NATURAL SCIENCES TRIPOS Part II

Tuesday 27 May 2014 9.00 am to 11.00 am

PHYSICS (2)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (2)

RELATIVITY — ANSWERS

1 (a) The 4-momentum of the first particle is $(m_1\gamma, m_1\gamma\boldsymbol{v})$ and of the second is $(m_2, \mathbf{0})$, so the composite system has a momentum $p \equiv (m_1\gamma + m_2, m_1\gamma\boldsymbol{v})$. This has an invariant $p^2 = m_1^2\gamma^2 + m_2^2 + 2m_1m_2\gamma - m_1^2\gamma^2\boldsymbol{v}^2$. As $\gamma^2(1-\beta^2) = 1$ this simplifies to $m_1^2 + m_2^2 + 2m_1m_2\gamma = m_\mathrm{c}^2$, where m_c is the rest mass of final system. The centre of mass is moving at a velocity $\boldsymbol{v}_\mathrm{c} = m_1\gamma\boldsymbol{v}/(m_1\gamma + m_2)$.

(b) The formula gives $\Gamma^x_{yy} = -f(x)f'(x)$; $\Gamma^y_{xy} = \Gamma^y_{yx} = f'(x)/f(x)$. Many will do this via the Lagrangian $\mathcal{L} = \dot{x}^2 + f^2(x)\,\dot{y}^2$. The required component of the Riemann tensor $R^x{}_{yxy} = -f(x)f''(x)$.

(c) It is extremely important that you are at the centre of mass of the spacecraft, or you will be splattered all over the hull... Since the encounter only takes 100 ms you might be able to withstand a good deal more than 100 s^{-1}. The tidal force is about $2GM/r^3$, so we estimate $r = 1,600$ km. If your original velocity is small, the tidal forces stretch/squash you by about your own radius in the time of the encounter $\approx \sqrt{r^3/GM}$. I'm not volunteering for this mission, but Beowolf Schaeffer (in Larry Niven's "Neutron Star") went in to $r = 12$ km. I hope he was already in Carlos Wu's autodoc...

(d) The gravitational bending of light is $\Delta\phi = 4GM/Rc^2$, which evaluates to 204 arcsec. You get [2] for GM/Rc^2, [1] for the 4 and [1] for working it out.

(e) The curve starts at the North Pole heading south at longitude 180°, heads east and sweeps by the equator at longitude 0° and continues to the South Pole at longitude −180°.

The metric is $\mathrm{d}s^s = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$, so $\mathrm{d}\theta = \mathrm{d}u$ and $\mathrm{d}\phi = 2\mathrm{d}u$, hence result ($L = 2\sqrt{5}E(2/\sqrt{5}) = 5.27$, where E is a complete elliptic integral).

2 (a) The answers will be very mathematical.

- The derivative of a vector (here contravariant) $A^i_{,j} \equiv \dfrac{\partial A^i}{\partial x^j}$ is not a tensor, as it transforms incorrectly under a change of coordinates $x \to \overline{x}$.
- We can correct this by making the covariant derivative

$$A^i_{;j} = \frac{\partial A^i}{\partial x^j} + \Gamma^i_{jk} A^k$$

where Γ^i_{jk} is the connection. All relevant formulae used to be in the formula book provided in the Exam, but are no longer there.

- Parallel transport of the vectors A^i and A_i uses the covariant derivative and is given by

$$\frac{\mathrm{d}A^i}{\mathrm{d}\tau} + \Gamma^i_{jk} A^j \dot{x}^k = 0 \; ; \quad \frac{\mathrm{d}A_i}{\mathrm{d}\tau} + \Gamma^j_{ik} A_j \dot{x}^k = 0 \; .$$

- Example of transport of vector around a sphere — rotates by amount equal to solid angle of loop.

(b) • This is not the answer to the question set, but contains some useful LaTex... Stuff about $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where A and ∇ are 4-vectors...
The bookwork requires persistence to do by hand, which I certainly didn't.
With $c = 1$ the definitions of $F_{\mu\nu}$ and L are

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix} \quad L = L^T = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1}$$

It's instructive to do in this way as you derive the Lorentz transformation of the E/M field:

$$\overline{E}_1 = E_1 \quad \overline{E}_2 = \gamma(E_2 - vB_3) \; ; \quad \overline{B}_1 = B_1 \quad \overline{B}_2 = \gamma(B_2 + vE_3/c^2) \tag{2}$$

i.e. longitudinal components unchanged and transverse ones like $\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}$ and $\boldsymbol{B} - \boldsymbol{v}\times\boldsymbol{E}/c^2$ with a Lorentz boost. However, since

$$\overline{F}_{\mu\nu} = \frac{\partial x^\alpha}{\partial \overline{x}^\mu} \frac{\partial x^\beta}{\partial \overline{x}^\nu} F_{\alpha\beta} \; ; \quad \overline{F}^{\mu\nu} = \frac{\partial \overline{x}^\mu}{\partial x^\alpha} \frac{\partial \overline{x}^\nu}{\partial x^\beta} F^{\alpha\beta} \; ; \quad \frac{\partial \overline{x}^\mu}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial \overline{x}^\nu} = \delta^\mu_\nu$$

it's not necessary to do the transformation at all...
This is how the last part should have been done:

$$F \equiv \nabla \wedge A = \boldsymbol{E} + ic\boldsymbol{B} \; ; \quad F^2 = \boldsymbol{E}^2 - c^2\boldsymbol{B}^2 + 2ic\boldsymbol{E} \cdot \boldsymbol{B}$$

This derivation gets the other invariant involving the Hodge dual as well.
They should have been asked to form $F^2 = ½F^{\mu\nu}F_{\nu\mu}$ in order to get the correct sign...

- Collapsed objects are the theoretically predicted end-points of stellar evolution as there is a well-defined limit to the size of object (white dwarf) that can be supported by electron degeneracy pressure (1.4 $M_\odot$, the Chandrasekhar limit).
- Even neutron degeneracy pressure only gets you to about 3 $M_\odot$ in neutron stars.
- There are many X-ray emitting objects known to have masses 5-10 $M_\odot$.
- Some objects (e.g. MGC-6-30) show spectral features of extreme Kerr black holes, presumably due to spin-up following accretion from binary companion.
- Rotation curve of Galaxy clearly indicates presence of 3.6×10^6 $M_\odot$ black hole.
- Extragalactic radio sources typically have black holes in range 10^8 to 10^9 $M_\odot$.
- Jets of radio sources show relativistic beaming effects and superluminal motions. These have to arise from potential wells that are very deep and can only exist in black holes, rather than other compact objects.
- We've actually no direct observational evidence at all. . .

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3 The Lorentz transform between frames S and S' in the standard configuration is

$$t' = \gamma(t - vx/c^2) \ ; \ \ x' = \gamma(x - vt); \ \ y' = y \ ; \ \ z' = z$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$. To get the velocities we use the inverse differential form

$$\mathrm{d}t = \gamma(\mathrm{d}t' + v\mathrm{d}x'/c^2) \ ; \ \mathrm{d}x = \gamma(x' + vt'); \ \ \mathrm{d}y = y' \ ; \ \mathrm{d}z = \ dz'$$

But $u_x = adx/\mathrm{d}t$, etc. so, dividing, we find

$$u_x = \frac{u_x + v}{1 + u_x v/c^2} \ ; \ \ u_y = \frac{u_y}{\gamma(1 + u_x v/c^2)} \ ; \ \ u_z = \frac{u_z}{\gamma(1 + u_x v/c^2)}$$

Use the relativistic addition of velocities formula for small increment $ad\tau$

$$v + \mathrm{d}v = \frac{v + a\mathrm{d}\tau}{1 + va\mathrm{d}\tau/c^2} \approx v + a\mathrm{d}\tau(1 - v^2/c^2) \implies \frac{\mathrm{d}v}{\mathrm{d}\tau} = \gamma^{-2}a$$

The substitutions $v = c\tanh u, \gamma = \cosh u$ imply that $v = c\tanh a\tau/c$ and integrating this up gives the required trajectory ($t = \frac{c}{a}\sinh(a\tau/c), x = \frac{c^2}{a}(\cosh(a\tau/c) - 1)$).

The momentum balance in the rest frame is $M\mathrm{d}v' = \mathrm{d}Mc$. So

$$\frac{\mathrm{d}M}{M} = \frac{\mathrm{d}v}{1 - v^2/c^2} \implies M = M_{\mathrm{i}}\left(\frac{c - v}{c + v}\right)^{1/2} \quad \text{(partial fractions).}$$

In S' the journey needs $L/\gamma = v\tau$, so $\beta\gamma = \sinh u = (100/10) = 10$, so $v = \tanh u = c10/\sqrt{101}$ and the final mass fraction (remembering we have to decelerate) is $(c - v)/(c + v) = 0.00249$. This has to be matter, so the fraction of anti-matter required is 0.49876. It had to be that big. Be generous here.

Obviously the passengers complained! To get to $\sinh u = 10$ takes $\tau = \sinh^{-1}(10)c/a = 2.86$ years (their calculators have arctanh on them, so that's an alternative) (used $365.25 \times 3600 \times 24$ for the year). They have travelled about $(\sqrt{101} - 1)c/a = 8.60$ light years at this time (see earlier). 5.70 years elapse going 17.19 light years, the remaining 82.81 light years take 8.28 years, so they arrive 3.99 years late.

4 The line element implies the Lagrangian ($c = 1$)

$$\mathcal{L} = \dot{t}^2(1 - 2\mu/r) - \dot{r}^2(1 - 2\mu/r)^{-1} - r^2\dot{\phi}^2 \tag{3}$$

The Lagrangian does not deped on the coordinates t or ϕ so there are conserved quantities:

$$k = \dot{t}(1 - 2\mu/r);\ \ h = r^2\dot{\phi}\ , \tag{4}$$

where k is the equivalent of the Special Relativity Lorentz factor γ, and h is the angular momentum per unit mass. The Lagrangian does not depend on the independent variable τ and, because τ is the proper time, we have

$$1 = \dot{t}^2(1 - 2\mu/r) - \dot{r}^2(1 - 2\mu/r)^{-1} - r^2\dot{\phi}^2 \tag{5}$$

Use the relations (4) to get the energy equation

$$\dot{r}^2 + V_{\text{eff}}(r) \equiv \dot{r}^2 + \frac{h^2}{r^2} - \frac{2\mu h^2}{r} - \frac{2\mu}{r} = k^2 - 1 \tag{6}$$

Circular motion with $\dot{r} = 0$ is possible when $\mathrm{d}V_{\text{eff}}/\mathrm{d}r = 0$:

$$-\frac{h^2}{r^3}(1 - 3\mu/r) + \frac{\mu}{r^2} = 0 \implies h^2 = \frac{\mu R^2}{R - 3\mu}$$

Substituting this into (6) gives

$$k^2 = \frac{(R - 2\mu)^2}{R(R - 3\mu)} \tag{7}$$

Alice is stationary and, since $v_A^2 = 1$, she must have 4-velocity $v_A^a = ((1 - 2\mu/R)^{-1/2}, 0, 0, 0)$. Bob is in a circular orbit so,using the above results,

$$v_B = (k(1 - 2\mu/R)^{-1}, 0, 0, h/R^2) = \left((1 - 3\mu/R)^{-1/2}, 0, 0, \sqrt{\frac{\mu}{R^3}}(1 - 3\mu/R)^{-1/2}\right) \tag{8}$$

From the general result $\gamma_{AB} = v_A{\cdot}v_B$, where $\gamma_{AB} = (1 - \beta_{AB}^2)^{-1/2}$ we get

$$v_A{\cdot}v_B = \gamma_{AB} = \sqrt{\frac{1 - 2\mu/R}{1 - 3\mu/R}} \implies \beta_{AB}^2 = 1 - \frac{1}{\gamma_{AB}^2} = \frac{\mu}{R - 2\mu} \tag{9}$$

Carol joins the party on a geodesic radial orbit with $k = 1$, with 4-velocity $v_C^a = ((1 - 2\mu/r)^{-1}, -\sqrt{2\mu/R}, 0, 0)$. Using the same formulae we get

$$v_A{\cdot}v_C = \gamma_{AC} = (1 - 2\mu/R)^{-1/2} \implies \beta_{AC} = \sqrt{\frac{2\mu}{R}}$$

Similarly, Carol sees Bob's velocity β_{BC} as

$$v_B{\cdot}v_C = \gamma_{BC} = (1 - 3\mu/R)^{-1/2} \implies \beta_{BC} = \sqrt{\frac{3\mu}{R}}$$

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From Alice's point of view the velocities are perpendicular, so the addition of velocities in special relativity gives

$$\beta_{CB}^2 = \beta_{AB}^2 + \beta_{AC}^2/\gamma_{AB}^2 = \frac{\mu}{R-2\mu} + \frac{2\mu}{R} \times \frac{R-3\mu}{R-2\mu} = \frac{3\mu}{R}$$

using $\beta_{CB}^2 = \beta_{AC}^2 + \beta_{AB}^2/\gamma_{AC}^2$ gives the same result.

Ted's photons at radius r have 4-momentum $h\nu_0((1-2\mu/r)^{-1}, -1, 0, 0)$. Using the general result $\omega_A = v_A \cdot k_A$ we get

$$\frac{\nu_A}{\nu_0} = (1-2\mu/R)^{-1/2}\ ; \quad \frac{\nu_B}{\nu_0} = (1-3\mu/R)^{-1/2}\ ; \quad \frac{\nu_C}{\nu_0} = \frac{1-\sqrt{2\mu/R}}{(1-2\mu/R)} = \frac{1}{1+\sqrt{2\mu/R}}\ .$$

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