UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

May–June 2020 **1 hour 15 minutes**

PHYSICS (3)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (3)

ADVANCED QUANTUM PHYSICS

*Candidates offering this paper should attempt a total of **four** questions: **three** questions from Section A and **one** question from Section B.*

*The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains **four** sides, including this coversheet. You may use the formula handbook for values of constants and mathematical formulae, which you may quote without proof.*

You have 75 minutes (plus any pre-agreed individual adjustment) to answer this paper. Do not start to read the questions on the subsequent pages of this question paper until the start of the time period.

Please treat this as a closed-book exam and write your answers within the time period. Downloading and uploading times should not be included in the allocated exam time. If you wish to print out the paper, do so in advance. You can pause your work on the exam in case of an external distraction, or delay uploading your work in case of technical problems.

Section A and the chosen section B question should be uploaded as separate pdfs. Please name the files 1234X_Qi.pdf, where 1234X is your examination code and i is the number of the question/section (A or 4 or 5).

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Master coversheet	Mathematical Formulae handbook
	Approved calculator allowed

SECTION A

*Attempt **all** questions in this Section. Answers should be concise and relevant formulae may be assumed without proof.*

1 Two spin-$\frac{1}{2}$ particles exhibit an exchange-like interaction between their spins

$$\widehat{H} = A[3(\widehat{S}_1^z \widehat{S}_2^z) - \widehat{\boldsymbol{S}}_1 \cdot \widehat{\boldsymbol{S}}_2],$$

where A is a constant. Show that the triplet and singlet states are eigenfunctions of the spin Hamiltonian with eigenvalues 0 for $|0,0\rangle$, $-\hbar^2 A$ for $|1,0\rangle$ and $\frac{\hbar^2 A}{2}$ for $|1,\pm 1\rangle$, where the first and second indices are quantum numbers of the total spin operators $\widehat{S}^2$ and $\widehat{S}^z$, respectively. [4]

2 Calculate the three lowest energy levels together with their degeneracies for a system of three distinguishable non-interacting spin-$\frac{1}{2}$ particles with mass m in a two-dimensional square box of area L^2. [4]

3 Using Hund's rules, find the spectroscopic term that describes the ground state of the phosphorus atom ($Z = 15$). [4]

SECTION B

Attempt one question from this section

4 An electron is in a constant magnetic field B_0 that points along the positive x-direction. At time $t = 0$ a constant magnetic field B_1 is applied in the z-direction.

(a) Briefly describe the phenomenon of spin precession in this magnetic field. [2]

(b) The electron is in its ground state $|0\rangle$ before B_1 is switched on. Show that at time $t = 0$, when B_1 is turned on, the probability of finding its spin component along the new direction of the total field $\boldsymbol{B}$ with values $\pm\frac{\hbar}{2}$ is $\frac{1}{2}(1 \mp \sin\theta)$, where θ is the angle between the total field $\boldsymbol{B}$ and the z-axis. [5]

(c) Show that the time-evolution operator, which describes $|\psi(t)\rangle = U(t)\,|\psi(0)\rangle$, where $|\psi(0)\rangle = |0\rangle$, can be written in the form

$$\widehat{U}(t,0) = \cos\frac{\omega t}{2}\,\widehat{I} - \mathrm{i}\sin\frac{\omega t}{2}(\sin\theta\,\widehat{\sigma}_x + \cos\theta\,\widehat{\sigma}_z),$$

where $\omega = \frac{egB}{2m_\mathrm{e}}$ is the precession frequency for electrons of mass m_e, $\widehat{I}$ is the identity matrix and $\widehat{\sigma}_i$ are the Pauli matrices. [5]

(d) What is the probability that at $t = T$ the electron is in the eigenstate of $\hat{S}_z$ with eigenvalue $-\hbar/2$? [4]

(e) At $t = 2T$ we want to bring the electron back to its initial state $|0\rangle$. Show that this can be done by switching the value of B_1 to $-B_1$ at $t = T$. [3]

[*The magnetic moment of an electron $\mu = \gamma S$, where S is the spin operator and $\gamma = -\frac{ge}{2m_\mathrm{e}}$ with $g \approx 2$ and e the absolute value of the electron charge.*
The Pauli matrices are $\widehat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $\widehat{\sigma}_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$; $\widehat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.]

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5 Consider the four degenerate levels of a non-relativistic hydrogen atom with $n = 2$. The perturbing Hamiltonian $\widehat{H}_1$ representing the application of an electric field in the z-direction can be written in the basis $|21\text{-}1\rangle$, $|211\rangle$, $|210\rangle$, $|200\rangle$ as

$$H_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\varepsilon \\ 0 & 0 & -\varepsilon & 0 \end{pmatrix} \qquad (\star)$$

where each state is of the form $|n\ell m\rangle$.

(a) Explain, without detailed calculation, why the only non-zero elements are $\langle 210|\widehat{H}_1|200\rangle$ and $\langle 200|\hat{H}_1|210\rangle$, with $\widehat{H}_1 = eE\widehat{z}$, and why perturbation theory cannot be applied in this case to find the corrections to the energy levels induced by the field. [4]

(b) What are the zeroth-order eigenstates and the first-order energy shifts induced by the electric field? [2]

(c) Suppose that the unperturbed states $|210\rangle$ and $|200\rangle$ were not degenerate but had energy levels $E_0 - \Delta$ and $E_0 + \Delta$. Calculate how these energy levels change, using second-order perturbation theory in the limit $\varepsilon \ll \Delta$. Compare your results with the exact eigenvalues. [5]

(d) What happens if $\varepsilon \gg \Delta$? [3]

(e) If the unperturbed states are all degenerate, how does the matrix in $(\star)$ change if the electric field points in the x-direction such that $\widehat{H}_1 = eE\widehat{x}$? Write the new elements of the matrix in terms of ε. Determine the eigenvalues and eigenstates. [5]

The degenerate n=2 states of the hydrogen atom in spherical polar coordinates are given by:

$$\psi_{200} = \sqrt{\frac{1}{32\pi a_0^3}} e^{-r/(2a_0)} (2 - \frac{r}{a_0})$$

$$\psi_{210} = \sqrt{\frac{1}{32\pi a_0^3}} e^{-r/(2a_0)} \frac{r}{a_0} \cos\theta$$

$$\psi_{21\pm1} = \sqrt{\frac{1}{64\pi a_0^3}} e^{-r/(2a_0)} \frac{r}{a_0} e^{\pm i\phi} \sin\theta$$

END OF PAPER