answers 09-10

20 October 2009
17:25

B1

(a) $A_{eff} = \frac{\lambda^2}{4\pi} G(\theta)$ [1]

$A_{max} = \frac{\lambda^2}{4\pi} \frac{3}{2}$ [1]

$= 0.5^2 \frac{3}{8\pi} \mu m^2$ [1]

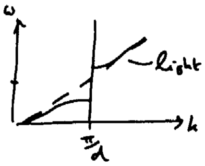
$= 0.03 \mu m^2$ [1]

[bookwork]

(b) a periodic arrangement of different n, on λ-scale.

[bookwork]

$n_1 = 1.4$, $n_2 = 1.6$
equal volumes, so $\bar{n} = 1.5$

[2] 

— light line $\frac{\omega}{k} = \frac{c}{\bar{n}}$

so $\omega = \frac{c}{\bar{n}} \frac{\pi}{d}$

so $\lambda = \frac{2\pi c}{\omega} = 2\bar{n}d = 600 nm$ [2]

(c) from $\phi = \frac{q}{\hbar}\int A.dr$, from $\hat{p} = -i\hbar\nabla - qA$

$\psi \sim \exp\left\{ i\left(\underline{k}.\underline{r} + \frac{qA.\underline{r}}{\hbar}\right)\right\}$ [2]

$\Delta_1 - \Delta_2 = \frac{q}{\hbar}\Phi_s$ — flux enclosed

need $\pi = \frac{e}{\hbar} BA$

so $B = \frac{\hbar}{er^2} \approx 7 \times 10^{-4} T$ [2]

[bookwork]

B2

(a) - sensing of wavefronts; relative phase
- mutual coherence function
$\Gamma(\tau) = \langle f(t) f^*(t-\tau)\rangle$ temporal coherence
- fringe visibility
$V(\tau) = \frac{|\Gamma(\tau)|}{I_0}$ $V(0) = 1$, $0 < V < 1$
- power spectrum
$P(\omega) = FT[\gamma(\tau)]$
- Michelson interferometer
- Spatial coherence lateral coherence
$V = |\gamma(u = kd)|$
$\gamma(u) = \frac{FT[I(\theta)]}{I_0}$
- Coherence volume
- [illegible]

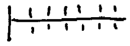
– stellar interferometer

(b) – Hertzian dipoles, quadrupoles, ... have specific θ, ϕ pattern
– antenna emit a combination of ED, MD, EQ, ...

– Power gain $= \dfrac{\text{flux in specific dir}^n}{\text{total flux}}$
$G(\theta,\phi)$
$\llcorner$ emission pattern

– flux $N = E \wedge H$

– from ED, $N = E_\theta B_\phi / \mu_0$ in far field

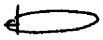
diagram of emission pattern
$G \propto \sin^4\theta \sin^2\phi$
– cancellation / interference of EM rad^n in some dir^{n}s
– phased array steering
– half-wave antenna
– stub antenna
–  explained: Yagi-Uda antenna

(c) – circular motion
$$\omega_B = \frac{qB}{\gamma m}$$

– power radiated
$$P = \frac{\mu_0 q^4 \gamma^2 B^2 u^2}{6\pi c m^2}$$

– radius $R = \dfrac{u}{\omega_B}$ velocity u

– $\lambda \sim \dfrac{2\pi c}{\omega_B} < R$ so not ED

–  in rest frame $\Rightarrow$  lab frame
$\Delta\theta \sim \dfrac{1}{\gamma}$

– pulsed appearance to observer

–  spectrum
cut off $\nu_c \sim \gamma^3 \omega_B$

– uses

B3

(a) $A = \left(\dfrac{\phi}{c}, \underline{A}\right)$ [1]
$J = (c\varrho, \underline{J})$ [1]

$\Box^2 A = \mu_0 J$ all Maxwell eq^{n}s manifestly indep. of frame [1]

$\Box . J$ is a Lorentz invariant $\Rightarrow$ cons. of charge (in all frames) [1]

$\Box . A$ [again scalar product of 4-vectors, so invariant] [1]
$\nwarrow$ gauge condition. [bookwork]

(b) $\phi' = \gamma(\phi - vA_x)$. $A_x' = \gamma(A_x - v\phi/c^2)$. $A_y' = A_y$. $A_z' = A_z$ [2]

(b) $\phi' = \gamma(\phi - vA_x)$, $A_x' = \gamma\left(A_x - v\phi/c^2\right)$, $A_y' = A_y$, $A_z' = A_z$ [2]

$B' = \nabla\wedge A'$, $E' = -\nabla'\phi' - \frac{\partial A'}{\partial t'}$

with $\frac{1}{c}\frac{\partial}{\partial t'} = \frac{1}{c}\left(\gamma\frac{\partial}{\partial t} + \gamma v\frac{\partial}{\partial x}\right)$ [2]

from chain rule $\frac{\partial}{\partial x'} = \gamma\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right)$

since $\Box$ is 4-vector

$B_x' = \frac{\partial A_z'}{\partial y'} - \frac{\partial A_y'}{\partial z'} = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = B_x$ [1]

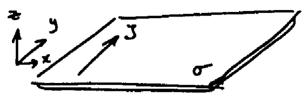
$B_y' = \frac{\partial A_x'}{\partial z'} - \frac{\partial A_z'}{\partial x'} = \gamma\left(\frac{\partial A_x}{\partial z} - \frac{v}{c^2}\frac{\partial \phi}{\partial z}\right) - \gamma\left(\frac{\partial A_z}{\partial x} + \frac{v}{c^2}\frac{\partial A_z}{\partial t}\right)$

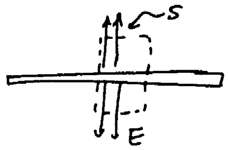
$= \gamma\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \gamma\frac{v}{c^2}\left(-\frac{\partial\phi}{\partial z} - \frac{\partial A_z}{\partial t}\right) = \gamma\left(B_y + \frac{v}{c^2}E_z\right)$ [1]

$E_x' = E_x$ | $B_x' = B_x$
$E_y' = \gamma(E_y - vB_z)$ | $B_y' = \gamma\left(B_y + \frac{v}{c^2}E_z\right)$
$E_z' = \gamma(E_z + vB_y)$ | $B_z' = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$

[bookwork]

(c)





$\int E.dS = \frac{Q}{\epsilon_0}$ [1]

$2ES = \frac{\sigma S}{\epsilon_0}$ outwards

$E = \frac{\sigma}{2\epsilon_0}$

$E = -\nabla\phi$

so $\phi = -\frac{\sigma z}{2\epsilon_0}$ [1]

[bookwork]

(d)

$\nabla^2 A = -\mu_0 J$ $\nabla . A = 0$ Coulomb gauge

$B = \nabla\wedge A$ [1]

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & -\frac{\mu_0 J z}{2} & 0 \end{vmatrix}$$ [1]

$= \left(\frac{\mu_0 J}{2}, 0, 0\right)$ [seen before]

indep. of height (field lines don't spread out)

(e) $E_\parallel = 0$, $E_\perp = \frac{\sigma}{2\epsilon_0}$, $B_\perp = 0$, $B_\parallel = \frac{\mu_0 J}{2}$

$E_\parallel' = E_\parallel = 0$ [1]

$E_\perp' = \gamma(E_\perp + v\wedge B_\perp) = \gamma\frac{\sigma}{2\epsilon_0}$ [1]

$B_\parallel' = B_\parallel = \mu_0 J/2$ [1]

$B_\perp' = \gamma\left(B_\perp - \frac{v}{c^2}\wedge E_\perp\right) = -\frac{v}{c^2}\frac{\sigma}{2\epsilon_0}\gamma$ [1]

[similar to problems]

(f) Lorentz contraction of plate by γ $\rightarrow\square\leftarrow$ [1]

squeezes up charge so charge density $\rightarrow \gamma\sigma$ [1]

$B_\parallel$ unchanged as current $\perp$ to velocity [1] but now $B_\perp$ as charge on plate moves.

$$J_{eff} = \frac{2B_\perp}{\mu_0} = \frac{v\sigma}{c^2}\frac{\gamma}{\epsilon_0\mu_0} = v\sigma\gamma \qquad [2]$$

Lorentz contracted charge $\gamma\sigma$ moves at v : ✓ [1]

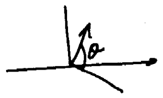
[seen in different form before]

B4

(a) Jones vector gives ratio of components of optical field along orthogonal axes. [1] [1]
State of light at a point [1]
Jones matrix gives transformation of Jones vector when passing through a component. [1]

[Bookwork]

(b)



field strength along polariser axes is
$(\cos\theta, \sin\theta)$ [1]

fraction transmitted is then $\cos\theta\,(\cos\theta,\ \sin\theta)$ [1]
$\sin\theta\,(\cos\theta,\ \sin\theta)$ [1]

giving $\underline{\underline{J}}_\theta$

(c) assume first polariser vertical so
$$\underline{V} = \begin{pmatrix}1\\0\end{pmatrix}$$

then
$$\underline{\underline{J}}_\theta \underline{V} = \begin{pmatrix}\cos^2\theta\\ \cos\theta\sin\theta\end{pmatrix} \qquad [1]$$

if $\theta = \pi/2$, $\underline{\underline{J}}_{\pi/2}\underline{V} = 0$ [1]

no light transmitted [seen in notes]

(d)
$$\underline{\underline{R}}(\theta)\,\underline{\underline{J}}(\theta) = \begin{pmatrix}c & s\\ -s & c\end{pmatrix}\begin{pmatrix}c^2 & cs\\ cs & s^2\end{pmatrix} \qquad [1]$$

$$= \begin{pmatrix}c^3+cs^2 & c^2s+s^3\\ -sc^2+c^2s & -cs^2+cs^2\end{pmatrix} \qquad [1]$$

$$= \begin{pmatrix}c & s\\ 0 & 0\end{pmatrix} \qquad [1]$$

where $c = \cos\theta$, $s = \sin\theta$

Many polarisers gives

$$(RJ)^N = \begin{pmatrix}c & s\\ 0 & 0\end{pmatrix}^N \qquad [1]$$

$$= \begin{pmatrix}c & s\\ 0 & 0\end{pmatrix}\begin{pmatrix}c & s\\ 0 & 0\end{pmatrix}\begin{pmatrix}c & s\\ 0 & 0\end{pmatrix}^{N-2}$$

$$= \begin{pmatrix}c^2 & cs\\ 0 & 0\end{pmatrix}\begin{pmatrix}c & s\\ 0 & 0\end{pmatrix}\begin{pmatrix}c & s\\ 0 & 0\end{pmatrix}^{N-3}$$

$$= \begin{pmatrix}c^3 & c^2s\\ 0 & 0\end{pmatrix}\begin{pmatrix}c & s\\ 0 & 0\end{pmatrix}^{N-3} \qquad [1]$$

$$= \begin{pmatrix} c^N & c^{N-1}s \\ 0 & 0 \end{pmatrix}$$

$$= \cos^N\theta \begin{pmatrix} 1 & \tan\theta \\ 0 & 0 \end{pmatrix}$$ [1]

[new problem]

(e) for N polarisers  $\frac{\pi}{2} = (N+1)\theta$ [1]

so $$(RJ)^N = \cos^N\left(\frac{\pi/2}{N+1}\right) \begin{pmatrix} 1 & \tan\frac{\pi/2}{N+1} \\ 0 & 0 \end{pmatrix}$$

For large N, small θ, $\cos\theta \rightarrow 1 - \frac{\theta^2}{2} + \dots$ [1]

Transmission $= (RJ)^N \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$= \cos^N\left(\frac{\pi/2}{N+1}\right)$$ [1]

$$\simeq \left[1 - \left(\frac{\pi/2}{N+1}\right)^2\right]^N$$

$$\simeq 1 - \frac{N\pi^2/4}{(N+1)^2}$$

$$\simeq 1 - \frac{\pi^2}{4N}$$ [1]

Intensity $= t^2 = \left(1 - \frac{\pi^2}{4N}\right)^2$ [1]

$$\simeq 1 - \frac{\pi^2}{2N}$$

$N = 20 \Rightarrow I_t = 75.3\%$ [1]

to get $I_t = 99\%$

$\frac{\pi^2}{2N} = \frac{1}{100}$ so $N = \frac{(10\pi)^2}{2} = 493$ [2]

[new problem]

(f) Problem is loss

- Fresnel loss because polarisers mean that $n_i \neq n_{i+1}$ [1]
- intrinsic loss of polarisers [1]

[new problem]