UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Wednesday 30 May 2018 9.00 am to 11.00 am

PHYSICS (3)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (3)

ADVANCED QUANTUM PHYSICS

Candidates offering this paper should attempt a total of **three** *questions. The questions to be attempted are* **1** *and* **two** *questions from Section B.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains **five** *sides, including this coversheet, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS
2 × 20 Page Answer Book
Metric graph paper
Rough workpad
Yellow master coversheet

SPECIAL REQUIREMENTS
Mathematical Formulae handbook
Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

draft: 17th May 2018

ADVANCED QUANTUM PHYSICS

SECTION A

Answers should be concise and relevant formulae may be assumed without proof.

1 *Attempt* **all** *parts of this question.*

(a) Two distinguishable spin-one particles combine into an overall $S = 0, m = 0$ singlet state. The overall spin state is a linear combination of products of spin states: $|S = 0, m = 0\rangle = \alpha\,|\chi_1\chi_2\rangle + \beta\,|\eta_1\eta_2\rangle + \gamma\,|\psi_1\psi_2\rangle$.
What are the three product states? By applying spin raising and lowering operators on the overall singlet state, find a possible combination of the coefficients α, β and γ. [*Note that* $\widehat{J^{\pm}}\,|j, m\rangle = \sqrt{j(j+1) - m(m \pm 1)}\,|j, m \pm 1\rangle$] [4]

(b) Why does an electric field applied across a 1-D quantum well with $V(x) = V(-x)$ give rise to a relative shift in energy levels proportional to the square of the electric field? [4]

(c) Sodium D-lines are produced by the transition of the single valence electron from a p- to an s-state. The lines split in an applied magnetic field. For the weak-field limit, draw an energy level diagram and indicate the allowed electric dipole transitions. Indicate which of the allowed transitions produces the shortest wavelength.
[*The Landé g-factor is* $g = \dfrac{3}{2} + \dfrac{S(S+1) - L(L+1)}{2J(J+1)}$] [4]

SECTION B
Attempt **two** *questions from this section*

B2 A particle with mass m and charge q constrained to move in one dimension is subject to a potential $V(x) = m\omega_0^2 x^2/2$, where ω_0 is the classical frequency of oscillation. Show that the matrix element of the position operator $\widehat{x}$ between ground state $|0\rangle$ and first excited state $|1\rangle$ has the magnitude

$$\left|\langle 1|\widehat{x}|0\rangle\right| = \sqrt{\frac{\hbar}{2m\omega_0}} \, .$$

[*An appropriate ladder operator is* $\widehat{a} = \widehat{x}\sqrt{\omega_0 m/(2\hbar)} + \mathrm{i}\widehat{p}/\sqrt{2\hbar\omega_0 m}$.] [5]

The particle starts out in its ground state for $t \to -\infty$ but is subjected to a weak perturbation $\widehat{V}_1(t)$. Writing the state of the particle in terms of its unperturbed eigenstates as

$$|\psi(t)\rangle = \sum_n c_n(t) \mathrm{e}^{-\mathrm{i}(n+1/2)\omega_0 t} \, |n\rangle \ ,$$

show that the coefficients c_n for $n \neq 0$ are approximately

$$c_n(t) = \frac{1}{\mathrm{i}\hbar} \int_{-\infty}^{t} \mathrm{e}^{\mathrm{i}n\omega_0 t'} \left\langle n \left| \widehat{V}_1(t') \right| 0 \right\rangle \mathrm{d}t' \, .$$ [7]

The perturbation is caused by a gradually increasing uniform electric field E: $\widehat{V}_1(t) = -qE\,\widehat{x}\,\mathrm{e}^{t/\tau}$, where τ is a time-scale. Show that the probability of finding the particle in the first excited state at $t = 0$ is

$$p_1(0) = \frac{q^2E^2}{2\hbar m\omega_0} \left| \int_{-\infty}^{0} \mathrm{e}^{\mathrm{i}\omega_0 t'} \mathrm{e}^{t'/\tau} \mathrm{d}t' \right|^2 \, .$$ [3]

Find how $p_1(0)$ depends on τ, illustrating your answer with a suitable sketch and analysing both limiting cases $\omega_0\tau \gg 1$ and $\omega_0\tau \ll 1$. [4]

B3 For spin-$\frac{1}{2}$ particles, a general spin state χ can be written as $|\chi\rangle = \alpha\,|\uparrow\rangle + \beta\,|\downarrow\rangle$, where α and β are complex numbers. Show that if χ is expressed in terms of angles θ, ϕ as $\alpha = \cos(\theta/2)$, $\beta = \mathrm{e}^{\mathrm{i}\phi}\sin(\theta/2)$, then θ and ϕ give the direction of $\langle\chi|\,\widehat{\boldsymbol{S}}\,|\chi\rangle$ in spherical polars. Explain why two angles contain enough information to specify the spin state, up to an undetermined phase factor. [5]

A spin-$\frac{1}{2}$ particle is described by a two-component spinor wavefunction $\psi(\boldsymbol{r}) = \phi(\boldsymbol{r})\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$. It is subject to the Hamiltonian

$$\widehat{H} = v\frac{\hbar}{\mathrm{i}}\left(\sigma_x\frac{\partial}{\partial x} + \sigma_y\frac{\partial}{\partial y} + \sigma_z\frac{\partial}{\partial z}\right),$$

where v is a constant and $\sigma_x, \sigma_y, \sigma_z$ are the Pauli matrices (see hint below). Show that travelling wave states

$$\psi_{\boldsymbol{k}}(\boldsymbol{r}) = \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\begin{pmatrix}\cos(\theta/2)\\ \mathrm{e}^{\mathrm{i}\phi}\sin(\theta/2)\end{pmatrix}$$

with $\boldsymbol{k} = \pm k\,(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ are eigenstates of $\widehat{H}$. Find the dispersion $E(\boldsymbol{k})$ and sketch it as a function of k. Sketch the spin orientation of eigenstates in the plane $k_z = 0$. [8]

A constant potential added to the Hamiltonian above has matrix elements V_{mn} between spinor states with the same wavevector, whereas the matrix elements are zero between states with different wavevectors. Explain why all Hermitian 2-by-2 matrices can be expressed as a linear superposition of the Pauli matrices and the identity matrix I: $V = a\sigma_x + b\sigma_y + c\sigma_z + dI$, with real coefficients a, b, c, d. Use this fact to sketch the dispersion $E(\boldsymbol{k})$ of the perturbed Hamiltonian. [6]

[*The Pauli matrices are*

$$\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\ \sigma_y = \begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix},\ \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$
]

B4 A ring of N atoms, each carrying an unpaired electron with spin $\frac{1}{2}$, has the Hamiltonian

$$\widehat{H} = -\frac{J}{\hbar^2} \sum_{j=1}^{N} \widehat{\boldsymbol{S}}_j \cdot \widehat{\boldsymbol{S}}_{j+1}$$

where an arbitrary atom has been labelled as atom 1, and we interpret the index $N+1$ as pointing back to atom 1. J is a positive constant. The eigenstates of any $\widehat{S}_z$ are $|\uparrow\rangle$ and $|\downarrow\rangle$. We denote the state in which all spins are up by $|\psi_0\rangle$, and by $|\psi_n\rangle$ the state in which all spins are up apart from that at position n, which is down. Show that

$$\widehat{\boldsymbol{S}}_j \cdot \widehat{\boldsymbol{S}}_{j+1} = \frac{1}{2}\left(\widehat{S}_j^+ \widehat{S}_{j+1}^- + \widehat{S}_j^- \widehat{S}_{j+1}^+\right) + \widehat{S}_j^z \widehat{S}_{j+1}^z .$$

[*Note that* $\widehat{J}^{\pm} |j, m\rangle = \sqrt{j(j+1) - m(m \pm 1)}\, |j, m \pm 1\rangle$] [2]

Hence, show that

$$\begin{aligned}
\widehat{\boldsymbol{S}}_j \cdot \widehat{\boldsymbol{S}}_{j+1} |\psi_n\rangle &= \frac{\hbar^2}{4} |\psi_n\rangle \quad \text{if}\, n \neq j\, \text{and}\, n \neq j+1, \text{ and} \\
\widehat{\boldsymbol{S}}_j \cdot \widehat{\boldsymbol{S}}_{j+1} \left|\psi_j\right\rangle &= \frac{\hbar^2}{4}\left[2\left|\psi_{j+1}\right\rangle - \left|\psi_j\right\rangle\right], \\
\widehat{\boldsymbol{S}}_j \cdot \widehat{\boldsymbol{S}}_{j+1} \left|\psi_{j+1}\right\rangle &= \frac{\hbar^2}{4}\left[2\left|\psi_j\right\rangle - \left|\psi_{j+1}\right\rangle\right] .
\end{aligned}$$

[5]

Find an expression for the energy E_0 of the state $|\psi_0\rangle$ in terms of J and N. [2]

Consider states

$$\left|\Psi_q\right\rangle = \sum_{n=1}^{N} \mathrm{e}^{\mathrm{i}qna} |\psi_n\rangle ,$$

where a is the straight-line distance between the atoms and q denotes a wavenumber. Find the values of q for which the translation of $\left|\Psi_q\right\rangle$ by N sites maps $\left|\Psi_q\right\rangle$ onto itself: $\widehat{T}_N \left|\Psi_q\right\rangle = \left|\Psi_q\right\rangle$. Show that these states are eigenstates of the Hamiltonian and calculate the associated energies $E(q)$. [7]

Sketch the dispersion $E(q) - E_0$ of these wave-like states for $-2\pi/a < q < 2\pi/a$, assuming large N. [3]

END OF PAPER