UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Friday 31 May 2013   13.30 to 15.30

EXPERIMENTAL AND THEORETICAL PHYSICS (3)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (3)

Advanced Quantum Physics — ANSWERS

1 (a) *For two fermions with spin $S_1 = S_2 = 3/2$, by considering the operator $(\boldsymbol{S}_1 + \boldsymbol{S}_2)^2$, or otherwise, find all eigenvalues of the operator $\boldsymbol{S}_1 \cdot \boldsymbol{S}_2$.* [4]

[Same method used several times in the course for LS coupling etc.]

•$(\boldsymbol{S}_1 + \boldsymbol{S}_2)^2 = \boldsymbol{S}_1^2 + 2\boldsymbol{S}_1 \cdot \boldsymbol{S}_2 + \boldsymbol{S}_2^2$

•$\boldsymbol{S}_1 \cdot \boldsymbol{S}_2 = (1/2)[S(S+1) - S_1(S_1+1) - S_2(S_2+1)]$

•$S = 3, 2, 1, 0$; plug in to get $\boldsymbol{S}_1 \cdot \boldsymbol{S}_2 = 9/4, -3/4, -11/4, -15/4$

(b) *A particle of mass m is confined in a smoothly varying one-dimensional potential. The wavefunction of the particle has an overall extent $\sim a$ and it has n approximately equally spaced nodes. Estimate the particle's kinetic energy.* [4]

[WKB and various sketches and estimates in lectures, examples, past papers.]

•$\int p\,dx \sim pa \sim (n + 1/2)h$ if want to refer to WKB, or simply say $\lambda \sim 2a/n$...

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$$E \sim \frac{n^2\pi^2\hbar^2}{2ma^2}$$

(only main scalings important, various factors of 2 etc are OK)

(c) *A particle is in the ground state of an infinitely deep one-dimensional rectangular potential well of width L. Find the unnormalised probability distribution, $p(k)$, for the particle's wavevector k.* [4]

[Fourier decomposition in lecture notes]

•$\psi(x) \sim \cos(\pi x/L)$ for $-L/2 \leq x \leq L/2$

•Fourier transform to get

$$\psi(k) \sim \frac{\cos(kL/2)}{k^2 - (\pi/L)^2}$$

(easiest with convolution theorem, but brute force also OK)

•$p(k) = |\psi(k)|^2$

2 *Write notes on* **two** *of the following:* *[each part carries equal credit]* [13]

(a) spatial and spin wavefunctions for identical particles;

(b) coherent and non-classical states of light;

(c) Stark effect.

[This is all standard bookwork.]

(a) Spatial and spin wavefunctions for identical particles

- For two or more identical particles the multi-particle wavefunction must be properly symmetrised under particle exchange - symmetric for bosons, anti-symmetric for fermions. [1]
- Formally - for fermions Slater determinant, for bosons the simplest example a trivial BEC wavefunction. [1]
- If (and only if) the Hamiltonian is separable in spatial and spin degrees of freedom, then we can also factorise the wavefunction, $\Psi = \phi(\boldsymbol{r}_1, \boldsymbol{r}_2...)\chi(\boldsymbol{s}_1, \boldsymbol{s}_2...)$. [1]
- Simple explicit example (maths) - singlet and triplet states (combined with symmetric/anti-symmetric spatial wavefunctions) for two identical spin-$1/2$ fermions. [1]
- Examples of physical consequences - (1) atomic structure (filling of orbitals, exchange energy, magnetic properties of atoms, Hund's rules), (2) ortho- vs. parahydrogen (symmetry of nuclear spins affects symmetries of spatial functions, rotational states, heat capacity...), (3) another good example is also OK (e.g. Fermi energy for a free electron gas, band insulators vs metals, BEC...)
  [It's enough to cover just one good example clearly/quantitatively, and mention some other(s) superficially.] [3]

(b) Coherent and non-classical states of light

- Coherent (Glauber) state is "the most classical state of light" (factual quote) [1]
- What this really means is that it is well described by a classical constant electric field amplitude (a c-number), because it has minimal relative fluctuations. The noise is always (independently of the field amplitude) only as big as in vacuum, so the more populated (on average) the mode is, the smaller the relative fluctuations are. [2]
- Some maths - at least state that a coherent state is an eigenstate of the annihilation operator, $a|\alpha\rangle = \alpha|\alpha\rangle$, but could also explicitly write $|\alpha\rangle \propto \exp(\alpha a^\dagger)|0\rangle$, normalise the state, or calculate the uncertainty in the electric field. [2]
- Non-classical - in principle anything that's not a coherent state, but of specific interest are squeezed states, where uncertainty in some observable is reduced at the expense of the uncertainty in the conjugate observable. Simplest

example is a Fock (“number") state. Experimentally, non-classical effects can become strong (and thus observable) in cavities - due to small mode volume single photons create a large field. [2]

(c) Stark effect

•Quadratic or linear shift of (atomic) energy levels due to an external electric field (enough to talk about static fields). [1]

•The perturbing Hamiltonian, $H' = qEz$, is odd, so can only connect states of different parity. Hence no state of definite parity can show linear Stark effect. [1]

•Some maths - e.g. could just quote the expression for the second order perturbation theory result, point out the sign of the shift for the ground state. [2]

•Linear Stark effect can arise if there are *degenerate states of different parity*. “Standard" second-order perturbation theory then fails and we address the problem in degenerate perturbation theory, hybridising (mixing) the states which then show linear Stark effect. It is crucial to stress that the hybridised states are *not* states of definite parity. [2]

•Classic example are the four (s and p) $n = 2$ states in hydrogen, where s and p_z (or $m_\ell = 0$) states get hybridised and show linear shift. [1]

•Bonus: What is really degenerate? The $n = 2$ states in hydrogen are also not perfectly degenerate (e.g. due to Lamb shift), so really one has to compare the size of the (off-diagonal) matrix element and the separation of energy levels - in general the Stark effect start off quadratic (for small E) and then turns linear.

3 [This is a standard examples-sheet and exam topic, with only the system and form of the perturbation varied. The first 9 marks are for the application of standard bookwork.]

A particle of mass m is trapped in a one-dimensional harmonic potential with a trapping frequency ω,

$$V = \frac{1}{2}m\omega^2(x - x_0)^2,$$

where x_0 *is the position of the trap centre. Sketch the wavefunctions of the particle,* $\psi_n(x)$*, in the ground state,* $|n = 0\rangle$*, and the first excited state,* $|n = 1\rangle$*. State the energies of the two states,* $\hbar\omega_0$ *and* $\hbar\omega_1$*. Show that, up to a dimensionless factor of order unity, the characteristic spatial extension of both states is given by* $a_0 = \sqrt{\hbar/(m\omega)}$. [5]

•Standard sketch of well known functions [2]

•$\omega_0 = (1/2)\omega$, $\omega_1 = (3/2)\omega$ [2]

•e.g. from potential energy $m\omega^2\langle x^2\rangle \sim \hbar\omega$, or equate potential and kinetic energy, or... [1]

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Initially $x_0 = 0$ and the particle is in the $|n = 0\rangle$ state. We want to excite it to the $|n = 1\rangle$ state by shaking the trap. Specifically, in the time interval $0 \le t \le T$, the position of the trap centre oscillates according to

$$x_0 = \varepsilon \cos(\Omega t),$$

where $\varepsilon \ll a_0$, $T \gg 1/\omega$ and $|\Omega - \omega| \ll \omega$. Writing the general time-dependent wavefunction in the form

$$|\Psi(t)\rangle = \sum_n c_n(t) e^{-i\omega_n t} |\psi_n\rangle,$$

within first order perturbation theory the coefficients c_n, for $n \neq 0$, are approximately

$$c_n(t) = \frac{1}{i\hbar} \int_{-\infty}^{t} e^{i(\omega_n - \omega_0)t'} \langle\psi_n|\hat{V}(t')|\psi_0\rangle dt',$$

where $\hat{V}(t')$ is the time-dependent perturbing potential.

Identify $\hat{V}(t')$ corresponding to the shaking of the trap in this problem and hence show that

$$c_1(T) = -\varepsilon \frac{m\omega^2}{i\hbar} \int_0^T e^{i\omega t} \langle\psi_1|x|\psi_0\rangle \cos(\Omega t) dt,$$

clearly stating all the steps in your derivation. [4]

- $V = (1/2)m\omega^2 x^2 - m\omega^2 x\varepsilon \cos(\Omega t) + (1/2)m\omega^2\varepsilon^2 \cos^2(\Omega t)$ [1]
- so $V(t) = -m\omega^2 x\varepsilon \cos(\Omega t) + (1/2)m\omega^2\varepsilon^2 \cos^2(\Omega t)$ [1]
- but the last, (spatially) constant term doesn't contribute to the matrix element [1]
- also plug in $\omega_1 - \omega_0 = \omega$ [1]

Explain why $c_n(T)$ is zero for any even n, why $|c_n(T)| \ll |c_1(T)|$ for all odd $n > 1$, and why for evaluating $c_1(T)$ it is a good approximation to replace $\cos(\Omega t)$ with $(1/2)e^{-i\Omega t}$. [6]

- Odd perturbation, so cannot connect states of same parity [2]
- States $n = 3, 5, 7...$ are far detuned [2]
- Similarly, the "counter-rotating" (they don't need to know the phrase) term $e^{i\Omega t}$ is far detuned from the $n = 0 \to 1$ transition [2]

Approximating $\langle\psi_1|x|\psi_0\rangle \approx a_0$, calculate the probability, p_1, that the particle is found in the state $|n = 1\rangle$ at time T. For a fixed T, sketch p_1 as a function of Ω. Comment on the main features of the function $p_1(\Omega)$. [10]

- $$c_1(T) = -\varepsilon a_0 \frac{m\omega^2}{i2\hbar} \int_0^T e^{i(\omega - \Omega)t} dt = \varepsilon a_0 \frac{m\omega^2}{2\hbar(\omega - \Omega)} \left[e^{i(\omega - \Omega)T} - 1\right]$$

[2]

- $p_1 = |c_1|^2$ [1]
-

$$p_1 = \left(\frac{\varepsilon a_0 m\omega^2}{\hbar}\right)^2 \frac{\sin^2((\omega - \Omega)T/2)}{(\omega - \Omega)^2},$$ [2]

Rewrite with nicer prefactor $\varepsilon a_0 m\omega^2 = (\varepsilon/a_0)\hbar\omega$:

$$p_1 = \left(\frac{\varepsilon\omega}{a_0}\right)^2 \frac{\sin^2((\omega - \Omega)T/2)}{(\omega - \Omega)^2}\ .$$ [1]

- Sketch: main resonance at $\Omega = \omega$, $(\sin x/x)^2$ behaviour with $(\omega - \Omega)$. [2]
- Sidebands and broadening from the Fourier components of the top-hat pulse. [2]

4 [The first 11 marks are for standard bookwork, straight from the problem set.]

With help of a sketch, explain what is meant by the Einstein's coefficients, B_{12}, B_{21}, and A_{21}, for radiative transitions in a two-level system with non-degenerate energy levels E_1 and E_2, with $E_2 > E_1$. [5]

- B_{12} describes the stimulated transition from 1 to 2, with rate per atom given by $B_{12}u(\omega_0)$, where $\hbar\omega_0 = E_2 - E_1$. [2]
- B_{21} is completely symmetric [1]
- A_{21} is spontaneous decay rate [2]

Given that in free space the energy density per unit ω of black-body radiation is

$$u(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3}\frac{1}{\exp(\hbar\omega/k_B T) - 1},$$

show that

$$B_{12} = B_{21}, \quad A_{21} = B_{21}\frac{\hbar\omega_0^3}{\pi^2 c^3},$$

where $\hbar\omega_0 = E_2 - E_1$. [6]

- In equilibrium $n_2[A_{21} + B_{21}u] = n_1 B_{12} u$, where $n_{1/2}$ are level populations [2]
- In equilibrium also $n_2/n_1 = \exp(-\hbar\omega_0/k_B T)$ [2]
- Combining these two equations and arguing that A_{21} is T-independent gives the required results. [2]

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Now consider two parallel conducting plates lying in the $x - y$ plane and separated along z by $d \ll c/\omega_0$. Between the plates, an electric field oscillating at angular frequency ω_0 must be linearly polarised along z and radiation can propagate only in the $x - y$ plane.

By considering only the modes propagating in the $x - y$ plane, show that, between the plates, for all frequencies $\omega \ll c/d$,

$$u(\omega) = \frac{\hbar\omega^2}{2\pi c^2 d} \frac{1}{\exp(\hbar\omega/k_\mathrm{B}T) - 1},$$

and comment on how this affects the ratio of the A and B coefficients. [8]

- Standard counting of k-states but remembering that there is only one polarisation, and that we have to divide by volume and not area, gives the above result. [4]
- By analogy with the result already derived above, the decay rate is now $A = B\frac{\hbar\omega_0^2}{2\pi c^2 d}$. [2]
- A/B is much larger than in free space, by a factor $\sim c/(\omega_0 d) \gg 1$. [2]

Now suppose that $\hbar\omega_0$ corresponds to the energy splitting between the ground state ($n = 1$) and the first excited state ($n = 2$) of the hydrogen atom. A hydrogen atom in the $2p$ state is placed inside the capacitor. Neglecting spin-orbit coupling and numerical factors of order unity, but clearly stating any assumptions or approximations you make, discuss how placing the atom inside the capacitor affects the spontaneous lifetime of the $2p$ state. By considering the relevant selection rules for dipolar transitions, distinguish between the cases of $2p$ states with quantum numbers $m_\ell = 1, 0$ and -1 (labelled with respect to the z-axis). [6]

- Selection rules $\Delta m_\ell = 1, 0, -1$, but different transitions correspond to different light polarisations σ^-, π, σ^+. [2]
- For the $m_\ell = 0$ state the decay to $1s$ is $\Delta m_\ell = 0$, so polarisation along z is required. B (given by the matrix elements) is essentially the same as in free space $\implies$ this state now decays much faster since A is larger. [2]
- For the $m_\ell = \pm 1$ states, polarisation in the $x - y$ plane is required, so these transitions are completely suppressed. [2]

END OF PAPER