UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Wednesday 25 May 2016 9.00 am to 11.00 am

PHYSICS (3)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (3)

ADVANCED QUANTUM PHYSICS

Candidates offering this paper should attempt a total of **three** *questions. The questions to be attempted are* **1**, **2** *and* **one** *other question.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains **four** *sides, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS
2×20 Page Answer Book
Rough workpad
Yellow master coversheet

SPECIAL REQUIREMENTS
Mathematical Formulae handbook
Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

ADVANCED QUANTUM PHYSICS

1 *Attempt* **all** *parts of this question. Answers should be concise and relevant formulae may be assumed without proof.*

(a) Find the magnitude of the energy splitting of the ground state of Ce^{3+}, which has electronic configuration $...(4f)^1$, in a magnetic field of 1 T.

[*Note that the Landé g-factor is* $g = \frac{3}{2} + \frac{S(S+1)-L(L+1)}{2J(J+1)}$] [4]

(b) A system has degenerate stationary states $|a\rangle$, $|b\rangle$ and $|c\rangle$ with energy E_0. A perturbation $\widehat{H}_1$ gives matrix elements $\langle a|\,\widehat{H}_1\,|b\rangle = \langle b|\,\widehat{H}_1\,|c\rangle = \langle c|\,\widehat{H}_1\,|a\rangle = t$ with t real, and $\langle a|\,\widehat{H}_1\,|a\rangle = \langle b|\,\widehat{H}_1\,|b\rangle = \langle c|\,\widehat{H}_1\,|c\rangle = 0$. Find the energy values of the perturbed system to first order in perturbation theory. [4]

(c) Make an order-of-magnitude comparison between (i) the ground-state energy density of electromagnetic waves in vacuum for wavelengths down to 0.1 nm and (ii) the binding energy of the electron in a hydrogen atom, divided by the volume of the hydrogen atom.

[*Note that in (i), each allowed wavevector and polarisation contributes* $\hbar\omega/2$ *to the ground-state energy.*] [4]

2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) the WKB method, with an example of its application;

(b) stimulated and spontaneous emission;

(c) Fermi's golden rule and its application to scattering within the Born approximation.

3 *Attempt* **either** *this question* **or** *question 4.*

The state of a system of two identical spin-$\frac{1}{2}$ particles, which occupy two spatial states $|a\rangle$, $|b\rangle$, can be expressed as a combination of products of normalised single-particle states such as $|1,\ a\uparrow\rangle\,|2,\ b\downarrow\rangle$. Show that the two-particle wavefunction can be written as a product of spatial and spin factors involving either singlet or triplet spin states. Comment on the exchange symmetry of the spin and spatial wavefunction in each case. [6]

[*Here, and for what follows, you may wish to use the short-hand notation* $|ab\rangle \equiv |1,\ a\rangle\,|2,\ b\rangle$, *and* $\langle ab| \equiv \langle 1,\ a|\,\langle 2,\ b|$.]

In a simple model for the H_2 molecule, consider two normalised spatial states $|a\rangle$, $|b\rangle$, which are orthogonal to each other and resemble the hydrogen 1s orbitals centred on the positions $\boldsymbol{R}_a$ and $\boldsymbol{R}_b$ of the two protons: $|a\rangle = \phi(\boldsymbol{r} - \boldsymbol{R}_a)$, $|b\rangle = \phi(\boldsymbol{r} - \boldsymbol{R}_b)$. The electronic Hamiltonian can be written as

$$\widehat{H} = \widehat{H}_1 + \widehat{H}_2 + \widehat{H}_{1,2} \quad ,$$

where $\widehat{H}_1$ acts only on electron 1, $\widehat{H}_2$ acts only on electron 2, and $\widehat{H}_{1,2}$ expresses the Coulomb repulsion between the two electrons.

Write out $\left\langle a\left|\widehat{H}_1\right|a\right\rangle$ and $\left\langle ab\left|\widehat{H}_{1,2}\right|ab\right\rangle$ in integral form. Show that $\left\langle aa\left|\widehat{H}_1\right|ab\right\rangle = 0$ and that $U = \left\langle aa\left|\widehat{H}_{1,2}\right|aa\right\rangle > 0$. [4]

Consider only singlet spin states, and introduce the following terms and simplifications from which the values of other matrix elements follow by symmetry.

$$\left\langle ab\left|\widehat{H}_1\right|ab\right\rangle = \left\langle a\left|\widehat{H}_1\right|a\right\rangle\langle b|b\rangle = E_0$$

$$\left\langle a\left|\widehat{H}_1\right|b\right\rangle = t < 0 \text{ is real}$$

$$\left\langle ab\left|\widehat{H}_{1,2}\right|ab\right\rangle \simeq 0 \text{ (neglect)}$$

$$\left\langle ab\left|\widehat{H}_{1,2}\right|ba\right\rangle \simeq 0 \text{ (neglect)}$$

$$\left\langle aa\left|\widehat{H}_{1,2}\right|bb\right\rangle \simeq 0 \text{ (neglect)}$$

Show that the energy of the trial state $|\psi\rangle = \frac{1}{\sqrt{2}}\,(|ab\rangle + |ba\rangle)$ is $E_\psi = 2E_0$. [5]

Show that the trial state formed from the bonding molecular orbital

$$|\phi\rangle = {}^1\!/_2\,(|1,\ a\rangle + |1,\ b\rangle)\,(|2,\ a\rangle + |2,\ b\rangle)$$

has energy $E_\phi = 2E_0 + 2t + U/2$. [6]

Sketch $(E_\phi - E_\psi)/U$ vs. $|t|\,/U$ and state how $|t|$ and U vary with the proton separation. Comment on the character of the ground state wavefunction, and how this affects the nature of the bond, as the proton separation is varied. [4]

4 *Attempt* **either** *this question* **or** *question 3.*

A spin-$\frac{1}{2}$ particle confined to move in the x–y plane is described by a wavefunction $\psi(\boldsymbol{r})\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$, where the spin factor corresponds to the spin state $\alpha|\uparrow\rangle + \beta|\downarrow\rangle$. The particle is subject to the Hamiltonian

$$\widehat{H} = c\frac{\hbar}{\mathrm{i}}\left(\sigma_x\frac{\partial}{\partial x} + \sigma_y\frac{\partial}{\partial y}\right) + V\sigma_z \quad ,$$

where c and V are constants and $\sigma_x, \sigma_y, \sigma_z$ are the Pauli spin matrices (see below). Show that travelling wave states $e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\begin{pmatrix}\alpha(\boldsymbol{k})\\ \beta(\boldsymbol{k})\end{pmatrix}$ are eigenstates of this Hamiltonian and find their dispersion $E(|\boldsymbol{k}|)$. Sketch $E(|\boldsymbol{k}|)$ vs. $|\boldsymbol{k}|$ for the three cases $V > 0$, $V = 0$ and $V < 0$. [8]

Consider the case $V = 0$. Show that the expectation value of the spin, $\left(\left\langle\widehat{S}_x(\boldsymbol{k})\right\rangle, \left\langle\widehat{S}_y(\boldsymbol{k})\right\rangle, \left\langle\widehat{S}_z(\boldsymbol{k})\right\rangle\right)$, is parallel to the wavevector $\boldsymbol{k}$. [6]

In the above Hamiltonian, V now is taken to depend on y:

$$\widehat{H} = c\frac{\hbar}{\mathrm{i}}\left(\sigma_x\frac{\partial}{\partial x} + \sigma_y\frac{\partial}{\partial y}\right) + V(y)\,\sigma_z \quad ,$$

with $V(y) < 0$ for $y < 0$ and $V(y) > 0$ for $y > 0$. Show that the state

$$\psi(x, y) = \mathrm{e}^{\mathrm{i}k_x x}\exp\left[-\frac{1}{\hbar c}\int_0^y V(y')\,\mathrm{d}y'\right]\begin{pmatrix}1\\ -1\end{pmatrix}$$

is an eigenstate of this Hamiltonian. Sketch $|\psi|$ along the y direction, if $V(y)$ takes positive and negative, constant, values for $y > 0$ and $y < 0$, respectively. [7]

Sketch the dispersion $E(k_x)$ of this eigenstate. Comment on the propagation along the x and y directions of wavepackets built from such eigenstates. [4]

[*The Pauli spin matrices are*

$$\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\ \sigma_y = \begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix},\ \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

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END OF PAPER