UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Saturday 2 June 2012 09.00 to 11.00

EXPERIMENTAL AND THEORETICAL PHYSICS (2)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (2)

Candidates offering this paper should attempt a total of **three** *questions. The questions to be attempted are* **1**, **2** *and* **one** *other question.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains **four** *sides, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS
2 × 20 Page Answer Book
Metric graph paper
Rough workpad
Yellow master coversheet

SPECIAL REQUIREMENTS
Mathematical Formulae handbook
Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

RELATIVITY

1 *Attempt* **all** *parts of this question. Answers should be concise, and relevant formulae may be assumed without proof.*

(a) Show that for an isolated free massive particle at rest in the laboratory frame, it is impossible for it to decay into a single photon. Show further that if the same particle is moving at any speed u, then decay to a single photon is also impossible. [4]

(b) A particle moves such that the 3-part of its 4-velocity is perpendicular to the 3-part of its acceleration. Show that it is moving at constant speed.

What is the acceleration 4-vector of a particle moving with 3-space coordinates

$$\boldsymbol{x}(t) = (r\cos\omega t, r\sin\omega t, 0),$$

where r and ω are constants, and t is laboratory time? [4]

(c) The metric for a 2-dimensional space is

$$\mathrm{d}s^2 = y^2\,\mathrm{d}x^2 + x^2\,\mathrm{d}y^2.$$

Using the Lagrangian method, find the geodesic equations in this space. Show that motion along a straight line through the origin, of the form $y(\lambda) = mx(\lambda)$, where m is a constant and λ is the affine parameter along the geodesic, is a possible solution. [*It is not necessary to find an explicit form for* $x(\lambda)$.] [4]

2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) the bending of light as a test of general relativity;

(b) tidal forces and geodesic deviation;

(c) the covariant formulation of electromagnetism in Minkowski spacetime and electromagnetic invariants.

3 *Attempt* **either** *this question* **or** *question 4.*

[*Note that throughout this question you may assume that the choice of units is such that the speed of light* $c = 1$.]

The Einstein equations in the presence of matter and a cosmological constant Λ are

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu},$$

where $\kappa = 8\pi G$. Define $G_{\mu\nu}$ in terms of the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R, and show how $R_{\mu\nu}$ is related to the Riemann curvature tensor. [2]

Demonstrate how in vacuum we may rewrite these equations as

$$R_{\mu\nu} = \Lambda g_{\mu\nu}. \quad (*)$$ [3]

For a metric of the form

$$\mathrm{d}s^2 = A(r)\,\mathrm{d}t^2 - \frac{1}{A(r)}\,\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right), \quad (**)$$

where (r, θ, ϕ) are standard spherical polar coordinates, it can be shown that the non-zero components of the Ricci tensor are

$$R_{00} = -A\left(\frac{A'}{r} + \frac{1}{2}A''\right), \quad R_{11} = -\frac{1}{A^2}R_{00}, \quad R_{22} = -\left(1 - A - rA'\right), \quad R_{33} = \sin^2\theta\,R_{22},$$

where primes denote derivatives with respect to r. The 22 component of the field equations (*) is thus

$$1 - A - rA' = r^2\Lambda. \quad (***)$$

By differentiating this equation, show that if a function $A(r)$ is found which solves (***), it will automatically solve the remaining field equations. [4]

Find such an $A(r)$, by integrating equation (***), and identify the resulting constant of integration in terms of the mass of the central body. [*You may find it helpful to note* $A + rA' = (rA)'$.] [4]

Treating A as a general function of r, apply the geodesic Lagrangian method to the metric (**) to demonstrate that for a massive particle moving in the $\theta = \pi/2$ plane

$$A\dot{t} = k, \quad r^2\dot{\phi} = h \quad \text{and} \quad \dot{r}^2 = k^2 - A\left(1 + \frac{h^2}{r^2}\right),$$

where a dot denotes differentiation with respect to the proper time of the particle, and give a physical interpretation of the constants k and h. [6]

Thus demonstrate that for radial motion

$$\ddot{r} = -\frac{1}{2}A',$$

and evaluate $\ddot{r}$ explicitly for the $A(r)$ you found above. [3]

What physical interpretation do you give this result, and what is its Newtonian limit? [3]

(TURN OVER

4 *Attempt* **either** *this question* **or** *question 3.*

The Schwarzschild metric for the vacuum around a spherically symmetric body of mass M is

$$ds^2 = \left(1 - \frac{2\mu}{r}\right) c^2\, dt^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right),$$

where $\mu = GM/c^2$. Using the Lagrangian method, obtain the geodesic equations for a photon moving in the $\theta = \pi/2$ plane in this metric, and in particular demonstrate

$$\left(1 - \frac{2\mu}{r}\right)\dot{t} = k, \quad c^2\left(1 - \frac{2\mu}{r}\right)\dot{t}^2 - \left(1 - \frac{2\mu}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = 0 \quad \text{and} \quad r^2\dot{\phi} = h, \quad (*)$$

where k and h are constants and a dot denotes differentiation with respect to an affine parameter along the worldline. [4]

Use these to construct the 'energy equation' for photon motion, in the form

$$\frac{\dot{r}^2}{h^2} + V_{\text{eff}}(r) = \frac{1}{b^2}, \quad (**)$$

where you should derive an explicit expression for $V_{\text{eff}}(r)$ as a function of r, and also identify the constant b in terms of k and h. [4]

Sketch $V_{\text{eff}}(r)$ and use this to show (without detailed proof) that a circular photon orbit is possible at $r = 3\mu$, but that this orbit is unstable. [5]

An alternative coordinate system for the spacetime around a spherical mass is known as advanced Eddington–Finkelstein coordinates. When using these coordinates the metric takes the form

$$ds^2 = \left(1 - \frac{2\mu}{r}\right) c^2\, dt'^2 - \frac{4\mu c}{r}\, dt'\, dr - \left(1 + \frac{2\mu}{r}\right) dr^2 - r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right),$$

where t' is a new time coordinate. Using the Lagrangian method, find the geodesic equations analogous to (*), which describe photon motion in this new metric. [4]

Use these equations to construct the photon 'energy equation' for the new metric, and show that the $V_{\text{eff}}(r)$ which appears in it has an identical form, as a function of r, as the $V_{\text{eff}}(r)$ you obtained in (**). [6]

Why would you expect this result on physical grounds? [2]

END OF PAPER