UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Tuesday 26 May 2015 9.00 am to 11.00 am

PHYSICS (2)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (2)

RELATIVITY

Candidates offering this paper should attempt a total of **three** *questions. The questions to be attempted are* **1**, **2** *and* **one** *other question.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains **four** *sides, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
2×20 Page Answer Book	Mathematical Formulae handbook
Rough workpad	Approved calculator allowed
Yellow master coversheet	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

RELATIVITY

1 *Attempt* **all** *parts of this question. Answers should be concise, and relevant formulae may be assumed without proof.*

(a) For the metric $ds^2 = a^2(d\theta^2 + \sin^2\theta d\phi^2)$, show that geodesics obey a relation $\frac{d\phi}{ds}\sin^2\theta = h$, where h is a constant. Write the geodesic equation in the form $\frac{d\theta^2}{d\phi^2} + V(\theta) = 0$ and obtain an expression for the maximum value of $|\theta|$ in terms of h. [4]

(b) The redshift z is defined in terms of wavelength λ as $1 + z = \lambda_{\text{received}}/\lambda_{\text{emitted}}$. The metric outside a neutron star of mass $M = 4 \times 10^{30}$ kg and radius 11 km is well described by the Schwarzschild metric

$$c^2d\tau^2 = \left(1 - \frac{2GM}{rc^2}\right)c^2dt^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

What is the redshift of light leaving the surface of the star as measured by an observer at infinity? Evaluate your answer to at least 3 significant figures. [4]

(c) Two particles have 3-space velocities $\boldsymbol{v}_1 = \boldsymbol{\beta}_1 c$ and $\boldsymbol{v}_2 = \boldsymbol{\beta}_2 c$. Show that their relative velocities satisfy

$$\beta_{\text{rel}}^2 = v_{\text{rel}}^2/c^2 = \frac{(\boldsymbol{\beta}_1 - \boldsymbol{\beta}_2)^2 - (\boldsymbol{\beta}_1 \times \boldsymbol{\beta}_2)^2}{(1 - \boldsymbol{\beta}_1 \cdot \boldsymbol{\beta}_2)^2}$$

[4]

(a) The Lagrangian is $\mathcal{L} = \dot\theta^2 + \sin^2\theta\dot\phi^2$. This does not depend on ϕ or s, so the geodesics satisfy

$$\sin^2\theta\dot\phi = h\ ; \quad \dot\theta^2 + \sin^2\theta\dot\phi^2 = 1$$

where h is a constant $|h| < 1$. Dividing these equations we get

$$\left(\frac{d\theta}{d\phi}\right)^2 + \sin^2\theta - \frac{\sin^4\theta}{h^2} = 0$$

The maximum values of θ satisfy $\sin\theta = \pm h$.

(b) There's nothing difficult about this, but you have to calculate accurately to get full marks. The gravitational redshift formula (there's enough in the question to derive it...) is

$$z = \left(1 - \frac{2GM}{rc^2}\right)^{-1/2} - 1 = 0.17039$$

(c) This is nice little problem. The 4-vector velocities are $v_i = (\gamma_i, \gamma_i \boldsymbol{\beta}_i)$. The Lorentz factor of the relative frame (emphasised and very useful in the example sheets...) is $\gamma_{\rm rel} = v_1 \cdot v_2 = \gamma_1 \gamma_2 (1 - \boldsymbol{\beta}_1 \cdot \boldsymbol{\beta}_2)$. We then use

$$\beta_{\rm rel}^2 = 1 - \frac{1}{\gamma_{\rm rel}^2} = 1 - \frac{(1-\beta_1^2)(1-\beta_2^2)}{(1-\boldsymbol{\beta}_1\cdot\boldsymbol{\beta}_2)^2} = \frac{\beta_1^2 + \beta_2^2 - 2\boldsymbol{\beta}_1\cdot\boldsymbol{\beta}_2 - \beta_1^2\beta_2^2 + (\boldsymbol{\beta}_1\cdot\boldsymbol{\beta}_2)^2}{(1-\boldsymbol{\beta}_1\cdot\boldsymbol{\beta}_2)^2}$$

We now need $(\boldsymbol{\beta}_1 \times \boldsymbol{\beta}_2)^2 = \beta_1^2\beta_2^2 - (\boldsymbol{\beta}_1\cdot\boldsymbol{\beta}_2)^2$ and we're done. I promise you they won't find it as easy as that...

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2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) tidal forces in Newtonian gravity and geodesic deviation in General Relativity;

(b) Friedmann-Robertson-Walker cosmological models;

(c) the covariant derivative and the parallel transport of vectors.

3 *Attempt* **either** *this question* **or** *question 4.*

Events in Minkowski spacetime have coordinates $\{x^a\}$ and $\{x'^a\}$ in frames S and S' respectively. Show that the contravariant components A^a $(a = \{0, 1, 2, 3\})$ and covariant components A_a of a vector A are related by

$$A'^a = \frac{\partial x'^a}{\partial x^b} A^b \; ; \quad A'_a = \frac{\partial x^b}{\partial x'^a} A_b \, . \tag*{[3]}$$

Explain why the gradient of the electromagnetic 4-potential $\partial_a A_b$ is not a tensor, and why the electromagnetic field $F_{ab} = \partial_a A_b - \partial_b A_a$ is a covariant tensor of rank 2. [4]

Using units where $c = 1$, show that the components of F_{ab} can be represented in the Cartesian frame $S = \{x^0, x^1, x^2, x^3\}$ by the matrix

$$F_{ab} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix},$$

where $\{E_1, E_2, E_3\}$, $\{B_1, B_2, B_3\}$ are the 3-vector components of the electric and magnetic fields. [4]

Give a corresponding formula for the contravariant components F^{ab} and demonstrate that $F_{ab}F^{ab}$ is a Lorentz invariant. Give a formula for $F_{ab}F^{ab}$ in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$. [4]

The isotropic alternating tensor ϵ^{abcd} is defined to be equal to 1 when $(abcd)$ is an even permutation of (0123), equal to -1 when $(abcd)$ is an odd permutation of (0123) and 0 otherwise. Evaluate the matrix representation in frame S of the dual tensor ${}^*F^{ab} = \epsilon^{abcd} F_{cd}$. Give a formula for ${}^*F^{ab} F_{ab}$ in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$. [5]

The electromagnetic stress-energy tensor is defined as

$$T^a_b = \frac{1}{\mu_0} \left(F^{ac} F_{cb} - \tfrac{1}{4} \delta^a_b F_{cd} F^{cd} \right)$$

Find a formula for T^0_a and interpret your results in terms of the electromagnetic energy density and energy flux. [5]

The contravariant components A^a and covariant components A_a of a vector A follow the transformations of

$$\mathrm{d}x'^a = \frac{\partial x'^a}{\partial x^b} \mathrm{d}x^b \; ; \quad \frac{\partial}{\partial x'^a} = \frac{\partial x^b}{\partial x'^a} \frac{\partial}{\partial x^b} \, .$$

The transformation laws are

$$A'_b = \frac{\partial x^c}{\partial x'^b} A_c \quad \Longrightarrow \quad \frac{\partial A'_b}{\partial x'^a} = \frac{\partial^2 x^c}{\partial x'^a \partial x'^b} A_c + \frac{\partial x^d}{\partial x'^a} \frac{\partial x^c}{\partial x'^b} \frac{\partial A_c}{\partial x^d}$$

The first term means that $A_{a,b}$ isn't a tensor, but when you form $F_{ab} = A_{a,b} - A_{b,a}$, the offending bits cancel and you have a covariant tensor of rank 2. You can also prove this by using the fact that $\nabla_a A_b = \partial_a A_b - \Gamma^c_{ab} A_c$ is a tensor. The connection terms cancel when you form F_{ab}.

Formation of the covariant components of F_{ab} is standard bookwork, which I'll copy from his notes…

The contravariant components of F have the signs of the electric fields reversed:

$$F^{ab} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix},$$

The following proves the invariance of $F^{ab}F_{ab} = 2(\boldsymbol{B}^2 - \boldsymbol{E}^2)$:

$$F'_{ab} = \frac{\partial x^\alpha}{\partial x'^a}\frac{\partial x^\beta}{\partial x'^b}F_{\alpha\beta}\ ;\ \ F'^{ab} = \frac{\partial x'^a}{\partial x^\alpha}\frac{\partial x'^b}{\partial x^\beta}F^{\alpha\beta}\ ;\ \ \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial x'^\nu} = \delta^\mu_\nu$$

The invariant $\boldsymbol{E}^2 - \boldsymbol{B}^2$ is the Lagrangian of the electromagnetic field.

[*This would be more straightforward is we asked for $F^{ab}F_{ba}$ so that the matrices can be multiplied without a transposition…But no one does it that way!)*]

The components of the dual tensor ${}^*F^{ab}$ are

$${}^*F^{ab} = \begin{pmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & E_3 & -E_2 \\ -B_2 & -E_3 & 0 & E_1 \\ -B_3 & E_2 & -E_1 & 0 \end{pmatrix},$$

The invariant ${}^*F^{ab}F_{ab}$ is $4\boldsymbol{E}\cdot\boldsymbol{B}$.

The stress-energy tensor has $T^0_0 = \frac{1}{2\mu_0}(\boldsymbol{E}^2 + \boldsymbol{B}^2)$ and $T^0_k = \frac{1}{\mu_0}(\boldsymbol{E}\times\boldsymbol{B})_k$.

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4 *Attempt* **either** *this question* **or** *question 3.*

The Schwarzschild metric has the form

$$\mathrm{d}s^2 = \left(1 - \frac{2\mu}{r}\right) c^2\mathrm{d}t^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} \mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)$$

where $\mu = GM/c^2$.

Show that, for geodesic motion in the equatorial plane $\theta = \pi/2$, the equations of motion can be written as

$$\begin{aligned} \left(1 - \frac{2\mu}{r}\right)\dot{t} &= k, \\ r^2\dot{\phi} &= h, \\ \dot{t}^2\left(1 - \frac{2\mu}{r}\right)c^2 - \dot{r}^2\left(1 - \frac{2\mu}{r}\right)^{-1} - r^2\dot{\phi}^2 &= \mathcal{E} \end{aligned}$$

where k and h are constants and dots denote differentiation with respect to some affine parameter. Explain why $\mathcal{E}$ takes the value c^2 for a material particle and 0 for a photon or other massless particle. [6]

Discuss why the value of k is arbitrary for a massless particle. What is the relationship between h and the impact parameter b for a massless particle when $k = 1$? [3]

Derive the shape equation for $u \equiv 1/r$ in the form

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 + u^2 = \frac{k^2}{h^2} + 2\mu u^3 - \mathcal{E}\frac{1 - 2\mu u}{h^2}.$$

[3]

Determine the radius at which a photon can travel in a circular orbit. Is this orbit stable? [3]

A photon coming from infinity has impact parameter b. Show that the photon will encounter the black hole if $b < 3\sqrt{3}\mu$. [3]

A photon with an impact parameter less than this value will encounter the black hole. Sketch the trajectory of the photon in the equatorial plane. By studying the form of the shape equation at large values of u, determine how r varies with ϕ as the singularity is approached. [3]

For the case of a material particle falling in from infinity, show that the particle encounters the hole if $h < 4\mu c$. [4]

[*First part varied from last year. The shape equation is lectured, but there's no example on it — there should be.*]

The shape equation can also be written as given or the second-order equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + u = \frac{\mu\mathcal{E}}{h^2} + 3\mu u^2$$

[4]

Haven't quite decided yet which way to do it — the lecturer thought it was more familiar as the second-order equation(though it's much more useful in the effective potential form).

The photon follows a null vector, so the affine parameter is arbitrary, and you can take $k = 1$.

With $k = 1$ we get $bc = h$. Best way of seeing that is with $\mu = 0$!

The radius of the unstable circular orbit is 3μ, which is obvious from the shape equation.

The limiting impact parameter is $b = 3\sqrt{3}\mu$. For b less than that, the photon spirals into the hole. The figures illustrates the "bunching" of the trajectory at $r = 3\mu$ before the photon goes in or out again.

The limiting form at $r = 0$ has $r \propto \phi^2$, i.e. it straightens up and goes in radially.

END OF PAPER