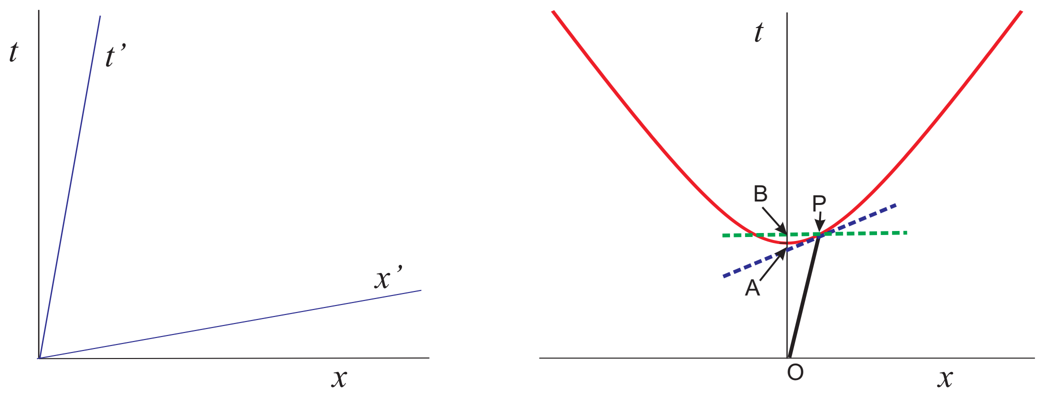
Part II Experimental and Theoretical Physics
Relativity — Examples 1-4 — Answers — 2013

Q1.1. Let the events be at $(0, 0, 0, 0)$ and $(t, x, 0, 0)$.
(a) If $c|t| > |x|$ then $x' = 0$ in the frame moving at $v = x/t$.
(b) If $c|t| < |x|$ then $t' = 0$ in the frame moving at $v = c^2t/x$.

Q1.2. (a) W.l.o.g. take events as A=$(0, 0, 0, 0)$ and B=$(t, 0, 0, 0)$ then $\Delta t' = \gamma t$ and is greater than zero.

(b) Causal events must have $c\Delta t \geq |\Delta r|$, i.e. in the forward light cone $\Delta s^2 \geq 0$. This is an invariant condition, so the events are causally connected in all frames.



Q1.3. (a) Diagram above. The fact that the angles are both $\tan^{-1}\beta$ is obvious from the Lorentz transformations

$$ct' = \gamma(ct - \beta x); \quad x' = \gamma(x - \beta ct) \tag{1}$$

(b) Diagram above. The invariant hyperboloid is $c^2t^2 - x^2 =$ constant, which gives $\mathrm{d}t/\mathrm{d}x = x/t$ as required. The time interval OB is t in S, but the proper time is $t(1 - \beta^2)^{1/2}$. This means the lab frame sees the moving one as time-dilated. The time interval OA is $t(1 - \beta^2)$, showing that the moving frame thinks that S is time-dilated. I don't really think of this as a geometric derivation, because we are using the algebra of Minkowski space…

To show the Lorentz contraction formula use a hyperboloid with $x^2 - c^2t^2 > 0$.

Q1.4. Thanks for improving the the hint, Mike. Thinking about relative scalars and vectors gets you there quickly. The t transform is linear, and a relative scalar, so can only involve $t, \boldsymbol{\beta}^2$ and $\boldsymbol{\beta}\cdot\boldsymbol{x}$. Since we know the usual Lorentz transform we have $t' = \gamma t - \gamma\boldsymbol{\beta}\cdot\boldsymbol{x}/c$. The $\boldsymbol{x}$ can only involve these scalars, $\boldsymbol{\beta}$ and $\boldsymbol{x}$ and we get

$$\boldsymbol{x}' = -\gamma\boldsymbol{\beta}ct + \boldsymbol{x} + \frac{(\gamma - 1)}{\boldsymbol{\beta}^2}\boldsymbol{\beta}\cdot\boldsymbol{x}\boldsymbol{\beta} \tag{2}$$

which is the required answer.

Q1.5. Method 1: From the velocity transformation law the components of the relative velocity are $(-v, v/\gamma, 0)$ and the magnitude is $v\sqrt{2-v^2/c^2}$.
Method 2: (More rigorous/fussy) Let the particles be at $(t, vt, 0, 0)$ and $(t, 0, vt, 0)$ in the lab frame. In the frame of the the first particle, the coordinates are

$$(\gamma t(1-v^2/c^2), 0, 0, 0) = (t/\gamma, 0, 0, 0) \equiv (t', 0, 0, 0)$$

and $(\gamma t, -\gamma vt, vt, 0)$ In terms of t' this is $(\gamma^2 t', -\gamma^2 vt', \gamma vt', 0)$, so the components of the velocity are $(-v, v/\gamma, 0)$ and the relative velocity is $v\sqrt{2-v^2/c^2}$.

Q1.6. (a) Method 1: In $\overline{\mathrm{O}}$ the longitudinal and transverse components are $l\cos\overline{\theta}$ and $l\sin\overline{\theta}$. The longitudinal component is contracted by factor γ, so in O we get $\tan\theta = \gamma\tan\overline{\theta}$.
Method 2: (More rigorous/fussy) In $\overline{\mathrm{O}}$ the ends of the rod are at $(\overline{t}, 0, 0, 0)$ and $(\overline{t}, l\cos\overline{\theta}, l\sin\overline{\theta}, 0)$, so that in O the ends are at $(\gamma\overline{t}, \gamma v\overline{t}, 0, 0)$ and $(\gamma(\overline{t} + vl\cos\overline{\theta}/c^2), \gamma(v\overline{t} + l\cos\overline{\theta}), l\sin\overline{\theta}, 0)$. At the moment $t = 0$, the first end is at $(0, 0, 0, 0)$ and other end has $\overline{t} = -vl\cos\overline{\theta}/c^2$, so that $x = \gamma l\cos\overline{\theta}(1-v^2/c^2) = l\cos\overline{\theta}/\gamma$. We get $\tan\theta = \gamma\tan\overline{\theta}$.
(b) The bullet in $\overline{\mathrm{O}}$ is at $(\overline{t}, \overline{u}\overline{t}\cos\overline{\theta}, \overline{u}\overline{t}\sin\overline{\theta}, 0)$, so that in O it is at

$$(\gamma\overline{t}(1+\overline{u}v\cos\overline{\theta}/c^2), \gamma\overline{t}(\overline{u}\cos\overline{\theta}+v), \overline{u}\overline{t}\sin\overline{\theta}, 0),$$

so that

$$\tan\theta = \frac{\overline{u}\sin\overline{\theta}}{\gamma(\overline{u}\cos\overline{\theta}+v)}$$

If $\overline{u} = c$ we get the usual aberration formula.

Q1.7. The null 4-vector in $\overline{\mathrm{O}}$ is related to the components in O by the Lorentz transformation

$$(\omega', \omega'\cos\theta', \omega'\sin\theta', 0) = (\omega\gamma(1-\beta\cos\theta), \omega\gamma(\cos\theta-\beta), \omega\sin\theta, 0) \tag{3}$$

so that $\cos\theta' = \dfrac{\cos\theta-\beta}{1-\beta\cos\theta}$. The rest-frame probability is $\Pr(\theta')\mathrm{d}\theta' = \frac{1}{2}\sin\theta'\mathrm{d}\theta' = \frac{1}{2}|\mathrm{d}\cos\theta'|$. This tidies up to have a $1-\beta^2$ in the numerator:

$$\Pr(\theta)\mathrm{d}\theta = \Pr(\theta')\mathrm{d}\theta' = \frac{\mathrm{d}\theta\ \sin\theta}{2\gamma^2(1-\beta\cos\theta)^2} \tag{4}$$

Q1.8 The velocity change $\mathrm{d}v' = g(\tau)\mathrm{d}\tau$ is related to the lab frame velocity v by

$$v + \mathrm{d}v = \frac{v+\mathrm{d}v'}{1-v\mathrm{d}v'/c^2)} = v + g\mathrm{d}\tau(1-v^2/c^2) \ \Rightarrow\ \frac{\mathrm{d}v}{1-v^2/c^2} = g(\tau)\mathrm{d}\tau \tag{5}$$

The substitution $v = c\tanh u$ gives the required form in terms of $\int_0^\tau \mathrm{d}s g(s)$. The complication involving the initial velocity can be derived (a) by the velocity addition law, or (b) from $\tanh(a+b) = \dfrac{\tanh a + \tanh b}{1 + \tanh a \tanh b}$. Clearly the velocity can't reach c because $\tanh(v/c) < 1$.

From $v = \mathrm{d}x/\mathrm{d}t$, $\mathrm{d}t = \gamma \mathrm{d}\tau$ and $\gamma = \cosh(v/c)$ we find that the trajectory is $x = g^{-1}c^2 \cosh(g\tau/c)$ and $t = g^{-1}c \sinh(g\tau/c)$. We need $\Delta x = g^{-1}c^2(\cosh(g\tau/c) - 1) = 10$ lt yr. So $(\cosh(g\tau/c) - 1) = 10.323$ and $g\tau/c = 3.118$. Therefore $\tau = 3.0205$ yr. If you can keep it up, you can really travel: 100 lt yr in 5.15 yr; 1000 lt yr in 7.40 yr; 10^6 lt yr in 14.09 yr.

Q1.9 The transformation is a a scale change of $\sqrt{2}$ and a 45° rotation in x^1-x^2 and a x^1x^2-dependent shift in x^3 that doesn't change the volume. Because of the scale change the volume element is 2. The matrix is horribly messy, but I'll write it down anyway:

$$\frac{\partial x^a}{\partial x'^b} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 2x'^2 & 2x'^1 & 1 \end{pmatrix} \Rightarrow g'_{ab} = \begin{pmatrix} 2+4(x'^2)^2 & 4x'^1x'^2 & 2x'^2 \\ 4x'^1x'^2 & 2+4(x'^1)^2 & 2x'^1 \\ 2x'^2 & 2'x^1 & 1 \end{pmatrix} \tag{6}$$

It's not orthogonal, but the determinant is 4, so the volume element is 2.

More details: the x frame is Cartesian so

$$\mathrm{d}s^2 = \mathrm{d}x^k\mathrm{d}x^k = g'_{ab}\mathrm{d}x'^a\mathrm{d}x'^b \quad \Rightarrow g'_{ab} = \frac{\partial x^k}{\partial x'^a}\frac{\partial x^k}{\partial x'^b} \tag{7}$$

Written carefully in matrix form as above $h_{kb} = \dfrac{\partial x^k}{\partial x'^b}$ this means $\underline{\underline{g}}' = \underline{\underline{h}}^T \cdot \underline{\underline{h}}$, not the other way around.

Q1.10 The parameterisation of the 3-sphere shows how to do it generally in N-dimensions:

$$w = a\cos\chi;\ \ z = a\sin\chi\cos\theta;\ \ x = a\sin\chi\sin\theta\cos\phi;\ \ a\sin\chi\sin\theta\sin\phi \tag{8}$$

Differentiating and simplifying, we find that the metric generates successive factors of $\sin^2\chi$ and $\sin^2\theta$:

$$\mathrm{d}s^2 = a^2\left(\mathrm{d}\chi^2 + \sin^2\chi\mathrm{d}\theta^2 + \sin^2\chi\sin^2\theta\mathrm{d}\phi^2\right) \tag{9}$$

The volume element is $\sin^2\chi\sin\theta\mathrm{d}\chi\mathrm{d}\theta\mathrm{d}\phi$. The limits of integration are 0 to π in χ and θ, but the last dimension has to go from $0 \le \phi \le 2\pi$. The surface $\chi = \chi_0$ has area $4\pi a^2\sin^2\chi_0$. The total surface area of the 3-sphere is $2\pi^2a^3$. Here's a derivation of the general result for Ω_N in N dimensions:

$$\int \mathrm{d}^N\boldsymbol{x}\ e^{-\boldsymbol{x}^2} = \pi^{N/2} = \Omega_N\int_0^\infty \mathrm{d}r\ r^{N-1}e^{-r^2} = \Omega_N\tfrac{1}{2}\Gamma(N/2) \tag{10}$$

where we have used the substitution $y \equiv r^2$. Hence $\Omega_N = \dfrac{2\pi^{N/2}}{\Gamma(N/2)}$. It has a maximum at $N = 7$ (a little after, actually).

Q2.1. From $\mathrm{d}\boldsymbol{x} = \boldsymbol{e}_a \mathrm{d}x^a$ and $\mathrm{d}s^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b$ we conclude $g_{ab} = \boldsymbol{e}_a\cdot\boldsymbol{e}_b$. The vectors $\boldsymbol{e}^a$ are dual to the $\boldsymbol{e}_a$ so $\boldsymbol{e}_a\cdot\boldsymbol{e}^b = \delta^b_a$. Thus $\boldsymbol{e}^a = g^{ab}\boldsymbol{e}_b$ and $\boldsymbol{e}_a = g_{ab}\boldsymbol{e}^b$.

From $\mathrm{d}\boldsymbol{x} = \boldsymbol{e}_1\mathrm{d}x^1 + \boldsymbol{e}_2\mathrm{d}x^2 + \boldsymbol{e}_3\mathrm{d}x^3$ and the coordinate definitions we have

$$\mathrm{d}\boldsymbol{x} = (\boldsymbol{e}_1 + \boldsymbol{e}_2 + 2x'^2\boldsymbol{e}_3)\mathrm{d}x'^1 + (\boldsymbol{e}_1 - \boldsymbol{e}_2 + 2x'^1\boldsymbol{e}_3)\mathrm{d}x'^2 + \boldsymbol{e}_3\mathrm{d}x'^3 \tag{11}$$

from which the $\boldsymbol{e}'_a$ can be read off and their dot products reproduce the results of Q1.9. The inverse is easy, actually:

$$\frac{\partial x'^a}{\partial x^b} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ -x^1 - x^2 & x^1 - x^2 & 1 \end{pmatrix} \Rightarrow g^{ab} = \begin{pmatrix} \frac{1}{2} & 0 & -x'^2 \\ 0 & \frac{1}{2} & -x'^1 \\ -x'^2 & -x'^1 & 1 + 2(x'^1)^2 + 2(x'^2)^2 \end{pmatrix} \tag{12}$$

so that

$$\boldsymbol{e}'^1 = \tfrac{1}{2}(\boldsymbol{e}_1 + \boldsymbol{e}_2);\ \boldsymbol{e}'^2 = \tfrac{1}{2}(\boldsymbol{e}_1 - \boldsymbol{e}_2);\ \boldsymbol{e}'^3 = -(x'^1 + x'^2)\boldsymbol{e}_1 + (x'^1 - x'^2)\boldsymbol{e}_2 + \boldsymbol{e}_3 \tag{13}$$

The covariant components of $\boldsymbol{v} = \boldsymbol{e}_1$ are $v_a = \boldsymbol{v}\cdot\boldsymbol{e}'_a = (1, 1, 0)$, and $v^a = (\frac{1}{2}, \frac{1}{2}, -(x'^1 + x'^2)$, hence result $v^a v_a = 1$.

Q2.2. (a) is obvious once you know what $T_{[ab]}$ and $T_{(ab)}$ mean.

(b) The transform laws are

$$v'_b = \frac{\partial x^c}{\partial x'^b}v_c \ \Rightarrow\ \frac{\partial v'_b}{\partial x'^a} = \frac{\partial^2 x^c}{\partial x'^a \partial x'^b}v_c + \frac{\partial x^d}{\partial x'^a}\frac{\partial x^c}{\partial x'^b}\frac{\partial v_c}{\partial x^d} \tag{14}$$

The first term means that $v_{a,b}$ isn't a tensor, but when you form $v_{a,b} - v_{b,a}$, the offending bits cancel and you have a tensor.

(c) The next bit is similar

$$\frac{\partial A'_{ab}}{\partial x'^c} = \frac{\partial x^r}{\partial x'^a}\frac{\partial x^s}{\partial x'^b}\frac{\partial x^t}{\partial x'^c}\frac{\partial A_{rs}}{\partial x^t} + \frac{\partial x^r}{\partial x'^a}\frac{\partial^2 x^s}{\partial x'^c \partial x'^b}A_{rs} + \frac{\partial x^r}{\partial x'^b}\frac{\partial^2 x^s}{\partial x'^c \partial x'^a}A_{rs} \tag{15}$$

Because $A_{ab} = -A_{ba}$, the unwanted bits vanish when you add the other 2 terms. $B_{abc} \equiv \partial_c A_{ab} + \partial_a A_{bc} + \partial_b A_{ca}$ is fully antisymmetric.

Some people assume that $A_{ab} = v_{a,b} - v_{b,a}$ as in the first part of the question, and so get zero…It's clear that the lecturer didn't mean that and A_{ab} is a general antisymmetric tensor.

Q2.3. (a) This needs the useful result $\partial_{g_{ab}} g = g g^{ab}$. To see that expand $\det \underline{\underline{g}}$ in a row (or column) containing g_{ab}. Then $\partial_{g_{ab}} g = M_{ab}$, where M_{ab} is the minor for g_{ab}. But $g^{-1}_{ab} = M_{ab}/g = g^{ab}$, hence result. Rest is chain rule: $\partial_c g = g g^{ab}\partial_c g_{ab}$.

(b) The covariant derivative is $\nabla_c g_{ab} = \partial_c g_{ab} - \Gamma^d_{cb} g_{ad} - \Gamma^d_{ac} g_{db}$. The Christoffel symbols are

$$\Gamma^m_{ij} = \tfrac{1}{2} g^{km}\left(\partial_i g_{kj} + \partial_j g_{ik} - \partial_k g_{ij}\right) \tag{16}$$

so that

$$\Gamma^d_{cb} g_{ad} + \Gamma^d_{ac} g_{db} = \tfrac{1}{2}\left(\partial_c g_{ab} + \partial_b g_{ca} - \partial_a g_{cb}\right) + \tfrac{1}{2}\left(\partial_a g_{bc} + \partial_c g_{ab} - \partial_b g_{ac}\right) = \partial_c g_{ab} \quad (17)$$

hence result.

(c) For a diagonal metric $g_{ab} = 0$, we clearly have $\Gamma^a_{bc} = 0$. For Γ^b_{aa} we get one term $-\frac{1}{2} g^{bc} g_{aa,c} = -\frac{1}{2}\partial_b g_{aa}/g_{bb}$. For Γ^a_{ab} or Γ^a_{ba} we still get one of the first terms e.g. $\frac{1}{2} g^{aa} g_{aa,b}$, which is indeed the answer given. For Γ^a_{aa} we get all three terms and the same answer because one has the opposite sign.

Q2.4. The surface of the unit sphere $\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2$ has non-zero Christoffel symbols

$$\Gamma^1_{22} = -\sin\theta\cos\theta \; ; \quad \Gamma^2_{12} = \Gamma^2_{21} = \cot\theta \quad (18)$$

(b) I'll start with a preamble about the geodesics and the variational principle

$$\delta \int \mathrm{d}\tau \; \tfrac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0$$

This variational principle implies the geodesic equation, the Euler-Lagrange equations being

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(g_{\lambda\mu}\dot{x}^\mu) = \tfrac{1}{2} g_{\mu\nu,\lambda}\dot{x}^\mu \dot{x}^\nu$$

where the symmetry $g^{\mu\nu} = g^{\nu\mu}$ has been used. But $\mathrm{d}/\mathrm{d}\tau g_{\lambda\mu} = \dot{x}^\nu g_{\lambda\mu,\nu}$ so

$$g_{\lambda\mu}\ddot{x}^\mu + (g_{\lambda\mu,\nu} - \tfrac{1}{2} g_{\mu\nu,\lambda})\dot{x}^\mu \dot{x}^\nu = 0 \Rightarrow \ddot{x}^\lambda + \Gamma^\lambda_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0$$

which is indeed the geodesic equation. The advantage of thinking about the variational principle is that it allow us to write down important first integrals of motion using symmetries of the Lagrangian. I prefer this variational principle rather than $S = \int \mathrm{d}\tau \sqrt{g_{ab}\dot{x}^a \dot{x}^b}$ because you get a parameter τ that is automatically affine. This becomes more important when working with null geodesics. Here we have $\mathcal{L} = \dot{\theta}^2 + \sin^2\theta\dot{\phi}^2$, so we get

$$\ddot{\theta} = \sin\theta\cos\theta\dot{\phi}^2 \; ; \quad \frac{\mathrm{d}}{\mathrm{d}\tau}\sin^2\theta\dot{\phi} = \ddot{\phi}\sin^2\theta + 2\sin\theta\cos\theta\dot{\theta}\dot{\phi} = 0 \quad (19)$$

This confirms the Christofell symbols found earlier. If $\dot{\theta} = 0$ the first equation implies $\theta = \pi/2$ (the other solutions $\theta = 0, \pi$ don't go anywhere, so they hardly count as geodesics!).

(c) I like John Baez's story of the Roman soldier with a javelin, who starts from the North Pole and marches to the equator with his weapon pointing North. He then turns West (not rotating the javelin, of course) and marches through $\pi/2$ of longitude. He then heads back up to the North Pole with his javelin still pointing North and finds that is rotated by $\pi/2$ when he gets there...

The parallel transport equation for a contravariant and covariant vector v^i and v_i along a path $x^i(\tau)$ is

$$\frac{\mathrm{d}v^i}{\mathrm{d}\tau} + \Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}\tau}v^k = 0\ ; \quad \frac{\mathrm{d}v_i}{\mathrm{d}\tau} + \Gamma^k_{ij}\frac{\mathrm{d}x^j}{\mathrm{d}\tau}v_k = 0 \tag{20}$$

We get

$$\dot{v}^\theta - \sin\theta\cos\theta\dot{\phi}v^\phi = 0\ ; \quad \dot{v}^\phi + \cot\theta(\dot{\phi}v^\theta + \dot{\theta}v^\phi) = 0 \tag{21}$$

This looks a trifle complicated, but we're only asked about the $\dot{\theta} = 0$ case. Clearly vectors rotate as ϕ changes, except at the equator $\cos\theta = 0$. However, the length of a vector $v^2 = (v^\theta)^2 + \sin^2\theta(v^\phi)^2$ remains the same:

$$\frac{\mathrm{d}v^2}{\mathrm{d}\tau} = -2\sin\theta\cos\theta\dot{\phi}v^\phi v^\theta + 2\sin^2\theta\cot\theta\dot{\phi}v^\theta v^\phi = 0 \tag{22}$$

This is also true when $\dot{\theta} \neq 0$.

We can do a bit better than that and derive the SHO equation for the components of v (since θ is constant we can express the result as a function of ϕ):

$$\frac{\mathrm{d}^2 v^{\theta,\phi}}{\mathrm{d}\phi^2} + \cos^2\theta v^{\theta,\phi} = 0$$

which shows that the components of the vector rotate at a uniform rate, once per revolution at the North Pole, and not at all at the equator.

Someone asked me why I used a contravariant vector rather than a covariant one. It doesn't matter, you get the same answer either way...

Q2.5. (a) $u{\cdot}v$ is invariant, so w.l.o.g take $u = (c, 0, 0, 0)$ and $v = (\gamma c, \gamma\beta c, 0, 0)$. In general $u{\cdot}v = \gamma_U c^2$. You can then get the relative velocity via $\beta^2 = 1 - 1/\gamma^2$, but this throws away the relative direction. The relative velocity is actually a bivector $u{\wedge}v$.

(b) Again, w.l.o.g specialise to $k = (k, k, 0, 0)$, so that $\omega_u = k{\cdot}u$. Hence the very important result given.

Q2.6. Take the velocity as $(\gamma c, \gamma\boldsymbol{u})$, so $\dot{v} = (\dot{\gamma}c, \dot{\gamma}\boldsymbol{u} + \gamma\dot{\boldsymbol{u}})$. We also need $\mathrm{d}t = \gamma\mathrm{d}\tau$, to show that $\dot{\boldsymbol{u}} = \gamma\boldsymbol{a}$ and $\dot{\gamma} = \gamma^4\boldsymbol{a}{\cdot}\boldsymbol{u}/c^2$. The invariant $\dot{v}^2$ tidies up to be

$$\dot{v}^2 = -\gamma^6(\boldsymbol{a}{\cdot}\boldsymbol{u})^2/c^2 - \gamma^4\boldsymbol{a}^2 \tag{23}$$

The acceleration of the particle moving in a circle is $\gamma^2\dfrac{\boldsymbol{u}^2}{r}$.

[People complain the the answer given has the wrong sign, as $\dot{v}^2 < 0$. However, the rest-frame acceleration is actually the bivector $\dot{v}{\wedge}v$, which has positive square (the wedge is unnecessary since $\dot{v}{\cdot}v = 0$).]

Q2.7. The composite system has 4-momentum $(E, \boldsymbol{p}) = (mc^2(1+\gamma), \gamma mv, 0, 0)$ so the c.o.m frame has $v_c = \gamma v/(1+\gamma)$ and $\gamma_c = \sqrt{(1+\gamma)/2}$. If the scattering angle in the c.o.m. frame is $\overline{\theta}$, the 4-momenta are $(\gamma_c mc^2, \pm\gamma_c mv_c \cos\overline{\theta}, \pm\gamma_c mv_c \sin\overline{\theta}, 0)$.

Transforming back, we get $(\gamma_c^2 m(1 \pm v_c^2/c^2 \cos\overline{\theta}), \gamma_c^2 mv_c(\pm\cos\overline{\theta}+1), \pm\gamma_c mv_c \sin\overline{\theta}, 0)$, which gives the rather nice result $\tan(\overline{\theta}/2) = \gamma_c \tan\theta$, though we don't need it... We find

$$\tan\theta_\pm = \frac{\sin\overline{\theta}}{\gamma_c(1 \pm \cos\overline{\theta})} \Rightarrow \tan\theta_+ \tan\theta_- = \frac{1}{\gamma_c^2} = \frac{2}{1+\gamma} \tag{24}$$

In the non-relativistic limit $\tan\theta_+ \tan\theta_- = 1$ so $\tan(\theta_+ + \theta_-) = (\tan\theta_+ + \tan\theta_-)/(1 - \tan\theta_+ \tan\theta_-) = \infty$ and the two particles are at right angles.

Q2.8. The input photon has 4-momentum (change in axes to be consistent) $h\nu/c(1, \cos\theta, \sin\theta, 0)$. In the frame of the mirror it is

$$\frac{h\nu}{c}\left(\gamma(1+\beta\cos\theta), \pm\gamma(\cos\theta+\beta), \sin\theta, 0)\right) \tag{25}$$

where the minus sign is the case after reflection. Transforming back we get

$$\frac{h\nu}{c}\Big(\gamma^2(1+\beta\cos\theta+\beta(\cos\theta+\beta)), -\gamma^2(\cos\theta+\beta+\beta(1+\beta\cos\theta)), \sin\theta, 0)\Big) \tag{26}$$

Tidying this up we find the frequency and direction

$$\nu' = \nu\gamma^2(1+\beta^2+2\beta\cos\theta)\ ; \quad \tan\theta' = \frac{\sin\theta}{\gamma^2(2\beta+\cos\theta(1+\beta^2))} \tag{27}$$

Q2.9. (a) The 4-momenta are $(\gamma m_\mathrm{p} c, \pm\gamma\beta m_\mathrm{p} c, 0, 0)$, so the invariant for the composite system is $4\gamma^2 m_\mathrm{p}^2 c^2$, which has to exceed $(2m_\mathrm{p} + m_\pi)^2 c^2$. The kinetic energy has to exceed $(\gamma - 1)m_\mathrm{p} c^2 > \frac{1}{2} m_\pi c^2$.

(b) The stationary proton contributes $(m_\mathrm{p} c, 0, 0, 0)$, so the composite system now has 4-momentum $((\gamma+1)m_\mathrm{p} c, \gamma\beta m_\mathrm{p} c, 0, 0)$ and invariant $2(\gamma+1)m_\mathrm{p}^2 c^2$. This again has to exceed $(2m_\mathrm{p} + m_\pi)^2 c^2$, so the kinetic energy has to be greater than $\dfrac{(4m_p m_\pi + m_\pi^2)}{2m_\mathrm{p}} c^2$.

Q2.10. Haven't typed in the first part yet...

The bookwork requires persistence to do by hand, which I certainly didn't. With $c = 1$ the definitions of $F_{\mu\nu}$ and L are

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix} \quad L = L^T = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{28}$$

It's instructive to do in this way as you derive the Lorentz transformation of the E/M field:

$$\overline{E}_1 = E_1 \quad \overline{E}_2 = \gamma(E_2 - vB_3)\ ; \ \ \overline{B}_1 = B_1 \quad \overline{B}_2 = \gamma(B_2 + vE_3/c^2) \tag{29}$$

i.e. longitudinal components unchanged and transverse ones like $\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}$ and $\boldsymbol{B} - \boldsymbol{v}\times\boldsymbol{E}/c^2$with a Lorentz boost. However, since

$$\overline{F}_{\mu\nu} = \frac{\partial x^\alpha}{\partial \overline{x}^\mu}\frac{\partial x^\beta}{\partial \overline{x}^\nu}F_{\alpha\beta}\ ;\ \ \overline{F}^{\mu\nu} = \frac{\partial \overline{x}^\mu}{\partial x^\alpha}\frac{\partial \overline{x}^\nu}{\partial x^\beta}F^{\alpha\beta}\ ;\ \ \frac{\partial \overline{x}^\mu}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial \overline{x}^\nu} = \delta^\mu_\nu$$

it's not necessary to do the transformation at all...

This is how the last part should have been done:

$$F \equiv \nabla \wedge A = \boldsymbol{E} + ic\boldsymbol{B}\ ; \quad F^2 = \boldsymbol{E}^2 - c^2\boldsymbol{B}^2 + 2ic\boldsymbol{E}\cdot\boldsymbol{B}$$

Oh dear... This derivation gets the other invariant involving the Hodge dual as well. They should have been asked to form $F^2 = \frac{1}{2}F^{\mu\nu}F_{\nu\mu}$ in order to get the correct sign...

Q3.1. The Earth clock has a gravitational blueshift $GM/Rc^2 - GM/rc^2$ relative to the satellite (the photon drops in the gravitational field), but it is also redshifted due to its transverse motion by $\frac{1}{2}v^2/c^2 = \frac{1}{2}GM/rc^2$. The ratio of rates is thus approximately $1 + GM/Rc^2 - 3GM/2rc^2$ (i.e. the orbiting one goes faster if $r > 3R/2$).

Q3.2. (a) The metric $\mathrm{d}s^2 = y^2\mathrm{d}x^2 + x^2\mathrm{d}y^2$ must come from a simple diffeomorphism of the plane. It took some doing. but the answer is

$$X = \frac{xy}{\sqrt{2}}\sin\left(\log(x/y)\right)\ ;\ \ Y = \frac{xy}{\sqrt{2}}\cos\left(\log(x/y)\right)\ ; \tag{30}$$

The non-zero Christoffel symbols are

$$\Gamma^1_{22} = -x/y^2\ ;\ \ \Gamma^1_{12} = \Gamma^1_{21} = 1/y\ ;\ \ \Gamma^2_{11} = -y/x^2\ ;\Gamma^2_{12} = \Gamma^2_{21} = 1/x$$

but the the Riemann tensor vanishes identically.

The metric $\mathrm{d}s^2 = y\mathrm{d}x^2 + x\mathrm{d}y^2$ has non-zero conection coefficients

$$\Gamma^1_{12} = \Gamma^1_{21} = -\Gamma^1_{22} = \frac{1}{2y}\ ;\ \ \Gamma^2_{12} = \Gamma^2_{21} = -\Gamma^2_{11} = \frac{1}{2x} \tag{31}$$

and non-zero Riemann and Ricci components

$$R^1_{212} = -R^2_{221} = R^2_2 = \frac{x+y}{4y^2x}\ ;\ \ R^1_{112} = -R^1_{121} = R^1_1 = \frac{x+y}{4x^2y}$$

The Ricci scalar is $R = (x+y)/2x^2y^2$. It's not flat...

Q3.3. (a) It's easier in coordinates where $(\Gamma^a{}_{bc})_P = 0$, so that

$$(\ddot{x}^a)_P = 0; \quad (\ddot{\overline{x}}^a)_Q = -\Gamma^a{}_{bc}\dot{x}^b\dot{x}^c \approx -(\partial_d\Gamma^a{}_{bc})_P\xi^d\dot{x}^b\dot{x}^c \tag{32}$$

The intrinsic derivative is then

$$\frac{D^2\xi^a}{Du^2} = \frac{\mathrm{d}}{\mathrm{d}u}\left(\dot{\xi}^a + \Gamma^a{}_{bc}\xi^b\dot{x}^c\right) = \ddot{\xi}^a + (\partial_d\Gamma^a{}_{bc})_P\xi^b\dot{x}^c\dot{x}^d \tag{33}$$

as the other terms are zero $(\ddot{x}^a)_P = 0, (\Gamma^b{}_{bc})_P = 0$. From the above we have $\ddot{\xi}^a + (\partial_d\Gamma^a{}_{bc})_P\xi^d\dot{x}^b\dot{x}^c = 0$ which all reassembles as

$$\frac{D^2\xi^a}{Du^2} + (\partial_b\Gamma^a{}_{cd} - \partial_d\Gamma^a{}_{bc})_P\xi^b\dot{x}^c\dot{x}^d = \frac{D^2\xi^a}{Du^2} + R^a{}_{bcd}\xi^b\dot{x}^c\dot{x}^d \tag{34}$$

(b) In Newtonian gravity we have $\ddot{\boldsymbol{\xi}} = -\boldsymbol{\xi}\cdot\boldsymbol{\nabla\nabla}\Phi$, or $\xi_i = -(\partial_i\partial_j\Phi)\ \xi^j$ in components.

(c) The Newtonian approximation is $g_{00} = 1 + 2\Phi(x, y, z)/c^2$. The Riemann tensor evaluates to $R^i{}_{00j} = -(\partial_i\partial_j\Phi)/c^2$.

(d) (For enthusiasts) A lot of you try this without $(\Gamma^a{}_{bc})_P = 0$. Here is a roadmap for masochists. Start with the geodesic equations at P and Q:

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0; \quad \ddot{\overline{x}}^a + \overline{\Gamma}^a_{bc}\dot{\overline{x}}^b\dot{\overline{x}}^c = 0 \tag{35}$$

Now use $\dot{\overline{x}}^a = \dot{x}^a + \dot{\xi}^a$ and $\overline{\Gamma}^a_{bc} \approx \Gamma^a_{bc} + (\partial_d\Gamma^a_{bc})\xi^d$ to show

$$\ddot{\xi}^a \approx -(\partial_d\Gamma^a_{bc})\xi^d\dot{x}^b\dot{x}^c - 2\Gamma^a_{bc}\dot{\xi}^b\dot{x}^c \tag{36}$$

The intrinsic derivative is

$$\frac{D^2\xi^a}{Du^2} = \frac{\mathrm{d}}{\mathrm{d}u}\left(\dot{\xi}^a + \Gamma^a_{bc}\xi^b\dot{x}^c\right) + \Gamma^a_{de}\dot{x}^e\left(\dot{\xi}^d + \Gamma^d_{bc}\xi^b\dot{x}^c\right) \tag{37}$$

Expanding this we get

$$\frac{D^2\xi^a}{Du^2} = \ddot{\xi}^a + (\partial_d\Gamma^a_{bc})\xi^b\dot{x}^d\dot{x}^c + \Gamma^a_{bc}(2\dot{\xi}^b\dot{x}^c + \xi^b\ddot{x}^c) + \Gamma^a_{de}\Gamma^d_{bc}\xi^b\dot{x}^e\dot{x}^c \tag{38}$$

We eliminate the $\ddot{\xi}^a$ and $\ddot{x}^a$ terms using (36) and (35) above. The $\dot{\xi}\dot{x}$ terms cancel and we reassemble the full Riemann tensor, hence result... OK, it doesn't look difficult when written out like this, but it was extremely tedious.

Q3.4. (a) The Schwarzschild metric ($c = G = 1$ is

$$\mathrm{d}s^2 = \left(1 - \frac{2\mu}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2\mu}{r}\right)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)$$

From the variational principle we find the Euler-Lagrange equations

$$\ddot{t} + \frac{2\mu\dot{t}\dot{r}}{r(r-2\mu)} = 0 \ ; \ \ddot{r} + \frac{\mu(r-2\mu)}{r^3}\dot{t}^2 - \frac{\mu}{r(r-2\mu)}\dot{r}^2 - (r-2\mu)(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) \ ;$$
$$\ddot{\theta} + 2\frac{\dot{r}\dot{\theta}}{r} - \sin\theta\cos\theta\dot{\phi}^2 = 0 \ ; \ \ddot{\phi} + 2\frac{\dot{r}\dot{\phi}}{r} + 2\cot\theta\dot{\theta}\dot{\phi} = 0 \tag{39}$$

from which the non-zero Christofell symbols are

$$\begin{aligned}
&\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{\mu}{r(r-2\mu)}; \Gamma^r_{tt} = \frac{\mu(r-2\mu)}{r^3} \ ; \quad \Gamma^r_{rr} = -\frac{\mu}{r(r-2\mu)} \ ; \\
&\Gamma^r_{\theta\theta} = -(r-2\mu) \ ; \ \Gamma^r_{\phi\phi} = -(r-2\mu)\sin^2\theta \ ; \ \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r} \ ; \\
&\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \ ; \ \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r} \ ; \ \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta
\end{aligned} \tag{40}$$

(b) This example is becoming tutorial, so I'll carry on with the general case, adopting Mike's notation where possible. For the Schwarzschild metric set $G = c = 1, \theta = \pi/2$ and get invariants

$$r^2\dot{\phi} = h \ ; \ k = (1 - 2\mu/r)\dot{t} \ ; \mathcal{E} = \frac{k^2 - \dot{r}^2}{1 - 2\mu/r} - \frac{h^2}{r^2}$$

Where $\mathcal{E}$ can be set to 1 for a material particle or $\mathcal{E} = 0$ for a photon. For a particle the dot can denote differentiation with respect to proper time $\mathrm{d}\tau = \mathrm{d}s$, which fixes the gauge. Manipulate it into the very Newtonian-looking (apart from a factor of 2, which is spurious in GR)

$$\dot{r}^2 + \frac{h^2}{r^2} - \frac{2\mu h^2}{r^3} - \frac{2\mu}{r} = k^2 - 1$$

The effective potential has stationary points at

$$h^2 = \frac{\mu r_*^2}{r_* - 3\mu} \ \Rightarrow r_* = \frac{h^2 \pm h\sqrt{h^2 - 12M^2}}{2M}$$

where the stable circular orbit takes the positive sign. This means that the closest circular orbit is at $h^2 = 12\mu^2$ or 6μ. Substituting back we get

$$k^2 = \frac{(r_* - 2\mu)^2}{r_*(r_* - 3\mu)}$$

(that means k is less than 1, of course). For large h (and the plus sign) we get the Newtonian result $Mr_* = h^2$.

Now let's do the problem by a systematic method that always works. The 4-velocity of the orbiting satellite is $u^\mu = ((1 - 3\mu/r_*)^{-1/2}, 0, \sqrt{M/r_*^3}(1 - 3\mu/r_*)^{-1/2}, 0)$. The 4-velocity of the Earth station is $v^\mu = ((1 - 2\mu/R)^{-1/2}, 0, 0, 0)$. We need a radial photon connecting them: the $k = \dot{t}(1-2\mu/r) = 1$ is arbitrary for a photon, and from the "energy" constraint $\dot{r} = \dot{t}(1-2\mu/r)$ so $k^\mu = k_0((1-2M/r)^{-1}, -1, 0, 0)$. Then the ratio of the frequencies is

$$\frac{\omega_u}{\omega_v} = \frac{u \cdot k}{v \cdot k} = \frac{(1-3\mu/r_*)^{-1/2}}{(1-2\mu/R)^{-1/2}} \tag{41}$$

as required.

While we're at it (and to compare with the Kerr case later), we'll prove the rather Newtonian result

$$\Omega^2 \equiv \left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 = \frac{h^2}{r_*^4}\left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^2 = \frac{h^2(1-2M/r)^2}{r_*^4\gamma^2} = \frac{M}{r_*^3}$$

It's actually quicker to get this from the geodesic equation:

$$\ddot{r} = 0 = -\Gamma^1_{00}\dot{t}^2 - \Gamma^1_{33}\dot{\phi}^2 = -\frac{\mu}{r^2}(1-2\mu/r)\dot{t}^2 + r(1-2\mu/r)\dot{\phi}^2$$

hence result.

Q3.5 (a) Alice's 4-velocity is $v_{\mathrm{A}}^\mu = ((1-2\mu/R)^{-1/2}, 0, 0, 0)$ as above. Bob is on a geodesic with $k = (1-2\mu/R)^{1/2}$, and 4-velocity

$$v_{\mathrm{B}}^\mu = ((1-2\mu/R)^{1/2}(1-2\mu/r)^{-1}, -(2\mu(1/r-1/R))^{1/2}, 0, 0) \tag{42}$$

Radial photon as above gives

$$\frac{\omega_{v_{\mathrm{A}}}}{\omega_{v_{\mathrm{B}}}} = \frac{v_{\mathrm{A}} \cdot k_R}{v_{\mathrm{B}} \cdot k_r} = \frac{(1-2\mu/R)^{-1/2}(1-2\mu/r)}{(1-2\mu/R)^{1/2} + (2\mu(1/r-1/R))^{1/2}} \tag{43}$$

which is infinitely redshifted at the horizon.

(b) The geodesic equation would be $\dot{r} = -\sqrt{k^2 + 2M/r - 1}$. For $k^2 = 0$ this integrates to $\tau = \pi M$ between $r = 2M$ and the origin. Any other value of k is worse because of the length contraction. A Maple plot confirms this. By way of comparison, the observer free-falling from infinity gets there in $4M/3$.

Proof that $k = 0$ geodesic is optimum is that any contribution from the terms $\mathrm{d}t^2(1-2M/r)$ and $-r^2\mathrm{d}\phi^2$ in the metric are negative and will decrease the value of $\mathrm{d}\tau/\mathrm{d}r$. You achieve $k = 0$ by hovering at the event horizon and then falling in...

Q3.6. From our previous results, $\mathcal{E} = 0$ for a photon, so the radial equation is

$$\dot{r}^2 = k^2 - \frac{h^2}{r^2}(1-2\mu/r)$$

We can, w.l.o.g, set $k = 1$, in which case $h = b$, the impact parameter. setting $\dot{r} = 0$ at $r = r_*$, we get

$$b = \sqrt{\frac{r_*^3}{r_* - 2\mu}} \tag{44}$$

which is the required result. For $\mu \ll r_*$ we get an apparent increase of diameter of $2GM/c^2$, or about 3 km for the Sun.

Q3.7. From above we have, for an observer falling radially in from infinity ($k = 1$) $v^\mu = ((1-2\mu/r)^{-1}, -\sqrt{2\mu/r}, 0, 0)$, so that $v_\mu = (1, \sqrt{2\mu/r}(1-2\mu/r)^{-1}, 0, 0)$, hence result $v_\mu = \partial_\mu cT$. The integral can be done, of course:

$$cT = ct + \sqrt{8\mu}\left(\sqrt{r} - \sqrt{2\mu}\tanh^{-1}\left(\sqrt{\frac{2\mu}{r}}\right)\right) \tag{45}$$

Assembling the metric from $\mathrm{d}t^2 = (\mathrm{d}T - \sqrt{2\mu/r}(1-2\mu/r)^{-1}\mathrm{d}r)^2$, we find the Painlevé- Gullstrand line element

$$\mathrm{d}s^2 = \left(1 - \frac{2\mu}{r}\right)\mathrm{d}T^2 - \sqrt{\frac{2\mu}{r}}\mathrm{d}T\mathrm{d}r - \mathrm{d}r^2 - r^2\mathrm{d}\Omega^2 \tag{46}$$

A surface of constant T is spatially flat.

Q3.8. The 4-D Milne metric is

$$\mathrm{d}s^2 = \mathrm{d}t^2 - t^2\left(\mathrm{d}\chi^2 + \sinh^2\chi(\mathrm{d}\theta^2 + \sin^2\theta\ \mathrm{d}\phi^2)\right) \tag{47}$$

looks like a FRW open universe but is, of course, actually flat space (the Riemann tensor is zero). The surfaces of constant t are hyperboloids that cannot correspond to the constant-time surfaces of any observer... (This is in contrast to the Rindler coordinates for an accelerating observer, which are useful.)

The transformation that proves all this is

$$T = t\cosh\chi\ ; \quad R = t\sinh\chi \tag{48}$$

for which $\mathrm{d}s^2 = \mathrm{d}T^2 - \mathrm{d}R^2 - R^2\mathrm{d}\Omega^2$.

Q3.9. The FRW metric can be taken here as

$$\mathrm{d}s^2 = \mathrm{d}t^2 - R(t)^2\left(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\right) \tag{49}$$

All comoving observers have a 4-velocity $(1, 0, 0, 0)$. For radial motion we find $\dot{r} = \lambda R^{-2}$, and a 4-velocity

$$v^\mu = \left(\sqrt{1 + \lambda^2/R^2}, \lambda/R^2, 0, 0\right) \tag{50}$$

We find $\gamma_v = \sqrt{1 + \lambda^2/R^2}$, and β_v from $\beta_v^2 = 1 - 1/\gamma_v^2$. It simplifies so that the momentum is

$$\gamma_v\beta_v = \lambda/R \tag{51}$$

Hence result. Photon is limiting case...

Q3.10. The cosmological equations are

$$\ddot{R} = -\frac{4\pi G}{3}\left(\rho + 3p/c^2\right) R + \tfrac{1}{3}\Lambda c^2 R\ ; \quad \dot{R}^2 = \frac{8\pi G}{3}\rho R^2 + \tfrac{1}{3}\Lambda c^2 R^2 - c^2 k \tag{52}$$

Setting $p = w\rho c^2$ we have, for $\Lambda = 0$ and a static solution $\dot{R} = \ddot{R} = 0$:
$(1+3w)\rho = 0, \quad \rho = \dfrac{3c^2 k}{8\pi G R^2}$, i.e. no solution if $w > 0$.
If $\Lambda \neq 0$ we can find a solution for $\Lambda > 0, k = +1$:

$$\Lambda c^2 = 4\pi G(1+3w)\rho\ ; \quad 4\pi G(1+w)\rho = c^2 k/R^2 \tag{53}$$

Now let $R = R_0 + \delta R$, so that, to first order in δR we have

$$\delta\ddot{R} = -\frac{4\pi G}{3}(1+3w)(\rho_0 \delta R + R_0 \delta\rho) + \tfrac{1}{3}\Lambda c^2 \delta R \tag{54}$$

We now use $\rho \propto R^{-3(1+w)} \Rightarrow \dfrac{\delta\rho}{\rho_0} = -3(1+w)\dfrac{\delta R}{R_0}$ to get

$$\delta\ddot{R} = \left(\frac{4\pi G}{3}(1+3w)(2+3w)\rho_0 + \tfrac{1}{3}\Lambda c^2\right)\delta R = \tfrac{1}{3}(1+w)\Lambda c^2 \delta R \tag{55}$$

The coefficient is positive if $\Lambda > 0$, so the static solution is unstable.