NATURAL SCIENCES TRIPOS Part II

Tuesday 26 May 2009 9.00 am to 12.00 noon

EXPERIMENTAL AND THEORETICAL PHYSICS (1)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (1)

Candidates offering the whole *of this paper should attempt a total of* six *questions, three from Section A* and *three from Section B. The questions to be attempted are* A1, A2 *and* one *other question from Section A and* B1, B2 *and* one *other question from Section B.*

Candidates offering half *of this paper should attempt a total of* three *questions,* either *three from Section A* or *three from Section B. The questions to be attempted are* A1, A2 *and* one *other question from Section A or* B1, B2 *and* one *other question from Section B. These candidates will leave after* 90 minutes.

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains 7 sides, and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.

Answers to each *section should be tied up separately, with the numbers of the questions attempted written clearly on the cover sheet.*

STATIONERY REQUIREMENTS
Script paper
Metric graph paper
Rough workpad
Blue coversheets (2)
Treasury tags

SPECIAL REQUIREMENTS
Mathematical formulae handbook
Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

THERMAL AND STATISTICAL PHYSICS

A1 *Attempt* **all** *parts of this question. Answers should be concise, and relevant formulae may be assumed without proof.*

(a) At room temperature, 1 gram of liquid water occupies a volume of $1\,\mathrm{cm}^3$. The water molecule has an atomic weight of 18. By considering the thermal de Broglie wavelength, show that the thermodynamics of liquid water molecules can be treated using classical statistics. [4]

(b) Derive the relationship between the fluctuations in the internal energy U of a system held at fixed temperature T and volume V and its heat capacity C_V. [4]

(c) Write down the partition function of an ideal classical spring whose energy is $\frac{1}{2}kx^2$ and use thermodynamics to show that the force on the spring is linearly proportional to its elongation x. [4]

A2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) the equipartition theorem, giving examples of its use;

(b) Brownian motion of a colloidal sphere;

(c) the radial distribution function of a classical liquid and its relation to the thermodynamic properties.

A3 *Attempt* either *this question* or *question A4.*

(a) Consider a uniform gas of ideal bosons in three dimensions. Sketch the form of the chemical potential $\mu(T)$ as a function of T. Explain why it is correct to set the chemical potential $\mu = 0$ in the temperature range where Bose-Einstein condensation occurs. [4]

(b) Consider a gas of bosons confined to move in two dimensions in a two-dimensional harmonic well. The energy levels are those of a simple harmonic oscillator (ignoring zero-point energy):

$$E_n = (n_x + n_y)\hbar\omega,$$

where n_x and n_y are positive integers and ω is the oscillator frequency. Show that for large n, where $n = n_x + n_y$, the number of states per unit energy is

$$g(E) \simeq \frac{E}{(\hbar\omega)^2}.$$

[3]

To investigate the possibility of Bose-Einstein condensation in this system, the number of particles N is divided into the number $N_0(T)$ in the lowest energy level and the number in the higher energy levels. Show that

$$N = N_0(T) + CT^2 \int_0^\infty \frac{x}{e^{x-\mu/(k_B T)} - 1} dx,$$

and determine the constant C. [5]

Show that Bose-Einstein condensation can occur in this system at a finite temperature T_c if $\mu/(k_B T)$ approaches zero as the temperature is reduced to T_c. [5]

Show that, for $T \leq T_c$, the heat capacity of the system is proportional to T^2. [3]

(c) Use similar arguments to investigate whether Bose-Einstein condensation is possible at finite temperature in a one-dimensional harmonic well. [5]

(TURN OVER

A4 *Attempt* **either** *this question* **or** *question A3.*
A simple model of a magnetic crystal consists of N non-interacting atoms with magnetic moments μs_i, where $s_i = \pm 1/2$. The crystal is placed in an external magnetic field B and is in equilibrium at a temperature T.

(a) Calculate the partition function for the spin states of the crystal. [2]

(b) Find an expression for S_{spin}, the contribution from the spin states to the entropy of the crystal. Evaluate S_{spin} in the strong and weak magnetic-field regimes. [6]

(c) The magnetisation is $M = \mu \left\langle \sum_{i=1}^{N} s_i \right\rangle$, and the susceptibility is $\chi = M/B$. Find expressions for M and χ, and evaluate them in the weak-field regime. [5]

Suppose, now, that the model is modified so that each atom interacts with n nearest neighbours in a way that favours ferromagnetism. To include this interaction approximately, we assume that the n nearest neighbours generate a "mean field" $\bar{B}$ at each atomic site, given by

$$\mu \bar{B} = \alpha \frac{n}{N} \left\langle \sum_{i=1}^{N} s_i \right\rangle ,$$

where α is the strength of the interaction between atoms.

(d) Use the mean field approximation, together with results from part (c), to calculate the susceptibility χ in the weak-field regime. At what temperature does χ become infinite? What phenomenon occurs at this temperature? [7]

(e) Write down an appropriate Landau expansion for the Helmholtz free energy near this phase transition, and comment on whether the susceptibility obtained from the Landau expansion would be expected to have the same form as obtained in part (d). [5]

SECTION B

RELATIVITY

Candidates are directed to pages 7 and 8 of the handbook of Mathematical Formulae for mathematical formulae in Relativity.

B1 *Attempt* all *parts of this question. Answers should be concise, and relevant formulae may be assumed without proof.*

(a) An atom of rest mass M_0, initially at rest, emits a photon, and in the process its rest mass decreases by m. Let $\boldsymbol{P}_a$ be the initial 4-momentum of the atom, $\boldsymbol{P}'_a$ be its 4-momentum after the emission and $\boldsymbol{P}_\gamma$ be the 4-momentum of the emitted photon. Show that

(1) $(\boldsymbol{P}'_a - \boldsymbol{P}_a) \cdot \boldsymbol{P}_\gamma = 0$,

(2) $\boldsymbol{P}_\gamma \cdot \boldsymbol{P}'_a = \frac{m}{2}(2M_0 - m)c^4$,

and thereby deduce an expression for the energy of the emitted photon in terms of m, M_0 and c. [4]

(b) For the metric

$$ds^2 = dz^2 + z^4 d\phi^2, \quad \begin{cases} 0 < z < \infty, \\ 0 \leq \phi < 2\pi, \end{cases}$$

calculate the non-zero affine connections $\Gamma^\phi_{\phi z}$, $\Gamma^\phi_{z\phi}$ and $\Gamma^z_{\phi\phi}$. Calculate the component $R^z_{\phi z\phi}$ of the Riemann tensor. Is it possible to find a coordinate transformation which transforms the metric to that of flat space? [4]

(c) A transparent liquid is moving with velocity v with respect to the laboratory frame. In the frame of the fluid light travels with velocity $u' = c/n$, where $n > 1$ is the refractive index of the fluid. Show that in the laboratory frame the light travelling in the fluid will be measured to have velocity

$$u \approx u' + \left(1 - \frac{1}{n^2}\right) v$$

to first order in v/c. [4]

(TURN OVER

B2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) the behaviour of test masses in a gravitational field;

(b) the transformation properties of electric and magnetic fields, using the concept of the electromagnetic field strength tensor $E_{\mu\nu}$;

(c) the bending of light by the Sun, and the precession of planetary perihelia as classical tests of General Relativity.

B3 *Attempt* **either** *this question* **or** *question B4.*

The spacetime around a point mass M is described by the Schwarzschild metric

$$ds^2 = c^2 d\tau^2 = \left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2.$$

What is the significance of the radius $r_S = 2GM/c^2$? [2]

A photon is emitted at position $r = r_0$ and moves radially towards a black hole. Starting from the Schwarzschild metric, show that the photon's coordinate speed is

$$\frac{dr}{dt} = -c\left(1 - \frac{2\mu}{r}\right),$$

where $\mu = GM/c^2$ and M is the mass of the black hole. [2]

Use this to derive an expression giving the coordinate time t at which the photon reaches radial distance r. (Assume $r_0 > r > r_S$.) [5]

How long, according to an external observer, does it take for the photon to reach r_S? Comment on whether a signal can in fact reach the interior of a black hole. [2]

Use the geodesic equations of motion to show that for a test particle moving in a spacetime described by the Schwarzschild metric, then $r^2 \frac{d\phi}{dp} = \text{const.}$, where p is an affine parameter. [*You may assume without derivation that the particle moves with* $\frac{d\theta}{dp} = 0$.] [12]

Give a physical interpretation of $r^2 \frac{d\phi}{dp}$. [2]

B4 *Attempt* **either** *this question* **or** *question B3.*

In General Relativity, the equation of motion of a test particle orbiting a point mass M in the plane $\theta = \pi/2$, can be written as

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2} + \frac{3GM}{c^2}u^2, \quad (*)$$

where $h = r^2\frac{d\phi}{d\tau}$, τ is the particle proper time, $u = 1/r$, G is the gravitational constant and (r, θ, ϕ) are the usual spacelike coordinates.

Show that circular orbits are not possible for $r < 3GM/c^2$. [3]

By multiplying through by $du/d\phi$ or otherwise, show that equation $(*)$ can be integrated to give the form

$$\left(\frac{dr}{d\tau}\right)^2 + V(r) = \text{const.},$$

and, assuming that $V(r) \to c^2$ as $r \to \infty$, give an explicit expression for $V(r)$ in terms of $\mu = GM/c^2$, and h. [8]

Transforming to dimensionless variables $x = r/\mu$ and $h' = h/(\mu c)$, give a sketch of the resulting potential $V(x)$, in units of c^2, for various suitable values of h'. [6]

A test particle is moving tangentially with velocity $v = 55\,\text{km}\,\text{s}^{-1}$ at distance $r_0 = 1$ parsec relative to a black hole with mass $M = 10^9 M_\odot$. What is the value of h' for this case? By comparing the value of $V(x)/c^2$ at $r = r_0$ with its values at smaller r, show that the test particle will cross the horizon of the black hole, despite its angular momentum. [1 parsec $= 3.086 \times 10^{16}$ m; $1M_\odot = 1.989 \times 10^{30}$ kg.] [8]

END OF PAPER