UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Monday 31 May 2021 11:00 to 13:00

PHYSICS (2)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (2)

RELATIVITY

Candidates offering this paper should attempt a total of ***five*** *questions:* ***three*** *questions from Section A and* ***two*** *questions from Section B.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains ***five*** *sides, including this coversheet, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
2×20 Page Answer Book	Mathematical Formulae handbook
Metric graph paper	Approved calculator allowed
Rough workpad	
Yellow master coversheet	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

*Attempt **all** questions in this Section. Answers should be concise and relevant formulae may be assumed without proof.*

1 For the two-dimensional metric $ds^2 = [dx^2 - c^2 dt^2] / (\alpha t^{-2})$, with α being a constant of appropriate dimensions, show that

$$\frac{\frac{dx}{dt}}{\sqrt{1 - \left(\frac{dx}{dt}\right)^2}}$$

is constant and hence, or otherwise, find all timelike geodesic curves. [4]

2 Consider a generalised form of the Einstein field equation

$$R_{\mu\nu} - \alpha g_{\mu\nu} R = -8\pi \frac{G}{c^4} T_{\mu\nu}$$

where $T^{\mu}{}_{\nu}$ is the energy-momentum tensor and α is some dimensionless constant. Show that $\nabla_\nu T_\mu{}^\nu \propto \partial_\mu T_\nu{}^\nu$ and argue which values of α are physically acceptable. [4]

3 Find the proper time elapsed for a circular orbit of radius r in the Schwarzchild metric,

$$ds^2 = c^2 \left(1 - \frac{2\mu}{r}\right) dt^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \,,$$

where $\mu = GM/c^2$. [4]

SECTION B
Attempt two questions from this section

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(a) Give the form of the Lorentz force law in general relativity. [3]

(b) A *five*-dimensional spacetime has metric

$$\mathrm{d}s^2 = \eta_{ab}\mathrm{d}x^a\mathrm{d}x^b + \left(A_a\mathrm{d}x^a + \mathrm{d}x^4\right)^2$$

where $\eta_{ab} = \mathrm{diag}(+1,-1,-1,-1)$ is the Minkowski metric of four-dimensional spacetime, and A_a is a dual 4-vector. Assume that A_a is independent of the fifth coordinate x^4. Show that the Euler–Lagrange equations for the geodesics imply

$$\frac{\mathrm{d}x^4}{\mathrm{d}\tilde{u}} + A_a\frac{\mathrm{d}x^a}{\mathrm{d}\tilde{u}} = \text{constant}, \qquad (\star)$$

where $\tilde{u}$ is an affine parameter in five-dimensional spacetime. [3]

(c) If, on a geodesic, the constant in $(\star)$ takes value ξ, show that $\tilde{u}$ is related to the interval u in four-dimensional spacetime by

$$\mathrm{d}\tilde{u}^2 = \frac{\mathrm{d}u^2}{1-\xi^2}.$$

[4]

(d) Show that the equation of motion for $x^a(\tilde{u})$ is:

$$\frac{\mathrm{d}^2x^a}{\mathrm{d}\tilde{u}^2} = \xi\eta^{ab}\left(\partial_c A_b - \partial_b A_c\right)\frac{\mathrm{d}x^c}{\mathrm{d}\tilde{u}}.$$

[5]

[*Hint: It is possible to simplify each term of the Euler–Lagrange equation for $x^a(\tilde{u})$ using $(\star)$ before combining them.*]

(e) Find an equation obeyed by the 4-velocity $u^a = \mathrm{d}x^a/\mathrm{d}\tau$ and interpret the result. [4]

5 A metric describing small deviations from Minkowski spacetime may be written as:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \qquad |h_{\mu\nu}| \ll 1.$$

In terms of $h_{\mu\nu}$ the linearized field equations in empty space take the form

$$\partial_\sigma\partial_\mu h^\sigma_\nu + \partial_\sigma\partial_\nu h^\sigma_\mu - \partial_\mu\partial_\nu h - \eta^{\sigma\rho}\partial_\sigma\partial_\rho h_{\mu\nu} - \eta_{\mu\nu}\partial_\rho\partial_\sigma h^{\rho\sigma} + \eta_{\mu\nu}\eta^{\sigma\rho}\partial_\sigma\partial_\rho h = 0,$$

where $h = \eta^{\mu\nu}h_{\mu\nu}$, and indices are raised and lowered with the Minkowski metric $\eta_{\mu\nu}$. This equation can be simplified by introducing

$$\phi^{\mu\nu} \equiv h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h,$$

and imposing the condition $\partial_\sigma\phi^{\sigma\nu} = 0$, which applies throughout this question.

(a) Show that the linearized field equations become

$$\partial_\lambda\partial^\lambda\phi^{\mu\nu} = 0\,. \qquad [4]$$

(b) For a trial solution of the form

$$\phi^{\mu\nu} = \epsilon^{\mu\nu}\cos(k_\alpha x^\alpha) \qquad (\star)$$

find and interpret the conditions that must be satisfied by $\epsilon^{\mu\nu}$ and k_α. [3]

For $k^\alpha = (\omega, 0, 0, \omega)$ one possible solution is

$$\epsilon^{\mu\nu}_\oplus = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_\oplus & 0 & 0 \\ 0 & 0 & -h_\oplus & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(c) The geodesics of stationary particles are unaffected in the linear approximation. Two such particles are separated by a small distance $\mathit{\Delta}$ in the $x-y$ plane at an angle θ to the x axis when $h_\oplus = 0$. Find the time dependence of their spatial separation when $h_\oplus \neq 0$. [4]

(d) An infinitesimal coordinate transformation $x^\mu \to x^\mu + \xi^\mu(x)$ gives rise to the transformation $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$ for some $\xi_\mu(x)$. Find an $\epsilon^{\mu\nu}$ such that $(\star)$ corresponds to a coordinate transformation, giving the corresponding ξ_μ. [5]

(e) Find an $\epsilon^{\mu\nu}$ distinct from $\epsilon^{\mu\nu}_\oplus$ that *does not* correspond to a coordinate transformation, which therefore describes a second propagating wave. [3]

6 This question concerns the following metric in 2 spacetime dimensions

$$ds^2 = (\alpha x)^2 dt^2 - dx^2 \,.$$

(a) Find the general form of the null geodesics, and sketch several examples on the $x - t$ plane. [4]

(b) The Schwarzchild metric is

$$ds^2 = c^2 \left(1 - \frac{2\mu}{r}\right) dt^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 \,.$$

Compare the geodesics from (a) with the radial null geodesics in the vicinity of the Schwarzchild radius $r_{\rm s} = 2\mu$, discussing both $r > r_{\rm s}$ and $r < r_{\rm s}$, explaining any similarities and differences. [4]

(c) Find the Christoffel symbols and show that the acceleration $a^\mu = \frac{Du^\mu}{D\tau}$ of a particle on the wordline $(x(\tau), t(\tau)) = (x_0, c\tau/(\alpha x_0))$ is constant. [4]

$\left[\Gamma^a_{bc} = \frac{1}{2} g^{ad} \left(\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc}\right)\right]$

(d) Express the metric in terms of the new coordinates

$$X = x \cosh(\alpha t)$$
$$T = (x/c) \sinh(\alpha t).$$

Interpret your answer to part (a) in terms of the new coordinates. [3]

(e) A spaceship has constant acceleration a in its own rest frame. A light signal is sent towards the departing spaceship from a distance d behind it, measured in the initial rest frame. Using the results from parts (a) and (c) (or otherwise), find the time until the signal catches up with the spaceship, as measured on the spaceship. [4]

END OF PAPER