b) convective stability

(ρ^*, p') —— ρ', p' —— stable if $\rho^* > \rho'$

$$\rho' = \rho + \frac{d\rho}{dz}\delta z$$

(ρ, p) —— ρ, p —— $p' = p + \frac{dp}{dz}\delta z$

$$p = K\rho^\gamma \Rightarrow \frac{\rho^*}{\rho} = \left(\frac{p'}{p}\right)^{1/\gamma}$$

$$\rho^* = \rho\left(1 + \frac{1}{p}\frac{dp}{dz}\delta z\right)^{1/\gamma} = \rho + \frac{\rho}{\gamma p}\frac{dp}{dz}\delta z$$

for stability $\quad \rho + \frac{\rho}{\gamma p}\frac{dp}{dz}\delta z > \rho + \frac{d\rho}{dz}\delta z$

$$\frac{\rho}{\gamma p}\frac{dp}{dz} > \frac{d\rho}{dz} > \frac{d\rho}{dT}\frac{dT}{dz}$$

$$p = \frac{R_* \rho T}{\mu} = K\rho^\gamma \Rightarrow T = \frac{K\mu}{R_*}\rho^{\gamma-1}$$

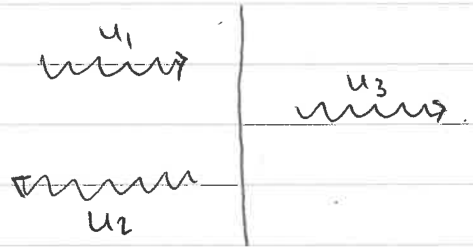
$$\frac{dT}{d\rho} = \frac{K\mu}{R_*}(\gamma-1)\rho^{\gamma-2} = \frac{\mu p}{R_*}(\gamma-1)\rho^{-2}$$

$$\frac{\cancel{\rho}}{\gamma \cancel{p}}\frac{dp}{dz} > \frac{R_*}{\mu \cancel{p}}\frac{1}{\gamma-1}\rho^{\cancel{2}}\frac{dT}{dz}$$

$$\frac{dp}{dz} > \frac{P}{T}\frac{\gamma}{\gamma-1}\frac{dT}{dz}$$

$$\left|\frac{dT}{dz}\right| < \left(1 - \frac{1}{\gamma}\right)\frac{T}{P}\left|\frac{dp}{dz}\right|$$

c) Sound wave crosses boundary between 2 media



ρ_A, T_A $\qquad$ ρ_B, T_B

Pressure equilibrium

$u_1 = e^{i(k_1 x - \omega t)}$

$u_2 = r e^{-i(k_1 x + \omega t)}$

$u_3 = t e^{i(k_2 x - \omega t)}$

u, $\partial u/\partial x$ continuous across boundary at $x=0$

$1 + r = t$

$k_1(1-r) = k_2 t = k_2(1+r)$

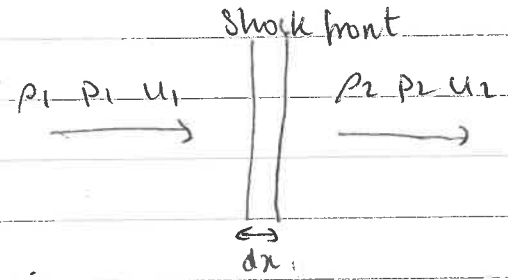
$$r = \frac{k_1 - k_2}{k_1 + k_2}, \quad t = \frac{2k_1}{k_1 + k_2}$$

$$c_s^2 = \frac{dp}{d\rho} \propto \frac{p}{\rho}$$

$k \propto 1/c_s \propto \sqrt{\rho}$

$$t = \frac{2\sqrt{\rho_A}}{\sqrt{\rho_A} + \sqrt{\rho_B}} = \frac{2}{1 + \sqrt{\delta}}, \quad \delta = \rho_A/\rho_B$$

3)a) Derive RH relations



continuity

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho u) = 0$$

$$\int_{-dx/2}^{dx/2} \frac{\partial \rho}{\partial t}\,dx + \left[\rho u\right]_{-dx/2}^{dx/2} = 0$$

limit $dx \to 0 \Rightarrow \rho_1 u_1 = \rho_2 u_2$

momentum equation

$$\rho\frac{\partial u}{\partial t} + \rho u_x u_x + p + \overset{\rho\,\partial\psi/\partial x}{\cancel{\rho\psi}} = 0$$

$$\int_{-dx/2}^{dx/2} \rho\frac{\partial u}{\partial t}\,dx + \left[\rho u_x u_x + p\right]_{-dx/2}^{dx/2} = 0$$ assuming ψ cts across boundary

limit $dx \to 0 \Rightarrow \rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$

energy equation

$$\frac{\partial E}{\partial t} + \nabla\cdot\left[(E+p)u\right] = -\rho\cancelto{0}{\dot{Q}_{cool}} + \rho\cancelto{0}{\frac{\partial\psi}{\partial t}}$$

$$\int_{-dx/2}^{dx/2} \frac{\partial E}{\partial t}\,dx + \left[(E+p)u\right]_{-dx/2}^{dx/2} = 0$$

limit $dx \to 0 \Rightarrow (E_1+p_1)u_1 = (E_2+p_2)u_2$

$$E = \rho\left(\tfrac{1}{2}u^2 + \varepsilon + \psi\right)$$

$$\varepsilon = c_v T$$

$$c_p = \gamma c_v = {R_*}/{\mu} + c_v \Rightarrow c_v = \frac{R_*/\mu}{\gamma-1} = \frac{1}{\gamma-1}\frac{P}{\rho T}$$

$$\varepsilon = \frac{1}{\gamma-1}\frac{P}{\rho}$$

$$E = \rho\left(\tfrac{1}{2}u^2 + \frac{1}{\gamma-1}\frac{P}{\rho} + \psi\right)$$

$$(E_1+p_1)\cancel{\rho_1}u_1 = \rho_1 u_1\left(\tfrac{1}{2}u_1^2 + \frac{1}{\gamma-1}\frac{p_1}{\rho_1} + \psi_1 + \frac{p_1}{\rho_1}\right) = \rho_1 u_1\left(\tfrac{1}{2}u_1^2 + \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1}\right)$$

assuming ψ continuous across boundary and using $\rho_1 u_1 = \rho_2 u_2$

$$\frac{1}{2}u_1^2 + \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1} = \frac{1}{2}u_2^2 + \frac{\gamma}{\gamma-1}\frac{p_2}{\rho_2}$$

3)b) $\frac{p_2}{p_1} = \frac{2M^2\gamma}{\gamma+1}$, $\frac{u_1}{u_2} = \frac{\rho_2}{\rho_1} = \frac{\gamma+1}{\gamma-1}$

$$T = \cancel{R\rho p}\ \frac{\mu p}{R_* \rho} \Rightarrow \frac{T_1}{T_2} = \frac{p_1 \rho_2}{\rho_1 p_2} = \frac{\gamma+1}{\gamma-1}\,\frac{\gamma+1}{2M^2\gamma} = \frac{(\gamma+1)^2}{2M^2\gamma(\gamma-1)}$$

heating rate = mass flow rate × $c_v \Delta T$ per unit mass

$$= \rho v S \times c_v (T_2 - T_1)$$

$$c_p = \gamma c_v = R_*/\mu + c_v \Rightarrow c_v = \frac{R_*/\mu}{\gamma - 1}$$

$$T_2 - T_1 = T_1\left(\frac{2M^2\gamma(\gamma-1)}{(\gamma+1)^2} - 1\right) \sim \frac{2M^2\gamma(\gamma-1)}{(\gamma+1)^2} T_1 \qquad (M \text{ large})$$

$$\text{heating rate } \dot{Q} = \rho v S \frac{R_*/\mu}{\gamma - 1} \frac{2M^2\gamma(\gamma-1)}{(\gamma+1)^2} T_1$$

$$\frac{R_* T_1}{\mu} = \frac{p_1}{\rho_1} = \frac{c_1^2}{\gamma} = \frac{v^2}{\gamma M^2}$$

$$\dot{Q} = \rho v S . \frac{v^2}{\gamma - 1} \frac{2\cancel{\gamma}(\gamma-1)}{(\gamma+1)^2} = \frac{2\rho S v^3 \cancel{\gamma}}{(\gamma+1)^2}$$

c) galaxy approximated by spherically symmetric system with $\psi = -Ar^{-1/2}$

isothermal, temperature T, in pressure equilibrium at radius r_0 with external medium with pressure p_0

find gas density in galaxy as a function p_0, T, r

$$\cancel{\frac{\partial u}{\partial t}} + \cancel{u \cdot \nabla u} = -\frac{1}{\rho}\nabla p - \nabla\psi$$

$$\frac{1}{\rho}\nabla p = -\nabla\psi$$

$$\nabla\psi = \tfrac{1}{2} A r^{-3/2}$$

$$\nabla p = \frac{dp}{d\rho}\nabla\rho = \frac{R_* T}{\mu}\nabla\rho$$

$$\int d\ln\rho = \frac{\mu A}{2R_*T}\int r^{-3/2}\,dr$$

$$\ln\rho = -\frac{\mu A}{R_*T} r^{-1/2} + c$$

$$p = p_0 \Rightarrow \rho = \frac{\mu p_0}{R_*T} \quad \text{at } r = r_0$$

$$\ln\left(\frac{\mu p_0}{R_*T}\right) = -\frac{\mu A}{R_*T} r_0^{-1/2} + c \quad \Rightarrow \quad c = \ln\frac{\mu p_0}{R_*T} + \frac{\mu A}{R_*T} r_0^{-1/2}$$

$$\ln\rho = -\frac{\mu A}{R_*T}\left(r^{-1/2} - r_0^{-1/2}\right) + \ln\frac{\mu p_0}{R_*T}$$

$$\ln\left(\frac{\rho R_*T}{\mu p_0}\right) = -\frac{\mu A}{R_*T}\left(r^{-1/2} - r_0^{-1/2}\right)$$

$$\rho = \frac{\mu p_0}{R_*T}\exp\left[-\frac{\mu A}{R_*T}\frac{1}{\sqrt{r_0}}\left(\sqrt{\frac{r_0}{r}} - 1\right)\right]$$

d) small galaxy collides radially with larger galaxy from c)
distance between small galaxy and centre of large galaxy is $d(t)$
neglecting shocks, find expression for ram pressure exerted by gas in large galaxy on small galaxy, as a function of d.

$$g = -\frac{1}{2} A d^{-3/2} = \ddot{d}$$

$$\dot{d}\frac{d\dot{d}}{dt} = -\frac{A}{2} d^{-3/2}$$

$$\frac{1}{2}\dot{d}^2 = A d^{-1/2} + c$$

$\dot{d} = 0$ at $d = r_0$

$c = -A r_0^{-1/2}$

$$\frac{1}{2}\dot{d}^2 = A(d^{-1/2} - r_0^{-1/2})$$

$$\dot{d} = \sqrt{2A}\left[d^{-1/2} - r_0^{-1/2}\right]^{1/2}$$

$$P_{ram} = \rho\dot{d}^2 = 2A\rho\left[d^{-1/2} - r_0^{-1/2}\right]$$

$$P_{ram} = \frac{2A\mu\rho_0}{R_* T}\left[\frac{\sqrt{r_0} - \sqrt{d}}{\sqrt{d r_0}}\right] \exp\left\{-\frac{\mu A}{R_* T}\frac{1}{\sqrt{r_0}}\left(\left(\frac{r_0}{d}\right)^{1/2} - 1\right)\right\}$$

4)a) Derive Bernoulli's equation for a gas with a barotropic EOS

momentum equation

$$\frac{\partial u}{\partial t} + u\cdot\nabla u = -\frac{1}{\rho}\nabla p - \nabla\psi$$

$$u\cdot\nabla u = \nabla(1/2\, u^2) - u\times(\nabla\times u).$$

$$\frac{d}{dx}\left(\int\frac{dp}{\rho}\right) = \frac{dp}{dx}\frac{d}{dp}\left(\int\frac{dp}{\rho}\right) = \frac{1}{\rho}\frac{dp}{dx}$$

$$\nabla\left(\int\frac{dp}{\rho}\right) = \frac{1}{\rho}\nabla p$$

$$\frac{\partial u}{\partial t} + \nabla(1/2\, u^2) - u\times(\nabla\times u) + \nabla\left(\int\frac{dp}{\rho}\right) + \nabla\psi = 0$$

take dot product with u:

$$u\cdot\frac{\partial u}{\partial t} - \underbrace{u\cdot[u\times(\nabla\times u)]}_{0} + u\cdot\nabla\left(1/2\, u^2 + \int\frac{dp}{\rho} + \psi\right) = 0$$

steady state $\partial u/\partial t = 0$

$$u\cdot\nabla\left(1/2\, u^2 + \int\frac{dp}{\rho} + \psi\right) = 0$$

$$H = 1/2\, u^2 + \int\frac{dp}{\rho} + \psi = \text{const along a streamline}$$

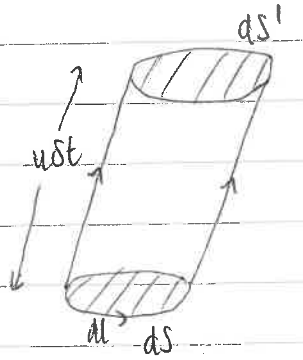
b) Show that the flux of the vorticity w over a surface S within a barotropic ~~equa~~ fluid is conserved, ie

$$\frac{D}{Dt}\int_S w\cdot dS = 0$$

$$\frac{D}{Dt}\int_S w\cdot dS = \int_S\frac{\partial w}{\partial t}\cdot dS + \int_S w\cdot\frac{D}{Dt}dS$$

$$\int_S\frac{\partial w}{\partial t}\cdot dS = \int_S[\nabla\times(v\times w)]\cdot dS$$ using Helmholtz equation

$$-\frac{\partial w}{\partial t} = \nabla\times(v\times w)$$



vector area of curved part

$$= \oint -u\delta t\times dl$$

vector area of whole surface $= 0$

$$dS' - dS = \delta t\oint u\times dl$$

$$\frac{D}{Dt}dS = \oint u\times dl$$

$$\int_S w \cdot \frac{D}{Dt} dS = \int_S \oint_c w \cdot (u \times dl) \cdot dS = \int_S \oint_C dl \cdot (w \times u) \cdot dS$$

internal loops cancel out

$$\int_S w \cdot \frac{D}{Dt} dS = \int_B (w \times u) \cdot dl = \int_S [\nabla \times (w \times u)] \cdot dS$$

$$\frac{D}{Dt} \int_S w \cdot dS = \int_S [\nabla \times (v \times w)] \cdot dS + \int_S [\nabla \times (w \times v)] \cdot dS = 0$$

as $w \times v = - v \times w$

c) Star of mass M produces isothermal wind with temperature T_0, mass outflow rate $\dot{M}$

max velocity at $r = r_m$

find density at r_m as a function of $T_0, \dot{M}, M$

$\dot{M} = 4\pi r^2 \rho u$ = const.

at $r = r_m$, $\dot{M} = 4\pi r_m^2 \rho_m u_{max}$, $u_m = u_{max}$, $\rho_m = \rho(r_m)$

$$\rho_m = \frac{\dot{M}}{4\pi r_m^2 u_m}$$

Steady state $u \cdot \nabla u = -\frac{1}{\rho} \nabla p + g$

$$u \frac{du}{dr} = -\frac{1}{\rho} \frac{dp}{d\rho} \frac{d\rho}{dr} + g \quad \text{- spherically symmetric}$$

$$u^2 \frac{d \ln u}{dr} = -c_s^2 \frac{d \ln \rho}{dr} - \frac{GM}{r^2}$$

$$\dot{M} = \text{const} \Rightarrow \frac{d \ln \dot{M}}{dr} = 0 = \frac{d \ln u}{dr} + \frac{d \ln \rho}{dr} + \frac{2}{r}$$

$$u^2 \frac{d \ln u}{dr} = c_s^2 \left(\frac{d \ln u}{dr} + \frac{2}{r} \right) - \frac{GM}{r^2}$$

$$(u^2 - c_s^2) \frac{d \ln u}{dr} = \frac{2 c_s^2}{r} \left(1 - \frac{GM}{2 c_s^2 r} \right)$$

Max velocity at r_m with $d \ln u / dr = 0 \Rightarrow r_m = GM / 2c_s^2$

$$\dot{M} = 4\pi r_m^2 \rho_m c_s = \pi \rho_m G^2 M^2 / c_s^3$$

$$c_s^2 = \frac{R_* T_0}{\mu} \Rightarrow \rho_M = \frac{\dot{M}}{\pi (GM)^2} \left(\frac{R_* T_0}{\mu} \right)^{3/2}$$

d) Now include radiation pressure

luminosity $= L_*$

radiation absorption cross section σ

Force exerted by radiation pressure

$F_{rad} = \frac{\sigma S}{c}$, S = radiation flux p u area $= \frac{L_*}{4\pi r^2}$

$$F_{rad} = \frac{\sigma L_*}{4\pi r^2 c}$$

Max velocity reached at r_m'

effective force $F_{eff} = -\frac{GM}{r^2} + \frac{\sigma L_*}{4\pi r^2 c}$

$$(u^2 - c^2)\frac{d\ln u}{dr} = \frac{2c_s^2}{r} - \frac{GM}{r^2} + \frac{\sigma L_*}{4\pi r^2 c}$$

$$= \frac{2c_s^2}{r}\left(1 - \frac{GM}{2c_s^2 r} + \frac{\sigma L_*}{8\pi c_s^2 c r}\right)$$

$$r_m' = \frac{GM}{2c_s^2} - \frac{\sigma L_*}{8\pi c c_s^2} = \frac{\mu}{2R_* T_0}\left(GM - \frac{\sigma L_*}{4\pi c}\right)$$

e) perfect gas, atomic hydrogen

at distance r_m hydrogen atoms ~~re~~combine ⇒ molecules

energy from formation of molecules radiated away

Changes in gas properties?

still isothermal if energy from molecule formation radiated away

velocity, density unaffected

$\mu \rightarrow 2\mu$

$p = \frac{R_* \rho T}{\mu}$ - decreases by factor of 2