UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Tuesday 4 June 2019 1.30 pm to 3.30 pm

PHYSICS (5)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (5)

ASTROPHYSICAL FLUID DYNAMICS

Candidates offering this paper should attempt a total of ***five*** *questions:* ***three*** *questions from Section A and* ***two*** *questions from Section B.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains ***five*** *sides, including this coversheet, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS
2 × 20 Page Answer Book
Metric graph paper
Rough workpad
Yellow master coversheet

SPECIAL REQUIREMENTS
Mathematical Formulae handbook
Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

*Attempt **all** questions in this Section. Answers should be concise and relevant formulae may be assumed without proof.*

1 Consider a uniform sphere of an ideal gas of particles of mass m with number density n at temperature T. By equating the work done against pressure with the change in gravitational potential energy when the sphere contracts, find the critical total mass M for gravitational collapse. [4]

2 A small-amplitude density disturbance $\rho_0(z) \to \rho_0(z) + \Delta\rho(z,t)$ in an isothermal atmosphere with equilibrium density $\rho_0(z) = \tilde{\rho}\exp(-z/H)$ obeys the equation

$$\frac{\partial^2 \Delta\rho}{\partial t^2} - c^2 \frac{\partial^2 \Delta\rho}{\partial z^2} - \frac{c^2}{H}\frac{\partial \Delta\rho}{\partial z} = 0.$$

Defining $\Delta\rho(z,t) = \Psi(z,t)\exp(-z/(2H))$, show that $\Psi(z,t)$ obeys

$$\frac{\partial^2 \Psi}{\partial t^2} - c^2 \frac{\partial^2 \Psi}{\partial z^2} - \omega_0^2 \Psi = 0.$$

Identify ω_0 and estimate the characteristic period $2\pi/\omega_0$ for the Earth's atmosphere. [4]

3 The dispersion relation of waves on the surface of a fluid of density ρ with surface tension σ in a gravitational field g is

$$\omega(k) = \sqrt{|k|\,(g + \sigma k^2/\rho)}.$$

How should this dispersion relation be modified to describe the instability of an *inverted* flat surface of water? Estimate the size of the largest container out of which water will not fall. Take $\sigma = 7.2 \times 10^{-2}\mathrm{Nm}^{-1}$ and $\rho = 1000\ \mathrm{kg\ m}^{-3}$. [4]

SECTION B

*Attempt **two** questions from this section*

4 Explain the meaning of the Lagrangian time derivative $\mathrm{D}\phi/\mathrm{D}t$ of a quantity ϕ. State the relation between the $\mathrm{D}\phi/\mathrm{D}t$ and the Eulerian time derivative $\partial\phi/\partial t$. [3]

Explain how the Lagrangian derivative leads to the Euler equation

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p,$$

where p is the pressure, ρ is the density and $\mathbf{u}$ is the velocity field. [4]

An incompressible fluid of uniform density ρ_0 rotates at angular velocity $\boldsymbol{\Omega}$. In a frame rotating with the fluid the acceleration $\mathbf{a}_\mathrm{r}$ of a fluid element is related to the acceleration $\mathbf{a}_\mathrm{l}$ in the lab frame by

$$\mathbf{a}_\mathrm{r} = \mathbf{a}_\mathrm{l} - 2\boldsymbol{\Omega}\times\mathbf{v}_\mathrm{r} - \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}),$$

where $\mathbf{v}_\mathrm{r}$ is the velocity in the rotating frame.

Show that in a coordinate frame rotating with the fluid the Euler equation takes the form

$$\rho_0\left(\frac{\partial \mathbf{u}}{\partial t} + 2(\boldsymbol{\Omega}\times\mathbf{u}) + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla\left(p - \frac{\rho_0}{2}|\boldsymbol{\Omega}\times\mathbf{r}|^2\right) \qquad (\star)$$

[4]

Consider a situation where the fluid velocity in the rotating frame is small. By taking the curl of the equation $(\star)$ and using the incompressibility condition find the linear equation describing small variations of $\mathbf{u}$. [3]

Taking $\boldsymbol{\Omega}$ to be along the z-axis, look for plane wave solutions of the form

$$\mathbf{u}(\mathbf{r},t) = \mathbf{u}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)},$$

where $\mathbf{u}_0$ is a constant vector. Find the dispersion relation $\omega(\mathbf{k})$ and describe how the frequency depends on the angle that the wavevector $\mathbf{k}$ makes with the z-axis. [5]

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5 Assuming adiabatic flow, the equations describing a spherically symmetric blast wave are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad (\star)$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial r} + \frac{2\rho u}{r} = 0$$

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial r}\right)(p\rho^{-\gamma}) = 0,$$

where u is the radial component of the velocity and other quantities have their usual meanings.

Explain the physical significance of the first two equations, and derive the third. [5]

Assume that after time t the radius R of the blast wave depends only on t, the energy E of the blast, and the resting density ρ_0 of the gas outside the blast wave. Explain why

$$R = \xi_0 E^{1/5} \rho_0^{-1/5} t^{2/5},$$

for some dimensionless constant ξ_0.

[3]

The density inside the blast wave can be written $\rho(r,t) = \rho_0 f_\rho(\xi)$, where $f_\rho(\xi)$ is a function of the dimensionless variable $\xi = r\left(\rho_0/Et^2\right)^{1/5}$. Show that if the velocity is written in terms of another function $f_u(\xi)$ as $u(r,t) = (r/t)f_u(\xi)$ then the kinetic energy inside the blast wave

$$E_{\mathrm{kin}} = \int_0^R 4\pi r^2 \frac{1}{2}\rho u^2 \mathrm{d}r$$

is constant in time. [4]

By substituting the given forms for $\rho(r,t)$ and $u(r,t)$ into $(\star)$, express $p(r,t)$ in terms of a third dimensionless function $f_p(\xi)$. Verify that the internal energy

$$E_{\mathrm{int}} = \int_0^R 4\pi r^2 \frac{p}{\gamma - 1} \mathrm{d}r$$

is constant. You do not need to determine the function $f_p(\xi)$. [7]

6 Consider a self-gravitating fluid with the gravitational potential Ψ related to the density ρ by

$$\nabla^2 \Psi = 4\pi G \rho,$$

with G being the constant of gravitation. Assume spherical symmetry.

Show that in static equilibrium

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}p}{\mathrm{d}r}\right) = -4\pi G r^2 \rho.$$

[4]

Find the radial distribution of pressure for an incompressible liquid (fixed ρ) sphere of radius R in vacuum. [4]

An ideal gas of particles at constant temperature satisfies $p = \rho c^2$, where c is the speed of sound. Show that $p(r) \propto 1/r^2$ is a possible equilibrium solution, and determine the constant of proportionality. Can this solution describe a gaseous sphere in vacuum? [5]

Now suppose that there is a steady outward radial flow $u(r)$ of gas from a spherical fluid body of mass M. Show that the radial velocity satisfies

$$\left(u - \frac{c^2}{u}\right)\frac{du}{dr} = \frac{2c^2}{r} - \frac{GM}{r^2}.$$

[3]

Show that at large distances the velocity behaves as

$$u(r) \to 2c\sqrt{\log(r/r_s)},$$

where $r_s = GM/(2c^2)$. [3]

END OF PAPER