4

B4

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu - p g^{\mu\nu}$$

Consider component in local cartesian inertial coordinate system $g^{\mu\nu} \to \eta^{\mu\nu}$. $u^\mu = (\gamma_u c, \gamma_u \underline{u})$

$$T^{00} = \gamma_u^2 \rho c^2 + (\gamma_u^2 - 1) p = \rho c^2$$ in rest frame

Energy density.

and in IRF only non-zero component $T^{ii} = p$

$$T^{i0} = \left(\rho + \frac{p}{c^2}\right) \gamma_u c\, u^i$$ – i^{th} component of the 3-momentum density $\times c$

$$T^{ij} = \left(\rho + \frac{p}{c^2}\right) \gamma_u^2 u^i u^j$$ – i^{th} component of 3-momentum in j direction

$\nabla_\mu T^{\mu\nu} = 0$ is continuity eqn

and gives conservation of energy & momentum.

when fluid is dust $p = 0$.

$$\nabla_\mu T^{\mu\nu} = 0 \quad \text{where} \quad T^{\mu\nu} = \rho u^\mu u^\nu$$

$$\Rightarrow \nabla_\mu (\rho u^\mu u^\nu) = \nabla_\mu (\rho u^\mu) u^\nu + \rho u^\mu \nabla_\mu u^\nu = 0 \quad (*)$$

contracting with u_ν and using $u_\nu u^\nu = c^2$

$$c^2 \nabla_\mu (\rho u^\mu) + \rho u_\nu u^\mu \nabla_\mu u^\nu = 0$$

now $\nabla_\mu u_\nu u^\nu = \nabla_\mu (c^2) = 0$

$\therefore$ 2nd term vanishes

and $c^2 \nabla_\mu (\rho u^\mu) = 0$

$\therefore$ back in $(*)$ gives

$$\underline{\underline{\rho u^\mu \nabla_\mu u^\nu = 0}}$$

Now $\dfrac{Du^\nu}{D\tau} = u^\mu \nabla_\mu u^\nu = 0$

intrinsic derivative along worldline

alternative form of geodesic eqn.

vanishes along a <u>geodesic</u>.

3 B

4-force $\quad f_\mu = q \, F_{\mu\nu} u^\nu$

~~eqn of motion~~ $\cancel{\frac{Dmu^\mu}{D\tau} = \frac{q}{m} F}$

(i) Require $f_\mu u^\mu = 0 \rightarrow$ Asymmetric.

ie $F_{\mu\nu} u^\mu u^\nu = 0 \implies F_{\mu\nu} = -F_{\nu\mu}$

raise indices $\underline{\underline{F^{\mu\nu} = -F^{\nu\mu}}}$

(ii) consider components of f_μ and relate EM

e.g. $f_0 = q \sum_i F_{0i} u^i = \frac{q}{c} \gamma_u \underline{E} \cdot \underline{u}$

$\rightarrow$ identify $F_{0i} = E^i/c$

similarly $f_i = q F_{i0} u^0 + q \sum_j F_{ij} u^j = -q\gamma_u \left(E^i + (\underline{u} \times \underline{B})^i\right)$

$\longrightarrow$ identify other components

$$F_{\mu\nu} = \begin{pmatrix} 0 & E^1/c & E^2/c & E^3/c \\ & 0 & -B^3 & B^2 \\ & & 0 & -B^1 \\ (-) & & & 0 \end{pmatrix}$$

(iii) Field Equations

Source $j^\mu = (c\rho, \underline{J})$

$$\underline{\underline{\nabla_\mu F^{\mu\nu} = \mu_0 . j^\nu}}$$

$$\underline{\underline{\nabla_{[\mu} F_{\nu\sigma]} = 0}}$$

together can recover Maxwell eqs in Minkowski space-time.

(iv) Transf

$$\underline{\underline{F'^{\mu\nu} = \Lambda^\mu{}_\sigma \Lambda^\nu{}_\rho F^{\sigma\rho}}}$$

(v)

$$\underline{\underline{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu}}$$

~~8~~

5

Dust particles carry charge q

~~Force eqn~~ ~~D~~

Eqn of motion

$$\frac{Du^\nu}{D\tau} = \frac{q}{m} F^\nu{}_\sigma u^\sigma$$

but $\frac{Du^\nu}{D\tau} = u^\mu \nabla_\mu u^\nu$ (earlier)

$$\therefore \quad u^\mu \nabla_\mu u^\nu = \frac{q}{m} F^\nu{}_\sigma u^\sigma$$

B5

$$ds^2 = c^2\left(1-\frac{2\mu}{r}\right)dt^2 - \left(1-\frac{2\mu}{r}\right)^{-1}dr^2$$

$$- r^2 d\theta^2 - r^2\sin^2\theta\, d\phi^2$$

Geodesic equations from Lagrangian $L = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$

$$L = c^2\left(1-\frac{2\mu}{r}\right)\dot{t}^2 - \left(1-\frac{2\mu}{r}\right)^{-1}\dot{r}^2$$

$$- r^2\dot{\theta}^2 - r^2\sin^2\theta\,\dot{\phi}^2$$

$$\frac{\partial L}{\partial x^\mu} = \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right)$$

t : $\frac{\partial L}{\partial \dot{t}} = \text{constant} \Rightarrow \left(1-\frac{2\mu}{r}\right)\dot{t} = k$

ϕ : $\frac{\partial L}{\partial \phi} = 0 \Rightarrow \frac{\partial L}{\partial \dot{\phi}} = \text{constant} \Rightarrow r^2\dot{\phi} = h$

with $\theta = \pi/2$

r: $-\left(1-\frac{2\mu}{r}\right)^{-1} 2\ddot{r} = \frac{2\mu c^2}{r^2}\dot{t}^2 - \left(1-\frac{2\mu}{r}\right)^{-2}\cdot\frac{2\mu}{r^2}\dot{r}^2$

$$- 2r\dot{\phi}^2 = 0$$

$$\Rightarrow \quad \left(1-\frac{2\mu}{r}\right)^{-1}\ddot{r} + \frac{\mu c^2}{r^2}\dot{t}^2 - \left(1-\frac{2\mu}{r}\right)^{-2}\frac{\mu\dot{r}^2}{r^2}$$

$$- r\dot{\phi}^2 = 0$$

Also note $\quad L = c^2 \quad$ for a massive particle

so

$$\left(1-\frac{2\mu}{r}\right)c^2\dot{t}^2 - \left(1-\frac{2\mu}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = c^2$$

eliminating $\dot{t}$ and $\dot{\phi}$

$$\dot{t}^2 = k^2\left(1-\frac{2\mu}{r}\right)^{-2} \qquad \dot{\phi}^2 = h^2/r^4$$

$$\therefore \quad c^2k^2\left(1-\frac{2\mu}{r}\right)^{-1} - \left(1-\frac{2\mu}{r}\right)^{-1}\dot{r}^2 - \frac{h^2}{r^2} = c^2$$

$$\Rightarrow \quad \dot{r}^2 + \frac{h^2}{r^2}\left(1-\frac{2\mu}{r}\right) - \frac{2\mu c^2}{r} = c^2(k^2-1)$$

Energy Equation $\quad V_{eff} = \dfrac{h^2}{2r^2}\left(1-\dfrac{2\mu}{r}\right) - \dfrac{\mu c^2}{r}$

Geodesic motion in circle with $r=R$

Circular orbit at radius R has $\dot{r}=0$ and

$$\frac{dV_{eff}}{dr} = 0$$

i.e.
$$\frac{d}{dr}\left(\frac{1}{2}\frac{h^2}{r^2}\left(1-\frac{2\mu}{r}\right) - \frac{\mu c^2}{r}\right) = 0$$

$$\Rightarrow \quad -\frac{h^2}{r^3} + \frac{3\mu h^2}{r^4} + \frac{\mu c^2}{r^2} = 0$$

or
$$-h^2 R + 3\mu h^2 + \mu c^2 R^2 = 0$$

$$\therefore \quad \underline{\mu c^2 R^2 = h^2(R-3\mu)}$$

and
$$\frac{1}{2}\cancel{c^2}(k^2-1) = \frac{1}{2}\frac{\mu \cancel{c^2} \cancel{R^2}}{(R-3\mu)} \cdot \frac{1}{\cancel{R^2}}\left(1-\frac{2\mu}{R}\right) - \frac{\mu \cancel{c^2}}{R}$$

$$\therefore \quad \cancel{\tfrac{1}{2}}(k^2-1) = \frac{\mu}{R}\left(\frac{R-2\mu}{R-3\mu} - 2\right)$$

$$= -\frac{\mu}{R}\left(\frac{R-4\mu}{R-3\mu}\right)$$

and

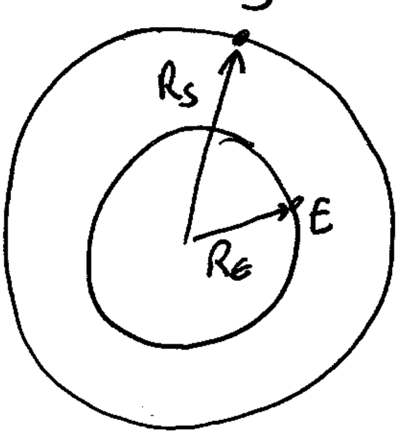
$$k^2 = 1 - \frac{\mu}{R}\left(\frac{R-4\mu}{R-3\mu}\right)$$

$$= \frac{1 - \frac{3\mu}{R} - \frac{\mu}{R} + 4\frac{\mu^2}{R^2}}{\left(1 - 3\mu/R\right)}$$

$$= \frac{\left(1 - 2\mu/R\right)^2}{\left(1 - 3\mu/R\right)} = \underline{\underline{\frac{(R-2\mu)^2}{R(R-3\mu)}}}$$

$$\text{and} \quad \underline{\underline{h^2 = \frac{\mu c^2 R^2}{R-3\mu}}}$$

GR test



$$h^2 = \frac{\mu c^2 R_S^2}{R_S - 3\mu}$$

$$\text{now} \quad h^{\cancel{2}} = r^2 \frac{d\phi}{d\tau}$$

$$\Rightarrow \Delta\tau_S = \frac{r^2 . 2\pi}{h}$$

$$= \underline{\underline{\frac{R_S^2 2\pi}{\mu^{1/2} c R_S} \left(R_S - 3\mu\right)^{1/2}}}$$

Coordinate time for S follows from

$$c^2 = \left(1 - \frac{2\mu}{r}\right)c^2\dot{t}^2 - r^2\dot{\phi}^2$$

$$\therefore \left(1 - \frac{2\mu}{R_S}\right)c^2\dot{t}^2 = c^2 + \frac{h^2}{R_S^2}$$

$$= \frac{c^2}{(R_S - 3\mu)}\left[R_S - 3\mu + \mu\right)$$

$$= c^2\left(\frac{R_S - 2\mu}{R_S - 3\mu}\right)$$

$$\therefore \quad \dot{t} = \frac{\Delta t}{\Delta\tau_S} = \left(\frac{R_S}{R_S - 3\mu}\right)^{1/2} = \frac{1}{\left(1 - 3\mu/R_S\right)^{1/2}}$$

But for static stationary observer $\dot{r} = \dot{\phi} = 0$

sees

$$\Delta\tau_E = \left(1 - \frac{2\mu}{R_E}\right)^{1/2}\Delta t$$

$$\Rightarrow \quad \frac{\Delta\tau_E}{\Delta\tau_S} = \frac{\left(1 - 2\mu/R_E\right)^{1/2}}{\left(1 - 3\mu/R_S\right)^{1/2}}$$

Estimate 3 hour period. $\quad m R_S \omega^2 = \frac{GMm}{R_S^2}$

$$\therefore \quad R_S^3 = \frac{GM}{\omega^2} = \frac{GM}{4\pi^2}T^2$$

c. $R_S = \left(\frac{3}{24}\right)^{2/3} 42.1$ km $= \underline{\underline{10.52 \text{ km}}}$ 6

Expand

$$\frac{\Delta\tau_S}{\Delta\tau_G} = \frac{(1 - 3\mu/R_S)^{1/2}}{(1 - 2\mu/R_E)^{1/2}}$$

$$\simeq 1 - \frac{3\mu}{2R_S} + \frac{\mu}{R_E}$$

$$= 1 - \frac{3GM}{2R_S c^2} + \frac{GM}{R_E c^2}$$

$$= 1 - \frac{3}{2}\frac{\omega^2 R_S^3}{R_S c^2} + \frac{\omega^2 R_S^3}{R_E c^2}$$

$\omega = \frac{2\pi}{3 \times 3600}$ rad s^{-1} $\Rightarrow$ $\omega^2 = 3.38 \times 10^{-7}$

$c^2 = 9 \times 10^{16}$

$R_S^2 = 1.1 \times 10^8$

$\frac{R_S^3}{R_E} = 1.8 \times 10^8$

and $\frac{\Delta\tau_S}{\Delta\tau_G} \simeq 1 - 6.2 \times 10^{-16} + 6.8 \times 10^{-16}$

$\simeq \underline{\underline{1 + 8 \times 10^{-17}}}$

6 |

Fundamental observers

= observers filling space who agree on what they observe (homogeneous)

- they must co-move with matter
- clocks synchronised
- must be on geodesics
- hypersurface of constant proper density must be orthogonal to the world lines

Adopt a synchronous coordinate system

- assign fixed spatial coordinates to each FO
- label surfaces of homogeneity ie the hypersurfaces of constant proper density with proper time as measured by FO.

  → synchronous or cosmic time

$$x^0 = t.$$

By homogeneity all FO see cosmic time pass at the same rate as their proper time.

$x^\mu = (t, x^i)$

K - curvature

t - proper time

χ - radial like coordinate such that area of 2 sphere χ = const is

$$A = 4\pi S_K^2(\chi)$$

Consider fundamental observer at $\chi = 0$ and consider $\Delta\chi$

proper separation $\ell(t) = a(t)\Delta\chi$

$$\therefore \frac{1}{\ell}\frac{d\ell}{dt} = \frac{1}{a}\frac{da}{dt} = H(t)$$

$H > 0 \rightarrow$ expanding universe.

$$\frac{dp_\mu}{d\lambda} = \frac{1}{2} \frac{\partial g_{\nu\rho}}{\partial x^\mu} p^\nu p^\rho$$

Radial motion

$p_\mu = (p_0, p_1, 0, 0)$ $\qquad p_0 = c^2 p^0 \quad p_1 = -a^2 p^1$

$$\frac{dp_1}{d\lambda} = \frac{1}{2}\left(\frac{\partial g_{00}}{\partial \chi} p^0 p^0 + \frac{\partial g_{11}}{\partial \chi} p^1 p^1 \right)$$

but $g_{00} = c^2 \qquad g_{11} = -a^2$

$\therefore$ RHS $= 0$ and $p_1 =$ constant.

Energy of photon p^μ measured by FO
with 4-velocity $u^\mu = \delta^\mu_0$

is $E = p_\mu u^\mu = p_0$

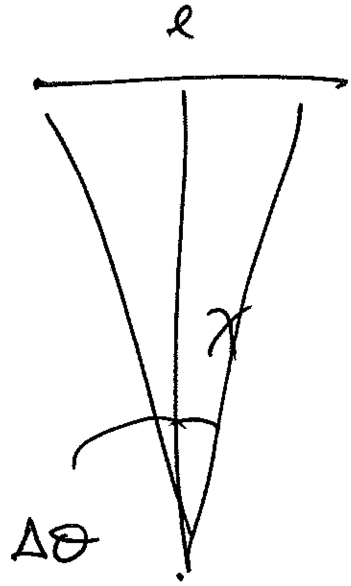
but $g_{\mu\nu} p^\mu p^\nu = 0 \Rightarrow c^2 p^{0\,2} - a^2 p^{1\,2} = 0$

$$\therefore \frac{c^2}{c^4} p_0^2 - a^2 \frac{1}{a^4} p_1^2 = 0$$

$$\Rightarrow p_1 = \frac{a}{c} p_0$$

$\therefore \quad a p_0 = e p_1 = \text{constant}$

$$\Rightarrow \quad \frac{\lambda_R}{\lambda_E} = \frac{p_{0E}}{p_{0R}} = \frac{a(t_R)}{a(t_E)} = 1+z$$



$$\ell = a(t_e)\, S(\chi)\, \Delta\theta$$

$$\text{but} \quad \frac{a(t_e)}{a(t_0)} = \frac{1}{1+z}$$

$$\therefore \quad \ell = \frac{a(t_0)\, S(\chi)}{1+z}\, \Delta\theta .$$