UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Friday 27 May 2016 9.00 am to 11.00 am

PHYSICS (5)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (5)

ASTROPHYSICAL FLUID DYNAMICS

Candidates offering this paper should attempt a total of **three** *questions. The questions to be attempted are* **1**, **2** *and* **one** *other question.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains **five** *sides, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS
2 × 20 Page Answer Book
Rough workpad
Yellow master coversheet

SPECIAL REQUIREMENTS
Mathematical Formulae handbook
Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

ASTROPHYSICAL FLUID DYNAMICS

1 *Answer* **all** *parts of this question. Answers should be concise and relevant formulae may be assumed without proof.*

(a) Demonstrate that an ideal gas in hydrostatic equilibrium, and in a uniform gravitational field, is stable for convection if

$$\frac{\rho}{p\gamma}\frac{\mathrm{d}p}{\mathrm{d}z} > \frac{\mathrm{d}\rho}{\mathrm{d}z}$$

where ρ is the density, γ is the adiabatic index of the gas, p is the pressure and z the spatial coordinate in the direction opposite to the gravitational force. [4]

(b) A black hole of mass M and Schwarzschild radius $R_s = 2GM/c^2$ is accreting steadily at a rate $\dot{m} = 1\ M_\odot\ \mathrm{yr}^{-1}$ through a thin viscous accretion disc which has inner and outer radii $6\ R_s$ and $30\ R_s$ respectively. Estimate the total luminosity of the disc. [4]

[*Assume that the gravitational field of the accretion disc is negligible relative to the black hole and neglect relativistic corrections.*]

(c) Monoatomic gas in free-fall from infinity accretes isotropically onto a protostar of radius r_0 and mass M_0. The gas undergoes a strong (adiabatic) shock at the impact with the protostar surface. Show that the temperature T of the gas after the shock is given by the relation

$$T = \frac{\mu}{R_*}\frac{3}{8}\frac{GM_0}{r_0}\ .$$ [4]

[*Assume that the temperature of the gas before the shock can be neglected and that the gas behaves as a perfect gas of mean molecular weight μ and modified gas constant R_*.*]

2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) Rayleigh–Taylor and Kelvin–Helmholtz instabilities;

(b) the formation and properties of shocks;

(c) the propagation of sound waves.

3 *Attempt* **either** *this question* **or** *question 4.*
Answer all parts of this question.

(a) Derive the equation of state $p = K\rho^{\gamma}$ for an adiabatic gas. [7]

(b) The Navier–Stokes equation is

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho}\nabla p - \nabla \Phi + \nu \left[\nabla^2 \boldsymbol{u} + \frac{1}{3}\nabla(\nabla \cdot \boldsymbol{u}) \right]$$

where ρ is the density, p is the pressure, Φ is the gravitational potential and ν is the kinematic viscosity, assumed to be constant. Show that for a barotropic fluid the vorticity, $\boldsymbol{w} = \nabla \times \boldsymbol{u}$, follows the relation

$$\frac{\partial \boldsymbol{w}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{w}) + \nu \nabla^2 \boldsymbol{w}.$$

[9]

(c) A protostar embedded in a molecular cloud of radius R, expels a collimated jet of incompressible fluid, of density ρ, with constant cross section of radius r_0. The flow is steady and is driven by the pressure p_0 at the base of the jet. The flow is viscous, with constant kinematic viscosity ν, and is such that at the sides of the jet the velocity of the flow is zero due to interaction with the confining cloud. Show that the mechanical power of the jet, P, i.e. the bulk kinetic energy transiting through the jet cross-section, has the form $P = Ar_0^8$. Find A in terms of R, p_0, ν and ρ. [9]

[*Neglect gravity and also neglect the pressure at the end of the jet.*]

4 *Attempt* **either** *this question* **or** *question 3.*
Answer both parts of this question.

(a) Demonstrate that a spherical system in hydrostatic equilibrium and with a polytropic equation of state of index n follows the Lane–Emden equation

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n,$$

where

$$\xi = \sqrt{\frac{4\pi G\rho_\mathrm{c}}{\psi_\mathrm{o} - \psi_\mathrm{c}}}\, r,$$

$$\theta = \frac{\psi_\mathrm{o} - \psi}{\psi_\mathrm{o} - \psi_\mathrm{c}},$$

and where r is the radius, ρ_c is the density at the centre of the system, ψ is the gravitational potential, and ψ_o and ψ_c are the gravitational potential at the surface and at the center of the system, respectively. [7]

By using the working in the previous part, show that for a spherically symmetric cloud of diatomic gas in hydrostatic equilibrium with adiabatic equation of state, sound speed and gravitational potential ψ are related by the equation

$$c_\mathrm{s} = \sqrt{-\frac{2}{5}\psi}$$

(assume the boundary condition that where the gas density $\rho \longrightarrow 0$ the gravitational potential $\psi_0 \longrightarrow 0$). [3]

Assume that stars behave as polytropes and that they have the same central temperature. Derive the relationship between stellar mass and radius. [6]

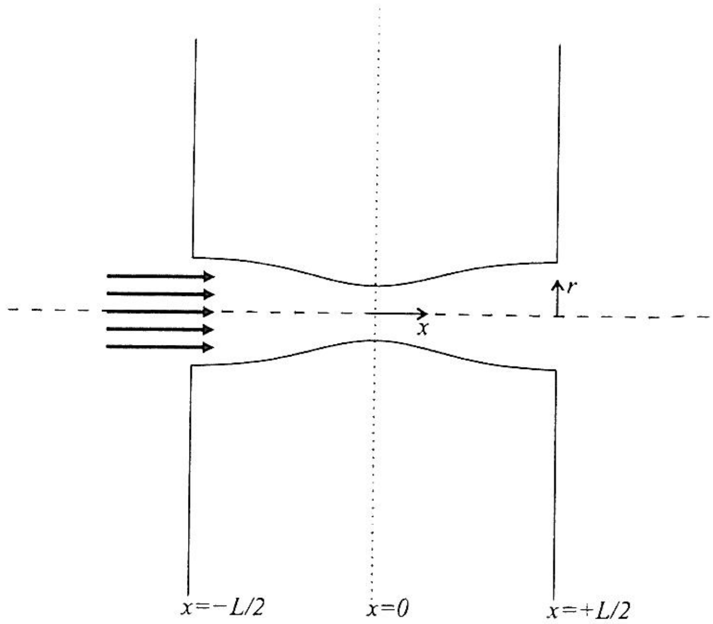
(b) A steady flow of gas with zero viscosity passes through a cavity in a slab of dense, incompressible medium of thickness L. The cavity is axisymmetric, perpendicular to the slab and has a cross-section whose radius varies as

$$r(x) = r_0\left(1 - \frac{1}{2}e^{-(4x/L)^2}\right),$$

where x is the coordinate perpendicular to the slab with origin at the midpoint of the slab (see the figure on the next page). At the entrance of the cavity ($x = -L/2$) the flow has a subsonic velocity and leaves the slab supersonically at the opposite side. Determine the position x_s where the jet becomes supersonic. [3]

In the previous part assume that the gas in the flow has an adiabatic equation of state $p = K\rho^\gamma$. The mass flow rate is $\dot{M}$. Determine the temperature of the gas T_{x_s} at the sonic point.

[6]



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ASTROPHYSICAL FLUID DYNAMICS

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(a) Demonstrate that an ideal gas in hydrostatic equilibrium, and in a uniform gravitational field, is stable for convection if

$$\frac{\rho}{p\gamma}\frac{\mathrm{d}p}{\mathrm{d}z} > \frac{\mathrm{d}\rho}{\mathrm{d}z}$$

where ρ is the density, p is the pressure and z the spatial coordinate in the direction opposite to the gravitational force. [4]

Bookwork from Handout.

Element of fluid with ρ and p in equilibrium with surrounding medium. Perturbation displaces the element of fluid by a quantity δz, in a position where the density and pressure of the surrounding medium are ρ' and p'. The perturbation is initially adiabatic and the pressure equilibrium requires that inside the element the pressure is now p'. ρ_* is the density of the perturbed element, and it will sink back (hence stable equilibrium) if $\rho_* > \rho'$, while the perturbation will grow (unstable equilibrium) if $\rho_* < \rho'$.

Since the perturbation is adiabatic, $p = K\,\rho^\gamma$ and $p' = K\,\rho_*^\gamma$, hence

$$\rho_* = \rho\left(\frac{p'}{p}\right)^{1/\gamma} \qquad (*)$$

To first order:

$$p' = p + \frac{dp}{dz}\,\delta z$$

hence by using $(*)$:

$$\rho_* = \rho\left(\frac{p + \frac{dp}{dz}\,\delta z}{p}\right)^{1/\gamma} = \rho\left(1 + \frac{1}{p}\frac{dp}{dz}\,\delta z\right)^{1/\gamma} \sim \rho + \frac{\rho}{p\gamma}\frac{dp}{dz}\,\delta z$$

For the surrounding medium:

$$\rho' = \rho + \frac{d\rho}{dz}\delta z$$

Medium stable if $\rho_* > \rho'$ hence:

$$\rho + \frac{\rho}{p\gamma}\frac{dp}{dz}\,\delta z > \rho + \frac{d\rho}{dz}\delta z$$

which implies the solution.

(b) A black hole of mass M_{BH} is accreting steadily at a rate $\dot{m}$ through a thin viscous accretion disc having surface density Σ. The innermost radius of the disc is 6 R_s, where $R_s = 2GM_{BH}/c^2$ is the Schwarzschild radius. Determine the inflow velocity u_r as a function of radius assuming constant viscosity ν. Estimate the total luminosity emitted by the disc assuming that it extends to 30 R_s and that the accretion rate is $\dot{m} = 1\ M_\odot\ yr^{-1}$. (Assume the gravitational field of the accretion disc negligible relative to the black hole and neglect relativistic corrections).

[*Recall that* $\nu\Sigma = \frac{\dot{m}}{3\pi}\left[1 - \left(\frac{3R_s}{r}\right)^{1/2}\right]$]

[4]

Partly seen in class.

The inflow velocity is simply given by the continuity equation (and by the fact that the accretion is steady): $\dot{m} = -u_r(r)\ 2\pi r\ \Sigma$.

Hence (since viscosity is constant)

$$u_r(r) = -\frac{\dot{m}}{2\pi r\ \Sigma} = \frac{3\nu}{2(r - \sqrt{3R_S r})}$$

The total luminosity is simply given by the accretion rate times half of the variation of gravitational energy between the internal and external boundaries of the accretion disc (seen in handouts or even more simply from the virial theorem):

$$L_{tot} = \frac{1}{2}\dot{m}\ GM_{BH}\left(\frac{1}{3R_S} - \frac{1}{30R_S}\right) \approx 0.076\ \dot{m}c^2 \approx 4.3\ 10^{45}\ erg\ s^{-1}$$

(c) A protostar of radius r_0 and mass M_0 accretes isotropically monoatomic gas in free-fall from infinity. The accreting gas undergoes a strong (adiabatic) shock at the impact with the protostar surface. Show that the temperature T of the gas after the shock is give by the relation

$$T_2 = \frac{\mu}{R_*}\frac{3}{8}\frac{GM_0}{r_0}$$

if the temperature of the gas before the shock can be ignored and assuming that the gas behaves as a perfect gas in which μ is the mean molecular weight and R_* is the modified gas constant. [4]

Subscripts 1 and 2 for quantities before and after the shock.

In the limit of strong shock for a monoatomic gas ($\gamma = 5/3$) the density ratio is:

$$\frac{\rho_2}{\rho_1} = \frac{\gamma + 1}{\gamma - 1} \Rightarrow \rho_2 = 4\rho_1 \quad (*)$$

PhD student interview Scheduled: 3 Mar 2016 09:00 to 09:30 hence, from first RH relation, the velocities are related by:

$$u_2 = \frac{1}{4}u_1 \quad (**)$$

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By using equations $(*)$ and $(**)$, and using the equation of state of the ideal gas on both sides of the shock, i.e. $p_1 = \frac{R_*}{\mu}\rho_1 T_1$ and $p_2 = \frac{R_*}{\mu}\rho_2 T_2$, the second RH relation becomes:

$$\rho_1 u_1^2 + \frac{R_*}{\mu}\rho_1 T_1 = 4\rho_1 \frac{u_1^2}{16} + \frac{R_*}{\mu} 4\rho_1 T_2$$

which gives

$$T_2 = \frac{1}{4}T_1 + \frac{\mu}{R_*}\frac{3}{16}u_1^2$$

Since the velocity of the gas in free fall from infinity is $u_1^2 = 2GM_0/r_0$, replacing into the previous equation gives the solution.

Alternatively, it can also be solved using the third RH equation, in a similar way.

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2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) Rayleigh-Taylor and Kelvin-Helmholtz instabilities;

(b) examples of shock waves in astrophysics;

(c) the propagation of sound waves.

3 *Attempt* **either** *this question* **or** *question 4.*

(a) Derive the equation of state $p = K\rho^{\gamma}$ for an adiabatic gas. [7]

Bookwork.

First law of thermodynamics

$$đQ = d\mathcal{E} + p dV = \frac{d\mathcal{E}}{dT} dT + p dV \quad (*)$$

replacing the specific heat capacity at constant volume $C_V = \frac{d\mathcal{E}}{dT}$ and using the equation of ideal gas (and $\rho = V^{-1}$ for a mass unit)

$$đQ = C_V dT + \frac{R_*}{\mu}\frac{T}{V} dV$$

for adiabatic cahnge $đQ = 0$ the previous equation translates into

$$C_V d\ln T + \frac{R_*}{\mu} d\ln V = 0$$

hence

$$V \propto T^{-\frac{C_V}{R_*/\mu}} \quad \Rightarrow \quad p \propto T^{1+\frac{C_V}{R_*/\mu}} \quad (**)$$

From the equation of ideal gases (and $\rho = V^{-1}$ for a mass unit)

$$p dV + V dp = \frac{R_*}{\mu} dT$$

replaced into $(*)$ gives

$$đQ = \left(\frac{d\mathcal{E}}{dT} + \frac{R_*}{\mu}\right) dT - V dp$$

the term in parenthesis is C_p, and $C_p - C_V = \frac{R_*}{\mu}$. Define $\gamma \equiv C_p/C_V$, then equations $(**)$ become:

$$V \propto T^{-1/(\gamma-1)} \quad \text{and} \quad p \propto T^{\gamma/(\gamma-1)}$$

hence $p \propto V^{-\gamma}$ or $p = k\rho^{\gamma}$.

(b) The Navier-Stokes equation is

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho}\nabla p - \nabla \Phi + \nu \left[\nabla^2 \boldsymbol{u} + \frac{1}{3}\nabla(\nabla \cdot \boldsymbol{u})\right]$$

where ρ is the density, p is the pressure, Φ is the gravitational potential and ν is the viscosity, assumed to be constant. Show that for a barotropic fluid the vorticity, $\boldsymbol{w} = \nabla \wedge \boldsymbol{u}$, follows the relation

$$\frac{\partial \boldsymbol{w}}{\partial t} = \nabla \wedge (\boldsymbol{u} \wedge \boldsymbol{w}) + \nu \nabla^2 w$$

[9]

Partly bookwork, but expanding the derivation of the last part which is not shown in the handout.

Take the curl, and take into account that $w = \nabla \wedge u$:

$$\frac{\partial w}{\partial t} + \nabla \wedge (u \cdot \nabla u) = \nabla \wedge \left(-\frac{1}{\rho}\nabla p - \nabla\Phi + \nu\left[\nabla^2 u + \frac{1}{3}\nabla(\nabla \cdot u)\right]\right) \qquad (*)$$

note that in the previous equation $\nabla \wedge \nabla\Phi = 0$ and that

$$u \cdot \nabla u = \frac{1}{2}\nabla u^2 - u \wedge (\nabla \wedge u) = \frac{1}{2}\nabla u^2 - u \wedge w \qquad (**)$$

Barotropic fluid, $p = p(\rho)$

$$\nabla \wedge \left(\frac{1}{\rho}\nabla p\right) = \nabla\left(\frac{1}{\rho}\right) \wedge \nabla p + \frac{1}{\rho}\nabla \wedge \nabla p$$

But the first term of the right hand side is zero because $\nabla\left(\frac{1}{\rho}\right) = -\frac{1}{\rho^2}\nabla\rho$ and $\nabla\rho$ is parallel to ∇p. The second term of the right hand side is also $\nabla \wedge \nabla p = 0$. Hence $\nabla \wedge \left(\frac{1}{\rho}\nabla p\right) = 0$.

Replacing $(**)$ into $(*)$ and taking into account the null terms:

$$\frac{\partial w}{\partial t} = -\nabla \wedge \left(\frac{1}{2}\nabla u^2 - u \wedge w\right) + \nabla \wedge \nu\left[\nabla^2 u + \frac{1}{3}\nabla(\nabla \cdot u)\right] \qquad (***)$$

In this equation take the term $\nabla \wedge \left(\frac{1}{2}\nabla u^2\right) = 0$ (curl of gradient is zero).

Also

$$\nabla \wedge \frac{\nu}{3}\nabla(\nabla \cdot u) = \frac{\nu}{3}\nabla \wedge \nabla(\nabla \cdot u) = 0$$

and

$$\nabla \wedge (\nu\nabla^2 u) = \nu\nabla \wedge \nabla^2 u = \nu\nabla \wedge [\nabla(\nabla \cdot u) - \nabla \wedge \nabla \wedge u] =$$
$$= -\nu\nabla \wedge \nabla \wedge w = -\nu[\nabla(\nabla \cdot w) - \nabla^2 w]$$

noting that $\nabla \cdot w = \nabla \cdot (\nabla \wedge u) = 0$ the previous equation becomes

$$\nabla \wedge (\nu\nabla^2 u)\nu\nabla^2 w$$

Replacing into $(***)$ gives the solution.

(c) A protostar embedded in a molecular cloud of radius R, expels a collimated jet of incompressible fluid, with constant section of radius r_0. The flow is steady and is driven by the pressure p_0 at the base of the jet. The flow is viscous with and is such that at the sides of the jet the velocity of the flow is zero due to interaction with the confining cloud. Show that the mechanical power of the jet, i.e. the bulk kinetic energy transiting through the jet cross section, is given by

$$P_K = \frac{\pi p_0^3}{512\,\rho^2 R^3 \nu^3} r_0^8 \tag{1}$$

where ρ is the gas density and ν is the viscosity, assumed to be constant.

[*Neglect the gravitational force by the star and also neglect the pressure surrounding the cloud at the end of the jet.*]

[8]

This is similar to an example in the handouts, but with some variations.

The velocity $\boldsymbol{u}$ depends on r, but the continuity equation requires that $\boldsymbol{u}$ does not depend on z, coordinate along the jet axis.

In the N-S equation the first term is zero: $\partial u/\partial t = 0$ because flux is steady, $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ because of axisymmetry. Also $\nabla \cdot \mathrm{u} = 0$ (because u only along z and constant with z).

Hence the N-S equation gives

$$\nu \nabla^2 \boldsymbol{u} = \frac{\nabla p}{\rho}$$

The radial and azimuthal (R and ϕ) components of ∇p are zero (as u has only a component along z).

The z component is

$$\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}z} = \nu \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}\right)$$

the first term is only function of z and the second term is only a function of r, as a consequence the first term is constant along z and equal to $\frac{1}{\rho}\frac{\Delta p}{R}$ or, since the pressure outside the cloud is zero, $\frac{1}{\rho}\frac{p_0}{R}$.

Integrating:

$$u(r) = -\frac{p_0}{4\rho\nu R} r^2 + c_1 \ln r + c_2 \quad (*)$$

but at $r = 0$ the velocity is not infinite, hence must be $c_1 = 0$. Also, at $r = r_0$ the velocity is zero, hence

$$c_2 = \frac{p_0}{4\rho\nu R} r_0^2$$

Therefore, equation $(*)$ becomes

$$u(r) = -\frac{p_0}{4\rho\nu R} r^2 + \frac{p_0}{4\rho\nu R} r_0^2 \quad (**)$$

Mechanical power. Energy passing through the annular element is $2\pi r\ \mathrm{d}r\ \rho\, u\, \frac{1}{2}u^2$, by integrating between 0 and r_0:

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$$P_K = \int_0^{r_0} \pi r \; \rho \, u^3 \, \mathrm{d}r$$

Which, by replacing u with equation $(**)$, can be easily solved analytically giving $P_K = Ar_0^8$, where A is a constant.

(d)

4 *Attempt* **either** *this question* **or** *question 3.*

(a) Demonstrate that a spherical system in hydrostatic equilibrium and with a polytropic equation of state with index n follows the Lane-Emden equation

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n$$

where

$$\xi = \sqrt{\frac{4\pi G\rho_c}{\psi_o - \psi_c}}\, r$$

$$\theta = \frac{\psi_o - \psi}{\psi_o - \psi_c}$$

and where r is the radius, ψ is the gravitational potential, ψ_o and ψ_c are the gravitational potential at the surface and at the center of the system, respectively. [7]

Bookwork. Equation of state

$$p = k\,\rho^{1+1/n}$$

Hydrostatic equilibrium:

$$\nabla p = -\rho\nabla\psi$$

replacing the equation of state

$$-\nabla\psi = \frac{1}{\rho}\,\nabla(k\rho^{1+1/n}) = (n+1)\,\nabla(k\rho^{1/n})$$

integrating

$$\rho = \left(\frac{\psi_T - \psi}{(n+1)k}\right)^n \qquad (*)$$

where ψ_T is the potential at the surface of the star, where $\rho = 0$.

hence

$$\rho = \rho_c\left(\frac{\psi_T - \psi}{\psi_T - \psi_c}\right)^n$$

where ρ_c and ψ_c are the density and potential at the star centre, respectively.

Poisson's equation becomes

$$\nabla^2\psi = 4\pi G\rho_c\left(\frac{\psi_T - \psi}{\psi_T - \psi_c}\right)^n \qquad (***)$$

From the definition of θ;

$$\psi = -(\psi_T - \psi_c)\theta + \psi_T$$

hence Poisson's equation becomes

$$\nabla^2\theta = -\frac{4\pi G\rho_c}{\psi_T - \psi_c}\theta^n$$

recalling that we are in spherical symmetry and that

$$\nabla^2 \theta = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\theta}{dr} \right)$$

whcih, by replacing r with ξ, gives the solution.

By using the working in the previous part, show that for a spherically symmetric cloud of diatomic gas in hydrostatic equilibrium with adiabatic equation of state, sound speed and gravitational potential ψ are related by the equation

$$c_s = \sqrt{-\frac{2}{5}\psi}$$

(assume the boundary condition that where the gas density $\rho \to 0$ the gravitational potential $\psi_0 \to 0$). [3]

Combine adiabatic equation of state $p = K\rho^\gamma$ with with equation of perfect gas, $p = \frac{R_*}{\mu} \rho T$, to obtain

$$T = K \frac{\mu}{R_*} \rho^{\gamma - 1} \qquad (***)$$

Taking into account that the politropic index $n = \frac{1}{\gamma-1}$ and that $\psi_T = 0$, equation $(*)$ in the previous part becomes

$$\rho = \left(\frac{1-\gamma}{\gamma k} \psi \right)^{\frac{1}{\gamma-1}}$$

replacing this into $(***)$ gives

$$T = \frac{\mu}{R_*} \frac{1-\gamma}{\gamma} \psi$$

replacing the expression of the sound speed for the adiabatic case $c_s = \sqrt{(\gamma\, R_*/\mu)\, T}$ and the adiabatic index for the diatomic gas $\gamma = 7/5$ gives the solution.

Assume that stars behave as polytropes and that they have the same central temperature. Derive the proportionality between stellar mass and radius.

[6]

Bookwork. Equation $(*)$ in the first part can be re-written as

$$\psi_0 - \psi_c = k(n+1)\rho_c^{1/n}$$

hence, replacing into the definition of ξ:

$$\xi = \sqrt{\frac{4\pi G \rho_c^{1-1/n}}{k(1+n)}}\, r \qquad (+)$$

also, using the definition of θ and equation $(**)$ in the first part:

$$\Rightarrow \quad \rho = \theta^n \qquad (++)$$

ξ_{max} is given by the condition that θ=0.

By using (+) and (++) the total mass is given by

$$M = \int_0^{r_{max}} 4\pi\rho r^2 dr = 4\pi\rho_c \left(\frac{4\pi G\rho_c^{1-1/n}}{k(1+n)}\right)^{-3/2} \int_0^{\xi_{max}} \theta^n \xi^2 d\xi \qquad (+++)$$

the last integral is a number independent of ρ_c.

Using the equation of state of perferct gases and using the polytropic equation of state implies that at the center of stars:

$$T_c = \frac{\mu k}{R_*}\rho_c^{1/n}$$

(where R_* is the modified constant of gases, not the stellar radius), hence if the central temperature is kept constant by the nuclear reactions this implies that

$$k \propto \rho_c^{-1/n}$$

hence, replacing into $(+++)$ gives

$$M \propto \rho_c^{-1/2}$$

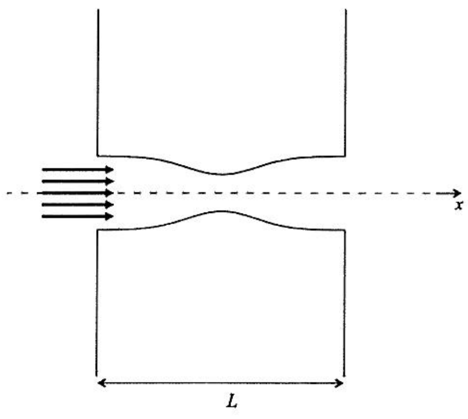
and replacing also into (+) gives

$$r \propto \rho_c^{-1/2}$$

therefore

$$M \propto r$$

(b) A steady flow of gas with zero viscosity passes thorugh an cavity in a slap of dense, incompressible medium of thickness L. The cavity is axisymmetric, perpendicular to the slab and has a cross-section whose radius r depends on the position x (which is the coordinate in the direction perpendicular to the slab) as $r(x) = r_0\,(1 - \frac{1}{2}e^{-(x-L/2)^2/(L/4)^2})$ (see figure). At the entrance of the cavity ($x = 0$) the flow has a subsonic velocity and leaves the slab supersonically at the opposite side. Determine the position x_s where the jet becomes supersonic. [3]



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The equation

$$(u^2 - c_s^2)\nabla \ln u = c_s^2 \nabla \ln A$$

implies that the fluid can make a sonic transition only where it has a minimum or a maximum, hence imposing that $dr/dx = 0$, it gives $x = L/2$.

In the previous part assume that the gas in the flow has an adiabatic equation of state $p = K\rho^\gamma$. The mass flow rate is $\dot{M}$. Determine the temperature of the gas T_{x_s} at the sonic point.

[6]

Adiabatic equation together with equation of perfect gas

$$T = k\frac{R_*}{\mu}\rho^{\gamma-1} \quad \Rightarrow \quad \rho = \left(\frac{\mu}{kR_*}T\right)^{\frac{1}{\gamma-1}} \qquad (*)$$

But also $c_s^2 = \gamma\frac{R_*}{\mu}T \quad (**)$

Conservation of mass (steady flow) implies:

$$\dot{M} = c_s(x_s)A(x_s)\rho(x_s)$$

by replacing in the former equation $(*)$ and $(**)$:

$$\dot{M} = \left(\gamma\frac{R_*}{\mu}T_{x_s}\right)^{1/2} A(x_s)\left(\frac{\mu}{kR_*}T_{x_s}\right)^{\frac{1}{\gamma-1}} \qquad (***)$$

also, from the expression of $r(x)$ and taking into account that $x_s = L/2$:

$$A(x_s) = \pi r_0^2\left(1 - \frac{1}{2}e^{-(x_s-L/2)^2/L^2}\right)^2 = \frac{\pi}{4}r_0^2 \qquad (****)$$

hence by combining $(***)$ and $(****)$

$$T_{x_s} = \left(\frac{4\dot{M}}{\pi r_0^2}\right)^{\frac{2(\gamma-1)}{\gamma+1}} k^{\frac{2}{\gamma+1}}\left(\frac{\mu}{R_*}\gamma\right)^{\frac{\gamma-1}{\gamma+1}}$$

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