UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Wednesday 31 May 2017 9.00 am to 11.00 am

PHYSICS (3)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (3)

ADVANCED QUANTUM PHYSICS

Candidates offering this paper should attempt a total of **three** *questions. The questions to be attempted are* **1**, **2** *and* **one** *other question.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains **four** *sides, including this coversheet, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
2 × 20 Page Answer Book	Mathematical Formulae handbook
Rough workpad	Approved calculator allowed
Yellow master coversheet	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

ADVANCED QUANTUM PHYSICS

1 *Attempt* **all** *parts of this question. Answers should be concise and relevant formulae may be assumed without proof.*

(a) A hydrogen atom is subjected to a steady magnetic field of strength $B_0 = 1$ T along the $z-$axis, with which the proton spin is aligned. A magnetic field $B_1 = 10^{-4}$ T modulated at angular frequency ω is applied perpendicular to z for a short duration τ. Afterwards, the proton spin is aligned in the $-z$ direction. Determine ω and τ.
[*The proton has* $g_p \simeq 5.6$ *and the nuclear magneton is* $\mu_N = 5.05 \times 10^{-27}$ J/T.] [4]

(b) For a system with two angular momentum operators $\widehat{\boldsymbol{J}_1}, \widehat{\boldsymbol{J}_2}$, which give total angular momentum $\widehat{\boldsymbol{J}} = \widehat{\boldsymbol{J}_1} + \widehat{\boldsymbol{J}_2}$, state the two maximal subsets of commuting operators among $\left\{\widehat{J}_{1z}, \widehat{J}_{2z}, \widehat{J}^2, \widehat{J}_z, \widehat{\boldsymbol{J}_1} \cdot \widehat{\boldsymbol{J}_2}, \widehat{J}_1^2, \widehat{J}_2^2\right\}$. [4]

(c) Using first-order perturbation theory, obtain the fractional change of the ground state energy of the hydrogen atom caused by the finite size of the proton, which we can model as a thin spherical shell with radius b.
[*The* 1s *wavefunction is* $\pi^{-1/2} a_0^{-3/2} \exp(-r/a_0)$ *with* a_0 *the Bohr radius.*] [4]

(a) (NMR was covered in the lectures, but doing an actual calculation like this may be unexpected).

The level separation is $\Delta E = g_p \mu_n B_0 = 2.82 \cdot 10^{-26}$ J, giving a Larmor frequency $\omega_L = \Delta E/\hbar = 2.69 \cdot 10^8$ s^{-1}. The high frequency field has to match this frequency to induce a spin-flip, so $\omega = \omega_L$. The resulting precession (or Rabi oscillation) occurs with rate $\omega_R = \frac{1}{2} g_p \mu_n B_1/\hbar = 10^{-4}\omega$, and for $\omega_R \tau = \pi$ we obtain $\tau = 2\pi/(2.69 \cdot 10^4 \text{ s}^{-1}) = 2.4 \cdot 10^{-4}$ s.

(b) (Should be bookwork, but it is harder than it looks).
Set 1: $\{J^2, J_1 \cdot J_2, , J_1^2, J_2^2, J_z\}$.
Set 2: $\{J_1^2, J_2^2, J_z, J_{1z}, J_{2z}\}$.

(c) (Standard first order perturbation theory calculation).

The change in the potential when assuming a spherical shell is $\frac{e^2}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{b}\right)$ for $r < b$ and 0 for $r > b$. This gives the first-order correction to the ground-state energy as $\Delta E = \frac{e^2}{4\pi\epsilon_0} \frac{1}{\pi a_B^3} \int_0^b 4\pi r^2 dr \left(\frac{1}{r} - 1b\right) e^{-2r/a_B}$. The ground state energy is $E_0 = -\frac{e^2}{4\pi\epsilon_0}\frac{1}{2a_B}$. The relative energy shift is therefore $\frac{\Delta E}{E_0} = -\frac{4b^2}{3a_b^2}$.

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2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) how to use the Wigner-Eckart theorem and Clebsch-Gordan coefficients to find the relative magnitudes of matrix elements. Examples could include the following cases:

$$M_1 = \left\langle j = \frac{3}{2},\ m = \frac{3}{2} \middle| \widehat{x} \middle| j = \frac{1}{2},\ m = \frac{1}{2} \right\rangle\ ,\ M_2 = \left\langle \frac{3}{2}, \frac{3}{2} \middle| \widehat{z} \middle| \frac{1}{2}, \frac{1}{2} \right\rangle\ ,\ M_3 = \left\langle \frac{3}{2}, \frac{1}{2} \middle| \widehat{z} \middle| \frac{1}{2}, \frac{1}{2} \right\rangle\ ;$$

(b) Landau quantisation for cyclotron orbits of electrons in a strong applied magnetic field, and examples of its experimental manifestation;

(c) coherent states for the 1D harmonic oscillator.

(a) Wigner-Eckart theorem:

- The *Wigner-Eckart theorem* helps us calculate matrix elements of scalar or vector operators.
- For *scalar* operators, it states that the matrix element between two angular momentum states is non-zero only if the two angular momentum states are the same, and that it does not depend on the projection into a particular direction: $\left\langle j'm' \middle| \widehat{K} \middle| jm \right\rangle = \delta_{j'j}\delta_{m'm} \left\langle j' \middle| |\widehat{K}| \middle| j \right\rangle$
- For *vector* operators, the situation is more complicated. A vector operator $\widehat{V}$ effectively carries one unit of angular momentum, $j = 1$. It can be expressed in terms of its spherical components $\widehat{V}_{+1} = -\frac{1}{\sqrt{2}}\left(\widehat{V}_1 + i\widehat{V}_2\right)$ which corresponds to $m = +1$, $\widehat{V}_{-1} = \frac{1}{\sqrt{2}}\left(\widehat{V}_1 - i\widehat{V}_2\right)$ $(m = -1)$, and $\widehat{V}_0 = \widehat{V}_3$ $(m = 0)$.
  The W-E theorem now states that the matrix element $\left\langle j''m'' \middle| \widehat{V} \middle| j'm' \right\rangle = \left\langle j'' \middle| |\widehat{V}| \middle| j' \right\rangle \langle j''m'' \mid j'm'; jm\rangle$, i.e. it can be expressed as the product between a factor that again does not depend on the projection into a particular direction (independent of m and m') and a factor that carries the dependence on the secondary angular momentum quantum numbers $m,\ m',\ m''$.
- The second factor above gives the overlap between a product state involving two angular momenta: $j,\ m$ and $j',\ m'$, and the state $j'',\ m''$. This means that the state produced by applying the vector operator $\widehat{V}$ on $j\ m'$ has the same angular momentum as the product state $|j = 1,\ m\rangle\,|j'\ m'\rangle$. The factor is given by a *Clebsch-Gordan coefficient* – a set of coefficients computed in the context of adding angular momenta.
- Working through the examples, we can illustrate how this works. Note, first, that the operators $\widehat{x}$ and $\widehat{z}$ are components of the vector operator $\widehat{V} = \boldsymbol{r}$. We find that $\widehat{x} = \frac{1}{\sqrt{2}}\left(\widehat{V}_{-1} - \widehat{V}_{+1}\right)$ and $\widehat{z} = V_3$. To find out the relative magnitudes of the three matrix elements, we need to work out the Clebsch-Gordan coefficients
  $c_1 = \left\langle j'' = \frac{3}{2},\ m'' = \frac{3}{2} \middle|\ j' = \frac{1}{2},\ m' = \frac{1}{2}; j = 1,\ m = +1 \right\rangle$ (from $\widehat{V}_{+1}$),
  $c_2 = \left\langle j'' = \frac{3}{2},\ m'' = \frac{3}{2} \middle|\ j' = \frac{1}{2},\ m' = \frac{1}{2}; j = 1,\ m = -1 \right\rangle$ (from $\widehat{V}_{-1}$),
  $c_3 = \left\langle j'' = \frac{3}{2},\ m'' = \frac{3}{2} \middle|\ j' = \frac{1}{2},\ m' = \frac{1}{2}; j = 1,\ m = 0 \right\rangle$ and
  $c_4 = \left\langle j'' = \frac{3}{2},\ m'' = \frac{1}{2} \middle|\ j' = \frac{1}{2},\ m' = \frac{1}{2}; j = 1,\ m = 0 \right\rangle$.
- For the brief notes question, only an explanation is needed about how the C-G coefficients are calculated, namely by using a single product state for the total angular momentum state with $m'' = j''$, and then using lowering operators to produce the lower total angular momentum states $m'' < j''$, as well as making use of

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orthogonality. It helps to show this explicitly on the example here. The first C-G coefficient is clearly $c_1 = 1$, as there is only one product state in $j'' = 3/2,\ m'' = 3/2$. The second is $c_2 = 0$, as $m + m' \neq m''$. The third is also zero, $c_3 = 0$, for the same reason. The fourth requires more work. Applying the total angular momentum lowering operator $\widehat{J}_- = \widehat{S}_- + \widehat{L}_-$ to both sides of the equation $\left|\frac{3}{2}\ \frac{3}{2}\right\rangle = \left|\frac{1}{2}\ \frac{1}{2}\right\rangle |11\rangle$ produces (using $\widehat{J}_- |jm\rangle = \sqrt{j(j+1) - m(m-1)}\,|jm-1\rangle$):

$$\sqrt{\frac{15}{4} - \frac{3}{4}}\left|\frac{3}{2}\ \frac{1}{2}\right\rangle = \sqrt{\frac{3}{4} - \frac{1}{4}}\left|\frac{1}{2}\ -\frac{1}{2}\right\rangle |1\ 1\rangle + \sqrt{2-0}\left|\frac{1}{2}\ \frac{1}{2}\right\rangle |1\ 0\rangle$$

We therefore find that

$$c_4 = \left\langle \frac{3}{2}\frac{1}{2}\ \middle|\ \frac{1}{2}\frac{1}{2};\ \ 1\ 0 \right\rangle = \sqrt{2/3}$$

•This gives the matrix elements as $M_1 = -\frac{1}{\sqrt{2}}M$, $M_2 = 0$, $M_3 = \sqrt{\frac{2}{3}}M$.

(b) Landau level quantisation:

•In strong magnetic fields, the quantum states of free electrons have a simple harmonic oscillator spectrum with energy level spacing $\hbar\omega_c$, where $\omega_c = \frac{eB}{m_e}$ is the cyclotron frequency.

•This can be shown by noting that the Hamiltonian $\widehat{H} = \frac{1}{2m_e}\left(\widehat{\boldsymbol{p}} + e\boldsymbol{A}\right)^2$ for a field along z and choosing the Landau gauge $\boldsymbol{A} = (-By, 0, 0)$ takes the form

$$\widehat{H} = \frac{1}{2m_e}\left[(\widehat{p}_x - eBy)^2 + \widehat{p}_y^2 + \widehat{p}_z^2\right]$$

Factorising the eigenstates into travelling wave solutions along x and z and a general y−dependence $\chi(y)$, we find that $\chi(y)$ satisfies an SHO Schrödinger equation

$$\left[\frac{\widehat{p}_y^2}{2m_e} + \frac{1}{2}m_e\omega_c^2\,(y - y_0)^2\right]\chi(y) + \frac{\widehat{p}_z^2}{2m_e}\chi(y) = E\chi(y)$$

giving quantised energy levels $E_{\nu,\ p_z} = \left(\nu + \frac{1}{2}\right)\hbar\omega_c + \frac{p_z^2}{2m_e}$. These states are centred on positions $y_0 = p_x/(eB)$.

•This spectrum is modified slightly by taking into account Zeeman splitting of levels due to the electron spin, which adds a term $-g_e\frac{\mu_B}{\hbar}B\widehat{S}_z$ to the Hamiltonian, giving a new spectrum

$$E_{\nu,\ p_z} = \left(\nu + \frac{1}{2} \pm \frac{1}{2}\right)\hbar\omega_c + \frac{p_z^2}{2m_e}$$

•Landau level quantisation leads to observable phenomena in metals. Particularly striking examples are found in two-dimensional electron gases, which can be produced with the tools of semiconductor physis. In a two-dimensional electron gas, we can show that the density of states is independent of energy.

- How many degenerate states are there for each Landau level? In a sample of extent L_y along y, the allowed spread of p_x given by the spread of y_0 falling within L_y is $\Delta p_x = eBL_y$. Using periodic boundary conditions, this produces $N_x = \frac{eBL_y}{2\pi\hbar/L_x} = \frac{eBA}{h}$ distinct states, where A is the area of the sample. We find the number of degenerate states per unit area as $n_B = N_x/A = Be/h$. If there are n electrons per unit area in total, and we include the spin degeneracy, the number of fully occupied Landau levels is $n_L = \text{int}\left[\frac{n}{2n_B}\right] = \text{int}\left[\frac{nh}{2eB}\right]$. (This is equal to the number of electrons divided by the number of flux-lines). The occupancy of the highest non-empty Landau-level is $f_L = \frac{nh}{2eB} - \text{int}\left[\frac{nh}{2eB}\right]$. This fraction changes between 0 and 1 as a function of B, with periodicity $\Delta(1/B) = \frac{2e}{nh}$.
- As the Fermi energy for $B = 0$ is $E_F = \frac{\hbar^2 k_F^2}{2m_e}$, and the number of $k-$ states up to the Fermi energy is $\pi k_F^2 A/(2\pi)^2$, we find that the number of electrons per unit area is $n = k_F^2/(2\pi) = \frac{\pi 2 m_e E_F}{2\pi\hbar^2}$, giving $\Delta(1/B) = \frac{e2\pi}{\hbar\pi k_F^2}$ (Onsager relation) or $\Delta(1/B) = \frac{\hbar e}{m_e E_F}$ or $\Delta\left(\frac{E_F}{\hbar\omega_c}\right) = 1$.
- As a consequence of this oscillatory behaviour in the filling of the highest Landau level, all bulk properties oscillate as a function of magnetic field.
- An extreme case is the quantum Hall effect. If the field is such that $f_L = 0$, i.e. all Landau levels are either filled or empty, then there is no scattering. This allows dissipationless current flow, $\rho_{xx} = 0$, although unfortunately $\sigma_{xx} = 0$, i.e. the longitudinal conductivity vanishes together with the longitudinal resistivity. The Hall resistivity ρ_{xy}, however, takes on quantised values $\rho_{xy} = \frac{B}{n_e e} = \frac{B}{n_L Bh/e} = \frac{h}{n_L e^2}$.
- This should be illustrated with sketches of ρ_{xy} and ρ_{xx} as a function of B, showing plateaus in one and spikes in the other.

(c) Coherent states:

- A coherent state is an eigenstate of the annihilation operator:

$$\widehat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$$

- We expand $|\alpha\rangle$ in terms of number states $|n\rangle$: $|\alpha\rangle = \sum c_n\,|n\rangle$.
- By forming $\langle n|\widehat{a}\,|\alpha\rangle$, we can obtain a recursion relation for c_n. The recursion relation is solved by $c_n = \frac{\alpha^n}{\sqrt{n!}} c_0$. Including normalisation, this gives

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum \frac{\alpha^n}{\sqrt{n!}}\,|n\rangle = e^{-|\alpha|^2/2} e^{\alpha\widehat{a}^\dagger}\,|0\rangle$$

- The probability of having n photons in the coherent state $|\alpha\rangle$ follows a Poisson distribution.
- A coherent state can be shown to have minimum uncertainty, $\Delta x\Delta p = \hbar/2$. It evolves with time without changing its form. The expectation values of x and p are solutions of the classical oscillator equations of motion. The width Δx remains constant. The radiation emitted by lasers can be represented as a coherent electromagnetic state, or Glauber state.

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3 *Attempt* **either** *this question* **or** *question 4.*

An atom with a single valence electron, which is in a state with $\ell = 1$, is subject to a magnetic field $\boldsymbol{B}$ along the $z-$axis, giving rise to the Hamiltonian

$$\widehat{H} = \frac{\mu_B B}{\hbar}\left(\widehat{L}_z + 2\widehat{S}_z\right) + \frac{2K}{\hbar^2}\widehat{\boldsymbol{L}} \cdot \widehat{\boldsymbol{S}}$$

Briefly outline the origin of the terms in this Hamiltonian. Show that $\widehat{\boldsymbol{L}} \cdot \widehat{\boldsymbol{S}} = \frac{1}{2}\left(\widehat{L}_+\widehat{S}_- + \widehat{L}_-\widehat{S}_+\right) + \widehat{L}_z\widehat{S}_z$. [6]

(The underlying idea of electronic states in a strong magnetic field has been disussed at length in the course, but doing the actual calculation in full is new.)

The first term handles coupling of magnetic field to orbital and spin angular momentum. The second term expresses the effect of spin-orbit coupling. Find expression for $L \cdot S = L_x S_x + L_y S_y + L_z S_z$ by substituting $S_+ = S_x + iS_y$ etc., and multiplying out.

Denote the eigenstates of $\widehat{L}_z$ as $|\phi_1\rangle$, $|\phi_0\rangle$, $|\phi_{-1}\rangle$, and the eigenstates of $\widehat{S}_z$ by $|\uparrow\rangle$, $|\downarrow\rangle$. This produces the six product states

$$|\psi_1\rangle = |\phi_1 \uparrow\rangle\,,\ |\psi_2\rangle = |\phi_1 \downarrow\rangle\,,\ |\psi_3\rangle = |\phi_0 \uparrow\rangle\,,\ |\psi_4\rangle = |\phi_0 \downarrow\rangle\,,\ |\psi_5\rangle = |\phi_{-1} \uparrow\rangle\,,\ |\psi_6\rangle = |\phi_{-1} \downarrow\rangle\ .$$

Show that the matrix elements $H_{mn} \equiv \langle\psi_m|\,\widehat{H}\,|\psi_n\rangle$ are given by

$$\boldsymbol{H} = \begin{pmatrix} 2a+b & 0 & 0 & 0 & 0 & 0 \\ 0 & -b & c & 0 & 0 & 0 \\ 0 & c & a & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & c & 0 \\ 0 & 0 & 0 & c & -b & 0 \\ 0 & 0 & 0 & 0 & 0 & -2a+b \end{pmatrix},$$

and determine a, b and c in terms of the constants in the Hamiltonian.
[*You may wish to use the identity* $\widehat{J}_\pm\,|j,\ m\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}\,|j,\ m\pm 1\rangle$.] [6]

Action of H on the product states:

$$\begin{aligned} H\,|\psi_1\rangle &= (2\mu_B B + K)\,|\psi_1\rangle \\ H\,|\psi_2\rangle &= -K\,|\psi_2\rangle + \sqrt{2}K\,|\psi_3\rangle \\ H\,|\psi_3\rangle &= \mu_B B\,|\psi_3\rangle + \sqrt{2}K\,|\psi_2\rangle \\ H\,|\psi_4\rangle &= -\mu_B B\,|\psi_4\rangle + \sqrt{2}K\,|\psi_5\rangle \\ H\,|\psi_5\rangle &= -K\,|\psi_5\rangle + \sqrt{2}K\,|\psi_4\rangle \\ H\,|\psi_6\rangle &= (-2\mu_B B + K)\,|\psi_6\rangle \end{aligned}$$

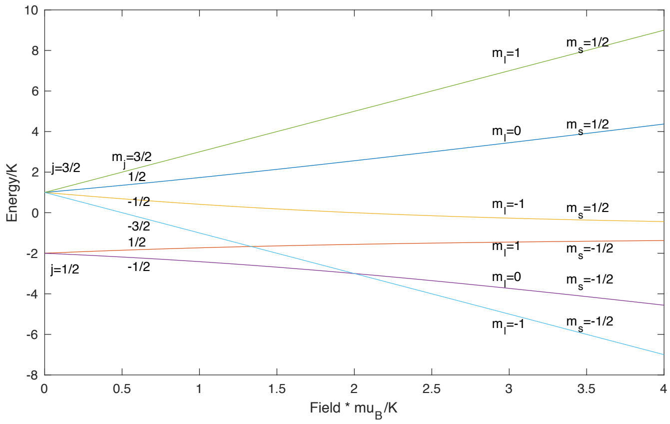
This produces the matrix given, if $a = \mu_B B$, $b = K$ and $c = \sqrt{2}K$.

Find the energies of the stationary states and sketch their dependence on the applied field $\boldsymbol{B}$. Label them with quantum numbers m_ℓ, m_s or j, m_j, as appropriate. [9]

Eigenvalues of the matrix give:

$$\begin{aligned} E_1 &= (2\mu_B B + K) \\ E_{2,3} &= \frac{1}{2}\left(\mu_B B - K \pm \sqrt{(\mu_B B + K)^2 + 8K^2}\right) \\ E_{4,5} &= \frac{1}{2}\left(-(\mu_B B + K) \pm \sqrt{(\mu_B B - K)^2 + 8K^2}\right) \\ E_6 &= (-2\mu_B B + K) \end{aligned}$$

Considering the limits for small and large B produces this sketch:



In the optical spectrum of sodium, the D_1 line arises at zero magnetic field from the transition $3^2p_{1/2} \to 3^2s_{1/2}$ and the D_2 line from $3^2p_{3/2} \to 3^2s_{1/2}$. The two lines differ in frequency by $\delta f = 5.16 \times 10^{11} s^{-1}$. Each line will split in an applied field. Estimate the magnetic field at which the split lines cross over. [$3^2p_{3/2}$ *means* $n = 3$, $2s + 1 = 2$, $\ell = 1$, $j = 3/2$. $\mu_B = 9.27 \times 10^{-24}$ J/T.] [4]

The crossing occurs roughly when $\mu_B B \simeq K$, and we get K from δf: $K = h\delta f = 2.1$ meV, giving $B \simeq 36$ T.

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4 *Attempt* **either** *this question* **or** *question 3.*

Particles with mass m are incident on a spherically symmetric potential

$$V(r) = \frac{A}{r} \mathrm{e}^{-\kappa r},$$

where A and κ are constants. Within the Born approximation, show that the differential scattering cross-section for the scattering wavevector $\boldsymbol{q} = \boldsymbol{k}_{\text{outgoing}} - \boldsymbol{k}_{\text{incoming}}$ is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{2mA}{\hbar^2(q^2 + \kappa^2)} \right)^2 .$$

[9]

[*You may find the following results useful:*
(i) $\int_0^\infty \sin(qr)\mathrm{e}^{-\kappa r}\mathrm{d}r = q/(q^2 + \kappa^2)$*;*
(ii) for $V(\boldsymbol{r}) = \delta(\boldsymbol{r})$, $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left\{ m/(2\pi\hbar^2) \right\}^2$.]

From this, show that α-particles of energy E incident on nuclei of atomic number Z will scatter by an angle θ according to

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{Ze^2}{8\pi\epsilon_0 E \sin^2(\theta/2)} \right)^2 .$$

[8]

A narrow beam of α-particles with energy $E = 7.68$ MeV is incident normally on a gold foil with thickness $t = 2.1 \times 10^{-7}$ m. The atomic weight of gold is 197, and its density is 19.3×10^3 kg/m^3. Particles scattered at an angle $\theta = 45°$ to the initial direction are counted by a detector, which has a transverse area of 1 mm^2 and is positioned 10 mm from the point where the α-particles hit the foil. It is found that a fraction $f = 3.7 \times 10^{-7}$ of incoming particles is scattered into the detector. What is the atomic number of gold according to these measurements? [8]

(First part has been done similarly in the lectures, the last part requires understanding scattering process practically and carrying out a numerical calculation)

In the Born approximation, the differential cross-section is $\frac{d\sigma}{d\Omega} = |f(\boldsymbol{q})|^2$ with

$$f(\boldsymbol{q}) = -\left(\frac{m}{2\pi\hbar^2} \right) \int V(\boldsymbol{r}) e^{i\boldsymbol{q}\cdot\boldsymbol{r}} d^3\boldsymbol{r}$$

Inserting the given screened potential and aligning the z-axis with $\boldsymbol{q}$, we evaluate the integral

$$\int V(r) e^{i\boldsymbol{q}\cdot\boldsymbol{r}} d^3\boldsymbol{r} = 2\pi \int_0^\infty V(r) r^2 dr \int_0^\pi e^{iqr\cos\theta} \sin\theta d\theta$$

Substituting $u = \cos\theta$ gives

$$f(q) = -\frac{m}{2\pi\hbar^2}\frac{4\pi A}{q}\int_0^\infty \sin(qr)e^{-\kappa r}dr$$

Using the result from the hint now gives the differential cross-section as:

$$\frac{d\sigma}{d\Omega} = \left(\frac{2mA}{\hbar^2(q^2+\kappa^2)}\right)^2$$

Now, for the Rutherford experiment, we let $\kappa \to 0$. The scattering angle θ is such that $\sin(\theta/2)k_0 = q/2$, where k_0 is the wavenumber of incoming particles of energy $E = \frac{\hbar^2 k_0^2}{2m}$. Hence, $\hbar^2 q^2/(2m) = (2\sin(\theta/2))^2\hbar^2 k_0^2/(2m) = 4\sin^2(\theta/2)E$. Noting that $A = 2eZe/(4\pi\epsilon_0)$, we obtain the Rutherford formula.

The fraction of particles scattered by 45° is
$f = 3.7\cdot 10^{-7} = \frac{d\sigma}{d\Omega}\cdot$ (no. of scattering centres) $\cdot\, d\Omega$. This gives

$$f = Z^2\left(\frac{e^2}{8\pi\epsilon_0 7.68\text{ MeV}\sin^2(\pi/8)}\right)^2 \frac{\rho_{Au}}{197\cdot 1.6\cdot 10^{-27}\text{ kg}}\cdot 2.1\cdot 10^{-7}\text{ m}\cdot\frac{1\text{ mm}^2}{(10\text{ mm})^2}$$

Finally, entering numbers (carefully...) we find an atomic number Z of about 84, which differs somewhat from the actual number for gold, $Z = 79$.

END OF PAPER