UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Tuesday 24 May 2016 9:00 am to 11:00 am

PHYSICS (2)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (2)

RELATIVITY

Candidates offering this paper should attempt a total of **three** *questions. The questions to be attempted are* **1**, **2** *and* **one** *other question.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains **four** *sides, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS
2 × 20 Page Answer Book
Rough workpad
Yellow master coversheet

SPECIAL REQUIREMENTS
Mathematical Formulae handbook
Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

1 *Attempt* **all** *parts of this question. Answers should be concise, and relevant formulae may be assumed without proof.*

(a) A two-dimensional Riemannian space has the metric

$$ds^2 = dr^2 + \alpha^2 r^4 d\phi^2$$

where α is a constant, $0 < r < \infty$ and $0 \leq \phi < 2\pi$. Using the Lagrangian method, find the geodesic equations for this metric. State what conserved quantity corresponds to the lack of dependence of the metric coefficients on ϕ, and show explicitly from your equations that this quantity is indeed constant. Also show explicitly that the Lagrangian itself is constant along a geodesic. [4]

Standard approach with $\mathcal{L} = \dot{r}^2 + \alpha^2 r^4 \dot{\phi}^2$, yields

$$\ddot{r} = 2\alpha^2 r^3 \dot{\phi}^2, \quad \ddot{\phi} = -\frac{4\dot{r}\dot{\phi}}{r}$$

Lack of dependence of coefficients of metric on ϕ means that

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = 2\alpha^2 r^4 \dot{\phi}$$

should be conserved. Differentiating $r^4\dot{\phi}$ to get $4r^3\dot{r}\dot{\phi} + r^4\ddot{\phi}$ we see immediately that this is 0.

Differentiating the Lagrangian itself we get

$$2\dot{r}\ddot{r} + 4\alpha^2 r^3 \dot{r}\dot{\phi}^2 + 2\alpha^2 r^4 \dot{\phi}\ddot{\phi}$$

and substituting our equations of motion in this we again get zero.

(b) For a given observer, the *kinetic energy* (K.E.) of a particle of rest mass m is defined as $(\gamma - 1)\, mc^2$, where γ is the Lorentz boost factor of the particle in the frame of the observer. A large particle disintegrates into a number of smaller particles which fly apart. Show that the total K.E. liberated in the explosion, i.e. the total K.E. after the explosion minus the total K.E. before the explosion, is a Lorentz invariant. [4]

If afterwards we have n particles with masses m_i and γ-factors γ_i, then the total KE is

$$\Sigma_{i=1}^{n} (\gamma_i - 1)\, m_i c^2 = \quad \text{Total energy} - \Sigma_{i=1}^{n} m_i c^2$$

Since the total energy must be conserved in the explosion, then if the rest mass of the initial mass is M (note we only need this, not any details of its initial motion), then the KE after minus the KE before evaluates to

$$Mc^2 - \Sigma_{i=1}^{n} m_i c^2$$

due to the total energy cancelling out. This only refers to the rest masses, and is therefore an invariant.

(c) A small radio transmitter radiates isotropically at a frequency ν_0 as measured in its rest frame. It is attached to the rim of a circular disc of radius r which is rotating at a constant angular velocity ω as measured in the laboratory frame. What are the maximum and minimum frequencies as measured by a stationary receiver located in the same plane as the disc and at a distance d ($> r$) from its centre?

A second receiver is attached to the rim of the disc at an angle α around the circumference with respect to the position of the transmitter, and rotates rigidly with the disc and the transmitter. What frequency will be measured at this receiver? [4]

[Hint: involved calculations are not needed in either case.]

The maximum and minimum frequencies will be received from the points on the disc's motion where the transmitter is moving tangentially towards or away from the fixed receiver. At these points its velocities relative to the receiver are $\pm r\omega$, hence from the standard SR head-on Doppler expression we have

$$\text{max. freq.} = \nu_0 \sqrt{\frac{1 + \frac{r^2\omega^2}{c^2}}{1 - \frac{r^2\omega^2}{c^2}}}, \quad \text{min. freq.} = \nu_0 \sqrt{\frac{1 - \frac{r^2\omega^2}{c^2}}{1 + \frac{r^2\omega^2}{c^2}}}$$

In the second case, since both source and receiver are on non-inertial trajectories, one would probably expect a bit more argument than saying that since they maintain a constant distance then there is no redshift. For example, the motion of one receiver relative to the other is motion in a circle, but we would still expect a shift here, due to the transverse Doppler effect, if this was a valid way of arguing. A good way to argue that there is no shift here, is to use the result that the redshift is given generally by

$$\frac{p \cdot v_{\text{observer}}}{p \cdot v_{\text{emitter}}}$$

where p is the photon 4-momentum, which in SR is the same at both source and receiver. The 4 velocities of source and receiver will be the same except for the direction of the spatial parts, but these will make the same angle with the spatial part of p, hence there is no shift, i.e. the measured frequency is ν_0.

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2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) the principle of equivalence;

[Topic 7, p4 onwards]

Initially introduced via considering particle motion in a freely-falling elevator. Particles released from rest remain floating 'weightless', while particles going from one

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side to other appear to move in a straight line at constant velocity, rather than a curved path. This follows from particle and elevator having same acceleration relative to Earth due to equivalence of gravitational and inertial mass.

These observations would hold exactly if the gravitational field of the Earth were truly uniform. But it actually acts radially inwards towards centre of mass with strength $\propto 1/r^2$. So, if elevator free falls for a long time, or 'elevator' is very large, particles released from rest in elevator would draw in horizontally (falling along radial lines), and draw out vertically (varying field strength with radius). These are the tidal forces from inhomogeneity in gravitational field, once main acceleration subtracted; always present in general for a finite elevator (laboratory).

If we make the provisos: time interval of observation is short, and elevator cabin is spatially small, then freely-falling elevator resembles a Cartesian inertial frame of reference. Thus the laws of special relativity hold inside the elevator.

We may incorporate these ideas into:

The Equivalence Principle: In a freely-falling (nonrotating) laboratory occupying a small region of spacetime, the laws of physics are those of special relativity

Basically asserts the equality of effects of acceleration and gravitational fields over limited regions. Has strong and weak forms.

In strong form extends to all the laws of physics — therefore difficult to test except in specific cases.

In weak form statement is restricted to equality of effects for motion of test masses. Here leads to identity of inertial and gravitational mass — the ratio of these has been found to be the same for all matter.

Equivalence principle amounts to: we can always find a local inertial system where the test particle moves in a straight line and the metric is SR. In this form provides a theoretical underpinning for GR, since it tells us how things behave in local frames.

As a specific application, the fact that the covariant derivative of the 4-momentum should vanish for a free particle, is one of the starting points for the geodesic equation, and this happens as the covariant generalisation of the SR law for particle motion, which we know must still apply locally due to the equivalence principle.

(b) the formation of a black hole as seen by a distant observer;

[Topic 11, p16 onwards. Note the discussion in the handout is divided into two parts, first a semi-qualitative description of the main features using Eddington-Finkelstein coordinates (which can reach inwards beyond the horizon), and then a more mathematical section in Schwarzschild coordinates, for which they already have the geodesic equations (for both massive particles and photons). A good answer would probably draw attention to the set of features in the first, qualitative section, which is sketched following, but maybe also illustrate some aspects with quantitative results from the second section (two examples given below).]

We can get an idea of this by following the collapse of a spherically symmetric ball of dust, which we assume starts from rest at infinity. We start with Eddington-Finkelstein coordinates and consider two observers, one riding the surface of the forming star down to

$r = 0$, and the other remaining fixed at some large radius. The infalling observer sends out light signals at equal intervals according to his or her clock. The distant observer receives pulses at later and later values of t' (his or her proper time), with the last light pulse reaching this observer emitted just before the surface of the star crosses $r = 2GM/c^2$.

Pulses emitted after the surface crosses this radius end up at the singularity at $r = 0$, so the distant observer never sees the star cross $r = 2GM/c^2$. Furthermore, pulses emitted at equal intervals by the falling observer arrive at increasingly longer intervals at the distant observer, which implies photons are also increasingly redshifted. As the star surface approaches $r = 2GM/c^2$, rate of arrival and redshift tend to infinity, so the distant observer sees the luminosity of star fall to zero.

In summary, the distant observer sees collapse slow down and approach a quasi-equilibrium object with radius $r = 2GM/c^2$, which becomes totally dark — this is the formation of a black hole.

Two more specific results highlighted in handout, are that the approach to the radius $r = 2GM/c^2 = 2\mu$ happens exponentially rapidly, with a characteristic time of $4\mu/c$, which is of the order of a few *micro*seconds for a solar mass star (very rapid for an astronomical event!) Also that the approach to zero luminosity also occurs exponentially, with a timescale of about 2μ, again very fast.

(c) the bending of light as a test of general relativity.

[Topic 10, p6 onwards]

The 'shape' equation for a photon trajectory in the equatorial plane of the Schwarzschild geometry is

$$\frac{d^2u}{d\phi^2} + u = \frac{3GM}{c^2}u^2 \tag{1}$$

where $u \equiv 1/r$. We can get a zeroth order solution to this by neglecting the r.h.s. This yields the straight line $u = \sin\phi/b$, where b is the impact parameter. Adding on a perturbation of the form:

$$u = \frac{\sin\phi}{b} + \Delta u,$$

and resubstituting into the original equation enables us to find a first order approximation, which is of the form

$$u = \frac{\sin\phi}{b} + \frac{3GM}{2c^2b^2}\left(1 + \tfrac{1}{3}\cos 2\phi\right)$$

(Exact form of this equation is not necessary.) Considering the limit as $r \to \infty$, i.e. $u \to 0$, we find the total deflection is

$$\Delta\phi = \frac{4GM}{c^2b}$$

(A diagram could be appropriate here to show how this comes about.) This is the famous GR result, and is a factor 2 bigger than would be predicted on Newtonian theory (first carried out by Soldner, 1804).

Eddington carried out the first measurement of this, in the eclipse expedition of 1919, where the expected deflection angle was $\sim 1.75°$. He obtained values consistent with GR (though there has been some controversy about this since).

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Later high precision tests use radio sources: these can be observed near the Sun, even when there is no lunar eclipse.

Modern radio experiments using VLBI (very long baseline interferometry) can measure gravitational deflection of positions of radio quasars as they are eclipsed by the Sun with an accuracy of better than $\sim 10^{-4}$ arcseconds. The results are in excellent agreement with the predictions of general relativity.

3 *Attempt* **either** *this question* **or** *question 4.*

Define the Ricci tensor $R_{\mu\nu}$ in terms of the Riemann tensor $R^{\mu}{}_{\nu\alpha\beta}$, and the Ricci scalar R in terms of $R_{\mu\nu}$. [2]

The Bianchi identity in General Relativity can be used to show that

$$\nabla_\mu \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) = 0$$

Discuss briefly the relationship between this identity and the conservation condition $\nabla_\mu T^{\mu\nu} = 0$, where $T^{\mu\nu}$ is the stress-energy tensor for a field or matter. [3]

[Bookwork: Topic 7, p21 and 8, p9 onwards]

$R_{\mu\nu} = R^{\alpha}{}_{\mu\nu\alpha}$ and $R = R^{\mu}_{\mu}$.

We have $G^{\mu\nu} \equiv \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right)$, and hence $\nabla_\mu G^{\mu\nu} = 0$. This is an automatic 'geometrical' identity, independent of anything to do with the matter or fields which are sources. The Einstein equations are $G^{\mu\nu} = -\kappa T^{\mu\nu}$, where the $T^{\mu\nu}$ here is the *total* stress-energy tensor for all the sources. Clearly this total SET must satisfy the conservation condition $\nabla_\mu T^{\mu\nu} = 0$, by virtue of the Bianchi identity. If there is only one type of source, then its SET is obliged to satisfy the conservation law individually.

The stress-energy tensor for a perfect fluid composed of dust (i.e. zero pressure) has the form

$$T^{\mu\nu} = \rho u^\mu u^\nu$$

Briefly describe the quantities ρ and u^μ, and by considering a local cartesian inertial frame, give a physical interpretation for the quantities T^{0i}, for $i = 1, 2, 3$. [3]

[Bookwork: p4 of Topic 8]

ρ is the density as measured in the instantaneous rest frame of an observer comoving with the fluid. u^μ is the 4-velocity of the fluid. In a local cartesian inertial frame this is $u^\mu = \gamma_u(c, \boldsymbol{u})$ and hence

$$T^{0i} = \rho u^0 u^i = \gamma_u^2 \rho c \boldsymbol{u}^i$$

which due to the length contraction by $1/\gamma_u$ in the direction of motion, is the perceived energy flux$/c$ in the i^{th} direction.

For such a fluid, show that the conservation condition $\nabla_\mu T^{\mu\nu} = 0$ leads to the requirement that each of the dust particles individually travels along a geodesic of the spacetime in which the fluid moves. [7]

[Bookwork: p23 of Topic 8]

Equation is

$$\nabla_\mu(\rho u^\mu u^\nu) = \nabla_\mu(\rho u^\mu) u^\nu + \rho u^\mu \nabla_\mu u^\nu \quad (*)$$

Now $u_\nu u^\nu = c^2$ means that $u_\nu \nabla_\mu u^\nu = 0$ and so contracting $(*)$ with u_ν yields

$$\nabla_\mu(\rho u^\mu) = 0$$

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which is the GR conservation equation. Inserting this into (*) then gives

$$u^\mu \nabla_\mu u^\nu = 0$$

as the equation of motion for the dust distribution under gravity. This is the intrinsic derivative along the worldline of the tangent vector to the worldline :

$$\frac{Du^\nu}{D\tau} = u^\mu \nabla_\mu u^\nu$$

and hence the dust particle motion is indeed along a geodesic.

Briefly describe the Faraday tensor $F^{\mu\nu}$, including its relationship with the electric and magnetic field vectors $\boldsymbol{E}$ and $\boldsymbol{B}$, and the Lorentz invariant quantities that can be constructed from $F^{\mu\nu}$. [4]

[Bookwork: Topic 6]

$F^{\mu\nu}$ is the Faraday tensor, which we can introduce via

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

where A^μ is the electromagnetic 4-potential. We can see that by construction $F^{\mu\nu}$ is antisymmetric. By noting that $A^\mu = (\phi/c, \boldsymbol{A})$, where ϕ and $\boldsymbol{A}$ are the ordinary electrostatic and magnetic potentials, we can relate $F^{\mu\nu}$ to the the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields, finding

$$[F_{\mu\nu}] = \begin{pmatrix} 0 & E^1/c & E^2/c & E^3/c \\ -E^1/c & 0 & -B^3 & B^2 \\ -E^2/c & B^3 & 0 & -B^1 \\ -E^3/c & -B^2 & B^1 & 0 \end{pmatrix}$$

The Lorentz invariants are $F^{\mu\nu}F_{\mu\nu}$ and $\epsilon_{\mu\nu\lambda\rho}F^{\mu\nu}F^{\lambda\rho}$, which evaluate to $\boldsymbol{E}^2 - \boldsymbol{B}^2$ and $\boldsymbol{E}\cdot\boldsymbol{B}$ respectively. Thus if $\boldsymbol{E}$ and $\boldsymbol{B}$ are orthogonal and of equal magnitude in one frame, they will be so in all frames.

If each of the dust particles in a fluid of density ρ carries charge q and has rest mass m, then it may be shown that the stress-energy tensor of the fluid now obeys the relation

$$\nabla_\mu T^{\mu\nu} = F^{\nu\alpha} j_\alpha$$

where $j^\alpha = (\rho/m)qu^\alpha$ is the 4-current of the fluid. Demonstrate from this that individual particles of the fluid now obey the equation of motion

$$u^\mu \nabla_\mu u^\nu = \frac{q}{m} F^{\nu\alpha} u_\alpha$$

and interpret this relation physically. [6]

[Extension of the bookwork (p23 of Topic 8) needed for part above]

Equation now is

$$\nabla_\mu(\rho u^\mu u^\nu) = \nabla_\mu(\rho u^\mu)u^\nu + \rho u^\mu \nabla_\mu u^\nu = F^{\nu\alpha} j_\alpha = F^{\nu\alpha}\frac{\rho}{m} q u_\alpha \quad (*)$$

We contract with u_ν again, and for the term on the r.h.s. this generates $(\rho q/m)F^{\nu\alpha}u_\alpha u_\nu$. We note this is 0 due to the antisymmetry of $F^{\nu\alpha}$.

Thus we still have

$$\nabla_\mu(\rho u^\mu) = 0$$

and inserting this into $(*)$ then gives

$$\rho u^\mu \nabla_\mu u^\nu = \frac{\rho q}{m} F^{\nu\alpha} u_\alpha$$

So now we have the intrinsic derivative along the worldline of the tangent vector to the worldline :

$$\frac{Du^\nu}{D\tau} = u^\mu \nabla_\mu u^\nu = \frac{q}{m} F^{\nu\alpha} u_\alpha$$

as required. This is the GR version of the Lorentz force law: particles would travel along geodesics, except that a force $q\boldsymbol{F}\cdot\boldsymbol{u}$ makes them deviate.

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4 *Attempt* **either** *this question* **or** *question 3.*

The Schwarzschild metric for the vacuum around a spherically symmetric body of mass M is

$$ds^2 = \left(1 - \frac{2\mu}{r}\right)c^2dt^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

where $\mu = GM/c^2$. Using the Lagrangian method, obtain the geodesic equations for a particle of non-zero mass moving in the $\theta = \pi/2$ plane in this metric, and in particular demonstrate

$$\left(1 - \frac{2\mu}{r}\right)\dot{t} = k, \quad c^2\left(1 - \frac{2\mu}{r}\right)\dot{t}^2 - \left(1 - \frac{2\mu}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = c^2 \quad \text{and} \quad r^2\dot{\phi} = h \quad (*)$$

where k and h are constants. [3]

Using the alternative form of the geodesic equation:

$$\dot{p}_\mu = \tfrac{1}{2}\left(\partial_\mu g_{\nu\sigma}\right)p^\nu p^\sigma$$

give a physical interpretation for the constants k and h. [2]

[Topic 9, p13]

Equations are

$$\frac{d}{d\sigma}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\right) = \frac{\partial\mathcal{L}}{\partial x^\mu}, \quad (\mu = 0, 1, 2, 3)$$

where

$$\mathcal{L} = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \left(1 - \frac{2GM}{c^2r}\right)\dot{t}^2 - \frac{1}{c^2}\left(1 - \frac{2GM}{c^2r}\right)^{-1}\dot{r}^2 - \frac{r^2}{c^2}\dot{\phi}^2$$

But $\mathcal{L}$ is *independent of t and ϕ* (i.e. $g_{\mu\nu}$ is). So we have

$$\frac{d}{d\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{t}}\right) = \frac{d}{d\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right) = 0,$$

i.e.

$$\left(1 - \frac{2GM}{c^2r}\right)\dot{t} = \text{constant} = k \text{ say,} \quad (2)$$

$$\text{and} \quad r^2\dot{\phi} = \text{constant} = h \text{ say.} \quad (3)$$

The radial equation is more complicated, but we can avoid the need for it by remembering that $\mathcal{L} = 1$.

Thus gives us the third equation we need, i.e.

$$c^2\left(1 - \frac{2\mu}{r}\right)\dot{t}^2 - \left(1 - \frac{2\mu}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = c^2$$

The alternative form of the geodesic equation tells us that the 'downstairs' momentum component p_μ is conserved if the metric is independent of the μth coordinate. We can therefore think of p_μ as the momentum 'conjugate' to the symmetry represented by $\partial_\mu g_{\nu\sigma} = 0$. Thus

$$p_0 = g_{0\nu} m \dot{x}^\nu = \left(1 - \frac{2GM}{c^2 r}\right) m\dot{t} = mk$$

identifies k as the (conserved) energy per init mass, and

$$p_3 = g_{3\nu} m \dot{x}^\nu = mr^2 \dot{\phi} = -mh$$

identifies h as minus the (conserved) angular momentum per unit mass.

By using the equations $(*)$, construct the 'energy equation' for particle motion, in the form

$$\dot{r}^2 + \frac{h^2}{r^2}\left(1 - \frac{2\mu}{r}\right) - \frac{2c^2\mu}{r} = c^2(k^2 - 1)$$

[2]

Substituting we get

$$\left(1 - \frac{2GM}{c^2 r}\right)^{-1} k^2 - \frac{1}{c^2}\left(1 - \frac{2GM}{c^2 r}\right)^{-1} \dot{r}^2 - \frac{r^2 h^2}{c^2 r^4} = 1,$$

i.e.

$$\dot{r}^2 + \frac{h^2}{r^2}\left(1 - \frac{2GM}{c^2 r}\right) - \frac{2GM}{r} = c^2(k^2 - 1).$$

Now consider the case of a particle projected radially outwards from a radius $r_0 > 2\mu$, which just escapes to infinity, and has zero velocity there. Using the energy equation, demonstrate that

$$r(\tau) = \left(r_0^{3/2} + a\tau\right)^{2/3}$$

gives the radius $r(\tau)$ reached by the particle after an elapse of proper time τ, and find the constant a.

[5]

[This is basically a rearrangement of the formula they see derived in Topic 9, p19.]

Radial motion means that $\dot{\phi} = 0$, and hence $h = 0$. Starting at rest at infinity means from putting $r \mapsto \infty$ and $\dot{r} = 0$ in the energy equation that $k = 1$. Hence this equation is just

$$\dot{r}^2 = \frac{2GM}{r}, \quad \text{or} \quad \dot{r} = \sqrt{\frac{2GM}{r}} \quad \text{since the motion is outwards}$$

We can now either get the solution constructively, e.g. by writing

$$r^{1/2} dr = \sqrt{2GM} d\tau \implies \frac{2}{3} r^{3/2} = \sqrt{2GM}\tau + \text{const.}$$

or substituting in the given form, obtaining

$$\dot{r} = \frac{2}{3} r^{-1/2} a = \sqrt{\frac{2GM}{r}} \quad \text{from which we deduce} \quad a = 3\sqrt{GM/2}$$

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By defining relative velocity in terms of rate of change of proper distance with respect to proper time, both as measured by a given observer, calculate the velocity of the particle described in the previous paragraph relative to a stationary observer at a general radius r at the moment the particle passes by the observer. [*Alternative methods of calculating the relative velocity will be accepted.*] [6]

[This part is from Topic 9, p20, adapted to an outgoing rather than infalling particle.]

For a stationary observer at radius r, then their proper time is given by

$$dt_p = \left(1 - \frac{2\mu}{r}\right)^{1/2} dt$$

while the proper radial distance measured is

$$dr_p = \left(1 - \frac{2\mu}{r}\right)^{-1/2} dr$$

Earlier we found that for the particle,

$$\frac{dt}{d\tau} = \left(1 - \frac{2\mu}{r}\right)^{-1}, \quad \frac{dr}{d\tau} = \left(\frac{2\mu c^2}{r}\right)^{1/2}$$

Hence putting these together we find that the speed of the radially outgoing particle as measured by the observer is

$$\frac{dr_p}{dt_p} = \left(1 - \frac{2\mu}{r}\right)^{-1} \frac{dr}{dt} = \left(\frac{2\mu c^2}{r}\right)^{1/2}$$

The particle is now projected from r_0 with an outward velocity such that it has a non-zero velocity u at infinity. Find the velocity of the particle as measured by a stationary observer at radius r at the moment the particle passes by them, in this new case, expressing your answer in terms of M, r and u. [*Note that an expression for the radial position of the particle as a function of proper time is not needed.*] [7]

[This part is new to them, but we can basically apply the same methods as in the preceding para.]

First from the $\dot{r}$ expression, evaluated at infinity, we now have that

$$k = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

Plugging these into the results for $\dot{t}$ and $\dot{r}$, we find

$$\frac{dt}{d\tau} = \left(1 - \frac{2\mu}{r}\right)^{-1} \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}, \quad \left(\frac{dr}{d\tau}\right)^2 = \frac{u^2}{1 - \frac{u^2}{c^2}} + \frac{2\mu c^2}{r}$$

Meanwhile, the expressions for dt_p and dr_p are the same as in the preceding part. So putting things together, we find (squaring for clarity)

$$\left(\frac{dr_p}{dt_p}\right)^2 = \left(1 - \frac{2\mu}{r}\right)^{-2}\left(\frac{dr}{d\tau}\right)^2 = \frac{2GM}{r} + u^2\left(1 - \frac{2GM}{rc^2}\right)$$

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