PAPER 2 (Relativity) – ANSWERS

(Note factors of c are omitted at some places in the answers, though they are meant to be right in the questions.)

1 (a) [*Unseen, though corresponds to a previous example sheet question.*] Equating energy and momentum for the particle and the photon, we have

$$\gamma mc^2 = h\nu$$
$$\gamma mu = \frac{h\nu}{c}$$

Assuming $u \neq 0$, then dividing we obtain $c^2/u = c$, from which $u = c$. This is impossible for a massive particle. If $u = 0$ then ν would have to be 0 to give zero momentum, but this contradicts the non-zero particle energy (mc^2).

(b) [*Unseen, though related to an example sheet question.*] $v = (\gamma, \gamma \boldsymbol{u})$ implies

$$a \equiv \frac{dv}{d\tau} = \left(\frac{d\gamma}{d\tau}, \frac{d}{d\tau}(\gamma \boldsymbol{u})\right)$$

so our information is that $\gamma \boldsymbol{u} \cdot \frac{d}{d\tau}(\gamma \boldsymbol{u}) = 0$. But quite generally, $v \cdot a = 0$ (since 4d length of v is constant), i.e. must have $\gamma \frac{d\gamma}{d\tau} - \gamma \boldsymbol{u} \cdot \frac{d}{d\tau}(\gamma \boldsymbol{u}) = 0$. Thus $\frac{d\gamma}{d\tau} = \gamma \frac{d\gamma}{dt} = 0$, i.e. γ, and therefore speed, is constant.

The particle is moving at constant speed $u = r\omega$ and hence $\gamma = \frac{1}{\sqrt{1-r^2\omega^2}}$. Then, since γ is constant, we have

$$a = \left(0, \gamma \frac{d}{d\tau}\left(\frac{d\boldsymbol{x}}{dt}\right)\right) = \left(0, \gamma^2 \frac{d^2\boldsymbol{x}}{dt^2}\right) = \left(0, -\frac{\omega^2}{1 - r^2\omega^2}\boldsymbol{x}\right)$$

(c) [*Unseen, though related to an example sheet question.*]

$$ds^2 = y^2\, dx^2 + x^2\, dy^2 \quad \Longrightarrow \quad \mathcal{L} = y^2\dot{x}^2 + x^2\dot{y}^2$$

Thus

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = 2\dot{x}y^2, \quad \frac{\partial \mathcal{L}}{\partial x} = 2x\dot{y}^2$$

and the x Euler-Lagrange equation is

$$\ddot{x}y^2 + 2\dot{x}\dot{y}y = x\dot{y}^2$$

By symmetry, the y Euler-Lagrange equation must be $\ddot{y}x^2 + 2\dot{x}\dot{y}x = y\dot{x}^2$.

If we put $y = mx$ in the second equation, we get $m\left(\ddot{x}x^2 + 2\dot{x}^2x\right) = mx\dot{x}^2$, which is $1/m$ times what we'd get by doing the same in the first. The two equations thus reduce to the single equation $x\ddot{x} + \dot{x}^2 = 0$ in this case, i.e. $y(\lambda) = mx(\lambda)$ is a possible motion. [*They are not asked for it, but a suitable solution would be* $y = m\sqrt{\lambda}, x = \sqrt{\lambda}$.]

2 (a) *The bending of light as a test of General Relativity*

The 'shape' equation for a photon trajectory in the equatorial plane of the Schwarzschild geometry is

$$\frac{d^2u}{d\phi^2} + u = \frac{3GM}{c^2}u^2 \tag{1}$$

where $u \equiv 1/r$. We can get a zeroth order solution to this by neglecting the r.h.s. This yields the straight line $u = \sin\phi/b$, where b is the impact parameter. Adding on a perturbation of the form:

$$u = \frac{\sin\phi}{b} + \Delta u,$$

and resubstituting into the original equation enables us to find a first order approximation, which is of the form

$$u = \frac{\sin\phi}{b} + \frac{3GM}{2c^2b^2}\left(1 + \tfrac{1}{3}\cos 2\phi\right)$$

(Exact form of this equation is not necessary.) Considering the limit as $r \to \infty$, i.e. $u \to 0$, we find the total deflection is

$$\Delta\phi = \frac{4GM}{c^2b}$$

(A diagram could be appropriate here to show how this comes about.) This is the famous GR result, and is a factor 2 bigger than would be predicted on Newtonian theory (first carried out by Soldner, 1804).

Eddington carried out the first measurement of this, in the eclipse expedition of 1919, where the expected deflection angle was $\sim 1.75°$. He obtained values consistent with GR (though there has been some controversy about this since).

Later high precision tests use radio sources: these can be observed near the Sun, even when there is no lunar eclipse.

Modern radio experiments using VLBI (very long baseline interferometry) can measure gravitational deflection of positions of radio quasars as they are eclipsed by the Sun with an accuracy of better than $\sim 10^{-4}$ arcseconds. The results are in excellent agreement with the predictions of general relativity.

(b) *Tidal forces and geodesic deviation*

An important consequence of curvature is that two nearby geodesics that are initially parallel will either converge or diverge, depending on the local curvature.

If we have two geodesics given by $x^a(u)$ and $\bar{x}^a(u)$ we can let $\xi^a(u)$ be the small 'vector' joining them. Then from

$$\begin{aligned}\frac{d^2x^a}{du^2} + \Gamma^a{}_{bc}\frac{dx^b}{du}\frac{dx^c}{du} &= 0,\\ \frac{d^2\bar{x}^a}{du^2} + \bar{\Gamma}^a{}_{bc}\frac{d\bar{x}^b}{du}\frac{d\bar{x}^c}{du} &= 0\end{aligned}$$

and expanding the Christoffel symbol values to first order in ξ^a, one can show

$$\frac{D^2\xi^a}{Du^2} + R^a{}_{cbd}\xi^b \dot{x}^c \dot{x}^d = 0$$

which is known as the *equation of geodesic deviation*, and gives us the relative acceleration of neighbouring test particles in terms of the Riemann curvature tensor.

This formulation allows us to give a quantitative treatment of tidal forces. From the above equation we can define a *tidal stress tensor*

$$S^\mu{}_\nu \equiv R^\mu{}_{\sigma\rho\nu} u^\sigma u^\rho$$

where u is 4-velocity, and by evaluation in the Instantaneous Rest Frame of the first particle (say), interpret the non-zero eigenvalues of $S^\mu{}_\nu$ as giving the forces that must be supplied to the particles in order to keep them moving parallel, as would be appropriate for a rigid body. For small velocities and fields, these in fact correspond to the standard Newtonian tidal forces.

We can apply these ideas near the horizon of a black hole, and there find a tension in the radial direction, and pressure in the transverse directions, with magnitudes proportional to $(+2GM/r^3, -GM/r^3, -GM/r^3)$. Comparing the magnitude of the tension with that which a human can endure, one finds that one needs black hole masses $\gtrsim 10^5 M_\odot$ in order for a human to be able to cross the event horizon without being torn apart.

(c) *The covariant formulation of electromagnetism in Minkowski spacetime and electromagnetic invariants*

The field strength tensor $F_{\mu\nu}$ is a covariant, tensorial way of combining the electric and magnetic fields into a single object.

From $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$, where A_μ is the four-vector potential (given by $(\phi/c, \boldsymbol{A})$, where ϕ is the electrostatic potential and $\boldsymbol{A}$ the usual 3d vector potential), together with the usual relations

$$\boldsymbol{B} = \nabla \wedge \boldsymbol{A}, \quad \boldsymbol{E} = \nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t}$$

we can deduce the form

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -cB_3 & cB_2 \\ -E_2 & cB_3 & 0 & -cB_1 \\ -E_3 & -cB_2 & cB_1 & 0 \end{pmatrix}$$

[Full credit will be given for any approximation to this — full accuracy is not expected, just the idea that $F_{\mu\nu}$ contains the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields arranged as an antisymmetric tensor.]

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The full set of Maxwell equations using this approach are

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= kj^\nu, \\ \partial_\sigma F_{\mu\nu} + \partial_\nu F_{\sigma\mu} + \partial_\mu F_{\nu\sigma} &= 0 \end{aligned}$$

where j^ν is the current 4-vector (equal to $(c\rho, \boldsymbol{j})$, where ρ is the charge density and $\boldsymbol{j}$ the usual 3d current).

Since $F_{\mu\nu}$ is a tensor, transformation to a new frame works like

$$F'_{\lambda\kappa} = F_{\mu\nu} \frac{\partial x^\mu}{\partial x'^\lambda} \frac{\partial x^\nu}{\partial x'^\kappa}$$

or in matrix form

$$\boldsymbol{F}' = \boldsymbol{\Lambda}^T \boldsymbol{F} \boldsymbol{\Lambda} \quad (*)$$

where Λ is a Lorentz transformation.

This arrangement now allows us to transform any set of $\boldsymbol{E}$ and $\boldsymbol{B}$ fields. We just carry out (*) and reidentify the components in the dashed frame via the matrix expression for $\boldsymbol{F}_{\mu\nu}$.

From the 4-vector A^μ or 4-tensor $F^{\mu\nu}$ and derivatives, one can construct many quantities invariant under Lorentz transformations.

Two important invariants containing only $F^{\mu\nu}$, which are therefore also gauge invariant, are

$$F^{\mu\nu} F_{\mu\nu} \qquad \text{and} \qquad \epsilon_{\mu\nu\lambda\rho} F^{\mu\nu} F^{\lambda\rho}$$

In terms of electric and magnetic fields, these invariants may be shown to be $E^2 - B^2$ and $\boldsymbol{E} \cdot \boldsymbol{B}$ respectively.

A physical consequence of this, is if $\boldsymbol{E}$ and $\boldsymbol{B}$ are orthogonal, or have equal magnitudes in one Cartesian inertial frame, then they are orthogonal or have equal magnitudes in all such frames.

3 [*Bookwork*] The Einstein tensor is

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

and the Ricci tensor is defined in terms of contraction of the curvature tensor as $R_{\mu\nu} = R^{\alpha}{}_{\mu\nu\alpha}$.

We rewrite the original equation given, with $T_{\mu\nu} = 0$, in *mixed form*, i.e.

$$R^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu} R + \Lambda \delta^{\mu}_{\nu} = 0$$

Contracting this yields $R - 2R + 4\Lambda = 0$, i.e. $R = 4\Lambda$, and reinserting this we get $R^{\mu}_{\nu} = \Lambda \delta^{\mu}_{\nu}$, i.e. $R_{\mu\nu} = \Lambda g_{\mu\nu}$ as stated.

[*Unseen in this form including a cosmological constant, but in fact simpler than the full Ricci expressions they have in the lectures (which have both A and B functions).*]

Differentiating $1 - A - rA' = r^2\Lambda$, we get $-A' - A' - rA'' = 2r\Lambda$, so

$$\frac{A'}{r} + \frac{1}{2}A'' = -\frac{1}{2r}(2r\Lambda) = -\Lambda$$

This gives (with a crucial bit being to understand that we have to insert the metric components in terms of A):

$$R_{00} = g_{00}\Lambda = A\Lambda = -A\left(\frac{A'}{r} + \frac{1}{2}A''\right)$$
$$R_{11} = g_{11}\Lambda = -\frac{1}{A}\Lambda = \frac{1}{A}\left(\frac{A'}{r} + \frac{1}{2}A''\right)$$

which verifies what we need.

For the integration, we get

$$1 - A - r\frac{dA}{dr} = r^2\Lambda \implies \frac{d}{dr}(rA) = 1 - r^2\Lambda \implies rA = r - \frac{1}{3}r^3\Lambda + C$$

i.e. we find $A = 1 - \frac{1}{3}r^2\Lambda + C/r$, where C is the constant of integration.

Examining this in relation to what we expect if $\Lambda = 0$, can see that we should take $C = -2GM$. Thus

$$A = 1 - \frac{2GM}{r} - \frac{1}{3}r^2\Lambda$$

[*Rest of question unseen, but parallel to derivations they already have in the lectures.*] The function in the Euler-Lagrange equations is

$$\mathcal{L} = A(r)\dot{t}^2 - \frac{1}{A(r)}\dot{r}^2 - r^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right)$$

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and we are working with θ fixed to $\pi/2$. The function is independent of t and ϕ so we immediately get

$$\frac{\partial \mathcal{L}}{\partial \dot{t}} = 2A\dot{t} = \text{constant} = 2k, \quad \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = -2r^2\dot{\phi} = \text{constant} = -2h$$

Here k is related to the particle energy (since energy is the 'momentum' conjugate to displacements in time),and h is the specific angular momentum. The remaining equation we can get by putting $\mathcal{L} \equiv 1$ for a massive particle, and this gives (substituting also for $\dot{t}$ and $\dot{\phi}$):

$$1 = A\left(\frac{k}{A}\right)^2 - \frac{1}{A}\dot{r}^2 - r^2\frac{h^2}{r^4}$$

which we can write as (reinstating factors of c)

$$\dot{r}^2 = c^2k^2 - A\left(c^2 - \frac{h^2}{r^2}\right)$$

For radial motion $h = 0$, and then differentiating the last equation w.r.t. proper time we get

$$\ddot{r} = -\frac{1}{2}c^2A'$$

Evaluating $\ddot{r}$ explicitly for the $A(r)$ above, i.e. for

$$A = 1 - \frac{2GM}{r} - \frac{1}{3}r^2\Lambda$$

we get

$$\ddot{r} = -\frac{GM}{r^2} + \frac{1}{3}\Lambda r$$

The physical interpretation is that we have an expected inward gravitational force going like $1/r^2$, but also an outward, *repulsive* force. This is the famous repulsive force due to the cosmological constant, which we can see goes $\propto r$, i.e. is proportional to distance. Our solution is thus a generalisation of the Schwarzschild solution to include a cosmological constant, and could be used e.g. to represent a black hole embedded in a de Sitter universe.

In the Newtonian limit neither force disappears, we just have proper time tending to ordinary coordinate time, and thus the Newtonian force per unit mass is the same as just discussed.

4 [*Bookwork through to where the geodesic equations in Advanced Eddington-Finkelstein coordinates are asked for.*]

Function in Euler-Lagrange is

$$\mathcal{L} = \left(1 - \frac{2\mu}{r}\right)\dot{t}^2 - \left(1 - \frac{2\mu}{r}\right)^{-1}\dot{r}^2 - r^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right)$$

and we are working with θ fixed to $\pi/2$. Function is independent of t and ϕ so immediately get

$$\frac{\partial\mathcal{L}}{\partial\dot{t}} = 2\left(1 - \frac{2\mu}{r}\right)\dot{t} = \text{constant} = 2k, \quad \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = -2r^2\dot{\phi} = \text{constant} = -2h$$

which does two of the equations. The third

$$\left(1 - \frac{2\mu}{r}\right)\dot{t}^2 - \left(1 - \frac{2\mu}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = 0$$

is just $\mathcal{L} = 0$, which is the appropriate value for a photon.

For the energy equation, we substitute in this last equation for $\dot{t}$ and $\dot{\phi}$ using k and h, respectively, and unwrap to get an expression for $\dot{r}$. This yields

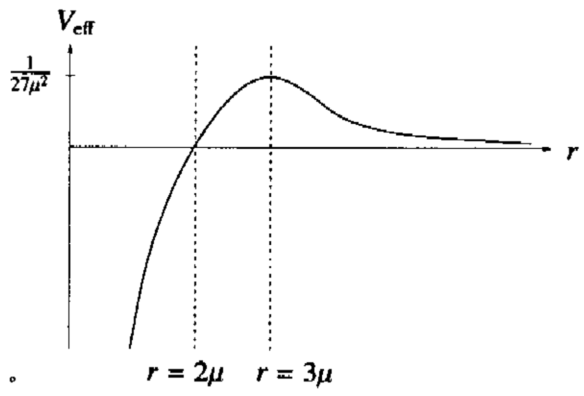
$$\dot{r}^2 + \frac{h^2}{r^2}\left(1 - \frac{2\mu}{r}\right) = c^2k^2$$

Thus see that $V_{\text{eff}}(r)$ for photons is given by

$$V_{\text{eff}}(r) = \frac{1}{r^2}\left(1 - \frac{2\mu}{r}\right)$$

and the constant b in the question is $h/(ck)$.

A plot of $V_{\text{eff}}(r)$ gives



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We can imagine the effective potential as a surface on which a 'ball bearing' type mass is moving, in 1d, such that its total energy is conserved. The photon's subsequent motion will be accurately modelled by this setup, since we've managed to get the equation to a Newtonian-like form, and it doesn't matter that the parameter is proper time rather than coordinate time. We can see that in principle if we place the 'ball bearing at the maximum occuring at $r = 3\mu$, then it doesn't have to immediately move, so a circular orbit at $r = 3\mu$ is possible (and not possible anywhere else, though it doesn't ask about this). However, the position is clearly unstable, and this corresponds to the instability of the photon orbit.

[*Rest of question unseen, though it parallels for AEF coordinates (which are discussed in the lectures) the bookwork development for Schwarzschild coordinates.*]

In the new coordinates, the Lagrangian for the geodesic method is now

$$\mathcal{L} = \left(1 - \frac{2\mu}{r}\right)\dot{t}'^2 - \frac{4\mu}{r}\dot{t}'\dot{r} - \left(1 + \frac{2\mu}{r}\right)\dot{r}^2 - r^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right)$$

From this, it is straightforward to read off, in exactly the same way as for the Schwarzschild coordinates case

$$\left(1 - \frac{2\mu}{r}\right)\dot{t}' - \frac{2\mu}{r}\dot{r} = k', \quad r^2\dot{\phi} = h'$$

$$\text{and} \quad \left(1 - \frac{2\mu}{r}\right)\dot{t}'^2 - \frac{4\mu}{r}\dot{t}'\dot{r} - \left(1 + \frac{2\mu}{r}\right)\dot{r}^2 - r^2\dot{\phi}^2 = 0$$

where again the last equation comes from setting $\mathcal{L} = 0$ for a photon. Note we have dashes on the k and h, since it hasn't told us to assume anything about the particle energy and angular momentum in the new metric.

We need the equation for $\dot{r}$ alone, so in the same way as in the Schwarzschild case, the obvious thing is to use the first equation to give $\dot{t}$ and plug this into the last equation, along with the replacement for $\dot{\phi}$ in terms of h'. This yields

$$\frac{\left(k' + \frac{2\mu}{r}\dot{r}\right)^2}{\left(1 - \frac{2\mu}{r}\right)} - \frac{4\mu\dot{r}}{r}\frac{\left(k' + \frac{2\mu}{r}\dot{r}\right)}{\left(1 - \frac{2\mu}{r}\right)} - \left(1 + \frac{2\mu}{r}\right)\dot{r}^2 - \frac{h^2}{r^2} = 0$$

which does indeed magically sort itself out into

$$\dot{r}^2 + \frac{h'^2}{r^2}\left(1 - \frac{2\mu}{r}\right) = k'^2.$$

meaning that $V_{\text{eff}}(r)$ is identical. We expect this since the r coordinate in each metric means the same thing physically (e.g. the proper area of a sphere at radius r is the same in each), and therefore the $V_{\text{eff}}(r)$ curve, since it too represents something physical, and is just a function of r, does not change.

END OF PAPER