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A1 (a) $v_{rms} = \sqrt{3k_BT/m}$ where $m = \frac{4}{3}\pi r^3 \rho$.

Assume $\rho = 1000 kgm^{-3} \doteq \rho_{water}$ (particle random in water),

$r = 0.5\times10^{-6} m$, $\therefore$ mass $m \doteq 5.2\times10^{-16} kg$.

At $T \doteq 300K$, $v_{rms} \doteq 0.0049 ms^{-1} = 4.9\times10^{-3} ms^{-1}$.

$x_{rms} = 4\times10^{-6} m$, $t = 100s$,

$\therefore v = x_{rms}/t = 4\times10^{-8} ms^{-1}$.

Assume (I'm not sure)

$v = v_{rms} e^{-t/\tau} \Rightarrow \tau \doteq \frac{t}{11.7} \doteq 8.5s$.

(b) $dF = -SdT - PdE \Rightarrow \left(\frac{\partial P}{\partial T}\right)_E = \left(\frac{\partial S}{\partial E}\right)_T$.

Also $C_E = T\left(\frac{\partial S}{\partial T}\right)_E$,

$\therefore$ RHS $= -\frac{T}{C_E}\left(\frac{\partial P}{\partial T}\right)_E$

$= -\frac{(\partial S/\partial E)_T}{(\partial S/\partial T)_E}$.

As $dS = \left(\frac{\partial S}{\partial T}\right)_E dT + \left(\frac{\partial S}{\partial E}\right)_T dE = 0$ at constant S.

$\therefore \left(\frac{\partial S}{\partial T}\right)_E dT = -\left(\frac{\partial S}{\partial E}\right)_T dE$,

$\therefore \left(\frac{dT}{dE}\right)_S = -\frac{(\partial S/\partial E)_T}{(\partial S/\partial T)_E} = $ LHS.

$\therefore$ LHS = RHS, $\therefore \left(\frac{\partial T}{\partial E}\right)_S = -\frac{T}{C_E}\left(\frac{\partial P}{\partial T}\right)_E$.

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(c) $\overline{\qquad}\ \varepsilon$ $\qquad U = \dfrac{\varepsilon e^{-\beta\varepsilon}}{1+e^{-\beta\varepsilon}} = \dfrac{\varepsilon}{e^{\beta\varepsilon}+1}$.

$\overline{\qquad}\ 0$

let $\varepsilon = k_B T_1$, then $U = \dfrac{k_B T_1}{1+e^{+T_1/T}} = k_B T_1 (1+e^{+T_1/T})^{-1}$.

At high T, $T \gg T_1$

$$U \doteq k_B T_1 \left(2 + \frac{T_1}{T}\right)^{-1} \doteq \frac{k_B T_1}{2}\left(1+\frac{T_1}{2T}\right)^{-1}$$

$$\doteq \frac{k_B T_1}{2}\left(1 \cancel{+} - \frac{T_1}{2T}\right)$$

$$= \frac{1}{2}k_B T_1 \cancel{+} - \frac{k_B T_1^2}{4T},$$

for C_{molar}.

$$\therefore C = \frac{\partial U}{\partial T} N_a = \frac{1}{4} R T_1^2 / T^2 = (7.5 \times 10^{-4}\,\mathrm{JK/mole})/T^2.$$

$$\therefore \frac{1}{4} R T_1^2 = 7.5 \times 10^{-4}\,\mathrm{JK/mole},$$

$$\therefore T_1 \doteq 0.019\,\mathrm{K}.$$

Thus at low T, $U = \dfrac{k_B T_1}{1+e^{T_1/T}}$,

$$\therefore C = \frac{\partial U}{\partial T} N_a = k_B T_1 (-1)(1+e^{T_1/T})^{-2} e^{T_1/T} \cdot T_1 \cdot (-1) T^{-2} N_a$$

$$= k_B T_1^2 \frac{e^{T_1/T}}{T^2 (1+e^{T_1/T})^2} \cdot N_a. \qquad (R = k_B N_a).$$

let $X = T_1/T$,

then $C = R X^2 \dfrac{e^X}{(1+e^X)^2}$

Schottky peak appears at $X \approx 2$, (by PC simulation)

$$\therefore T \doteq T_1/2 \doteq 10\,\mathrm{mK}.$$

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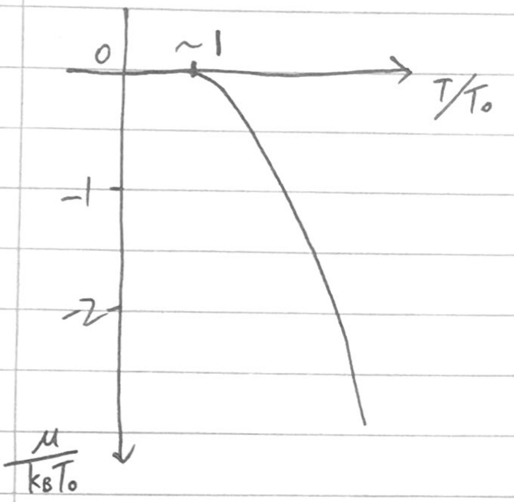
A3 (a) $n(\varepsilon_k) = \dfrac{1}{e^{\beta(\varepsilon_k - \mu)} - 1} = \langle n_k \rangle$



$$\boxed{N = \int_0^\infty g(\varepsilon)\, n(\varepsilon)\, d\varepsilon}$$

$$= \frac{\sigma V m^{3/2}}{\sqrt{2}\pi^2 \hbar^3} \int_0^\infty \frac{\sqrt{\varepsilon}\, d\varepsilon}{e^{\beta(\varepsilon - \mu)} + 1}$$

When $T \to 0$, $\langle n_{\varepsilon=0} \rangle = N \Rightarrow \mu \doteq -\frac{k_B T}{N} \approx 0$.

When $0 < T < T_0$,

$N_{\varepsilon>0} = N(T/T_0)^{3/2}$ by setting $\mu = 0$.

$\therefore \underset{\substack{\uparrow \\ \text{condensate}}}{N_0} = N - N_{\varepsilon>0} = N[1 - (T/T_0)^{3/2}]$.

$T_0 = T_c$.

no. not in the condensate. $\mu = 0$.

(b) $n = N/V$. $\quad g(k)\,dk = \left(\frac{a}{\pi}\right)^3 \frac{4\pi k^2 dk}{8} = \frac{Vk^2}{2\pi^2} dk$.

$\therefore N = \int_0^\infty \frac{Vk^2}{2\pi^2} dk \cdot \frac{1}{e^{\frac{\alpha k}{k_B T_0}} - 1}$, $\quad \underset{\sim\sim\sim}{n = \frac{1}{2\pi^2} \int_0^\infty \frac{k^2\, dk}{e^{\frac{\alpha k}{k_B T_0}} - 1}}$.

let $x = \frac{\alpha}{k_B T_c} k$, $\quad dk = \frac{k_B T_c}{\alpha} dx$,

$$\therefore N = \frac{1}{2\pi^2} \left(\frac{k_B T_c}{\alpha}\right)^3 V \underbrace{\int_0^\infty \frac{x^2\, dx}{e^x - 1}}_{2.40}.$$

$$\therefore n = \frac{1}{2\pi^2} \left(\frac{k_B T_c}{\alpha}\right)^3 \cdot 2.40,$$

$$\therefore \underset{\sim\sim\sim}{T_c = \frac{2.02\, \alpha n^{1/3}}{k_B}}.$$

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(c) $F = U - TS$, $dF = -SdT - pdV$, ①

$$\therefore p = -\left(\frac{\partial F}{\partial V}\right)_T = -\left(\frac{\partial (U-TS)}{\partial V}\right)_T$$

$$= -\left(\frac{\partial U}{\partial V}\right)_T + T\left(\frac{\partial S}{\partial V}\right)_T,$$

From ①, $\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V$,

$$\therefore p - T\left(\frac{\partial p}{\partial T}\right)_V = -\left(\frac{\partial U}{\partial V}\right)_T \quad ②$$

(d) Similar to 'N' in (b), $T < T_c$.

$$U = \int_0^\infty \frac{Vk^2}{2\pi^2} dk \frac{\alpha k}{e^{\frac{\alpha k}{k_B T}} - 1}$$

$$= \frac{V}{2\pi^2} \alpha \int_0^\infty k^3 \frac{1}{e^{\frac{\alpha k}{k_B T}} - 1} dk,$$

let $x = \frac{\alpha k}{k_B T}$, $\therefore U = \frac{\alpha V}{2\pi^2}\left(\frac{k_B T}{\alpha}\right)^4 \underbrace{\int_0^\infty \frac{x^3}{e^x - 1} dx}_{6.49}$.

$$\therefore U = \frac{6.49}{2\pi^2} \frac{(k_B T)^4}{\alpha^3} V \doteq \underbrace{0.329 \frac{(k_B T)^4}{\alpha^3}}_{C} V = CV \propto V \text{ at const. } T.$$

Use ②, $p = T\left(\frac{\partial p}{\partial T}\right)_V - C$,

rearrange with an integrating factor:

$$\frac{1}{T}\left(\frac{\partial p}{\partial T}\right) - \frac{1}{T^2} p = \frac{C}{T^2} \Rightarrow \frac{\partial (p/T)}{\partial T} = \frac{C}{T^2},$$

$$\therefore \frac{\partial (p/T)}{\partial T} \doteq 0.33 \frac{k_B^4}{\alpha^3} T^2, \quad \frac{p}{T} = 0.11 \frac{k_B^4}{\alpha^3} T^3 + \text{constant}.$$

At $T=0$, $p=0$, $\therefore$ constant $= 0$. Thus $p = 0.11 \frac{k_B^4 T^4}{\alpha^3}$.

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(e) p is independent of n due to BEC.
particles in BEC don't exert p.
For $N_{\varepsilon>0}$, $\mu = 0$ ~ ∴ similar p expression as photon gas.
However photon gas doesn't have to be $T < T_c$.

A4 (a)

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A4 (a) · order parameter

· $F(T,M) = F_0(T) + A(T)M + B(T)M^2 + \cdots$

· Symmetry to simplify the expansion.

· Broken symmetry at the transition temp T_c.

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(b) Most probable n means minimum F,

$$F(M,n) = f(M) + \alpha n^2 + \beta M^2 n,$$

$$\frac{\partial F}{\partial n} = 2\alpha n + \beta M^2 = 0 \Rightarrow n = -\frac{\beta M^2}{2\alpha}.$$

$$\therefore F(M,n) = f(M) + \alpha \frac{\beta^2 M^4}{4\alpha^2} + \beta M^2\left(-\frac{\beta M^2}{2\alpha}\right)$$

$$= f(M) - \frac{\beta^2 M^4}{4\alpha}.$$

$$= f(M) - \frac{\beta^2}{4\alpha} M^4 \text{ at most prob. } n.$$

(c) Now $F(M) = \gamma(T-T_0)M^2 + \left(b - \frac{\beta^2}{4\alpha}\right)M^4 + cM^6 - MB$

At condition $T > T_0$, $B = 0$, $b < \frac{\beta^2}{4\alpha}$, all others > 0.

$F(M) = XM^2 - YM^4 + ZM^6$ for +ve X, Z & Y. ~ const.

$$\begin{cases} X = \gamma(T-T_0). \\ Y = -\left(b - \frac{\beta^2}{4\alpha}\right). \\ Z = c. \end{cases}$$

$F(0) = 0$. $\partial F/\partial M = 2XM \overset{-}{\cancel{+}} 4YM^3 + 6ZM^5 = F'$

$\partial^2 F/\partial M^2 = 2X \overset{-}{\cancel{+}} 12YM^2 + 30ZM^4 = F''$

$F' = 0 \Rightarrow M = 0$, or $6ZM^4 \overset{-}{\cancel{+}} 4YM^2 + 2X = 0$.

$\Downarrow$ 2 solutions for $M^2 \Rightarrow 4$ for M but symmetrical.

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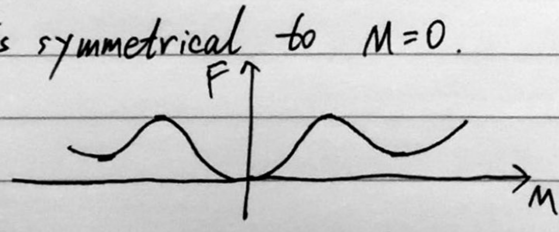
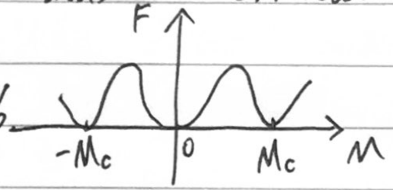
$\because F(M)$ is symmetrical to $M = 0$.

$\therefore$



For the 1st order transition (discontinuous),
~~'M' has more than one value for the min F~~, $\boxed{F_{min} < F_0}$.
thus the critical situation is when $F = 0 = F'$.

P103. or Q25/26 (example sheet 4).



~ equal F at the transition.

$$\therefore \begin{cases} XM^2 - YM^4 + ZM^6 = 0 \\ 2XM - 4YM^3 + 6ZM^5 = 0. \end{cases}$$

For $M \neq 0$ solutions,

$$\begin{cases} ZM^4 - YM^2 + X = 0 \quad ① \quad \cancel{\sim 6ZM^4 - 6YM^2 + 6X = 0} \\ 6ZM^4 - 4YM^2 + 2X = 0. \quad ② \end{cases}$$

$$\cancel{\Rightarrow 2YM^2 - 4X = 0}$$

$$\cancel{\therefore M^2 = \frac{2X}{Y}}$$

$$\cancel{\Rightarrow M_c^2 = \frac{2\gamma(T - T_0)}{-(b - \frac{\beta^2}{4\alpha})}}$$

$(② - ①\times 2)$. $4ZM^4 - 2YM^2 = 0$,

$$\therefore 2ZM^2 - Y = 0,$$

$$\therefore M_c^2 = \frac{Y}{2Z} = \left(\frac{\beta^2}{4\alpha} - b\right)/(2c),$$

(d) $$\therefore M_c = \left(\frac{\beta^2}{8\alpha c} - \frac{b}{2c}\right)^{\frac{1}{2}}.$$

$\because F(M_c) = 0$,

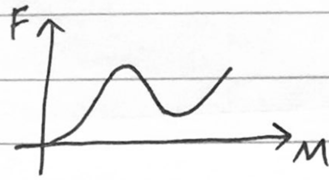
$$\therefore ZM_c^4 - YM_c^2 + X = 0.$$

$$\therefore c\left(\frac{\beta^2}{8\alpha c} - \frac{b}{2c}\right)^2 + \left(b - \frac{\beta^2}{4\alpha}\right)\left(\frac{\beta^2}{8\alpha c} - \frac{b}{2c}\right) + \gamma(T_c - T_0) = 0.$$

(c) $$\therefore T_c = T_0 + \cancel{4\alpha}\frac{1}{4\gamma c}\left(\frac{\beta^2}{4\alpha} - b\right)^2.$$ ↙ I calculated....

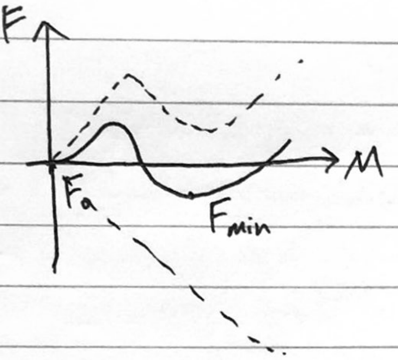
(e) $T > T_c$.

∵ symmetry, only draw +ve M.



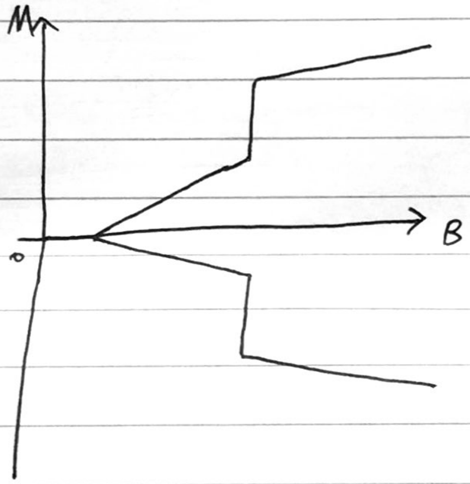
Also $B \neq 0$. (and $B > 0$ as given in condition in (c)).

∴ a '$-BM$' term is added to original F.

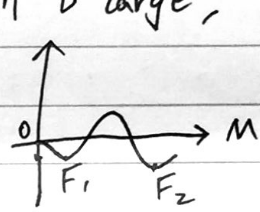


At the point $F_{min} < F_0$, M can jump like 1st order transition.

Thus



correction.
if B large,



$F_2 > F_1 \Rightarrow$ jump too.

(f) Large no. of particles ⇒ central limit theorem can be used.