[Markscheme]

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Q.1 (a) $\sum m_j$ = good quantum number m_J [1]

Max m_J defines max J [1]

Max m_J due to exclusion principle
$= 5/2 + 3/2 + 1/2 + (-1/2) = 8/2$

$\Rightarrow$ **Max $J = 4$** [2]

(b) Two possible approaches

(i) Due to centripetal barrier $V^{eff}(r) = \dfrac{L(L+1)\hbar^2}{2mr^2}$ [2]

only $l=0$ partial wave reaches scatterer in low energy limit. [2]

OR ii) Range of scattering vector $0 < \Delta k < 2|k_i|$ [1] k_i incident wavevector

if $\Delta k\, a << 1$ phase of all scatterings [1] within potential are the same (0)

$\Rightarrow$ all scatterings in phase irrespective of direction of outgoing wave $\rightarrow$ isotropic

requires $k_i\, a << 1$ - low energy [2]

(c)

↑	↓	m_l	
↑	↓	1	Max $S = 1$ [1]
↑		0	Max L (given S) $= 1$ [1]
↑		-1	$> 1/2$ full $J = L + S = 2$ [1]

$^3P_2$ [1]

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Q2 (a) Variational method in QM.

i) Theorem $\frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} \geq E_0$ [1]

(ii) Proof [2]

(iii) How to apply - variational parameter [1]

(iv) Excited states - not bound, in general, for E_1, E_2 except for symmetry, [1]
or fixed basis set for $|\psi_0\rangle + |\psi_1\rangle$ [1]

(v) Use to prove existence of bound state [1]

b) Landau levels

(i) 2D electron gas + ⊥ magnetic field. [1]

(ii) Different gauges - different description of states. [1]

(iii) Massive degeneracy in levels of energy $(n + 1/2)\hbar\omega_c$ $\omega_c = \frac{eB}{m_e}$ [1]

(iv)

(v) in 3D normal dispersion along field [1]

(vi) Further splitting due to spin magnetic energy [1]

(vii) Number of states per level $= \frac{eBA}{h}$ [2]

(c) ~~Spectrum of~~ ~~Groundstate and~~ ~~excited electron state of~~ Helium atom.

i) 2 electron atom - angular momentum states no longer degenerate. ~ Each electron in ~~hydrogen-like state~~. [1]

ii) Identical particle symmetry

$S = 1$ symm requires antisymm spatial [1]

$S = 0$ antisymm requires symm spatial

iii) Spatial state - antisymmetry requires electrons in different single particle states - same state forbidden [1]

Q2 cont

iv) Antisym spatial state lower in energy [1]
. exchange interaction

v) Further splitting of J states [1]
(for S = 1 states only)

vi) Example of lower states) [2]

OR

Diagrammatically — All information

S = 0 [1] | S = 1 [1]

$^1S_0$ $^1P_1$ [1] $^1D_2$ etc | $^3S_1$ $^3P_{2,1,0}$ [1] $^3D_{3,2,1}$ etc

(1s)(3s) (1s)(3p) (1s)(3d) | (1s)(3s) (1s)(3p) (1s)(3d)

(1s)(2p) [1] | (1s)(2p)

(1s)(2s) — | (1s)(2s) } exchange splitting [1]

$(1s)^2$ — [1] | No such state

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Q3 (i)

use λ to designate order

$$\hat{H} = \hat{H}_0 + \lambda \hat{H}_1 \quad [1] \qquad |\phi_n\rangle = |\psi_n\rangle + \lambda \sum_j c_j |\psi_j\rangle \quad [1]$$

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$$E_n^T = E_n + \lambda E_n'$$

$$\hat{H} |\phi_n\rangle = \mathcal{E}_n |\phi_n\rangle \quad [1]$$

$$(\hat{H}_0 + \lambda \hat{H}_1)\left(|\psi_n\rangle + \lambda \sum_j c_j |\psi_j\rangle\right) = (E_n + \lambda E_n')\left(|\psi_n\rangle + \lambda \sum_j c_j |\psi_j\rangle\right)$$

[1]

Terms in λ

$$\hat{H}_1 |\psi_n\rangle + \hat{H}_0 \sum_j c_j |\psi_j\rangle = E_n' |\psi_n\rangle + E_n \sum_j c_j |\psi_j\rangle$$

$$\hat{H}_0 \sum_j c_j |\psi_j\rangle \to \sum_j c_j E_j |\psi_j\rangle$$

$\langle \psi_{j'}|$ [1] $\delta_{j j'}$, $j' \neq n$

$$\langle \psi_{j'} | \hat{H}_1 | \psi_n \rangle + c_{j'} E_{j'} = E_n c_{j'} \quad [1]$$

$$c_{j'} = \frac{\langle \psi_{j'} | \hat{H}_1 | \psi_n \rangle}{E_n - E_j} \quad [1]$$

$$\rightarrow |\phi_n\rangle = |\psi_n\rangle + \sum_{j \neq n} \frac{\langle \psi_j | \hat{H}_1 | \psi_n \rangle}{E_n - E_j} |\psi_j\rangle \quad [1]$$

$$\hat{H}_1 = \hat{V} = -\mathcal{E} q x \quad [1]$$

$$|\phi_0\rangle = |\psi_0\rangle + \frac{\langle \psi_1 | -\mathcal{E} q x | \psi_0 \rangle}{-\hbar\omega} |\psi_1\rangle \quad [1]$$

$$\langle \psi_1 | x | \psi_0 \rangle = \frac{\langle \psi_1 | \psi_1 \rangle}{\sqrt{2\alpha}} = \sqrt{\frac{\hbar}{2m\omega}} \quad [1]$$

$$\therefore |\phi_0\rangle = |\psi_0\rangle + \frac{\mathcal{E} q}{\hbar\omega} \sqrt{\frac{\hbar}{2m\omega}} |\psi_1\rangle \quad [1]$$

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[1] $\langle \psi_0 | x | \psi_0 \rangle = 0$ etc

$$\text{D.pol} = \langle \phi_0 | q x | \phi_0 \rangle \quad [1]$$

[1]

$$= \frac{\mathcal{E} q^2}{\hbar\omega} \sqrt{\frac{\hbar}{2m\omega}} \left(\langle \psi_0 | x | \psi_1 \rangle + \langle \psi_1 | x | \psi_0 \rangle \right)$$

$$= \frac{2 \mathcal{E} q^2}{\hbar\omega} \frac{\hbar}{2m\omega} = 2 \mathcal{E} q^2 \frac{1}{2k} = \frac{\mathcal{E} q^2}{k}$$

[1]

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[1]

$$\frac{1}{2} k x^2 - \mathcal{E} q x = \frac{1}{2} k \left(x - \frac{\mathcal{E} q}{k} \right)^2 - \frac{\mathcal{E}^2 q^2}{2k}$$

[1]

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Q3 (ii) For S.H.O $\langle \psi_n | x | \psi_0 \rangle = 0$

except for $n = 1$

$\therefore$ Only $n=1$ term in perturbation theory is non-zero so expression is exact to 1st order even though have only included $j=1$ term. [2]

However larger ε - get higher order terms ie scattering from $\psi_1 \to \psi_2$ etc and these are not included in our expression [2]

→ We do not see this as for small ε do not need to worry about normalisation but clearly expression will eventually break down

Normalisation of perturbed state also a problem. When expression is correctly normalised the $\langle p \rangle_{max}$ for pure state $|\psi_0\rangle, |\psi_1\rangle$ [2]

is then $|\phi\rangle = \frac{1}{\sqrt{2}}\left(|\psi_0\rangle + |\psi_1\rangle\right)$

and $p_{max} = q \cdot \langle \phi | x | \phi \rangle$

$= q \langle \psi_0 | x | \psi_1 \rangle = q\sqrt{\frac{\hbar}{2m\omega}}$

Unseen, general principle, discussed technique

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Q4. $\left[H_0 + H' \cos(\omega t)\right]\left(c_1(t)\exp(i\omega_1 t)|\psi_1\rangle + c_2(t)\exp(i\omega_2 t)|\psi_2\rangle\right)$

$= i\hbar \frac{\partial}{\partial t}\left(c_1(t)\exp(i\omega_1 t)|\psi_1\rangle + c_2(t)\exp(i\omega_2 t)|\psi_2\rangle\right)$ [2]

$H_0|\psi_1\rangle\, c_1(t)\exp(i\omega_1 t) = E_1|\psi_1\rangle c_1(t)\exp(i\omega_1 t)$

$i\hbar \frac{\partial \exp(i\omega_1 t)}{\partial t}\left[c_1(t)|\psi_1\rangle\right] = \hbar\omega_1 \exp(i\omega_1 t)|\psi_1\rangle c_1(t)$

Terms cancel, similarly for $|\psi_2\rangle$ [1]

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$\Rightarrow H'\cos(\omega t)\left[c_1(t)\exp(i\omega_1 t)|\psi_1\rangle + c_2(t)\exp(-i\omega_2 t)|\psi_2\rangle\right]$ [2]

$= i\hbar\frac{\partial c_1(t)}{\partial t}\exp(-i\omega_1 t)|\psi_1\rangle + i\hbar\frac{\partial c_2(t)}{\partial t}\exp(-i\omega_2 t)|\psi_2\rangle$

$\langle\psi_2| \longrightarrow \hbar\omega'^{*}\frac{1}{2}\left(\exp(i\omega t)+\exp(-i\omega t)\right) c_1(t)\exp(-i\omega_1 t)$

$\langle\psi_2|H'|\psi_1\rangle = (\hbar\omega')^{*}$ $\quad = i\hbar\frac{\partial c_2}{\partial t}\exp(-i\omega_2 t)$ [1]

$\longrightarrow \frac{\omega'^{*}}{2} c_1(t)\left[\exp(i(\omega+\omega_0)t) + \exp(i(\omega_0-\omega)t)\right]$

$= i\frac{\partial c_2(t)}{\partial t}$ [1]

[1]

$\langle\psi_1| \longrightarrow \hbar\omega'\frac{1}{2}\left(\exp(i\omega t)+\exp(-i\omega t)\right)c_2(t)\exp(-i\omega_2 t)$

$= i\hbar\frac{\partial c_1}{\partial t}\exp(-i\omega_1 t)$

$\longrightarrow \frac{\omega'}{2}c_2(t)\left[\exp(i(\omega-\omega_0)t) + \exp(-i(\omega+\omega_0)t)\right]$

$= i\frac{\partial c_2}{\partial t}$ [1]

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QL (cont)

At $\omega \sim \omega_0$ $\omega - \omega_0$ very small

$\omega + \omega_0 \gg \omega - \omega_0$

Terms in $(\omega + \omega_0)$ rapidly oscillating [1]
~~do not~~ tend to produce destructive interference

Say $c_1 \sim 1$ $c_2 \sim 0$ at $t = 0$

$$c_2(t) \sim \frac{1}{i(\omega + \omega_0)} \left[\exp i(\omega + \omega_0)t\right]_0^t \frac{(\omega')^*}{2}$$

oscillates [1]

whereas $\omega - \omega_0$ term can produce steady increase in $c_2(t) \sim \frac{1}{i}\frac{(\omega')^*}{2} t$ [1]

Spin ½ nuclear energy levels $\pm \frac{1}{2}\gamma\hbar B$

$\omega_0 = \gamma B$ B ↑ $|\psi_1\rangle$

↓ $|\psi_2\rangle$

For $\boxed{\omega = \omega_0}$

$$\rightarrow \frac{(\omega')^*}{2} c_1(t) = i\frac{\partial c_2(t)}{\partial t}$$

$$\frac{(\omega')}{2} c_2(t) = i\frac{\partial c_1(t)}{\partial t}$$

$$\rightarrow \frac{|\omega'|^2}{4} c_1(t) = -\frac{\partial^2 c_1}{\partial t^2}$$ [2]

Soln $c_1(t) = c_1(0)\cos\left(\frac{|\omega'|}{2}\right)t$

$$c_2(t) = \frac{(\omega')^*}{i|\omega'|} \sin\left(\frac{|\omega'|}{2}t\right)$$

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Q4 cont

Unseen

Spin in x direcn $= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

$\therefore \quad \frac{|\omega'|}{2} t = \frac{\pi}{4}$ [1]

$\frac{(\omega')^*}{i|\omega'|} = 1$ (reqd)

[1]

$\frac{(\omega')^*}{|\omega'|} = -i$ → Field in y direcn

[1]

$|\omega'| = \gamma B \frac{1}{2}\hbar$

Choice of B, t but $\gamma B \hbar t = \pi$

Spin in y direcn $= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$ [1]

$\frac{(\omega')^*}{i|\omega'|} = -1$ → Field in $-x$ direction

t, B etc as before [1]

Seen in lectures as spin precession [1] x, y plane

$-z$ spin $= \begin{pmatrix}0\\1\end{pmatrix}$

Field anywhere in x y plane [1]

with $\frac{|\omega'|}{2} t = \frac{\pi}{2}$. $\gamma B \hbar t = 2\pi$

[1]

Unseen, as is lecture

If $\omega \approx \omega_0$ but $\omega \neq \omega_0$

Then $|c_2(t)|$ never become 1 [1]

x y rotation still possible [2]

but not $-z$. [1]