UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Friday 27 May 2022 1.30 pm to 3.30 pm

PHYSICS (2)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (2)

RELATIVITY

Candidates offering this paper should attempt a total of ***five*** *questions: all* ***three*** *questions from Section A and* ***two*** *questions from Section B.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains ***six*** *sides, including this coversheet, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS
2 × 20 Page Answer Book
Rough workpad
Yellow master coversheet

SPECIAL REQUIREMENTS
Mathematical Formulae handbook
Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

*Attempt **all** questions in this Section. Answers should be concise and relevant formulae may be assumed without proof.*

1 In Minkowski spacetime, two momentarily coincident observers travel towards a small and distant planet. To one observer the planet appears to have twice the angular diameter as to the other observer. What is the relative speed of the observers? [4]

2 Using the formulae given below, or otherwise, calculate the non-zero connection coefficients Γ^a_{bc} for the metric

$$ds^2 = dx^2 + f^2(x)dy^2,$$

where f is a twice-differentiable function of x. Calculate also the coefficient R^x_{yxy}. [4]

[*You may assume without proof the formulae*

$$\Gamma^a_{bc} = \frac{1}{2}g^{ad}(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc})$$

$$R^d_{abc} = \partial_b \Gamma^d_{ac} - \partial_c \Gamma^d_{ab} + \Gamma^e_{ac}\Gamma^d_{eb} - \Gamma^e_{ab}\Gamma^d_{ec}.$$
]

3 Calculate the gravitational deflection of a light ray passing close to a dwarf star of mass 2×10^{30} kg and radius 6000 km. [4]

[*You may quote appropriate formulae or use dimensional methods to provide an estimate.*]

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SECTION B

Attempt two questions from this section

4 (a) Discuss the concept of fundamental observers in cosmology and how these relate to the cosmological principles of homogeneity and isotropy. Further discuss the concept of cosmic time and how this leads to a metric of the form [6]

$$ds^2 = c^2 dt^2 + g_{ij}(t, \boldsymbol{x}) dx^i dx^j.$$

(b) The Friedman-Robertson-Walker metric may be written as

$$ds^2 = c^2 dt^2 - a^2(t) \left[d\chi^2 + S_K^2(\chi) d\Omega^2 \right]$$

where $S_K(\chi) = \sin\chi, \chi$, or $\sinh\chi$ depending on whether the universe is spatially closed, flat or open respectively.

By considering an appropriate Lagrangian or otherwise show from the geodesic equations, and assuming without proof that $\dot{\phi} = \dot{\theta} = 0$, that

$$\begin{aligned} \frac{d(c^2\dot{t})}{d\tau} + a\frac{da}{dt}\dot{\chi}^2 &= 0 \\ a^2\dot{\chi} &= \text{constant} \end{aligned}$$

where the dot indicates a derivative with respect to the affine parameter τ.

Show also that [5]

$$\dot{t}^2 = \begin{cases} 1 + a^2\dot{\chi}^2/c^2 & \text{for a massive particle} \\ a^2\dot{\chi}^2/c^2 & \text{for a massless particle.} \end{cases}$$

(c) By considering the photon 4-momentum, show that the cosmological redshift z is given by

$$1 + z = \frac{a(t_R)}{a(t_E)}$$

where t_R and t_E refer to received and emitted cosmic times respectively. [3]

(d) Show that the velocity of a massive particle moving with respect to a fundamental observer can be written as

$$v = a\frac{d\chi}{dt}.$$

Hence find how this velocity depends on the scale factor a. [5]

5 (a) The spacetime around a black hole is described by the Schwarzschild metric which has the form

$$ds^2 = c^2 d\tau^2 = \left(1 - \frac{2\mu}{r}\right) c^2 dt^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

where $\mu = GM/c^2$. Show that, for geodesic motion in the equatorial plane $\theta = \pi/2$, the equations of motion can be written as

$$\begin{aligned} \left(1 - \frac{2\mu}{r}\right)\dot{t} &= k\,, \\ r^2\dot{\phi} &= h\,, \\ \dot{t}^2\left(1 - \frac{2\mu}{r}\right)c^2 - \dot{r}^2\left(1 - \frac{2\mu}{r}\right)^{-1} - r^2\dot{\phi}^2 &= \mathcal{E}\,, \end{aligned}$$

where k and h are constants and dots denote differentiation with respect to the affine parameter τ. Explain why $\mathcal{E}$ takes the value c^2 for a massive particle and 0 for a photon and discuss the possible values k may take for a massless particle. [6]

(b) Show that for a photon or other massless particle the radial motion may be described by an equation of the form

$$\frac{1}{2}\dot{r}^2 + V_{\text{eff}}(r) = \frac{1}{2}c^2k^2$$

and find the effective potential $V_{\text{eff}}(r)$. [2]

(c) Sketch $V_{\text{eff}}(r)$ and find the radii of any circular photon orbits, commenting on their stability. [3]

(d) Find the condition on k such that a photon coming from infinity reaches the black hole. By finding an expression for the impact parameter b, show that this condition is equivalent to $b < 3\sqrt{3}\mu$. [3]

(e) A photon is emitted in the region $2\mu \leq r \leq 3\mu$ with the photon wave vector making an angle α to the outward radial direction. Find an expression for the maximum angle α that the wave vector can make as a function of r and μ so that the photon escapes to infinity. Comment on your answer for $r = 2\mu$ and $r = 3\mu$. [5]

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6 (a) Discuss briefly the concept of the local reference frame of a general observer and show that such a frame has the form of Minkowski spacetime. Define the relationship between the position X^μ, velocity U^μ, and acceleration A^μ 4-vectors in a general coordinate system, together with the four-momentum P^μ and four-force f^μ. Evaluate the Lorentz invariants, valid in any frame, $U^\mu U_\mu$, $U^\mu A_\mu$ and $A^\mu A_\mu$. [4]

(b) For the remainder of this question we adopt a system of units in which $c = 1$. The coordinates (t, x) define an inertial frame S in a $(1 + 1)$-dimensional Minkowski spacetime with line element $\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2$. The lightcone coordinates (p, q) in S are defined by $p = t - x$ and $q = t + x$.

(i) Find the form of the line-element and the components of the metric in these new coordinates. [2]

(ii) The inertial frame S' with coordinates (t', x') moves along the x-axis in S at a constant speed v and with the axes of S and S' coinciding at $t = t' = 0$. Show that the lightcone coordinates in S' are related to those in S by

$$p' = \alpha p, \qquad q' = \frac{q}{\alpha}$$

where α is a constant and find an expression for α in terms of v. [2]

(iii) A spaceship is moving along the x-axis of S and an astronaut on board measures the spaceship to have a constant acceleration a. Show that the lightcone coordinates of the spaceship must satisfy

$$\dot{p}\dot{q} = 1 \qquad \text{and} \qquad \ddot{p}\ddot{q} = -a^2,$$

where a dot denotes differentiation with respect to the astronaut's proper time τ. [2]

(iv) If the spaceship has coordinates $t = 0$ and $x = 1/a$ at $\tau = 0$, obtain an expression for the lightcone coordinates $p(\tau)$ and $q(\tau)$ along the spaceship's worldline and show that the corresponding coordinates (x, t) are given by [5]

$$t(\tau) = \frac{1}{a}\sinh a\tau, \qquad x(\tau) = \frac{1}{a}\cosh a\tau.$$

(v) A photon of frequency ν_0 is emitted at a time t_0 by a stationary star at $x = L$, where $0 < L < 1/a$ and travels in the positive x direction. If the photon is observed by the astronaut at a proper time τ, obtain an expression for the measured frequency. What is the latest time t_0 at which the photon can be emitted if it is to be observed by the astronaut? [4]

END OF PAPER