AFD 2018

1) a) $\dot{Q} = A\rho T^{\alpha} - H$

constant pressure, $\rho = \frac{\mu p}{R_* T}$

$$\dot{Q} = \frac{A\mu p}{R_*} T^{\alpha-1} - H$$

$$\frac{\partial \dot{Q}}{\partial T} = \frac{A\mu p}{R_*}(\alpha-1)T^{\alpha-2} < 0 \quad \text{for } \alpha = \tfrac{1}{2}$$

$\therefore$ unstable - need $\partial \dot{Q}/\partial T > 0$ for stability so that small increases in T can be countered by increase in cooling rate to bring system back into equilibrium

b) $j = \sigma(E + u \times B)$

$\nabla \times B = \mu_0 j + \cancel{\frac{1}{c^2}\dot{E}}$ neglect - small

$\nabla \times B = \mu_0 \sigma (E + u \times B)$

$\nabla \times (\nabla \times B) = -\nabla^2 B = \mu_0 \sigma (\nabla \times E + \nabla \times (u \times B))$

$\nabla^2 B = \mu_0 \sigma \dot{B} - \mu_0 \sigma \nabla \times (u \times B)$

$$\frac{\partial B}{\partial t} = \nabla \times (u \times B) + \frac{1}{\mu_0 \sigma} \nabla^2 B$$

c) $p = k\rho^2$, hydrostatic equilibrium

vertical density structure

$$\frac{1}{\rho}\nabla p = g = \frac{1}{\rho}\frac{dp}{d\rho}\frac{d\rho}{dz} = 2k\frac{d\rho}{dz}$$

$$\int g \cdot ds = -4\pi G \int \rho \, dV$$

$$2Ag = -4\pi G . 2A \int \rho \, dz$$

$$g = -4\pi G \int \rho \, dz$$

$$2k\frac{d\rho}{dz} + 4\pi G \int \rho \, dz = 0$$

$$\frac{d^2\rho}{dz^2} + \frac{2\pi G}{k}\rho = 0$$

3) Show that $\frac{DQ}{Dt} = \frac{\partial Q}{\partial t} + u \cdot \nabla Q$

for $Q(r,t)$, $\delta Q = \frac{\partial Q}{\partial t}\delta t + \nabla Q \cdot \delta r$

$$\frac{DQ}{Dt} = \lim_{\delta t \to 0} \frac{\delta Q}{\delta t} = \lim_{\delta t \to 0} \frac{\partial Q}{\partial t} + \frac{\delta r}{\delta t} \cdot \nabla Q$$

$$= \frac{\partial Q}{\partial t} + u \cdot \nabla Q$$

Origin of Euler's equation

Forces - pressure and gravity

pressure force $= -\int p \cdot dS = -p\delta S = -\nabla p \,\delta V$

pressure force $-\nabla p$ per unit volume

gravity : $F_g = -\nabla\psi\rho$ per unit volume

$m\frac{Du}{Dt} = \sum$ forces $= \rho\frac{Du}{Dt}$ ρ u vol

$$\rho\frac{Du}{Dt} = \rho\frac{\partial u}{\partial t} + \rho u \cdot \nabla u = -\nabla p - \rho\nabla\psi$$

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\frac{1}{\rho}\nabla p - \nabla\psi$$

Show that the quantity $H = \frac{1}{2}u^2 + \int\frac{dp}{\rho} + \psi = \text{const}$ along a streamline

$$\frac{d}{dx}\int\frac{dp}{\rho} = \frac{dp}{dx}\frac{d}{dp}\int\frac{dp}{\rho} = \frac{dp}{dx}\frac{1}{\rho} \Rightarrow \nabla\int\frac{dp}{\rho} = \frac{1}{\rho}\nabla p$$

$u \cdot \nabla u = \nabla(\tfrac{1}{2}u^2) - u \times (\nabla \times u)$

momentum equation for steady state $\frac{\partial u}{\partial t} = 0$

$$\nabla(\tfrac{1}{2}u^2) - u \times (\nabla \times u) + \nabla\int\frac{dp}{\rho} + \nabla\psi = 0$$

dot product with $u \rightarrow$ quantity along streamline

$$u \cdot \nabla\left(\frac{1}{2}u^2 + \int\frac{dp}{\rho} + \psi\right) - \underbrace{u \cdot [u \times (\nabla \times u)]}_{0} = 0$$

$$u\cdot\nabla\left(\tfrac{1}{2}u^2 + \int\frac{dp}{\rho} + \psi\right) = 0$$

$$\hookrightarrow \quad H = \tfrac{1}{2}u^2 \int\frac{dp}{\rho} + \psi = \text{const along a streamline}$$

Spherical accretion - show that

$$(u^2 - c_s^2)\frac{d\ln u}{dr} = \frac{2c_s^2}{r}\left(1 - \frac{GM}{2c_s^2 r}\right).$$

Steady state $\quad \frac{1}{\rho}\nabla p = g - u\cdot\nabla u$

$$g = -\frac{GM}{r^2}, \quad \frac{1}{\rho}\nabla p = \frac{1}{\rho}\frac{dp}{d\rho}\frac{d\rho}{dr} = c_s^2\frac{d\ln\rho}{dr}$$

$$u\cdot\nabla u = u\frac{du}{dr} = u^2\frac{d\ln u}{dr}$$

$$u^2\frac{d\ln u}{dr} + c_s^2\frac{d\ln\rho}{dr} = -\frac{GM}{r^2}$$

Steady state $-\dot{M} = \rho u A = 4\pi r^2 \rho u = \text{const}$

$$\frac{d\ln\dot{M}}{dr} = 0 = \frac{d\ln u}{dr} + \frac{d\ln\rho}{dr} + \frac{2}{r}$$

$$u^2\frac{d\ln u}{dr} - c_s^2\left(\frac{d\ln u}{dr} + \frac{2}{r}\right) = -\frac{GM}{r^2}$$

$$(u^2 - c_s^2)\frac{d\ln u}{dr} = \frac{2c_s^2}{r}\left(1 - \frac{GM}{2c_s^2 r}\right)$$

isothermal - show that $u^2 = 2c_s^2\ln\rho_\infty/\rho + 2GM/r$

$$H = \tfrac{1}{2}u^2 + \int\frac{dp}{\rho} + \psi = \tfrac{1}{2}u^2 + c_s^2\ln\rho - \frac{GM}{r}$$

at ∞ $\quad H = c_s^2\ln\rho_\infty \quad$ as $u = 0$

$$\tfrac{1}{2}u^2 + c_s^2\ln\rho - \frac{GM}{r} = c_s^2\ln\rho_\infty$$

$$u^2 = 2c_s^2\ln(\rho_\infty/\rho) + 2GM/r$$

find accretion rate in terms of M, ρ_∞, c_s

at sonic radius $r_s = \frac{GM}{2c_s^2}$ $\quad (u = c_s)$

$$H = \frac{1}{2}c_s^2 + c_s^2 \ln\rho_s - 2c_s^2 = c_s^2\ln\rho_s - 3/2\, c_s^2$$

$$c_s^2 \ln\rho_s - 3/2\, c_s^2 = c_s^2 \ln\rho_\infty$$

$$\rho_s = \rho_\infty e^{3/2}$$

$$\dot{M} = 4\pi r_s^2 \rho_s c_s = 4\pi \frac{(GM)^2}{4c_s^4}\rho_\infty e^{3/2} c_s = \frac{\pi (GM)^2 \rho_\infty e^{3/2}}{c_s^{\,3}}$$

4) $\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho u) = 0$

$$\frac{\partial u}{\partial t} + u\cdot\nabla u = -\frac{1}{\rho}\nabla p - \nabla\Psi$$

$$\nabla^2\Psi = 4\pi G\rho$$

problem with infinite static uniform medium

$\rho = \text{const}$

↳ momentum equation with $\frac{\partial u}{\partial t}$, $u = 0$ (static) gives

$$\frac{1}{\rho}\nabla p = -\nabla\Psi = 0$$

↳ $\nabla^2\Psi = 0 \Rightarrow \rho = 0$

Jeans ignored problem and introduced small perturbations in p, ρ, u, Ψ to solve governing equations using PT

linearised equations:

$$\frac{\partial}{\partial t}(\rho_0 + \rho_1) + \nabla\cdot((\rho_0+\rho_1)u_1) = 0$$

$$\frac{\partial\rho_1}{\partial t} + \rho_0\nabla\cdot u_1 + \nabla\cdot(\rho_1 u_1) = 0$$

(1) $\frac{\partial\rho_1}{\partial t} + \rho_0\nabla\cdot u_1 = 0$ $\quad$ ignoring second order term

$$\frac{\partial}{\partial t} u_1 + u_1 \cdot \nabla u_1 = -\frac{1}{\rho}\frac{dp}{d\rho}\nabla(\rho_0+\rho_1) - \nabla(\Psi_0+\Psi_1)$$

② $\frac{\partial u_1}{\partial t} = -\frac{1}{\rho_0} c_s^2 \nabla \rho_1 - \nabla \Psi_1$ ignoring second order term

③ $\nabla^2 \Psi_1 = 4\pi G \rho_1$

wavelike solutions $\propto e^{i(k \cdot x - \omega t)}$

① $\omega \rho_1 = \rho_0 k u_1$

② $\omega u_1 = \frac{c_s^2}{\rho_0} k \rho_1 + k \Psi_1$

③ $-k^2 \Psi_1 = 4\pi G \rho_1$

$$\Psi_1 = -\frac{4\pi G \rho_1}{k^2} = \frac{1}{k}\left(\omega u_1 - \frac{c_s^2}{\rho_0} k \rho_1\right)$$

$$\rho_1\left(\frac{c_s^2}{\rho_0} - \frac{4\pi G}{k^2}\right) = \frac{\omega}{k} u_1$$

$$u_1 = \frac{\omega \rho_1}{k \rho_0} \Rightarrow \frac{c_s^2}{\rho_0} - \frac{4\pi G}{k^2} = \frac{\omega^2}{k^2 \rho_0}$$

$$\omega^2 = c_s^2 k^2 - 4\pi G \rho_0 = c_s^2\left(k^2 - \frac{4\pi G \rho_0}{c_s^2}\right) = c_s^2 (k^2 - k_J^2)$$

$$k_J = \sqrt{\frac{4\pi G \rho_0}{c_s^2}}$$

Jeans length $l_J = \frac{2\pi}{k_J} = \frac{2\pi c_s}{\sqrt{4\pi G \rho_0}} = c_s \sqrt{\frac{\pi}{G\rho_0}}$

growing unstable modes
if ω is purely imaginary ($= i\tilde{\omega}$), $\rho_1 \propto e^{ik \cdot x} e^{\tilde{\omega} t}$
- growing mode
ω purely imaginary if $k < \sqrt{\frac{4\pi G \rho_0}{c_s^2}}$

$\tilde{\omega} = c_s \sqrt{\frac{4\pi G \rho_0}{c_s^2} - k^2}$ - fastest growing modes (largest $\tilde{\omega}$) have smaller k

length scale over which sound-crossing time is the same as the free-fall time under gravity

$v_{ff}^2 = \frac{2GM}{R}$, $t_{ff} = R\sqrt{\frac{R}{2GM}} \sim \frac{1}{\sqrt{G\rho_0}}$

sound-crossing time $t_s = R/c_s$

$\frac{R}{c_s} \propto \frac{1}{\sqrt{G\rho_0}}$

length scale $R \sim \frac{c_s}{\sqrt{G\rho_0}} \sim L_J$ - same dependence on c_s, G, ρ_0