UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Tuesday 24 May 2016 1.30 pm to 3.30 pm

PHYSICS (1)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (1)

THERMAL AND STATISTICAL PHYSICS

Candidates offering this paper should attempt a total of **three** *questions. The questions to be attempted are* **1**, **2** *and* **one** *other question.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains **four** *sides, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS
2 × 20 Page Answer Book
Rough workpad
Yellow master coversheet

SPECIAL REQUIREMENTS
Mathematical Formulae handbook
Approved calculator allowed

THERMAL AND STATISTICAL PHYSICS

1 *Attempt* **all** *parts of this question. Answers should be concise and relevant formulae may be assumed without proof.*

(a) Define the *second virial coefficient*, $B_2(T)$, for a non-ideal gas. Sketch the dependence of $B_2(T)$ on temperature for inert gases, and comment briefly on the origin of the main features. [4]

(b) Show that, for a system consisting of N spin-half particles of magnetic dipole moment μ in a uniform magnetic field B, the magnetic contribution to the entropy is

$$S = Nk_B\left[\ln(2\cosh x) - x\tanh x\right] ,$$

where $x = \mu B/(k_B T)$. Suggest how the fact that the entropy depends only on the ratio B/T might allow a paramagnetic sample to be cooled by suitably varying the strength of an external magnetic field. [4]

(c) Write down the Fermi–Dirac expression for the average occupancy $\langle n(\epsilon)\rangle$ of a single-particle state of energy ϵ for a system of identical fermions in thermal equilibrium at temperature T. By considering the approximate extent of the spreading of the Fermi edge at temperature T, or otherwise, explain why the electronic contribution to the heat capacity of a metal depends linearly on temperature. [4]

2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) thermodynamic ensembles;

(b) Brownian motion;

(c) the Landau theory of phase transitions.

3 *Attempt* **either** *this question* **or** *question 4.*

Sketch the chemical potential $\mu(T)$ as a function of temperature for a system of identical bosons, and explain what is meant by a *Bose–Einstein condensate*. Outline the conditions under which such a condensate might be produced in a gas, and give an example of an experimental signature for its production. [5]

By applying suitable boundary conditions on the surface of a cube of arbitrary size, show that the density of states for a spinless free particle of mass m is $g(k) = k^2/(2\pi^2)$ per unit k and volume, where k is related to the particle energy ϵ as $\hbar k = \sqrt{2m\epsilon}$. [3]

Show that applying this result to the classically allowed region for a particle of mass m confined by an isotropic potential $U(r)$ gives the density of states (per unit energy) as

$$g(\epsilon) = \frac{(2m)^{3/2}}{\pi\hbar^3} \int_0^{r_\epsilon} r^2 \sqrt{\epsilon - U(r)}\, \mathrm{d}r \, ,$$

where ϵ is the total energy of the particle, and the upper limit r_ϵ is defined by the equation $\epsilon = U(r_\epsilon)$. [5]

For a harmonic potential $U(r) = m\omega^2 r^2/2$, by using a suitable trigonometric substitution or otherwise, show that

$$g(\epsilon) = \frac{\epsilon^2}{2(\hbar\omega)^3} \, .$$ [5]

For a system of N identical bosons confined by such a harmonic potential, obtain an expression for the number of particles in the ground state (of energy $\epsilon = 0$) at low temperature, and show that the Bose–Einstein condensation temperature is

$$T_0 \approx 0.94 \frac{\hbar\omega}{k_\mathrm{B}} N^{1/3} \, .$$

Sketch the ground state occupancy as a function of temperature. [7]

[*You may wish to use the standard integral* $\int_0^\infty x^2/(\mathrm{e}^x - 1)\,\mathrm{d}x \approx 2.404$.]

4 *Attempt* **either** *this question* **or** *question 3.*

Show that the chemical potential, μ, of a classical gas of N particles at temperature T is given by

$$\mu = -k_B T\,[\ln Z_1 - \ln N]\ ,$$

where Z_1 is the partition function for a single particle. [4]

Define the conditions for equilibrium in an open system and show that, for a system at constant temperature, pressure and particle number, its equilibrium state is determined by minimising the Gibbs function G. [6]

Show that

$$\mathrm{d}G = -S\ \mathrm{d}T + V\,\mathrm{d}p + \mu\,\mathrm{d}N\ ,$$

and that $G = \mu N$. [4]

A system consists of a mixture of two gases, A and B, which can undergo the chemical reaction A $\rightleftharpoons$ 2B. Show that, in equilibrium, the chemical potentials of A and B are related by $\mu_A = 2\mu_B$. [5]

The single-particle partition functions for the gases A and B have the form $Z_1^{(i)} = C_i V T^{3/2}$ $(i = \mathrm{A}, \mathrm{B})$, where C_i is a constant. Show that the quantity $K_N(V,T) \equiv N_B^2/N_A$ is a function of V and T only, and that at fixed V and T the fraction of molecules of type A dissociated into molecules of type B is reduced as the density of the mixture increases. Explain on physical grounds why dissociation can be expected to decrease with increasing density. [6]

[*You may wish to use the following identities:* $U = TS - pV + \mu N$, $G = U - TS + pV$ *and* $\mathrm{d}F = -S\ \mathrm{d}T - p\,\mathrm{d}V + \mu\,\mathrm{d}N$.]

END OF PAPER