Part-II Physics/Astrophysics, Michaelmas Term 2021 **Anastasia Fialkov**

Relativity: Example Sheet 1

1. (a) Show that if two events are separated by a timelike interval, then there is a frame in which they occur at the same spatial location. (b) Similarly, if two events are separated by a spacelike interval, show there is a frame in which they are simultaneous.

2. (a) Show that if an event A precedes an event B in some frame S at the same spatial location, then the event A precedes event B in all frames. (b) Two general events A and B are separated in S by a spatial distance Δr. If event A causes event B, determine an inequality for the time difference between the events, $\Delta t = t_B - t_A$. Hence show that the events are causally related in all frames.

3. (a) On a spacetime diagram with the x and ct axes of an inertial frame S horizontal and vertical, respectively, construct the lines of constant x' and ct', where these coordinates refer to the frame S' in standard configuration with S (i.e., where S' moves at a speed v along the positive x-direction and the two frames coincide at $t = t' = 0$). Show that the angle between the x- and x'- axes is the same as that between the ct- and ct'- axes and has the value $\tan^{-1}(v/c)$.

(b) Sketch on your diagram the loci of events separated from the spacetime origin $x = ct = 0$ by a constant invariant interval $\Delta s^2 = c^2t^2 - x^2$ for positive (timelike) and negative (spacelike) values of Δs^2. Show that the tangents to these curves where they intersect the x'- and ct'-axes are parallel to the ct'- and x'-axes, respectively. How can these curves be used to calibrate lengths along the axes of the S and S' frames?

(c) Use your diagram to illustrate graphically why a rod at rest in S' is *contracted* as measured in S, and the time on a clock at rest in S' is *dilated* as observed in S.

4. An inertial frame S' is related to the frame S by a boost of $\vec{v}$ whose components in S are (v_x, v_y, v_z). Show that the coordinates (ct', x', y', z') and (ct, x, y, z) of an event are related by

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1+\alpha\beta_x^2 & \alpha\beta_x\beta_y & \alpha\beta_x\beta_z \\ -\gamma\beta_y & \alpha\beta_y\beta_x & 1+\alpha\beta_y^2 & \alpha\beta_y\beta_z \\ -\gamma\beta_z & \alpha\beta_z\beta_x & \alpha\beta_z\beta_y & 1+\alpha\beta_z^2 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix},$$

where $\vec{\beta} = \vec{v}/c$, $\gamma = (1 - |\vec{\beta}|^2)^{-1/2}$ and $\alpha = (\gamma - 1)/|\vec{\beta}|^2$. (*Hint: resolve the 3-vector position with components* (x, y, z) *into parallel and perpendicular parts with respect to* $\vec{\beta}$*, and similarly in the* S' *frame.*)

5. In a given inertial frame, two particles are shot out simultaneously from a given point, with equal speeds v in orthogonal directions. What is the speed of each particle relative to the other?

6. (a) Frame S' moves with speed v relative to frame S in standard configuration. A rod at rest in frame S' makes an angle θ' with respect to the forward direction of motion. What is the angle θ measured in S? (b) If a bullet is fired in S' at speed u' at an angle θ' with respect to the forward direction of motion, what is the angle θ measured in S? What if the bullet is a photon?

7. Frame S' moves with speed v relative to frame S in standard configuration. Neutral π-mesons at rest in S' decay into two photons that are emitted isotropically. Show that the angular distribution of photons in S is

$$P(\theta)\, d\theta = \frac{\sin\theta\, d\theta}{2\gamma^2(1-\beta\cos\theta)^2}\,.$$

8. (a) A spaceship travels in a straight line at a variable speed $u(t)$ in some inertial frame S. An observer on the spaceship measures his acceleration to be $f(\tau)$, where τ is his proper time. If at $\tau = 0$ the spaceship has a speed u_0 in S show that

$$\frac{u(\tau) - u_0}{1 - u(\tau)u_0/c^2} = c\tanh\psi(\tau)\,,$$

where $c\psi(\tau) = \int_0^\tau f(\tau')\, d\tau'$. Show that the speed of the spaceship can never reach c.

(b) If the spaceship leaves base at time $t = \tau = 0$ with initial speed $u_0 = 0$ and travels forever in a straight line with constant acceleration g (for comfort), how long by the spaceship clock does it take to reach a star 10 light years from the base?

9. In 3D Euclidean space, coordinates x'^a are related to Cartesian coordinates x^a by

$$x^1 = x'^1 + x'^2\,, \qquad x^2 = x'^1 - x'^2\,, \qquad x^3 = 2x'^1x'^2 + x'^3\,.$$

Describe the coordinate surfaces in the primed system. Obtain the metric functions g'_{ab} in the primed system and hence show that these coordinates are not orthogonal. Calculate the volume element dV in the primed coordinate system.

10. Show that the line element of a 3-sphere of radius a embedded in 4D Euclidean space can be written in the form

$$ds^2 = a^2[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)]\,.$$

Hence, in this 3D non-Euclidean space, calculate the area of the 2-sphere defined by $\chi = \chi_0$. Also find the total volume of the 3D space.