UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Monday 30 May 2022 9.00 am to 11.00 am

PHYSICS (3)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (3)

ADVANCED QUANTUM PHYSICS

*Candidates offering this paper should attempt a total of* ***five*** *questions: all* ***three*** *questions from Section A and* ***two*** *questions from Section B.*

*The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains* ***six*** *sides, including this coversheet, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

| STATIONERY REQUIREMENTS | SPECIAL REQUIREMENTS |
| --- | --- |
| $2 \times 20$ Page Answer Book | Mathematical Formulae handbook |
| Rough workpad | Approved calculator allowed |
| Yellow master coversheet | |

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

*Attempt **all** questions in this Section. Answers should be concise and relevant formulae may be assumed without proof.*

1 Three distinguishable spin-1/2 particles have Hamiltonian

$$\hat{H} = \frac{\varepsilon_{\mathrm{t}}}{\hbar^2}\hat{\boldsymbol{S}}^2 + \frac{\varepsilon_z}{\hbar}\hat{S}^{(z)},$$

with $\varepsilon_{\mathrm{t}}$ and $\varepsilon_z$ constants of dimension energy. Here, $\hat{\boldsymbol{S}} = \hat{\boldsymbol{S}}_1 + \hat{\boldsymbol{S}}_2 + \hat{\boldsymbol{S}}_3$ is the total spin operator and $\hat{S}^{(z)} = \hat{S}_1^{(z)} + \hat{S}_2^{(z)} + \hat{S}_3^{(z)}$ is its $z$ component, with $\hat{\boldsymbol{S}}_i$ the spin-1/2 operator for particle $i$. Obtain the energy levels of $\hat{H}$ and their degeneracies. [4]

2 Consider a two-state system with density matrix

$$\hat{\rho} = C\frac{\hat{I} + \boldsymbol{r}\cdot\hat{\boldsymbol{\sigma}}}{2},$$

where $C$ is a constant, $\boldsymbol{r} = (r_x, r_y, r_z)$ is a three-dimensional vector and $\hat{\boldsymbol{\sigma}} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ is the vector of Pauli matrices. Find the values of $C$ and $\boldsymbol{r}$ such that $\langle\hat{\sigma}_x\rangle = 1/2$ and $\langle\hat{\sigma}_{j\neq x}\rangle = 0$. Show that the corresponding $\hat{\rho}$ is a mixed state. [4]

[*You may want to use that the Pauli matrices* $\hat{\sigma}_x = \left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$, $\hat{\sigma}_y = \left(\begin{smallmatrix}0 & -i\\ i & 0\end{smallmatrix}\right)$, $\hat{\sigma}_z = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ *satisfy* $\hat{\sigma}_j^2 = \hat{I}$ *and* $\hat{\sigma}_1\hat{\sigma}_2 = i\hat{\sigma}_3$, $\hat{\sigma}_2\hat{\sigma}_3 = i\hat{\sigma}_1$, $\hat{\sigma}_3\hat{\sigma}_1 = i\hat{\sigma}_2$.]

3 Sketch, as a function of magnetic field $B_z$, the lowest five energy levels for two indistinguishable spin-1/2 fermions in one dimension, in a harmonic potential, experiencing the magnetic field via $\hat{H}_{\mathrm{B}} = (2\mu_{\mathrm{B}}B_z/\hbar)(\hat{S}_1^{(z)} + \hat{S}_2^{(z)})$, and interacting via $\hat{H}_{\mathrm{I}} = \lambda\delta(\hat{x}_1 - \hat{x}_2)$. (Here, $\hat{x}_j$ and $\hat{S}_j^{(z)}$ are the position and $z$ component of the spin for the $j^{\mathrm{th}}$ fermion, respectively.) You may assume that $\hat{H}_{\mathrm{I}}$ and $\hat{H}_{\mathrm{B}}$ are small perturbations to the harmonic oscillator Hamiltonian. [4]

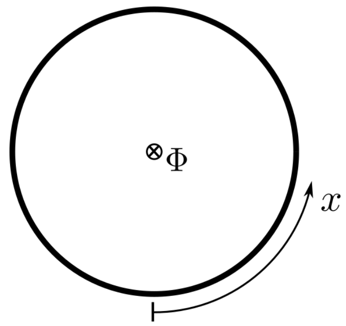
SECTION B

*Attempt two questions from this section*

4 An electron moving on a ring of circumference $L$ has Hamiltonian

$$\hat{H}_a = \frac{(\hat{p} - a)^2}{2m},$$

with constant $a$. Being on the ring imposes periodic boundary conditions for the wavefunction, $\psi(x) = \psi(x + L)$.




[*The coordinate $x$ is along the circumference of the ring. Also shown is the magnetic flux $\Phi$ referred to in part (a) below.*]

(a) Show that $\hat{H}_a$ applies when a magnetic flux $\Phi$ passes through the ring without magnetic field being present at the ring itself. Relate $a$, $L$, and the charge of an electron to $\Phi$. [3]

(b) Construct a gauge transformation that results in Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m}.$$

State the boundary conditions for the wavefunctions acted on by $\hat{H}$. [4]

(c) Sketch the lowest two energy levels $\varepsilon_0(a)$ and $\varepsilon_1(a)$ of $\hat{H}_a$ as a function of $a$. [2]

(d) Consider now the system in a potential $V(x) = -\lambda \cos(2\pi x/L)$. State the conditions on $\lambda$ for $\hat{V}(x)$ to be a small perturbation to $\hat{H}_a$. Considering such small $\lambda$, sketch the lowest two energy levels $\tilde{\varepsilon}_{0,1}(a)$ of $\hat{H}_a + \hat{V}(x)$ as a function of $a$ and calculate $\tilde{\varepsilon}_{0,1}(a = \pi\hbar/L)$ to first order in $\lambda$. [6]

(e) Consider now the opposite limit of large $\lambda$. Sketch the corresponding $\tilde{\varepsilon}_0(a)$ and $\tilde{\varepsilon}_1(a)$. Give an approximate expression for $\tilde{\varepsilon}_1(0) - \tilde{\varepsilon}_0(0)$ using a parabolic approximation of $V(x)$ near its minimum. [4]

5 An electron in a one-dimensional quantum well has Hamiltonian $\hat{H}_0 = \frac{\hat{p}^2}{2m} + \hat{V}(x)$ with potential

$$V(x) = \begin{cases} 0 & \text{for } -a < x < a, \\ \infty & \text{otherwise.} \end{cases}$$

(a) An electric field $\boldsymbol{E}$ of magnitude $E$ is applied to the system along the $x$ direction, resulting in a perturbation

$$V_1(x) = eEx,$$

where $-e$ is the charge of an electron. Qualitatively explain why the lowest-order corrections to the unperturbed energy levels are quadratic in $E$. [4]

(b) Obtain these lowest-order energy corrections $\delta\varepsilon_n^{(2)} \propto E^2$ to the $n^{\text{th}}$ energy level. You may leave the resulting expression in the form of an infinite sum. [4]

[*The following integral may be useful:*

$$\frac{1}{a}\int_{-a}^{a} dx\, x \sin\left[\frac{m\pi}{2a}(x+a)\right] \sin\left[\frac{n\pi}{2a}(x+a)\right] = \frac{8amn[(-1)^{m+n} - 1]}{\pi^2(m^2 - n^2)^2}.$$
]

(c) Qualitatively explain why the ground-state dipole moment $\boldsymbol{d}$ depends linearly on $\boldsymbol{E}$ to leading order in perturbation theory, and then use the corresponding perturbed wavefunction to relate $\boldsymbol{d}$ to the ground-state energy correction $\delta\varepsilon_0^{(2)}$. [5]

(d) Approximate the ground-state energy variationally using the ansatz

$$|\psi(c)\rangle \propto |\phi_0\rangle + c|\phi_1\rangle,$$

where $|\phi_0\rangle$ and $|\phi_1\rangle$ are the unperturbed ground and first excited state, respectively, and $c$ is a real variational parameter. In the variational calculation, you may work up to the lowest order in $E$, i.e., assume that $\hat{V}_1$ is a small perturbation. Compare the variational ground-state energy to the perturbative result and comment. [6]



(TURN OVER

6 Consider a spin-1/2 system (Hamiltonian $\hat{H}_1$) and a collection of oscillators (Hamiltonian $\hat{H}_2$), between which an interaction $\hat{V}$ is turned on at time $t = 0$. Here

$$\hat{H}_1 = \frac{\hbar\Omega}{2}\hat{\sigma}_z, \quad \hat{H}_2 = \sum_j \hbar\omega_j\left(\hat{a}_j^\dagger\hat{a}_j + \frac{1}{2}\right), \quad \hat{V} = \lambda\hat{\sigma}_x\sum_j(\hat{a}_j^\dagger + \hat{a}_j), \quad \Omega, \omega_j > 0,$$

and $\hat{\sigma}_{z,x}$ are Pauli matrices. At $t = 0$, the system is in an eigenstate $|\sigma, \{n\}\rangle \equiv |\sigma\rangle|\{n\}\rangle$ where $\hat{\sigma}_z|\sigma\rangle = s_\sigma|\sigma\rangle$ with $\sigma =\uparrow, \downarrow$ and $s_\uparrow = -s_\downarrow = 1$, and we denote $|\{n\}\rangle \equiv \prod_j |n_j\rangle$ where $\{n\}$ is the set of oscillator occupation numbers $n_j$ such that $\hat{a}_j^\dagger\hat{a}_j|\{n\}\rangle = n_j|\{n\}\rangle$.

(a) At time $t$, the state of the system in interaction picture ($I$) is $|\psi_\sigma^I(t)\rangle = c_{\uparrow\sigma}(t)|\phi_\uparrow(t)\rangle + c_{\downarrow\sigma}(t)|\phi_\downarrow(t)\rangle$, where $c_{\eta\sigma}$ are coefficients, and $|\phi_\eta(t)\rangle$ are normalised and satisfy $\hat{\sigma}_z|\phi_\eta(t)\rangle = s_\eta|\phi_\eta(t)\rangle$. Show that

$$\langle\psi_\sigma^I(t)|\hat{\sigma}_z^I(t)|\psi_\sigma^I(t)\rangle = |c_{\uparrow\sigma}(t)|^2 - |c_{\downarrow\sigma}(t)|^2.$$ [3]

(b) Express $c_{\eta\sigma}(t)|\phi_\eta(t)\rangle$ in terms of $|\sigma, \{n\}\rangle$, the scattering operator $\hat{S}(t)$, and a suitable projector. Hence show that $|c_{\eta\sigma}(t)|^2 = \sum_{\{n'\}} P_{\sigma,\{n\}\to\eta,\{n'\}}$, where the probability $P_{\sigma,\{n\}\to\eta,\{n'\}} = |\langle\eta, \{n'\}|\hat{S}(t)|\sigma, \{n\}\rangle|^2$. [5]

(c) Fermi's golden rule implies that, to leading order in $\hat{V}$, and for $\eta \neq \sigma$,

$$P_{\sigma,\{n\}\to\eta,\{n'\}} = \frac{2\pi t}{\hbar^2}\,|\langle\eta, \{n'\}|\hat{V}|\sigma, \{n\}\rangle|^2\,\delta\left(\omega_{\eta\{n'\}}^{(\mathrm{tot})} - \omega_{\sigma\{n\}}^{(\mathrm{tot})}\right),$$

where $\hbar\omega_{\sigma\{n\}}^{(\mathrm{tot})}$ is the energy of the unperturbed system in eigenstate $|\sigma, \{n\}\rangle$. Hence show that, for $\eta \neq \sigma$

$$|c_{\eta\sigma}(t)|^2 = \frac{2\pi\lambda^2 t}{\hbar^2} \times \begin{cases} g(\Omega)(n_\Omega + 1) & \text{if } |\sigma\rangle = |\uparrow\rangle, \\ g(\Omega)n_\Omega & \text{if } |\sigma\rangle = |\downarrow\rangle, \end{cases} \qquad (\star)$$

where $g$ is the oscillators' density of states and $n_\Omega$ is the occupation number at frequency $\Omega$ in the initial state $|\sigma, \{n\}\rangle$. [6]

$\left[\textit{Recall that} \quad \hat{a}_j^\dagger|n_j\rangle = \sqrt{n_j + 1}|n_j + 1\rangle, \qquad \hat{a}_j|n_j\rangle = \sqrt{n_j}|n_j - 1\rangle.\right]$

(d) What must $t$ satisfy for Equation ($\star$) to hold? State your answer in terms of the scale $\Delta\omega$ over which variations of $g$ around $\Omega$ are negligible and the validity of perturbation theory. [2]

(e) Equation ($\star$) also holds when the oscillators are in a thermal state with $n_\Omega = [\exp(\beta\hbar\Omega) - 1]^{-1}$, where $\beta = (k_B T)^{-1}$ is the inverse temperature. Consider an ensemble of spin-1/2 systems such that the state $|\sigma\rangle$ has probability $p_\sigma$. Show that for this to be in equilibrium, i.e., where spins flipping at rate $\Gamma_{\eta\sigma} = |c_{\eta\sigma}(t)|^2/t$ do not change $p_\sigma$, the probabilities must be thermal: $p_\uparrow = e^{-\beta\hbar\Omega}p_\downarrow$. [3]

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