UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Tuesday 29 May 2012 13.30 to 15.30

EXPERIMENTAL AND THEORETICAL PHYSICS (1)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (1)

Candidates offering this paper should attempt a total of **three** *questions. The questions to be attempted are* **1**, **2** *and* **one** *other question.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains **four** *sides, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS
2 × 20 Page Answer Book
Metric graph paper
Rough workpad
Yellow master coversheet

SPECIAL REQUIREMENTS
Mathematical Formulae handbook
Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

THERMAL AND STATISTICAL PHYSICS

1 *Attempt* **all** *parts of this question. Answers should be concise and relevant formulae may be assumed without proof.*

(a) Estimate the temperature at which the typical de Broglie wavelength is of the order of the interparticle spacing for electrons in solids, for atomic gases and for neutron stars. [4]

(b) The heat capacity of a material at low temperatures is of the form $C = (220 \text{ J K}^{-5/2} \text{ m}^{-3})T^{3/2}$. If this is due to elementary bosonic excitations with an energy-momentum relation $\epsilon = \alpha p^n$, find n and α. [4]

[*Note that* $\displaystyle\int_0^\infty \frac{x^4 \, \mathrm{d}x}{\mathrm{e}^x - 1} \approx 25$.]

(c) The entropies of two phases of a given material are equal on a coexistence line in the temperature-pressure plane. Show that the slope of the coexistence line is given by

$$\frac{\mathrm{d}T}{\mathrm{d}P} = \frac{TV\,\Delta\alpha_T}{\Delta C_P},$$

where $\Delta\alpha_T$ and ΔC_P are, respectively, the differences in the volume thermal expansion coefficient and heat capacity between the two phases and V is the volume. [4]

2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) the Landau model of phase transitions;

(b) Brownian motion of free and confined particles;

(c) Bose–Einstein condensation.

3 *Attempt* **either** *this question* **or** *question 4.*

The grand potential for non-interacting fermions can be written as

$$\Phi(T,\mu) = k_{\mathrm{B}}T \int_0^\infty \mathrm{d}\epsilon\, g(\epsilon) \ln[1 - n_{\mathrm{F}}(\epsilon,\mu)],$$

where T is the temperature, μ is the chemical potential, $g(\epsilon)$ is the single-particle density of states and $n_{\mathrm{F}}(\epsilon,\mu)$ is the Fermi–Dirac distribution function. Give an outline of the derivation of this result. [6]

By means of a double integration by parts, or otherwise, show that

$$\Phi(T,\mu) = \int_0^\infty \mathrm{d}\epsilon\, \Phi(0,\epsilon)\left[-\frac{\partial n_{\mathrm{F}}(\epsilon,\mu)}{\partial \epsilon}\right],$$

where $\Phi(0,\epsilon) = -\int_0^\epsilon \mathrm{d}\epsilon' \int_0^{\epsilon'} \mathrm{d}\epsilon''\, g(\epsilon'')$. [7]

[*Note that* $n_{\mathrm{F}}(\epsilon,\mu) = -\frac{\partial}{\partial\epsilon}\{k_{\mathrm{B}}T \ln[1 - n_{\mathrm{F}}(\epsilon,\mu)]\}$.]

If $g(\epsilon)$ has the form

$$g(\epsilon) = 1 + \alpha \cos\left(\frac{2\pi\epsilon}{\varDelta}\right),$$

where α and $\varDelta$ are, respectively, the amplitude and the period of oscillations in ϵ, show that for $k_{\mathrm{B}}T$ and $\varDelta$ much smaller than μ,

$$\Phi(T,\mu) = \Phi(0,\mu) - \frac{\pi^2}{6}k_{\mathrm{B}}^2T^2 + \frac{\alpha\varDelta^2}{4\pi^2}\left(\frac{X}{\sinh X} - 1\right)\cos\left(\frac{2\pi\mu}{\varDelta}\right),$$

where $X = 2\pi^2 k_{\mathrm{B}}T/\varDelta$. [7]

Sketch the temperature variation of the ratio of the amplitude of the oscillatory to the non-oscillatory components in $\Phi(T,\mu) - \Phi(0,\mu)$ for fixed μ. [5]

[*Note that* $\int_{-\infty}^{\infty} \mathrm{d}\epsilon\, [-\partial n_{\mathrm{F}}(\epsilon,\mu)/\partial\epsilon]\, \mathrm{e}^{\mathrm{i}2\pi(\epsilon-\mu)/\varDelta} = X/\sinh X$.]

4 *Attempt* **either** *this question* **or** *question 3.*

Write brief notes, with examples, on the thermodynamic description of two-level systems in thermal equilibrium. [5]

Recalling that a magnetic atom with angular momentum J in an applied magnetic field B has $2J + 1$ levels, show that its partition function at temperature T is given by

$$Z = \sum_{m=-J}^{J} \exp\left(m\frac{T_B}{T}\right),$$

where $T_B = g\mu_B B/k_B$ and g is the Landé g-factor. [3]

Prove that the entropy depends on T and B only through the ratio B/T and that in the high-temperature limit, $T \gg T_B$, the entropy (per atom) reduces to

$$s(T, B) = k_B \left[\ln(2J + 1) - J(J + 1)\frac{T_B^2}{6T^2}\right].$$

[6]

Let N non-interacting magnetic atoms be in thermal equilibrium with a second system with M particles and a heat capacity $Mk_B T/\theta$ for $T \ll \theta$, where θ is a constant. Show that the minimum final temperature, $T_{\min}$, that can be reached by a process of isothermal magnetization followed by adiabatic demagnetization, is given by $Ns(T_i, B_i) = Mk_B(T_i - T_{\min})/\theta$, where B_i and T_i are the initial magnetic field and temperature at the beginning of the demagnetization process, and hence

$$T_{\min} = T_i - \frac{N\theta J(J + 1)g^2\mu_B^2 B_i^2}{6Mk_B^2 T_i^2}.$$

[6]

Discuss the range of validity of your result. Discuss qualitatively how interactions between the magnetic moments may be expected to modify the calculation of the entropy. [5]

[*Note that* $\sum_{m=-J}^{J} m^2 = \frac{1}{3}J(J + 1)(2J + 1)$.]

END OF PAPER