(\epsilon+pv)/k_BT} - 1}$$

$$G = E(n) - TS(n) + p(n+N)v$$

$$= n\epsilon - k_BT\ln\frac{(N+n)!}{n!N!} + p(n+N)v$$

$$= n\epsilon - k_BT\left[(N+n)\ln(N+n) - N\ln N - n\ln n\right] + p(n+N)v$$

$$\frac{\partial G}{\partial n} = 0 = \epsilon - k_BT\ln\left(\frac{N+n}{n}\right) + pv$$

$$\frac{N+n}{n} = e^{(\epsilon+pv)/k_BT}$$

$$n = \frac{N}{e^{(\epsilon+pv)/k_BT} - 1}$$

average volume $= (n+N)v = \dfrac{Ne^{(\epsilon+pv)/k_BT}}{e^{(\epsilon+pv)/k_BT} - 1} = \dfrac{N}{1 - e^{-(\epsilon+pv)/k_BT}}$

4) Spherical trap

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + ar$$

Single particle partition function

$$Z_1 = \frac{1}{(2\pi\hbar)^3}\int e^{-\beta p^2/2m}\, d^3p \int e^{-\beta a r}\, d^3r$$

$$= \frac{2}{\pi\hbar^3}\int_0^\infty p^2 e^{-\beta p^2/2m}\, dp \int_0^\infty r^2 e^{-\beta a r}\, dr$$

$$= \frac{\cancel{2}}{\pi\hbar^3}\frac{1}{\cancel{2}}\left(\frac{m}{\beta}\right)^{3/2}\sqrt{2\pi}\int_0^\infty t^2 e^{-t}\, dt \Big/ (\beta a)^3$$

$$= \frac{2}{\pi}\left(\frac{k_BT}{\hbar a}\right)^3\left(\frac{m}{\beta}\right)^{3/2}\sqrt{2\pi}$$

For N particles:

$$Z_N = \frac{1}{N!}Z_1^N = \frac{1}{N!}\left(\frac{8}{\pi}\right)^{1/2 N}(k_BT)^{9/2 N}\frac{1}{(\hbar a)^{3N}}m^{3/2 N}$$

indistinguishable, classical stats apply

Find F and show that $S = Nk_B\left(\ln\frac{Z_1}{N} + \frac{11}{2}\right)$

$$F = -k_BT\ln Z_N = k_BT(N\ln N - N) - Nk_BT\left(\frac{1}{2}\ln\frac{8}{\pi} + \frac{9}{2}\ln(k_BT) - 3\ln\hbar a + \frac{3}{2}\ln m\right)$$

$$F = -Nk_BT\ln Z_1 + Nk_BT(\ln N - 1)$$

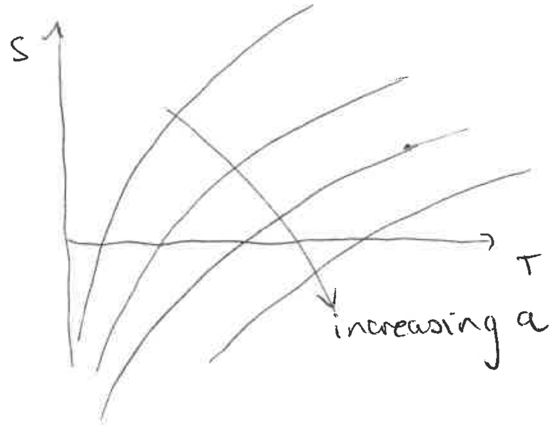
$$S = -\frac{\partial F}{\partial T} = Nk_B\ln Z_1 + Nk_B(\ln N - 1) - Nk_BT\frac{\partial \ln Z_1}{\partial T}$$

$$Z_1 \propto T^{9/2} \Rightarrow \frac{\partial \ln Z_1}{\partial T} = \frac{9}{2T}$$

$$S = Nk_B\left(\ln\frac{Z_1}{N} - 1\right) - \frac{9}{2}k_BN$$

$$= Nk_B\left(\ln\frac{Z_1}{N} - \frac{11}{2}\right)$$

$$S = Nk_B\left(\frac{9}{2}\ln T - \ln Na^3\right)$$ + constants



Show that by decreasing a in adiabatic conditions the gas can be cooled reversibly