Part-II Physics/Astrophysics, Michaelmas Term 2021 **Anastasia Fialkov**

Relativity: Example Sheet 3

1. A particle of rest mass m with speed u collides elastically with a stationary particle of equal mass. If, after the collision, the two particles travel in directions making angles θ and ϕ, respectively, with the incident particle's original direction, show that

$$\tan\theta\,\tan\phi = \frac{2}{\gamma_u + 1}\,.$$

Show that this result tends to the correct Newtonian limit as $u \to 0$. (*Hint: you may find it useful to transform to and from the zero-momentum frame, but take care in determining the relative velocity of this frame.*)

2. A mirror moves in the x-direction perpendicular to its plane with speed $-v$. A photon travelling in the (x, y)-plane hits the mirror with incident angle θ relative to the normal of the mirror plane. By considering the 4-momentum of the photon before and after the impact, find the angle relative to the normal at which the photon is reflected and the change in the photon's frequency.

3. Show that it is impossible for an isolated free electron to absorb or emit a single photon. Show also that it is impossible for an isolated free massive particle moving with any speed u to decay into a single photon.

4. Two protons of mass m_p are collided to produce a π-meson of mass m_π via the reaction $p + p \to p + p + \pi^0$. Derive an expression for the minimum required total kinetic energy of the incident protons if (a) the protons are travelling in opposite directions with equal speeds; and (b) one of the protons is stationary.

5. In a Cartesian inertial coordinate system in Minkowski spacetime the field equations of electromagnetism can be written

$$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu\,,$$
$$\partial_\sigma F_{\mu\nu} + \partial_\nu F_{\sigma\mu} + \partial_\mu F_{\nu\sigma} = 0\,.$$

(a) Show that the second field equation can be written as $\partial_{[\sigma}F_{\mu\nu]} = 0$. (b) Show that the above equations are equivalent to the standard form of Maxwell's equations. (c) Two Cartesian inertial frames S and S' are in standard configuration. Using $F^{\mu\nu}$, calculate how the components of the electric and magnetic fields in the two frames are related. (d) Show that $c^2|\vec{B}|^2 - |\vec{E}|^2$ is Lorentz-invariant. How is this invariant related to $F_{\mu\nu}$?

6. A satellite is in circular polar orbit of radius r around the Earth (radius R, mass M). A standard clock C on the satellite records the proper time taken to perform each orbit, $\Delta\tau_C$. An identical clock C_0 at rest at the North Pole on Earth records the time, $\Delta\tau_{C_0}$, between successive observations of the satellite being overhead. Show that, in the Newtonian limit of a weak gravitational field and slow-moving objects, the ratio of times is approximately

$$\frac{\Delta\tau_C}{\Delta\tau_{C_0}} \approx 1 - \frac{3GM}{2rc^2} + \frac{GM}{Rc^2} \,.$$

7. Show that the line element $ds^2 = y^2\, dx^2 + x^2\, dy^2$ represents the Euclidean plane, but the line element $ds^2 = y\, dx^2 + x\, dy^2$ represents a curved 2D manifold.

8. (a) Show that the only independent component of the curvature tensor on the unit 2-sphere is $R_{\theta\phi\theta\phi} = \sin^2\theta$, where θ and ϕ are spherical polar coordinates. (*Hint: use the connection coefficients that you derived in Question 5 on Example Sheet 2.*) (b) Consider the two affinely-parameterised geodesics $\theta(u) = \pi u$, $\phi(u) = 0$ and $\bar{\theta}(u) = \pi u$, $\bar{\phi}(u) = \delta$, where δ is infinitesimal and $0 \leq u \leq 1$. Verify directly that the connecting vector $\xi^a(u) = \bar{x}^a(u) - x^a(u)$, which has components $\xi^\theta(u) = 0$ and $\xi^\phi(u) = \delta$, satisfies the equation of geodesic deviation

$$\frac{D}{Du}\left(\frac{D\xi^a}{Du}\right) = R_{dbc}{}^{a}\frac{dx^b}{du}\frac{dx^c}{du}\xi^d \,. \qquad (*)$$

9. (a) In Newtonian gravity, consider two nearby particles with trajectories $x^i(t)$ and $\bar{x}^i(t)$ $(i = 1, 2, 3)$, respectively, in Cartesian coordinates. Show that the components of the separation vector $\zeta^i(t) = \bar{x}^i(t) - x^i(t)$ evolve as

$$\frac{d^2\zeta^i}{dt^2} = -\left(\frac{\partial^2\Phi}{\partial x^i \partial x^j}\right)\zeta^j ,$$

where Φ is the Newtonian gravitational potential.

(b) In curved spacetime, two particles travel on neighbouring timelike geodesics $x^\mu(\tau)$ and $\bar{x}^\mu(\tau)$, where τ is proper time, with separation vector $\xi^\mu(\tau) = \bar{x}^\mu(\tau) - x^\mu(\tau)$. Let the particle with worldline $x^\mu(\tau)$ parallel transport four orthonormal vectors $\{\hat{\boldsymbol{e}}_\alpha(\tau)\}$ $(\alpha = 0, 1, 2, 3)$, with $c\hat{\boldsymbol{e}}_0(\tau)$ equal to the 4-velocity of the particle. By writing the connecting vector as $\boldsymbol{\xi} = \xi^{\hat{\alpha}}\hat{\boldsymbol{e}}_\alpha$, show that the equation of geodesic deviation can be written as

$$(\hat{e}_\alpha)^\mu \frac{d^2\xi^{\hat{\alpha}}}{d\tau^2} = c^2 R_{\nu\alpha\beta}{}^{\mu}\,(\hat{e}_0)^\alpha\,(\hat{e}_0)^\beta\,\xi^{\hat{\rho}}\,(\hat{e}_\rho)^\nu \,, \qquad (\dagger)$$

where $(\hat{e}_\alpha)^\mu$ are the coordinate components of $\hat{\boldsymbol{e}}_\alpha$.

(c) In the weak-field Newtonian limit of general relativity, we may choose coordinates such that $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $|h_{\mu\nu}| \ll 1$, and we assume that all particle speeds are small compared with c. If the $h_{\mu\nu}$ are time independent, show that the spatial components of $(\dagger)$, i.e., $\mu = 1, 2, 3$, reduce to the Newtonian result in (a). (*Hint: you only require the components of the* $\hat{\boldsymbol{e}}_\alpha$ *to zero order in* $h_{\mu\nu}$ *and the particle speed; you may assume that at this order, the components* $(\hat{e}_\alpha)^\mu = \delta^\mu_\alpha$.)

10. (*Optional: for enthusiasts.*) The construction of Fermi-normal coordinates is discussed in Handout VII. A free-falling observer parallel-transports orthogonal vectors $\{\hat{\boldsymbol{e}}_\alpha(\tau)\}$ along their wordline $\mathcal{C}$, where τ is proper time. The vector $\hat{\boldsymbol{e}}_0(\tau)$ is chosen to be along the observer's 4-velocity. At each τ, a family of spacelike geodesics are constructed with initial tangent vectors on $\mathcal{C}$ of the form $\boldsymbol{t} = \sum_i \hat{n}^i \hat{\boldsymbol{e}}_i(\tau)$, where $\sum_i (\hat{n}^i)^2 = 1$. Coordinates $x^\mu = (c\tau, s\hat{n}^i)$ are assigned to events that lie on this family of geodesics, where s is distance along the geodesics from $\mathcal{C}$. (a) Explain why, along $\mathcal{C}$, the coordinate components of the $\{\hat{\boldsymbol{e}}_\alpha\}$ are $(\hat{e}_\alpha)^\mu = \delta^\mu_\alpha$, and hence that the metric takes the Minkowski form along $\mathcal{C}$.

(b) Noting that the $\hat{\boldsymbol{e}}_\alpha(\tau)$ are parallel transported along $\mathcal{C}$, show that $\Gamma^\mu_{0\nu} = 0$ along $\mathcal{C}$.

(c) By applying the geodesic equation to the spacelike geodesic $x^\mu = (c\tau, s\hat{n}^i)$ with constant τ and $\hat{n}^i$, show that $\Gamma^\mu_{ij} = 0$ on $\mathcal{C}$. Hence conclude that $\Gamma^\mu_{\nu\rho} = 0$ all along $\mathcal{C}$.