UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Tuesday 01 June 2021 11.00 am to 13.00

PHYSICS (3)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (3)

ADVANCED QUANTUM PHYSICS

Candidates offering this paper should attempt a total of ***five*** *questions:* ***three*** *questions from Section A and* ***two*** *questions from Section B.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains ***six*** *sides, including this coversheet, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS
2 × 20 Page Answer Book
Metric graph paper
Rough workpad
Yellow master coversheet

SPECIAL REQUIREMENTS
Mathematical Formulae handbook
Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

*Attempt **all** questions in this Section. Answers should be concise and relevant formulae may be assumed without proof.*

1 The normalised state of a one dimensional harmonic oscillator of mass m and angular frequency ω is written at $t = 0$ as a linear combination of eigenstates as

$$|\alpha(0)\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle.$$

α is a complex number and $\hat{a}|\alpha(0)\rangle = \alpha|\alpha(0)\rangle$, where $\hat{a}$ is the lowering operator. Show that the expectation value of the displacement varies sinusoidally in time with amplitude $\sqrt{\frac{2\hbar N}{m\omega}}$, where $N = \langle\alpha(0)|\hat{a}^+\hat{a}|\alpha(0)\rangle$. Comment on this result by comparing it to the case of a classical oscillator with energy $E = \hbar\omega N$. [4]

[*The definition of the lowering and raising operators for a 1-D harmonic oscillator might be useful:* $\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \frac{i}{m\omega}\hat{p})$; $\hat{a}^+ = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} - \frac{i}{m\omega}\hat{p})$]

2 In a one-electron atom the electron occupies the p-orbital ($L = 1$) and its eigenstates are $|J, J_z\rangle$, where J quantifies the total angular momentum while J_z its projection along a direction $\hat{z}$. A beam is initially prepared with an equal number of atoms in each of the four degenerate states $|\frac{3}{2}, \frac{3}{2}\rangle$, $|\frac{3}{2}, \frac{1}{2}\rangle$, $|\frac{3}{2}, -\frac{1}{2}\rangle$ and $|\frac{3}{2}, -\frac{3}{2}\rangle$. A static magnetic field B_z is then switched on and the electron's Hamiltonian is approximated by

$$H = \frac{\mu_B B_z}{\hbar}(L_z + 2S_z).$$

Describe how the energy spectrum of the atoms changes and specify the relative number of atoms with each energy. [4]

[*The following relations for the ladder operators might be useful:*

$$L^+|l, m\rangle = \hbar\sqrt{l(l+1) - m(m+1)}|l, m+1\rangle$$

$$L^-|l, m\rangle = \hbar\sqrt{l(l+1) - m(m-1)}|l, m-1\rangle$$
]

3 A particle is in a potential of the form $V = kx^4$, where k is a constant. Use the variational technique to find an upper bound value of the ground state energy within the family of trial functions $f_\alpha(x) \propto e^{-\alpha x^2/2}$. What trial function could you use to estimate the energy of the first excited state? Explain your reasoning. [4]

SECTION B
Attempt two questions from this section

4 The Hamiltonian of an electron in a time-dependent magnetic field $B(t) = (B_x \cos(\omega t), 0, B_z)$ is

$$H = H_0 + V(t) = -\gamma B_z S_z - \gamma B_x S_x \cos(\omega t),$$

where γ is the gyromagnetic ratio. At time $t = 0$, $|\psi(0)\rangle = |\uparrow\rangle$, where $|\uparrow\rangle$ is the eigenstate of S_z with eigenvalue $\frac{\hbar}{2}$.

(a) Show that in the interaction picture $|\psi_I(t)\rangle = e^{\frac{iH_0 t}{\hbar}}|\psi(t)\rangle$ satisfies

$$i\hbar \frac{d}{dt}|\psi_I(t)\rangle = V_I(t)|\psi_I(t)\rangle$$

$$V_I(t) = e^{\frac{iH_0 t}{\hbar}} V(t) e^{-\frac{iH_0 t}{\hbar}}.$$

[2]

(b) Hence show that if $|\psi_I(t)\rangle = c_1(t)|\uparrow\rangle + c_2(t)|\downarrow\rangle$ the coefficients $c_1(t)$ and $c_2(t)$ satisfy

$$\begin{pmatrix} \frac{dc_1(t)}{dt} \\ \frac{dc_2(t)}{dt} \end{pmatrix} = \frac{i\gamma B_x}{4} \begin{pmatrix} 0 & e^{i\delta t} \\ e^{-i\delta t} & 0 \end{pmatrix} \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix},$$

where $\delta = \omega - \gamma B_z$ is small (close to resonance condition). [5]

A solution for this system of differential equations is

$$c_1(t) = e^{\frac{i\delta t}{2}} \left[\cos\left(\frac{\omega_R t}{2}\right) - \frac{i\delta}{\omega_R} \sin\left(\frac{\omega_R t}{2}\right) \right]$$

$$c_2(t) = \frac{i\gamma B_x}{2\omega_R} e^{-\frac{i\delta t}{2}} \sin\left(\frac{\omega_R t}{2}\right),$$

where $\omega_R = \sqrt{\delta^2 + \left(\frac{\gamma B_x}{2}\right)^2}$.

(c) At what time is the probability of finding an electron in the state $|\downarrow\rangle$ maximal? Plot this maximum probability as a function of ω and show that $\frac{\omega_{res}}{\triangle\omega} \propto \frac{B_z}{B_x}$, where ω_{res} and $\triangle\omega$ are respectively the resonance frequency and resonance width for the transition $|\uparrow\rangle \rightarrow |\downarrow\rangle$. [5]

(d) Calculate $\langle\psi(t)|\mathbf{S}|\psi(t)\rangle$ at resonance ($\delta = 0$) and describe the motion of the spin in space for $B_x \ll B_z$ and $B_z \ll B_x$. [7]

5 A spinless particle of mass m and charge $-e$ interacts with a monochromatic radiation field described by the vector potential $\mathbf{A}(\mathbf{r}, t)$. Its Hamiltonian is

$$H = \frac{1}{2m}\left[\mathbf{p} + e\mathbf{A}(\mathbf{r}, t)\right]^2 + V(\mathbf{r}).$$

(a) Show that in the Coulomb gauge ($\nabla \cdot \mathbf{A} = 0$) this can be simplified at first order in the vector potential to $H = H_0 + H_1(t)$, where $H_1(t) = \frac{e}{m}\mathbf{p} \cdot \mathbf{A}(\mathbf{r}, t)$.

[3]

If the particle is initially in the ground state of H_0, $\psi(0) = |0\rangle$, at time t the state can be written in terms of the unperturbed eigenstates $|n\rangle$ and unperturbed energy levels E_n as

$$\psi(t) = e^{-\frac{iE_0 t}{\hbar}}|0\rangle + \sum_{n\neq 0} e^{-\frac{iE_n t}{\hbar}}\left(\frac{1}{i\hbar}\int_0^t e^{\frac{i(E_n - E_0)t'}{\hbar}}\langle n|H_1(t')|0\rangle dt'\right)|n\rangle.$$

(b) Determine which transitions are allowed at $t > 0$ and explicitly calculate the coefficients in the above expansion in the case in which

$$H_0 = \frac{p^2}{2m} + \frac{m\omega_0^2}{2}z^2$$

$$\mathbf{A}(\mathbf{r}, t) = \begin{cases} 2A_0\hat{\mathbf{z}}\cos(ky - \omega t) & \text{at} \quad t > 0 \\ 0 & \text{at} \quad t \leq 0. \end{cases}$$

[6]

(c) Show that the induced dipole moment $\mathbf{P}$ is

$$\mathbf{P} = -\frac{e^2 A_0}{m}\hat{\mathbf{z}}\mathrm{Re}\left[e^{iky}\left(\frac{e^{-i\omega t} - e^{-i\omega_0 t}}{i(\omega_0 - \omega)} - \frac{e^{-i\omega t} - e^{i\omega_0 t}}{i(\omega_0 + \omega)}\right)\right].$$

[6]

(d) Now show that the term oscillating at ω, $\mathbf{P}_\omega$, is proportional to the electric field component of the radiation $\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}$ as

$$\mathbf{P}_\omega = \frac{e^2}{m}\frac{\mathbf{E}}{\omega_0^2 - \omega^2}.$$

[4]

[*The definition of the lowering and raising operators for a 1-D harmonic oscillator might be useful:* $\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \frac{i}{m\omega}\hat{p})$; $\hat{a}^{+} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} - \frac{i}{m\omega}\hat{p})$]

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6 Two identical particles of spin 1/2 and mass m are confined to move in a 1-D box such that $V = 0$ for $|x| < a/2$ and $V = \infty$ for $|x| \geq a/2$. Their spins interact via an exchange term such that the total Hamiltonian is

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - \lambda \mathbf{S}_1 \cdot \mathbf{S}_2,$$

where p_i and $\mathbf{S}_i$ are respectively the momentum and spin of particle i.

(a) Explain why the eigenfunctions can be labelled as $|n_1, n_2, S, S_z\rangle$, where n_1 and n_2 label the spin-independent part of the wavefunction, while S and S_z represent the total spin and its component along the $\hat{z}$ direction.

[2]

(b) Calculate the energy of the three lowest levels in the case in which $\lambda \ll \frac{\pi^2}{ma^2}$. For each level specify its degeneracy and write down the corresponding eigenfunctions.

[6]

(c) If we now include a small correction to the Hamiltonian of the form $\alpha x_1 \cdot x_2$, where x_1 and x_2 are the coordinates of the two particles, use perturbation theory to determine the first order correction to the energy of the three lowest levels.

[5]

(d) Repeat parts (b)-(c) in the case in which the particles were distinguishable and show that the first order correction to the energy levels is zero in this case.

[6]

END OF PAPER