UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Wednesday 1 June 2011 1.30 pm to 3.00 pm

EXPERIMENTAL AND THEORETICAL PHYSICS (3B)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (3B)

Candidates offering this paper should attempt a total of **three** *questions. The questions to be attempted are* **1**, **2** *and* **one** *other question.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains **4** *sides, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS
20 Page Answer Book
Metric graph paper
Rough workpad
Yellow master coversheet

SPECIAL REQUIREMENTS
Mathematical Formulae handbook
Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

ASTROPHYSICAL FLUID DYNAMICS

1 *Attempt* **all** *parts of this question. Answers should be concise, and relevant formulae may be assumed without proof.*

(a) Draw a labelled sketch showing the streamlines through a de Laval nozzle. [4]

(b) A fluid of conductivity σ moves at a non-relativistic velocity $\boldsymbol{u}$. In this limit the current density is given by $\boldsymbol{j} = \sigma(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B})$ where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic fields in the fluid. Show that the magnetic field satisfies the following differential equation:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \frac{1}{\mu_0 \sigma} \nabla^2 \boldsymbol{B}.$$

[4]

[*You may find useful the vector identity*

$$\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}.$$
]

(c) The speed u of an incompressible viscous fluid undergoing laminar flow between two infinite parallel plates of separation d when subject to a pressure gradient P' is governed by the Navier-Stokes equation in the form $\nabla^2 u = -(1/\eta)P'$. What is the maximum speed in the flow? [4]

2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) blast waves associated with supernova remnants;

(b) thermal instabilities in a gas;

(c) viscous accretion discs.

3 *Attempt* **either** *this question* **or** *question 4.*

Euler's equation for the motion of a fluid in a gravitational potential Φ is

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\frac{1}{\rho}\nabla p - \nabla\Phi,$$

where $\boldsymbol{u}$, p and ρ are the velocity, pressure and density fields respectively. Show that the quantity

$$H = \frac{1}{2}u^2 + \int \mathrm{d}p/\rho + \Phi$$

is constant along a streamline. [4]

Spherically-symmetric accretion of gas occurs onto a star of mass M. If the accretion is steady show that at a radius r

$$(u^2 - c_\mathrm{s}^2)\frac{\mathrm{d}\ln u}{\mathrm{d}r} = \frac{2c_\mathrm{s}^2}{r}\left(1 - \frac{GM}{2c_\mathrm{s}^2 r}\right),$$

where $c_\mathrm{s} = \sqrt{\mathrm{d}p/\mathrm{d}\rho}$ is the sound-speed in the gas, and find an expression for the sonic radius, r_s. [4]

If the accretion is isothermal show that

$$u^2 = 2c_\mathrm{s}^2 \ln\left(\frac{\rho_\infty}{\rho}\right) + \frac{2GM}{r},$$

where ρ_∞ is the density at infinity. [4]

By finding an expression for the density of the gas at the sonic radius, or otherwise, find an expression for the accretion rate $\dot{M}$ in terms of M, ρ_∞ and c_s. [5]

Evaluate r_s and $\dot{M}$ for a solar-mass star accreting from a medium with a number density of hydrogen atoms of $10^6\ \mathrm{m}^{-3}$ and a temperature 200 K. [2]

By comparing these results to those obtained if the gas were in free-fall, derive an expression for the luminosity of the accreting gas outside the sonic radius. Estimate also the additional surface luminosity at the star due to the accreting gas and comment on your answer. [6]

[*You may use the vector identity*

$$(\boldsymbol{A}\cdot\nabla)\,\boldsymbol{A} = \nabla\left(\frac{1}{2}A^2\right) - \boldsymbol{A}\times(\nabla\times\boldsymbol{A})\,.$$

The mass of the Sun is 2×10^{30} kg.]

(TURN OVER

4 *Attempt* **either** *this question* **or** *question 3.*

The equations governing the behaviour of a self-gravitating fluid are:

$$\begin{aligned}\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) &= 0,\\ \frac{\partial\boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} &= -\frac{1}{\rho}\nabla p - \nabla\Phi,\\ \nabla^2\Phi &= 4\pi G\rho,\end{aligned}$$

where $\boldsymbol{u}$, p and ρ are the velocity, pressure and density fields respectively and Φ is the gravitational potential. Write down the equations for the hydrostatic equilibrium of a fluid of density ρ_0, pressure p_0 and gravitational potential Φ_0. [2]

Briefly discuss why there is a problem when we consider an infinite, static, uniform medium, and the approach adopted by Jeans to circumvent this problem. [3]

By following this approach, and considering a small perturbation of the form $p = p_0 + p_1$, $\rho = \rho_0 + \rho_1$, $\Phi = \Phi_0 + \Phi_1$ and $\boldsymbol{u} = \boldsymbol{u}_1$, find the linearised forms of the above equations which are first order in the perturbed quantities. You should assume that the sound speed c_s is constant. [5]

Assuming a wave-like solution of the form

$$\rho_1 = \rho_{1,0}\,\mathrm{e}^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)},$$

and similarly for the other perturbed quantities, show that the dispersion relation has the form

$$\omega^2 = c_\mathrm{s}^2(k^2 - k_\mathrm{J}^2)$$

and find an expression for the Jeans wave number k_J. [6]

What is the criterion for a growing unstable mode which leads to gravitational collapse? Which modes, and hence physical scales, grow fastest? [3]

A similar analysis performed for an infinite disc with a surface density σ_0 gives the following dispersion relation

$$\omega^2 = c_\mathrm{s}^2\left(k^2 - \frac{2\pi G\sigma_0|k|}{c_\mathrm{s}^2}\right),$$

where k is the wavenumber. For this case determine the criterion for unstable modes. What is the scale size of the fastest-growing instabilities? [3]

How does this result help explain the relatively small mass of stars compared to their parent clouds? [3]

END OF PAPER