Astrophysical Fluid Dynamics Paper 1 (i) 2014

(a) • Sound waves are generated when there are low amplitude disturbance that propagate in a fluid (linear regime)

2 • Shocks occur when the disturbances generated in the fluid (eg. compression, acceleration) lead to velocities larger than local sound speed

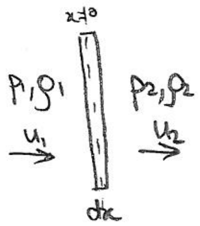
(b) • Structure formation shocks that heat IGM to high temperatures ensuring gas is virialised within dark matter halos

2 • Supernovae blast waves that heat surrounding ISM, regulating its thermodynamics and the efficiency of star formation

• Collisions of molecular clouds that impact ISM structure

• Bow shocks ahead of infalling satellite galaxies

(c)



• Continuity: $\partial\rho/\partial t + \partial(\rho u_x)/\partial x = 0$

2 Integrate over dx: $\partial/\partial t \int \rho\, dx + \rho u_x|_{dx/2} - \rho u_x|_{-dx/2} = 0$

Mass flux in = that out, so $\underline{\rho_1 u_1 = \rho_2 u_2}$

• Momentum: $\partial(\rho u_x)/\partial t = -\partial/\partial x(\rho u_x^2 + p) - \rho\, \partial\Phi/\partial x$

2 Integrate over dx: $\underbrace{\partial/\partial t \cancel{\int \rho u_x\, dx}}_{\to 0 \text{ (continuity)}} = -(\rho u_x^2 + p)_{dx/2} + (\rho u_x^2 + p)_{-dx/2} - \rho\, \cancel{d\Phi}^{\,0}$ (Φ continuous)

$\therefore \underline{\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2}$

(d) • Energy: $\partial E/\partial t + \partial/\partial x((E+p)u_x) = -\rho\cancel{\dot{Q}}^{\,0}$ as adiabatic

Integrate over dx noting Energy doesn't accumulate

$\therefore (E_1 + p_1)u_1 = (E_2 + p_2)u_2$

2 As $E = \rho(\frac{1}{2}u^2 + \varepsilon + \Phi)$ and Φ is continuous

$\therefore \underline{\frac{1}{2}u_1^2 + \varepsilon_1 + p_1/\rho_1 = \frac{1}{2}u_2^2 + \varepsilon_2 + p_2/\rho_2}$

Astrophysical Fluid Dynamics Paper 1 (ii) 2014

(a) • For an isothermal shock the first two RH relations are unchanged

$$\rho_1 u_1 = \rho_2 u_2 \qquad \text{RH1}$$

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2 \qquad \text{RH2}$$

However the gas loses heat, so get third relation from

4 $$T_2 = T_1 \Rightarrow c_{s2} = c_{s1} = c_s = (p/\rho)^{1/2}$$

Put into RH2: $\rho_1(u_1^2 + c_s^2) = \rho_2(u_2^2 + c_s^2)$

Combine with RH1: $(u_2 - u_1)(c_s^2 - u_1 u_2) = 0 \rightarrow c_s^2 = u_1 u_2$

$$\therefore \underline{\rho_2/\rho_1} = u_1/u_2 = (u_1/c_s)^2 \underline{= M_1^2}$$

So for a strong shock, ρ_2/ρ_1 can be very large

(b) • For an adiabatic shock, $\varepsilon = \left(\frac{1}{\gamma-1}\right)p/\rho$

So RH3 $\rightarrow \frac{1}{2}u_1^2 + \left(\frac{\gamma}{\gamma-1}\right)p_1/\rho_1 = \frac{1}{2}u_2^2 + \left(\frac{\gamma}{\gamma-1}\right)p_2/\rho_2$

Let $j = \rho_1 u_1 = \rho_2 u_2$

RH2 $\rightarrow p_1 + j^2/\rho_1 = p_2 + j^2/\rho_2$ and so $j^2 = (p_2 - p_1)\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)^{-1}$

RH3 $\rightarrow \frac{1}{2}j^2/\rho_1^2 + \left(\frac{\gamma}{\gamma-1}\right)p_1/\rho_1 = \frac{1}{2}j^2/\rho_2^2 + \left(\frac{\gamma}{\gamma-1}\right)p_2/\rho_2$

$$\therefore \frac{1}{2}(p_2 - p_1)\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)^{-1}\left(\frac{1}{\rho_1^2} - \frac{1}{\rho_2^2}\right) = \left(\frac{\gamma}{\gamma-1}\right)\left(\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1}\right)$$

6 $$\therefore \frac{1}{\rho_2}\left[p_2\left(\frac{1}{2} - \frac{\gamma}{\gamma-1}\right) - \frac{1}{2}p_1\right] = \frac{1}{\rho_1}\left[p_1\left(\frac{1}{2} - \frac{\gamma}{\gamma-1}\right) - \frac{1}{2}p_2\right]$$

$$\therefore \rho_2/\rho_1 = \frac{(\gamma+1)p_2 + (\gamma-1)p_1}{(\gamma+1)p_1 + (\gamma-1)p_2}$$

If $p_2 \gg p_1$, $\underline{\rho_2/\rho_1 \simeq (\gamma+1)/(\gamma-1)}$

So density contrast is limited, because as $M_1 \uparrow$ thermal pressure behind the shock increases preventing it from being compressed too much, unless it can radiate away its internal energy as in the isothermal case

(c) • RH2 $\Rightarrow \rho_2 u_2^2(\rho_2/\rho_1 - 1) = p_2 - p_1$

$$\therefore \rho_2 u_2^2/p_2 = (1 - p_1/p_2)(\rho_2/\rho_1 - 1)^{-1} \quad = \text{ratio of ram pressure to thermal pressure}$$

For adiabatic strong shock $\rho_2/\rho_1 \simeq (\gamma+1)/(\gamma-1)$

$$\therefore \underline{\rho_2 u_2^2/p_2} = \frac{1}{2}(\gamma-1)(1 - p_1/p_2) \underline{\leqslant \frac{1}{2}(\gamma-1)}$$

For isothermal strong shock $\rho_2/\rho_1 = M_1^2 \gg 1$

5 As $p \propto \rho$, $p_1/p_2 = \rho_1/\rho_2$

$$\therefore \underline{\rho_2 u_2^2/p_2} = (1 - \rho_1/\rho_2)(\rho_2/\rho_1 - 1)^{-1}$$

$$= (1 - M_1^{-2})(M_1^2 - 1)^{-1}$$

$$\simeq M_1^{-2} \;\underline{\ll 1}$$

Since $\rho_2 u_2^2/p_2 \leqslant \frac{1}{2}(\gamma-1)$ for both adiabatic and isothermal cases, this is also valid for gas that gradually cools

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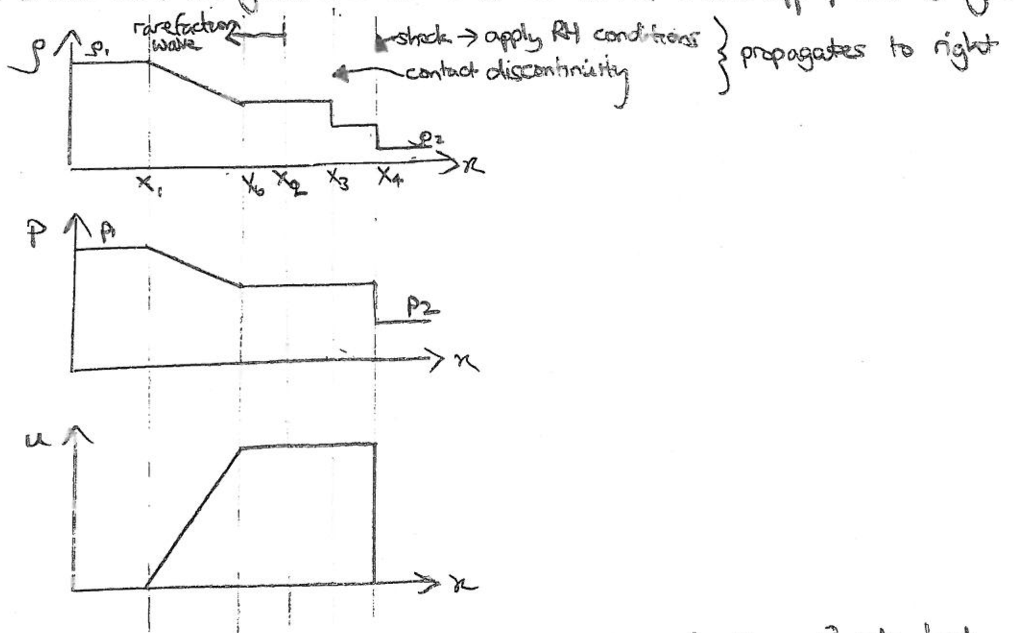
(d) This is the standard Sod shock tube problem.

A shock will be generated at $t=0$ because of the sharp pressure gradient.



5

In the limit $p_1 = p_2$ there is no pressure gradient to form shock, but there will be diffusion.

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1 ⓐ • The forces acting on the gas are gravity and pressure

ⓑ • Isothermal $\rightarrow T = \text{const}$

$$\therefore\ p = \left(\frac{R_*}{\mu}T\right)\rho$$

Consider atmosphere as a plane s.t. $\underline{g} = -g\,\hat{\underline{z}}$

Geometry $\Rightarrow \nabla = \partial/\partial z$ and $\rho = \rho(z)$

4 Hydrostatic equilibrium $\Rightarrow \frac{1}{\rho}\nabla p = -\nabla\Psi$

$$\therefore\ \frac{1}{\rho}\left(\frac{R_*}{\mu}T\right)d\rho/dz = -g$$

$$\therefore\ \underline{\rho = \rho_0 e^{-\left(\frac{\mu g}{R_* T}\right)z}}$$

ⓒ • The equation breaks down at heights above which either the assumption that gravity is constant is no longer valid

3 (ie. $g(z) \not\approx g(z=0)$), or where gas density is so low that it is no longer collisional, or where isothermal assumption is invalid

• Let $h = \mu g / R_* T$ s.t. $\rho = \rho_0 e^{-z/h}$

Mean free path $\lambda = \frac{1}{n\sigma} = \frac{\mu}{\rho\sigma} = \left(\frac{\mu}{\rho_0\sigma}\right)e^{z/h}$

2 $\therefore$ collisional assumption broken when $\lambda > h$

$$\therefore\ \underline{z > h \ln\left(\rho_0 \sigma h/\mu\right)}$$

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(a) • Ideal ⇒ $p = \frac{R_*}{\mu}\rho T$

Hydrostatic eqn ⇒ $\frac{1}{\rho}\nabla p = -\nabla\Psi$

$\therefore \frac{1}{\rho}\nabla\left(\frac{R_*}{\mu}\rho T\right) = -\nabla\Psi$

Spherical symmetry ⇒ $d\Psi/dr = GM(r)/r^2$ where $M(r)$ = total enclosed mass

6 $\therefore GM(r)/r^2 = -\frac{1}{\rho}\nabla\left(\frac{R_*}{\mu}\rho T\right)$

$= -\frac{1}{\rho}\frac{R_*}{\mu}\left[\rho\nabla T + T\nabla\rho\right]$

$\underline{M(r) = -\left(\frac{R_*T}{\mu}\right)\left(\frac{r}{G}\right)\left[d\ln\rho/d\ln r + d\ln T/d\ln r\right]}$

(b) • T = const → $M(r) = -\left(\frac{R_*T}{\mu}\right)\left(\frac{r^2}{G}\right)\frac{d\ln\rho}{dr}$, so to get $\rho(r)$, need $M(r)$

Neglect gas self gravity → $\underline{M(r)} = \int_0^r 4\pi r^2\rho_{tot}\,dr \simeq \int_0^r 4\pi r^2\rho_{DM}\,dr$

$= \int_0^r 4\pi r^2\delta_c\rho_{crit}\,dr \Big/ \left(\frac{r}{r_s}\right)\left(1+\frac{r}{r_s}\right)^2$

$= C\int_0^{r/r_s} x(1+x)^{-2}\,dx$

5 where $x = r/r_s$, $dx = dr/r_s$, $\underline{C = 4\pi\delta_c\rho_{crit}r_s^3}$

$= C\left[\left[-\frac{x}{1+x}\right]_0^{r/r_s} + \int_0^{r/r_s}(1+x)^{-1}\,dx\right]$

$\underline{= C\left[-\frac{r/r_s}{1+r/r_s} + \ln(1+r/r_s)\right]}$

• $\therefore \int_{\rho_0}^{\rho} d\ln\rho = -\int_0^r \frac{G\mu}{R_*T}Cr^{-2}\left[-\frac{r/r_s}{1+r/r_s} + \ln(1+r/r_s)\right]dr$

$\therefore \ln\rho/\rho_0 = D\left[\int_0^{r/r_s}[x(1+x)]^{-1} - x^{-2}\ln(1+x)\,dx\right]$

where $\underline{D = 4\pi\delta_c\rho_{crit}r_s^2G\mu/R_*T}$

5 As $\int_0^{r/r_s}x^{-2}\ln(1+x)\,dx = \left[-x^{-1}\ln(1+x)\right]_0^{r/r_s} + \int_0^{r/r_s}x^{-1}(1+x)^{-1}\,dx$

$\therefore \ln\rho/\rho_0 = D\left[(r_s/r)\ln(1+r/r_s) - 1\right]$ as $\lim_{x\to0}\frac{\ln(1+x)}{x} = \lim_{x\to0}\frac{1}{1+x} = 1$

$\therefore \underline{\rho(r) = \rho_0e^{-D}(1+r/r_s)^{Dr_s/r}}$

(c) • As $r\to0$, $\rho_{DM}\propto r^{-1}$ → central cusp

3 $\ln(\rho/\rho_0) \to D\left[\frac{1}{x}(x - x^2/2 + O(x^3)) - 1\right] \simeq -x/2 \to 0$

→ core of radius that depends on D and r_s

• For a given T, gas can only be compressed to certain densities as it is

1 collisional giving it pressure

Astrophysical Fluid Dynamics Paper 3 (i) 2014

(a) Continuity: $\cancel{\partial\rho/\partial t}^{\text{steady}} + \nabla\cdot(\rho\underline{u}) = 0$

Spherical symmetry $\frac{1}{r^2}\frac{d}{dr}(r^2\rho u) = 0$

$\therefore 4\pi r^2 \rho u = \dot{M} = \text{const.}$

Momentum: $\rho\cancel{\partial \underline{u}/\partial t} + \rho\underline{u}\cdot\nabla\underline{u} = -\nabla p - \nabla\Phi$

$\therefore u\,du/dr = -\frac{1}{\rho}dp/dr - GM/r^2$

6 $\therefore u^2\, d\ln u/dr = -c_s^2\, d\ln\rho/dr - GM/r^2$

but from continuity $d\ln\rho/dr + d\ln u/dr + 2/r = 0$

$\therefore (u^2 - c_s^2)\, d\ln u/dr = 2c_s^2/r - GM/r^2$

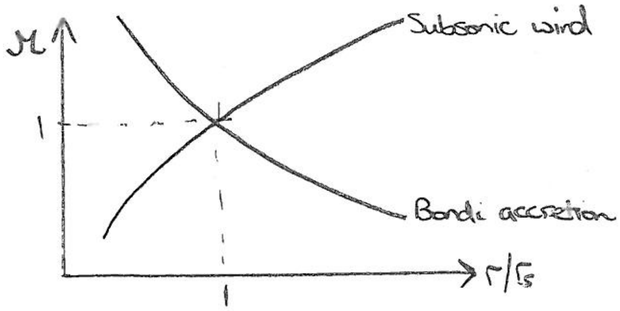
For $\mathcal{M} = u/c_s$, $d\ln u/dr = \frac{1}{\mathcal{M}}\, d\mathcal{M}/dr$

$\therefore (\mathcal{M}^2 - 1)\frac{1}{\mathcal{M}}\, d\mathcal{M}/dr = 2/r - GM/c_s^2 r$

$\therefore \underline{\left(\frac{1}{\mathcal{M}} - \mathcal{M}\right) d\mathcal{M}/dr = GM/c_s^2 r - 2/r}$

(b) • $\mathcal{M}^2 - \ln\mathcal{M}^2 = 4\ln r/r_s + 4\, r_s/r + C$ where $r_s = GM/2c_s^2$

2



• Accretion: flow starts at low velocity at $r = \infty$, but becomes supersonic for $r < r_s$

1 wind: subsonic flow is accelerated to high velocities for $r > r_s$

(c) • These solutions have $M = 1$ at $r = r_s$

1 $\therefore 1 = 4 + C$

$\therefore \underline{C = -3}$

Astrophysical Fluid Dynamics Paper 3 (ii) 2014

(a) • Continuity $\Rightarrow \dot{M} = 4\pi r^2 \rho \mathcal{M} c_s$

where $c_s = \sqrt{\frac{R_*}{\mu} T} = 1.2\times10^5$ m/s

4 Sonic radius $r_s = GM_\odot/2c_s^2 = 4.8\times10^9$ m $= 6.9 R_\odot$

Eq. (*) with $C=-3$ is $\mathcal{M}_\odot^2 - \ln\mathcal{M}_\odot^2 = 4\ln\frac{R_\odot}{r_s} + 4\frac{r_s}{R_\odot} - 3$ at Solar surface

$\therefore \mathcal{M}_\odot^2 - \ln\mathcal{M}_\odot^2 = 16.8$

Solve iteratively with calculator, given $\mathcal{M}_\odot < 1 \rightarrow \mathcal{M}_\odot \simeq 2.2\times10^{-4}$

$\therefore \dot{M} = 4\pi R_\odot^2 \mu n_p m_p \mathcal{M}_\odot c_s = \underline{1.6\times10^7 \text{ kg/s}}$

(b) • Steady $\Rightarrow r^2\rho\mathcal{M} = $ const

$\therefore n_{1AU}/n_\odot = (R_\odot/R_{1AU})^2(\mathcal{M}_\odot/\mathcal{M}_{1AU})$

3 But $\mathcal{M}_{1AU}^2 - \ln\mathcal{M}_{1AU}^2 = 4\ln\frac{R_{1AU}}{r_s} + 4\frac{r_s}{R_{1AU}} - 3 = 10.9$

$\therefore \mathcal{M}_{1AU} = 3.7$ (again noting that $\mathcal{M}_{1AU} > 1$)

$\therefore \underline{n_{1AU} = 0.13 \text{ cm}^{-3}}$

1 (c) • Binding energy per unit mass $E_b/M_c = \underline{\frac{1}{2}V^2 - GM_{BH}/r}$

1 (d) • At pericentre $\frac{1}{2}V_{peri}^2 - GM_{BH}/r_{peri} = +5.1\times10^{11}$ m²/s² < 0 so $\underline{\text{G2 is unbound}}$

(e) • Hydrostatic eqm, spherically symmetric

$\therefore dp_{hot}/dr = -\rho_{hot} GM_{BH}/r^2$

But $p_{hot} = \frac{R_*}{\mu}\rho_{hot}T_{hot}$ and $\rho_{hot} = \rho_0 r_0/r$

4 $\therefore \frac{R_*}{\mu}\rho_0 r_0\, d/dr\,(T_{hot}/r) = -\rho_0 r_0 GM_{BH} r^{-3}$

$\therefore T_{hot}(r)/r - T_0/r_0 = \frac{1}{2}GM_{BH}\frac{\mu}{R_*}[r^{-2} - r_0^{-2}]$

boundary condition $-T_0/r_0 = -\frac{1}{2}GM_{BH}\frac{\mu}{R_*}r_0^{-2}$

$\therefore \underline{T_{hot} = T_0 r_0/r}$ where $\underline{T_0 = \frac{1}{2}GM_{BH}\frac{\mu}{R_*}r_0^{-1}}$

(f) • $c_{s\,ad}^2 = \gamma\frac{R_*}{\mu}T_{hot} = \frac{1}{2}\gamma GM_{BH}/r$

2 $\therefore c_{s\,ad} = 2.2\times10^{13}\, r^{-1/2}$ m/s

At pericentre $c_{s\,ad} \simeq 3400$ km/s, so $\underline{\text{supersonic}}$

(g) • $p_c = \left(\frac{R_*}{\mu}\right)\rho_c T_c = p_{hot}$

As $T_c = $ const, $\underline{\rho_c} \propto p_{hot} \propto \rho_{hot}T_{hot} \propto \underline{r^{-2}}$

2

• $M_c \propto \rho_c R_c^3 = $ const

$\therefore \underline{R_c} \propto \rho_c^{-1/3} \propto \underline{r^{2/3}}$

(h) • Ablation

Evaporation due to thermal conduction

3 Ram pressure (important as $V/c_s > 1$)

Rayleigh-Taylor instability

Kelvin-Helmholtz instability

Tidal disruption by BH

Astrophysical Fluid Dynamics Paper 4 (i) 2014

(a) • Vertical component of momentum eqn in cylindrical polar coords

$$\frac{1}{\rho} dp/dz = \frac{d}{dz} GM/\sqrt{R^2+z^2}$$

$$\simeq -GMz/R^3 \quad \text{for a thin disc}$$

At $z=h$, $dp/dz \simeq -p/h$

3 $= -\rho c_s^2/h$

$\therefore -c_s^2/h \simeq -GMh/R^3$

$\therefore \underline{h \simeq c_s R\sqrt{R/GM}}$ $*$

(b) • At large R, $T_{eff}^4 \propto R^{-3}$

If $T \simeq T_{eff}$, $T \propto R^{-3/4}$

$\therefore c_s \propto T^{1/2} \propto R^{-3/8}$

2 $\therefore \underline{h/R \propto R^{1/8}}$

(c) • Mass in annulus is $\Sigma 2\pi R\Delta R$

2 Mass crossing inner edge per time $\dot{m} = -\Sigma 2\pi R\Delta R\, u_R/\Delta R$

$\underline{= -2\pi\Sigma R u_R}$

(d) • $\nu\Sigma = -\frac{2\cancel{\pi}\Sigma R u_R}{3\cancel{\pi}}\left[1-\sqrt{R_*/R}\right]$

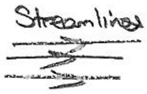
1 $\therefore \underline{u_R = -\frac{3}{2}\left(\frac{\nu}{R}\right)\left[1-\sqrt{R_*/R}\right]^{-1}}$

• At large R, $u_R \simeq -\frac{3}{2}\frac{\nu}{R} = -\frac{3}{2}\alpha c_s h/R$

2 $\therefore \underline{u_R/c_s \ll 1}$ as $h/R \ll 1$

• As $u_\phi \simeq \sqrt{GM/R}$, $\underline{u_\phi/c_s} \simeq R/h \underline{\gg 1}$ from $*$

Astrophysical Fluid Dynamics Paper 4 (ii) 2014

(a) • Linear shear flow 

$\uparrow \nabla \underline{v}$

3 Due to thermal motion there is a random motion component $\perp$ to streamlines $\sim \sqrt{\frac{kT}{m}}$
Thus momentum is transferred over a scale length $\sim$ mean free path
Viscosity $\uparrow$ with T $\uparrow$ as rate of momentum transfer increases with T

(b) • The stress tensor σ_{ij} = force per unit area in i direction acting on a surface with its normal in the j direction
It is $\sigma_{ij} = \rho u_i u_j + p\delta_{ij} + \sigma_{ij}'$

4 = momentum advected with fluid + force due to pressure differentials + viscous stress tensor due to differential motion of neighbouring fluid elements

• Continuity $\partial\rho/\partial t + \partial_j(\rho u_j) = 0$
Momentum $\partial(\rho u_i)/\partial t = -\partial_j\sigma_{ij} + \rho g_i$

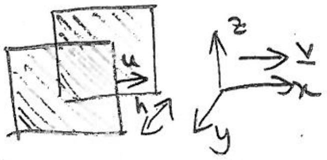
(c) • Momentum: $u_i\partial\rho/\partial t + \rho\partial u_i/\partial t = -\partial_j\rho u_i u_j - \partial_j p\delta_{ij} + \partial_j[\eta(\partial_j u_i + \partial_i u_j - \frac{2}{3}\delta_{ij}\partial_k u_k) + \zeta\delta_{ij}\partial_k u_k] + \rho g_i$

3 $\therefore -u_i\cancel{\partial_j(\rho u_j)} + \rho\partial u_i/\partial t = -u_i\cancel{\partial_j\rho u_j} - \rho u_j\partial_j u_i -$ "

$\therefore \underline{\rho(\partial u_i/\partial t + u_j\partial_j u_i) = -\partial_j p\delta_{ij} + \partial_j[\eta(\partial_j u_i + \partial_i u_j)] - \frac{2}{3}\partial_j(\eta\delta_{ij}\partial_k u_k) + \partial_j(\zeta\delta_{ij}\partial_k u_k) + \rho g_i}$

(d) • ζ = coefficient of bulk viscosity, diagonal elements of σ_{ij}', associated with momentum transfer due to bulk compression of flow, depends on ρ, T

3 η = shear viscosity coefficient, non diagonal elements, associated with momentum transfer in shear flows, depends on T

(e) • 

1

Symmetry $\Rightarrow$ fluid velocity $\underline{V}$ is only in x dir^n, ie. $V_y = V_z = 0$
Boundary conditions: $\underline{V_x(y=0) = 0}$ and $\underline{V_x(y=h) = u}$

(f) • Steady $\Rightarrow \partial/\partial t = 0$
Nongravitating $\Rightarrow g_i = 0$
Symmetry $\Rightarrow u_j\partial_j u_i = 0$ and $\partial_k u_k = 0$

4 Constant $\eta \Rightarrow$ $\hat{x}$ of NS: $\eta\, d^2V_x/dy^2 = 0$
$\hat{y}$ of NS: $dp/dy = 0$

$\therefore \underline{p = \text{const}}$
$\underline{V_x} = ay + b = \underline{uy/h}$

(g) • If $dp/dx \neq 0$ then $\hat{x}$ of NS is $\eta\,\partial^2 V_x/\partial y^2 = \partial p/\partial x$
Since LHS dep on y and RHS on x, both must be constant

$\therefore V_x = \frac{1}{2\eta}\left(\frac{dp}{dx}\right)y^2 + ay + b$

2 BC $\Rightarrow b = 0$, $u = \frac{1}{2\eta}\frac{dp}{dx}h^2 + ah$

$\therefore \underline{V_x = \frac{1}{2\eta}\frac{dp}{dx}y(y-h) + uy/h}$