AFD Solutions

1)

$$\nabla^2 \psi = 4\pi G\rho$$

For slab

$$\frac{d^2\psi}{dz^2} = 4\pi G\rho$$

$$\frac{d\psi}{dz} = 4\pi G\rho z + A$$

Force must be zero at $z=0 \implies A=0$

Hydrostatic equ^{n}:

$$\frac{1}{\rho}\frac{dp}{dz} = -\frac{d\psi}{dz}$$

$$\therefore \quad \frac{dp}{dz} = -4\pi G\rho^2 z$$

$$\implies \quad p = -2\pi G\rho^2 z^2 + B$$

B.C. $p=0$ at $z=a \implies B = 2\pi G\rho^2 a^2$

$$\implies \quad p = \frac{\rho k T}{\mu m_H} = 2\pi G\rho^2(a^2 - z^2)$$

$$\therefore \quad T = \frac{2\pi G\rho\mu m_H}{k}(a^2 - z^2)$$

atomic hydrogen $\mu = 1$.

2) Supernova blast wave

Energy	$\mathcal{E}$	$[M.L^2T^{-2}]$
Density	ρ	$[ML^{-3}]$
length	r	$[L]$
time	t	$[T]$

$\therefore$ $\dfrac{\mathcal{E}t^2}{\rho}$ has dimensions L^5

$$\therefore \quad \xi_0^{\,5} = \frac{R^5\rho_0}{\mathcal{E}t^2}$$

$$\text{or} \quad R = \xi_0 \left(\frac{\mathcal{E}t^2}{\rho_0}\right)^{1/5}$$

$\rho = 10^6 \times 1.6 \times 10^{-27}$ kg m^{-3}

$\simeq 1.6 \times 10^{-21}$ kg m^{-3}

take $\xi_0 \simeq 1$ $\quad t = 1000$ yrs $= 3 \times 10^{10}$ s

$$\Rightarrow \quad R \sim \left(\frac{10^{44} \times 9 \times 10^{20}}{1.6 \times 10^{-21}}\right)^{1/5}$$

$$\sim \left(5.6 \times 10^{85}\right)^{1/5} = 1.4 \times 10^{17}\,\text{m}$$

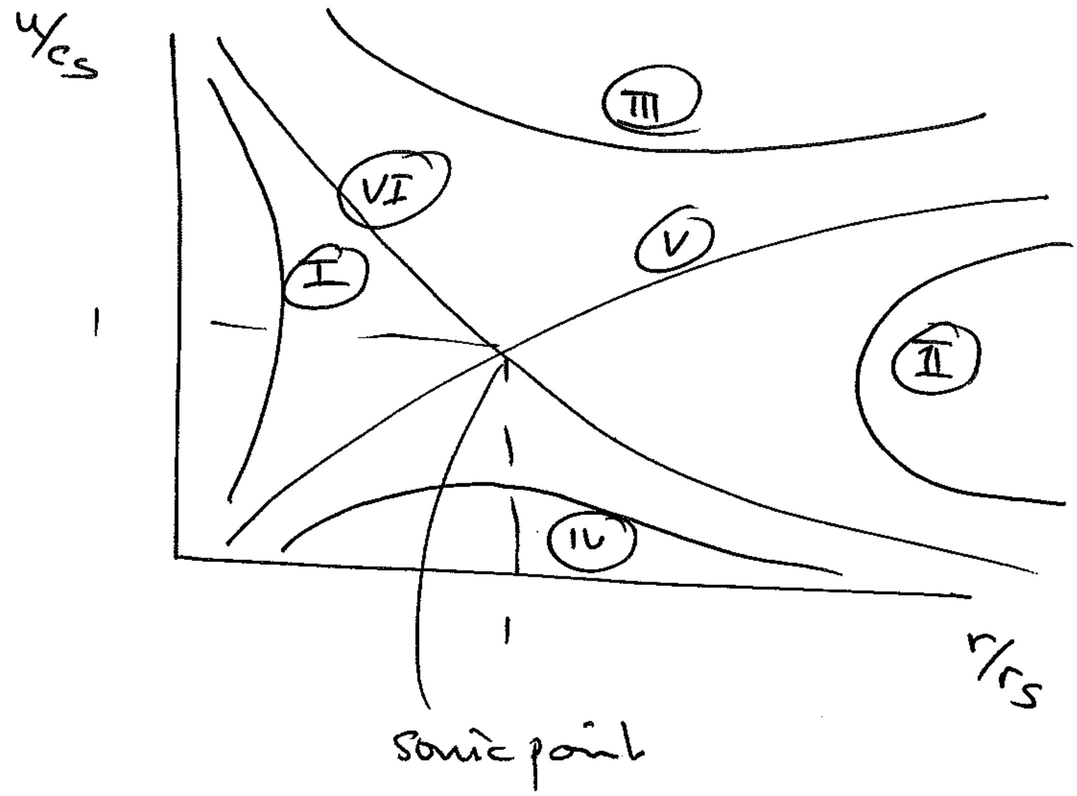
$$\simeq \underline{\underline{5\,\text{pc}}}$$

3]

Wind / accretion flow isothermal limit

$$\left(\frac{u}{c_s}\right)^2 - \ln\left(\frac{u}{c_s}\right)^2 = 4\ln\frac{r}{r_s} + \frac{4r_s}{r} + C$$

Guessing they will recall diagram



- I & III not physical - double values for given r.
- II supersonic & IV subsonic everywhere
- V & VI have sonic points V is wind initially subsonic accelerates to large r. VI is spherical (Bondi) accretion. C = 3
- If initially in I or II solution unstable. Evolve (with dissipation) to stable solution.

5] Q4

Hydrostatic eqn

$$\nabla p = -\rho \nabla \psi \qquad \nabla^2 \psi = 4\pi G \rho$$

Spherical symmetry

$$\frac{dp}{dr} = -\rho \frac{d\psi}{dr} \qquad (1)$$

Now $\rho > 0$ everywhere $\therefore$ p monotonic function of ψ and $\frac{dp}{d\psi} = -\rho$

$$\therefore \quad p = p(\psi) \quad \text{and} \quad \rho = \rho(\psi)$$

This implies $\underline{\underline{p = P(\rho)}}$ a barotrope [2]

Bookwork

Assume eqn of state of form $p = K\rho^{1+1/n}$

in (1) $$-\frac{d\psi}{dr} = \frac{1}{\rho}\frac{d}{dr}\left(K\rho^{1+1/n}\right)$$

let $y = \rho^{1/n}$ $\quad \rho^{1+1/n} = y^{n+1}$

$$\therefore \frac{d}{dr}\rho^{1+1/n} = \frac{d}{dr}y^{n+1} = y^n(n+1)\frac{dy}{dr}$$

$$\Rightarrow \quad -\frac{d\psi}{dr} = \frac{1}{\rho}.K.\rho(n+1)\frac{d\rho^{1/n}}{dr}$$

Integrating $$\int_0^\rho d(\rho^{1/n}) = -\int_{\psi_T}^{\psi} \frac{d\psi}{K(n+1)}$$

ψ_T is ψ at $\rho = 0$

$$\Rightarrow \rho^{1/n} = \frac{\psi_T - \psi}{K(n+1)}$$

[4], bookwork

Lane-Emden follow

$$\nabla^2 \psi = 4\pi G \rho = 4\pi G \left(\frac{\psi_T - \psi}{(n+1)K} \right)^n$$

Define ρ_c and ψ_c at centre when $r=0$

$$\rho_c = \left(\frac{\psi_T - \psi_c}{(n+1)K} \right)^n$$

$$\Rightarrow \rho/\rho_c = \left(\frac{\psi_T - \psi}{\psi_T - \psi_c} \right)^n .$$

Let $\theta = \dfrac{\psi_T - \psi}{\psi_T - \psi_c}$

$$\Rightarrow \psi = -(\psi_T - \psi_c)\,\theta + \psi_T$$

$$\therefore \quad \nabla^2 \theta = -\frac{4\pi G \rho_c}{\psi_T - \psi_c}\, \theta^n$$

$\dfrac{1}{r^2}\dfrac{d}{dr} r^2 \dfrac{d\theta}{dr}$ let $\xi = \alpha r$

mostly Bookwork + new form for α.

$$\therefore \quad \alpha^2 \frac{1}{\xi^2}\frac{d}{d\xi} \xi^2 \frac{d\theta}{d\xi} = -\frac{4\pi G \rho_c}{\psi_T - \psi_c}\,\theta^n$$

$$\therefore \quad \alpha^2 = \frac{4\pi G \rho_c}{\psi_T - \psi_c} \quad \text{but} \quad \rho_c^{1/n} = \frac{\psi_T - \psi_c}{(n+1)K}$$

$$\therefore \quad \alpha^2 = \frac{4\pi G \rho_c}{(n+1)K \rho_c^{1/n}} = \frac{4\pi G \rho_c^2}{(n+1) P_c}$$

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$n = 5$ $\qquad \theta^5 = \left(1 + \frac{\xi^2}{3}\right)^{-5/2}$

$$\frac{d\theta}{d\xi} = -\left(1+\frac{\xi^2}{3}\right)^{-3/2} \cdot \frac{2\xi}{3}$$

and

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = -\frac{1}{\xi^2}\frac{d}{d\xi}\left(\frac{1}{3}\xi^3\left(1+\frac{\xi^2}{3}\right)^{-3/2}\right)$$

$$= -\frac{1}{\cancel{\xi^2}}\left(\cancel{\xi^2}\left(1+\frac{\xi^2}{3}\right)^{-3/2} - \frac{\xi^{\cancel{4}\,2}}{3}\left(1+\frac{\xi^2}{3}\right)^{-5/2}\right)$$

$$= -\left(1+\frac{\xi^2}{3}\right)^{-5/2}$$

✓

2

unseen

Total Mass

$$M = \int_{r=0}^{\infty} 4\pi r^2 \rho\, dr = 4\pi\rho_c \alpha^{-3}\int_{\xi=0}^{\infty} \xi^2\theta^5\, d\xi$$

let $\frac{\xi^2}{3} = x^2$

$$\Rightarrow \quad M = 4\pi\rho_c\alpha^{-3} \cdot 3\sqrt{3}\ \underbrace{\int_0^{\infty} x^2(1+x^2)^{-5/2}\,dx}_{=1/3}$$

3

unseen

$$= 4\pi\rho_c \frac{\rho_c^{3/2}\, 3^{3/2} \cdot \sqrt{3}}{(2\pi G)^{3/2}\rho_c^{3}} = \frac{18}{\sqrt{2\pi}}\left(\frac{\rho_c}{G}\right)^{3/2} \cdot \frac{1}{\rho_c^2}$$

SU

Perfect gas law

$$P_c = \frac{\rho_c k T_c}{\mu m_H}$$

$$\therefore \; T_c \propto P_c / \rho_c$$

$$\xi = 1 \Rightarrow \alpha = 1/R \propto \frac{G^{1/2} \rho_c}{P_c^{1/2}}$$

now $$M \propto \frac{P_c^{3/2}}{G^{3/2}} \cdot \frac{1}{\rho_c^{2}}$$

$$\therefore \; \frac{P_c}{\rho_c} \propto T_c \propto \frac{G^{3/2} M \rho_c}{P_c^{1/2}} = \frac{GM}{R}$$

3

unseen.

Q5

Equilibrium fluid $\rho_0, p_0, \psi_0 \quad u_0 = 0$

$$\therefore \quad \frac{1}{\rho_0}\nabla p_0 - \nabla\psi_0 = 0$$

$$\nabla^2\psi_0 = 4\pi G\rho_0$$

Problem $\rho_0 \neq 0 \Rightarrow \nabla^2\psi_0 \neq 0$

$\therefore \quad \nabla\psi_0 \neq 0 \Rightarrow$ pressure gradients

Jeans swindle. $\longrightarrow$ ignore & proceed,

Bookwork. 2

Perturbation analysis $p = p_0 + \Delta p$, $\rho = \rho_0 + \Delta\rho$,
$\psi = \psi_0 + \Delta\psi \quad u = \Delta u$

Linearised
$$\frac{\partial\Delta\rho}{\partial t} + \underline{\nabla}\cdot(\rho_0\underline{\Delta u}) = 0$$

$$\frac{\partial\underline{\Delta u}}{\partial t} + \frac{1}{\rho_0}\underline{\nabla}\Delta p + \underbrace{\cancel{\frac{1}{\rho_0}\Delta p_0} - \cancel{\nabla\psi_0}}_{\text{from equilib}} - \underline{\nabla}\Delta\psi = 0$$

and
$$\nabla^2(\psi_0 + \Delta\psi) = 4\pi G(\rho_0 + \Delta\rho)$$

$$\Rightarrow \quad \nabla^2\Delta\psi = 4\pi G\Delta\rho$$

Constant
~~adiabatic~~ sound speed

$$\Delta p = c_s^2\Delta\rho$$

Collecting terms

$$\frac{\partial \Delta\rho}{\partial t} + \rho_0 \nabla \cdot \Delta\underline{u} = 0$$

$$\frac{\partial \Delta u}{\partial t} = -\frac{c_s^2}{\rho_0} \underline{\nabla} \Delta\rho - \nabla \Delta\psi$$

$$\nabla^2 \Delta\psi = 4\pi G \Delta\rho$$

Look for wavelike solutions $\Delta\rho = \rho_1 e^{i(k \cdot x - \omega t)}$ etc.

$\therefore$ ~~...~~

$$-i\omega\rho_1 + \rho_0 i \underline{k} \cdot \underline{u}_1 = 0$$

$$-\rho_0 \omega \underline{u}_1 = -c_s^2 \underline{k} \rho_1 - \rho_0 \underline{k} \psi_1$$

$$-k^2 \psi_1 = 4\pi G \rho_1$$

eliminate u_1

$$-\rho_0 \omega \underline{k} \cdot \underline{u}_1 = -c_s^2 k^2 \rho_1 - \rho_0 k^2 \psi_1$$

but $\rho_0 \underline{k} \cdot \underline{u}_1 = \omega \rho_1$

$$\therefore \quad -\omega^2 \rho_1 = -c_s^2 k^2 \rho_1 - 4\pi G \rho_0 \rho_1$$

or $$\omega^2 = c_s^2 \left(k^2 - \frac{4\pi G \rho_0}{c_s^2} \right)$$

$$\therefore \quad k_J^2 = \frac{4\pi G \rho_0}{c_s^2}$$

$$l_J = \frac{2\pi}{k_J} = 2\pi \frac{c_s}{(4\pi G\rho_0)^{1/2}}$$

$$\therefore \quad l_J = \frac{\pi^{1/2} c_s}{(G\rho_0)^{1/2}}$$

$$M_J \sim l_J^3 \rho_0 = \frac{\pi^{3/2} c_s^3}{G^{3/2} \rho_0^{1/2}}$$

Bookwork 9

Criterion for growing modes is $\omega^2 < 0$ ie $k < k_J$ and ~~largest~~ fastest growing mode is $k \to 0$.

Jeans argument. 2

Galaxy:

Problem $M \gg M_J$ galaxy size R_0

take $k \sim 2\pi / R_0$

$\therefore$ fastest growing mode is the whole galaxy. based on basic Jeans analysis.

Consider dispersion relation given

$$\omega^2 = c_s^2 k^2 - 2\pi G |k| \sigma_0$$

this has a minimum when

$$\frac{\partial}{\partial k} \omega^2 = 0$$

i.e. $2 c_s^2 |k| - 2\pi G \sigma_0 = 0$

$$\therefore \; k = \frac{\pi G \sigma_0}{c_s^2}$$

and $$\omega_m^2 = - \frac{\pi^2 G^2 \sigma_0^2}{c_s^2}$$

$\therefore$ there is a preferred scale of collapse on the scale of a galaxy of

$$l = \frac{2\cancel{\pi} c_s^2}{\cancel{\pi} G \sigma_0}$$

note

$$\frac{l_J^2}{l} = \frac{\pi c_s^2}{G \rho_0} \cdot \frac{G \sigma_0}{c_s^2} = \frac{\sigma_0}{\rho_0} = \frac{h \rho_0}{\rho_0}$$

disc ~~thick~~ ↓ (above h)

$$\therefore \; l = \frac{l_J^2}{h}.$$

$\therefore$ much smaller scales preferred.

Once collapse starts use stuck Jeans as much smaller than galaxy.

If initially ~~size~~ ~~is~~ collapse region is $\sim \ell$, then if # Jeans mass contained in N_0

then $N_0 = \left(\frac{\ell_J}{h}\right)^3$

As cloud collapses, say at constant T and uniform density

$$M_J = \frac{\pi^{3/2} c_s^3}{G^{3/2} \rho^{1/2}}$$

and $M = \text{constant} = r^3 \rho = \ell^3 \rho_0$

$$\therefore \frac{\rho}{\rho_0} = \left(\ell/r\right)^3$$

$$\therefore M_J \propto \left(r/\ell\right)^{3/2}$$

Unseen

and $N_J = N_0 \left(\frac{\ell}{r}\right)^{3/2}$

$\therefore$ number of Jeans mass increase

$\longrightarrow$ fragmentation into clouds of different sizes.

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