UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

01 June 2021 11.00 am to 13.00

PHYSICS (3)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (3)

ADVANCED QUANTUM PHYSICS

Candidates offering this paper should attempt a total of ***five*** *questions:* ***three*** *questions from Section A and* ***two*** *questions from Section B.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains ***five*** *sides, including this coversheet, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS
2 × 20 Page Answer Book
Metric graph paper
Rough workpad
Yellow master coversheet

SPECIAL REQUIREMENTS
Mathematical Formulae handbook
Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

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SECTION A

*Attempt **all** questions in this Section. Answers should be concise and relevant formulae may be assumed without proof.*

1 The normalised state of a one dimensional harmonic oscillator of mass m and angular frequency ω is written at $t = 0$ as a linear combination of eigenstates as

$$|\alpha(0)\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

α is a complex number and $\hat{a}|\alpha(0)\rangle = \alpha|\alpha(0)\rangle$, where $\hat{a}$ is the lowering operator. Show that the expectation value of the displacement varies sinusoidally in time with amplitude $\sqrt{\frac{2\hbar N}{m\omega}}$, where $N = \langle\alpha(0)|\hat{a}^+\hat{a}|\alpha(0)\rangle$. Comment on this result by comparing it to the case of a classical oscillator with energy $E = \hbar\omega N$. [4]

$$\langle\alpha(t)|\hat{x}|\alpha(t)\rangle = e^{-|\alpha|^2} \sqrt{\frac{\hbar}{2m\omega}} \sum_{n,m} \frac{\alpha^n \alpha^m}{\sqrt{n!m!}} e^{\frac{i(E_m - E_n)t}{\hbar}} \langle m|\hat{a}^+ + \hat{a}|n\rangle =$$

$$e^{-|\alpha|^2} \sqrt{\frac{\hbar}{2m\omega}} [e^{i\omega t} \sum_n \frac{\alpha^n \alpha^{n+1}}{\sqrt{n!(n+1)!}} \sqrt{n+1} + e^{-i\omega t} \sum_n \frac{\alpha^n \alpha^{n-1}}{\sqrt{n!(n-1)!}} \sqrt{n}] =$$

$$\sqrt{\frac{\hbar}{2m\omega}} [e^{i\omega t} \langle\alpha(0)|\hat{a}^+|\alpha(0)\rangle + e^{-i\omega t} \langle\alpha(0)|\hat{a}|\alpha(0)\rangle] =$$

$$\sqrt{\frac{2\hbar}{m\omega}} Re[|\alpha| e^{i(\omega t + \lambda)}] = \sqrt{\frac{2\hbar}{m\omega}} |\alpha| \cos(\omega t + \lambda)$$

where I have taken $\alpha = |\alpha| e^{-\lambda i}$. Now, $N = \langle\alpha(0)|\hat{a}^+\hat{a}|\alpha(0)\rangle = |\alpha|^2$. It follows that

$$\langle\alpha(t)|\hat{x}|\alpha(t)\rangle = \sqrt{\frac{2\hbar N}{m\omega}} \cos(\omega t + \lambda)$$

For a classical oscillator with energy $E = \hbar\omega N$ we have $\frac{1}{2}kA^2 = \hbar\omega N$, where A is the oscillation amplitude and $k = m\omega^2$. It follows that $A = \sqrt{\frac{2\hbar N}{m\omega}}$, as found earlier.

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2 In a one-electron atom the electron occupies the p-orbital ($L = 1$) and its eigenstates are $|J, J_z\rangle$, where J quantifies the total angular momentum while J_z its projection along a direction $\hat{z}$. A beam is initially prepared with an equal number of atoms in each of the four degenerate states $|\frac{3}{2}, \frac{3}{2}\rangle$, $|\frac{3}{2}, \frac{1}{2}\rangle$, $|\frac{3}{2}, -\frac{1}{2}\rangle$ and $|\frac{3}{2}, -\frac{3}{2}\rangle$. A static magnetic field B_z is then switched on and the electron's Hamiltonian is approximated by

$$H = \frac{\mu_B B_z}{\hbar}(L_z + 2S_z).$$

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Describe how the energy spectrum of the atoms changes and specify the relative number of atoms with each energy. [4]

We can decompose the states $|\frac{3}{2}, J_z\rangle$ in the basis of states $|L_z, S_z\rangle$ by applying the ladder operators on both sides as:

$$J^-|\frac{3}{2},\frac{3}{2}\rangle = (L^- + S^-)|1,\frac{1}{2}\rangle$$

$$\hbar\sqrt{3}|\frac{3}{2},\frac{1}{2}\rangle = \hbar(|1,-\frac{1}{2}\rangle + \sqrt{2}|0,\frac{1}{2}\rangle)$$

It follows that:

$$|\frac{3}{2},\frac{1}{2}\rangle = \frac{1}{\sqrt{3}}(|1,-\frac{1}{2}\rangle + \sqrt{2}|0,\frac{1}{2}\rangle)$$

If we keep doing the same, we get:

$$|\frac{3}{2},-\frac{1}{2}\rangle = \frac{1}{\sqrt{3}}(|-1,\frac{1}{2}\rangle + \sqrt{2}|0,-\frac{1}{2}\rangle)$$

$$|\frac{3}{2},-\frac{3}{2}\rangle = |-1,-\frac{1}{2}\rangle$$

In summary, the beam $|\frac{3}{2},\frac{3}{2}\rangle$, which corresponds to a value of the magnetic moment $\mu = 2\mu_B$ is unsplit and keeps making 1/4 of the original beam. The beam $|\frac{3}{2},\frac{1}{2}\rangle$ is split into two beams, $|1,-\frac{1}{2}\rangle$, which corresponds to a value of the magnetic moment $\mu = 0$, and $|0,\frac{1}{2}\rangle$, with $\mu = \mu_B$. The beam $|\frac{3}{2},-\frac{1}{2}\rangle$ is split into two beams, $|-1,\frac{1}{2}\rangle$, with $\mu = 0$, and $|0,-\frac{1}{2}\rangle$, with $\mu = -\mu_B$. Finally, $|\frac{3}{2},-\frac{3}{2}\rangle$ is unsplit and corresponds to $\mu = -2\mu_B$. So, there are 5 energy levels that correspond to $\mu = 2\mu_B$ (1/4), $\mu = \mu_B$ (1/6), $\mu = 0$ (1/6), $\mu = -\mu_B$ (1/6), $\mu = -2\mu_B$ (1/4).

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3 A particle is in a potential of the form $V = kx^4$, where k is a constant. Use the variational technique to find an upper bound value of the ground state energy within the family of trial functions $f_\alpha(x) \propto e^{-\alpha x^2/2}$. What trial function could you use to estimate the energy of the first excited state? Explain your reasoning. [4]

The expectation value of the energy on the trial function constitutes an upper bound for the ground state energy. The trial function that is given is not normalised and we need to take the normalisation into account when finding $\langle H\rangle$.

$$\langle H\rangle = \frac{\int_{-\infty}^{\infty} dx e^{-\alpha x^2/2}[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + kx^4]e^{-\alpha x^2/2}}{\int_{-\infty}^{\infty} dx e^{-\alpha x^2}}$$

Integrating by parts $\langle H\rangle = \frac{\hbar^2\alpha}{4m} + \frac{3k}{4\alpha^2}$, which is minimised by $\alpha = \sqrt[3]{\frac{6km}{\hbar^2}}$.

For the first excited state we would need to pick an odd function of the form $f_\alpha(x) \propto xe^{-\alpha x^2/2}$. The potential $V = kx^4$ is symmetric and the wavefunctions will need to be eigenstates of the parity operator, i.e. even or odd.

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SECTION B

Attempt two questions from this section

4 The Hamiltonian of an electron in a time-dependent magnetic field $B(t) = (B_x \cos(\omega t), 0, B_z)$ is

$$H = H_0 + V(t) = -\gamma B_z S_z - \gamma B_x S_x \cos(\omega t)$$

γ is the gyromagnetic ratio. At time $t = 0$, $|\psi(0)\rangle = |\uparrow\rangle$, where $|\uparrow\rangle$ is the eigenstate of S_z with eigenvalue $\frac{\hbar}{2}$.

(a) Show that in the interaction picture $|\psi_I(t)\rangle = e^{\frac{iH_0 t}{\hbar}}|\psi(t)\rangle$ satisfies

$$i\hbar \frac{d}{dt}|\psi_I(t)\rangle = V_I(t)|\psi_I(t)\rangle$$

$$V_I(t) = e^{\frac{iH_0 t}{\hbar}} V(t) e^{-\frac{iH_0 t}{\hbar}}$$

[2]

$|\psi_I(t)\rangle = e^{\frac{iH_0 t}{\hbar}}|\psi(t)\rangle$, it follows that:

$$i\hbar \frac{d}{dt}|\psi_I(t)\rangle = i\hbar \frac{d}{dt}(e^{\frac{iH_0 t}{\hbar}}|\psi(t)\rangle) = -e^{\frac{iH_0 t}{\hbar}} H_0|\psi(t)\rangle + i\hbar e^{\frac{iH_0 t}{\hbar}} \frac{d|\psi(t)\rangle}{dt} =$$

$$-e^{\frac{iH_0 t}{\hbar}} H_0|\psi(t)\rangle + e^{\frac{iH_0 t}{\hbar}}[H_0 + V(t)]|\psi(t)\rangle = e^{\frac{iH_0 t}{\hbar}} V(t)|\psi(t)\rangle = V_I(t)|\psi_I(t)\rangle$$

(b) Hence show that if $|\psi_I(t)\rangle = c_1(t)|\uparrow\rangle + c_2(t)|\downarrow\rangle$ the coefficients $c_1(t)$ and $c_2(t)$ satisfy

$$\begin{pmatrix} \frac{dc_1(t)}{dt} \\ \frac{dc_2(t)}{dt} \end{pmatrix} = \frac{i\gamma B_x}{4} \begin{pmatrix} 0 & e^{i\delta t} \\ e^{-i\delta t} & 0 \end{pmatrix} \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix}$$

where $\delta = \omega - \gamma B_z$ is small (close to resonance condition). [5]

By simple substitution we get

$$i\hbar[\dot{c}_1(t)|\uparrow\rangle + \dot{c}_2(t)|\downarrow\rangle] = e^{\frac{iH_0 t}{\hbar}} V(t)[c_1(t)e^{-\frac{iE^0_\uparrow t}{\hbar}}|\uparrow\rangle + c_2(t)e^{-\frac{E^0_\downarrow t}{\hbar}}|\downarrow\rangle]$$

where $H_0|\uparrow\rangle = E^0_\uparrow|\uparrow\rangle = -\frac{\gamma B_z \hbar}{2}|\uparrow\rangle$ and $H_0|\downarrow\rangle = E^0_\downarrow|\downarrow\rangle = \frac{\gamma B_z \hbar}{2}|\uparrow\rangle$. If we multiply by the bra $\langle\uparrow|$ we obtain

$$i\hbar\dot{c}_1(t) = -e^{\frac{iE^0_\uparrow t}{\hbar}}\langle\uparrow|S_x|\downarrow\rangle c_2(t)e^{-\frac{iE^0_\downarrow t}{\hbar}}\gamma B_x \cos(\omega t) = -\frac{\hbar}{2}e^{-i\gamma B_z}\gamma B_x c_2(t)\cos(\omega t)$$

Similarly, if we multiply by the bra $\langle\downarrow|$ we obtain

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$$i\hbar\dot{c}_2(t) = -\frac{\hbar}{2}e^{i\gamma B_z}\gamma B_x c_1(t)\cos(\omega t)$$

In the near-resonance condition we can exclude the fast oscillating terms and

$$i\hbar\dot{c}_1(t) = \frac{i\gamma B_x}{4}e^{\delta t i}c_2(t)$$
$$i\hbar\dot{c}_2(t) = \frac{i\gamma B_x}{4}e^{-\delta t i}c_1(t)$$

where $\delta = \omega - \gamma B_z$.

A solution for this system of differential equations is

$$c_1(t) = e^{\frac{i\delta t}{2}}\left[\cos\left(\frac{\omega_R t}{2}\right) - \frac{i\delta}{\omega_R}\sin\left(\frac{\omega_R t}{2}\right)\right]$$

$$c_2(t) = \frac{i\gamma B_x}{2\omega_R}e^{-\frac{i\delta t}{2}}\sin\left(\frac{\omega_R t}{2}\right)$$

where $\omega_R = \sqrt{\delta^2 + \left(\frac{\gamma B_x}{2}\right)^2}$.

(c) At what time is the probability of finding an electron in the state $|\downarrow\rangle$ maximal? Plot this maximum probability as a function of ω and show that $\frac{\omega_{res}}{\triangle\omega} \propto \frac{B_z}{B_x}$, where ω_{res} and $\triangle\omega$ are respectively the resonance frequency and resonance width for the transition $|\uparrow\rangle \to |\downarrow\rangle$. [5]

The transition probability is $|c_2(t)|^2$ and oscillates with time. The maximum probability is

$$P^{max}_{\uparrow\longrightarrow\downarrow} = \frac{\gamma^2 B_x^2}{4\omega_R^2} = \frac{\gamma^2 B_x^2}{4[(\frac{\gamma B_x}{2})^2 + (\omega - \gamma B_z)^2]}$$

which is a Lorentzian with maximum in γB_z and width $\frac{\gamma B_x}{2}$.

(d) Calculate $\langle\psi(t)|\mathbf{S}|\psi(t)\rangle$ at resonance ($\delta = 0$) and describe the motion of the spin in space for $B_x \ll B_z$ and $B_z \ll B_x$. [7]

At resonance the solutions for the system of differential equations is found by substituting $\delta = 0$. It follows that $c_1(t) = \cos(\frac{\gamma B_x}{4}t)$ and $c_2(t) = i\sin(\frac{\gamma B_x}{4}t)$. By calculating the expectation value of each component of the spin on the state $|\psi(t)\rangle = c_1(t)e^{-\frac{iE^0_\uparrow t}{\hbar}}|\uparrow\rangle + c_2(t)e^{-\frac{iE^0_\downarrow t}{\hbar}}|\downarrow\rangle$ we get

$$\langle S_x\rangle = \hbar Re[c_1^*(t)c_2(t)e^{-i\gamma B_z t}] = \frac{\hbar}{2}\sin(\frac{\gamma B_x}{2}t)\sin(\gamma B_z t)$$

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$$\langle S_y \rangle = -i\hbar Im[\hbar c_1^*(t) c_2(t) e^{-i\gamma B_z t}] = \frac{\hbar}{2}\sin(\frac{\gamma B_x}{2}t)\cos(\gamma B_z t)$$
$$\langle S_z \rangle = \frac{\hbar}{2}(|c_1(t)|^2 - |c_2(t)|^2) = \frac{\hbar}{2}\cos(\frac{\gamma B_x}{2}t)$$

The motion of the spin is a superposition of two different rotations, one around the $\hat{z}$ axis, with angular frequency γB_z and one around a rotating axis in the $\hat{x}\hat{y}$ plane, with angular frequency $\frac{\gamma B_x}{2}$.

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5 A spinless particle of mass m and charge $-e$ interacts with a monochromatic radiation field described by the vector potential $\mathbf{A}(\mathbf{r}, t)$. Its Hamiltonian is

$$H = \frac{1}{2m}[\mathbf{p} + e\mathbf{A}(\mathbf{r}, t)]^2 + V(\mathbf{r})$$

(a) Show that in the Coulomb gauge ($\nabla \cdot \mathbf{A} = 0$) this can be simplified at first order in the vector potential to $H = H_0 + H_1(t)$ where $H_1(t) = \frac{e}{m}\mathbf{p} \cdot \mathbf{A}(\mathbf{r}, t)$.

[3]

$$H = \frac{1}{2m}(\mathbf{p} + e\mathbf{A}(\mathbf{r}, t))^2 - eV(\mathbf{r}) = H = \frac{p^2}{2m} + \frac{e}{2m}(\mathbf{p} \cdot \mathbf{A} + \mathbf{A} \cdot \mathbf{p}) - eV(\mathbf{r})$$

to the first approximation in A. If $\nabla \cdot \mathbf{A}$, $[\mathbf{p}, \mathbf{A}] = 0$ and we get to the expected result.

If the particle is initially in the ground state of H_0, $\psi(0) = |0\rangle$, at time t the state can be written in terms of the unperturbed eigenstates $|n\rangle$ and unperturbed energy levels E_n as

$$\psi(t) = e^{-\frac{iE_0 t}{\hbar}}|0\rangle + \sum_{n\neq 0} e^{-\frac{iE_n t}{\hbar}}\left(\frac{1}{i\hbar}\int_0^t e^{\frac{i(E_n - E_0)t'}{\hbar}}\langle n|H_1(t')|0\rangle dt'\right)|n\rangle$$

(b) Determine which transitions are allowed at $t > 0$ and explicitly calculate the coefficients in the above expansion in the case in which

$$H_0 = \frac{p^2}{2m} + \frac{m\omega_0^2}{2}z^2$$

$$\mathbf{A}(\mathbf{r}, t) = \begin{cases} 2A_0\hat{\mathbf{z}}\cos(ky - \omega t) & \text{at} \quad t > 0 \\ 0 & \text{at} \quad t \leq 0 \end{cases}$$

[6]

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The coefficients in the given expansion depend on

$$\langle n|H_1(t')|0\rangle = \frac{2eA_0}{m}\langle n|p_z|0\rangle \cos(ky-\omega t') = \frac{2eA_0}{m} i\sqrt{\frac{\hbar m\omega_0}{2}}\delta_{n1}\cos(ky-\omega t')$$

Only the transition $|0\rangle \to |1\rangle$ is allowed. It follows that

$$\psi(t) = e^{-\frac{iE_0 t}{\hbar}}|0\rangle + eA_0\sqrt{\frac{\omega_0}{2\hbar m}}e^{-\frac{iE_1 t}{\hbar}}|1\rangle \int_0^t (e^{iky}e^{i(\omega_0-\omega)t'} + e^{-iky}e^{i(\omega_0+\omega)t'})dt' =$$

$$e^{-\frac{iE_0 t}{\hbar}}\left\{|0\rangle + eA_0\sqrt{\frac{\omega_0}{2\hbar m}}e^{-i\omega_0 t}|1\rangle\left[e^{iky}(\frac{e^{i(\omega_0-\omega)t}-1}{i(\omega_0-\omega)}) + e^{-iky}(\frac{e^{i(\omega_0+\omega)t}-1}{i(\omega_0+\omega)})\right]\right\}$$

(c) Show that the induced dipole moment $\mathbf{P}$ is

$$\mathbf{P} = -\frac{e^2 A_0}{m}\hat{\mathbf{z}}\mathrm{Re}\left[e^{iky}\left(\frac{e^{-i\omega t}-e^{-i\omega_0 t}}{i(\omega_0-\omega)} - \frac{e^{-i\omega t}-e^{i\omega_0 t}}{i(\omega_0+\omega)}\right)\right]$$

[6]

The induced dipole moment $\mathbf{P} = -e\langle\psi(t)|\hat{z}|\psi(t)\rangle$. By substituting the above expression for $|\psi(t)\rangle$ and taking into account that $\hat{z} = \frac{1}{2}\sqrt{\frac{2\hbar}{m\omega_0}}(\hat{a}+\hat{a}^+)$ we get to the expression

$$P^z = -\frac{e^2 A_0}{2m}\left[e^{iky}(\frac{e^{-i\omega t}-e^{-i\omega_0 t}}{i(\omega_0-\omega)}) + e^{-iky}(\frac{e^{i\omega t}-e^{-i\omega_0 t}}{i(\omega_0+\omega)}) - e^{-iky}(\frac{e^{i\omega t}-e^{i\omega_0 t}}{i(\omega_0-\omega)}) - e^{iky}(\frac{e^{-i\omega t}-e^{i\omega_0 t}}{i(\omega_0+\omega)})\right] =$$

$$-\frac{e^2 A_0}{m}Re\left[e^{iky}(\frac{e^{-i\omega t}-e^{-i\omega_0 t}}{i(\omega_0-\omega)} - \frac{e^{-i\omega t}-e^{i\omega_0 t}}{i(\omega_0+\omega)})\right]$$

(d) Now show that the term oscillating at ω, $\mathbf{P}_\omega$, is proportional to the electric field component of the radiation $\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}$ as

$$\mathbf{P}_\omega = \frac{e^2}{m}\frac{\mathbf{E}}{\omega_0^2-\omega^2}$$

[4]

If we only retain those terms that oscillate with frequency ω

$$P^z_\omega = -\frac{e^2 A_0}{m}Re\left[2\omega\frac{e^{i(ky-\omega t)}}{i(\omega_0^2-\omega^2)}\right] = -2\frac{e^2 A_0}{m}\omega\frac{\sin(ky-\omega t)}{\omega_0^2-\omega^2} = \frac{e^2}{m}\frac{E^z}{\omega_0^2-\omega^2}$$

where $E^z = -2A_0\hat{z}\omega\sin(ky-\omega t)$

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6 Two identical particles of spin 1/2 and mass m are confined to move in a 1-D box such that $V = 0$ for $|x| < a/2$ and $V = \infty$ for $|x| \geq a/2$. Their spins interact via an exchange term such that the total Hamiltonian is

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - \lambda \mathbf{S}_1 \cdot \mathbf{S}_2$$

where p_i and $\mathbf{S}_i$ are respectively the momentum and spin of particle i.

(a) Explain why the eigenfunctions can be labelled as $|n_1, n_2, S, S_z\rangle$, where n_1 and n_2 label the spin-independent part of the wavefunction, while S and S_z represent the total spin and its component along the $\hat{z}$ direction.

[2]

The students should easily recognise that $\mathbf{S}_1 \cdot \mathbf{S}_2 = \frac{1}{2}[(\mathbf{S}_1 + \mathbf{S}_2)^2 - \mathbf{S}_1^2 - \mathbf{S}_2^2]$, hence that Hamiltonian commutes with the total spin operator $(\mathbf{S}_1 + \mathbf{S}_2)$,$\mathbf{S}_1^2$ and $\mathbf{S}_2^2$.

(b) Calculate the energy of the three lowest levels in the case in which $\lambda \ll \frac{\pi^2}{ma^2}$. For each level specify its degeneracy and write down the corresponding eigenfunctions.

[6]

The energy of the wavefunction $|n_1, n_2, S, S_z\rangle$ is $E_{n_1,n_2,s,s_z} = \frac{\pi^2\hbar^2}{2ma^2}(n_1^2 + n_2^2) - \frac{\lambda\hbar^2}{2}[s(s+1) - \frac{3}{2}]$. Because the particles are undistinguishable, the total wavefunction must be asymmetric. The lowest energy state, which corresponds to $n_1 = n_2 = 1$ must therefore correspond to a singlet state. The wavefunction and the energy for this state are

$$\psi_0(\mathbf{x}_1, \mathbf{x}_2) = \psi_1(\mathbf{x}_1)\psi_1(\mathbf{x}_2)|0, 0\rangle$$
$$E_0 = \frac{\pi^2\hbar^2}{ma^2} + \frac{3\lambda\hbar^2}{4}$$

The next energy levels are

$$\psi_1(\mathbf{x}_1, \mathbf{x}_2) = \frac{\psi_1(\mathbf{x}_1)\psi_2(\mathbf{x}_2) - \psi_2(\mathbf{x}_1)\psi_1(\mathbf{x}_2)}{\sqrt{2}} \begin{matrix} |1, 1\rangle \\ |1, 0\rangle \\ |1, -1\rangle \end{matrix}$$
$$E_1 = \frac{5\pi^2\hbar^2}{2ma^2} - \frac{\lambda\hbar^2}{4}$$
$$\psi_2(\mathbf{x}_1, \mathbf{x}_2) = \frac{\psi_1(\mathbf{x}_1)\psi_2(\mathbf{x}_2) + \psi_2(\mathbf{x}_1)\psi_1(\mathbf{x}_2)}{\sqrt{2}}|0, 0\rangle$$
$$E_2 = \frac{5\pi^2\hbar^2}{2ma^2} + \frac{3\lambda\hbar^2}{4}$$

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(c) If we now include a small correction to the Hamiltonian of the form $\alpha x_1 \cdot x_2$, where x_1 and x_2 are the coordinates of the two particles, use perturbation theory to determine the first order correction to the energy of the three lowest levels.

[5]

The first order corrections to the lowest level is

$$\Delta E_0^1 = \alpha\langle\psi_1(\mathbf{x}_1)\psi_1(\mathbf{x}_2)|\mathbf{x}_1\cdot\mathbf{x}_2|\psi_1(\mathbf{x}_1)\psi_1(\mathbf{x}_2)\rangle = \alpha\langle\psi_1(\mathbf{x}_1)|\mathbf{x}_1|\psi_1(\mathbf{x}_1)\rangle\langle\psi_1(\mathbf{x}_2)|\mathbf{x}_2|\psi_1(\mathbf{x}_2)\rangle$$

Both integrals correspond to the integration of an odd function over the whole space and are therefore zero.

The correction of the other energy levels involves calculating the integrals

$$\langle\psi_1(\mathbf{x})|\mathbf{x}|\psi_1(\mathbf{x})\rangle = 0$$

$$\langle\psi_2(\mathbf{x})|\mathbf{x}|\psi_2(\mathbf{x})\rangle = 0$$

$$\langle\psi_1(\mathbf{x})|\mathbf{x}|\psi_2(\mathbf{x})\rangle = \frac{16}{9}\frac{a}{\pi^2}$$

It follows that $\Delta E_1^1 = -\alpha(\frac{16}{9})^2\frac{a^2}{\pi^4}$ and $\Delta E_2^1 = \alpha(\frac{16}{9})^2\frac{a^2}{\pi^4}$

(d) Repeat parts (b)-(c) in the case in which the particles were distinguishable and show that the first order correction to the energy levels is zero in this case.

[6]

When the particles are distinguishable there is no need to impose the asymmetry of the wavefunction and the lowest energy levels are

$$\psi_0(\mathbf{x}_1,\mathbf{x}_2) = \psi_1(\mathbf{x}_1)\psi_1(\mathbf{x}_2) \quad \begin{matrix}|1,1\rangle \\ |1,0\rangle \\ |1,-1\rangle\end{matrix}$$

$$E_0 = \frac{\pi^2\hbar^2}{ma^2} - \frac{\lambda\hbar^2}{4}$$

three times degenerate.

$$\psi_1(\mathbf{x}_1,\mathbf{x}_2) = \psi_1(\mathbf{x}_1)\psi_1(\mathbf{x}_2)|0,0\rangle$$

$$E_1 = \frac{\pi^2\hbar^2}{ma^2} + \frac{3\lambda\hbar^2}{4}$$

one time degenerate

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$$\psi_2(\mathbf{x}_1, \mathbf{x}_2) = \begin{array}{c} \psi_1(\mathbf{x}_1)\psi_2(\mathbf{x}_2) \\ \psi_2(\mathbf{x}_1)\psi_1(\mathbf{x}_2) \end{array} \quad \begin{array}{c} |1,1\rangle \\ |1,0\rangle \\ |1,-1\rangle \end{array}$$

$$E_2 = \frac{5\pi^2\hbar^2}{2ma^2} - \frac{\lambda\hbar^2}{4}$$

six times degenerate.

It is strightforward to show that at the first order the correction tot he energy is always zero since no 'mixed' integrals of the form $\langle\psi_1(\mathbf{x})|\mathbf{x}|\psi_2(\mathbf{x})\rangle$ appear.

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