UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Friday 29 May 2015 9.00 am to 11.00 am

PHYSICS (5)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (5)

ASTROPHYSICAL FLUID DYNAMICS

Candidates offering this paper should attempt a total of **three** *questions. The questions to be attempted are* **1**, **2** *and* **one** *other question.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains **four** *sides, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS
2 × 20 Page Answer Book
Rough workpad
Yellow master coversheet

SPECIAL REQUIREMENTS
Mathematical Formulae handbook
Approved calculator allowed

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ASTROPHYSICAL FLUID DYNAMICS

1 *Answer* **all** *parts of this question. Answers should be concise and relevant formulae may be assumed without proof.*

(a) A black hole of mass 2×10^{37}kg is located in a the central region of a galaxy where it is steadily accreting gas, composed of atomic hydrogen only. Assuming that at a distance of 3×10^{16} m from the black hole the gas number density is 10^6 m^{-3} and that the gas at this location has the free-fall velocity (from infinity), derive the accretion rate onto the black hole. [4]

(b) A sound wave crosses a discontinuity from a medium with density ρ_1 and temperature T_1 to a medium with density ρ_2 and temperature T_2. The two media are in pressure equilibrium. Show that the ratio of the velocity perturbations of the reflected wave to the incident wave is

$$r = \frac{\rho_1\sqrt{T_1} - \rho_2\sqrt{T_2}}{\rho_1\sqrt{T_1} + \rho_2\sqrt{T_2}}$$

(assume that both fluids have the same adiabatic index γ and that both fluids can be approximated as perfect gases of the same mean molecular weight). [4]

(c) Demonstrate the Virial Theorem $2T + \Omega = 0$ for a cloud of particles, where T is the total kinetic energy and Ω is the total gravitational potential energy. [4]

2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) magneto-hydrodynamic waves;

(b) the Bernoulli equation and its applications;

(c) mass-radius scaling relations for stars.

3 *Attempt* **either** *this question* **or** *question 4.*

Explain the physical meaning of the three Rankine-Hugoniot relations:

$$\rho_1 u_1 = \rho_2 u_2;$$

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2;$$

$$\frac{1}{2}u_1^2 + \mathcal{E}_1 + \frac{p_1}{\rho_1} = \frac{1}{2}u_2^2 + \mathcal{E}_2 + \frac{p_2}{\rho_2} \quad , \qquad (*)$$

where ρ is density, u is velocity measured in the frame of the shock, p is pressure, $\mathcal{E}$ is the internal energy per unit mass, and the subscripts 1,2 refer to upstream (pre-shock) and downstream (post-shock) conditions respectively. [3]

Show that, for an adiabatic shock in a perfect gas with adiabatic index γ, equation (*) can be rewritten as

$$\frac{c_{s,1}^2}{\gamma - 1} + \frac{1}{2}u_1^2 = \frac{c_{s,2}^2}{\gamma - 1} + \frac{1}{2}u_2^2 \quad ,$$

where $c_{s,1}$, $c_{s,2}$ are the speeds of sound in the up-stream and down-stream gas. Hence or otherwise show that

$$\frac{\rho_2}{\rho_1} = \frac{\gamma + 1}{\gamma - 1 + 2/M_1^2} \quad ,$$

where M_1 is the upstream Mach number. [7]

Discuss the limit of the density contrast for a strong adiabatic shock. [3]

Show that for an isothermal shock in a perfect gas the density contrast is $\rho_2/\rho_1 = M_1^2 = (u_1/c_{s,1})^2$ where $c_s = \sqrt{p/\rho}$ is now the isothermal sound speed. Discuss the implications of this relation for the physics of isothermal shocks in comparison to the adiabatic case. [6]

Suppose that the pre-shocked medium is made of molecular hydrogen and that the shock dissociates all molecules. Derive, in the isothermal shock case, the relation between density contrast ρ_2/ρ_1 and the velocities u_1 and u_2 of the pre-shocked and post-shocked gas, respectively, relative to the shock front. (Assume that both media behave as a perfect gas.) [6]

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4 *Attempt* **either** *this question* **or** *question 3.*

(a) Show that, for a self gravitating cloud of a perfect gas with uniform density ρ and uniform temperature T, the maximum stable mass (Jeans mass) is proportional to $T^{3/2}\rho^{-1/2}$. [6]

(b) Briefly explain why in a gas thermal stability requires $\left(\partial\dot{Q}/\partial T\right)_P > 0$, where $\dot{Q}$ is the net cooling rate. [2]

A hydrogen gas cloud is photoionized by a constant ultraviolet flux, which produces a constant heat input Γ to the gas. The cooling of the gas is given by the radiation resulting from recombination, and has the form $\dot{Q}_{\rm rad} = \frac{3}{2}n^2\alpha_B k_B T$, where $n = n_{\rm e} = n_{\rm p}$ is the gas number density, T is the gas temperature (assuming $T_{\rm e} = T_{\rm p} = T$), and α_B is the recombination coefficient. Assuming that the medium behaves like a perfect gas, determine whether the system is thermally stable. [4]

(c) A young star generates a spherically symmetric wind that reaches the escape velocity $v_{\rm esc}$ at a distance $R_{\rm esc}$ from the star. The gas in the outflow behaves adiabatically and the gas motion is dominated by the stellar gravitational potential. Determine the gas velocity as a function of radius. [6]

(d) Assume in the previous part that $R_{\rm esc} = 3 \times 10^{11}$ m, $M_* = 4 \times 10^{30}$ kg and the gas number density is $n(R_{\rm esc}) = 10^8\ {\rm m}^{-3}$ (assume the wind to be composed of atomic hydrogen). Derive a formula for the mass outflow rate of the wind. [3]

(e) Assume that the wind of the previous part behaves as a perfect gas and expands adiabatically. Derive the variation of pressure and of temperature as a function of radius. [4]

END OF PAPER

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ASTROPHYSICAL FLUID DYNAMICS

1 *Answer* **all** *parts of this question. Answers should be concise and relevant formulae may be assumed without proof.*

(a) Demonstrate the Virial Theorem $2T + \Omega = 0$ for a cloud of particles, where T is the total kinetic energy and Ω is the total gravitational potential energy. [4]

Bookwork from Handout.

Force on each particle $\boldsymbol{F} = m\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}$. Second time derivative of the moment of inertia:

$$\frac{1}{2}\frac{\mathrm{d}^2(mr^2)}{\mathrm{d}t^2} = m\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}\cdot\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right) = m\boldsymbol{r}\cdot\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} + m\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)^2 = \boldsymbol{r}\cdot\boldsymbol{F} + m\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)^2$$

the last term is twice the kinetic energy T.

Summing over all particles:

$$\frac{1}{2}\frac{\mathrm{d}^2I}{\mathrm{d}t^2} = \Sigma(\boldsymbol{r}\cdot\boldsymbol{F}) + 2T \qquad (*)$$

Consider any two particles with masses m_i and m_j located at $\boldsymbol{r}_i$ and $\boldsymbol{r}_j$; since $\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji}$, their contribution to the first term of the right hand side in eq.$(*)$ is $\boldsymbol{F}_{ij}\cdot(\boldsymbol{r}_i - \boldsymbol{r}_j)$, hence (in absence of external fields)

$$\Sigma(\boldsymbol{r}\cdot\boldsymbol{F}) = \Sigma_i\Sigma_{j>i}\boldsymbol{F}_{ij}\cdot(\boldsymbol{r}_i - \boldsymbol{r}_j)$$

If the ideal gas laws apply (no collisions except for $\boldsymbol{r}_i = \boldsymbol{r}_j$) all forces except for gravitation can be neglected, hence

$$\boldsymbol{F}_{ij} = -G\frac{m_i m_j}{r_{ij}^3}\,\boldsymbol{r}_{ij}$$

therefore

$$\Sigma(\boldsymbol{r}\cdot\boldsymbol{F}) = -\Sigma_i\Sigma_{j>i}\frac{Gm_i m_j}{r_{ij}}$$

The latter is the gravitational potential energy. Therefore eq.$(*)$ becomes

$$\frac{1}{2}\frac{\mathrm{d}^2I}{\mathrm{d}t^2} = 2T + \Omega$$

If the system is in steady state, then $I = const.$, hence $2T + \Omega = 0$.

(b) A sound wave crosses a discontinuity from a medium with density ρ_1 and temperature T_1 to a medium with density ρ_2 and temperature T_2. The two media are in pressure equilibrium. Show that the ratio of the velocity perturbations of the reflected wave to the incident wave is

$$r = \frac{\rho_1\sqrt{T_1} - \rho_2\sqrt{T_2}}{\rho_1\sqrt{T_1} + \rho_2\sqrt{T_2}}$$

(assume that both fluids have the same adiabatic index $\gamma = C_P/C_V$ – where C_P is the specific heat for constant pressure, and C_V is the specific heat for constant volume – and that both fluids can be approximated as perfect gases). [4]

The students have seen another solution to a similar problem giving r in terms of wavenumber ratios, which can be transformed into the required relation through a few transformations.

The following gives a slightly different approach, leading directly to the solution.

Adiabatic perturbations $\Delta p = c_s^2 \Delta \rho$

For the incident wave: $v = A_i\, e^{i(k_1 x - \omega t)}$; where A_i is the amplitude of the incident wave, ω is the frequency of the wave, k_1 is the wavenumber in the medium 1 given by $k_1 = \omega/c_1$, where c_1 is the sound speed in medium 1.

The reflected wave has the same frequency as the incident wave, the same wavenumber and amplitude A_r. The transmitted wave has the same frequency as the incident wave, wavenumber $k_2 = \omega/c_2$ and amplitude A_t.

At the boundary the velocity v is continuous, therefore $A_i + A_r = A_t$ (*).

Momentum equation (no gravity)→ velocity perturbations satisfy (neglecting second order terms): $\frac{\partial v}{\partial t} = \frac{1}{\rho}\frac{\partial \Delta p}{\partial x}$, therefore $\Delta p \propto \rho c_s v$.

Pressure continuos at boundary: $\rho_1 c_1 A_i - \rho_1 c_1 A_r = \rho_2 c_2 A_t$.

Hence, using (*), $\rho_1 c_1 (A_i - A_r) = \rho_2 c_2 (A_i + A_r)$.

$r = A_r/Ai$, hence $r = \frac{\rho_1 c_1 - \rho_2 c_2}{\rho_1 c_1 + \rho_2 c_2}$ (***)

as it is a perfect gas: $c_i = \sqrt{\frac{R_*}{\mu} T_i}$; replacing the latter into (***) gives the solution.

(c) A black hole of mass 2×10^{37}kg is located in a the central region of a galaxy where it is steadily accreting gas, composed of atomic hydrogen only. Assuming that at a distance of 3×10^{16} m from the black hole the gas number density is 10^6 m^{-3} and that the gas at this location has the free-fall velocity, derive the accretion rate onto the black hole. [4]

The free fall velocity is given by

$$v_{ff}^2 = 2GM_R/R$$

where M_R is the total mass contained within radius R, it is easy to show that the gas mass within 0.1 pc is negligible, hence $M_R = M_{BH}$.

For steady accretion:

$$\dot{M} = 4\pi R^2 \rho v_{ff} = 4\pi R^{3/2} n m_H (2GM_{BH})^{1/2}$$

replacing numerical values: $\dot{M} \sim 3 \times 10^{-3} M_\odot\, yr^{-1}$

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Write brief notes on **two** of the following: [13]

(a) magneto-hydrodynamic waves;

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3 *Attempt* **either** *this question* **or** *question 4.*
Explain the physical meaning of the three Rankine-Hugoniot relations:

$$\rho_1 u_1 = \rho_2 u_2;$$

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2;$$

$$\frac{1}{2}u_1^2 + \varepsilon_1 + \frac{p_1}{\rho_1} = \frac{1}{2}u_2^2 + \varepsilon_2 + \frac{p_2}{\rho_2} \quad ,$$

where ρ is density, u is velocity measured in the frame of the shock, p is pressure, ε is the internal energy per unit mass, and the subscripts 1,2 refer to upstream (preshock) and downstream (postshock) conditions respectively. [3]

The students are expected to explain that the first one gives the conservation of mass across the shock; the second the conservation of momentum, which shows that ram pressure upstream is transformed by the shock into thermal pressure downstream; the third is the conservation of energy flux, showing that the kinetic energy upstream is converted by the shock into enthalpy downstream.

Show that for an adiabatic shock

$$\frac{\rho_2}{\rho_1} = \frac{\gamma + 1}{\gamma - 1 + 2/M_1^2}$$

where M_1 is the upstream Mach number. [7]

For an ideal gas the internal energy per unit volume is $\varepsilon = c_v T = \frac{c_v}{R_*/\mu}\frac{p}{\rho}$.

For the adiabatic case, $\gamma = c_p/c_v$, together with $c_p - c_v = R_*/\mu$ gives $c_v(\gamma - 1) = R_*/\mu$, hence $\varepsilon = \frac{1}{1-\gamma}\frac{p}{\rho} = \frac{c_s^2}{\gamma(1+\gamma)}$. Replacing this into the third (energy) Rankine-Hugoniot relation gives:

$$\frac{\gamma p_1}{\gamma - 1} + \frac{1}{2}u_1^2 = \frac{\gamma p_2}{\gamma - 1} + \frac{1}{2}u_2^2 \ (*)$$

Replacing u_1 from the first R-H equation and p_1 from the second R-H equation gives (a few algebraic steps) the solution.

Discuss the limit of the density contrast for a strong adiabatic shock. [3]

The students have seen this in the course, although inferred from a relation different than the one shown in the previous part.

In the strong shock limit, $M \gg 1$, the previous equation yields

$$\frac{\rho_2}{\rho_1} \rightarrow \frac{\gamma + 1}{\gamma - 1}$$

.

For the case of monatomic gas $\gamma = 5/3 \Rightarrow \rho_2 = 4\rho_1$. Therefore, the density contrast has a maximum possible value in the case of an adiabatic shock. This is because in strong adiabatic

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shock the thermal pressure increases and prevents further compression of the gas (energy cannot be dissipated).

Moreover, since $p_2 \gg p_1$ (from the previous part), while $\rho_2 = 4\rho_1$,

$$\Rightarrow \frac{p_1}{\rho_1^\gamma} \neq \frac{p_2}{\rho_2^\gamma}$$

i.e. the gas passes from an adiabat to another. This means that despite the gas being adiabatic on both sides of the shock (hence has reversible properties on either side) dissipation of entropy at the shock front results into a non-reversible change.

Show that for a strong isothermal shock the density contrast is $\rho_2/\rho_1 = M_1^2$. Discuss the implications of this relation for the physics of the isothermal shocks in comparison to the adiabatic case. [6]

Mostly bookwork.

The third (energy) R-H relation does not hold, since in the isothermal shock energy is radiated away. However the first two R-H relations still hold. Therefore:

$$\rho_1 u_1 = \rho_2 u_2$$
$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$$
$$T_1 = T_2$$

where u_1 and u_2 are the velocities of the two media relative to the shock front, p_1 and p_2 are the pressure in the two media, and $T_1 = T_2 = T$ the temperature. The isothermal sound speed is the same in the two media $c_{s,1} = c_{s,2} = c_s = \sqrt{R_* T/\mu} = \sqrt{p/\rho}$, hence $p = c_s^2 \rho$. Replacing in the second R-H relation:

$$\rho_1(u_1^2 + c_s^2) = \rho_2(u_2^2 + c_s^2)$$

dividing both terms by $\rho_1 u_1$ $(= \rho_2 u_2)$:

$$u_1 + \frac{c_s^2}{u_1} = u_2 + \frac{c_s^2}{u_2}$$

hence

$$c_s^2\left(\frac{1}{u_1} - \frac{1}{u_2}\right) = u_2 - u_1$$

hence

$$c_s^2 = u_1 u_2 \qquad (*)$$

therefore

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \left(\frac{u_1}{c_s}\right)^2 = M_1^2$$

In contrast to the adiabatic case, in which the density contrast is limited to a maximum value of $(\gamma + 1)/(\gamma - 1)$, in the isothermal case the density contrast can be very high (energy can be dissipated).

Also, since $u_1 > c_s$ (condition for a shock), $c_s^2 = u_1 u_2$ (eq. $(*)$) implies that $u_2 < c_s$, i.e. the flow of the gas after the shock is subsonic.

Suppose that the pre-shocked medium is made of molecular hydrogen and that the shock dissociates all molecules. Derive, in the strong isothermal shock case, the relation between density contrast ρ_2/ρ_1 and the velocities u_1 and u_2 of the pre-shocked and post-shocked gas, respectively, relative to the shock front. Discuss this case relative to the case of monatomic gas discussed in the previous part. [6]

In this case $\mu_2 = \mu_1/2$, hence $c_{s,2} = \sqrt{2}\, c_{s,1}$.

Replacing this into the derivation of the velocity for the isothermal case, as in the previous part, i.e. the generic equation $\rho_1(v_1^2 + c_{s,1}^2) = \rho_2(v_2^2 + c_{s,2}^2)$, gives (a few steps):

$$c_s^2 = v_1 v_2 \left(\frac{v_2 - v_1}{v_2 - 2v_1}\right)$$

.

Using the first R-H relation gives:

$$\frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} = \left(\frac{v_1}{c_s}\right)^2 \left(\frac{v_2 - 2v_1}{v_2 - v_1}\right) = M_1^2 \left(\frac{v_2 - 2v_1}{v_2 - v_1}\right)$$

Since $v_1 > c_s > v_2$ (always true), from the previous exercise it implies that for a dissociative shock the density contrast is always lower than for the monoatomic case.

4 *Attempt* **either** *this question* **or** *question 3.*

(a) Show that, for a self gravitating cloud of uniform density ρ and uniform temperature T, the maximum stable mass (Jeans mass) is proportional to $T^{3/2}\rho^{-1/2}$. [6]

Mostly bookwork (but where, in addition, they have to replace the expression of the sound speed).

Sound wave passing through a uniform medium with density ρ_0 and pressure p_0. Perturbation analysis: $p = p_0 + \Delta p$, $\rho = \rho_0 + \Delta\rho$, velocity $u = \Delta u$, gravitational potential $\psi = \psi_0 + \Delta\psi$. The hydrodynamical equations for the perturbed quantities become, continuity equation:

$$\frac{\partial \Delta\rho}{\partial t} + \rho_0 \nabla \cdot \Delta\boldsymbol{u} = 0 \quad (*)$$

momentum equation:

$$\frac{\partial \Delta\boldsymbol{u}}{\partial t} = -\frac{\mathrm{d}p}{\mathrm{d}\rho}\frac{\nabla\Delta\rho}{\rho_0} - \nabla\Delta\psi = -c_s^2\frac{\nabla\Delta\rho}{\rho_0} - \nabla\Delta\psi \quad (**)$$

Poisson's equation:

$$\nabla^2\Delta\psi = 4\pi G\Delta\rho \quad (***)$$

Search wave-like solutions $\Delta\rho = \rho_1\, e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$, $\Delta\psi = \psi_1\, e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$, $\Delta\boldsymbol{u} = \boldsymbol{u_1}\, e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$.

The three equations $(*)$, $(**)$, $(***)$ become:

$$-\omega\rho_1 + \rho\boldsymbol{k}\cdot\boldsymbol{u_1} = 0 \quad (\#)$$

$$-\rho_0\omega\boldsymbol{u_1} = -c_s^2\rho_1\boldsymbol{k} - \rho_0\psi_1\boldsymbol{k} \quad (\#\#)$$

$$-k^2\psi_1 = 4\pi G\rho_1 \quad (\#\#\#)$$

Combining (#) and (##), by eliminating $\boldsymbol{u_1}$:

$$\rho_1\omega^2 = k^2(\rho_0 c_s^2 + \rho_0\psi_1)$$

By replacing the latter into (###):

$$\omega^2 = c_s^2\left(k^2 - \frac{4\pi G\rho_0}{c_s^2}\right) = c_s^2(k - k_J)$$

where $k_J^2 = 4\pi G\rho_0/c_s^2$.

If $k_J > k$ then $\omega^2 < 0$, implies that the solutions grow exponentially, i.e. gravitational instability (collapse). Hence, the maximum stable wavelength (Jeans length) is

$$\lambda_J = \frac{2\pi}{k_J} = \sqrt{\frac{\pi c_s^2}{G\rho_0}}$$

Hence the Jeans mass is $M_J \propto \rho_0\lambda_J^3$.

By replacing $c_s = (R_* T/\mu)^{1/2}$, the former implies $M_J \propto T^{3/2}\rho^{-1/2}$.

(b) Briefly explain why in a gas thermal stability requires $\left(\partial \dot{Q}/\partial T\right)_P > 0$, where $\dot{Q}$ is the net cooling rate. [2]

The student should simply explain that if a system in thermal equilibrium, i.e.:

$$cooling - heating = \dot{Q}_{eq} = 0$$

is perturbed by a small increase in temperature ΔT, then

$$\dot{Q} \approx \dot{Q}_{eq} + \left(\frac{\partial \dot{Q}}{\partial T}\right)_p \Delta T$$

hence if $\left(\frac{\partial \dot{Q}}{\partial T}\right)_P > 0$ the net cooling is positive, reducing the temperature; on the contrary, if $\left(\frac{\partial \dot{Q}}{\partial T}\right)_P < 0$ the net cooling rate becomes negative, yielding to further temperature increase, in a runaway process.

A gas cloud is photoionized by a constant ultraviolet flux, which produces a constant heat input Γ to the gas. The cooling of the gas is given by the radiation resulting from recombination, and has the form $\dot{Q}_{rad} = \frac{3}{2}n^2\alpha_B kT$, where n is the gas number density (assuming $n_e = n_p$), T is the gas temperature (assuming for simplicity $T_e = T_p = T$), and α_B is the recombination coefficient. Determine whether the system is thermally stable or not. [4]

This is similar to a problem the students have seen in one of the examples, but the cooling function is different.

Cooling by recombination is given by $\dot{Q}_{rad} = (\rho/m_p)^2\alpha_B\frac{3}{2}kT$.

Effective cooling:

$$\dot{Q}_{cool} = \dot{Q}_{rad} - \Gamma = (\rho/m_p)^2\alpha_B\frac{3}{2}kT - \Gamma$$

Replacing equation of state for perfect gas $p = (R_*/\mu)\rho T$, implies that the effective cooling is:

$$\dot{Q}_{cool} = \left(\frac{\mu}{R_* m_p}\right)^2 \alpha_B k\frac{3}{2}\frac{p^2}{T} - \Gamma$$

$$\Rightarrow \left(\frac{\partial \dot{Q}}{\partial T}\right)_p = -A(\mu/R_*)^2\alpha_B k\frac{p^2}{T^2} < 0$$

i.e. the system is thermally unstable.

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(c) A young star generates a spherically symmetric wind that reaches the escape velocity v_e at a distance R_e from the star. The gas in the outflow behaves adiabatically, is monatomic and the gas motion is dominated by the stellar gravitational potential. Determine the gas velocity as a function of radius. [6]

Since steady wind ($\partial v/\partial t = 0$) and since dominated by gravity, and since in radial symmetry, the momentum equation becomes:

$$v\frac{dv}{dr} = -\frac{GM_*}{r^2} \qquad (*)$$

where M_* is the mass of the star, which is given by

$$M_* = (R_e v_e^2)/(2G) \qquad (**)$$

Integrating (*) between R_e and generic r:

$$\int_{v_e}^{v} v\, dv = \int_{R_e}^{r} (-GM_*/r^2)\, dr$$

therefore

$$\frac{1}{2}(v^2 - v_e^2) = GM_*\left(\frac{1}{r} - \frac{1}{R_e}\right)$$

hence, by replacing (**),

$$v^2 = v_e^2\frac{R_e}{r} \qquad (***)$$

(d) Assume in the previous part that $R_e = 3 \times 10^{11}$ m, $M_* = 2\ M_\odot$ (where $M_\odot = 1.989 \times 10^{30}$ kg is the mass of the Sun) and the gas number density is $n(R_e) = 10^8\ \mathrm{m}^{-3}$ (assume the wind to be composed of atomic hydrogen). Derive a formula for the outflow rate of the wind. [3]

In a steady outflow, and specifically in this case, must be:

$$\dot{M} = v_e\ n\ m_H\ 4\pi R_e^2$$

Inverting (**) above gives $v_e = \sqrt{2GM_*/R_e}$. Hence

$$\dot{M} = n\ m_H\ 4\pi R_e^{3/2}\sqrt{2GM}$$

.

Replacing numerical values gives $\dot{M} = 1.3\ 10^{-13}\ M_\odot yr^{-1}$.

(e) Assume that the wind of the previous part expands adiabatically. Derive the variation of pressure and of temperature as a function of radius. [4]

Continuity equation with radial symmetry gives:

$$\rho = \rho_e \frac{v_e}{v} \left(\frac{R_e}{r}\right)^2$$

Replacing (***) from above gives: $\rho = \rho_e \left(\frac{R_e}{r}\right)^{3/2}$
Adiabatic monatomic gas: $p = k\rho^{5/3}$, hence

$$p = p_e \left(\frac{R_e}{r}\right)^{5/2}$$

By using equation of state for ideal gas gives, $T \propto \rho^{2/3}$, hence:

$$T = T_e (R_e/r)^{5/3}$$

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