UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Tuesday 31 May 2011 9.00 am to 12.00 noon

EXPERIMENTAL AND THEORETICAL PHYSICS (1)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (1H)

Candidates offering the **whole** *of this paper should attempt a total of* **six** *questions, three from Section A* **and** *three from Section B. The questions to be attempted are* **A1***,* **A2** *and* **one** *other question from Section A and* **B1***,* **B2** *and* **one** *other question from Section B.*

Candidates offering **half** *of this paper should attempt a total of* **three** *questions,* **either** *three from Section A* **or** *three from Section B. The questions to be attempted are* **A1***,* **A2** *and* **one** *other question from Section A or* **B1***,* **B2** *and* **one** *other question from Section B. These candidates will leave after* **90 minutes***.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains **8** *sides, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

A separate Answer Book should be used for each section.

STATIONERY REQUIREMENTS
2 × 20 Page Answer Book
Metric graph paper
Rough workpad
Yellow master coversheet

SPECIAL REQUIREMENTS
Mathematical Formulae handbook
Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

THERMAL AND STATISTICAL PHYSICS

A1 *Attempt* **all** *parts of this question. Answers should be concise, and relevant formulae may be assumed without proof.*

(a) A pollen particle moving at random in water at room temperature is found to travel typically 4 μm in 100 seconds. If the particle's diameter is 1 μm, estimate the particle's root mean square velocity and velocity relaxation time. [4]

(b) Show that at constant entropy S, the change in temperature T of a dielectric body in an applied electric field E is given by

$$\left(\frac{\partial T}{\partial E}\right)_S = -\frac{T}{C_E}\left(\frac{\partial P}{\partial T}\right)_E,$$

where C_E is the heat capacity at constant E, and P is the electric polarisation. [4]

(c) The molar heat capacity of a material can be approximated by $C = (7.5 \times 10^{-4}\,\mathrm{JK/mole})/T^2$ in the range $0.1\,\mathrm{K} < T < 1\,\mathrm{K}$. If this represents the high temperature limit of the contribution of two closely spaced energy levels for each atom, estimate the temperature at which you expect to observe a peak in C versus T. [4]

A2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) the pair interaction potential and the virial expansion for a classical gas;

(b) the experimental evidence for the Bose-Einstein and Fermi-Dirac distribution functions;

(c) critical phenomena and universality.

A3 *Attempt* **either** *this question* **or** *question A4.*

Sketch the temperature dependence of the chemical potential μ of a gas of non-interacting Bose particles and show that a Bose-Einstein condensation occurs when $\mu \to 0$. [5]

Consider non-interacting spinless bosons of fixed density n and energy spectrum $\epsilon = \alpha k$, where k is the wavenumber and α is a positive constant. Show that the Bose-Einstein condensation temperature in three dimensions is given approximately by

$$T_\mathrm{c} = \frac{2.02\alpha n^{1/3}}{k_\mathrm{B}},$$ [5]

where k_B is the Boltzmann constant.

From the Helmholtz free energy $F = U - TS$ or otherwise, show that the pressure p satisfies

$$p - T\left(\frac{\partial p}{\partial T}\right)_V = -\left(\frac{\partial U}{\partial V}\right)_T,$$ [4]

where U is the internal energy and V is the volume. Hence, prove that if α is independent of volume and $T < T_\mathrm{c}$, the pressure of the gas is approximately given by

$$p = \frac{0.11 k_\mathrm{B}^4 T^4}{\alpha^3}.$$ [7]

Explain why p is independent of particle density. Compare and contrast this behaviour to that of the pressure of a photon gas. [4]

[*Note that*

$$\int_0^\infty \frac{x^n dx}{e^x - 1}$$

is approximately 2.40 for $n = 2$ *and 6.49 for* $n = 3$.]

A4 *Attempt* **either** *this question* **or** *question A3.*

Give a brief outline of Landau's theory of symmetry breaking and phase transitions. [5]

The free-energy density for a magnetic material is given by

$$F(M,n) = f(M) + \alpha n^2 + \beta n M^2,$$

where M is the magnetization, n is the atomic density, α is a positive constant and β is a coupling parameter. Show that for the most probable n, $F(M,n)$ reduces to

$$F(M) = f(M) - \frac{\beta^2}{4\alpha}M^4.$$ [3]

Consider the form

$$f(M) = \gamma(T - T_0)M^2 + bM^4 + cM^6 - MB,$$

where T is the temperature, B is the applied magnetic field and all other parameters are positive constants. Sketch $F(M)$ versus M for $T > T_0$, $B = 0$ and $b < \beta^2/(4\alpha)$ and show that in this limit a first-order transition occurs at

$$T_c = T_0 + \frac{1}{4\gamma c}\left(\frac{\beta^2}{4\alpha} - b\right)^2,$$ [5]

with a magnetization jump of magnitude

$$M_c = \left(\frac{\beta^2}{8\alpha c} - \frac{b}{2c}\right)^{1/2}.$$ [4]

Sketch the form of $F(M)$ versus M for $B \neq 0$, and hence the form of the most probable magnetization versus field for $T > T_c$. [5]

Under what conditions is the most probable value of n or M equal to the corresponding average value? [3]

SECTION B

RELATIVITY

B1 *Attempt* **all** *parts of this question. Answers should be concise, and relevant formulae may be assumed without proof.*

(a) By considering the effective Lagrangian $\mathcal{L} = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$, or otherwise, show that the geodesic equations for a massive particle in a Riemannian spacetime with metric $g_{\mu\nu}$ may be written as

$$\frac{\mathrm{d}u_\mu}{\mathrm{d}\tau} = \tfrac{1}{2}(\partial_\mu g_{\rho\sigma})u^\rho u^\sigma,$$

where $u^\rho = \dot{x}^\rho$ are the contravariant components of the particle's four-velocity and τ is its proper time. Comment on the physical implication of this result for metrics possessing symmetries. [4]

(b) In Minkowski spacetime, two observers, A and B, are moving at uniform speeds u and v, respectively, along different trajectories, each parallel to the y-axis of some inertial frame S. Observer A emits a photon with frequency ν_A that travels in the x-direction in S and is received by observer B with frequency ν_B. Show that the Doppler shift ν_B/ν_A in the photon frequency is independent of whether A and B are travelling in the same direction or opposite directions. [4]

(c) In Minkowski spacetime, a spaceship is moving along the x-axis of some inertial frame S. An astronaut on board measures the spaceship to have a constant acceleration g. Show that the components u^μ of the spaceship's four-velocity in S satisfy

$$\frac{\mathrm{d}^2u^\mu}{\mathrm{d}\tau^2} = \frac{g^2u^\mu}{c^2},$$

where τ is the proper time of the astronaut. [4]

B2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) the equivalence principle and local inertial coordinates;

(b) the formation of a black hole as perceived by a distant observer;

(c) the dragging of inertial frames.

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B3 *Attempt* **either** *this question* **or** *question B4.*

Explain briefly why in general relativity one requires $\nabla_\mu T^{\mu\nu} = 0$, where $T^{\mu\nu}$ is the energy-momentum tensor of all matter and fields in the spacetime. [2]

The energy-momentum tensor of a perfect fluid is

$$T^{\mu\nu}_{\text{fluid}} = \left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu - p g^{\mu\nu},$$

where ρ is its proper density, p is the isotropic pressure in the instantaneous rest frame and u^μ are the components of its 4-velocity.

Show that for any fluid

$$u_\nu \nabla_\mu u^\nu = 0.$$

[2]

Hence show that a perfect fluid in a gravitational field must satisfy the equations

$$\begin{aligned} \nabla_\mu(\rho u^\mu) + \frac{p}{c^2}\nabla_\mu u^\mu &= 0, \\ \left(\rho + \frac{p}{c^2}\right) u^\mu \nabla_\mu u^\nu &= \left(g^{\mu\nu} - \frac{u^\mu u^\nu}{c^2}\right)\nabla_\mu p. \end{aligned}$$

[7]

If the fluid is dust ($p = 0$), show that the worldline $x^\mu(\tau)$ of any one of the dust particles satisfies

$$\frac{\mathrm{d}^2 x^\nu}{\mathrm{d}\tau^2} + \Gamma^\nu{}_{\mu\sigma}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} = 0,$$

and interpret this result physically. [4]

If the dust particles each carry an electric charge q, show that, in the presence of both a gravitational and an electromagnetic field,

$$\nabla_\mu T^{\mu\nu}_{\text{dust}} = F^\nu{}_\mu j^\mu,$$

where j^μ is the four-current-density of the dust and $F_{\mu\nu}$ is the electromagnetic field tensor. [4]

The energy-momentum tensor of the electromagnetic field is

$$T^{\mu\nu}_{\text{em}} = -\mu_0^{-1}\left(F^\mu{}_\rho F^{\nu\rho} - \tfrac{1}{4} g^{\mu\nu} F_{\rho\sigma}F^{\rho\sigma}\right),$$

and Maxwell's equations in a curved spacetime are $\nabla_\mu F^{\mu\nu} = \mu_0 j^\nu$ and $\nabla_\mu F_{\nu\sigma} + \nabla_\sigma F_{\mu\nu} + \nabla_\nu F_{\sigma\mu} = 0$. Hence show that

$$\nabla_\mu\left(T^{\mu\nu}_{\text{dust}} + T^{\mu\nu}_{\text{em}}\right) = 0.$$

[6]

B4 *Attempt* **either** *this question* **or** *question B3.*

The Schwarzschild metric is

$$ds^2 = c^2\left(1 - \frac{2\mu}{r}\right)dt^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2.$$

Show that the geodesic equations for the worldline $x^\mu(\tau)$ of a massive particle in the equatorial plane $\theta = \pi/2$ are

$$\begin{aligned} \left(1 - \frac{2\mu}{r}\right)\dot{t} &= k, \\ r^2\dot{\phi} &= h, \\ \left(1 - \frac{2\mu}{r}\right)^{-1}\ddot{r} + \frac{\mu c^2}{r^2}\dot{t}^2 - \left(1 - \frac{2\mu}{r}\right)^{-2}\frac{\mu}{r^2}\dot{r}^2 - r\dot{\phi}^2 &= 0, \end{aligned}$$

where k and h are constants along the geodesic and dots denote $d/d\tau$, and give a physical interpretation of k and h. [4]

Hence show that for such a particle

$$\dot{r}^2 + \frac{h^2}{r^2}\left(1 - \frac{2\mu}{r}\right) - \frac{2\mu c^2}{r} = c^2(k^2 - 1),$$

and give a brief physical interpretation of this equation. [3]

For geodesic motion of a massive particle in the circle $r = R$, show that

$$k^2 = \frac{(R - 2\mu)^2}{R(R - 3\mu)} \qquad \text{and} \qquad h^2 = \frac{\mu c^2 R^2}{R - 3\mu}.$$

[6]

Show further that for the circular orbit to be stable one requires $R \geq 6\mu$. [4]

Alice and Bob are astronauts in a space capsule in a geodesic circular orbit at $r = R$ (where $R \geq 6\mu$). At some point in the orbit, Bob leaves the capsule, uses his rocket-pack to maintain a hovering position at that fixed point in space, and then rejoins the capsule after it has completed one orbit. If $\Delta\tau_A$ is the proper time interval measured by Alice between Bob leaving and rejoining the capsule and $\Delta\tau_B$ is the corresponding proper time interval measured by Bob, show that

$$\frac{\Delta\tau_B}{\Delta\tau_A} = \left(\frac{R - 2\mu}{R - 3\mu}\right)^{1/2}.$$

[4]

If Bob chooses not to rejoin the capsule, but instead observes it fly past him, show that he will measure the capsule to have a speed

$$v = \left(\frac{\mu c^2}{R - 2\mu}\right)^{1/2}.$$

[4]

END OF PAPER