UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Monday 31 May 2021   11:00 to 13:00

PHYSICS (2)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (2)

RELATIVITY

Candidates offering this paper should attempt a total of ***five*** *questions:* ***three*** *questions from Section A and* ***two*** *questions from Section B.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains ***five*** *sides, including this coversheet, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS
2 × 20 Page Answer Book
Metric graph paper
Rough workpad
Yellow master coversheet

SPECIAL REQUIREMENTS
Mathematical Formulae handbook
Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

*Attempt **all** questions in this Section. Answers should be concise and relevant formulae may be assumed without proof.*

1 For the two-dimensional metric $ds^2 = \left[dx^2 - c^2 dt^2\right]/(\alpha t^{-2})$, with α being a constant of appropriate dimensions, show that

$$\frac{t\,\frac{dx}{dt}}{\sqrt{c^2 - \left(\frac{dx}{dt}\right)^2}}$$

is constant and hence, or otherwise, find all timelike geodesic curves. [4]

Given the metric in the question, the Lagrangian with t as a parameter is

$$L = t\sqrt{\dot{x}^2 - c^2}\,,$$

up to an irrelevant multiplicative constant α. [1]

Since this is independent of x, we have the conserved quantity [1]

$$\frac{\partial L}{\partial \dot{x}} = \frac{t\dot{x}}{\sqrt{\dot{x}^2 - c^2}}\,.$$

In fact, if we want time-like curves, we should use $L = t\sqrt{c^2 - \dot{x}^2}$, in which case the constant (let us call it τ) is [1]

$$\tau = \frac{t\dot{x}}{\sqrt{c^2 - \dot{x}^2}}\,.$$

Solving for $\dot{x}$,

$$\dot{x} = \frac{c}{\sqrt{1 + t^2/\tau^2}},$$

we can finally integrate to obtain

$$x(t) = c\tau \,\mathrm{arcsinh}(t/\tau) + x_0 \qquad \text{or} \qquad t = \tau \sinh\left(\frac{x - x_0}{c\tau}\right)$$

for some integration constant x_0. [1]

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2 Consider a generalised form of the Einstein field equation

$$R_{\mu\nu} - \alpha g_{\mu\nu} R = -8\pi \frac{G}{c^4} T_{\mu\nu}$$

where $T^{\mu}{}_{\nu}$ is the energy-momentum tensor and α is some dimensionless constant. Show that $\nabla_\nu T_\mu{}^\nu \propto \partial_\mu T_\nu{}^\nu$ and argue which values of α are physically acceptable. [4]

From Bianchi's identity, we know that $R_{\mu\nu} - (1/2)g_{\mu\nu}R$ is divergenceless. Thence, [1]

$$\left(\frac{1}{2} - \alpha\right)\partial_\mu R = -8\pi\frac{G}{c^4}\nabla_\nu T^\nu_\mu\,.$$

If we first contract and then differentiate the equation given in the question, we obtain instead [1]

$$(1-4\alpha)\,\partial_\mu R = -8\pi\frac{G}{c^4}\partial_\mu T\,,$$

where $T = T^\mu_\mu$.

Combining the two equations, we can get rid of R and obtain [1]

$$\nabla_\nu T^\nu_\mu = \kappa\partial_\mu T\,,$$

where $\kappa = [\frac{1}{2} - \alpha]/[1-4\alpha]$. The only physically acceptable choice is the one where T^ν_μ is divergenceless, i.e., $\alpha = 1/2$. [1]

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3 Find the proper time elapsed for a circular orbit of radius r in the Schwarzchild metric,

$$ds^2 = c^2\left(1 - \frac{2\mu}{r}\right)dt^2 - \left(1 - \frac{2\mu}{r}\right)^{-1}dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2\,,$$

where $\mu = GM/c^2$. [4]

From the Schwarzchild solution, one obtains the Lagrangian L and observes that it is independent of ϕ. Therefore, the angular momentum $h = \partial L/\partial\dot\phi$ is a constant. Choosing for convenience the reference frame where $\theta = \pi/2$, we obtain $h = r^2 d\phi/d\tau$. [1]

Substituting into the Lagrangian, one can show that it is then equivalent to motion in an effective central potential:

$$V_{\text{eff}}(r) = -\frac{GM}{r} + \frac{h^2}{2r^2}\left(1 - \frac{2\mu}{r}\right)\,.$$

Stable circular orbits require $dV_{\text{eff}}/dr = 0$ and $d^2V_{\text{eff}}/d^2r > 0$. Following the lecture notes, this gives

$$r_\pm = \frac{h}{2\mu c^2}\left(h \pm \sqrt{h^2 - 12\mu^2c^2}\right)\,,$$

where only $r = r_+$ is stable for $h > \sqrt{12}\mu c$. [1]

We can then solve for h:

$$\begin{aligned}2\mu c^2 r - h^2 &= h\sqrt{h^2 - 12\mu^2c^2}\\(r-3\mu)h^2 &= \mu c^2 r^2\\h &= cr\sqrt{\frac{\mu/r}{1-3\mu/r}}\end{aligned}$$

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and substitute into the expression for the angular momentum to find [1]

$$\frac{d\phi}{d\tau} = \frac{h}{r^2} = \frac{c}{r}\sqrt{\frac{\mu/r}{1-3\mu/r}}\,.$$

Finally, we obtain the proper time of a circular Schwarzchild orbit: [1]

$$\frac{2\pi}{d\phi/d\tau} = \frac{2\pi r}{c}\sqrt{\frac{1-\frac{3\mu}{r}}{\frac{\mu}{r}}}\,.$$

SECTION B
Attempt two questions from this section

4

(a) Give the form of the Lorentz force law in general relativity. [3]

(b) A *five*-dimensional spacetime has metric

$$ds^2 = \eta_{ab}\mathrm{d}x^a\mathrm{d}x^b + \left(A_a\mathrm{d}x^a + \mathrm{d}x^4\right)^2$$

where $\eta_{ab} = \mathrm{diag}(+1,-1,-1,-1)$ is the Minkowski metric of four-dimensional spacetime, and A_a is a dual 4-vector. Assume that A_a is independent of the fifth coordinate x^4. Show that the Euler–Lagrange equations for the geodesics imply

$$\frac{\mathrm{d}x^4}{\mathrm{d}\tilde{u}} + A_a\frac{\mathrm{d}x^a}{\mathrm{d}\tilde{u}} = \text{constant}, \qquad (\star)$$

where $\tilde{u}$ is the interval in five-dimensional spacetime. [3]

(c) If, on a geodesic, the constant in $(\star)$ takes value ξ, show that $\tilde{u}$ is related to the interval u in four-dimensional spacetime by

$$\mathrm{d}\tilde{u}^2 = \frac{\mathrm{d}u^2}{1-\xi^2}.$$

[4]

(d) Show that the equation of motion for $x^a(\tilde{u})$ is:

$$\frac{\mathrm{d}^2x^a}{\mathrm{d}\tilde{u}^2} = \xi\eta^{ab}\left(\partial_b A_c - \partial_c A_b\right)\frac{\mathrm{d}x^c}{\mathrm{d}\tilde{u}}.$$

[5]

[*Hint: It is possible to simplify each term of the Euler–Lagrange equation for $x^a(\tilde{u})$ using $(\star)$ before combining them.*]

(e) Find an equation obeyed by the 4-velocity $u^a = \mathrm{d}x^a/\mathrm{d}\tau$ and interpret the result. [4]

[Note that the model answers below, parts (c) and (d) in particular, are written for a generic four-dimensional metric g_{ab} instead of the simpler case of the Minkowski metric η_{ab} asked in the exam question.]

(a) In terms of the 4-velocity u^a, the Lorentz force law is [2]

$$\frac{Du^a}{Dt} = \frac{q}{m}F^a{}_b u^b,$$

where $F_{ab} = -F_{ba}$ is the field strength tensor. [1]

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(b) Since x^4 is by assumption a cyclic coordinate (it does not appear in the Lagrangian) the corresponding EL equation has the form

$$\frac{d}{d\tilde{u}}\frac{\partial L}{\partial \dot{x}^4} = 0\,.$$

Thus [3]

$$\frac{\partial L}{\partial \dot{x}^4} = \frac{dx^4}{d\tilde{u}} + A_a\frac{dx^a}{d\tilde{u}} = \xi, \text{ constant}\,.$$

(c) Considering the five-dimensional interval $\tilde{u}$: [1]

$$\begin{aligned} d\tilde{u}^2 = g_{AB}dx^A dx^B &= (g_{ab} + A_a A_b)dx^a dx^b + 2A_a dx^a dx^4 + (dx^4)^2 \\ &= du^2 + (dx^4 + A_a dx^a)^2 = du^2 + \xi^2 d\tilde{u}^2, \end{aligned} \tag{1}$$

the result follows. [3]

(d) EL equations for $x^a(\tilde{u})$ are [1]

$$\frac{\partial L}{\partial x^a} - \frac{d}{d\tilde{u}}\frac{\partial L}{\partial \dot{x}^a} = 0\,. \tag{2}$$

Evaluating the terms gives [2]

$$\begin{aligned} \frac{\partial L}{\partial x^c} &= \partial_c(g_{ab} + A_a A_b)\dot{x}^a\dot{x}^b + 2(\partial_c A_a)\dot{x}^a\dot{x}^4 && (3)\\ &= \dot{x}^a\dot{x}^b\partial_c g_{ab} + 2\xi\dot{x}^a\partial_c A_a && (4)\\ \frac{d}{d\tilde{u}}\frac{\partial L}{\partial \dot{x}^c} &= 2\frac{d}{d\tilde{u}}(g_{ac}\dot{x}^a + A_a A_c\dot{x}^a + A_c\dot{x}^4) && (5)\\ &= 2\frac{d}{d\tilde{u}}(g_{ac}\dot{x}^a + \xi A_c) && (6)\\ &= 2\frac{d}{d\tilde{u}}(g_{ac}\dot{x}^a) + 2\xi\dot{x}^a\partial_a A_c\,. && (7) \end{aligned}$$

We have used the x^4 equation ($\star$). The terms involving g_{ab} will give rise to the standard Geodesic equation with the Christoffel symbols defined in terms of g_{ab} (not required). Putting it all together gives [2]

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = \xi g^{ab}\left(\partial_b A_c - \partial_c A_b\right)\dot{x}^c.$$

(e) Finally, writing in terms of the 4D proper time $\tau = u/c$ we end up with the equation for the 4-velocity $u^a = dx^a/d\tau$ [2]

$$\begin{aligned} \frac{Du^a}{D\tau} &= \frac{\xi c}{\sqrt{1-\xi^2}}g^{ab}\left(\partial_b A_c - \partial_c A_b\right)u^c && (8)\\ &= \frac{q}{m}F^a{}_b u^b, && (9) \end{aligned}$$

where we identify $F_{ab} = (\partial_a A_b - \partial_b A_a)$ and [1]

$$\frac{q}{m} = \frac{\xi c}{\sqrt{1-\xi^2}}.$$

Note that q/m has dimension of velocity because A_a is dimensionless instead of having units of momentum / charge. [1]

5 A metric describing small deviations from Minkowski spacetime may be written as:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \qquad |h_{\mu\nu}| \ll 1.$$

In terms of $h_{\mu\nu}$ the linearized field equations in empty space take the form

$$\partial_\sigma\partial_\mu h^\sigma_\nu + \partial_\sigma\partial_\nu h^\sigma_\mu - \partial_\mu\partial_\nu h - \eta^{\sigma\rho}\partial_\sigma\partial_\rho h_{\mu\nu} - \eta_{\mu\nu}\partial_\rho\partial_\sigma h^{\rho\sigma} + \eta_{\mu\nu}\eta^{\sigma\rho}\partial_\sigma\partial_\rho h = 0,$$

where $h = \eta^{\mu\nu}h_{\mu\nu}$, and indices are raised and lowered with the Minkowski metric $\eta_{\mu\nu}$. This equation can be simplified by introducing

$$\phi^{\mu\nu} \equiv h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h,$$

and imposing the condition $\partial_\sigma\phi^{\sigma\nu} = 0$, which applies throughout this question.

(a) Show that the linearized field equations become

$$\partial_\lambda\partial^\lambda\phi^{\mu\nu} = 0\,.$$ [4]

(b) For a trial solution of the form

$$\phi^{\mu\nu} = \epsilon^{\mu\nu}\cos(k_\alpha x^\alpha) \qquad (\star)$$

find and interpret the conditions that must be satisfied by $\epsilon^{\mu\nu}$ and k_α. [3]

For $k^\alpha = (\omega, 0, 0, \omega)/c$ one possible solution is

$$\epsilon^{\mu\nu}_{\oplus} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_\oplus & 0 & 0 \\ 0 & 0 & -h_\oplus & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(c) The geodesics of stationary particles are unaffected in the linear approximation. Two such particles are separated by a small distance Δ in the $x-y$ plane at an angle θ to the x axis when $h_\oplus = 0$. Find the time dependence of their spatial separation when $h_\oplus \neq 0$. [4]

(d) An infinitesimal coordinate transformation $x^\mu \to x^\mu + \xi^\mu(x)$ gives rise to the transformation $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$ for some $\xi_\mu(x)$. Find an $\epsilon^{\mu\nu}$ such that $(\star)$ corresponds to a coordinate transformation, giving the corresponding ξ_μ. [5]

(e) Find an $\epsilon^{\mu\nu}$ distinct from $\epsilon^{\mu\nu}_\oplus$ that *does not* correspond to a coordinate transformation, which therefore describes a second propagating wave. [3]

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(a) Starting from the condition

$$\partial_\sigma \phi^{\sigma\nu} = \partial_\sigma h^{\sigma\nu} - \frac{1}{2}\partial^\nu h = 0,$$

lower the ν index, apply ∂_μ and symmetrize $\mu \leftrightarrow \nu$ to give [1]

$$\partial_\mu \partial_\sigma h^\sigma_\nu + \partial_\nu \partial_\sigma h^\sigma_\mu - \partial_\mu \partial_\nu h = 0.$$

This eliminates the first three terms of the field equations. We also have [1]

$$\partial_\sigma \partial_\nu \phi^{\sigma\nu} = \partial_\sigma \partial_\nu h^{\sigma\nu} - \frac{1}{2}\partial_\nu \partial^\nu h = 0,$$

which shows that the fifth and sixth terms are related. The result follows. [2]

(b) By substitution find $k_\alpha k^\alpha = 0$, and $\epsilon^{\mu\nu} k_\nu = 0$. [2]

Interpretation: (i) waves moving at the speed of light without dispersion and (ii) waves are transverse. [1]

(c) Since the geodesics of the particles are unchanged, we can simply evaluate the spatial separation in the rest frame [2]

$$\sqrt{(1 + h_\oplus \cos(kz - \omega t))\Delta x^2 + (1 - h_\oplus \cos(kz - \omega t))\Delta y^2} \tag{10}$$

$$= \sqrt{(1 + h_\oplus \cos(kz - \omega t))\Delta^2 \cos^2\theta + (1 - h_\oplus \cos(kz - \omega t))\sin^2\theta \Delta^2} \tag{11}$$

$$\sim \Delta \left(1 + \frac{1}{2} h_\oplus \cos(kz - \omega t)\cos(2\theta)\right). \tag{12}$$

The time dependent part is therefore $\frac{\Delta h_\oplus}{2}\cos(kz - \omega t)\cos(2\theta)$. [2]

(d) Simplest way is to work backwards. From a ξ_μ of the form [1]

$$\xi_\mu = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \cos(k_\alpha x^\alpha),$$

we find [2]

$$h_{\mu\nu} = k_\mu \xi_\nu + k_\nu \xi_\mu = \begin{pmatrix} 2a & b & c & d-a \\ b & 0 & 0 & -b \\ c & 0 & 0 & -c \\ d-a & -b & -c & -2d \end{pmatrix} \cos(k_\alpha x^\alpha).$$

Raising the indices gives

$$h^{\mu\nu} = k^\mu \xi^\nu + k^\nu \xi^\mu = \begin{pmatrix} 2a & -b & -c & a-d \\ -b & 0 & 0 & -b \\ -c & 0 & 0 & -c \\ a-d & -b & -c & -2d \end{pmatrix} \cos(k_\alpha x^\alpha),$$

or [2]

$$\phi^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h = \begin{pmatrix} a-d & -b & -c & a-d \\ -b & -a-d & 0 & -b \\ -c & 0 & -a-d & 0 \\ a-d & -b & -c & a-d \end{pmatrix} \cos(k_\alpha x^\alpha),$$

The transversality condition is satisfied for all members of this four component family.

Full marks are awarded for just finding one component, although the complete analysis would make the next part trivial.

(e) The other mode is [3]

$$\epsilon^{\mu\nu}_{\otimes} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & h_\otimes & 0 \\ 0 & h_\otimes & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This is the only possibility in light of the above. Educated guesswork suffices. Even without the complete analysis one should be able to argue that a transformation with dependence $\cos(k_\alpha x^\alpha)$ cannot produce entries in the xy and yx blocks.

6 This question concerns the following metric in 2 spacetime dimensions

$$ds^2 = (\alpha x)^2 dt^2 - dx^2 \,.$$

(a) Find the general form of the null geodesics, and sketch several examples on the $x - t$ plane. [4]

(b) The Schwarzchild metric is

$$ds^2 = c^2 \left(1 - \frac{2\mu}{r}\right) dt^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 \,.$$

Compare the geodesics from (a) with the radial null geodesics in the vicinity of the Schwarzchild radius $r_{\mathrm{s}} = 2\mu$, discussing both $r > r_{\mathrm{s}}$ and $r < r_{\mathrm{s}}$, explaining any similarities and differences. [4]

(c) Find the Christoffel symbols and show that the acceleration $a^\mu = \frac{Du^\mu}{D\tau}$ of a particle on the wordline $(x(\tau), t(\tau)) = (x_0, c\tau/(\alpha x_0))$ is constant. [4]

$\left[\Gamma^a_{bc} = \frac{1}{2} g^{ad} \left(\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc}\right)\right]$

(d) Express the metric in terms of the new coordinates

$$X = x \cosh(\alpha t)$$
$$T = (x/c) \sinh(\alpha t).$$

Interpret your answer to part (a) in terms of the new coordinates. [3]

(e) A spaceship has constant acceleration a in its own rest frame. A light signal is sent towards the departing spaceship from a distance d behind it, measured in the initial rest frame. Using the results from parts (a) and (c) (or otherwise), find the time until the signal catches up with the spaceship, as measured on the spaceship. [4]

(a) Null geodesics satisfy $dx/dt = \pm\alpha x$, so have the form $x = x_0 e^{\pm\alpha t}$. [2]

A sketch should show $\pm$ geodesics with $x_0 > 0$ and $x_0 < 0$. [2]

(b) Radial null geodesics in the Schwarzchild metric satisfy [1]

$$\frac{dr}{dt} = \pm c \left(1 - \frac{2\mu}{r}\right),$$

with solution [1]

$$ct = \pm r \pm 2\mu \log \left| \frac{r}{2\mu} - 1 \right| + \text{const.}$$

Close to r_{s}, we can write these as [1]

$$ct = \pm 2\mu \log |r - r_{\mathrm{s}}| + \text{const.},$$

or

$$r = r_{\rm s} \pm A e^{\pm ct/2\mu},$$

which have the same form as in part (a).

The *difference* between the two metrics is that the causal structure of the Schwarzchild metric changes at $r_{\rm s}$, on acount of the change of sign of the metric coefficients, while in the Rindler metric it does not. [1]

(c) Using the given formula, [2]

$$\Gamma^x_{tt} = \alpha^2 x \qquad \Gamma^t_{xt} = \Gamma^t_{xt} = x^{-1}.$$

The velocity is $u^\mu = (0, c/(\alpha x_0))$, which gives the acceleration [2]

$$\frac{Du^\mu}{D\tau} = \frac{du^\mu}{d\tau} + \Gamma^\mu_{\sigma\rho} u^\sigma u^\rho = (c^2/x_0, 0)\,.$$

(d) We have [1]

$$dX = dx \cosh(\alpha t) + \alpha x dt \sinh(\alpha t) \tag{13}$$
$$cdT = dx \sinh(\alpha t) + \alpha x dt \cosh(\alpha t)\,, \tag{14}$$

therefore [2]

$$c^2 dT^2 - dX^2 = (\alpha x)^2 dt^2 - dx^2.$$

This is the Minkowski metric. $x > 0$ corresponds to the wedge $X > 0$, $-X < T < X$, and $x < 0$ to $X < 0$, $X < T < -X$. The geodesics are straight lines, with the null geodesics being parallel to $X = \pm T$. The ingoing null geodesics of part (a) reach $X = \pm T$ after a finite increment of the affine parameter.

(e) *Using the Rindler coordinates.* If the spaceship leaves from $x_{\rm ship}$ it has acceleration $a = c^2/x_{\rm ship}$ (from part (c)). The light signal leaves from $x_{\rm light}$ and follows $x(t) = x_{\rm light} e^{\alpha t}$ (from part (a)), catching the spaceship when $x(t) = x_{\rm ship}$ (the spaceship is stationary in the Rindler frame). [2]

Thus $\alpha t = \log(x_{\rm ship}/x_{\rm light})$. The initial separation $x_{\rm ship} - x_{\rm light} = d$, giving [1]

$$\alpha t = -\log\left(1 - \frac{ad}{c^2}\right).$$

On $x = x_{\rm ship}$ constant the proper time elapsed is $\tau = \alpha x_{\rm ship} t/c$, so [1]

$$\tau = -\frac{c}{a}\log\left(1 - \frac{ad}{c^2}\right),$$

which diverges when $ad = c^2$ i.e. the ship accelerates too fast or starts too far ahead to be caught.

'First principles' i.e. not taking the help. The spaceship moves on a path $(X(\tau), T(\tau))$ in Minkowski space. The 4-acceleration is $a^\mu = (X'', T'')$. Transforming to the instantaneous rest frame

$$\gamma(X'' - vT'') = a \tag{15}$$
$$\gamma(T'' - vX''/c^2) = 0 \tag{16}$$

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where $v = X'/T'$ and $\gamma = (1 - v^2/c^2)^{-1/2}$. The solution is

$$X(\tau) = (c^2/a)\cosh(a\tau/c) \tag{17}$$

$$T(\tau) = (c/a)\sinh(a\tau/c). \tag{18}$$

A light ray leaving from d behind the spaceship has $(X - c^2/a + d)/T = c$ or $X - cT = c^2/a - d$ and hits the spaceship when

$$X - cT = \frac{c^2}{a} e^{-a\tau/c} = c^2/a - d,$$

which gives the same result

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