UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

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Tuesday 27 May 2014 1.30 pm to 3.30 pm

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PHYSICS (1)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (1)

THERMAL AND STATISTICAL PHYSICS

*Candidates offering this paper should attempt a total of* **three** *questions. The questions to be attempted are* **1**, **2** *and* **one** *other question.*

*The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains* **four** *sides, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS
2 × 20 Page Answer Book
Rough workpad
Yellow master coversheet

SPECIAL REQUIREMENTS
Mathematical Formulae handbook
Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

THERMAL AND STATISTICAL PHYSICS

1 *Attempt* **all** *parts of this question. Answers should be concise and relevant formulae may be assumed without proof.*

(a) Make an annotated sketch of the temperature dependence of the internal energy and of the heat capacity for a quantum harmonic oscillator with natural frequency $\Omega$. [4]

(b) Show that for a system with the equation of state $p = a(V)T + b(V)$ the heat capacity at constant volume, $C_{\mathrm{V}}$, is independent of volume $V$.
[*Note that* $(\partial p/\partial T)_V = (\partial S/\partial V)_T$ .] [4]

(c) Derive the equation of state and molar heat capacity at constant volume of a classical ideal gas from the volume and temperature dependence of its single-particle partition function $Z_1 \propto VT^{3/2}$. [4]

2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) Brownian motion of a free particle, including the use of the Langevin equation to obtain Einstein's law of diffusion;

(b) Landau theory and its description of first and second-order phase transitions;

(c) the heat capacity of a Fermi gas at low temperature.

3 *Attempt* **either** *this question* **or** *question 4.*

State the average occupation number of a quantum state with energy $\epsilon$, in contact with a reservoir at chemical potential $\mu$, for fermions and bosons. In both cases, find the occupation number in the non-degenerate limit $\epsilon - \mu \gg k_B T$. [3]

Show that for a Bose gas in three dimensions with $\mu = 0$, the particle number density $n$ is proportional to $T^{3/2}$ and that for a non-degenerate ideal gas in three dimensions, $n$ is proportional to $T^{3/2} e^{\mu/(k_B T)}$. [6]

Explain the term *Bose–Einstein condensation* and show that for a Bose gas in three dimensions the condensation temperature $T_c$ depends on $n$ as $T_c \propto n^{2/3}$. [5]

In a certain magnetic insulator, which features electronic spin singlets on every lattice site, the excited states are spin triplets. The triplets can move through the lattice, forming a dilute gas of bosonic *triplons*. In an applied magnetic field $H$ the triplon energy $\epsilon(\boldsymbol{k})$ depends on wavevector $\boldsymbol{k}$ as

$$\epsilon(\boldsymbol{k}) = \frac{\hbar^2 k^2}{2m} + \nu n - \eta(H - H_g),$$

where the term $\nu n$, in which $\nu$ is a constant and $n$ is the triplon density, models the effect of repulsion between the triplons, $\eta$ is a constant and $H_g$ is a threshold field. Because triplon number is not conserved, the chemical potential $\mu$ is zero.

Sketch the triplon dispersion for $H = 0$, $H = H_g$ and $H = H_g + n\nu/\eta$. Explain why a form of Bose–Einstein condensation will occur when $H$ reaches the critical field $H_c \equiv H_g + n\nu/\eta$. [5]

Demonstrate that $H_c$ depends on temperature as

$$H_c - H_g \propto T^{3/2}$$

and sketch the expected $H$–$T$ phase diagram for this material. Explain how the dispersion $\epsilon(\boldsymbol{k})$ develops for $H > H_c$ and find the field dependence of the condensate number density. [6]

4 *Attempt* **either** *this question* **or** *question 3.*

The van der Waals equation of state for one mole of a real gas is

$$\left(p + \frac{a}{V^2}\right)(V - b) = RT,$$

where $a$ and $b$ are constants. Sketch isotherms $p(V)$, illustrating gas, vapour and liquid regions, the coexistence range, and the critical isotherm. Explain the physical origin of the constants $a$ and $b$. [8]

Sketch the $p{-}T$ phase diagram for the van der Waals equation of state, indicating the gas, vapour and liquid regions, the coexistence line and the critical end point. [3]

Find expressions for the volume, temperature and pressure at the critical point in terms of $a$, $b$ and $R$. [4]

Show that a thermodynamic variable $x$ at equilibrium undergoes statistical fluctuations with variance

$$\left\langle (x - \langle x\rangle)^2 \right\rangle = k_{\mathrm{B}}T \Big/ \left(\frac{\partial^2 A}{\partial x^2}\right),$$

where $A$ is the availability. [5]

One mole of a van der Waals gas is held at the critical volume and is cooled towards the liquid-gas critical point. Consider a small quantity of gas with volume $V$ inside the cylinder. Calculate the temperature dependence of its volume fluctuations, $\left\langle (V - \langle V\rangle)^2 \right\rangle$ and extract the critical exponent on approaching the critical temperature. [*Note that $(\partial F/\partial V)_T = -p$, where $F$ is the Helmholtz free energy.*] [5]

END OF PAPER