UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Wednesday 27 May 2015 9.00 am to 11.00 am

PHYSICS (3)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (3)

ADVANCED QUANTUM PHYSICS

Candidates offering this paper should attempt a total of **three** *questions. The questions to be attempted are* **1**, **2** *and* **one** *other question.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains **four** *sides, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
2 × 20 Page Answer Book	Mathematical Formulae handbook
Rough workpad	Approved calculator allowed
Yellow master coversheet	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

ADVANCED QUANTUM PHYSICS

1 *Attempt* **all** *parts of this question. Answers should be concise and relevant formulae may be assumed without proof.*

(a) What is the maximum total angular momentum J for four identical fermions each with angular momentum $j = 5/2$, assuming that all other non-J quantum numbers are the same? [4]

(b) The scattering from a short-ranged potential (one where the potential is 0 beyond a finite radius a) becomes isotropic as the kinetic energy of the incident particle $\to 0$. Give a physical explanation. [4]

(c) Use Hund's rules to determine the ground state of the oxygen atom, which has electron configuration $...(2p)^4$. [4]

2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) the variational method in quantum mechanics;

(b) Landau levels;

(c) the ground state and excited electronic states of the helium atom.

3 *Attempt* **either** *this question* **or** *question 4.*

A Hamiltonian $\hat{H}_0$ has eigenstates $|\psi_n\rangle$ which have non-degenerate energies E_n. Show that, to first order in the perturbation $\hat{H}_1$, the eigenfunctions $|\phi_n\rangle$ of the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_1$ are given by

$$|\phi_n\rangle \;=\; |\psi_n\rangle \;+\; \sum_{j\neq n} \frac{\langle\psi_j|\hat{H}_1|\psi_n\rangle}{E_n - E_j}\;|\psi_j\rangle\;.$$

[8]

A particle of mass m and charge q in a one-dimensional simple harmonic potential, $\frac{1}{2}kx^2$, is subject to an electric field of magnitude ϵ in the x direction.

Using the perturbation theory expression above, and considering only the states $|\psi_0\rangle$ and $|\psi_1\rangle$, calculate the dipole induced by the electric field to first order in ϵ. [8]

By rewriting $\hat{H}$ as a simple harmonic oscillator Hamiltonian calculate the exact value of the induced dipole in the ground state and compare it with the value calculated previously. Comment on your answer. [4]

Explain why the perturbed groundstate calculated using the above formula is not correct for large values of ϵ. [5]

[*You may assume that the eigenstates of the simple harmonic oscillator have energies $(n+\frac{1}{2})\hbar\omega$ where $\omega = \sqrt{\frac{k}{m}}$ and that the normalised wavefunctions of the ground state and first excited state are given by*

$$\psi_0 = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} \exp\left(-\alpha x^2/2\right)\,,\;\; \psi_1 = \sqrt{2\alpha}\,x\psi_0.$$

where $\alpha = m\omega/\hbar$]

4 *Attempt* **either** *this question* **or** *question 3.*

A two-state system has Hamiltonian $\hat{H}_0$ which has eigenstates $|\psi_1\rangle$ and $|\psi_2\rangle$ with energies $E_1 = \hbar\omega_1$ and $E_2 = \hbar\omega_2$, respectively. A time-dependent perturbation $\hat{H}' \cos(\omega t)$ is applied to the system from $t = 0$, where the perturbation has matrix elements

$$\langle\psi_1|\hat{H}'|\psi_1\rangle = \langle\, \psi_2|\hat{H}'|\psi_2\rangle = 0,\ \langle\psi_1|\hat{H}'|\psi_2\rangle = \hbar\omega',$$

where ω' is a complex constant.

If the state of the system for $t \geq 0$ is written

$$|\psi(t)\rangle \ = \ c_1(t)\mathrm{e}^{(-\mathrm{i}\omega_1 t)}\,|\psi_1\rangle \ + \ c_2(t)\mathrm{e}^{(-\mathrm{i}\omega_2 t)}\,|\psi_2\rangle\,,$$

show that $c_1(t)$ and $c_2(t)$ obey the following equations

$$\frac{(\omega')^*}{2}c_1(t)\left[\mathrm{e}^{(\mathrm{i}(\omega+\omega_0)t)} \ + \ \mathrm{e}^{(-\mathrm{i}(\omega-\omega_0)t)}\right] \ = \ \mathrm{i}\frac{dc_2(t)}{dt}$$

$$\frac{\omega'}{2}c_2(t)\left[\mathrm{e}^{(\mathrm{i}(\omega-\omega_0)t)} \ + \ \mathrm{e}^{(-\mathrm{i}(\omega+\omega_0)t)}\right] \ = \ \mathrm{i}\frac{dc_1(t)}{dt}$$

where $\omega_0 = \omega_2 - \omega_1$. [9]

Explain why, for $\omega \approx \omega_0$, the terms involving $\omega + \omega_0$ can be ignored in these equations. [3]

A spin $\frac{1}{2}$ nucleus has magnetic dipole $\gamma\hat{\boldsymbol{s}}$, where $\hat{\boldsymbol{s}}$ is the spin operator. It is placed in a uniform magnetic field $(0, 0, B)$ with the spin in an eigenstate $s_z = \hbar/2$.

Using time-varying magnetic fields of frequency $\omega = \omega_0$, show how the spin can be rotated to be an eigenstate in: (i) the x direction; (ii) the y direction; and (iii) the $-z$ direction. Give full mathematical details of the procedure. [9]

Without mathematical details, discuss the effect of using harmonic perturbations of frequency $\omega \approx \omega_0$ but $\omega \neq \omega_0$. Which of the above spin rotations can still be carried out in this regime? [4]

[*The Pauli spin matrices are*

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \hat{\sigma}_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix},\ \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
]

END OF PAPER