UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Tuesday 29 May 2018 1.30 pm to 3.30 pm

PHYSICS (1)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (1)

THERMAL AND STATISTICAL PHYSICS

Candidates offering this paper should attempt a total of **three** *questions. The questions to be attempted are* **1** *and* **two** *questions from Section B.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains **five** *sides, including this coversheet, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS
2 × 20 Page Answer Book
Metric graph paper
Rough workpad
Yellow master coversheet

SPECIAL REQUIREMENTS
Mathematical Formulae handbook
Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

draft: 17th May 2018

THERMAL AND STATISTICAL PHYSICS

SECTION A

Answers should be concise and relevant formulae may be assumed without proof.

A1 *Attempt* **all** *parts of this question.*

(a) The equation of state of an imperfect gas containing N particles is given by

$$pV = Nk_{\mathrm{B}}T - NA(T)p,$$

where the function A only depends on temperature. What is the change in the Gibbs free energy and entropy if the pressure p of the system is increased from p_0 to p_1 while the temperature and particle number are held constant? [4]

(b) The resistivity ρ of metals at room temperature is primarily due to electrons scattering off lattice vibrations: $\rho = AN_{\mathrm{ph}}(T)$, where A is temperature independent and $N_{\mathrm{ph}}(T)$ is the number of phonons. Assuming that the Debye temperature Θ_{D} is far below room temperature, how does ρ depend on T in the room temperature regime? [4]

(c) A container of a classical ideal gas at temperature T contains N atoms, each of which is either of type A or type B. Both types have the same single-particle partition function, $Z_1 \propto VT^{3/2}$. What are the partition function and the entropy of the system if there are N_A atoms of type A and N_B atoms of type B? [*For the entropy, you may make use of the Sackur–Tetrode expression without derivation.*] [4]

SECTION B
Attempt **two** *questions from this section*

B2 Write down the expression for the average fermionic occupancy $\langle n(\varepsilon)\rangle$ of a single-particle state of energy ε in equilibrium at chemical potential μ and temperature T. By considering the range of energies over which $\langle n(\varepsilon)\rangle$ changes for a fixed temperature T, or otherwise, show that the low-temperature electronic contribution to the specific heat of metals depends linearly on temperature. [5]

Consider a 2-D electronic conductor with dispersion relation

$$\varepsilon_{\boldsymbol{k}} = \hbar v|\boldsymbol{k}|\,,$$

where v is a material dependent velocity. Show that when $\mu = 0$, the electronic specific heat satisfies

$$C_{\mathrm{el}} \sim \frac{A k_{\mathrm{B}}^3 T^2}{\hbar^2 v^2}\,,$$

where A is the area of the system. [4]

Consider now the $\mu > 0$ case for this 2-D conductor. In what temperature regime (relative to μ/k_{B}) would you expect the specific heat to display the linear temperature dependence characteristic of a metal? Explain your reasoning. [2]

The dispersion relation above provides a simplified description of the conducting 2-D surface of certain 3-D insulating materials. Compare the $\mu = 0$ surface electronic contribution C_{el} calculated above to the specific heat contribution C_{ph} from the phonons of the bulk 3-D crystal: by calculating C_{ph} up to an overall numerical factor, and focusing on temperatures much less than the Debye temperature Θ_{D}, show that

$$\frac{C_{\mathrm{el}}}{C_{\mathrm{ph}}} \sim \frac{A k_{\mathrm{B}}^2 \Theta_{\mathrm{D}}^3}{\hbar^2 v^2 N T}\,,$$

where N is the number of atoms (proportional to that of vibrational modes). Considering the scaling with temperature and system size, under what conditions does C_{el} dominate over C_{ph}? [5]

With numerical factors included, the expression above becomes

$$\frac{C_{\mathrm{el}}}{C_{\mathrm{ph}}} \approx \frac{5}{12\pi^2}\frac{A k_{\mathrm{B}}^2 \Theta_{\mathrm{D}}^3}{\hbar^2 v^2 N T}\,.$$

Use this to estimate the C_{el} to C_{ph} ratio for a cubic 1 mm^3 topological insulator sample with $v = 5 \times 10^5\ \mathrm{m\,s^{-1}}$, lattice constant $a = 1.5$ nm, $\Theta_{\mathrm{D}} = 500$ K, at temperature $T = 1$ K. [3]

B3 Describe the conditions under which a many-particle system may be treated using classical statistical mechanics, formulating your answer (i) in terms of the particle density versus the thermal de Broglie wavelength $\lambda = \sqrt{2\pi\hbar^2/(mk_B T)}$, and (ii) in terms of the average occupancy $n(\varepsilon)$ of an energy level. Using the Fermi–Dirac and the Bose–Einstein expressions for $n(\varepsilon)$, show that the condition on the latter is equivalent to $e^{\beta\mu} \ll 1$, where $\beta = (k_B T)^{-1}$ and μ is the chemical potential. [5]

Consider a 3-D gas of noninteracting bosons or fermions, obeying the dispersion relation $\varepsilon_{\boldsymbol{k}} = \hbar^2 \boldsymbol{k}^2/(2m)$. By obtaining the grand partition function or otherwise, show that the particle density N/V and pressure p are given by

$$\beta p = -\eta \int \frac{d^3k}{(2\pi)^3} \ln\left[1 - \eta z \exp\left(-\frac{\beta\hbar^2 k^2}{2m}\right)\right],$$

$$\frac{N}{V} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{z^{-1}\exp\left(\beta\hbar^2 k^2/2m\right) - \eta},$$

where $\eta = 1$ for bosons, $\eta = -1$ for fermions, and $z = e^{\beta\mu}$. [3]

The rest of the question explores the first quantum correction to the classical ideal gas equation of state. In preparation for this, expand N/V in series for $z \ll 1$, and hence show that

$$\frac{\lambda^3 N}{V} \approx z + \eta\frac{z^2}{2^{3/2}},$$

and therefore

$$z \approx \frac{\lambda^3 N}{V} - \frac{\eta}{2^{3/2}}\left(\frac{\lambda^3 N}{V}\right)^2.$$

[*You may use the result* $\frac{2}{\sqrt{\pi}}\int_0^\infty dx\,\sqrt{x}e^{-\alpha x} = \alpha^{-3/2}$ *for* $\alpha > 0$.] [4]

By expanding βp in series for $z \ll 1$ and using the results above, show that the equation of state can be expressed as

$$pV = Nk_B T\left[1 - \frac{\eta}{2^{5/2}}\left(\frac{\lambda^3 N}{V}\right) + \ldots\right],$$

and identify the first quantum correction in this expression. [3]

State the definition of the virial coefficients $B_n(T)$ and identify the quantity formally playing the role of $B_2(T)$ in the above equation of state. [2]

By considering how the probability of finding two particles together depends on their fermionic/bosonic nature, provide a physical interpretation for the sign of $B_2(T)$ above, and thus for the sign of the first quantum correction. [2]

B4 Give a brief account of Landau theory, illustrating the concepts for the example of a magnetic phase transition: interpret the magnetisation M as an "order parameter"; explain why one may use a free energy $F(M, B)$ with *both* M and the magnetic field B as variables. Explain under what conditions the free energy can be expressed as

$$F(T, M) = F_0(T) + b(T - T_c)M^2 + dM^4 ,$$

with constants b and d, and define the critical temperature T_c. Sketch $F(T, M)$ for $T > T_c$ and $T < T_c$, indicating in each case the average magnetisation. [5]

Show that for $T < T_c$ the minimum of the free energy is $F = F_0(T) - [b(T - T_c)]^2/(4d)$ and use this to find an expression for the heat capacity jump at T_c. [5]

Under certain conditions, external parameters of the system may be tuned to a so-called "tricritical" point, characterised by $d = 0$, i.e., the absence of the quartic term in the Landau expansion. Explain why the Landau expansion at this point should include a $d'M^6$ term. What is the sign of the coefficient d' and why? [2]

Considering now the Landau expansion at the tricritical point, show that for $T < T_c$ the minimum of the free energy is $F = F_0(T) - 2b^{3/2}(T_c - T)^{3/2}/\sqrt{27d'}$. [4]

Using the result above or otherwise, show that the heat capacity near T_c has a singular power-law contribution, $C_V^{\text{sing}} \propto (T_c - T)^{-\alpha}$, and obtain the value of the critical exponent α. [3]

END OF PAPER