NATURAL SCIENCES TRIPOS Part II

Tuesday 25 May 2010 9.00 am to 12.00 noon

EXPERIMENTAL AND THEORETICAL PHYSICS (1)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (1)

Candidates offering the **whole** *of this paper should attempt a total of* **six** *questions, three from Section A* **and** *three from Section B. The questions to be attempted are* **A1***,* **A2** *and* **one** *other question from Section A and* **B1***,* **B2** *and* **one** *other question from Section B.*

Candidates offering **half** *of this paper should attempt a total of* **three** *questions,* **either** *three from Section A* **or** *three from Section B. The questions to be attempted are* **A1***,* **A2** *and* **one** *other question from Section A or* **B1***,* **B2** *and* **one** *other question from Section B. These candidates will leave after* **90 minutes***.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains **7** *sides, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

A separate Answer Book should be used for each section.

STATIONERY REQUIREMENTS
2 × 20 Page Answer Book
Metric graph paper
Rough workpad
Yellow master coversheet

SPECIAL REQUIREMENTS
Mathematical Formulae handbook
Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

THERMAL AND STATISTICAL PHYSICS

A1 *Attempt* **all** *parts of this question. Answers should be concise, and relevant formulae may be assumed without proof.*

(a) Consider a system of N identical but distinguishable particles, each of which has two energy levels with energy 0 or $\epsilon > 0$. The upper energy level has degeneracy g while the lower level is non-degenerate. Give an expression for the occupation number of the upper level in terms of the temperature of the system, and sketch its temperature dependence. [4]

(b) A system has entropy

$$S = A(NVU)^{1/3},$$

where A is a constant, V the volume, N the number of particles, and U the internal energy. Derive expressions for $U(S, V, N)$ and the temperature $T(S, V, N)$, and hence show that the pressure is

$$p(V, N, T) = \sqrt{\frac{N}{V}}\left(\frac{AT}{3}\right)^{3/2}.$$ [4]

(c) The density of states for spin $\frac{1}{2}$ fermions of mass m confined to a two-dimensional planar surface of area A is

$$g(\epsilon) = \frac{Am}{\pi\hbar^2}.$$

Find the chemical potential of a gas of N such fermions in the zero-temperature limit. [4]

A2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) the pressure of a Fermi gas at low temperature;

(b) the escape of a Brownian particle over a potential barrier;

(c) the thermodynamics of a quantum oscillator, including a discussion of the free energy and the entropy in the high and low temperature limits.

A3 *Attempt* **either** *this question* **or** *question A4.*

For a Fermi–Dirac energy distribution with chemical potential μ, show that the probability that a state with energy $\mu + \delta$ is occupied is equal to the probability that a state with energy $\mu - \delta$ is not occupied. [4]

Graphene monolayers formed by peeling single atomic sheets off graphite have an electronic band structure consisting of two symmetric branches which, at low energies, can be approximated by

$$E_k = \pm\hbar v_F|\boldsymbol{k}|,$$

where $\boldsymbol{k}$ is the two-dimensional wave-vector and $v_F = 10^6$ m s^{-1} is the Fermi velocity. The quasi-particles described by this relation behave as massless fermions with spin degeneracy $g = 2$. There is one electron per site.

(a) At zero temperature, all negative-energy states are occupied and all positive-energy states are empty; particle-hole symmetry then enforces $\mu(T) = 0$ at all temperatures. Show that, in this case, the mean excitation energy per unit area of a monolayer is given by

$$E(T) - E(0) = 4\int \frac{d^2\boldsymbol{k}}{(2\pi)^2}\frac{|E_k|}{\exp(\beta|E_k|) + 1},$$

where $\beta \equiv 1/k_B T$. [5]

Hence calculate the quasi-particle contribution to the heat capacity of the system. [6]

(b) Calculate the contribution of 2D phonons (in-plane lattice vibrations) to the heat capacity of a free-standing monolayer in the low-temperature limit. The typical sound velocity in graphite is of the order of $v_p = 2 \times 10^4$ m s^{-1}; explain which of phonons or electronic quasi-particles provides the dominant contribution to the heat capacity of graphene monolayers at low temperature. [10]

[*The following integrals may be useful:*

$$\int_0^\infty dx \frac{x^2}{e^x - 1} \simeq 2.4, \qquad \int_0^\infty dx \frac{x^2}{e^x + 1} \simeq 1.8.$$
]

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A4 *Attempt* **either** *this question* **or** *question A3.*

Explain why, in the canonical ensemble, the partition function of a dilute classical system of N particles of mass m interacting through a pair potential $\phi(r)$ can be written as

$$Z = \frac{1}{N!}\left(\frac{mk_{\mathrm{B}}T}{2\pi\hbar^2}\right)^{3N/2} Z_\phi,$$

where

$$Z_\phi = \int \exp\left(-\sum_{j>i}\beta\phi(r_{ij})\right) \mathrm{d}^3r_1 \ldots \mathrm{d}^3r_N,$$

and $\beta \equiv 1/k_{\mathrm{B}}T$. [5]

Find the difference between the internal energy of a dilute classical gas described by a pair potential $\phi(r)$ and the internal energy of an ideal gas of the same pressure, volume and temperature. For a pair potential of the form

$$\phi(r) = \begin{cases} \infty & \text{for } r < a \\ \psi(r) & \text{for } r > a, \end{cases}$$

where $\psi(r)$ converges rapidly to zero for $r \to \infty$, show that at high temperature this energy difference is equal to the pair potential averaged over the relative positions of all pairs of molecules. [8]

A system is described by a pair potential of the form above with $\psi(r) = -J/r^6$, where J is a positive constant. Describe qualitatively the behaviour of the system at high and low temperatures, and sketch the pair correlation function $g(r)$ in each case. [5]

Calculate the second virial coefficient

$$B_2(T) = \frac{1}{2}\int \mathrm{d}^3r\,[1 - \exp(-\beta\phi(r))]$$

for this system and obtain the Boyle temperature T^*. [7]

$$\left[\textit{The integral } \int_b^\infty (1 - \exp(1/x^6))x^2\,\mathrm{d}x \approx -\frac{1}{3b^3} \quad \textit{for } b > 1.\right]$$

SECTION B

RELATIVITY

Candidates are directed to pages 7 and 8 of the handbook of Mathematical Formulae for mathematical formulae in Relativity.

B1 *Attempt* **all** *parts of this question. Answers should be concise, and relevant formulae may be assumed without proof.*

(a) A moving relativistic particle of mass m collides head on with a stationary particle of mass $2m$. What is the minimum energy needed to produce a final state consisting of two particles each of mass αm, where $\alpha > 3/2$? [4]

(b) Calculate the gravitational deflection of a light ray passing close to a black dwarf star of mass 2×10^{30} kg and radius 6000 km. [4]

(c) For the metric

$$ds^2 = d\alpha^2 + \sinh^2 \alpha \, d\beta^2 \qquad (0 < \alpha < \infty),$$

calculate the non-zero affine connection coefficients $\Gamma^{\beta}_{\beta\alpha}$, $\Gamma^{\beta}_{\alpha\beta}$ and $\Gamma^{\alpha}_{\beta\beta}$, and the components $R^{\alpha}_{\beta\alpha\beta}$ and $R^{\beta}_{\alpha\beta\alpha}$ of the Riemann tensor. [4]

B2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) inertial and gravitational mass and the Principle of Equivalence;

(b) the covariant derivative and the parallel transport of vectors;

(c) the observational evidence for the existence of black holes.

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B3 *Attempt* **either** *this question* **or** *question B4.*

In Newton's theory of gravity, the equation of motion of a test particle orbiting a point mass M in the plane $\theta = \pi/2$ can be written as

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u = \frac{GM}{h^2}, \qquad h = r^2 \frac{\mathrm{d}\phi}{\mathrm{d}t},$$

where $u = 1/r$, G is the gravitational constant and (r, θ, ϕ) are the usual spherical polar coordinates. What is the physical significance of the parameter h? Show that a test particle can move in a circular orbit for any radius R, and determine h and the orbital period T as a function of R. [5]

In General Relativity, the equation of motion becomes

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u = \frac{GM}{h^2} + \frac{3GMu^2}{c^2}, \qquad h = r^2 \frac{\mathrm{d}\phi}{\mathrm{d}\tau},$$

where τ is the proper time. Show that there is no stable circular orbit with $R < 6GM/c^2$ and that there is no circular orbit, stable or unstable, with $R < 3GM/c^2$. Show also that, in the limit $GM/hc \ll 1$, the radius of the stable circular orbit approaches the Newtonian value. [9]

Show that, for a circular orbit of radius R, the parameter h is given by

$$h^2 = \frac{GMR^2}{R - 3GM/c^2}.$$

[5]

A test particle is initially at a distance 1 AU from a black hole of mass $3 \times 10^6\ \mathrm{M}_\odot$ and has zero radial velocity and transverse velocity $0.1c$. Show that the test particle will cross the horizon of the black hole. [3]

Calculate the velocity of a test particle in a circular orbit of radius 1 AU. [3]

[*You may assume that, in General Relativity, for a particle moving transversely with velocity v at radius R, the parameter h is given by $h = Rv/(1 - v^2/c^2)^{1/2}$.*
The mass of the Sun is $1\ \mathrm{M}_\odot = 2 \times 10^{30}$ kg. 1 AU $= 1.5 \times 10^{11}$ m.]

B4 *Attempt* **either** *this question* **or** *question B3.*

A spacecraft falls radially from rest at infinity towards a black hole of mass M, in a spacetime described by the Schwarzschild metric

$$ds^2 = \left(1 - \frac{2\mu}{r}\right)c^2 dt^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,,$$

where $\mu = GM/c^2$, G is the gravitational constant and (r, θ, ϕ) are the usual spherical polar coordinates.

Show that the radial coordinate r of the spacecraft satisfies

$$\frac{dr}{dt} = -c\left(1 - \frac{2\mu}{r}\right)\sqrt{\frac{2\mu}{r}}.$$ [7]

Show that an observer at fixed r, θ and ϕ, momentarily coincident with the spacecraft, measures the radial velocity v of the spacecraft as

$$v = -c\sqrt{\frac{2\mu}{r}}.$$ [9]

While it is falling, the spacecraft sends a radio signal radially outwards to an observer at infinity. If the spacecraft emits the signal at frequency ν_0 and is at radial coordinate r, at what frequency will the observer at infinity receive the signal? [9]

END OF PAPER