UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Tuesday 30 May 2017 9:00 am to 11:00 am

PHYSICS (2)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (2)

RELATIVITY

Candidates offering this paper should attempt a total of **three** *questions. The questions to be attempted are* **1**, **2** *and* **one** *other question.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains **six** *sides, including this coversheet, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
2 × 20 Page Answer Book	Mathematical Formulae handbook
Rough workpad	Approved calculator allowed
Yellow master coversheet	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

RELATIVITY

1 *Attempt* **all** *parts of this question. Answers should be concise, and relevant formulae may be assumed without proof.*

(a) The metric in a 2d space is given by

$$\mathrm{d}s^2 = \mathrm{e}^{2\chi}\left(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2\right)$$

where (r,θ) are polar coordinates, and $\chi = \chi(r)$. Calculate the Christoffel symbols $\Gamma^r{}_{rr}$, $\Gamma^r{}_{\theta\theta}$, $\Gamma^\theta{}_{\theta r}$ and $\Gamma^\theta{}_{r\theta}$.
[*You may use the formula* $\Gamma_{\mu\alpha\beta} = \frac{1}{2}\left(\partial_\alpha g_{\mu\beta} + \partial_\beta g_{\alpha\mu} - \partial_\mu g_{\alpha\beta}\right)$.] [4]

(b) A particular example of a *scalar field* in spacetime has a stress-energy tensor

$$T^{\mu\nu} = (\partial^\mu\phi)\,(\partial^\nu\phi) - \tfrac{1}{2}g^{\mu\nu}\left[(\partial_\sigma\phi)\,(\partial^\sigma\phi) - m^2\phi^2\right]$$

and obeys the equation of motion

$$\nabla_\mu\nabla^\mu\phi + m^2\phi = 0\,.$$

Here ϕ, a scalar quantity, is the field, m is the mass of the particle it represents, the ∂_μ are partial derivatives and ∇_μ is the covariant derivative in a general curved space. (Note natural units are being used used in which $\hbar = c = 1$.)

Demonstrate explicitly that the stress-energy tensor is conserved, i.e. $\nabla_\mu T^{\mu\nu} = 0$. [4]

(c) In units with $c = 1$, the Faraday tensor is given by

$$[F_{\mu\nu}] = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & -B^3 & B^2 \\ -E^2 & B^3 & 0 & -B^1 \\ -E^3 & -B^2 & B^1 & 0 \end{pmatrix}.$$

Use this to demonstrate that $\boldsymbol{E}^2 - \boldsymbol{B}^2$ and $\boldsymbol{E}\cdot\boldsymbol{B}$ are Lorentz invariants. [2]

For an electromagnetic field in Minkowski space, the stress-energy tensor takes the form

$$T^{\mu\nu}_{\mathrm{em}} = -\mu_0^{-1}\left(F^\mu{}_\rho F^{\nu\rho} - \tfrac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right),$$

Show that the vector formed from the T^{0i}_{em} components of this tensor (where $i = 1,2,3$) is proportional to the Poynting vector, $\mu_0^{-1}\boldsymbol{E}\times\boldsymbol{B}$. [2]

2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) classic tests of General Relativity;

(b) tidal forces and geodesic deviation;

(c) advanced Eddington–Finkelstein coordinates in the Schwarzschild spacetime.

(TURN OVER

3 *Attempt* **either** *this question* **or** *question 4.*
This question has two parts and both should be answered.

(a) The Lagrangian method for finding geodesics is based upon a Lagrangian $\mathcal{L}$ defined by

$$\mathcal{L} = g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\,.$$

Describe the quantities appearing in the right hand side of this expression, and state the Euler-Lagrange equations which $\mathcal{L}$ satisfies. [2]

When the equations are for the motion of a particle of mass m, show that the covariant components of the particle 4-momentum are related to $\mathcal{L}$ via

$$p_{\mu} = \frac{m}{2}\frac{\partial \mathcal{L}}{\partial \dot{x}^{\mu}}\,,$$

and describe why the quantity p_{μ} is conserved if the metric is independent of the μ^{th} coordinate. [3]

The metric in a cylindrical coordinate system, (t, r, ϕ, z), describing a rotating frame, is

$$\mathrm{d}s^2 = \left(1 - \frac{\omega^2}{c^2}r^2\right)c^2\mathrm{d}t^2 - 2\omega r^2\mathrm{d}t\,\mathrm{d}\phi - \mathrm{d}r^2 - r^2\mathrm{d}\phi^2 - \mathrm{d}z^2\,.$$

Show that the covariant momentum components p_t, p_ϕ and p_z, for a particle of mass m moving in this frame, are conserved, and find explicit expressions for each in terms of $\dot{t}$, $\dot{\phi}$, $\dot{z}$, m, ω, c and r. [4]

Show also that $\dot{t}$ is constant, and by explicitly forming the $\ddot{r}$ Euler–Lagrange equation, or otherwise, demonstrate that

$$\ddot{r} = \frac{p_{\phi}^2}{m^2 r^3}\,.$$ [4]

Discuss the physical interpretation of this result in relation to centrifugal force. [2]

(b) Derive the *velocity addition formula*

$$(u_x, u_y, u_z) = \left(\frac{u'_x + v}{1 + \frac{u'_x v}{c^2}}, \frac{u'_y}{\gamma_v\left(1 + \frac{u'_x v}{c^2}\right)}, \frac{u'_z}{\gamma_v\left(1 + \frac{u'_x v}{c^2}\right)}\right),$$

where you should define the quantities involved, and make clear their relation to the underlying reference frames. [4]

A particle of mass m_1 collides elastically with a stationary particle of mass m_2, where $m_2 < m_1$. Show that the angle θ, by which the particle of mass m_1 is deflected from its original line of motion, satisfies $\sin\theta < m_2/m_1$. [6]

[*You may wish to work initially in the zero momentum frame of the collision. Also, you may assume that for constants a and b which satisfy* $b > a > 0$,

$$\frac{\sin\phi}{a\cos\phi + b},$$

considered as a function of ϕ, *attains a maximum value of* $1/\sqrt{b^2 - a^2}$.]

(TURN OVER

4 *Attempt* **either** *this question* **or** *question 3.*

[*Note units with $c = 1$ are used in this question.*]

The Riemann curvature tensor of a maximally symmetric 3-space has the form

$$R_{ijkl} = K\left(g_{ik}g_{jl} - g_{il}g_{jk}\right) ,$$

where K is a constant, and i, j, k and l range over the spatial indices 1 to 3. Show from this that the Ricci tensor $R_{jk} = g^{il}R_{ijkl}$ of the 3-space is given by $R_{jk} = -2Kg_{jk}$. [3]

Similarly to this, the requirement for maximal symmetry of a 4 dimensional space leads to the Riemann curvature tensor having the form

$$R_{\alpha\beta\gamma\delta} = -\frac{\Lambda}{3}\left(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}\right) ,$$

where Λ is a constant, and α, β, γ and δ range over the full set of spacetime indices. Show from this that the Ricci tensor is now given by

$$R_{\mu\nu} = \Lambda g_{\mu\nu} \quad (*)$$

and show that this equation corresponds to the vacuum Einstein field equations of General Relativity in the presence of a cosmological constant Λ. [4]

For a metric of the form

$$\mathrm{d}s^2 = A(r)\,\mathrm{d}t^2 - \frac{1}{A(r)}\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) ,$$

where (r, θ, ϕ) are standard spherical polar coordinates, it can be shown that the non-zero components of the Ricci tensor are

$$R_{00} = -A\left(\frac{A'}{r} + \frac{1}{2}A''\right), \quad R_{11} = -\frac{1}{A^2}R_{00}, \quad R_{22} = -\left(1 - A - rA'\right), \quad R_{33} = \sin^2\theta\,R_{22} ,$$

where dashes denote derivatives with respect to r. Use this to show that the choice of A given by $A = 1 + \lambda^2 r^2$, where $\lambda > 0$ is a constant, provides a solution of equation $(*)$ corresponding to a *negative* cosmological constant, the value of which you should find. [5]

In a space with this choice for A, a massive particle moves freely in the $\theta = \pi/2$ plane. Using the Lagrangian method for geodesic motion, or otherwise, show that $k = \dot{t}\left(1 + \lambda^2 r^2\right)$ and $h = r^2\dot{\phi}$ are both conserved during the motion, and identify these quantities physically. [3]

Use these results to find an equation for $\dot{r}^2$, and show that for purely radial motion the trial solution $r(\tau) = a\sin(b\tau)$ provides an expression for r as a function of proper time τ, and find the values of the constants a and b in terms of k and λ. [5]

Use the expression you find for r to show that a particle launched radially outwards from the origin returns to the origin in a proper time which is independent of the speed with which it is launched, and find this time. [2]

Circular orbits are also possible in this space. Without detailed calculation, but giving reasons, discuss whether or not you think the period of an orbit will depend on its radius. [3]

END OF PAPER