UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

May–June 2020 **1 hour 15 minutes**

PHYSICS (5)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (5)

ASTROPHYSICAL FLUID DYNAMICS

Candidates offering this paper should attempt a total of ***four*** *questions:* ***three*** *questions from Section A and* ***one*** *question from Section B.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains ***four*** *sides, including this coversheet. You may use the formula handbook for values of constants and mathematical formulae, which you may quote without proof.*

You have 75 minutes (plus any pre-agreed individual adjustment) to answer this paper. Do not start to read the questions on the subsequent pages of this question paper until the start of the time period.

Please treat this as a closed-book exam and write your answers within the time period. Downloading and uploading times should not be included in the allocated exam time. If you wish to print out the paper, do so in advance. You can pause your work on the exam in case of an external distraction, or delay uploading your work in case of technical problems.

Section A and the chosen section B question should be uploaded as separate pdfs. Please name the files 1234X_Qi.pdf, where 1234X is your examination code and i is the number of the question/section (A or 4 or 5).

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Master coversheet	Mathematical Formulae handbook
	Approved calculator allowed

SECTION A

*Attempt **all** questions in this Section. Answers should be concise and relevant formulae may be assumed without proof.*

1 The Galactic disc in the direction perpendicular to the disc can be approximated as a static slab of gas in equilibrium with a total thickness $2a$ and uniform gas density ρ. Calculate the temperature variation with height above the mid-plane z. [4]

2 A supernova injects 10^{44}J into the interstellar medium, which has a density of 10^6 hydrogen atoms per cubic metre. Assuming the initial mass of the explosion can be ignored, use dimensional analysis to find an expression for the radius of the blast wave as a function of time and estimate its radius one thousand years after the supernova. [4]

3 The radial flow velocity, u of an isothermal stellar wind (or accretion flow) with sound speed c_s can be described by the Bondi solution

$$\left(\frac{u}{c_\mathrm{s}}\right)^2 - \ln\left(\frac{u}{c_\mathrm{s}}\right)^2 = 4\ln\frac{r}{r_\mathrm{s}} + 4\frac{r_\mathrm{s}}{r} + C,$$

where r_s is the sonic radius and C a constant. On a single plot, sketch the different regimes for this solution for different values of C. Briefly discuss the solution for each regime noting the unphysical regimes. What would happen if the initial conditions placed a flow into one of the unphysical regimes initially? [4]

SECTION B

Attempt one question from this section

4 Hydrostatic equilibrium of a self-gravitating gaseous object is described by the equations

$$\begin{aligned} \nabla p &= -\rho \nabla \psi \\ \nabla^2 \psi &= 4\pi G \rho, \end{aligned}$$

where p is the pressure, ρ the density and ψ the gravitational potential.

(a) Explain why for a spherically symmetric system $p = p(\psi)$ and $\rho = \rho(\psi)$ and hence we must have a barotropic equation of state. [2]

(b) Assuming a spherically symmetric system with a barotropic equation of state of the form $p = K\rho^{1+1/n}$, and assuming suitable boundary conditions, show that

$$\rho = \left(\frac{\psi_T - \psi}{(n+1)K} \right)^n .$$

Hence derive the Lane-Emden equation for such a system

$$\frac{1}{\xi^2} \frac{\mathrm{d}}{\mathrm{d}\xi} \left(\xi^2 \frac{\mathrm{d}\theta}{\mathrm{d}\xi} \right) + \theta^n = 0,$$

where $\theta^n = \rho/\rho_\mathrm{c}$, ρ_c is the central density, $\xi = \alpha r$ is a dimensionless radial coordinate and α is a constant. Show also that

$$\alpha^2 = \frac{4\pi G \rho_\mathrm{c}^2}{(n+1)p_c},$$

where p_c is the central pressure. [9]

(c) For $n = 5$, show that

$$\frac{\rho}{\rho_\mathrm{c}} = \left(1 + \frac{\xi^2}{3} \right)^{-5/2}$$

is a solution of the Lane-Emden equation. [2]

(d) Hence show that the total mass of the system is given by

$$M = \frac{18}{\sqrt{2\pi}} \left(\frac{p_\mathrm{c}}{G} \right)^{3/2} \frac{1}{\rho_\mathrm{c}^2} .$$ [3]

(e) Let R be the radius such that $\alpha R = 1$, and assuming the gas obeys the ideal gas law, show that the central temperature T_c scales as GM/R. [3]

[*You may wish to use the following result*

$$\int_0^\infty x^2 (1 + x^2)^{-5/2} \, \mathrm{d}x = 1/3 .$$]

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5 The equations governing the behaviour of a self-gravitating fluid are

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) &= 0 \ , \\ \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} &= -\frac{1}{\rho} \boldsymbol{\nabla} p - \boldsymbol{\nabla} \psi \ , \\ \nabla^2 \psi &= 4\pi G \rho \ ,\end{aligned}$$

where $\boldsymbol{u}$, p and ρ are respectively the velocity pressure and density fields and ψ is the gravitational potential.

(a) Write down the equations for a stationary fluid of density ρ_0, pressure p_0 and gravitational potential ψ_0 which is in hydrostatic equilibrium. Briefly discuss why there is a problem when we consider an infinite static uniform medium, and note the approach adopted by Jeans to circumvent this problem. [2]

(b) By following this approach and considering small changes of the form $p = p_0 + \Delta p$, $\rho = \rho_0 + \Delta \rho$, $\psi = \psi_0 + \Delta \psi$ and $u = \Delta u$, find the linearised forms of the above equations that are valid to first order in the perturbed quantities—assume the sound speed $c_{\rm s}$ is constant. Assuming a wave-like solution of the form $\Delta p = p_1 \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}$ and similarly for the other perturbed quantities, show that the dispersion relation has the form

$$\omega^2 = c_{\rm s}^2 (k^2 - k_{\rm J}^2) \ ,$$

and find an expression for the Jeans wavenumber, $k_{\rm J}$, Jeans length $l_{\rm J} = 2\pi / k_{\rm J}$ and Jeans mass $M_{\rm J}$. [9]

(c) What is the criterion for a growing unstable mode that leads to gravitational collapse? Which modes, and hence which physical scales, grow fastest? [2]

(d) Discuss why a large object such as a galaxy, which has a mass that significantly exceeds the Jeans mass for the gas conditions in the galaxy, does not collapse to form a single large collapsed object, but instead gravitational collapse leads to the formation of stars with a range of masses. In your discussion consider the effect of the galaxy being in the form of a disc in which the dispersion relation for a Jeans-like analysis in this geometry is

$$\omega^2 = c_{\rm s}^2 k^2 - 2\pi G |k| \sigma_0 \ ,$$

where σ_0 is the initial surface density of gas. Consider also the number of Jeans masses contained within the initial object and how this scales as the object collapses. [6]

END OF PAPER