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Lent/Easter Terms 2022

Part II Particle and Nuclear Physics
Examples Sheet 4

Basic Nuclear Properties

28. SEMF, Tripos B-style question.
The Semi-Empirical mass formula (SEMF) for *nuclear* masses may be written in the form

$$M(A,Z) = Zm_{\mathrm{p}} + (A-Z)m_{\mathrm{n}} - a_{\mathrm{V}}A + a_{\mathrm{S}}A^{\frac{2}{3}} + a_{\mathrm{C}}\frac{Z^2}{A^{\frac{1}{3}}} + a_{\mathrm{A}}\frac{(A-2Z)^2}{A} + \delta(A,Z)$$

where m_{p} and m_{n} are the masses of the proton and neutron respectively. Fitted values for the coefficients are given at the end of the question.

(a) Explain the physical significance and functional form of the various terms.

(b) Treating the nucleus as a sphere of uniform charge density, show that the constant

$$a_{\mathrm{C}} = \frac{3e^2}{20\pi\epsilon_0 R_0} = 0.72\ \mathrm{MeV}\ \ .$$

(c) By reference to the SEMF explain in physical terms why nuclear fission and fusion are possible.

(d) Use the Semi-Empirical mass formula to predict that the value of Z of the most stable isobar of mass number A is

$$Z = \frac{m_{\mathrm{n}} - m_{\mathrm{p}} + 4a_{\mathrm{A}}}{2a_{\mathrm{C}}A^{-\frac{1}{3}} + 8a_{\mathrm{A}}/A}\ \ .$$

Predict the Z for the most stable nuclei with $A = 101$ and $A = 191$. Compare with nuclear data, which you can find on the web (e.g. `http://www.nndc.bnl.gov/chart/`). Predict the most stable super-heavy nucleus with mass number 300.

(e) Show that the effect of gravitational binding in the nucleus may be accounted for by adding a term $-a_{\mathrm{G}}A^{\frac{5}{3}}$ to the SEMF (neglecting the proton-neutron mass difference). Show that a_{G} is given by

$$a_{\mathrm{G}} = \frac{3Gm_{\mathrm{n}}^2}{5R_0} \approx 5.8\times 10^{-37}\ \mathrm{MeV}.$$

Use the SEMF, so modified, to estimate that the lightest "nucleus" consisting entirely of neutrons (i.e. a neutron star) has a mass approximately 10% that of the sun. *Amazingly enough, this reckless extrapolation over more than 50 orders of magnitude turns out to give more or less the right answer!*

[Take the mass of the sun to be 2 $\times 10^{30}$ kg]

[Use the following data:
m_p = 938.3 MeV, m_n = 939.6 MeV, m_e= 0.511 MeV; a_V = 15.8 MeV, $a_\mathrm{S} = 18.0$ MeV, a_A= 23.5 MeV. Nuclear radius $R = R_0 A^{1/3}$ with R_0 = 1.2 fm]

29. The Fermi Gas model, Tripos B-style question.
We can predict the form of the asymmetry term, and estimate its magnitude, using the Fermi Gas model. This involves treating the N neutrons and Z protons as free fermions of mass m moving in a box of volume $V = \frac{4}{3}\pi R_0^3 A$. The model therefore only accounts for the kinetic energy of the nucleons, and not their potential energy.

A standard calculation, which you have done before (at least for a cubic box), gives the density of states for each species (including spin degeneracy) as

$$g(\epsilon) = BA\epsilon^{\frac{1}{2}} \quad \text{where} \quad B = \frac{4\sqrt{2}m^{\frac{3}{2}}R_0^3}{3\pi\hbar^3} .$$

You need not prove this unless you want to practice. Show that the Fermi energy for the neutrons is

$$\epsilon_\mathrm{F} = \left(\frac{3N}{2BA}\right)^{\frac{2}{3}} .$$

Calculate the Fermi energy $\bar{\epsilon}_\mathrm{F}$ and the corresponding nucleon momentum for the symmetric case $N = Z = \frac{1}{2}A$.

Show that the total kinetic energy of the nucleons is given by

$$\frac{3}{5}\left(\frac{3}{2BA}\right)^{\frac{2}{3}}\left(N^{\frac{5}{3}} + Z^{\frac{5}{3}}\right) .$$

Expand about the symmetric point $N = Z = \frac{1}{2}A$ by writing $N = \frac{1}{2}A(1+\alpha)$ and $Z = \frac{1}{2}A(1-\alpha)$ to show that the asymmetry energy has the form:

$$a_\mathrm{A}\frac{(N-Z)^2}{A} \quad \text{where} \quad a_\mathrm{A} = \frac{1}{3}\bar{\epsilon}_\mathrm{F} .$$

Note that this underestimates the empirical value, because this model does not take account of the potential energy, which also depends on $(N - Z)$.

One contribution to the pairing energy can also be estimated from this model, reflecting the stepwise increase of the kinetic energy resulting from the exclusion principle. This would be expected to be approximately equal to the energy spacing of levels at the Fermi level, i.e. $1/g(\epsilon_\mathrm{F})$. Show that this is, for the $N = Z = \frac{1}{2}A$ case:

$$\frac{4\bar{\epsilon}_\mathrm{F}}{3A} .$$

Evaluate and compare with the fitted value in the SEMF for a typical value of A. Note that the Fermi Gas model again gives an underestimate because it does not take account of the additional potential energy arising from the spatial overlap of two nucleons in the same energy level.

30. Nuclear size, Tripos B-style question.
A spherically symmetric nucleus has a radial charge density $\rho(r)$ which is normalised such that $\int \rho(r)\mathrm{d}^3\mathbf{r} = 1$. Show that in this case the form factor is given by:

$$F(q^2) = \frac{4\pi}{q}\int_0^\infty r \sin qr\,\rho(r)\mathrm{d}r$$

which can then be approximated by

$$F(q^2) \simeq 1 - \frac{1}{6}q^2\overline{R^2} + \ldots$$

in natural units, where $\overline{R^2}$ is the mean square radius of the charge distribution. When elastic scattering of 200 MeV electrons from a gold foil is observed at 11°, it is found that the scattered intensity is 70% of that expected for a point nucleus. Calculate the r.m.s. radius of the gold nucleus.

For larger scattering angles ($> 50°$) it is found that the scattered intensity, instead of falling off monotonically with angle, exhibits definite (oscillatory) structure. What does this suggest about the form of $\rho(r)$?

31. Mirror Nuclei, Tripos B-style question.
One method of estimating nuclear radii is from the mass difference between a pair of odd-A mirror nuclei (A, Z) and $(A, Z+1)$ for which $Z = (A \pm 1)/2$. Use the Semi-Empirical mass formula to show that the difference in Coulomb energies between these nuclides, ΔE_C, can be written as

$$\Delta E_\mathrm{C} = \frac{3}{5}\frac{A\alpha}{R}$$

where the fine structure constant $\alpha = e^2/4\pi\epsilon_0$ in natural units and R is the nuclear radius.

The *atomic* mass difference between two mirror nuclei can be determined from the β^+ spectra of the $(A, Z+1)$ member of the pair,

$$M(A, Z+1) - M(A, Z) = 2m_\mathrm{e} + E_\mathrm{max}$$

where m_e is the mass of an electron and E_max is the maximum kinetic energy of the positron. Calculate the radius of the $(A, Z+1)$ member of each of the following pairs of mirror nuclei

(a) $({}^{11}_{5}\mathrm{B},\ {}^{11}_{6}\mathrm{C})$, $E_\mathrm{max} = 0.98$ MeV;
(b) $({}^{23}_{11}\mathrm{Na},\ {}^{23}_{12}\mathrm{Mg})$, $E_\mathrm{max} = 2.95$ MeV;
(c) $({}^{39}_{19}\mathrm{K},\ {}^{39}_{20}\mathrm{Ca})$; $E_\mathrm{max} = 5.49$ MeV;

and comment on the results.
Be careful when using atomic mass vs nuclear mass.

32. Deuteron, Tripos A-style question
The deuteron has spin-parity $J^P = 1^+$. Assuming that the deuteron is dominated by the orbital angular momentum state $\ell = 0$, and noting that no excited states exist, what can be concluded about the nature of the $n-p$ force and about the existence of nn and pp bound states?

The Nuclear Shell Model

33. Spin & Parity, Tripos A-style question.
What are *magic numbers* ? Outline the basis of the Nuclear Shell Model and show how it accounts for magic numbers. How can the shell model be used to predict the spins and parities of nuclear ground states ?

Using the shell model determine the spins and parities of the ground states of the nuclides listed below and compare them with the experimental values given. Comment on any discrepancies you find.

$$\begin{array}{cccccccccc} {}^{3}_{2}\mathrm{He} & {}^{9}_{4}\mathrm{Be} & {}^{7}_{3}\mathrm{Li} & {}^{12}_{6}\mathrm{C} & {}^{13}_{6}\mathrm{C} & {}^{15}_{7}\mathrm{N} & {}^{17}_{8}\mathrm{O} & {}^{23}_{11}\mathrm{Na} & {}^{131}_{54}\mathrm{Xe} & {}^{207}_{82}\mathrm{Pb} \\ \frac{1}{2}^{+} & \frac{3}{2}^{-} & \frac{3}{2}^{-} & 0^{+} & \frac{1}{2}^{-} & \frac{1}{2}^{-} & \frac{5}{2}^{+} & \frac{3}{2}^{+} & \frac{3}{2}^{+} & \frac{1}{2}^{-} \end{array}$$

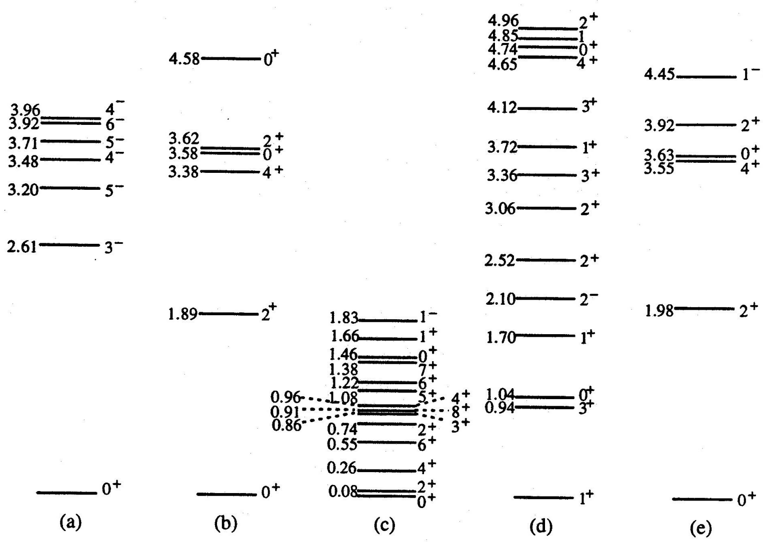
Assume the following ordering of levels:

$$1s_{\frac{1}{2}}\ 1p_{\frac{3}{2}}\ 1p_{\frac{1}{2}}\ 1d_{\frac{5}{2}}\ 1d_{\frac{3}{2}}\ 2s_{\frac{1}{2}}\ 1f_{\frac{7}{2}}\ 1f_{\frac{5}{2}}\ 2p_{\frac{3}{2}}\ 2p_{\frac{1}{2}}\ 1g_{\frac{9}{2}}\ 1g_{\frac{7}{2}}\ 2d_{\frac{5}{2}}\ 2d_{\frac{3}{2}}\ 1h_{\frac{11}{2}}\ 3s_{\frac{1}{2}}\ 1h_{\frac{9}{2}}\ 2f_{\frac{7}{2}}\ 3p_{\frac{3}{2}}\ 1i_{\frac{13}{2}}\ 3p_{\frac{1}{2}}\ 2f_{\frac{5}{2}}\cdots$$

34. Energy Levels, Tripos A-style question.
The diagram below shows the low-lying energy levels for the nuclides:

$${}^{18}_{10}\mathrm{Ne}\quad {}^{166}_{68}\mathrm{Er}\quad {}^{18}_{9}\mathrm{F}\quad {}^{208}_{82}\mathrm{Pb}\quad {}^{18}_{8}\mathrm{O}$$

The schemes are drawn to the same scale, with energies (in MeV) with respect to the ground state and the spin and parity (J^P) values given for each level. Identify which scheme corresponds to each nuclide and explain as fully as you can which features of the levels support your choices.



Nuclear Decay

35. Alpha Decay, Tripos B-style question.
Discuss the factors which affect the lifetimes of nuclei decaying by α-decay. You should include a description of the Geiger-Nuttall law.

   (a) A nucleus with A=200 can decay by the emission of α particles. The α particles are observed with two different energies, 4.687 MeV and 4.650 MeV. Neither of these decays populates the ground state of the daughter nucleus, but each is followed by the emission of a γ ray, whose energies are 266 and 305 keV respectively. No other γ rays are seen.

   - From this information construct an energy level diagram for the states involved, and indicate on it the decay scheme.
   - The decaying parent state has $\mathrm{J^P} = 1^-$ and the daughter has ground state $\mathrm{J^P} = 0^-$. Explain why there is no direct α decay to the ground state.

   (b) The values of the energy release, Q, and the measured α-decay half-lives of some isotopes of Thorium are as follows:

Isotope	Q / MeV	Half-life / s
$^{220}_{90}$Th	8.95	1.0×10^{-5}
$^{222}_{90}$Th	8.13	2.8×10^{-3}
$^{226}_{90}$Th	6.45	1.9×10^{3}
$^{228}_{90}$Th	5.52	6.0×10^{7}
$^{230}_{90}$Th	4.77	2.5×10^{12}
$^{232}_{90}$Th	4.08	4.4×10^{17}

   Estimate the α-decay half-life of $^{224}_{90}$Th, given that the Q-value for this decay is 7.31 MeV. What is the approximate uncertainty in your estimate?
   (Part II 1995)

36. Beta Decay, Tripos A-style question.
Outline the Fermi theory of β decay and explain the principal assumptions made.

   Explain the difference between Fermi and Gamow-Teller transitions and between super-allowed, allowed and forbidden decays. Explain the significance of ft values.

   Classify each of the following examples of β decay according to whether the decay is super-allowed, allowed, 1$^{\text{st}}$ forbidden etc., and whether Fermi or Gamow-Teller matrix elements are involved.

(i) $\mathrm{n} \rightarrow \mathrm{p}$

(ii) ${}^{6}_{2}\mathrm{He}(0^+) \rightarrow {}^{6}_{3}\mathrm{Li}(1^+)$ (ft=830s)

(iii) ${}^{14}_{6}\mathrm{C}(0^+) \rightarrow {}^{14}_{7}\mathrm{N}^*(0^+)$ (ft=3300s)

(iv) ${}^{35}_{16}\mathrm{S}(\frac{3}{2}^+) \rightarrow {}^{35}_{17}\mathrm{Cl}(\frac{3}{2}^+)$ (ft=1x10^5s)

(v) ${}^{36}_{17}\mathrm{Cl}(2^-) \rightarrow {}^{36}_{18}\mathrm{Ar}(0^+)$

(vi) ${}^{76}_{35}\mathrm{Br}(1^-) \rightarrow {}^{76}_{34}\mathrm{Se}(0^+)$

(vii) ${}^{137}_{55}\mathrm{Cs}(\frac{7}{2}^+) \rightarrow {}^{137}_{56}\mathrm{Ba}(\frac{3}{2}^+)$

37. Fermi Theory, Tripos A-style question.
Show that the electron momentum spectrum in β-decay using Fermi theory can be written as

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}p_\mathrm{e}} = \frac{G_\mathrm{F}^2}{2\pi^3}\left(E_0 - E_\mathrm{e}\right)^2 p_\mathrm{e}^2$$

where G_F is the Fermi constant, E_e and p_e are the energy and momentum of the electron and E_0 is the total energy released. You may treat the electron and neutrino as massless.

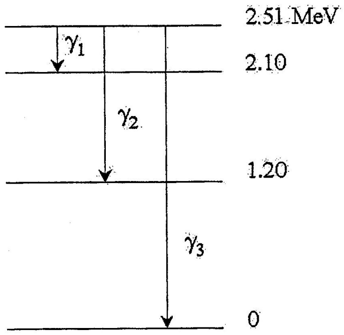
Show that the average kinetic energy carried off by the electron in β decay is $E_0/2$ when the electron is highly relativistic, and $E_0/3$ when the electron is non-relativistic.

When the electron is highly relativistic, show that the total decay rate is given approximately by

$$\Gamma = \frac{G_\mathrm{F}^2 E_0^5}{60\pi^3}$$

The E_0^5 dependence is sometimes known as Sargent's Rule.

38. Gamma Decay, Tripos B-style question.
A nucleus in an excited state at 2.51 MeV can decay by emission of three γ rays as shown below. The transitions labelled γ_1, γ_2 and γ_3 are known to proceed via magnetic dipole, electric dipole and electric quadrupole transitions respectively. No other transitions to the ground state, of comparable intensities, are observed. Given that the ground state has a spin of $\frac{3}{2}^+$ what are the most probable spins and parities of these three excited states?



Fission and Fusion

39. Fission, Tripos B-style question.

(a) Graphite (i.e. ^{12}C) is sometimes used as a moderator in nuclear reactors. Compute the maximum fractional energy loss which a non-relativistic neutron can undergo in a single elastic collision with a ^{12}C nucleus. Hence calculate the minimum number of collisions which would be required in order to bring a 2.5 MeV fission neutron down to a thermal energy of 0.025 eV.

(b) Experiments are being performed using a mixture of ^{235}U and graphite. The graphite contains a fraction of 10^{-6} by weight of ^{10}B. The neutron absorption cross-sections for ^{12}C, ^{10}B, and ^{235}U at thermal energies are 0.04 b, 3800 b and 700 b respectively, where fission accounts for 580 b of the cross-section in ^{235}U. What is the maximum fraction by weight of ^{235}U which could be allowed in the mixture if the multiplication factor at infinite volume is not to exceed unity? Assume that 2.5 neutrons are produced per fission, and that all reactions take place at thermal energies.

40. Fusion, Tripos A-style question.
Estimate the size of the Coulomb barrier between two $^{16}_{8}$O nuclei which needs to be overcome before they can undergo fusion, and thus estimate the temperature needed to bring about fusion in this case.

Numerical answers

28. (b) 0.7 MeV; (d) 44 ($^{101}_{44}$Ru is stable); 77 ($^{191}_{77}$Ir is stable); 114 ; (e) 150 MeV.

30. 5 fm.

31. (a) 3.4 fm; (b)4.2 fm; (c)4.6 fm.

35. ~ 10 s

36. (i) superallowed, F/GT; (ii) superallowed GT; (iii) superallowed F; (iv) allowed F/GT; (v) 1$^{\text{st}}$ forbidden GT; (vi) 1$^{\text{st}}$ forbidden F/GT; (vii) 2$^{\text{nd}}$ forbidden F/GT.

38. From the highest level downwards: $\frac{7}{2}^+$, $\frac{9}{2}^+$, $\frac{9}{2}^-$ or $\frac{7}{2}^-$.

Suggested Tripos Questions

Semi-Empirical Mass Formula: 2018 B2, 2010 A1(a)

Nuclear Forces & Scattering: 2016 4, 2012 1(a)

Shell Model: 2017 1(b), 2016 1(a)

Nuclear excitations: 2017 3 (last part), 2015 1(a)

Nuclear decay: 2017 3, 2013 1(c)

α-decay: 2010 A1(b), 2007 A3

β-decay: 2016 3(b)(c)(d), 2015 4

γ-decay: 2015 A1(b), 2014 4

Fission and Fusion: 2018 B2 (last part), 2011 4