Relativity: Example Sheet 4

1. Suppose that a manifold possesses a symmetry such that under an infinitesimal coordinate transformation $x'^a = x^a + \xi^a(x)$, the metric components g'_{ab} have the same functional dependence on the x'^a coordinates as the original components g_{ab} do on the x^a coordinates. Show that the vector field ξ^a satisfies *Killing's equation*

$$\nabla_a \xi_b + \nabla_b \xi_a = 0 \,. \qquad (*)$$

Such vector fields are called *Killing vectors.* If the spacetime metric is independent of the x^0 coordinate, show that the vector field $\boldsymbol{e}_0 \equiv \partial/\partial x^0$ is a Killing vector. If $\boldsymbol{t}$ is the tangent vector to an affinely-parameterised geodesic, show directly from $(*)$ that $\xi_a t^a$ is constant along the geodesic.

2. (a) The Schwarzschild line-element is

$$ds^2 = c^2 \left(1 - \frac{2\mu}{r}\right) dt^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2 \,.$$

By considering the 'Lagrangian' $L = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$, where overdots denote differentiation with respect to an affine parameter λ, calculate the connection coefficients $\Gamma^\mu_{\nu\sigma}$.

(b) Repeat Question 6 on Example Sheet 3 using the full Schwarzschild metric (assuming the Earth is not rotating) to show that

$$\frac{\Delta\tau_C}{\Delta\tau_{C_0}} = \left(1 - \frac{3\mu}{r}\right)^{1/2} \left(1 - \frac{2\mu}{R}\right)^{-1/2} .$$

Verify that this reduces to the result given there in the weak-field limit.

3. Alice and Bob are astronauts holding on to the outside of a spaceship at rest at coordinate radius $r = R$ in Schwarzschild spacetime. Bob lets go of the spaceship and free-falls radially. When he reaches the coordinate radius $r = r_e$, he emits a photon radially outwards. What is the redshift z of the photon when received by Alice? Show that $z \to \infty$ as $r_e \to 2GM/c^2$.

4. All massive objects look larger than they really are. Show that a light ray grazing the surface of a massive sphere of coordinate radius $r > 3GM/c^2$ will arrive at infinity with impact parameter

$$b = r \left(1 - \frac{2GM}{c^2 r}\right)^{-1/2} .$$

Hence show that the apparent diameter of the Sun ($M_\odot = 2 \times 10^{30}\,\mathrm{kg}$, $R_\odot = 7 \times 10^8\,\mathrm{m}$) exceeds the coordinate diameter by nearly 3 km.

5. Show that once an infalling observer crosses the radius $r = 2\mu$ in the Schwarzschild metric, they will reach the origin in a proper time $\tau \leq \pi\mu/c$ no matter what they do to try and avoid it.

6. Show that, for an empty universe with vanishing cosmological constant, a solution of the cosmological field equations is

$$ds^2 = c^2 dt^2 - c^2 t^2 \left[d\chi^2 + \sinh^2\chi \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right] .$$

What is the geometry of the spatial hypersurfaces? Show that this metric describes Minkowski spacetime. What is the geometry of the spatial hypersurfaces in this case? Reconcile your answers. (*Hint: you may wish to consider the family of wordlines followed in Minkowski spacetime by free massive particles, which are emitted from a common spacetime origin in all directions and with all speeds* $v = c \tanh \psi$ *in the range* $0 \leq v < c$. *Use as coordinate labels the proper time along the wordlines, the rapidity* ψ, *and the angular coordinates* θ *and* ϕ.)

7. At cosmic time t_1, a massive particle is shot out into an expanding FRW universe with velocity v_1 relative to comoving cosmological observers. At a later cosmic time t_2 the particle has a velocity v_2 with respect to comoving cosmological observers. Show that

$$\frac{\gamma_{v_2} v_2}{\gamma_{v_1} v_1} = \frac{a(t_1)}{a(t_2)} ,$$

where $\gamma_v = (1 - v^2/c^2)^{-1/2}$ and $a(t)$ is the scale factor at cosmic time t. By considering the particle momentum, show that as $v_1 \to c$ the photon redshift formula is recovered.

8. Show that for a physically reasonable perfect fluid (i.e., density $\rho > 0$ and pressure $p \geq 0$) there is no static, isotropic and homogeneous solution to Einstein's equations with $\Lambda = 0$. Show that it is possible to obtain a static, pressureless solution if $\Lambda > 0$, but that this solution is unstable.