UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Wednesday 31 May 2017 9.00 am to 11.00 am

PHYSICS (3)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (3)

ADVANCED QUANTUM PHYSICS

Candidates offering this paper should attempt a total of **three** *questions. The questions to be attempted are* **1**, **2** *and* **one** *other question.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains **four** *sides, including this coversheet, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
2×20 Page Answer Book	Mathematical Formulae handbook
Rough workpad	Approved calculator allowed
Yellow master coversheet	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

ADVANCED QUANTUM PHYSICS

1 *Attempt* **all** *parts of this question. Answers should be concise and relevant formulae may be assumed without proof.*

(a) A hydrogen atom is subjected to a steady magnetic field of strength $B_0 = 1$ T along the z−axis, with which the proton spin is aligned. A magnetic field $B_1 = 10^{-4}$ T modulated at angular frequency ω is applied perpendicular to z for a short duration τ. Afterwards, the proton spin is aligned in the $-z$ direction. Determine ω and τ.
[*The proton has* $g_p \simeq 5.6$ *and the nuclear magneton is* $\mu_N = 5.05 \times 10^{-27}$ J/T.] [4]

(b) For a system with two angular momentum operators $\widehat{\boldsymbol{J}}_{\mathbf{1}}$, $\widehat{\boldsymbol{J}}_{\mathbf{2}}$, which give total angular momentum $\widehat{\boldsymbol{J}} = \widehat{\boldsymbol{J}}_{\mathbf{1}} + \widehat{\boldsymbol{J}}_{\mathbf{2}}$, state the two maximal subsets of commuting operators among $\left\{\widehat{J}_{1z},\ \widehat{J}_{2z},\ \widehat{J}^{\,2}, J_z,\ \widehat{\boldsymbol{J}}_{\mathbf{1}} \cdot \widehat{\boldsymbol{J}}_{\mathbf{2}},\ \widehat{J}_1^{\,2},\ \widehat{J}_2^{\,2}\right\}$. [4]

(c) Using first-order perturbation theory, obtain the fractional change of the ground state energy of the hydrogen atom caused by the finite size of the proton, which we can model as a thin spherical shell with radius b.
[*The* 1s *wavefunction is* $\pi^{-1/2} a_0^{-3/2} \exp(-r/a_0)$ *with* a_0 *the Bohr radius.*] [4]

2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) how to use the Wigner-Eckart theorem and Clebsch-Gordan coefficients to find the relative magnitudes of matrix elements. Examples could include the following cases:

$$M_1 = \left\langle j = \frac{3}{2},\ m = \frac{3}{2} \middle| \widehat{x} \middle| j = \frac{1}{2},\ m = \frac{1}{2} \right\rangle\ ,\ M_2 = \left\langle \frac{3}{2}, \frac{3}{2} \middle| \widehat{z} \middle| \frac{1}{2}, \frac{1}{2} \right\rangle\ ,\ M_3 = \left\langle \frac{3}{2}, \frac{1}{2} \middle| \widehat{z} \middle| \frac{1}{2}, \frac{1}{2} \right\rangle\ ;$$

(b) Landau quantisation for cyclotron orbits of electrons in a strong applied magnetic field, and examples of its experimental manifestation;

(c) coherent states for the 1D harmonic oscillator.

3 *Attempt* **either** *this question* **or** *question 4.*
An atom with a single valence electron, which is in a state with $\ell = 1$, is subject to a magnetic field $\boldsymbol{B}$ along the z−axis, giving rise to the Hamiltonian

$$\widehat{H} = \frac{\mu_B B}{\hbar}\left(\widehat{L}_z + 2\widehat{S}_z\right) + \frac{2K}{\hbar^2}\widehat{\boldsymbol{L}} \cdot \widehat{\boldsymbol{S}}$$

Briefly outline the origin of the terms in this Hamiltonian. Show that
$\widehat{\boldsymbol{L}} \cdot \widehat{\boldsymbol{S}} = \frac{1}{2}\left(\widehat{L}_+\widehat{S}_- + \widehat{L}_-\widehat{S}_+\right) + \widehat{L}_z\widehat{S}_z$. [6]

Denote the eigenstates of $\widehat{L}_z$ as $|\phi_1\rangle$, $|\phi_0\rangle$, $|\phi_{-1}\rangle$, and the eigenstates of $\widehat{S}_z$ by $|\uparrow\rangle$, $|\downarrow\rangle$. This produces the six product states

$$|\psi_1\rangle = |\phi_1 \uparrow\rangle\,,\ |\psi_2\rangle = |\phi_1 \downarrow\rangle\,,\ |\psi_3\rangle = |\phi_0 \uparrow\rangle\,,\ |\psi_4\rangle = |\phi_0 \downarrow\rangle\,,\ |\psi_5\rangle = |\phi_{-1} \uparrow\rangle\,,\ |\psi_6\rangle = |\phi_{-1} \downarrow\rangle\ .$$

Show that the matrix elements $H_{mn} \equiv \langle\psi_m|\,\widehat{H}\,|\psi_n\rangle$ are given by

$$\boldsymbol{H} = \begin{pmatrix} 2a+b & 0 & 0 & 0 & 0 & 0 \\ 0 & -b & c & 0 & 0 & 0 \\ 0 & c & a & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & c & 0 \\ 0 & 0 & 0 & c & -b & 0 \\ 0 & 0 & 0 & 0 & 0 & -2a+b \end{pmatrix},$$

and determine a, b and c in terms of the constants in the Hamiltonian.
[*You may wish to use the identity* $\widehat{J}_\pm\,|j,\ m\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}\,|j,\ m\pm 1\rangle$.] [6]

Find the energies of the stationary states and sketch their dependence on the applied field $\boldsymbol{B}$. Label them with quantum numbers m_ℓ, m_s or j, m_j, as appropriate. [9]

In the optical spectrum of sodium, the D_1 line arises at zero magnetic field from the transition $3^2\mathrm{p}_{1/2} \to 3^2\mathrm{s}_{1/2}$ and the D_2 line from $3^2\mathrm{p}_{3/2} \to 3^2\mathrm{s}_{1/2}$. The two lines differ in frequency by $\delta f = 5.16 \times 10^{11}\,s^{-1}$. Each line will split in an applied field. Estimate the magnetic field at which the split lines cross over. [$3^2\mathrm{p}_{3/2}$ *means* $n = 3$, $2s+1 = 2$, $\ell = 1$, $j = 3/2$. $\mu_B = 9.27 \times 10^{-24}$ J/T.] [4]

4 *Attempt* **either** *this question* **or** *question 3.*

Particles with mass m are incident on a spherically symmetric potential

$$V(r) = \frac{A}{r} e^{-\kappa r},$$

where A and κ are constants. Within the Born approximation, show that the differential scattering cross-section for the scattering wavevector $\boldsymbol{q} = \boldsymbol{k}_{\text{outgoing}} - \boldsymbol{k}_{\text{incoming}}$ is

$$\frac{d\sigma}{d\Omega} = \left(\frac{2mA}{\hbar^2 (q^2 + \kappa^2)} \right)^2 .$$

[9]

[*You may find the following results useful:*
(i) $\int_0^\infty \sin(qr) e^{-\kappa r} dr = q/(q^2 + \kappa^2)$;
(ii) for $V(\boldsymbol{r}) = \delta(\boldsymbol{r})$, $\frac{d\sigma}{d\Omega} = \left\{ m/(2\pi\hbar^2) \right\}^2$.]

From this, show that α-particles of energy E incident on nuclei of atomic number Z will scatter by an angle θ according to

$$\frac{d\sigma}{d\Omega} = \left(\frac{Ze^2}{8\pi\epsilon_0 E \sin^2(\theta/2)} \right)^2 .$$

[8]

A narrow beam of α-particles with energy $E = 7.68$ MeV is incident normally on a gold foil with thickness $t = 2.1 \times 10^{-7}$ m. The atomic weight of gold is 197, and its density is 19.3×10^3 kg/m^3. Particles scattered at an angle $\theta = 45°$ to the initial direction are counted by a detector, which has a transverse area of 1 mm^2 and is positioned 10 mm from the point where the α-particles hit the foil. It is found that a fraction $f = 3.7 \times 10^{-7}$ of incoming particles is scattered into the detector. What is the atomic number of gold according to these measurements? [8]

END OF PAPER