UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Friday 27 May 2016 1.30 pm to 3.30 pm

PHYSICS (7)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (7)

QUANTUM CONDENSED MATTER PHYSICS

Candidates offering this paper should attempt a total of **three** *questions. The questions to be attempted are* **1**, **2** *and* **one** *other question.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains **five** *sides, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
2 × 20 Page Answer Book	Mathematical Formulae handbook
Rough workpad	Approved calculator allowed
Yellow master coversheet	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

QUANTUM CONDENSED MATTER PHYSICS

1 *Attempt* **all** *parts of this question. Answers should be concise and relevant formulae may be assumed without proof.*

(a) Sketch the behaviour of the lattice heat capacity $C_{\mathrm{m}}(T)$ in the Debye approximation, indicating the asymptotic forms in the low temperature and high temperature limits. Given the speed of sound in silver, $v \simeq 3600$ m s^{-1}, and the number of atoms per unit volume, $n \simeq 5.9 \times 10^{28}$ m^{-3}, estimate the Debye temperature θ_{D}. [4]

(b) Sketch the square of the absolute value of the frequency dependent Drude conductivity $\sigma(\omega)$ as a function of ω, in a metal with scattering rate τ^{-1}. Compute the value of the frequency where $|\sigma(\omega)|^2$ drops to one half of its maximum for a metal with plasma frequency $\omega_p = 5.6 \times 10^{15}$ s^{-1} and DC conductivity $\sigma_0 \simeq 9.9 \times 10^6\ \Omega^{-1}$ m^{-1}. [4]

(c) A semiconductor at 350 K and chemical potential 0.9 eV lower than the bottom of the conduction band has electron carrier density $n = 2.3 \times 10^{16}$ cm^{-3}. A crossover from intrinsic to extrinsic behaviour is observed at 280 K. Obtain the extrinsic electron carrier density n_{ext} of the semiconductor. [4]

2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) the junction field effect transistor and pinch-off;

(b) direct exchange magnetic interaction;

(c) quantum oscillations.

3 *Attempt* **either** *this question* **or** *question 4.*

Discuss the notion of *effective mass* for electrons moving in a periodic potential. [6]

Show how Bloch's theorem follows from the fact that, in a lattice, the Hamiltonian $\widehat{H}$ commutes with the translation operator $\widehat{T}_{\boldsymbol{a}}$, where $\boldsymbol{a}$ is a Bravais lattice vector. Hence, or otherwise, show that Bloch's theorem implies that a wave function $\psi_{n\boldsymbol{k}}$ of wave vector $\boldsymbol{k}$ and band index n must take the form $\psi_{n\boldsymbol{k}} = e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\, u_{n\boldsymbol{k}}(\boldsymbol{r})$, where $u_{n\boldsymbol{k}}(\boldsymbol{r})$ has the same periodicity as the lattice. [8]

The wave function $\psi_{n\boldsymbol{k}}$ satisfies the equation $\left[\widehat{\boldsymbol{p}}^2/(2m) + \widehat{V}(\boldsymbol{r})\right]\psi_{n\boldsymbol{k}} = \varepsilon_n(\boldsymbol{k})\psi_{n\boldsymbol{k}}$, where $\widehat{\boldsymbol{p}} = -\mathrm{i}\hbar\nabla$. Show that this reduces to

$$\left[\frac{1}{2m}\left(\widehat{\boldsymbol{p}} + \hbar\boldsymbol{k}\right)^2 + \widehat{V}(\boldsymbol{r})\right] u_{n\boldsymbol{k}}(\boldsymbol{r}) = \varepsilon_n(\boldsymbol{k}) u_{n\boldsymbol{k}}(\boldsymbol{r})\ .$$ [2]

Divide the Hamiltonian into the two terms

$$\widehat{H}_0 = \frac{1}{2m}\widehat{\boldsymbol{p}}^2 + \widehat{V}(\boldsymbol{r}) \qquad \text{and} \qquad \widehat{H}_1 = \frac{\hbar^2 k^2}{2m} + \frac{\hbar\boldsymbol{k}\cdot\widehat{\boldsymbol{p}}}{m}\ .$$

At $\boldsymbol{k} = 0$, $u_{n0}(\boldsymbol{r})$ is the solution of $\widehat{H}_0\, u_{n0} = \varepsilon_n(0)\, u_{n0}$. For a general (small) value of $\boldsymbol{k}$, $\widehat{H}_1$ can be treated as a perturbation of the $\boldsymbol{k} = 0$ case. Use the results from second order perturbation theory to show that the energy at small $\boldsymbol{k}$ is

$$\varepsilon_n(\boldsymbol{k}) = \varepsilon_n(0) + \frac{\hbar^2 k^2}{2m} + \frac{\hbar^2}{m^2}\sum_{n'\neq n}\frac{\left|\langle u_{n0}|\boldsymbol{k}\cdot\widehat{\boldsymbol{p}}|u_{n'0}\rangle\right|^2}{\varepsilon_n(0) - \varepsilon_{n'}(0)}\ ,$$

where the sum is over all other states $\psi_{n'\boldsymbol{k}}$ at $\boldsymbol{k} = 0$ (you may assume that the n-th state is nondegenerate; you may also quote second order perturbation theory formulae without deriving them). [5]

Derive the ratio m/m^* of the electron mass to the effective mass at $\boldsymbol{k} = 0$. You may use the formula $m^* = \hbar^2[\nabla_k^2 \varepsilon_n(\boldsymbol{k})]^{-1}$. [4]

(TURN OVER

4 *Attempt* **either** *this question* **or** *question 3.*

Briefly state the assumptions used in Drude theory. [2]

Within Drude theory, show that the resistivity tensor for currents within the x-y plane and for an applied magnetic field B along the z-direction is:

$$\begin{pmatrix} \frac{1}{\sigma_0} & -\frac{B}{nq} \\ \frac{B}{nq} & \frac{1}{\sigma_0} \end{pmatrix}$$

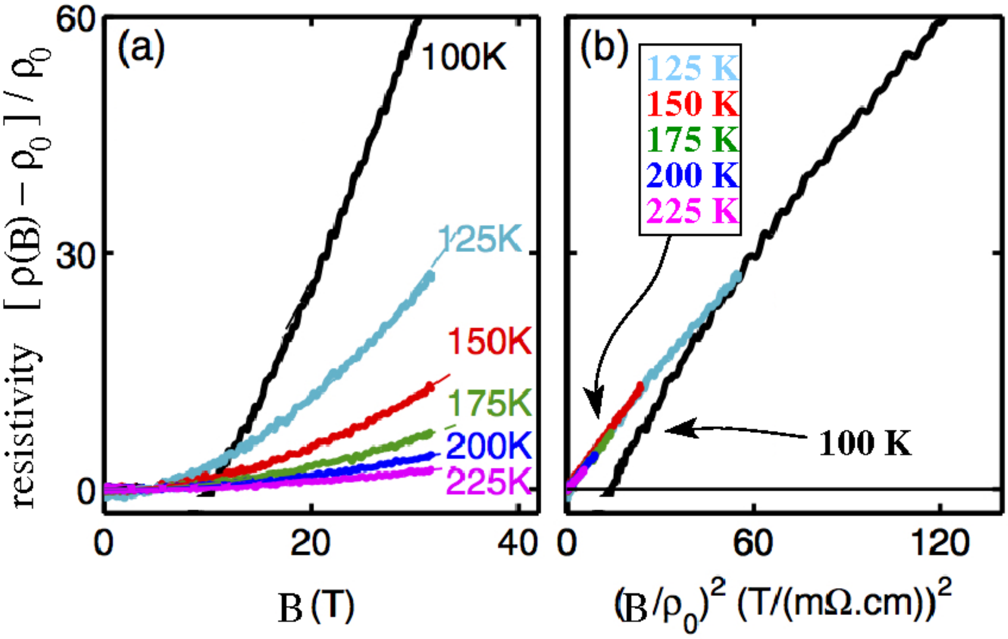
and give an expression for σ_0 in terms of n, q, m and τ. Explain the meaning of each of these five symbols. [10]

A material has equal numbers of electron and hole carriers. Assuming that the *conductivity* tensors associated with the two carrier types add, and that both carrier types contribute the same DC conductivity in zero applied field σ_0, find the resistivity tensor of the material. [4]

Show that the diagonal components ρ_{xx} and ρ_{yy} of the *resistivity* tensor of the material approach the value $\sigma_0 B^2/(2n^2q^2)$ in the limit of large magnetic fields. Compare the value of B needed to achieve this limit to that required to observe quantum oscillations caused by the quantisation of cyclotron orbits. [5]

Now consider the resistivity $\rho_{xx}(B)$ of the material introduced above for general values of the field B. Assuming that the carrier density n is constant in the range of temperatures and fields of interest, show that $\rho_{xx}(B)$ divided by the longitudinal zero-field resistivity $\rho_{xx}(B = 0)$ depends on the parameters of the system only via the product $B\sigma_0$: i.e., $\rho_{xx}/\rho_{xx}(B = 0) = F(B\sigma_0)$, where F is a scaling function. Derive the functional form of F. [2]

Consider the two plots at the end of the question, showing the same resistivity data as a function of the applied magnetic field B (left panel) and as a function of $(B/\rho_0)^2$, where $\rho_0 = 1/\sigma_0$. Are the data consistent with the resistivity depending on the parameters of the system only via the product $B\sigma_0$? Explain the reasons for your answer. For what range of temperatures does this dependence hold? [2]



(a)
(b)
100K
125K
150K
175K
200K
225K
125 K
150 K
175 K
200 K
225 K
100 K
resistivity [ρ(B) − ρ0] / ρ0
B (T)
(B/ρ0)2 (T/(mΩ.cm))2
0
30
60
20
40
120

END OF PAPER