UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Tuesday 29 May 2018 9:00 am to 11:00 am

PHYSICS (2)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (2)

RELATIVITY

Candidates offering this paper should attempt a total of **three** *questions. The questions to be attempted are* **1** *and* **two** *questions from Section B.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains **five** *sides, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
2×20 Page Answer Book	Mathematical Formulae handbook
Rough workpad	Approved calculator allowed
Yellow master coversheet	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

draft: 17th May 2018

RELATIVITY

SECTION A

Answers should be concise and relevant formulae may be assumed without proof.

1 *Attempt* **all** *parts of this question. Answers should be concise, and relevant formulae may be assumed without proof.*

(a) A particle of rest mass m and Lorentz factor γ collides head-on with a stationary particle of rest mass $2m$. Find the value of γ needed to produce a final state consisting of two particles each of rest mass αm, where $\alpha > 3/2$. [4]

(b) The electromagnetism field-strength tensor in an inertial frame is given by

$$[F_{\mu\nu}] = \begin{pmatrix} 0 & E^1/c & E^2/c & E^3/c \\ -E^1/c & 0 & -B^3 & B^2 \\ -E^2/c & B^3 & 0 & -B^1 \\ -E^3/c & -B^2 & B^1 & 0 \end{pmatrix},$$

where E^i and B^i are the Cartesian components of the electric and magnetic fields $\vec{E}$ and $\vec{B}$.

Using $[F_{\mu\nu}]$, show that $c^2|\vec{B}|^2 - |\vec{E}|^2$ is Lorentz invariant. [4]

(c) The Robertson-Walker (RW or FRW) spacetime line element for cosmology can be written as

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - a^2(t)\left(\frac{\mathrm{d}r^2}{1 - Kr^2} + r^2\mathrm{d}\Omega^2\right).$$

Explain: why part of the metric has a form somewhat similar to spherical polar coordinates; what $a(t)$ does and why; what K represents; and what t, r and Ω have in common given the cosmological nature of the metric. [4]

SECTION B

Attempt **two** *questions from this section*

B2 State the two fundamental postulates of special relativity (SR) – the first postulate concerns the laws of physics and the second concerns the propagation of light. [2]

Two inertial frames S and S' are in standard configuration, with S' moving at speed v with respect to S along the x-axis. In S, a particle with non-zero mass has velocity components $u_x = \mathrm{d}x/\mathrm{d}t$, $u_y = \mathrm{d}y/\mathrm{d}t$, and $u_z = \mathrm{d}z/\mathrm{d}t$. Show clearly that the particle's velocity components in S' are given by

$$u'_x = \frac{\mathrm{d}x'}{\mathrm{d}t'} = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} ,$$

$$u'_y = \frac{\mathrm{d}y'}{\mathrm{d}t'} = \frac{u_y}{\gamma_v(1 - \frac{u_x v}{c^2})} ,$$

$$u'_z = \frac{\mathrm{d}z'}{\mathrm{d}t'} = \frac{u_z}{\gamma_v(1 - \frac{u_x v}{c^2})} .$$

[4]

Answer **both** of the following parts.

(a) In Cartesian coordinates, the acceleration 4-vector is given by $a^\mu = \mathrm{d}u^\mu/\mathrm{d}\tau$, where u^μ is the 4-velocity $\gamma_u(c, \vec{u})$, $\vec{u}$ denotes the 3-velocity, and τ is proper time. Show that

$$\frac{\mathrm{d}\gamma_u}{\mathrm{d}t} = \frac{\gamma_u^3 \vec{u}\cdot\vec{a}}{c^2} ,$$

and that

$$a^\mu = \gamma_u^2 \left[\frac{\gamma_u^2 \vec{u}\cdot\vec{a}}{c},\ \vec{a} + \frac{\gamma_u^2 (\vec{u}\cdot\vec{a})\vec{u}}{c^2} \right].$$

[3]

An observer begins to accelerate such that, at some instant, the observer's 3-velocity and 3-acceleration in S are $\vec{u}$ and $\vec{a}$. Show that in the observer's instantaneous rest-frame, the (proper) acceleration experienced by the observer is given by

$$\alpha^2 = \frac{\gamma_u^6 (\vec{u}\cdot\vec{a})^2}{c^2} + \gamma_u^4 \vec{a}\cdot\vec{a} .$$

[4]

(b) Examine SR's second fundamental postulate by replacing c in the Lorentz transform (LT) with another speed a to create an 'a-LT', and then show that a must have a unique value, as follows.

If S' moves in S in standard configuration at velocity v along the x-axis and the a-LT applies, find x' and t' in terms of x and t. Then, if S'' moves in S' in standard configuration at velocity u and the transform is now a b-LT, relate x'' and t'' to x and t, and from these expressions explain why b^2 must equal a^2. [4]

Explain why a^2 cannot be infinite. [1]

Explain why Einstein required the second fundamental postulate to concern c. [1]

B3 The Schwarzschild metric is

$$\mathrm{d}s^2 = c^2\left(1 - \frac{2\mu}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2\mu}{r}\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\theta^2 - r^2\sin^2\theta\,\mathrm{d}\phi^2,$$

where $\mu \equiv GM/c^2$. Starting from a suitable "Lagrangian", or otherwise, determine the geodesic equations for a particle orbiting with $\theta = \pi/2$ in this metric, showing clearly that

$$\left(1 - \frac{2\mu}{r}\right)\dot{t} = k\,,$$

$$r^2\dot{\phi} = h\,,$$

and $$c^2\left(1 - \frac{2\mu}{r}\right)\dot{t}^2 - \left(1 - \frac{2\mu}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = \kappa,$$

where k, h and κ are constant, and a dot implies differentiation with respect to an affine parameter along the particle's world line. Write down the values of κ for a particle with mass and for a photon, give the physical interpretation of h, and indicate what k is associated with. [5]

For a particle with mass, show that

$$c^2(k^2 - 1) = \dot{r}^2 - \frac{2\mu c^2}{r} + \frac{h^2}{r^2}\left(1 - \frac{2\mu}{r}\right).$$

[3]

For this particle, identify the relativistic effective potential $V_{\mathrm{eff}}(r)$ for the radial motion and show that for a circular orbit, with coordinate radius $r = R$,

$$h^2 = \frac{\mu c^2 R^2}{R - 3\mu}.$$

Find the orbital speed of this particle, as measured at a point at infinity and at rest in this coordinate system, when $R = 4\mu$. [5]

Discuss the perihelion advance of the of the planet Mercury as a classic test of general relativity (GR). If you wish, you may make use of:

i. the relevant GR orbit-shape equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + u - 3\mu u^2 = \frac{GM}{h^2},$$

where $u \equiv \frac{1}{r}$ (marks can be obtained for a derivation of this orbit-shape equation);

ii. the Newtonian shape-orbit equation

$$u = \frac{GM}{h^2}(1 + e\cos\phi),$$

where e is the orbit eccentricity. [6]

B4 Discuss the Principle of Equivalence; you may wish to include, for example, inertial and gravitational mass, the relative motion of free particles in a freely falling lift (elevator), and geodesic deviation. [7]

In a weak gravitational field, the metric is close to the Minkowski metric $g_{\mu\nu}$:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,,$$

where $|h_{\mu\nu}| \ll 1$. For non-relativistic particle speeds in this field, the geodesic equation is

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{00}\, c^2 \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 \approx 0,$$

and the relevant connection coefficients are

$$\Gamma^\mu_{00} = \frac{1}{2}\, g^{\mu\nu}\left(2\frac{\partial g_{\mu 0}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^\nu}\right).$$

Assuming the metric is stationary in time and that metric derivatives are first order in $h_{\mu\nu}$, show explicitly that

$$\Gamma^0_{00} \approx 0\,,$$

and that

$$\Gamma^i_{00} \approx \frac{1}{2}\frac{\partial h_{00}}{\partial x^i}\,.$$

Then show that $g_{00} \approx (1 + 2\Phi/c^2)$, and make clear the meaning of Φ. [5]

The spacetime around a compact, dark star of mass M is Schwarzschild. The star is at rest with respect to us and is very distant from us. A blob of gas is in circular orbit around the star at a radius r at which the star's gravitational field is weak. We see the orbit of the gas blob to be in the plane of our sky (that is, the displacement vector from us to the star is perpendicular to the plane of the gas-blob orbit). The gas blob emits photons of frequency ν_{em}, which we observe at frequency ν_{ob}. Explaining your steps, find an approximate expression for ν_{em} to lowest order in R_{S}/r, where R_{S} is the Schwarzschild radius of the star. [7]

END OF PAPER