PAPER 4 (Optics and Electrodynamics) – ANSWERS

1 (a) *As the Sun rises over a still lake, an angle of elevation is reached where the reflected image of the sun is completely linearly polarised in a plane parallel to the lake's surface. Explain the effect and calculate the angle, given that the refractive index of water is 1.33.* [4]

No reflection of the other polarisation at the Brewster angle θ_B. [1]

$\theta_B = \tan^{-1}\frac{n_1}{n_2}$. $n_1 = 1, n_2 = 1.33$. [1]

So $\theta_B = \tan^{-1} 1.33 = 53.06° \approx 53°$ to vertical (normal to surface), so angle is $90 - 53 = 37°$ to horizontal. [2]

(b) *Two Hertzian dipoles are driven by alternating current sources of the same amplitude. The first dipole is 1 cm long and operates at a wavelength of 20 cm, while the second is 0.5 cm long and operates at a wavelength of 25 cm. What is the ratio of powers emitted by the two dipoles?* [4]

Power P radiated from a dipole p is $P \propto [\ddot{p}]^2 \propto \omega^4 p^2$.

$p = qL$ where L is length of dipole and q is the (oscillating) charge on it.

Current $I \propto \dot{q} \propto q\omega$ so $q \propto I/\omega$ and amplitude of I is the same for each dipole (given).

So $P \propto (I^2/\omega^2)L^2\omega^4 \propto I^2L^2/\lambda^2$.

$L_1 = 1\text{cm}, L_2 = 0.5\text{cm}, \lambda_1 = 20\text{cm}, \lambda_2 = 25\text{cm}$.

$$\frac{P_1}{P_2} = \frac{L_1^2}{L_2^2}\frac{\lambda_2^2}{\lambda_1^2} = \left(\frac{1}{0.5}\right)^2\left(\frac{25}{20}\right)^2 = \frac{25}{4} = 6.25 \approx 6.$$

(c) *Estimate the magnitude of the aberration (i.e. the apparent angular shift) of a star due to the Earth's orbital motion around the Sun, when the star's light is incident normal to the Earth's trajectory.* [4]

[*A simple geometrical argument is sufficient.*
The Sun–Earth distance is 1.5×10^{11}*m.*]

Geometry - non-relativistic,

(Notes p262-4): Formula for aberration (incoming light at angle θ):
$\tan\frac{\theta}{2} = \left(\frac{1-\beta}{1+\beta}\right)^{1/2}\tan\frac{\theta'}{2}$. [1]

Maximum shift when $\theta' = \pi/2$ ($\tan\frac{\theta'}{2} = 1$). [1]

Orbit: $v = R\omega = R2\pi/(1\text{yr}) = \frac{2\pi\times1.5\times10^{11}}{365\times24\times3600} = 29886 \approx 30000\text{ms}^{-1} \approx c \times 10^{-4}$.
So $\beta \approx 10^{-4}$. [1]

So $\tan\frac{\theta}{2} \approx (1-\beta/2)(1+\beta/2).1 \approx 1 - \beta^2/4 \approx 1 - 10^{-8}/4$, so $\theta/2 = \pi/4 - 1.25 \times 10^{-9}$ rad using calculator or small-angle approximation which gives $\tan(A+B) \approx 1 + 2B$, for $A = \pi/4$ rad, B in radians.

So $\theta \approx= \pi/2 - 2.5 \times 10^{-9}$ rad, so magnitude of aberration $\approx 2.5 \times 10^{-9}$ rad $= 1.43 \times 10^{-7}$ °.

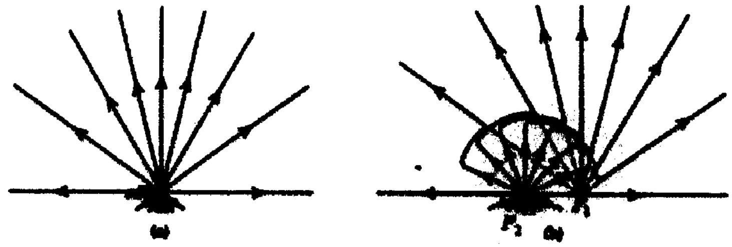

Figure 1: Charge that suddenly stops.

2 *Write brief notes on* **two** *of the following:* [13]

(a) *Radiation from an accelerating point charge;*

[Bookwork p306] Single moving charge suddenly stops. Beforehand, when moving uniformly, the electric field lines pointed straight out from the instantaneous position of the charge. When the particle suddenly changes velocity, e.g. it stops, the field lines nearby still point straight out from the instantaneous position, but those further away than the distance light can travel in the time since the particle stopped point towards where the particle would have been if it had not changed its velocity (see fig. 1). There is a (transverse) kink in the field lines at the distance light has travelled in the time since the change, and this kink propagates outwards at the speed of light. Hence this acceleration has caused radiation, and is an example of how radiation happens for any accelerating (or decelerating) charge. [2]

A charge q moving at a non-relativistic speed in an oscillating electric field behaves like an oscillating dipole (since each charge of the dipole contributes half the radiated field, or that due to a dipole $p = qz$ where z is the displacement from the average position). So the power loss P is the same as for a dipole, $P = \frac{\mu_0|\ddot{p}|^2}{6\pi c}$. $\ddot{\boldsymbol{p}} = q\boldsymbol{a}$ where $\boldsymbol{a}$ is the acceleration (in the particle's IRF) (Larmor formula). [Power loss is Lorentz invariant.] [2]

[Bookwork p318-327] A free charge moving in a magnetic field moves in a cyclotron orbit. Because its direction of motion is changing, it is accelerating and so radiates. $\omega_B = qB/\gamma m$, acceleration $a_\perp$ perpendicular to motion (at speed u) is $\omega_B u$, so use this in formula for P above, using $a^2 = \gamma^6 a_\parallel^2 + \gamma^4 a_\perp^2$ ($a_\parallel = 0$). [2]

Synchrotron radiation: when particle's speed in magnetic field is relativistic, get beaming of radiation in forward direction, so see a pulse of radiation each orbit. Duration $\Delta t \sim 1/\gamma^3\omega_B$, frequency $\sim 1/\Delta t$. Diagrams. [2]

(b) *Temporal coherence and the power spectrum, giving two examples;*

[Bookwork p82-4] Power spectrum $P(\omega)d\omega \sim |F(\omega)|^2 d\omega$ where $F(\omega)$ is the FT of the electric field $f(t)$. [2]

Temporal coherence (p98): autocorrelation (temporal coherence function) $\Gamma t =< f(t)f^*(t-\tau) >$. $I(\tau) = 2I_0 + 2\Re[\Gamma(\tau)]$. [2]

Wiener-Khinchine theorem (p103-4): $P(\omega) \sim \mathrm{FT}[\gamma(\tau) \equiv V(\tau)]$. [1]

Examples (p85-7): Lifetime broadening (Lorentzian linewidth of power spectrum); thermal broadening (Gaussian). [3]

(c) *The gauge choice for the electromagnetic potentials.*

[Bookwork p133-138 etc): Magnetic vector potential $\boldsymbol{A}$, electrostatic potential ϕ, $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, $\boldsymbol{E} = -\dot{\boldsymbol{A}} - \nabla\phi$. $\boldsymbol{A}$ and ϕ not completely determined by these equations. Can add any $\nabla\chi$ to $\boldsymbol{A}$ provided also subtract $\partial\chi/\partial t$ from ϕ. This choice is a "gauge". Physical fields (E and B etc) unchanged by a gauge transformation. [2]

Can choose $\nabla \cdot \boldsymbol{A}$ freely; Coulomb gauge: $\nabla \cdot \boldsymbol{A} = 0$. [1]

If choose Coulomb gauge, can write formula for $\boldsymbol{A}$ in terms of currents. [1]

[p150-163] $\boldsymbol{A}$ appears in QM. It is somehow more important than the magnetic field. Aharonov-Bohm effect shows that a particle's phase depends on $\boldsymbol{A}$ rather than $\boldsymbol{B}$—can exclude $\boldsymbol{B}$ entirely from regions where the particle wavefunction is non-zero (AB experiment, oscillations). Total phase change $\frac{q}{\hbar}\oint \boldsymbol{A}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \frac{q}{\hbar}\int \boldsymbol{B} \cdot d\boldsymbol{S}$, so independent of choice of gauge (since is a measurable value). SQUID as magnetometer. [3]

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3 *Explain briefly how the polarisation state of a light beam can be described by a Jones vector.* [1]

A Jones vector $\begin{pmatrix} a \\ b \end{pmatrix}$ represents the electric field vectors of the two polarisations of an electromagnetic wave with respect to the chosen axes. [1]

Describe the concept of birefringence and the effect of a quarter-wave plate on plane-polarised light incident normal to its surface, as a function of the angle between the Jones vector and one of the plate's transverse principal axes Ox and Oy. Calculate the thickness d of the plate for light at angular frequency ω_0, explaining any quantities you introduce. [6]

[Bookwork, related Q on sheet] A birefringent material has two different refractive indices, n_o and n_e corresponding to light travelling polarised with its transverse E field along one of two different principal axes. The 'fast' axis is the one with the smallest refractive index. A QWP is made of such a material, and is cut so that the two different axes lie in the plane of incidence. Its thickness is chosen such that it causes a phase change of $\pi/2$ between these two waves. When linearly polarised light is incident at 45° to both axes, the components along the two axes are the same and end up with a $\pi/2$ phase difference, thus linear polarisation is converted to circular polarisation. So, as the angle θ between the incident polarisation and the QWP's fast axis increases from zero, the polarisation goes from linear, to elliptical and then circular at $\theta = 45°$, back to elliptical and then linear at $\theta = 90°$. [2] [2]

The phase change through thickness d is $\Delta\phi = (k_e - k_o)d = \frac{\omega d(n_e - n_o)}{c} = \pi/2$ for a $\lambda/4$ (QW) plate, so $d = \frac{\pi c}{2\omega|n_e - n_o|}$. [2]

Consider now a frequency ω that differs slightly from ω_0. Show that the Jones matrix Q for the quarter-wave plate, with its fast axis along Ox, can be written as $Q = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i}e^{\mathrm{i}\epsilon} \end{pmatrix}$, where $\epsilon \ll 1$, and write down an expression for ϵ. Say why the Jones matrix H for a similar half-wave plate is $H = \begin{pmatrix} 1 & 0 \\ 0 & -e^{2\mathrm{i}\epsilon} \end{pmatrix}$. [3]

[Straightforward extension of bookwork] Ignoring prefactors that are the same for both components, [1]

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & e^{\mathrm{i}\Delta\phi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i}e^{\mathrm{i}\epsilon} \end{pmatrix}$$

where $\epsilon = \Delta\phi - \pi/2$ (since $e^{\mathrm{i}\pi/2} = \mathrm{i}$). H gives twice this phase change, hence it is the square of Q. [1] [1]

In an attempt to make a quarter-wave plate that works over the whole visible spectrum, a combination of Q and H is used. Light plane-polarised along the x-axis is passed through a half-wave plate that has been rotated through an angle $\alpha = 15°$ from that axis. It then passes into a quarter-wave plate that has been rotated through a further *angle $\beta = 60°$. Calculate the Jones vector of the resulting wave and show that it is proportional to $\begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix}$ if $\epsilon = 0$. What type of polarisation is this?* [6]

[Unseen calculation] Incoming light has Jones vector $A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. [1]

Resultant Jones vector is: [3]

$$
\begin{aligned}
A' &= Q\,R(\beta = 60^\circ)\,H\,R(\alpha = 15^\circ)\,A \\
&= \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i}e^{\mathrm{i}\epsilon} \end{pmatrix}\begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -e^{2\mathrm{i}\epsilon} \end{pmatrix}\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} & \\ & \end{pmatrix}\begin{pmatrix} & \\ & \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -e^{2\mathrm{i}\epsilon} \end{pmatrix}\begin{pmatrix} \cos\alpha \\ -\sin\alpha \end{pmatrix} \\
&= \begin{pmatrix} & \\ & \end{pmatrix}\begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix}\begin{pmatrix} \cos\alpha \\ e^{2\mathrm{i}\epsilon}\sin\alpha \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i}e^{\mathrm{i}\epsilon} \end{pmatrix}\begin{pmatrix} \cos\alpha\cos\beta + e^{2\mathrm{i}\epsilon}\sin\alpha\sin\beta) \\ -\cos\alpha\sin\beta + e^{2\mathrm{i}\epsilon}\sin\alpha\cos\beta \end{pmatrix} \\
&= \begin{pmatrix} \cos\alpha\cos\beta + e^{2\mathrm{i}\epsilon}\sin\alpha\sin\beta) \\ \mathrm{i}e^{\mathrm{i}\epsilon}(-\cos\alpha\sin\beta + e^{2\mathrm{i}\epsilon}\sin\alpha\cos\beta) \end{pmatrix}
\end{aligned}
$$

If $\epsilon = 0$, [1]

$$
A' = \begin{pmatrix} \cos(\alpha - \beta) \\ \mathrm{i}\sin(\alpha - \beta) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix} \quad \text{since } (\alpha - \beta) = 15 - 60 = -45^\circ
$$

This is circular polarisation. [1]

Show that this result is unchanged to first order in ϵ, by calculating the ratio of the two components (work to 3 significant figures). [4]

$$
\begin{aligned}
A' &= \begin{pmatrix} \cos 15\cos 60 + e^{2\mathrm{i}\epsilon}\sin 15\sin 60) \\ \mathrm{i}e^{\mathrm{i}\epsilon}(-\cos 15\sin 60 + e^{2\mathrm{i}\epsilon}\sin 15\cos 60 \end{pmatrix} \\
&\approx \begin{pmatrix} 0.483 + 0.224 \times (1 + 2\mathrm{i}\epsilon)) \\ \mathrm{i}(1 + \mathrm{i}\epsilon)(-0.837 + 0.130 \times (1 + 2\mathrm{i}\epsilon) \end{pmatrix} \\
&= \begin{pmatrix} 0.707 + 0.448\mathrm{i}\epsilon \\ \mathrm{i}(1 + \mathrm{i}\epsilon)(-0.707 + 0.260\mathrm{i}\epsilon) \end{pmatrix} \\
&= \begin{pmatrix} 0.707 + 0.448\mathrm{i}\epsilon \\ \mathrm{i}(-0.707 + 0.260\mathrm{i}\epsilon - 0.707\mathrm{i}\epsilon + O(\epsilon^2)) \end{pmatrix} \\
&\approx \begin{pmatrix} 0.707 + 0.448\mathrm{i}\epsilon) \\ \mathrm{i}(-0.707 - 0.447\mathrm{i}\epsilon) \end{pmatrix} \approx (0.707 + 0.448\mathrm{i}\epsilon)\begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix}
\end{aligned}
$$

So this is circularly polarised still, even to first order in ϵ. [4]

In order to make a quarter-wave plate that can be switched on and off rapidly, consider using (a) an electric field (Kerr effect) and (b) a magnetic field (Faraday effect) applied to some appropriate material. Explain the effect of each type of polariser on linearly-polarised light and say why only one of these would be suitable. [5]

[Application of bookwork] The Kerr effect causes an isotropic material to become uniaxially birefringent, so it would be suitable. An electric field can be switched on and [2]

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off quickly as required. The Faraday effect causes linearly-polarised light to be rotated, [1] as the two directions of circular polarisation have different refractive indices and linearly polarised light is a combination of the two circular polarisations, so it is impossible to [1] obtain a circular polarisation using the Faraday effect. [1]

[*The matrix that rotates the axes through an angle θ is*

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

]

4 *Define the terms 4-current, 4-potential and 4-gradient, and use them to show that the conservation of charge holds in all frames.* [5]

[Bookwork] The 4-current is $\mathsf{J} = (c\rho, \boldsymbol{J})$, 4-potential is $\mathsf{A} = (\phi/c, \boldsymbol{A})$ and 4-gradient is $\square = (\frac{1}{c}\frac{\partial}{\partial t}, -\nabla)$. [3]

We know charge conservation, $\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = 0 = \square \cdot \mathsf{J}$ in lab frame. [1]

$\square \cdot \mathsf{J}$ is a Lorentz invariant (scalar product), so the charge conservation equation must hold in all frames. [1]

The transformation of the **E** *and* **B** *fields between two frames S and S′ in standard configuration (i.e. whose origins and axes coincide at $t = t' = 0$, and where the S′ frame moves at velocity* $\boldsymbol{u}$ *in the positive x-direction) can be written as:*

$$E'_x = E_x\,, \qquad E'_{y,z} = \gamma(\boldsymbol{E} - \boldsymbol{u} \times \boldsymbol{B})_{y,z}\,,$$
$$B'_x = B_x\,, \qquad B'_{y,z} = \gamma(\boldsymbol{B} - \boldsymbol{u} \times \boldsymbol{E}/c^2)_{y,z}\,.$$

An observer at rest in frame S′ sees a stationary line charge, distributed along the x'-axis with uniform charge per unit length λ'.

(a) *Calculate the electric and magnetic fields seen by this observer at a distance r' from the line of charge and derive, with the aid of the transformation laws quoted above, the corresponding fields seen by an observer who is at rest in S.* [7]

[Application of bookwork] In S′, use Gauss's law for a cylinder of length L around the line, E radial by symmetry: [1]

$$\begin{aligned}\int \boldsymbol{E}'(r') \cdot d\boldsymbol{S}' &= \frac{\lambda' L}{\epsilon_0} \\ &= E'(r') 2\pi r' L\end{aligned}$$
$$\therefore E'(r') = \frac{\lambda'}{2\pi\epsilon_0 r'}$$

$\boldsymbol{B}' = 0$ since no movement of charge. [1]

Inverse LT to S using equations given: [2]

$$\begin{aligned}E_x &= E'_x = 0 \\ E_\perp &= \gamma(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B})_\perp \\ &= \gamma \boldsymbol{E}_\perp + 0 = \frac{\gamma\lambda'}{2\pi\epsilon_0 r}\end{aligned}$$

using $r = (y^2 + z^2)^{1/2} = r'$.

$B_x = B'_x = 0$ and

$$\begin{aligned}B_\perp &= \gamma(\boldsymbol{B}_\perp - \boldsymbol{u} \times \boldsymbol{E}/c^2)_\perp \\ \therefore B_y &= \tfrac{\gamma}{c^2}(-uE'_z) = -\frac{\gamma u \lambda'(z/r)}{2\pi\epsilon_0 c^2 r} = -\frac{\gamma u \lambda' z}{2\pi\epsilon_0 c^2 r^2} \\ \textit{and}\quad B_z &= \tfrac{\gamma}{c^2}(uE'_y) = \frac{\gamma u \lambda'(y/r)}{2\pi\epsilon_0 c^2 r} = \frac{\gamma u \lambda' y}{2\pi\epsilon_0 c^2 r^2}\end{aligned}$$

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using $E_y = E_\perp \cos\theta = E_\perp y/r$ and similarly for z. [3]

(b) *Find the charge density λ in S, and also the current that flows in this frame. Show that your results are consistent with the fields derived in (a).* [5]

[Application of bookwork] In S, $\lambda = \gamma\lambda'$ since length is contracted by factor of $1/\gamma$ and this is charge per unit length. [1]

Current I is the charge passing a point per unit time $= \frac{\lambda ut}{t} = \lambda u = \gamma\lambda' u$. [1]

From part (a),

$$B_{y,z} = B_\phi = \frac{\gamma\lambda' ur}{2\pi\epsilon_0 c^2 r^2} = \frac{\mu_0\gamma\lambda' u}{2\pi r} = \frac{\mu_0 I}{2\pi r}$$

since $\epsilon_0 c^2 = 1/\mu_0$. [1]

This agrees with Ampère's law. [1]

For a line charge λ,

$$E_\perp = \frac{\lambda}{2\pi\epsilon_0 r} = \frac{\gamma\lambda'}{2\pi\epsilon_0 r}$$

as in (a). [1]

(c) *In S, another line charge, which is stationary and has charge density $-\lambda$, is now superposed on the original line charge. Use the results derived in (b) to write down the total **E** and **B** fields and the linear charge density and current in S′, and show that these too are consistent.* [5]

[Application of bookwork] Instead of λ and u in S′, use $-\lambda$ in S and the results in (b):

$$\begin{aligned} E''_\perp &= \frac{-\gamma\lambda}{2\pi\epsilon_0 r} &= \frac{-\gamma^2\lambda'}{2\pi\epsilon_0 r} \\ B''_\phi &= \frac{\mu_0\gamma\lambda u}{2\pi r} &= \frac{\mu_0\gamma^2\lambda' u}{2\pi r} \\ -\lambda'' &= -\gamma\lambda &= -\gamma^2\lambda'. \end{aligned}$$

In S′, total fields are the sum of those from λ' and λ'':

$$\begin{aligned} E'''_\perp &= \frac{-(\gamma^2-1)\lambda'}{2\pi\epsilon_0 r} \\ B'''_\phi &= B''_\phi \\ -\lambda''' &= -\lambda'' + -\gamma\lambda' = -\lambda'(1-\gamma^2) \end{aligned}$$

Current $I''' = (-u)(-\lambda''') = u\gamma^2\lambda'$ (only consider moving charge).

Gives $B'''_\phi = \frac{\mu_0\gamma^2\lambda' u}{2\pi r}$, consistent with the above.

(d) *Explain how the situation in (c) is related to an ordinary wire.* [3]

[Understanding of bookwork] In S there is a current flowing but the wire is neutral. This is what happens for an ordinary wire carrying a current, with static ions and moving carriers. It produces a magnetic field but no E field. [2]

In S′, the ions are now moving and appear closer together. Conversely, the electrons were closer together in S that in S′. Thus there is a net line charge [and a factor of γ^2]. [1]

END OF PAPER