University of Cambridge, Physics Part II, AFD

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Basics

Lagrangian derivative	$\frac{\mathrm{D}Q}{\mathrm{D}t} = \frac{\partial Q}{\partial t} + \mathrm{u} \cdot \nabla Q$
Eulerian Contiunuity Equation	$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0$
Lagrangian Continuity Equation	$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \boldsymbol{\nabla} \cdot \mathbf{u} = 0$
Lagrangian Momentum Equation	$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \rho \mathrm{g}$
Eulerian Momentum Equation	$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \rho \mathbf{g}$
Gravitational acceleration	$g = -\nabla \Psi$
Poisson's Equation	$\nabla \cdot \mathrm{g} = -\nabla^2 \Psi = -4\pi G \rho$
The Virial Theorem	$2E_{kinetic} + E_{potential} = 0$

Energy

EoS for ideal gas	$p = \frac{\rho}{\mu} kT$
Adiabatic	$p \propto \rho^{\gamma}$

Internal energy per unit mass $\mathcal{E} = \frac{1}{\gamma - 1}\frac{p}{\rho}$

Energy Density $E = \rho\left(\frac{1}{2}u^2 + \Psi + \mathcal{E}\right)$

Energy Equation $\frac{\partial E}{\partial t} + \nabla \cdot [(E + p)\mathbf{u}] = \rho\frac{\partial \Psi}{\partial t} - \rho\dot{Q}_{\mathrm{cool}}$

Entropy Energy Equation $\frac{1}{K}\frac{\mathrm{D}K}{\mathrm{D}t} = -(\gamma - 1)\frac{\rho\dot{Q}}{p}$

$(p = K\rho^\gamma)$

Hydrostatic

Equation of Hydrostatic Eqm $\frac{1}{\rho}\nabla p = -\nabla\Psi$

Polytropes $p = K\rho^{1+1/n}$

Lane-Emden Eqn. of Index n $\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n$

Mass-Radius Relation, Polytropic Stars $M \propto R^{\frac{3-n}{1-n}}$

Waves

Wave Equation $\frac{\partial^2(\Delta\rho)}{\partial t^2} = \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_{\rho=\rho_0}\nabla^2(\Delta\rho)$

Sound Speed $c_s = \sqrt{\left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_{\rho=\rho_0}}$

Adiabatic Speed $c_{s,\ \mathrm{A}} = \sqrt{\frac{\gamma kT}{\mu}}$

Isothermal Speed $$c_{s,\text{ A}} = \sqrt{\frac{kT}{\mu}}$$

Stratified Atmosphere Dispersion $$\rho_0(z) = \tilde{\rho}e^{-z/H}$$

$$\omega^2 = c_u^2\left(k^2 - \frac{\mathrm{i}k}{H}\right)$$

Reflection and Transmission $$t = \frac{2k_i}{k_i + k_t} \quad r = \frac{k_i - k_t}{k_i + k_t}$$

Mach Angle $$M \equiv \frac{v}{c_s} \quad \sin\alpha = \frac{1}{M}$$

Rankine-Hugoniot Relations $$\rho_1 u_1 = \rho_2 u_2$$

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$$

$$\frac{1}{2}u_1^2 + \mathcal{E}_1 + \frac{p_1}{\rho_1} = \frac{1}{2}u_2^2 + \mathcal{E}_2 + \frac{p_2}{\rho_2}$$

R-H combined $$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma+1)p_2 + (\gamma-1)p_1}{(\gamma+1)p_1 + (\gamma-1)p_2}$$

Strong Shock Limit ($p_2 \gg p_1$) $$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} \to \frac{\gamma+1}{\gamma-1}$$

Supernova Explosions $$R \propto t^{2/5}, \quad u_0 \propto t^{-3/5}, \quad p_1 \propto t^{-6/5}$$

$$R(t) = \left(\frac{E}{\rho_0}\right)^{1/5} t^{2/5}, \quad u_0(t) = \frac{2}{5}\frac{R}{t}$$

Transonic Flows

Vorticity $$\mathbf{w} = \boldsymbol{\nabla} \times \mathbf{u}$$

Bernoulli's $$H = \frac{1}{2}u^2 + \int \frac{\mathrm{d}p}{\rho} + \Psi = Constant$$

Helmholtz's equation $$\frac{\partial \mathbf{w}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{w})$$

Spherical Accretion $$\left(u^2 - c_s^2\right) \frac{\mathrm{d}}{\mathrm{d}r} \ln u = \frac{2c_s^2}{r} \left(1 - \frac{GM}{2c_s^2 r}\right)$$

Instability

Schwarzschild stability criterion $$\frac{\mathrm{d}T}{\mathrm{d}z} > \left(1 - \frac{1}{\gamma}\right) \frac{T}{p} \frac{\mathrm{d}p}{\mathrm{d}z}$$

Jean's stability $$\omega^2 = c_s^2 \left(k^2 - \frac{4\pi G\rho_0}{c_s^2}\right) = c_s^2 \left(k^2 - k_J^2\right)$$

Jeans Length $$\lambda_J = \frac{2\pi}{k_J} = \sqrt{\frac{\pi c_s^2}{G\rho_0}}$$

Jeans Mass $$M_J \sim \rho_0 \lambda_J^3$$

Interface dipersion $$\rho(kU - \omega)^2 + \rho' \left(kU' - \omega\right)^2 = kg\left(\rho - \rho'\right)$$

1. $U = U' = 0, \quad \rho' < \rho$ **(denser on bottom)**

$\Rightarrow$ Surface gravity waves (stable) $$\frac{\omega}{k} = \pm\sqrt{\frac{g}{k}\frac{\rho - \rho'}{\rho + \rho'}}$$

2. $U = U' = 0, \quad \rho' > \rho$ **(denser on top)**

$\Rightarrow$ Rayleigh-Taylor instability if: $$\frac{\omega}{k} = \pm\mathrm{i}\sqrt{\frac{g}{k}\frac{\rho' - \rho}{\rho + \rho'}}$$

3. $U \neq 0, \quad \rho' < \rho$ **(denser on bottom)**

Disperion: $$\frac{\omega}{k} = \frac{\rho U + \rho' U'}{\rho + \rho'} \pm \sqrt{\frac{g}{k}\frac{\rho - \rho'}{\rho + \rho'} - \frac{\rho\rho' \left(U - U'\right)^2}{\left(\rho + \rho'\right)^2}}$$

$\Rightarrow$ Kelvin-Helmholtz Instability if: $$\frac{g}{k}\frac{\rho-\rho'}{\rho+\rho'}-\frac{\rho\rho'\left(U-U'\right)^2}{\left(\rho+\rho'\right)^2}<0$$

Thermal Instability if: $$\left.\frac{\partial \dot{Q}}{\partial T}\right|_p<0$$

Viscous Fluid

kinematic viscosity $$\nu=\frac{\eta}{\rho}$$

Navier-Stokes Equation $$\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}=-\frac{1}{\rho}\boldsymbol{\nabla}p-\nabla\Psi+\frac{\eta}{\rho}\left[\nabla^2\mathbf{u}+\frac{1}{3}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u})\right]$$

Vorticity $$\frac{\partial \mathbf{w}}{\partial t}=\nabla\times(\mathbf{u}\times\mathbf{w})+\frac{\eta}{\rho}\nabla^2\mathbf{w}$$

Surface Density $$\Sigma\equiv\int_{-\infty}^{\infty}\rho\mathrm{d}z$$

Accretion disk $$\frac{\partial\Sigma}{\partial t}=\frac{3}{R}\frac{\partial}{\partial R}\left[R^{1/2}\frac{\partial}{\partial R}\left(\nu\Sigma R^{1/2}\right)\right]$$

Plasma

Alfven speed $$v_A=\sqrt{\frac{B^2}{\rho\mu_0}}$$

Magnetosonic waves $$c=\sqrt{c_s^2+v_A^2}$$

Magnetic R-T $$\omega^2=-kg\frac{\rho_1-\rho_2}{\rho_1+\rho_2}+\frac{2}{\mu_0}\frac{(\mathbf{k}\cdot\mathbf{B})^2}{\rho_1+\rho_2}$$