UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Friday 27 May 2016 9.00 am to 11.00 am

PHYSICS (5)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (5)

ASTROPHYSICAL FLUID DYNAMICS

Candidates offering this paper should attempt a total of **three** *questions. The questions to be attempted are* **1**, **2** *and* **one** *other question.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains **five** *sides, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
2 × 20 Page Answer Book	Mathematical Formulae handbook
Rough workpad	Approved calculator allowed
Yellow master coversheet	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

ASTROPHYSICAL FLUID DYNAMICS

1 *Answer* **all** *parts of this question. Answers should be concise and relevant formulae may be assumed without proof.*

(a) Demonstrate that an ideal gas in hydrostatic equilibrium, and in a uniform gravitational field, is stable for convection if

$$\frac{\rho}{p\gamma}\frac{\mathrm{d}p}{\mathrm{d}z} > \frac{\mathrm{d}\rho}{\mathrm{d}z}$$

where ρ is the density, γ is the adiabatic index of the gas, p is the pressure and z the spatial coordinate in the direction opposite to the gravitational force. [4]

(b) A black hole of mass M and Schwarzschild radius $R_s = 2GM/c^2$ is accreting steadily at a rate $\dot{m} = 1\ M_\odot\ \mathrm{yr}^{-1}$ through a thin viscous accretion disc which has inner and outer radii 6 R_s and 30 R_s respectively. Estimate the total luminosity of the disc. [4]

[*Assume that the gravitational field of the accretion disc is negligible relative to the black hole and neglect relativistic corrections.*]

(c) Monoatomic gas in free-fall from infinity accretes isotropically onto a protostar of radius r_0 and mass M_0. The gas undergoes a strong (adiabatic) shock at the impact with the protostar surface. Show that the temperature T of the gas after the shock is given by the relation

$$T = \frac{\mu}{R_*}\frac{3}{8}\frac{GM_0}{r_0}\ .$$ [4]

[*Assume that the temperature of the gas before the shock can be neglected and that the gas behaves as a perfect gas of mean molecular weight μ and modified gas constant R_*.*]

2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) Rayleigh–Taylor and Kelvin–Helmholtz instabilities;

(b) the formation and properties of shocks;

(c) the propagation of sound waves.

3 *Attempt* **either** *this question* **or** *question 4.*
Answer all parts of this question.

(a) Derive the equation of state $p = K\rho^{\gamma}$ for an adiabatic gas. [7]

(b) The Navier–Stokes equation is

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho}\nabla p - \nabla \Phi + \nu \left[\nabla^2 \boldsymbol{u} + \frac{1}{3}\nabla(\nabla \cdot \boldsymbol{u}) \right]$$

where ρ is the density, p is the pressure, Φ is the gravitational potential and ν is the kinematic viscosity, assumed to be constant. Show that for a barotropic fluid the vorticity, $\boldsymbol{w} = \nabla \times \boldsymbol{u}$, follows the relation

$$\frac{\partial \boldsymbol{w}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{w}) + \nu \nabla^2 w.$$

[9]

(c) A protostar embedded in a molecular cloud of radius R, expels a collimated jet of incompressible fluid, of density ρ, with constant cross section of radius r_0. The flow is steady and is driven by the pressure p_0 at the base of the jet. The flow is viscous, with constant kinematic viscosity ν, and is such that at the sides of the jet the velocity of the flow is zero due to interaction with the confining cloud. Show that the mechanical power of the jet, P, i.e. the bulk kinetic energy transiting through the jet cross-section, has the form $P = A r_0^8$. Find A in terms of R, p_0, ν and ρ. [9]

[*Neglect gravity and also neglect the pressure at the end of the jet.*]

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4 *Attempt* **either** *this question* **or** *question 3.*
Answer both parts of this question.

(a) Demonstrate that a spherical system in hydrostatic equilibrium and with a polytropic equation of state of index n follows the Lane–Emden equation

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n,$$

where

$$\xi = \sqrt{\frac{4\pi G\rho_\mathrm{c}}{\psi_\mathrm{o}-\psi_\mathrm{c}}}\, r,$$

$$\theta = \frac{\psi_\mathrm{o}-\psi}{\psi_\mathrm{o}-\psi_\mathrm{c}},$$

and where r is the radius, ρ_c is the density at the centre of the system, ψ is the gravitational potential, and ψ_o and ψ_c are the gravitational potential at the surface and at the center of the system, respectively. [7]

By using the working in the previous part, show that for a spherically symmetric cloud of diatomic gas in hydrostatic equilibrium with adiabatic equation of state, sound speed and gravitational potential ψ are related by the equation

$$c_\mathrm{s} = \sqrt{-\frac{2}{5}\psi}$$

(assume the boundary condition that where the gas density $\rho \to 0$ the gravitational potential $\psi_0 \to 0$). [3]

Assume that stars behave as polytropes and that they have the same central temperature. Derive the relationship between stellar mass and radius. [6]

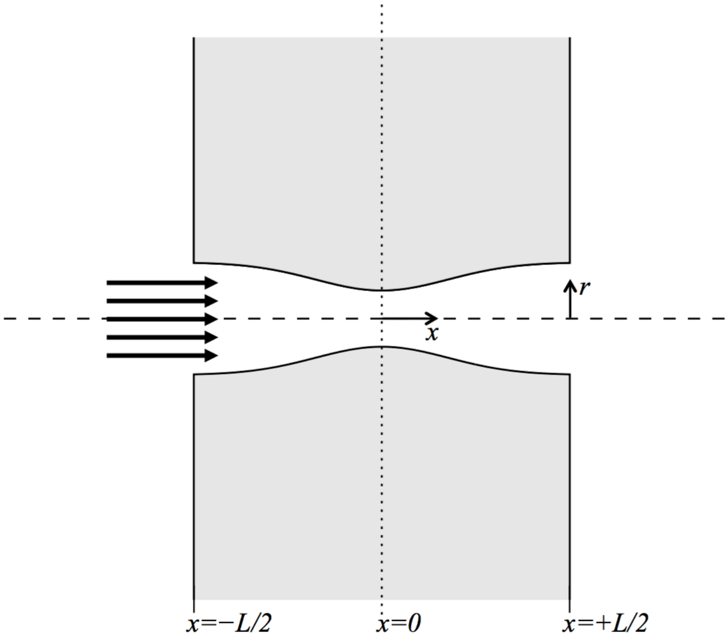
(b) A steady flow of gas with zero viscosity passes through a cavity in a slab of dense, incompressible medium of thickness L. The cavity is axisymmetric, perpendicular to the slab and has a cross-section whose radius varies as

$$r(x) = r_0\left(1-\frac{1}{2}e^{-(4x/L)^2}\right),$$

where x is the coordinate perpendicular to the slab with origin at the midpoint of the slab (see the figure on the next page). At the entrance of the cavity ($x = -L/2$) the flow has a subsonic velocity and leaves the slab supersonically at the opposite side. Determine the position x_s where the jet becomes supersonic. [3]

In the previous part assume that the gas in the flow has an adiabatic equation of state $p = K\rho^\gamma$. The mass flow rate is $\dot{M}$. Determine the temperature of the gas T_{x_s} at the sonic point.

[6]



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