Part II: Michaelmas 2021

Advanced Quantum Mechanics Question Sheet IV

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Question 1. Time evolution of spin

A spin 1/2 particle has gyromagnetic ratio γ, so that its magnetic moment is given by $\hat{\boldsymbol{\Gamma}} = \gamma \hat{\boldsymbol{S}}$ where $\hat{\boldsymbol{S}}$ is the spin operator.

Using Schrödinger's equation, show that the equation of motion for the spin state $|\psi(t)\rangle$ of such a particle in a magnetic field $\boldsymbol{B}$ is

$$-\frac{1}{2}\gamma\left(\boldsymbol{B}\cdot\hat{\boldsymbol{\sigma}}\right)|\psi(t)\rangle = i\frac{\partial}{\partial t}|\psi(t)\rangle\,,$$

where $\hat{\boldsymbol{\sigma}}$ is a vector with the Pauli matrices $\hat{\sigma}_i$ as components.

$\boldsymbol{B}$ is a constant field in the z-direction with magnitude B_0, and we choose

$$|\psi(0)\rangle = \cos(\theta/2)|\uparrow\rangle + \sin(\theta/2)|\downarrow\rangle\,.$$

By representing the spin states as column vectors, show that at time t,

$$|\psi(t)\rangle = \cos(\theta/2)\exp(i\omega_0 t/2)|\uparrow\rangle + \sin(\theta/2)\exp(-i\omega_0 t/2)|\downarrow\rangle\,,$$

where $\omega_0 = \gamma B_0$, and find the expectation values of the components of the magnetic moment $\hat{\boldsymbol{\mu}}$ at time t.

Using the general result

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{A}\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{A}]\rangle$$

for the time evolution of the expectation value of an operator $\hat{A}$, show that for an arbitrarily varying magnetic field $\boldsymbol{B}(t)$ the magnetic dipole moment operator satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{\boldsymbol{\mu}}\rangle = \gamma\langle\hat{\boldsymbol{\mu}}\times\boldsymbol{B}(t)\rangle\,,$$

and demonstrate explicitly that the expectation values found above for the constant field satisfy this relation.

Interpret your results physically.

Question 2. Photon momentum

The total linear momentum operator $\hat{\boldsymbol{P}}$ for an electromagnetic field is

$$\hat{\boldsymbol{P}} = \sum_{\boldsymbol{k},\lambda} \hbar\boldsymbol{k}\, \hat{a}^{\dagger}_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda} \ .$$

Describe each of the terms in this expression. On what vector space does it act?

By generating a single photon state from the vacuum, show that a photon of wave vector $\boldsymbol{k}$ (in any polarisation state λ) has linear momentum $\hbar\boldsymbol{k}$.

Similarly, the intrinsic spin angular momentum operator $\hat{\boldsymbol{J}}_{\mathrm{s}}$ is given by

$$\hat{\boldsymbol{J}}_{\mathrm{s}} = \hbar \sum_{\boldsymbol{k}} \frac{\boldsymbol{k}}{|\boldsymbol{k}|} \left[\hat{a}^{\dagger}_{\boldsymbol{k},L}\hat{a}_{\boldsymbol{k},L} - \hat{a}^{\dagger}_{\boldsymbol{k},R}\hat{a}_{\boldsymbol{k},R}\right] \ .$$

Again by generating a single photon state from the vacuum, show that for left-handed (right-handed) photons, the spin is oriented parallel to the photon direction of motion, with spin projection $+\hbar$ $(-\hbar)$.

Question 3. Time evolution of coherent states

Consider the time evolution of a coherent state in the case where the Hamiltonian is time-independent. Using the time evolution operator $\hat{U}(t)$, show that a coherent state at $t = 0$ always evolves into another coherent state at some subsequent time.

If $|\beta\rangle$ is a coherent state, and the creation and annihilation operators are in the Heisenberg picture, derive expressions for the following:

(a) $\langle\beta|\hat{a}(t)|\beta\rangle$ and $\langle\beta|\hat{a}^{\dagger}(t)|\beta\rangle$

(b) $\langle\beta|\hat{x}(t)|\beta\rangle$ where $\hat{x}(t) = \sqrt{\hbar/2m\omega}\left(\hat{a}(t) + \hat{a}^{\dagger}(t)\right)$

(c) $\langle\beta|\hat{p}(t)|\beta\rangle$ where $\hat{x}(t) = -i\sqrt{2m\hbar\omega}\left(\hat{a}(t) - \hat{a}^{\dagger}(t)\right)$

Show that a coherent state is a minimum uncertainty state at all time.

Using these results, explain why coherent states are the quantum mechanical equivalents of classical simple harmonic motion.

Question 4. Creating coherent states

Show that

$$\frac{\partial}{\partial\beta}\left(e^{-\beta\hat{a}^{\dagger}}\hat{a}e^{\beta\hat{a}^{\dagger}}\right) = 1$$

.
By subsequently integrating the result, show that

$$e^{-\beta\hat{a}^\dagger}\hat{a}e^{\beta\hat{a}^\dagger} = \beta + \hat{a}\,.$$

Using this expression, show that $|\beta\rangle = Ne^{\beta\hat{a}^\dagger}|0\rangle$ is a coherent state, i.e. $\hat{a}|\beta\rangle = \beta|\beta\rangle$, where N is a normalisation factor. Show that N is given by $N = e^{-|\beta|^2/2}$.

Why is $e^{\beta\hat{a}^\dagger}$ sometimes called the 'displacement operator'.

Calculate the expectation values, $x_0 = \langle\hat{x}\rangle$ and $p_0 = \langle\hat{p}\rangle$, with respect to $|\beta\rangle$ and, by considering $\langle\hat{x}^2\rangle$ and $\langle\hat{p}^2\rangle$, show that

$$(\Delta p)^2(\Delta x)^2 = \frac{\hbar^2}{4}\,,$$

where $(\Delta p)^2 = \langle(\hat{p} - \langle\hat{p}\rangle)^2\rangle$ (and similarly $(\Delta x)^2$).

[Hint: Remember how the creation and annihilation operators are related to the phase space operators $\hat{x}$ and $\hat{p}$. Also, note that taking the Hermitian conjugate of the eigenvalue equation $\hat{a}|\beta\rangle = \beta|\beta\rangle$ leads to the relation $\langle\beta|\hat{a}^\dagger = \langle\beta|\beta^*$.]

Show that the eigenvalue equation $\hat{a}|\beta\rangle = \beta|\beta\rangle$ translates to the equation

$$\sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)\psi(x) = \beta\psi(x)\,.$$

for the coordinate representation, $\psi(x)$, of the coherent state. Show that this equation has the solution

$$\psi(x) = N\exp\left[-\frac{(x-x_0)^2}{4(\Delta x)^2} + i\frac{p_0 x}{\hbar}\right]\,,$$

where x_0 and p_0 are defined in part (b) above.

By expressing $|\beta\rangle$ in the number basis, show that

$$|\beta(t)\rangle = e^{-i\omega t/2}|\beta e^{-i\omega t}\rangle\,.$$

As a result, deduce expressions for $x_0(t)$ and $p_0(t)$ and show they represent solutions to the classical equations of motion. How does the width of the coherent state wavepacket evolve with time?

Question 5. Addition of angular momenta

Consider the addition of two angular momenta, $\ell_1 = 2$ and $\ell_2 = 1$. By drawing a diagram similar to that of Figure 9 of Handout VI, tabulate the possible values of the corresponding quantum numbers m_1, m_2 and $M = m_1 + m_2$ (relating to $\hat{L}_z$), and show that the values of M correspond to the expected values $L = 3, 2, 1$ of the total angular momentum quantum number L.

Repeat for the case $\ell_1 = 3$, $\ell_2 = 1$.

(†) For the case $\ell_1 = 2$ and $\ell_2 = 1$, the state $|L, M\rangle = |3, 3\rangle$ can be written down straightforwardly as $|\ell_1, m_1\rangle \otimes |\ell_2, m_2\rangle = |2, 2\rangle \otimes |1, 1\rangle$. Use ladder operators to construct the state $|L, M\rangle = |3, 2\rangle$ as a linear combination of the product states $|\ell_1, m_1\rangle \otimes |\ell_2, m_2\rangle$, and then orthogonality to construct the state $|L, M\rangle = |2, 2\rangle$.

[The angular momentum ladder operators $\hat{L}_\pm$ act as
$\hat{L}_\pm |L, m_L\rangle = \hbar\sqrt{L(L+1) - m_L(m_L \pm 1)}\,|L, m_L \pm 1\rangle$.]

Verify that the states obtained in (b) are the same as would be written down using the tables of Clebsch-Gordan coefficients appended to this examples sheet (see the table labelled $2 \otimes 1$).

Using the $2 \otimes 1$ table, write down the state $|L, M\rangle = |1, -1\rangle$ as a linear combination of the $|\ell_1, m_1\rangle \otimes |\ell_2, m_2\rangle$ states.

Show that the scalar product $\hat{\boldsymbol{L}}_1 \cdot \hat{\boldsymbol{L}}_2$ of two angular momentum operators can be expressed as

$$\hat{\boldsymbol{L}}_1 \cdot \hat{\boldsymbol{L}}_2 = \tfrac{1}{2}(\hat{L}_1)_+(\hat{L}_2)_- + \tfrac{1}{2}(\hat{L}_1)_-(\hat{L}_2)_+ + (\hat{L}_1)_z(\hat{L}_2)_z \ ,$$

where $(\hat{L}_{1,2})_\pm = (\hat{L}_{1,2})_x \pm i(\hat{L}_{1,2})_y$ are ladder operators. By operating directly with $(\hat{\boldsymbol{L}}_1 + \hat{\boldsymbol{L}}_2)^2$ and $(\hat{L}_1)_z + (\hat{L}_2)_z$, verify that the linear combination of product states written down in (†) does indeed have total angular momentum quantum numbers $L = 1$ and $M = -1$.

Convince yourself that each table of Clebsch-Gordan coefficients corresponds to a unitary (in fact, orthogonal) matrix. For the cases $j_1 \otimes j_2 = (1/2) \otimes (1/2)$, $1 \otimes 1$, $(3/2) \otimes (3/2)$ and $2 \otimes 2$ (for which $j_1 = j_2$), what is the symmetry of the total angular momentum eigenstates $|j, m_j\rangle$ for each possible value of j under interchange of the labels 1 and 2?

Question 6. Rotational symmetry

Write down the Wigner-Eckart theorem for matrix elements of the form $\langle \alpha_1 j_1 m_1 | \hat{V}_m | \alpha_2 j_2 m_2 \rangle$, where the operators $\hat{V}_m$ ($m = \pm 1, 0$) are the spherical components of a vector operator $\hat{\boldsymbol{V}}$, and the α_i represent any other quantum numbers needed to uniquely identify the total angular momentum eigenstates $|\alpha_i j_i m_i\rangle$ of the system.

(a) For the case $j_1 = 1$, $j_2 = 0$, identify the matrix elements $\langle \alpha_1 j_1 m_1 | \hat{V}_m | \alpha_2 j_2 m_2 \rangle$ which can be non-zero, and show that they are all equal. Hence show that the Cartesian components $(\hat{V}_x, \hat{V}_y, \hat{V}_z)$ of $\hat{\boldsymbol{V}}$ have matrix elements of the form

$$\langle \alpha_1 10 | (\hat{V}_x, \hat{V}_y, \hat{V}_z) | \alpha_2 00 \rangle = A(0, 0, 1)$$

$$\langle \alpha_1 1, \pm 1 | (\hat{V}_x, \hat{V}_y, \hat{V}_z) | \alpha_2 00 \rangle = \frac{A}{\sqrt{2}}(\mp 1, i, 0)$$

where A is a constant.

(b) Verify this result explicitly for the matrix elements of the position operator $\hat{\boldsymbol{r}} = (\hat{x}, \hat{y}, \hat{z})$ for a system such as the hydrogen atom for which the $j = 0$ and $j = 1$ angular momentum eigenstates $|\alpha j m\rangle$ are spatial wavefunctions of the form

$$|\alpha 00\rangle = R_{\alpha 0}(r) Y_{00}(\theta, \phi) \ ; \qquad |\alpha 1 m\rangle = R_{\alpha 1}(r) Y_{1m}(\theta, \phi) \ .$$

[The $\ell = 0$ and $\ell = 1$ spherical harmonics $Y_{\ell m}(\theta, \phi)$ are

$$Y_{00} = \sqrt{\frac{1}{4\pi}} \ , \qquad Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta \ , \qquad Y_{1,\pm 1} = \mp \sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi} \ . \,]$$

(c) What is the equivalent result to part (a) for the case $j_1 = j_2 = 0$?

Question 7. Linear Stark effect

A hydrogen atom is placed in an external electric field of strength $\mathcal{E}$, resulting in shifts in the atomic energy levels which are large relative to atomic fine structure. The effect of the electric field on the level with principal quantum number $n = 3$ is to be analysed using first-order degenerate perturbation theory, with a perturbation $\hat{H}' = e\mathcal{E}z$, and working in the basis of states $|n\ell m_\ell\rangle$ ordered as

$$|300\rangle, \quad |310\rangle, \quad |320\rangle, \quad |311\rangle, \quad |321\rangle, \quad |31, -1\rangle, \quad |32, -1\rangle, \quad |322\rangle, \quad |32, -2\rangle \ .$$

The reduced matrix elements for the electron position operator $\hat{\boldsymbol{r}}$ for $n = 3$ are $\langle 3\mathrm{s} \| \hat{\boldsymbol{r}} \| 3\mathrm{p}\rangle = 9\sqrt{2}a_0$, and $\langle 3\mathrm{d} \| \hat{\boldsymbol{r}} \| 3\mathrm{p}\rangle = -(9/\sqrt{2})a_0$, where a_0 is the Bohr radius.

(a) Show that the matrix representation of $\hat{H}'$ in the basis above is block diagonal, with sub-matrices of the form

$$H'_0 = \begin{pmatrix} 0 & a & 0 \\ a & 0 & b \\ 0 & b & 0 \end{pmatrix} \ , \qquad H'_{+1} = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix} \ , \qquad H'_{-1} = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix} \ ,$$

where a, b, c are constants such that $a = \sqrt{2}b = \sqrt{8/3}c$, and $c = -(9/2)ea_0\mathcal{E}$.

[You may find the $1 \otimes 1$ table of Clebsch-Gordan coefficients useful: see attached table.]

(b) Show that the electric field splits the $n = 3$ level into five equally spaced levels with energy separation $(9/2)ea_0\mathcal{E}$. State the values of the quantum number m_ℓ associated with each of these five levels.

(c) Show that, in the electric field, the $n = 3$ level of highest energy corresponds to the zeroth-order eigenstate

$$|\psi\rangle = \sqrt{\frac{1}{3}}|300\rangle - \sqrt{\frac{1}{2}}|310\rangle + \sqrt{\frac{1}{6}}|320\rangle \ ,$$

and (*optionally*) determine the zeroth-order eigenstates for the other four levels.

Question 8. Identical particles

Two non-interacting, indistinguishable particles of mass m move in the one-dimensional potential $V(x)$ given by

$$V(x) = \begin{cases} 0 & 0 < x < L \\ \infty & \text{otherwise} \end{cases} .$$

Show that the energy of the system is of the form $E = (n_1^2 + n_2^2)\varepsilon$, where n_1 and n_2 are integers, and find an expression for ε.

Consider the state with $E = 5\varepsilon$ for each of the following three cases:

(a) spin-zero particles;

(b) spin-1/2 particles in a spin-singlet state;

(c) spin-1/2 particles in a spin-triplet state.

In each case, state the symmetries of the spin and spatial components of the two-particle wavefunction. Write down the spatial wavefunction $\psi(x_1, x_2)$, and sketch the probability density $|\psi(x_1, x_2)|^2$ in the (x_1, x_2) plane.

Describe qualitatively how the energies of these states would change if the particles carried electric charge and hence interacted with each other.

Question 9. Aharonov-Bohm effect

A ring-shaped semiconductor device is fabricated from a high mobility two-dimensional electron gas, Fig. 1, and cooled in a cryostat to 0.3 K. On the figure, the lighter grey is the conducting region.

A voltage is applied across the ring (between points at the bottom and the top of the image) and the current flow is measured as a function of a magnetic field applied perpendicular to the plane of the ring.

Explain why the oscillations in conductance occur, account for their periodicity, and obtain a value for the average diameter of the ring. (1 Tesla = 10^4 Gauss.)

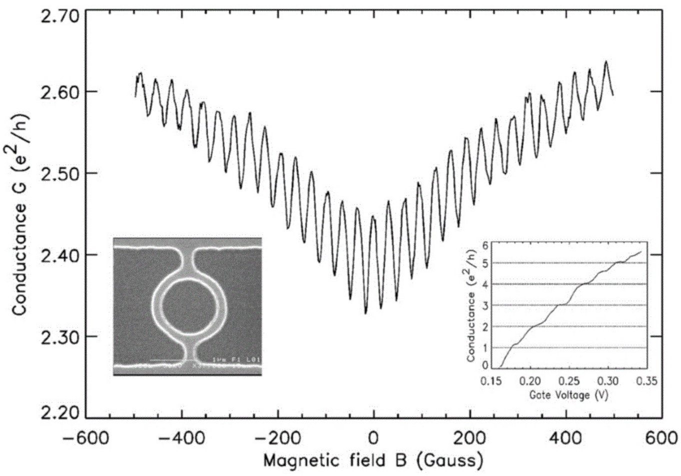

Figure 1: Aharonov-Bohm effect in a semiconductor quantum ring. [From S. Pedersen *et al.*, Phys. Rev. B **61** (2000) 5457.]

43. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND *d* FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

m_1	m_2	J	J	...
		M	M	...
m_1	m_2	Coefficients		
.	.			
.	.			

1/2×1/2

m_1	m_2	1
		+1
+1/2	+1/2	1

m_1	m_2	1	0
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	1
		−1
−1/2	−1/2	1

1×1/2

m_1	m_2	3/2
		+3/2
+1	+1/2	1

m_1	m_2	3/2	1/2
		+1/2	+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

m_1	m_2	3/2	1/2
		−1/2	−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

m_1	m_2	3/2
		−3/2
−1	−1/2	1

2×1

m_1	m_2	3
		+3
+2	+1	1

m_1	m_2	3	2
		+2	+2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

m_1	m_2	3	2	1
		+1	+1	+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

m_1	m_2	3	2	1
		0	0	0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

m_1	m_2	3	2	1
		−1	−1	−1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

m_1	m_2	3	2
		−2	−2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

m_1	m_2	3
		−3
−2	−1	1

1×1

m_1	m_2	2
		+2
+1	+1	1

m_1	m_2	2	1
		+1	+1
+1	0	1/2	1/2
0	+1	1/2	−1/2

m_1	m_2	2	1	0
		0	0	0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

m_1	m_2	2	1
		−1	−1
0	−1	1/2	1/2
−1	0	1/2	−1/2

m_1	m_2	2
		−2
−1	−1	1

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^{\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

2×1/2

m_1	m_2	5/2
		+5/2
+2	+1/2	1

m_1	m_2	5/2	3/2
		+3/2	+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

m_1	m_2	5/2	3/2
		+1/2	+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

m_1	m_2	5/2	3/2
		−1/2	−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

m_1	m_2	5/2	3/2
		−3/2	−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

m_1	m_2	5/2
		−5/2
−2	−1/2	1

3/2×1/2

m_1	m_2	2
		+2
+3/2	+1/2	1

m_1	m_2	2	1
		+1	+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

m_1	m_2	2	1
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	2	1
		−1	−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

m_1	m_2	2
		−2
−3/2	−1/2	1

3/2×1

m_1	m_2	5/2
		+5/2
+3/2	+1	1

m_1	m_2	5/2	3/2
		+3/2	+3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

m_1	m_2	5/2	3/2	1/2
		+1/2	+1/2	+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

m_1	m_2	5/2	3/2	1/2
		−1/2	−1/2	−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

m_1	m_2	5/2	3/2
		−3/2	−3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

m_1	m_2	5/2
		−5/2
−3/2	−1	1

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^j_{m',m} = (-1)^{m-m'} d^j_{m,m'} = d^j_{-m,-m'}$$

3/2×3/2

m_1	m_2	3
		+3
+3/2	+3/2	1

m_1	m_2	3	2
		+2	+2
+3/2	+1/2	1/2	1/2
+1/2	+3/2	1/2	−1/2

m_1	m_2	3	2	1
		+1	+1	+1
+3/2	−1/2	1/5	1/2	3/10
+1/2	+1/2	3/5	0	−2/5
−1/2	+3/2	1/5	−1/2	3/10

m_1	m_2	3	2	1	0
		0	0	0	0
+3/2	−3/2	1/20	1/4	9/20	1/4
+1/2	−1/2	9/20	1/4	−1/20	−1/4
−1/2	+1/2	9/20	−1/4	−1/20	1/4
−3/2	+3/2	1/20	−1/4	9/20	−1/4

m_1	m_2	3	2	1
		−1	−1	−1
+1/2	−3/2	1/5	1/2	3/10
−1/2	−1/2	3/5	0	−2/5
−3/2	+1/2	1/5	−1/2	3/10

m_1	m_2	3	2
		−2	−2
−1/2	−3/2	1/2	1/2
−3/2	−1/2	1/2	−1/2

m_1	m_2	3
		−3
−3/2	−3/2	1

2×3/2

m_1	m_2	7/2
		+7/2
+2	+3/2	1

m_1	m_2	7/2	5/2
		+5/2	+5/2
+2	+1/2	3/7	4/7
+1	+3/2	4/7	−3/7

m_1	m_2	7/2	5/2	3/2
		+3/2	+3/2	+3/2
+2	−1/2	1/7	16/35	2/5
+1	+1/2	4/7	1/35	−2/5
0	+3/2	2/7	−18/35	1/5

m_1	m_2	7/2	5/2	3/2	1/2
		+1/2	+1/2	+1/2	+1/2
+2	−3/2	1/35	6/35	2/5	2/5
+1	−1/2	12/35	5/14	0	−3/10
0	+1/2	18/35	−3/35	−1/5	1/5
−1	+3/2	4/35	−27/70	2/5	−1/10

m_1	m_2	7/2	5/2	3/2	1/2
		−1/2	−1/2	−1/2	−1/2
+1	−3/2	4/35	27/70	2/5	1/10
0	−1/2	18/35	3/35	−1/5	−1/5
−1	+1/2	12/35	−5/14	0	3/10
−2	+3/2	1/35	−6/35	2/5	−2/5

m_1	m_2	7/2	5/2	3/2
		−3/2	−3/2	−3/2
0	−3/2	2/7	18/35	1/5
−1	−1/2	4/7	−1/35	−2/5
−2	+1/2	1/7	−16/35	2/5

m_1	m_2	7/2	5/2
		−5/2	−5/2
−1	−3/2	4/7	3/7
−2	−1/2	3/7	−4/7

m_1	m_2	7/2
		−7/2
−2	−3/2	1

2×2

m_1	m_2	4
		+4
+2	+2	1

m_1	m_2	4	3
		+3	+3
+2	+1	1/2	1/2
+1	+2	1/2	−1/2

m_1	m_2	4	3	2
		+2	+2	+2
+2	0	3/14	1/2	2/7
+1	+1	4/7	0	−3/7
0	+2	3/14	−1/2	2/7

m_1	m_2	4	3	2	1
		+1	+1	+1	+1
+2	−1	1/14	3/10	3/7	1/5
+1	0	3/7	1/5	−1/14	−3/10
0	+1	3/7	−1/5	−1/14	3/10
−1	+2	1/14	−3/10	3/7	−1/5

m_1	m_2	4	3	2	1	0
		0	0	0	0	0
+2	−2	1/70	1/10	2/7	2/5	1/5
+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	18/35	0	−2/7	0	1/5
−1	+1	8/35	−2/5	1/14	1/10	−1/5
−2	+2	1/70	−1/10	2/7	−2/5	1/5

m_1	m_2	4	3	2	1
		−1	−1	−1	−1
+1	−2	1/14	3/10	3/7	1/5
0	−1	3/7	1/5	−1/14	−3/10
−1	0	3/7	−1/5	−1/14	3/10
−2	+1	1/14	−3/10	3/7	−1/5

m_1	m_2	4	3	2
		−2	−2	−2
0	−2	3/14	1/2	2/7
−1	−1	4/7	0	−3/7
−2	0	3/14	−1/2	2/7

m_1	m_2	4	3
		−3	−3
−1	−2	1/2	1/2
−2	−1	1/2	−1/2

m_1	m_2	4
		−4
−2	−2	1

$$d^1_{0,0} = \cos\theta \qquad d^{1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^1_{1,1} = \frac{1+\cos\theta}{2} \qquad d^1_{1,0} = -\frac{\sin\theta}{\sqrt{2}} \qquad d^1_{1,-1} = \frac{1-\cos\theta}{2}$$

$$d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,1/2} = -\sqrt{3}\,\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-1/2} = \sqrt{3}\,\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^2_{2,2} = \Big(\frac{1+\cos\theta}{2}\Big)^2$$

$$d^2_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^2_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^2_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^2_{2,-2} = \Big(\frac{1-\cos\theta}{2}\Big)^2$$

$$d^2_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^2_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^2_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^2_{0,0} = \Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

Figure 43.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974).