UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Friday 30 May 2014 9.00 am to 11.00 am

PHYSICS (5)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (5)

ASTROPHYSICAL FLUID DYNAMICS

Candidates offering this paper should attempt a total of **three** *questions. The questions to be attempted are* **1**, **2** *and* **one** *other question.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains **four** *sides, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS
2×20 Page Answer Book
Rough workpad
Yellow master coversheet

SPECIAL REQUIREMENTS
Mathematical Formulae handbook
Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

ASTROPHYSICAL FLUID DYNAMICS

1 *Answer* **all** *parts of this question. Answers should be concise and relevant formulae may be assumed without proof.*

(a) Analysis of the Lane–Emden equations for stars which have a polytropic equation of state of the form $p = K\rho^{1+1/n}$ leads to the following scaling relations for the stellar mass and radius

$$M \propto \rho_c^{\frac{1}{2}(3/n-1)} K^{\frac{3}{2}}$$
$$R \propto \rho_c^{\frac{1}{2}(1/n-1)} K^{\frac{1}{2}}$$

where ρ_c is the central density. Assuming that stellar nuclear reactions in stars ensure that the central temperature is the same in all stars, find the mass–radius relation. [4]

(b) High pressure monatomic gas, with sound speed c_s, is initially at rest and escapes from the surface of an asteroid of mass M and radius r. Find an expression for the maximum speed attained by the gas at large distances from the asteroid. [4]

(c) A supernova injects 10^{44} J into the interstellar medium which has a density 10^6 hydrogen atoms per cubic metre. Assuming the initial mass of the explosion can be ignored, estimate the radius of the blast wave one thousand years later. [4]

2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) stellar winds;

(b) viscous accretion discs;

(c) the thermal and Rayleigh–Taylor instabilities in a gas.

3 *Attempt* **either** *this question* **or** *question 4.*

The equations governing the behaviour of a self-gravitating fluid are:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) &= 0, \\ \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} &= -\frac{1}{\rho} \boldsymbol{\nabla} p - \boldsymbol{\nabla} \Psi, \\ \nabla^2 \Psi &= 4\pi G \rho, \end{aligned}$$

where $\boldsymbol{u}$, p and ρ are the velocity, pressure and density fields respectively and Ψ is the gravitational potential. The Jeans analysis for the gravitational stability of a fluid considers a small gravitational perturbation to an infinite, static uniform medium of the form $p = p_0 + p_1$, $\rho = \rho_0 + \rho_1$, $\Psi = \Psi_0 + \Psi_1$ and $\boldsymbol{u} = \boldsymbol{u}_1$, where p_0, ρ_0 are the pressure and density fields and Ψ_0 the gravitational potential of the unperturbed medium. Explain why in this case $p_1 = c_{\rm s}^2 \rho_1$ where $c_{\rm s}$ is the sound speed, and why $c_{\rm s}$ may be assumed constant. [2]

By assuming wave-like solutions of the form $\rho_1 = \bar{\rho} e^{i(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)}$ for the perturbed density and similarly for all perturbed quantities, show that the dispersion relation for the growth of instabilities in the gas has the form

$$\omega^2 = c_{\rm s}^2 (k^2 - k_{\rm J}^2)$$

and find an expression for the Jeans wave number $k_{\rm J}$. Sketch the dispersion relation. [7]

Explain the criteria for growing unstable modes, and hence discuss the significance of the Jeans wavelength and Jeans mass. Find also an expression for the Jeans mass $M_{\rm J}$. [3]

An initially stationary cloud of gas with a uniform initial density ρ_0, radius r_0 and negligible pressure is unstable to gravitational collapse. By considering the forces on a test particle at the edge of the cloud, and assuming the pressure in the gas is negligible throughout collapse, show that this test particle will have a speed at a radius r as the cloud collapses given by $\dot{r} = -\sqrt{2GM}\,(1/r - 1/r_0)^{1/2}$ where $M = \frac{4}{3}\pi r_0^3 \rho_0$. By making the substitution $r = r_0 \sin^2\theta$ or otherwise show that the time taken for the cloud to collapse, the so-called free-fall time $t_{\rm ff} = (3\pi/32 G\rho_0)^{1/2}$. [5]

For a cloud of non-negligible pressure we still define the free-fall time in the same way. Find the mass of an initially uniform spherical cloud of radius r_0 and sound speed $c_{\rm s}$ for which the free fall-time and sound-crossing time ($r_0/c_{\rm s}$) are equal. Give your answer in terms of ρ_0 and $c_{\rm s}$. [2]

A cloud is predominantly supported against collapse by its internal magnetic field B. By assuming that disturbances in the cloud travel at the Alfvén speed $v_{\rm A} = B/\sqrt{\mu_0 \rho_0}$, find an expression for the magnetic Jeans mass. Calculate the number of Jeans masses or magnetic Jeans masses contained by collapsing isothermal clouds which are initially supported by thermal pressure and magnetic fields, respectively. Hence describe qualitatively the difference in the cloud collapse processes in these two cases. [6]

(TURN OVER

4 *Attempt* **either** *this question* **or** *question 3.*

Define the Mach number M and using the concept of the Mach cone discuss the origin of shocks in supersonic gas flow. [4]

Derive the Rankine–Hugoniot conditions at a normal adiabatic shock front

$$\begin{aligned}
\rho_2 u_2 &= \rho_1 u_1, \\
p_2 + \rho_2 u_2^2 &= p_1 + \rho_1 u_1^2, \\
\frac{\gamma}{\gamma - 1}\frac{p_2}{\rho_2} + \frac{1}{2}u_2^2 &= \frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} + \frac{1}{2}u_1^2,
\end{aligned}$$

where ρ is the density, p the pressure, γ the adiabatic index and u the normal velocity. The subscripts 1 and 2 refer to the upstream and downstream conditions respectively. [5]

The limit of a strong shock is when the upstream Mach number $M_1 \gg 1$. Explain why this implies $\rho_1 u_1^2 \gg p_1$ and hence show that in this limit

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{\gamma + 1}{\gamma - 1}.$$ [4]

By considering the ratio p_2/p_1 or otherwise show that for a strong shock

$$\frac{T_2}{T_1} = 2\gamma \frac{(\gamma - 1)}{(\gamma + 1)^2} M_1^2.$$ [3]

The entropy per unit volume of a gas obeying the ideal equation of state can be written as $s = c_V \ln\left(\kappa p/\rho^\gamma\right)$, where κ is a constant and c_V is the heat capacity at constant volume. Find an expression for the change in entropy per unit volume of gas across a strong shock and comment on your result. [2]

Explain briefly why in the downstream gas cooling is likely to be more efficient than in the upstream gas and sketch the expected variation of T_2/T_1 through the shock identifying important regimes and length scales. [2]

For an isothermal shock the final gas temperature of the downstream gas returns to the upstream value. Find an expression for the downstream density in this case. Hence sketch the variation of the entropy through the shock. [5]

END OF PAPER