UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Tuesday May 30 2017 1.30 pm to 3.30 pm

PHYSICS (1)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (1)

THERMAL AND STATISTICAL PHYSICS

Candidates offering this paper should attempt a total of **three** *questions. The questions to be attempted are* **1**, **2** *and* **one** *other question.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains **four** *sides, including this coversheet, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
2×20 Page Answer Book	Mathematical Formulae handbook
Rough workpad	Approved calculator allowed
Yellow master coversheet	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

THERMAL AND STATISTICAL PHYSICS

1 *Attempt* **all** *parts of this question. Answers should be concise and relevant formulae may be assumed without proof.*

(a) The electronic density of states $g(\epsilon)$ above the superconducting transition temperature in a certain metal can be taken to be independent of energy ϵ near the Fermi energy ϵ_{F}, i.e., $g(\epsilon) = g(\epsilon_{\mathrm{F}})$. As the metal is cooled, it becomes superconducting. Electronic states which had energy ϵ in the non-superconducting metal now correspond to excitations from the ground state of the superconductor with energy $E(\epsilon) = \sqrt{(\epsilon - \epsilon_{\mathrm{F}})^2 + |\Delta|^2}$, where $|\Delta|$ is an energy gap. Show that the density of these excitations for $E > |\Delta|$ is given by

$$D(E) = \frac{g(\epsilon_{\mathrm{F}})E}{\sqrt{E^2 - |\Delta|^2}}.$$

[5]

(b) How many permutations are possible for the letters in the word “quagga”? [1]

A system of particles is divided into M cells of unit volume. Determine the number of ways that N particles can be distributed among M cells (with $0 \leq N \leq M$), such that each cell is either empty or filled by one particle, for the cases of both distinguishable and indistinguishable particles. [2]

(c) Explain the meaning of the Gibbs probability p_i for a system in a Grand Canonical Ensemble:

$$p_i = \frac{\mathrm{e}^{-(E_i - \mu N_i)/k_{\mathrm{B}}T}}{\Sigma_j\, \mathrm{e}^{-(E_j - \mu N_j)/k_{\mathrm{B}}T}} .$$

Use p_i to determine the Fermi-Dirac and Bose-Einstein distributions for identical non-interacting particles. [4]

2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) the radial distribution function, internal energy, and the virial expansion of a classical liquid;

(b) Bose–Einstein condensation in a non-interacting gas;

(c) elementary excitations, with particular reference to spin waves, phonons, and the internal energy.

3 *Attempt* **either** *this question* **or** *question 4.*
This question has two parts and both should be answered.

(a) A classical ideal gas of spin-½ particles at temperature T exists in a magnetic field of strength $B(\boldsymbol{r})$, which gives rise to two distinct energy levels for the particles, $E = \pm\mu_B B(\boldsymbol{r})$, where μ_B is the Bohr magneton. The concentrations of spin-up and spin-down particles are $n_\uparrow(\boldsymbol{r})$ and $n_\downarrow(\boldsymbol{r})$.

By considering spin-up and spin-down particles as separate sub-systems in thermal equilibrium, show that the spin-up and spin-down chemical potentials are given by:

$$\mu_\uparrow = k_B T \ln\left(\frac{n_\uparrow(\boldsymbol{r})}{n_Q(T)}\right) - \mu_B B(\boldsymbol{r}) \qquad \text{and} \qquad \mu_\downarrow = k_B T \ln\left(\frac{n_\downarrow(\boldsymbol{r})}{n_Q(T)}\right) + \mu_B B(\boldsymbol{r}),$$

where $n_Q(T)$ is the quantum concentration. [4]

What is the equilibrium value of the total chemical potential of the system, μ? [1]

Explain why in equilibrium the two chemical potentials must be independent of position $\boldsymbol{r}$ and must take the same value. [3]

Show that the magnetisation $M(\boldsymbol{r})$ (magnetic moment per unit volume) of the equilibrated gas is given by

$$M(\boldsymbol{r}) = 2\mu_B n_Q(T) \sinh\left(\frac{\mu_B B(\boldsymbol{r})}{k_B T}\right) e^{\mu/k_B T}.$$ [4]

Calculate the magnetisation $M(\boldsymbol{r})$ at high temperatures, i.e., $k_B T \gg \mu_B B(\boldsymbol{r})$, and comment on its relationship to the Curie law. [2]

Calculate the particle density $n(\boldsymbol{r})$ in the equilibrated gas. [1]

(b) A solid consists of N atoms and n vacancies on a lattice with $N + n$ sites. Vacancies are generated in thermal equilibrium at temperature T. The volume occupied by an atom or vacancy is v, the pressure is p, and the vacancy formation energy is ϵ.

Calculate the minimum Gibbs free energy of the system. [2]

Show that the average number of vacancies in the system is

$$n = \frac{N}{e^{(\epsilon+pv)/k_B T} - 1}.$$ [5]

Hence determine the average volume of the system as a function of temperature T. [3]

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4 *Attempt* **either** *this question* **or** *question 3.*

An atom is held at temperature T in thermal equilibrium within a spherical trap. The Hamiltonian of the system is

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + ar \,,$$

where r is the radial distance from the origin, and a is the trap strength.

Calculate the single-particle partition function Z_1 of the system.
[*You may use the result:* $\int_0^\infty t^2 \mathrm{e}^{-t}\, dt = 2$.] [6]

Calculate the partition function Z_N of a gas of N indistinguishable non-interacting atoms in the same trap at temperature T. The gas can be assumed to be sufficiently dilute that classical statistics apply. [4]

Find an expression for the free energy, F. Show that the entropy S of the system, expressed in terms of N and $Z_1(T, a)$, takes the form:

$$S = Nk_B \left(\ln \frac{Z_1}{N} + \frac{11}{2} \right) .$$ [6]

Sketch $S(T)$ for two different values of a. Hence or otherwise demonstrate that by decreasing a in adiabatic conditions, this gas can be cooled reversibly. [6]

A trap contains a variable number of atoms. Find an expression for the chemical potential, μ of the system. By applying a suitable threshold criterion to μ, or otherwise, estimate the number of atoms required for quantum statistics to become important. [3]

END OF PAPER