PAPER 5 (Astrophysical Fluid Dynamics) – ANSWERS 2012

1 (a) (UNSEEN) Given dispersion relation applies when $\boldsymbol{k}||\boldsymbol{v}_A$

Mode: (A) if $\boldsymbol{v}_A \cdot \boldsymbol{u} = 0$, we have motion transverse to $\boldsymbol{v}_A$ and we get a wave with:

$$\frac{\omega^2}{k^2} = \boldsymbol{v}_A^2$$

i.e. a pure MHD wave [2]

Mode (B): if $\boldsymbol{u}||\boldsymbol{k}$ then we have

$$(k^2 v_A^2 - \omega^2)\boldsymbol{u} + \left(\frac{c_s^2}{v_A^2} - 1\right) k^2 v_A^2 \boldsymbol{u} = 0$$

or

$$(k^2 c_s^2 - \omega^2) u = 0$$

i.e. a normal compressible cound wave [2]

(b) (UNSEEN, BUT SIMILAR TO EXAMPLE SHEET QUESTION)

Cooling rate $\dot{Q} = A\rho T^\alpha - H = \frac{\mu A p T^{\alpha-1}}{k_B} - H$ [2]

therefore

$$\dot{Q} = \dot{Q}\Big)_{T_0} + \frac{\partial \dot{Q}}{\partial T}\Big)_p \Delta T$$

$\frac{\partial \dot{Q}}{\partial T}\Big)_p < 0$ implies instablility and

$$\frac{\partial \dot{Q}}{\partial T}\Big)_p = (\alpha - 1)\frac{\mu A \rho}{k_B} T^{\alpha-1}$$

therefore unstable when $\alpha < 1$ i.e. Bremsstrahlung is unstable. [2]

(c) (UNSEEN, BUT SIMILAR TO EXAMPLE SHEET QUESTION)

Sound crossing time $t_c = R/c_s$

Estimate the free-fall collapse time:

Energy

$$\frac{GM^2}{R} \sim \frac{M\dot{R}}{2}$$

therefore

$$\dot{R} \sim \left(\frac{2GM}{R}\right)^{1/2} = \frac{R}{t_{ff}}$$

therefore

$$t_{ff} = \frac{R}{\dot{R}} \sim \left(\frac{R^3}{2GM}\right)^{1/2} \sim \left(\frac{3}{8\pi G\rho_0}\right)^{1/2}$$

or quote exact result:

$$t_{ff} = \left(\frac{3\pi}{32G\rho_0}\right)^{1/2}$$

equating

$$R^2 = c_s^2\left(\frac{3}{8\pi G\rho_0}\right)$$

[3]

if $t_{ff} < t_c$ will collapse. Could also note result is approximately Jeans length. [1]

2 Brief notes:

(a) stellar winds;

- Spherical symmetry and similarity to accretion (time reversed equations)
- Consider steady state
- Equations which govern the flow are:
 conservation of mass $\dot{M} = 4\pi r^2 \rho u$
 Momentum

$$u\frac{\mathrm{d}}{\mathrm{d}r} = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}r} - \frac{GM}{r^2}$$

 Continuity

$$\frac{\mathrm{d}\ln\rho}{\mathrm{d}r} = -\frac{2}{r} - \frac{\mathrm{d}\ln u}{\mathrm{d}}$$

- Can derive key equation

$$(u^2 - c_s^2)\frac{\mathrm{d}\ln u}{\mathrm{d}r} = \frac{2c_s^2}{r}\left[1 - \frac{GM}{2c_s^2 r}\right]$$

 This defines point where flow transitions between subsonic and supersonic flow $r_s = GM/2c_s^2$.

- Stellar winds are driven into ISM from stellar surface – provide energy input to the ISM
- Able to blow bubbles in the ISM – could give examples or more astrophysical insights here
- Proceeding with analysis need to consider equation of state; isothermal or adiabatic are most obvious
- Need a solution with speed going to zero as we get to infinity – could discuss general solution and select appropriate solution either mathematially or via a sketch of the solutions; either would get full credit.

(b) the Rayleigh-Taylor instability;

- Unstable stratified configuration of fluid under gravity
- Relationship to convective instability. Discussion of the effects of the entropy gradient and the requirements for how the entropy gradient stabilises or not the fluid.
- Might mention at this point astrophysical applications to the convective stability of stars
- Discuss general stratified fluid and how one would approach the analysis – no etails are required for full credit just diagrams indicating the general considerations.
- RT instability is the static case – either analysis of physical insight will give full credit. Physical insight is that a heavier fluid sits above a lighter one in a static gravitational field – any instability in the layer moves heavy gas to

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lower potential and forces cannot act to reverse the process due to having an equilibrium situation in which the stratified fluid is maintained.

- discuss astrophysical examples – e.g. filament formation in swept up gas behind a shock etc.

(c) Bernoulli's equation.

- Outline derivation of Bernoulli from time-dependent Euler equation
- Noting that

$$\boldsymbol{u}\cdot\nabla\boldsymbol{u} = \nabla\frac{1}{2}u^2 - \boldsymbol{u}\times(\nabla\times\boldsymbol{u})$$

and that the second term is the curl of the vorticity.

- We get to the critical equation

$$\frac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{u}\times(\nabla\times\boldsymbol{u}) = -\nabla\left(\int\frac{\mathrm{d}p}{\rho} + \frac{1}{2}u^2 + \phi\right)$$

- The term in the parentheses

$$H = \int\frac{\mathrm{d}p}{\rho} + \frac{1}{2}u^2 + \phi$$

is a constant if:

  - We consider flow along a streamline which is steady - dotting with the velocity loses the curl term. H is then constant along a streamline
  - We have vorticity free steady flow everywhere in which case H is the same constant throughout the fluid

- Examples of its use include: aircraft wings; motion of a shower curtain when you turn on the tap or any other relevant examples
- In the notes under this heading is the discussion of the conservation of vorticity in a fluid. Credit will be given if this is reproduced.

[13]

3 (BOOKWORK) Briefly describe how shock waves arise in interstellar space.
Answer: Arises from supersonic flow which is intrinsically unstable – gas moving faster than the sound speed means pressure waves cannot stabilise (also faster than molecular men speed). result is a discontinuous change in which the kinetic energy of the gas is thermlised in a shock with the post shock gas moving subsonically (spped goes down and sound speed increases substantially. [3]

Rankine-Hugoniot conditions at a normal adiabatic shock:

Conservation of mass

$$\rho_1 u_1 = \rho_2 u_2$$

or integrate the continuity equation across the sock front under stationary conditions. [1]

Momentum Integrate Euler

$$\frac{\partial \rho u}{\partial t} = -\frac{\partial}{\partial x}(\rho u_x u_x + p)$$

again stationary condition across shock surface or use physical arguent to give pressure balance directly. [2]

Energy:

Conserve flow of total energy allowing for the internal energy and pdV work

$$\rho_1 u_1(\frac{1}{2}u_1^2 e_1 + p_1/\rho_1) = \rho_2 u_2(\frac{1}{2}u_2^2 e_2 + p_2/\rho_2)$$

but $e = 1/(1-\gamma)p/\rho$ and use mass conservation to give

$$\frac{\gamma}{\gamma-1}\frac{p_2}{\rho_2} + \frac{1}{2}u_2^2 = \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1} + \frac{1}{2}u_1^2$$

Together we have [3]

$$\begin{aligned} \rho_2 u_2 &= \rho_1 u_1 \\ p_2 + \rho_2 u_2^2 &= p_1 + \rho_1 u_1^2 \\ \frac{\gamma}{\gamma-1}\frac{p_2}{\rho_2} + \frac{1}{2}u_2^2 &= \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1} + \frac{1}{2}u_1^2 \end{aligned}$$

(UNSEEN, BUT SIMILAR TO EXAMPLE SHEET QUESTION)

Strong shock the upstream pressure can be neglected – terms in p_1 can be neglected. Hence we have to this approximation:

$$\begin{aligned} \rho_2 u_2 &= \rho_1 u_1 && (1) \\ p_2 + \rho_2 u_2^2 &\approx \rho_1 u_1^2 && (2) \\ \frac{2\gamma}{\gamma-1}\frac{p_2}{\rho_2} + u_2^2 &\approx u_1^2 && (3) \end{aligned}$$

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From (2) $p_2 = \rho_1 u_1^2 - \rho_2 u_2^2 = \rho_2 u_2(u_1 - u_2)$ therefore

$$\frac{2\gamma}{\gamma - 1}\frac{\rho_2 u_2(u_1 - u_2)}{\rho_2} = u_1^2 - u_2^2 = (u_1 - u_2)(u_1 + u_2)$$

cancelling ρ_2 and $(u_1 - u_2)$ gives

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{\gamma + 1}{\gamma - 1}$$

[6]

(UNSEEN) Colliding clouds:

- Work in zero-momentum frame of colliding clouds – interface between clouds is stationary
- shock is driven back into each coliding cloud
- Shocked gas must have $u = 0$ since $u = 0$ at the interface between the clouds
- Unshocked gas must be moving at the original speed towards the interface – doesn't know the collision has occured since no sound waves travel ahead of the shock

[3]

Speed of shock – transform from the frame considered in first part of question to the frame in which the shocked gas is stationary. Define shock speed at V_s

- V_s is the relative speed between the frames since shock was stationary in original frame
- Must have $u_2 = V_s$ to have shocked gas stationary
- Therefore $u_1 = V + V_s$

But

$$u_1 = \frac{\gamma + 1}{\gamma - 1}u_2 = \frac{\gamma + 1}{\gamma - 1}V_s$$

therefore

$$V = \left(\frac{\gamma + 1}{\gamma - 1} - 1\right)V_s$$

Hence

$$V_s = \left(\frac{\gamma - 1}{2}\right)V$$

[4]

- As shock reaches back of cloud need to consider boundary conditions
- Cloud must be embedded in an external medium, think of shock as a wave – it will propogate into external gas
- There must be some form of relected discontunuity into the cloud
- In fact a shock propagates and a rarefaction wave is reflected into the cloud but not expecting this level of detail

[3]

4 (BOOKWORK) Lagrangian derivative of some quantity Q: Consider $Q = Q(\boldsymbol{r}, t)$ then

$$\Delta Q = \frac{\partial Q}{\partial t}\Delta t + \Delta \boldsymbol{r} \cdot \nabla Q$$

Hence

$$\frac{\Delta Q}{\Delta t} = \frac{\mathrm{D}Q}{\mathrm{D}t} = \frac{\partial Q}{\partial t} + \boldsymbol{u} \cdot \nabla Q$$

where $\boldsymbol{u} = \Delta \boldsymbol{r}/\Delta t$ is the fluid velocity. [3]

Euler's equation for fluid flow: Consider a small volume $\delta\tau$ then

$$\rho\delta\tau\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \text{forces}$$

Gravitational forces:

$$\boldsymbol{g} = -\nabla\phi$$

and

$$\boldsymbol{F}_g = -\rho\delta\tau\nabla\phi$$

Pressure:

$$\boldsymbol{F}_p = \oint_S p\mathrm{d}\boldsymbol{S} = -\int_V \nabla p \mathrm{d}\tau$$

therefore $-\nabla p\mathrm{d}\tau$ is the force on $\mathrm{d}\tau$. Hence

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho}\nabla p - \nabla\phi$$

[4]

Hydrostatic equilibrium:

$$\rho\nabla\phi = -\nabla p$$

Spherical cloud

$$\phi = -\frac{GM_r}{r}$$

$$M_r = \int_{-}^{r} 4\pi r'^2 \rho \mathrm{d}r$$

Therefore [3]

$$\rho\frac{GM_r}{r^2} = -\frac{\mathrm{d}p}{\mathrm{d}r} = -c_s^2\frac{\mathrm{d}\rho}{\mathrm{d}r}$$

Trial solution $\rho = A/r^2$ find $M_r = 4\pi A r$ and

$$\frac{\mathrm{d}\rho}{\mathrm{d}r} = -\frac{2A}{r^3}$$

Therefore [3]

$$\frac{G4\pi A r}{r^2} = -c_s^2\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} = c_s^2\frac{r^2}{A}\frac{2A}{r^3}$$

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works if

$$A = \frac{c_s^2}{2\pi G}$$

[2]

Cloud of mass $M = 4\pi A r_0$ from integral above hence:

$$M = \frac{2c_s^2 r_0}{G}$$

and must have $p_0 = p(r_0)$ as a boundary condition hence

$$p_0 = c_s^2 \frac{A}{r_0^2} = \frac{c_s^4}{2\pi G r_0^2}$$

[2]

(UNSEEN) Virial analysis:
Gravitational PE

$$\Omega = -\int \frac{GM_r}{r} \rho \mathrm{d}\tau = -2c_s^2 . \int \rho \mathrm{d}\tau = -2c_s^2 M$$

Internal energy

$$T = \int \frac{1}{\gamma - 1} p \mathrm{d}\tau = \frac{3}{2} c_s^2 \int \rho \mathrm{d}\tau = \frac{3}{2} c_s^2 M$$

hence

$$2T + \Omega = 3c_s^2 M - 2c_s^2 M = c_s^2 M$$

Now

$$4\pi r_0^3 p_0 = 4\pi r_0^3 \frac{c_s^4}{2\pi G r_0^2} = \frac{2c_s^4 r_0}{G} = c_s^2 M$$

Hence shown modified form of virial. [6]

The extra term is an Area, times a pressure times a distance - it is telated to the work done in embedding the cloud in the external medium – i.e. it is work term which is the most important thing to note. [2]

END OF PAPER