NATURAL SCIENCES TRIPOS Part II

Wednesday 28th May 2008 9.00 to 11.00

EXPERIMENTAL AND THEORETICAL PHYSICS (2)

Attempt the **whole** *of Section A, and* **two** *questions from Section B. Answers from Section A should be tied up in a single bundle, with the letter A written clearly on the cover sheet. Answers to* **each** *question from Section B should be tied up separately, with the number of the question written clearly on the cover sheet.*

Section A carries approximately a quarter of the total marks. The **approximate** *number of marks allocated to each part of a question in Section B is indicated in the right margin. This paper contains 4 sides, and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS
Script paper
Metric graph paper
Rough work paper
Blue coversheets
Tags

SPECIAL REQUIREMENTS
Mathematical formulae handbook
Approved calculators allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

Answers should be concise, and relevant formulae may be assumed without proof. All questions carry an equal amount of credit.

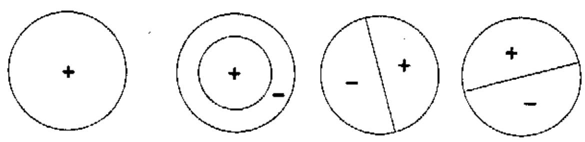
A1 Make an annotated sketch of the real-space wavefunction $\psi(x)$ for a quantum harmonic oscillator in its tenth excited energy eigenstate. Include in your sketch a graph showing the potential $V(x)$ with the energy E of the eigenstate marked. Indicate the principal features of $\psi(x)$ on your sketch.

A2 At time $t = 0$ the electron in a hydrogen atom is placed in a superposition of states,

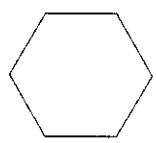
$$\psi = \frac{1}{\sqrt{2}}|1s\rangle + \frac{1}{\sqrt{2}}|2s\rangle.$$

Describe the probability distribution of the electron's location at that time, and at subsequent times.

A3 The ground state and three independent excited states of a particle constrained to move inside a circular region in a two-dimensional plane can be qualitatively represented as shown below.

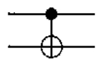


The potential in the bounded region is uniform. All lines in these diagrams indicate nodes of the wavefunctions, and the signs $+$ and $-$ indicate the sign of the wavefunction. Make similar qualitative predictions for the ground state and three independent excited states of a particle constrained to move in a hexagonal region.

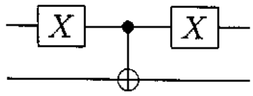


Explain whether you would predict any degeneracies among the states you describe.

A4 In quantum computing, the gate X is represented by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The CNOT gate, depicted in a quantum circuit by

,

takes a 2-qubit input and flips the target qubit (the lower wire) if and only if the control qubit (the upper wire) is set to $|1\rangle$. Describe the effect of the circuit shown below.



SECTION B

B5 Write brief notes on **three** of the following: [22]

(a) the optical spectrum of sodium;

(b) the Stern–Gerlach experiment;

(c) Fermi's golden rule;

(d) quantum teleportation.

B6 Prove that, for a Hermitian Hamiltonian $\hat{H}$ with lowest eigenvalue E_0, any wavefunction ψ satisfies

$$\frac{\langle\psi|\hat{H}|\psi\rangle}{\langle\psi|\psi\rangle} \geq E_0.$$

How is this result applied in the variational method for estimating eigenvalues? [5]

In some three-dimensional systems, the screened Coulomb potential

$$V(r) = -\frac{e^2}{4\pi\epsilon_0 r}\exp(-\alpha r)$$

is a better approximation to the true potential than the Coulomb potential. A trial wavefunction

$$\psi(r) = \left(\frac{b^3}{\pi}\right)^{1/2} \exp(-br)$$

is to be used to estimate the ground state energy of an electron in such a system.

(TURN OVER

Taking the Hamiltonian as

$$-\frac{\hbar^2}{2m_e}\nabla^2 + V(r),$$

with $V(r)$ as above, show that the expected value of the energy in the state $\psi(r)$ is

$$E = \frac{\hbar^2 b^2}{2m_e} - \frac{e^2 b^3}{\pi\epsilon_0(\alpha + 2b)^2}.$$

[8]

Expand this energy to first order in α and show that, in this approximation, the minimum in E occurs at $b = 1/a_0$, where $a_0 = 4\pi\epsilon_0\hbar^2/(m_e e^2)$ is the Bohr radius. Thus obtain an estimate of the ground state energy of an electron in a screened Coulomb potential having $\alpha = 0.2/a_0$, giving your answer as a multiple of $\hbar^2/(ma_0^2)$.

[9]

[*The following formulae may be useful.*]

$$\int_0^\infty e^{-\beta r} r\,dr = \frac{1}{\beta^2}$$

$$\int_0^\infty e^{-\beta r} r^2\,dr = \frac{2}{\beta^3}$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial f}{\partial r}\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

B7 Describe the method of degenerate perturbation theory. [5]

Explain the origin of the linear and quadratic Stark effects in the hydrogen atom. [3]

Taking $n = 2$ hydrogenic states of the form:

$$\begin{aligned}
\psi_{2\mathrm{s}} &= \frac{1}{2\sqrt{2\pi}} a_0^{-3/2} (1 - r/2a_0) \exp(-r/2a_0), \\
\psi_{2\mathrm{p}0} &= \frac{1}{4\sqrt{2\pi}} a_0^{-3/2} (r/a_0) \exp(-r/2a_0) \cos\theta, \\
\psi_{2\mathrm{p}\pm1} &= \mp\frac{1}{8\sqrt{\pi}} a_0^{-3/2} (r/a_0) \exp(-r/2a_0) \sin\theta \exp(\pm i\phi),
\end{aligned}$$

show that the linear Stark effect due to an electric field $\mathcal{E}$ brings about a shift of energy of magnitude $3e\mathcal{E}a_0$ for two of these states, and give a normalised form for the wavefunctions of all four perturbed states. e is the electron charge, a_0 the Bohr radius. [7]

[*The integral* $\langle \psi_{2\mathrm{s}} | r\cos\theta | \psi_{2\mathrm{p}0} \rangle$ *is equal to* $-3a_0$.]

Sketch the energies of these four states, and of the 1s state, as a function of the electric field $\mathcal{E}$ for both positive and negative $\mathcal{E}$. Describe, with the help of sketches, how the mean position of the electron would vary if the atom is initially in the 2s state and the field is zero, *then* the field is gradually made positive, and *then* gradually reduced and made negative. [4]

Explain why only a small fraction of atomic states have a permanent electric dipole. [3]

END OF PAPER