UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Monday 27 May 2019 9.00 am to 11.00 am

PHYSICS (3)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (3)

ADVANCED QUANTUM PHYSICS

Candidates offering this paper should attempt a total of ***five*** *questions:* ***three*** *questions from Section A and* ***two*** *questions from Section B.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains ***six*** *sides, including this coversheet, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
2×20 Page Answer Book	Mathematical Formulae handbook
Metric graph paper	Approved calculator allowed
Rough workpad	
Yellow master coversheet	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

*Attempt **all** questions in this Section. Answers should be concise and relevant formulae may be assumed without proof.*

1 The electronic structure of carbon is $1s^2\ 2s^2\ 2p^2$. Use Hund's rules to determine the electronic ground state. What are the total angular momenta J of the first two excited levels? [4]

2 A gas of two-level atoms, with non-degenerate atomic energy levels E_1 and E_2, is subjected to continuous radiation that can drive transitions between these two levels. The system reaches a steady state in which the populations of the two levels are N_1 and N_2. What is the ratio of the rates of spontaneous and stimulated emission of radiation at the resonant frequency $(E_2 - E_1)/h$? [4]

3 A container of sodium gas is subjected to a weak magnetic field along the z axis. Linearly polarised light at a frequency close to the D1 transition ($3S_{1/2} \rightarrow 3P_{1/2}$) is sent into the gas along a direction perpendicular to the magnetic field. Assuming that the magnetic field is sufficiently strong to allow the Zeeman splitting to be spectrally resolved, describe how the absorption spectrum differs when the polarization axis of the light is (i) parallel to and (ii) perpendicular to the z axis. [4]

SECTION B
*Attempt **two** questions from this section*

4 Explain how the variational principle can be used to approximate the ground state energy E_0 of a Hamiltonian $\hat{H}$. As part of your answer provide a proof of the inequality [8]

$$E_0 \leq \frac{\langle\psi|\hat{H}|\psi\rangle}{\langle\psi|\psi\rangle}\,.$$

A quantum particle moves in one dimension, with Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\mathrm{d}}{\mathrm{d}x^2} + ax^2 = \hat{T} + \hat{V}\,.$$

By considering the wavefunction $\psi(x) = \phi_0(\lambda x)$, where $\phi_0(x)$ is the exact ground state of $\hat{H} = \hat{T} + \hat{V}$, use the variational theorem to show that the ground state expectation values of the kinetic and potential energies $\hat{T}$ and $\hat{V}$ satisfy [6]

$$\langle\hat{T}\rangle = \langle\hat{V}\rangle = E_0/2\,.$$

Consider now the situation in which the particle experiences an attractive potential

$$\hat{U}(x) = -\frac{a}{|x|^p}\,.$$

By considering the wavefunction $\psi(x) = \phi_0(\lambda x)$ where $\phi_0(x)$ is the exact ground state of the new Hamiltonian $\hat{H} = \hat{T} + \hat{U}$, use the variational theorem to show that there can be no bound state in the potential for $p > 2$. [5]

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5 Two electrons are placed in a spherically symmetric harmonic potential. Neglecting interactions between the electrons, the total Hamiltonian is

$$\hat{H}_0 = \sum_{i=1,2} \left[\frac{|\hat{\boldsymbol{p}}_i|^2}{2m} + \frac{1}{2} m\omega_0^2 |\hat{\boldsymbol{r}}_i|^2 \right] .$$

By considering the single-particle spectrum in terms of the quantum numbers of the three oscillators directed along each of the three Cartesian directions (n_x, n_y, n_z) show that the degeneracies of the two lowest energy two-particle states are 1 and 12. [6]

Explain why the total angular momentum $\hat{\boldsymbol{L}}^2$ and the total spin $\hat{\boldsymbol{S}}^2$ of the two-particle system are conserved quantities. [2]

The single-particle states with quantum numbers $(n_x, n_y, n_z) = (1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ have wavefunctions

$$\Phi(x, y, z) \propto \begin{cases} x\ \mathrm{e}^{-(x^2+y^2+z^2)/2a_0^2} \\ y\ \mathrm{e}^{-(x^2+y^2+z^2)/2a_0^2} \\ z\ \mathrm{e}^{-(x^2+y^2+z^2)/2a_0^2} \end{cases}$$

where a_0 is the oscillator length scale. By rewriting these wavefunctions in spherical polar coordinates (r, θ, ϕ) show that they can be used to construct three states of angular momentum $\ell = 1$ and projection $m_\ell = -1, 0$ and 1. [3]

[*You may quote* $Y_{1,0}(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos\theta$, $Y_{1,\pm1}(\theta, \phi) = \mp\sqrt{\frac{3}{8\pi}} \mathrm{e}^{\pm i\phi} \sin\theta$.]

By considering the single-particle states in terms of their angular momentum ℓ and projection m_ℓ, determine the possible values of total angular momentum quantum number L and total spin quantum number S of the two lowest energy two-particle states. Show that these reproduce the degeneracies determined above. [4]

Determine the energy-level splittings of the first excited state induced by a spin-orbit interaction $\Delta\hat{H} = \lambda\hat{\boldsymbol{S}} \cdot \hat{\boldsymbol{L}}$, to first order in the coupling λ. [4]

6 A quantum system with Hamiltonian $\hat{H}_0$ has energy eigenstates $|i\rangle$ and associated eigenvalues E_i. It is subjected to a time-dependent perturbation $\Delta\hat{V}(t)$ that vanishes for both $t \to -\infty$ and $t \to \infty$.

Using time-dependent perturbation theory show that the amplitude for a system initially in state $|\psi(t = -\infty)\rangle = |i\rangle$ to be in a state $|j\rangle$ at time $t = \infty$ is

$$\gamma_{ji} \equiv \langle j|\psi(t = +\infty)\rangle = \frac{1}{\mathrm{i}\hbar} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega_{ji}t'} \langle j|\hat{V}(t')|i\rangle \,\mathrm{d}t' ,$$

where $\omega_{ji} = (E_j - E_i)/\hbar$. [8]

A fast-moving ion of charge Ze passes through a material at a constant speed v along the straight line trajectory $\boldsymbol{R}(t) = (b, vt, 0)$. It causes a one-electron atom located at the origin to experience the additional potential energy

$$\Delta\hat{V}(t) \simeq -\frac{Ze^2}{(b^2 + v^2t^2)^{3/2}}(b\hat{x} + vt\hat{y}) ,$$

where $\hat{x}$ and $\hat{y}$ are two Cartesian components of the position of the electron within the atom.

Using first-order perturbation theory, show that, for $\omega_{ji}b/v \ll 1$, the probability for the ion to induce an atomic transition from $|i\rangle \to |j\rangle$ is [6]

$$P_{ji} \simeq \left(\frac{2Ze^2}{\hbar bv}\right)^2 |\langle j|\hat{x}|i\rangle|^2 .$$

The figure on the next page shows the energy loss δE experienced by an ion on passing through a material as a function of the ion's kinetic energy E. The results are for different isotopes of boron, carbon and nitrogen. By considering the possible excitations that the ion can produce in the material, explain the dependence of δE on the kinetic energy E. (For simplicity consider only the regime where the energy loss is sufficiently small that the ion moves at approximately constant velocity.) Comment on the dependence of δE on atomic species and atomic mass. [5]

[*You may use that, for* $\beta \ll 1$,

$$\int_0^{\infty} \cos(u)/(\beta^2 + u^2)^{3/2} du \simeq 1/\beta^2 ,$$

$$\int_0^{\infty} u \sin(u)/(\beta^2 + u^2)^{3/2} du \simeq c - \log(\beta) ,$$

with c a numerical constant.]

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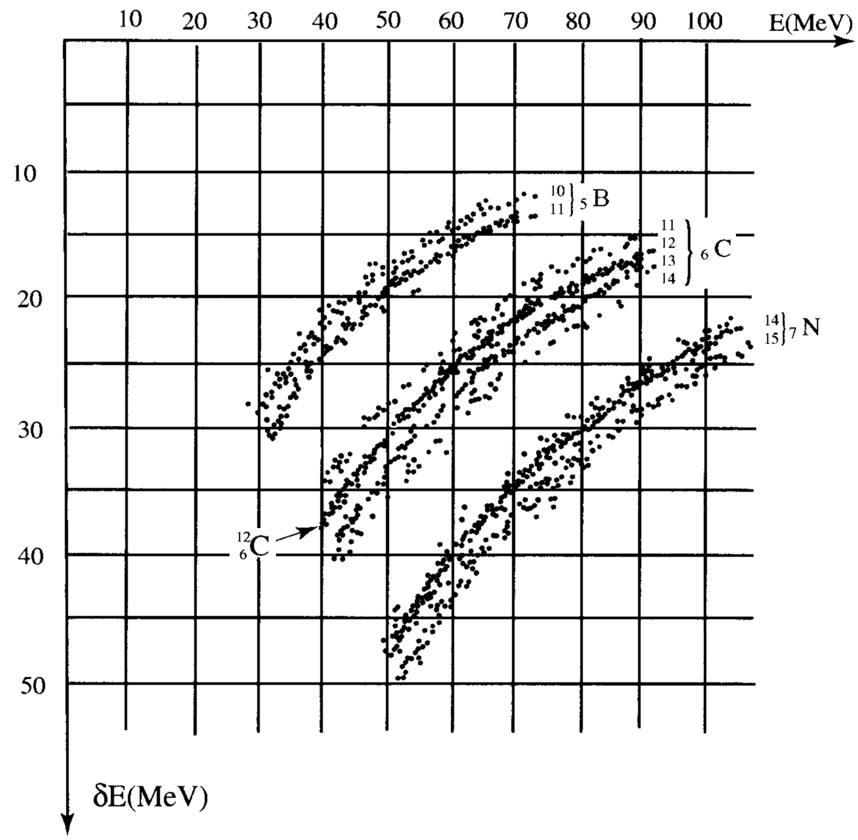

Figure: Energy loss δE experienced by an ion on passing through a material as a function of its kinetic energy E. The results are for different isotopes of boron, carbon and nitrogen, as labelled on the right hand side of the data using the notation $^{A}_{Z}N$ for a nucleus N of atomic number Z and atomic mass A (e.g. there are four isotopes of C with masses $A = 11, 12, 13, 14$, *of which* $A = 12$ *is highlighted to the left of the data).*

END OF PAPER