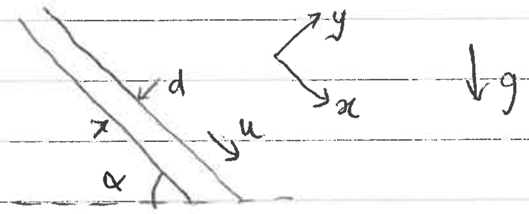
AFD 2013

1a) incompressible fluid, viscosity η, density ρ flows steadily under gravity down plane inclined at angle α to horizontal
depth of flow $= h$

$$u \cdot \nabla u = -\frac{1}{\rho}\nabla p + g + \frac{\eta}{\rho}\nabla^2 u$$

reduces to $0 = g + \frac{\eta}{\rho}\nabla^2 u$

incompressible $\Rightarrow \nabla \cdot u = 0$
flows under gravity - can neglect pressure term



find max speed
$$\nabla^2 u = -\frac{g\rho}{\eta}$$

component of g in y direction
$= g\sin\alpha$

$$\frac{\partial^2 u_x}{\partial y^2} = -\frac{g\rho\sin\alpha}{\eta}$$

BCs : $\frac{\partial u_x}{\partial y} = 0$ at $y = h$, $u_x = 0$ at $y = 0$

$$\frac{\partial u_x}{\partial y} = -\frac{g\rho}{\eta}\sin\alpha\, y + c$$

$c = \frac{g\rho}{\eta}h\sin\alpha \Rightarrow \frac{\partial u_x}{\partial y} = \frac{g\rho}{\eta}\sin\alpha(h-y)$

$u_x = \frac{g\rho}{\eta}\sin\alpha(hy - \frac{1}{2}y^2) + c'$, $c' = 0$

$$u_x = \frac{g\rho}{2\eta}\sin\alpha(2hy - y^2)$$

b) static slab of gas of total thickness $2a$, uniform density ρ
how long does it take a star falling from rest from height a above midplane to pass through midplane?

$$\int g \cdot dS = -4\pi G\rho \int \rho dV$$

$$g = -4\pi G \int \rho dz = -4\pi G \rho z$$

$$g = \ddot{z} = -4\pi G \rho z$$

$$z = A\sin\omega t + B\cos\omega t \quad , \omega = \sqrt{4\pi G\rho}$$

$$z(0) = a, \ \dot{z}(0) = 0 \Rightarrow A = 0, B = a$$

$$z = a\cos\omega t$$

1/4 cycle to reach midplane

$$t = T/4 = \frac{2\pi}{4\omega} = \frac{\pi}{2\sqrt{4\pi G\rho}} = \frac{1}{4}\sqrt{\frac{\pi}{G\rho}}$$

c) strong adiabatic shock

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{\gamma+1}{\gamma-1}$$

Show that $\frac{k_B T_2}{m} = \frac{2(\gamma-1)}{(\gamma+1)^2} u_1^2$

$$p_2 = \rho_1 u_1^2 - \rho_2 u_2^2$$

$$= \rho_2 \left(\frac{\gamma-1}{\gamma+1}\right) u_1^2 - \rho_2 u_2^2$$

$$= \rho_2 \left(\frac{\gamma-1}{\gamma+1}\right) u_2^2 \left(\frac{\gamma+1}{\gamma-1}\right)^2 - \rho_2 u_2^2$$

$$p_2 = \rho_2 \left(\frac{\gamma-1}{\gamma+1}\right) u_1^2 - \rho_2 u_1^2 \left(\frac{\gamma-1}{\gamma+1}\right)^2$$

$$T_2 = \cancel{\frac{R_*}{\mu}} \frac{\mu}{R_*} \frac{p_2}{\rho_2} = \frac{\mu}{R_*} u_1^2 \left(\frac{\gamma-1}{\gamma+1} - \left(\frac{\gamma-1}{\gamma+1}\right)^2\right)$$

$$T_2 = \frac{2}{\gamma+1} \frac{\gamma-1}{\gamma+1} \frac{\mu u_1^2}{R_*}$$

$$\frac{\mu}{R_*} = \frac{m}{k_B} \Rightarrow \frac{k_B T}{m} = \frac{2(\gamma-1)}{(\gamma+1)^2} u_1^2$$

$$c_s^2 = \frac{dp}{d\rho} \Rightarrow c_s|_{A_m} = (k\gamma\rho^{\gamma-1})^{1/2}|_{A_m}$$

$$\dot{m} = (k\gamma)^{1/2} \rho^{\frac{1}{2}(\gamma+1)}|_{A_m} A_m$$

at sonic transition $\rho^{1/2(\gamma+1)} = \frac{\dot{m}}{A_m(k\gamma)^{1/2}}$

$$\rho = \left[\frac{\dot{m}}{A_m(k\gamma)^{1/2}}\right]^{\frac{2}{\gamma+1}} = \left[\left(\frac{\dot{m}}{A_m}\right)^2 \frac{1}{k\gamma}\right]^{1/\gamma+1} \quad \text{at } A = A_m$$

Show that $\frac{1}{2}u^2 + \frac{k\gamma}{\gamma-1} \left(\frac{\dot{m}}{Au}\right)^{\gamma-1} = \frac{1}{2}\left(\frac{\gamma+1}{\gamma-1}\right) c_{s,m}^2$

Bernoulli $\frac{1}{2}u^2 + \int \frac{dp}{\rho} = \text{const}$

$$\int \frac{dp}{\rho} = \gamma k \int \rho^{\gamma-2} d\rho = \frac{\gamma k}{\gamma-1}\rho^{\gamma-1}$$

$$\int \frac{dp}{\rho} = \frac{\gamma k}{\gamma-1}\left(\frac{\dot{m}}{Au}\right)^{\gamma-1}$$

$$H = \frac{1}{2}u^2 + \frac{\gamma k}{\gamma-1}\left(\frac{\dot{m}}{Au}\right)^{\gamma-1}$$

at sonic transition, $u = c_{s,m}$

$$\int \frac{dp}{\rho} = \frac{\gamma k}{\gamma-1}\rho_m^{\gamma-1} = \frac{\gamma}{\gamma-1}\frac{p_m}{\rho_m} = \frac{c_{s,m}^2}{\gamma-1}$$

$$H = \frac{1}{2}c_{s,m}^2 + \frac{c_{s,m}^2}{\gamma-1} = \frac{1}{2}c_{s,m}^2\left(\frac{\gamma+1}{\gamma-1}\right)$$

$$\frac{1}{2}u^2 + \frac{k\gamma}{\gamma-1}\left(\frac{\dot{m}}{Au}\right)^{\gamma-1} = \frac{1}{2}c_{s,m}^2\left(\frac{\gamma+1}{\gamma-1}\right)$$

At large distances, $A \propto r^2$

find u at large r and how ρ depends on r

$$\frac{1}{2}u^2 = \frac{1}{2}c_{s,m}^2\left(\frac{\gamma+1}{\gamma-1}\right) \Rightarrow u = c_s\sqrt{\frac{\gamma+1}{\gamma-1}}$$

- neglect term with A

density variation:

$\dot{m} = \rho u A = \text{const.}$

$u = \text{const.}$ at large r, $A \propto r^2$

$\rho r^2 = \text{const.} \Rightarrow \rho \propto \frac{1}{r^2}$

4) hydrostatic equilibrium of self-gravitating gaseous object

$\nabla p = -\rho \nabla \Psi$, $\nabla^2 \Psi = 4\pi G \rho$

Explain why for a spherically symmetric system,

$p = p(\Psi)$, $\rho = \rho(\Psi)$

$\frac{dp}{dr} = -\rho \frac{d\Psi}{dr}$, $\rho > 0$ within object $\Rightarrow p = p(\Psi)$

$\frac{dp}{dr} = \frac{dp}{d\Psi}\frac{d\Psi}{dr} = -\rho \frac{d\Psi}{dr} \Rightarrow \rho = -\frac{dp}{d\Psi} \Rightarrow \rho = \rho(\Psi)$

For a barotropic equation of state $p = K\rho^{1+1/n}$, show that

$\rho = \frac{\Psi_T - \Psi}{(n+1)K}$

$\frac{1}{\rho}\nabla p = \frac{1}{\rho}\nabla(K\rho^{1+1/n}) = K(n+1)\nabla(\rho^{1/n})$

$K(n+1)\nabla(\rho^{1/n}) = -\nabla\Psi$

$K(n+1)\rho^{1/n} = -\Psi + c$

at surface, $\rho = 0$, $\Psi = \Psi_T \Rightarrow \Psi_T = c$

$\Psi_T - \Psi = K(n+1)\rho^{1/n}$

at core, $\Psi = \Psi_c$, $\rho = \rho_c$

$\Psi_T - \Psi_c = K(n+1)\rho_c^{1/n}$

$\frac{\rho}{\rho_c} = \left(\frac{\Psi_T - \Psi}{\Psi_T - \Psi_c}\right)^n$

$\Psi_T - \Psi = K(n+1)\rho^{1/n}$

$\rho = \left(\frac{\Psi_T - \Psi}{K(n+1)}\right)^n$

Poisson's equation in dimensionless form

$$\rho = \rho_c \left(\frac{\psi_T - \psi}{\psi_T - \psi_c}\right)^n = \theta^n \quad , \quad \vartheta = \frac{\psi_T - \psi}{\psi_T - \psi_c}$$

$$\nabla^2 \theta = \frac{-\nabla^2 \psi}{\psi_T - \psi_c} = \frac{-4\pi G \rho}{\psi_T - \psi_c} = \frac{-4\pi G \rho_c}{\psi_T - \psi_c} \theta^n$$

$$\xi = \alpha r \quad , \quad \alpha^2 = \frac{4\pi G \rho_c}{\psi_T - \psi_c} = \frac{4\pi G \rho_c^{1-1/n}}{(n+1)K}$$

$$\nabla^2 \theta = -\frac{\xi^2}{r^2} \theta^n$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\theta}{dr}\right) = -\frac{\xi^2}{r^2}\theta^n$$

$$\frac{d}{dr} = \alpha \frac{d}{d\xi} \Rightarrow \alpha \frac{d}{d\xi}\left(\frac{\xi^2}{\alpha^2} \alpha \frac{d\theta}{d\xi}\right) = -\xi^2 \theta^n$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) + \theta^n = 0$$

Show that $M \propto \rho_c^{1/2(3/n-1)} K^{3/2}$

$$M = \int 4\pi r^2 \rho \, dr$$

$$r^2 \propto \frac{1}{\alpha^2} , \; dr \propto \frac{1}{\alpha} , \; \rho \propto \rho_c$$

$$\alpha \propto \rho_c^{1/2(1-1/n)} K^{-1/2}$$

$$M \propto \frac{\rho_c}{\alpha^3} \propto \rho_c \rho_c^{3/2(1/n-1)} K^{3/2}$$

$$M \propto \rho_c^{1/2(3/n-1)} K^{3/2}$$

scaling relationship for r:

$$r \propto \frac{1}{\alpha} \propto \rho_c^{1/2(1/n-1)} K^{1/2}$$

under what conditions is the relationship $M \propto R$ recovered?

$$\frac{M}{R} = \frac{\rho_c^{1/2(3/n-1)} K^{3/2}}{\rho_c^{1/2(1/n-1)} K^{1/2}} = \rho_c^{1/n} K$$

for $M \propto R$, $\rho_c^{1/n} K = 1 \Rightarrow K \propto \rho_c^{-1/n}$

- stars don't share the same polytropic index K

clouds of cold monatomic hydrogen radiate inefficiently - well-modelled as adiabatic systems
find mass-radius scaling relationship

$P = K\rho^{\gamma}$, $\gamma = 1 + 1/n = 5/3 \Rightarrow n = 3/2$
K = const for adiabatic systems
$M \propto \rho_c^{1/2(3/n - 1)} \propto \rho_c^{1/2}$
$R \propto \rho_c^{1/2(1/n - 1)} \propto \rho_c^{-1/6}$
$M \propto R^{-3}$

isothermal - $P = \frac{R_* \rho T}{\mu}$

$T_c = \frac{\mu P_c}{R_* \rho_c} = \frac{K \rho_c^{1+1/n} \mu}{R_* \rho_c} = \text{const.}$

$K\rho_c^{1/n} = \text{const.} \Rightarrow K \propto \rho_c^{-1/n}$
$M \propto R$