NATURAL SCIENCES TRIPOS Part II

Saturday 30 May 2009 9.00 am to 12.00 noon

EXPERIMENTAL AND THEORETICAL PHYSICS (2)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (2)

Candidates offering the whole *of this paper should attempt a total of* six *questions, three from Section A* and *three from Section B. The questions to be attempted are* A1, A2 *and* one *other question from Section A and* B1, B2 *and* one *other question from Section B.*

Candidates offering half *of this paper should attempt a total of* three *questions,* either *three from Section A* or *three from Section B. The questions to be attempted are* A1, A2 *and* one *other question from Section A or* B1, B2 *and* one *other question from Section B. These candidates will leave after* 90 minutes.

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains 7 sides, and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.

Answers to each *section should be tied up separately, with the numbers of the questions attempted written clearly on the cover sheet.*

STATIONERY REQUIREMENTS
Script paper
Metric graph paper
Rough workpad
Blue coversheets (2)
Treasury tags

SPECIAL REQUIREMENTS
Mathematical formulae handbook
Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

ADVANCED QUANTUM PHYSICS

A1 *Attempt* **all** *parts of this question. Answers should be concise, and relevant formulae may be assumed without proof.*

(a) Assuming Hund's rules apply, determine the spectroscopic term of the ground state of nitrogen, with configuration $(1s)^2(2s)^2(2p)^3$. [4]

(b) Two non-interacting spinless particles move in a one-dimensional potential possessing orthonormal single-particle eigenstates $|a\rangle = \psi_a(x)$ and $|b\rangle = \psi_b(x)$. Show that the mean square separation $\langle\psi|(x_1 - x_2)^2|\psi\rangle$ of the particles is reduced by an amount $2\,|\langle a|x|b\rangle|^2$ when the particles are in the state $|\psi\rangle = (1/\sqrt{2})\,[|a,b\rangle + |b,a\rangle]$ compared to the state $|\psi\rangle = |a,b\rangle$, where $|a,b\rangle = \psi_a(x_1)\psi_b(x_2)$ and $|b,a\rangle = \psi_b(x_1)\psi_a(x_2)$. Comment on this result. [4]

(c) In single-electron atoms, the electronic eigenstates have azimuthal dependence $e^{im\phi}$. Show that electric dipole transitions in single-electron atoms obey the selection rule $\Delta m = 0, \pm 1$. [4]

A2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) four-level laser operation;

(b) molecular bonding in H_2^+ and H_2;

(c) the Born approximation in scattering theory.

A3 *Attempt* **either** *this question* **or** *question A4.*

When lowest-order relativistic corrections are included, the hydrogen atom is described by a Hamiltonian

$$\hat{H} = \frac{\hat{\boldsymbol{p}}^2}{2m_e} + V(r) - \frac{\hat{\boldsymbol{p}}^4}{8m_e^3c^2} + \frac{1}{2m_e^2c^2}\frac{1}{r}\frac{\mathrm{d}V}{\mathrm{d}r}\hat{\boldsymbol{L}}\cdot\hat{\boldsymbol{S}} - \frac{e\hbar^2}{8m_e^2c^2}\nabla^2 V ,$$

where $V(r) = -\beta/r$, with $\beta = e^2/4\pi\epsilon_0$.

Show that, for the $2S_{1/2}$, $2P_{1/2}$ and $2P_{3/2}$ states of atomic hydrogen, the expectation values of the operator $\hat{\boldsymbol{L}}\cdot\hat{\boldsymbol{S}}$ are equal to 0, $-\hbar^2$ and $\frac{1}{2}\hbar^2$, respectively. Hence show that the spin-orbit term produces an energy correction $-E'/3$ for the $2P_{1/2}$ state and $E'/6$ for the $2P_{3/2}$ state, where $E' = \frac{1}{16}m_e c^2\alpha^4$ and $\alpha = e^2/4\pi\epsilon_0\hbar c \approx 1/137$. [8]

By expressing the $\hat{\boldsymbol{p}}^4$ term in terms of $\hat{H}_0$ and V, with $\hat{H}_0 = (\hat{\boldsymbol{p}}^2/2m_e) + V$, show that the expectation value of the $\hat{\boldsymbol{p}}^4$ term for an eigenstate of $\hat{H}_0$ with energy E_0 is

$$-\frac{1}{2m_e c^2}\left(E_0^2 + 2E_0\beta\left\langle\frac{1}{r}\right\rangle + \beta^2\left\langle\frac{1}{r^2}\right\rangle\right) .$$

Given that $E_0 = -\frac{1}{8}m_e c^2\alpha^2$ for the $2S$ and $2P$ states, show that the $\hat{\boldsymbol{p}}^4$ term results in an energy correction $-\frac{13}{8}E'$ for the $2S_{1/2}$ state and $-\frac{7}{24}E'$ for the $2P$ states. [7]

Given that $\nabla^2(1/r) = -4\pi\delta(\boldsymbol{r})$, explain why the energy correction due to the $\nabla^2 V$ term is non-zero only for S states. [3]

Given that the $\nabla^2 V$ term results in an energy correction equal to E' for the $2S_{1/2}$ state, compare the energy level splittings predicted by the Hamiltonian $\hat{H}$ for the $2S$ and $2P$ states with the measured energy levels of 10.117845 eV, 10.117849 eV, and 10.117890 eV. Comment on how any residual discrepancies can be understood. [7]

[The $2S$ and $2P$ states of hydrogen have expectation values

$$\left\langle\frac{1}{r}\right\rangle = \frac{1}{4a_0}, \quad \left\langle\frac{1}{r^2}\right\rangle = \frac{1}{8a_0^2(\ell+\frac{1}{2})}, \quad \left\langle\frac{1}{r^3}\right\rangle = \frac{1}{8a_0^3(\ell+1)(\ell+\frac{1}{2})\ell},$$

where a_0 is the Bohr radius:

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{m_e e^2} .$$

Also

$$m_e c^2 = 511\,\mathrm{keV} . \quad]$$

(TURN OVER

A4 *Attempt* **either** *this question* **or** *question A3.*

A system is described by a Hamiltonian $\hat{H}(t) = \hat{H}_0 + \hat{H}' \cos \omega t$, where the operators $\hat{H}_0$ and $\hat{H}'$ are time-independent. The operator $\hat{H}_0$ has two and only two eigenstates ψ_1 and ψ_2, with energies $E_1 = \hbar\omega_1$ and $E_2 = \hbar\omega_2$, respectively, where $E_2 > E_1$. The operator $\hat{H}'$ has matrix elements

$$\langle \psi_1 | \hat{H}' | \psi_1 \rangle = \langle \psi_2 | \hat{H}' | \psi_2 \rangle = 0 \ , \qquad \langle \psi_1 | \hat{H}' | \psi_2 \rangle = \hbar\omega' \ . \qquad (*)$$

The time evolution of the system is described by a wavefunction $\psi(t) = c_1(t) e^{-i\omega_1 t} \psi_1 + c_2(t) e^{-i\omega_2 t} \psi_2$.

Show that, for $\omega \approx \omega_0 \equiv \omega_2 - \omega_1$, the coefficients $c_1(t)$ and $c_2(t)$ satisfy the equations

$$i \frac{\mathrm{d}c_1}{\mathrm{d}t} = \frac{\omega'}{2} c_2(t) e^{i(\omega - \omega_0)t} \ , \qquad i \frac{\mathrm{d}c_2}{\mathrm{d}t} = \frac{\omega'}{2} c_1(t) e^{-i(\omega - \omega_0)t} \ .$$

[5]

Obtain a linear, second-order differential equation for $c_2(t)$, and hence show that, if the system is in the state ψ_1 at time $t = 0$, then $c_2(t) \propto e^{-\frac{1}{2} i(\omega - \omega_0)t} \sin\left(\frac{1}{2}\omega_R t\right)$, where $\omega_R = \sqrt{(\omega - \omega_0)^2 + \omega'^2}$. [5]

Obtain the constant of proportionality, and hence show that the probability that the system is found in the state ψ_2 at time $t > 0$ is equal to $(\omega'^2 / \omega_R^2) \sin^2\left(\frac{1}{2}\omega_R t\right)$. Sketch the maximum transition probability as a function of ω. [6]

A beam of hydrogen molecules travelling in the $+x$ direction passes through a region containing an oscillating magnetic field $(B_x, 0, 0) \cos \omega t$ of fixed frequency 5.662 MHz superimposed on a uniform, static magnetic field $(0, 0, B_z)$. Inhomogeneous magnetic fields upstream and downstream of this region ensure that the hydrogen molecules enter with each proton spin aligned along $\pm z$, and that molecules are subsequently lost from the beam if either of the proton spin states changes. The field strength B_z is varied, and a sharp minimum in the number of molecules detected downstream is observed for $B_z = 0.1336$ T.

Write down operators $\hat{H}_0$ and $\hat{H}'$ that correspond to this experiment, and explain why the matrix elements of $\hat{H}'$ have the form $(*)$ above. Determine the magnitude of the proton magnetic moment. [9]

SECTION B

OPTICS AND ELECTRODYNAMICS

B1 *Attempt* **all** *parts of this question. Answers should be concise, and relevant formulae may be assumed without proof.*

(a) Why does a charge moving uniformly in vacuum not radiate? [4]

(b) A transparent garnet crystal of length L and Verdet coefficient $134\,\mathrm{radians}\,\mathrm{T}^{-1}\,\mathrm{m}^{-1}$, within a 10-turn solenoid also of length L and carrying a current of 1 A, is placed between crossed polarizers. What fraction of the light passing through the first polarizer is transmitted by the second? (Ignore end effects on the solenoid field.) [4]

(c) Why is scattered light polarised? Use a diagram to describe the polarisation of visible light scattered from the solar corona as seen during a solar eclipse. [4]

B2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) the formation of photonic band gaps;

(b) the Aharanov-Bohm effect;

(c) the propagation of light through calcite.

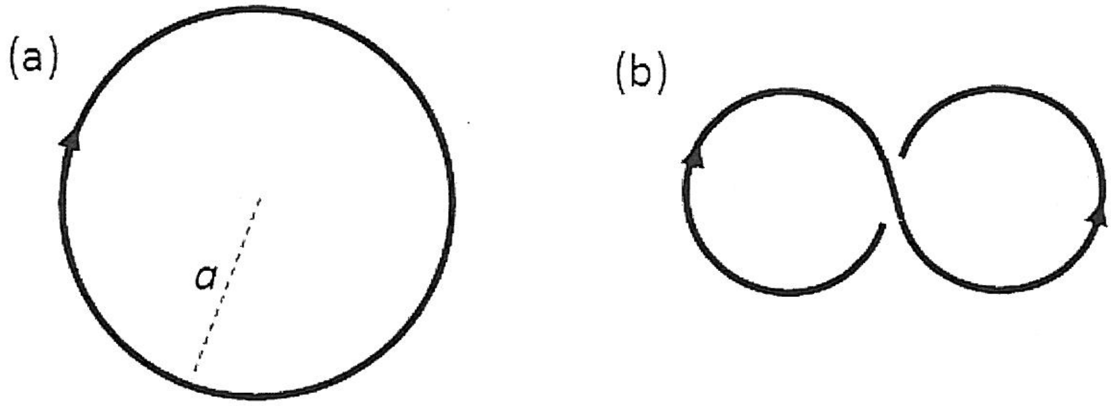
(TURN OVER

B3 *Attempt* **either** *this question* **or** *question B4.*

A circular loop carrying an alternating current (as in Figure (a)) emits magnetic dipole (MD) radiation. The magnetic field $B^{MD}(r, \theta, \phi)$ is given by

$$B_\theta^{MD} = \frac{\mu_0 \sin\theta}{4\pi} \frac{[\ddot{m}]}{rc^2} \hat{\theta}.$$

State the assumptions made in obtaining this expression. Include a sketch of the radiation pattern from this dipole, and the expected polarisation of the radiated electric field. [7]



The same wire loop is now twisted into a figure of eight as in Figure (b), without shorting at the crossover. By summing up the radiation from the two resulting wire loops, show that the total magnetic quadrupole (MQ) field found at distances $r \gg a$ is

$$B_\theta^{MQ} = \frac{\mu_0 \sin\theta}{4\pi} \frac{[\dddot{m}]}{rc^2} \frac{a \sin\theta \sin\phi}{2c} \hat{\theta},$$

explaining why the separation of the dipoles along the viewed direction is $a \sin\theta \sin\phi/2$. $\left[\right.$*It may be assumed that* $\frac{\delta[X]}{\delta r} = \frac{-1}{c}[\dot{X}]$ *where* $[X]$ *is a retarded quantity.*$\left.\right]$ [6]

What is meant by the *radiation resistance* of an antenna emitting at frequency ω? By estimating the ratio of the power emitted before and after the loop is twisted, calculate the fractional change in the radiation resistance caused by twisting the loop. [7]

Discuss the radiation pattern of the twisted loop, and explain for which direction and for which polarisation the effective area is largest. Is the maximum possible effective area of the antenna larger or smaller after it is twisted? [5]

B4 *Attempt* either *this question* or *question B3.*

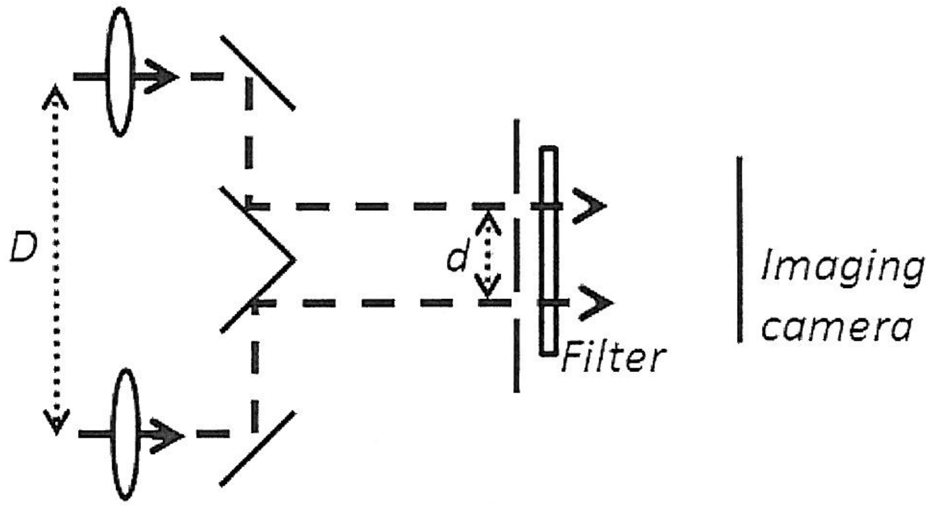
Explain what is meant by the *degree of mutual coherence* between two locations within an optical field. [4]

Explain briefly how one would measure the coherence width and coherence length of an optical field. [4]

A star emits a broad spectrum of light but its surrounding gas cloud absorbs a single narrow spectral line from this spectrum. Explain what will be observed in an experiment that measures the coherence length of the light transmitted by the gas cloud. [5]

Write down and explain the relation between a distant emitting source of particular angular intensity distribution and the fringe visibility it can produce. [4]

Each aperture of a double slit of spacing $d = 155\,\mu$m is independently illuminated by light from a double aperture telescope (as in the Figure below) pointed at a double star which is 8.6 light years from Earth, with each star of equal brightness. The resulting interference fringes are observed on an imaging camera.



The slits are combined with a spectral filter which transmits only a very narrow spectral width around $\lambda_0 = 589.0\,$nm. Explain what would be observed as the spacing of the receivers D is increased if the stars are 3×10^9 m apart (perpendicular to the line of sight), being as quantitative as possible. What would happen if the spectral filter transmits a wider bandwidth? [8]

END OF PAPER