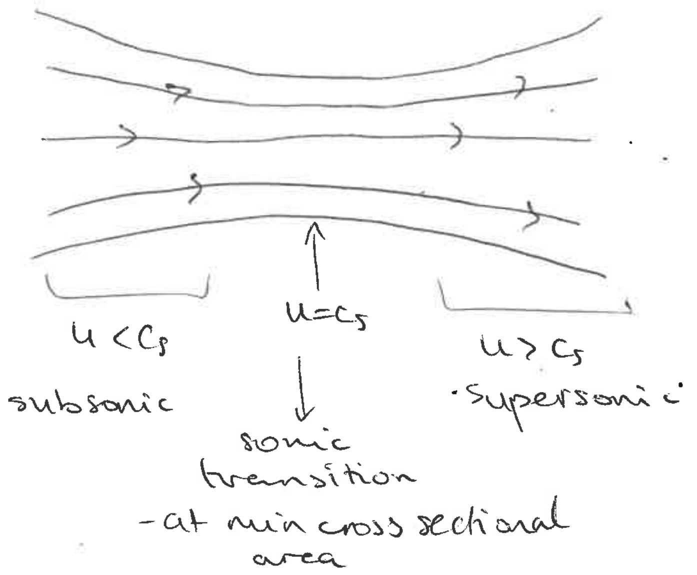
AFD 2011

20

1) a) Streamlines through a de Laval nozzle



b) $j = \sigma(E + u \times B)$

$\nabla \times B = \mu_0 j = \mu_0 \sigma (E + u \times B)$

$\nabla \times (\nabla \times B) = \nabla(\nabla \cdot B) - \nabla^2 B = \mu_0 \sigma (\nabla \times E + \nabla \times (u \times B))$

Maxwell $- \nabla \cdot B = 0$, $\nabla \times E = -\frac{\partial B}{\partial t}$

$-\nabla^2 B = \mu_0 \sigma \left(-\frac{\partial B}{\partial t} + \nabla \times (u \times B)\right)$

$$\frac{\partial B}{\partial t} = \nabla \times (u \times B) + \frac{1}{\mu_0 \sigma} \nabla^2 B$$

c) laminar flow between infinite parallel plates

$\nabla^2 u = -\frac{1}{\eta} p'$ $\qquad$ $u = 0$ at $y = \pm d/2$, $\frac{\partial u}{\partial y} = 0$ at $y = 0$

$$\frac{\partial^2 u_x}{\partial y^2} = -\frac{p'}{\eta}$$

$$\frac{\partial u_x}{\partial y} = -\frac{p'}{\eta} y + c \quad , \quad c = 0$$

$$u_x = -\frac{p'}{2\eta} y^2 + c \quad , \quad c = \frac{p'}{8\eta} d^2$$

$$u_x = \frac{p'}{8\eta}(d^2 - 4y^2)$$

max u_x at $y = 0 \Rightarrow u_{max} = \frac{p' d^2}{8\eta}$

3) $$\frac{\partial u}{\partial t} + (u\cdot\nabla)u = -\frac{1}{\rho}\nabla p - \nabla\phi$$

show that the quantity $H = \frac{1}{2}u^2 + \int\frac{dp}{\rho} + \phi$

is constant along a streamline

$$\frac{d}{dx}\int\frac{dp}{\rho} = \frac{dp}{dx}\frac{d}{dp}\int\frac{dp}{\rho} = \frac{1}{\rho}\frac{dp}{dx} \Rightarrow \nabla\int\frac{dp}{\rho} = \frac{1}{\rho}\nabla p$$

$$(u\cdot\nabla)u = \nabla(1/2\,u^2) - u\times(\nabla\times u)$$

steady state - $\frac{\partial u}{\partial t} = 0$

$$\nabla(1/2\,u^2) - u\times(\nabla\times u) + \nabla\int\frac{dp}{\rho} + \nabla\phi = 0$$

dot product with u:

$$u\cdot\nabla\left(\frac{1}{2}u^2 + \int\frac{dp}{\rho} + \phi\right) - u\cdot[u\times\cancelto{0}{(\nabla\times u)}] = 0$$

$$u\cdot\nabla\left(\frac{1}{2}u^2 + \int\frac{dp}{\rho} + \phi\right) = 0$$

$$\frac{1}{2}u^2 + \int\frac{dp}{\rho} + \phi = \text{const.}$$ along streamline

Spherically symmetric accretion onto star of mass M

steady - $\frac{d\ln\dot{M}}{dr} = 0$

Show that $(u^2 - c_s^2)\frac{d\ln u}{dr} = \frac{2c_s^2}{r}\left(1 - \frac{GM}{2c_s^2 r}\right)$

$$u\cdot\nabla u = -\frac{1}{\rho}\frac{dp}{d\rho}\nabla\rho - \nabla\phi$$

$$\frac{dp}{d\rho} = c_s^2, \quad \phi = -\frac{GM}{r} \Rightarrow \nabla\phi = \frac{GM}{r^2}$$

$$u\frac{du}{dr} = -c_s^2\frac{1}{\rho}\nabla\rho - \frac{GM}{r^2}$$

$$u^2\frac{d\ln u}{dr} = -c_s^2\frac{d\ln\rho}{dr} - \frac{GM}{r^2}$$ - spherical symmetry

$$\frac{d\ln\dot{m}}{dr} = 0 = \frac{d\ln u}{dr} + \frac{d\ln\rho}{dr} + \frac{2}{r}$$

$$u^2\frac{d\ln u}{dr} = c_s^2\left(\frac{d\ln u}{dr} + \frac{2}{r}\right) - \frac{GM}{r^2}$$

$$(u^2 - c_s^2)\frac{d\ln u}{dr} = \frac{2c_s^2}{r}\left(1 - \frac{GM}{2c_s^2 r}\right)$$

sonic radius - at $u = c_s$ - $r_s = \frac{GM}{2c_s^2}$

isothermal accretion - show that $u^2 + 2c_s^2 \ln\frac{\rho_\infty}{\rho} + \frac{2GM}{r}$

Bernoulli: $H = \frac{1}{2}u^2 + \int\frac{dp}{\rho} + \phi = \text{const.}$

$$\frac{1}{2}u^2 + c_s^2 \ln\rho - \frac{GM}{r} = \frac{1}{2}c_s^2 + c_s^2 \ln\rho_s - \underbrace{\frac{GM}{r_s}}_{2c_s^2}$$

$$\frac{1}{2}u^2 + c_s^2 \ln\rho/\rho_s = -3/2\, c_s^2 + \frac{GM}{r}.$$

$$u^2 = 2c_s^2 (\ln\rho/\rho_s - 3/2) + \frac{2GM}{r}$$

at ∞, $u \to 0$, $\ln\rho_s/\rho_\infty = 3/2$

$$\rho_\infty = \rho_s e^{-3/2} \quad ; \rho_s = \rho_\infty e^{3/2}$$

$$u^2 = 2c_s^2 \left(\ln\frac{\rho_\infty e^{3/2}}{\rho} - 3/2\right) + \frac{2GM}{r}$$

$$= 2c_s^2 \ln\rho_\infty/\rho + \frac{2GM}{r}$$

Find ρ at sonic radius

$$\rho_s = \rho_\infty e^{3/2}$$

$$\dot{m} = 4\pi r_s^2 \rho_s c_s = 4\pi\left(\frac{GM}{2c_s^2}\right)^2 \rho_\infty e^{3/2} c_s = \frac{\pi G^2 M^2 \rho_\infty e^{3/2}}{c_s^3}$$

for $M = M_\odot$, $\rho_\infty = 10^6 m_H\, m^{-3}$, $T = 200K$

$$c_s^2 = \frac{R_* T}{\mu} = 8300 \times 200 \quad (\mu = 1) \quad = 1.66 \times 10^6 (m s^{-1})^2$$

$$r_s = \frac{GM_\odot}{2c_s^2} = 4.02 \times 10^{13}$$

$$\dot{m} = \frac{\pi (GM_\odot)^2 . 10^6 m_H e^{3/2}}{c_s^3} = 1.96 \times 10^{11} \text{ kg s}^{-1}$$

If gas is in free fall, $v_f^2 = \frac{2GM}{r}$

loses KE $\frac{1}{2}v_f^2 - \frac{1}{2}c_s^2$ at sonic radius $= \frac{3}{2}c_s^2$

$$L = \frac{dE}{dt} = \frac{dE}{dm}\dot{m} = 4.9 \times 10^{17} \text{ W}$$

4.) $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0$

$\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\frac{1}{\rho} \nabla p - \nabla \phi$

$\nabla^2 \phi = 4\pi G \rho$

hydrostatic equilibrium - $\frac{\partial u}{\partial t}, (u \cdot \nabla) u = 0$ ($u = 0$) $\Rightarrow \frac{1}{\rho} \nabla p = -\nabla \phi$

infinite, static, uniform medium

$\nabla p = 0 \xrightarrow{\text{hydrostatic equilibrium}} \nabla \phi = 0 \Rightarrow \nabla^2 \phi = 0 \xrightarrow{\text{Poisson}} \rho = 0$ - no mass

avoid problem - ignore and introduce small perturbations in p, ρ, ϕ, u - formulate governing equations in terms of these perturbations

$\frac{\partial}{\partial t}(\rho_0 + \rho_1) + \nabla \cdot ((\rho_0 + \rho_1) u_1) = 0$

$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot u_1 = 0$ ignoring 2nd order term

$\frac{\partial u_1}{\partial t} + (u_1 \cdot \nabla) u_1 = -\frac{1}{\rho_0} \frac{dp}{d\rho} \nabla(\rho_0 + \rho_1) - \nabla(\phi_0 + \phi_1)$

$\frac{\partial u_1}{\partial t} = -\frac{c_s^2}{\rho_0} \nabla \rho_1 - \nabla \phi_1$

$\nabla^2 \phi_1 = 4\pi G \rho_1$

assuming wavelike solutions of the form $\rho_1 = \rho_{1,0} e^{i(k \cdot x - \omega t)}$

$-i\omega \rho_1 + \rho_0 i k u_1 = 0 \Rightarrow \omega \rho_1 = \rho_0 k u_1$

$-i\omega u_1 = -\frac{c_s^2}{\rho_0} i k \rho_1 - i k \phi_1 \Rightarrow \omega u_1 = k c_s^2 \frac{\rho_1}{\rho_0} + k \phi_1$

$-k^2 \phi_1 = 4\pi G \rho_1$

$\phi_1 = \frac{-4\pi G \rho_1}{k^2} = -\frac{1}{k}\left(k c_s^2 \frac{\rho_1}{\rho_0} - \omega u_1\right)$

$\frac{4\pi G \rho_1}{k^2} = \frac{c_s^2 \rho_1}{\rho_0} - \frac{\omega u_1}{k} = \frac{c_s^2 \rho_1}{\rho_0} - \frac{\omega}{k} \frac{\omega \rho_1}{\rho_0 k}$

$\frac{4\pi G}{k^2} - \frac{c_s^2}{\rho_0} = -\frac{\omega^2}{k^2} \frac{1}{\rho_0}$

$\omega^2 = c_s^2 k^2 - \frac{4\pi G}{\rho_0} = c_s^2 (k^2 - k_J^2), \quad k_J^2 = \frac{4\pi G}{c_s^2 \rho_0}$

criterion for growing unstable mode which leads to collapse

for $k^2 < k_J^2$, ω is purely imaginary

$\omega = i\tilde{\omega}$ $\Rightarrow$ $\rho \propto e^{\tilde{\omega}t} e^{ik\cdot x}$

↳ density perturbations grow with time ⇒ gravitational collapse

criterion $k^2 < \frac{4\pi G}{c_s^2 \rho_0}$ ...

modes with smallest k grow most quickly

- ~~$\tilde{\omega}^2 =$~~ let $\tilde{\omega} = \sqrt{k_J^2 - k^2}$ - largest ω → fastest growth if k is smaller.

Infinite disc with surface density σ_0

$$\omega^2 = c_s^2\left(k^2 - \frac{2\pi G\sigma_0 |k|}{c_s^2}\right)$$

unstable modes for imaginary ω

- $\omega^2 < 0$ - $\frac{2\pi G\sigma_0 |k|}{c_s^2} > k^2$

$k < \frac{2\pi G\sigma_0}{c_s^2}$

fastest growing instabilitees:

$$\tilde{\omega}^2 = \frac{2\pi G\sigma_0 |k|}{c_s^2} - k^2$$

$\frac{\partial \tilde{\omega}}{\partial k} = 0$ for fastest growth

$$\frac{\partial(\tilde{\omega}^2)}{\partial k} = \frac{2\pi G\sigma_0}{c_s^2} - 2k = 0$$

$k = \frac{\pi G\sigma_0}{c_s^2}$ for fastest growing modes

when cloud collapses, λ_J decreases → k increases

σ_0 increases - instabilities grow more quickly