

NATURAL SCIENCES TRIPOS Part II

May–June 2020 **1 hour 15 minutes**

PHYSICS (2)

PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (2)

RELATIVITY

*Candidates offering this paper should attempt a total of **four** questions: **three** questions from Section A and **one** question from Section B.*

*The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains **four** sides, including this coversheet. You may use the formula handbook for values of constants and mathematical formulae, which you may quote without proof.*

You have 75 minutes (plus any pre-agreed individual adjustment) to answer this paper. Do not start to read the questions on the subsequent pages of this question paper until the start of the time period.

Please treat this as a closed-book exam and write your answers within the time period. Downloading and uploading times should not be included in the allocated exam time. If you wish to print out the paper, do so in advance. You can pause your work on the exam in case of an external distraction, or delay uploading your work in case of technical problems.

Section A and the chosen section B question should be uploaded as separate pdfs. Please name the files 1234X_Qi.pdf, where 1234X is your examination code and i is the number of the question/section (A or 4 or 5).

STATIONERY REQUIREMENTS

Master coversheet

SPECIAL REQUIREMENTS

Mathematical Formulae handbook
Approved calculator allowed

SECTION A

Attempt **all** questions in this Section. Answers should be concise and relevant formulae may be assumed without proof.

1 An observer detects light from a source moving at speed v . In the frame of the source, there is an angle θ between the direction of motion and rays reaching the observer. What angle does the observer measure between the direction of motion and the light rays? [4]

2 General relativity predicts that the centre of the Earth is younger than its surface. Assuming a uniform distribution of mass, estimate the size of this effect. The radius of the Earth is $R = 6371$ km and its age is $T = 4.54 \times 10^9$ years (ignore rotation of the Earth). [4]
[In the weak-field limit, $g_{00} \approx 1 + \frac{2\Phi}{c^2}$, where Φ is the gravitational potential.]

3 The Schwarzschild metric is

$$ds^2 = c^2 \left(1 - \frac{2\mu}{r}\right) dt^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2 - r^2 d\Omega^2$$

where $\mu = GM/c^2$ and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. Show that the geodesic equation for the radial coordinate implies that, for circular orbits, the square of the period is proportional to the cube of the radius.

SECTION B

Attempt one question from this section

- 4 (a) State the symmetries of the Riemann curvature tensor R_{abcd} and show that there are six independent components in three dimensions. [6]
- (b) In three dimensions,

$$R_{abcd} = 2(g_{b[c}R_{d]a} - g_{a[c}R_{d]b}) + Rg_{a[c}g_{d]b} ,$$

where $R_{ab} = R_{cab}{}^c$ is the Ricci tensor and $R = R_a{}^a$ is the scalar curvature. Show that this R_{abcd} satisfies the required symmetries, and how this fixes the form uniquely. [4]

- (c) Find the metric of a conical surface with opening angle Θ in terms of the angular coordinate ϕ and distance from the tip r . Show that this metric coincides with that of the plane written in terms of polar coordinates (r, ϕ') , where ϕ' has period different from 2π . Find the period in terms of Θ . [3]

- (d) Because $R_{abcd} = 0$ in free space in two space and one time dimensions, the most general static metric describing a point mass at the origin has the form

$$ds^2 = c^2 dt^2 - \{\text{conical metric}\}$$

parameterised by Θ . Describe the orbits of a test particle. [3]

- (e) The Einstein equations are

$$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R = -\kappa T_{ab}$$

for some constant κ . The static spherically symmetric metric

$$ds^2 = c^2 dt^2 - r_0^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

has $R_{\theta\phi\theta\phi} = r_0^2 \sin^2 \theta$, with the remaining components not related by symmetry equal to zero. Find the form of T_{ab} that gives rise to this metric. [3]

(TURN OVER)

- 5 (a) By considering the geodesic equation for the tangent vector t^a in the form

$$\frac{dt_a}{du} = \frac{1}{2}(\partial_a g_{bc})t^b t^c,$$

show that g_{ab} and $\tilde{g}_{ab} = g_{ab}/A$ have the same null geodesics for any function of spacetime $A(x)$. [4]

- (b) Find the differential equations obeyed by geodesics of a metric of the form

$$ds^2 = c^2 dt^2 - n(\mathbf{x})^2 d\ell^2, \quad (\star)$$

where $d\ell^2 = dx^2 + dy^2 + dz^2$. [5]

- (c) As a function of arc length ℓ , the curvature $k(\ell)$ of a planar curve $\mathbf{x}(\ell)$ is defined in terms of the unit tangent vector $\mathbf{T}(\ell) = d\mathbf{x}/d\ell$ and unit normal vector $\mathbf{N}(\ell)$ by

$$\frac{d\mathbf{T}}{d\ell} = k(\ell)\mathbf{N}(\ell).$$

(Arc length and unit vectors are defined with respect to the Euclidean metric.) Show, for a light ray, that $k(\ell) = \mathbf{N} \cdot \nabla \log n$ for a metric of the form \star . [6]

[$ds^2 = 0$ so $d\ell/dt = c/n(\mathbf{x})$.]

There is a coordinate system in which the Schwarzschild metric takes the form

$$ds^2 = c^2 \left(\frac{1 - \frac{\mu}{2r}}{1 + \frac{\mu}{2r}} \right)^2 dt^2 - \left(1 + \frac{\mu}{2r} \right)^4 d\ell^2$$

where $\mu = GM/c^2$. By the result of part (a), this has the same null geodesics as a metric of the form (\star) .

- (d) The curvature of a curve can also be expressed as $k(\ell) = d\phi/d\ell$, where ϕ is the angle to a fixed axis. By integrating $k(\ell)$ along a straight light ray that passes at a distance R from a point mass M , obtain an approximate expression for the angle of deflection $\Delta\phi$ of the ray. Work to first order in M . [4]

END OF PAPER