Part-II Physics/Astrophysics, Michaelmas Term 2021 Anastasia Fialkov

Relativity: Example Sheet 2

1. In 3D Euclidean space, coordinates x'^a are related to Cartesian coordinates x^a by

$$x^1 = x'^1 + x'^2 , \qquad x^2 = x'^1 - x'^2 , \qquad x^3 = 2x'^1 x'^2 + x'^3 .$$

(a) Express the coordinate basis vectors $\boldsymbol{e}'_a \equiv \partial/\partial x'^a$ for the primed coordinates in terms of those for the Cartesian coordinates. How are these related to the intersections of the coordinate surfaces that you sketched in Question 9 of Examples Sheet 1? By considering $\boldsymbol{g}(\boldsymbol{e}'_a, \boldsymbol{e}'_b)$ obtain the components of the metric g'_{ab}. (*Hint: since the original coordinates are Cartesian,* $\boldsymbol{g}(\boldsymbol{e}_a, \boldsymbol{e}_b) = \delta_{ab}$.) (b) Let the vector $\boldsymbol{v} \equiv \boldsymbol{e}_1$. Write down the components v^a and those of the associated dual vector v_a. Calculate the components of the same vector $\boldsymbol{v}$ and its associated dual vector in the primed coordinates.

2. (a) If the tensor A_{ab} is an antisymmetric tensor, S_{ab} is a symmetric tensor and T_{ab} is a general tensor, show that $A^{ab}T_{ab} = A^{ab}T_{[ab]}$ and $S^{ab}T_{ab} = S^{ab}T_{(ab)}$. (b) If v_a are the components of a dual vector, show that in an *arbitrary* coordinate system $A_{ab} = \partial_b v_a - \partial_a v_b$ are the components of a type-$(0,2)$ tensor. Show further, for a general antisymmetric tensor A_{ab}, that $B_{abc} = \partial_c A_{ab} + \partial_a A_{bc} + \partial_b A_{ca}$ are the components of a type-$(0,3)$ tensor. What are the symmetry properties of B_{abc}?

3. (a) If $g = \det(g_{ab})$ is the determinant of the metric, show that $\partial_c g = g g^{ab}(\partial_c g_{ab})$. (b) Verify directly, in a general coordinate system, that $\nabla_c g_{ab} = 0$ for the covariant derivative constructed with the metric connection. (c) For a diagonal metric g_{ab}, show that the connection coefficients are given by (*with $a \neq b \neq c$ and no summation over repeated indices*)

$$\Gamma^a_{bc} = 0 , \qquad \Gamma^b_{aa} = -\frac{1}{2g_{bb}}\frac{\partial g_{aa}}{\partial x^b} , \qquad \Gamma^a_{ba} = \Gamma^a_{ab} = \frac{\partial}{\partial x^b}\left(\ln\sqrt{|g_{aa}|}\right) .$$

4. In 2D Euclidean space, the line element in plane-polar coordinates is

$$ds^2 = d\rho^2 + \rho^2 d\phi^2 .$$

(a) Obtain the non-zero connection coefficients

$$\Gamma^\phi_{\rho\phi} = \Gamma^\phi_{\phi\rho} = 1/\rho , \qquad \Gamma^\rho_{\phi\phi} = -\rho .$$

(b) If the coordinate components v^a of a vector $\boldsymbol{v}$ are written as v^ρ and v^ϕ, show that the divergence of $\boldsymbol{v}$ is

$$\nabla_a v^a = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho v^\rho) + \frac{\partial v^\phi}{\partial\phi} .$$

What would be the equivalent result in terms of the components of $\boldsymbol{v}$ in an orthonormal basis aligned with the coordinate directions?

(c) Show that the Laplacian of a scalar field f is

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2} .$$

5. On the surface of a unit sphere $ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$. (a) Calculate the connection coefficients in the (θ, ϕ) coordinate system directly from the metric. (b) By considering the 'Lagrangian' $L = g_{ab}\dot{x}^a\dot{x}^b$, derive the equations for an affinely-parameterised geodesic on the surface of a sphere in the coordinates (θ, ϕ) and thereby verify your answer to (a). Hence show that, of all the circles of constant latitude on a sphere, only the equator is a geodesic. (c) A vector $\boldsymbol{v}$ of unit length is defined at the point $(\theta_0, 0)$ and is parallel to the circle $\phi = 0$. Calculate the components of $\boldsymbol{v}$ after it has been parallel transported around the circle $\theta = \theta_0$. Hence show that, in general, after parallel transport, the direction of $\boldsymbol{v}$ is different, but its length is unchanged.

6. A hypersurface $\mathcal{H}$ within a manifold $\mathcal{M}$ contains a non-null curve $\mathcal{C}$. Give a geometric argument showing that if $\mathcal{C}$ is a geodesic in $\mathcal{M}$, it is also a geodesic in $\mathcal{H}$. Give an example to show that the converse is not necessarily true.

7. (*Optional: for enthusiasts.*) A surface $\mathcal{H}$ of M dimensions is embedded in ND Euclidean space ($N > M$). The surface is specified in terms of coordinates u^I ($I = 1, \ldots, M$) in the surface by the N functions $x^a(u)$, where x^a are Cartesian coordinates in the embedding space. (a) Show, by considering $ds^2 = \delta_{ab}dx^a dx^b$, that the metric induced on $\mathcal{H}$ is given

$$g_{IJ} = \delta_{ab}\frac{\partial x^a}{\partial u^I}\frac{\partial x^b}{\partial u^J},$$

where implicit summation over repeated indices should be assumed throughout.

(b) Show that the metric connection on $\mathcal{H}$ satisfies

$$g_{IL}\Gamma^L_{JK} = \delta_{ab}\frac{\partial x^a}{\partial u^I}\frac{\partial^2 x^b}{\partial u^J \partial u^K}.$$

(c) A vector $\boldsymbol{A}$ lies in the tangent space to $\mathcal{H}$ at some point P. By considering the relation between the coordinate basis vectors $\partial/\partial u^I$ in $\mathcal{H}$ and those in the embedding space, $\partial/\partial x^a$, show that the coordinate components A^I and A^a are related by

$$A^a = A^I \left.\frac{\partial x^a}{\partial u^I}\right|_P.$$

(d) A neighbouring point Q in $\mathcal{H}$ is displaced from P by infinitesimal coordinate differentials δu^I. A vector $\boldsymbol{A}_\parallel$ is defined in the tangent space to $\mathcal{H}$ at Q by displacing the vector $\boldsymbol{A}$ from P to Q in the embedding space, keeping its components A^a fixed, and then taking the projection into the surface at Q. Show that

$$A^I(P)\left.\frac{\partial x^a}{\partial u^I}\right|_P = A^I_\parallel(Q)\left.\frac{\partial x^a}{\partial u^I}\right|_Q + A^a_\perp(Q),$$

where $A^a_\perp(Q)$ are the Cartesian components of the projection of the displaced vector normal to the surface at Q, so that

$$\delta_{ab}A^a_\perp(Q)\left.\frac{\partial x^b}{\partial u^I}\right|_Q = 0.$$

By writing $A^I_{\parallel}(Q) = A^I(P) + \delta A^I$, and expanding the $\left(\partial x^a/\partial u^I\right)|_Q$ about the point P, show that to first-order in small quantities

$$g_{IK}(P)\delta A^K = -\delta_{ab} \left.\frac{\partial x^a}{\partial u^I}\right|_P \left.\frac{\partial^2 x^b}{\partial u^J \partial u^K}\right|_P A^K(P)\delta u^J \, .$$

Hence show that the change δA^K is the same as would be obtained by parallel-transporting in the surface from P to Q: $\delta A^K = -\Gamma^K_{JL}(P)A^L(P)\delta u^J$. (*This question shows that infinitesimal parallel transport in the curved surface is equivalent to parallel transport in the Euclidean embedding space followed by projection into the surface.*)

8. In Minkowski spacetime, two uniformly-moving observers $\mathcal{E}$ and $\mathcal{R}$ have 4-velocities $\boldsymbol{u}$ and $\boldsymbol{v}$, respectively. (a) Show that $u^\mu v_\mu = c^2\gamma_V$, where V is their relative speed. (b) If $\mathcal{E}$ emits a photon that is subsequently received by $\mathcal{R}$, show that the ratio of the emitted and received photon frequencies is given by

$$\frac{\nu_{\mathcal{E}}}{\nu_{\mathcal{R}}} = \frac{u^\mu p_\mu}{v^\nu p_\nu} \, ,$$

where $\boldsymbol{p}$ is the photon 4-momentum.

9. Suppose an observer $\mathcal{O}$ begins to accelerate in Minkowski spacetime such that, at some instant, his 3-velocity and 3-acceleration in an inertial frame S are $\vec{u}$ and $\vec{a}$, respectively. Show that the (proper) acceleration α measured by $\mathcal{O}$ at this instant is given by

$$\alpha^2 = \frac{\gamma_u^6(\vec{u}\cdot\vec{a})^2}{c^2} + \gamma_u^4\, \vec{a}\cdot\vec{a} \, .$$

Find an expression for α if the motion in S is circular with radius r.