UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Thursday 23 May 2019 9.00 am to 11.00 am

PHYSICS (1)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (1)

THERMAL AND STATISTICAL PHYSICS

Candidates offering this paper should attempt a total of ***five*** *questions:* ***three*** *questions from Section A and* ***two*** *questions from Section B.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains ***five*** *sides, including this coversheet, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS
2 × 20 Page Answer Book
Metric graph paper
Rough workpad
Yellow master coversheet

SPECIAL REQUIREMENTS
Mathematical Formulae handbook
Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

*Attempt **all** questions in this Section. Answers should be concise and relevant formulae may be assumed without proof.*

1 Show that the isothermal compressibility $\kappa_{\mathrm{T}} = -V^{-1}\,(\partial V/\partial p)_{T,N}$ may be expressed as

$$\kappa_{\mathrm{T}} = \frac{1}{n^2}\left(\frac{\partial n}{\partial \mu}\right)_T ,$$

where $n = N/V$ is the particle density and μ is the chemical potential. [4]

2 Consider a d-dimensional gas of non-interacting bosons described by dispersion relation $\varepsilon_{\boldsymbol{k}} \propto k^{\alpha}$ $(\alpha > 0)$. Under what conditions on α and d will the system display Bose-Einstein condensation below some nonzero temperature T_{c}? [4]

3 For a dilute gas of molecules in three dimensions, the inter-molecular potential takes the form

$$\phi(r) = \begin{cases} \varepsilon\left(1 - \frac{r^3}{a^3}\right) & r < a \\ 0 & r \geq a \end{cases},$$

with the constant $\varepsilon > 0$. Evaluate the second virial coefficient of the gas and show that the Boyle temperature is infinite. [4]

SECTION B

Attempt two questions from this section

4 Describe the conditions for the canonical ensemble to apply and derive the probability P_i of a microstate with energy E_i,

$$P_i = \frac{\exp(-\beta E_i)}{Z}.$$

[4]

Consider a magnetic moment $\boldsymbol{m}$ of fixed magnitude m in a magnetic field $\boldsymbol{B}$ of magnitude B. The energy is

$$E(\theta) = -\boldsymbol{m} \cdot \boldsymbol{B} = -mB\cos(\theta),$$

where θ is the angle between $\boldsymbol{m}$ and $\boldsymbol{B}$. Considering that $\boldsymbol{m}$ may point in any direction, show that the specific heat is

$$C = k_\mathrm{B}\left(1 - \left[\frac{\beta mB}{\sinh(\beta mB)}\right]^2\right).$$

[5]

State and prove the equipartition theorem. [4]

Discuss how the equipartition theorem applies to the low-temperature ($\beta mB \gg 1$) behaviour of the magnetic moment above, with energy still given by $E(\theta) = -mB\cos(\theta)$. [2]

Now consider a quantum mechanical description where $m_z = \gamma\hat{S}_z$ and hence the Hamiltonian is

$$\mathcal{H} = -\gamma B\hat{S}_z,$$

where $\hat{S}_z$ is the z-component of a spin operator of total spin $S = 1$, and γ is the gyromagnetic ratio. Obtain the specific heat in the low-temperature ($\beta\hbar|\gamma|B \gg 1$) regime and interpret the results. [4]

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5 Consider a system of a fixed number N of particles that is thermally and mechanically coupled to a large reservoir at temperature $T_{\rm R}$ and pressure $p_{\rm R}$. Explain how the availability A allows one to express the change in the total (system + reservoir) entropy $S_{\rm tot}$ in terms of the changes in the internal energy U and volume V of the system. Hence show that the availability is minimised in thermal equilibrium. [3]

Show that the probability density $P(x)$ of a variable x of a system in contact with a reservoir is given by

$$P(x) = \aleph \exp\left(-\frac{A(x)}{k_{\rm B}T_{\rm R}}\right),$$

where $T_{\rm R}$ is the reservoir temperature and $\aleph$ is a normalisation constant. Relate the appropriate derivative of A to the fluctuations $\langle (x - x_0)^2 \rangle$ of x around its equilibrium value x_0. [4]

A bubble of air is in a large container of liquid. The work corresponding to increasing the surface area of the bubble is $dW = \mathit{\Gamma}\, d\mathit{\Omega}$ where $\mathit{\Gamma}$ is the surface tension and $\mathit{\Omega} = \alpha V^{2/3}$ is the surface area. (Other contributions from the surface, e.g., related to entropy or particle number, may be neglected.) Taking the liquid to be at pressure $p_{\rm R}$ and temperature $T_{\rm R}$, show that the equilibrium volume of the bubble satisfies

$$V_0 = \left[\frac{2\mathit{\Gamma}\alpha}{3(p_{\rm air} - p_{\rm R})}\right]^3 .$$

[4]

Show that the volume fluctuations around the equilibrium value V_0 are given by

$$\langle \mathit{\Delta} V^2 \rangle = \frac{k_{\rm B}T_{\rm R}V_0}{\kappa_{\rm T}^{-1} - \frac{2}{9}\mathit{\Gamma}\alpha V_0^{-1/3}},$$

where $\mathit{\Delta} V = V - V_0$ and $\kappa_{\rm T}^{-1} = -V_0\left(\partial p_{\rm air}/\partial V\right)_{V=V_0, T=T_{\rm R}}$ is the inverse isothermal compressibility of the air in the bubble. (You may assume that the air temperature is fixed and equal to its equilibrium value $T = T_{\rm R}$.) [4]

Obtain a numerical estimate for the fractional volume fluctuations for a spherical ($\alpha \approx 4.84$) air bubble, of equilibrium volume $V_0 = 10^{-3}$mm^3, described as an ideal gas in water at room temperature ($T_{\rm R} = 25$C$^\circ$) and atmospheric pressure ($p_{\rm R} \approx 10^5$N/m^2). The surface tension is $\mathit{\Gamma} = 72 \times 10^{-3}$N/m. [4]

6 Write down the expression for the average fermionic occupancy $n(\varepsilon)$ of a single-particle state of energy ε in equilibrium at chemical potential μ and temperature T. Explain the role of μ in describing a system with a given average number of fermions. Obtain μ at zero temperature for a system of N fermions in two dimensions with energy $\varepsilon_{\boldsymbol{k}} = \frac{\hbar^2 k^2}{2m}$ (k is the magnitude of the momentum $\boldsymbol{k}$ and m is the particle mass). [5]

Consider a single electronic level in a magnetic field B. The energy is

$$\varepsilon_\sigma = \varepsilon + \sigma \mu_{\mathrm{B}} B,$$

where μ_{B} is the Bohr magneton and $\sigma = \pm 1$ labels the electron spin. Obtain the chemical potential at nonzero temperature for the case when this system is occupied (on average) by one electron. Hence evaluate the magnetic susceptibility

$$\chi = -\mu_{\mathrm{B}} \left(\frac{\partial [N_+ - N_-]}{\partial B} \right)_{B=0},$$

where $N_\pm$ is the average number of up- and down-spin electrons. [5]

The rest of the question explores the more general case of the chemical potential and the contribution of the spin to the magnetic susceptibility for an ideal Fermi gas with density of states $g_\pm = g(\varepsilon \mp \mu_{\mathrm{B}} B)$ for the two spin directions.

Show that

$$\chi = 2\mu_{\mathrm{B}}^2 \int_{-\infty}^{\infty} g'(\varepsilon) n(\varepsilon)_{B=0} d\varepsilon,$$ [2]

where $n(\varepsilon)_{B=0}$ indicates $n(\varepsilon)$ at the zero-field chemical potential and $g' = dg/d\varepsilon$.

Next show that, for zero magnetic field, the low-temperature chemical potential is

$$\mu = \varepsilon_{\mathrm{F}} - \frac{a}{2} \frac{g'(\varepsilon_{\mathrm{F}})}{g(\varepsilon_{\mathrm{F}})} (k_{\mathrm{B}} T)^2,$$

up to second order in T, where ε_{F} is the Fermi energy and a is a numerical coefficient of order unity. [4]

[*You may want to use that $n' = dn/d\varepsilon$ satisfies*

$$-\int_{-\infty}^{\infty} n'(\varepsilon) d\varepsilon = 1, \quad \int_{-\infty}^{\infty} \varepsilon \left(-n'(\varepsilon)\right) d\varepsilon = \mu, \quad \int_{-\infty}^{\infty} (\varepsilon - \mu)^2 \left(-n'(\varepsilon)\right) d\varepsilon = a (k_{\mathrm{B}} T)^2.$$
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Using the above, show that at low temperatures, up to second order in T,

$$\frac{\chi}{2\mu_{\mathrm{B}}^2} = g(\varepsilon_{\mathrm{F}}) - \frac{a}{2} \left(\frac{g'^2(\varepsilon_{\mathrm{F}})}{g(\varepsilon_{\mathrm{F}})} - g''(\varepsilon_{\mathrm{F}}) \right) (k_{\mathrm{B}} T)^2.$$ [3]

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