UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Tuesday 27 May 2014 9.00 am to 11.00 am

PHYSICS (2)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (2)

RELATIVITY

Candidates offering this paper should attempt a total of **three** *questions. The questions to be attempted are* **1**, **2** *and* **one** *other question.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains **four** *sides, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
2 × 20 Page Answer Book	Mathematical Formulae handbook
Rough workpad	Approved calculator allowed
Yellow master coversheet	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

RELATIVITY

1 *Attempt* **all** *parts of this question. Answers should be concise, and relevant formulae may be assumed without proof.*

(a) A particle of rest mass m_1 and 3-velocity v collides with a stationary particle of mass m_2 and is absorbed by it. Find the rest mass and velocity of the compound system. [4]

(b) Using the formulae given below, or otherwise, calculate the non-zero connection coefficients $\Gamma^a_{\ bc}$ for the metric

$$\mathrm{d}s^2 = \mathrm{d}x^2 + f^2(x)\,\mathrm{d}y^2\,,$$

where f is a twice-differentiable function of x. Calculate also the coefficient $R^x_{\ yxy}$ of the Riemann tensor. [4]

[*You may assume without proof the formulae* $\Gamma^a_{\ bc} = \frac{1}{2}g^{ad}(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc})$ *and* $R^d_{\ abc} = \partial_b \Gamma^d_{\ ac} - \partial_c \Gamma^d_{\ ab} + \Gamma^e_{\ ac}\Gamma^d_{\ eb} - \Gamma^e_{\ ab}\Gamma^d_{\ ec}$.]

(c) Given that an astronaut can briefly withstand a gravity gradient of 100 s^{-2}, estimate the closest it is possible to approach a neutron star of mass 3×10^{30} kg and radius 10 km without suffering serious damage from tidal forces. [4]

2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) inertial and gravitational mass and the Principle of Equivalence;

(b) the transformation properties of electric and magnetic fields, using the concept of the electromagnetic field strength tensor $F_{\mu\nu}$;

(c) the observational evidence for the existence of black holes.

3 *Attempt* **either** *this question* **or** *question 4.*

Write down the Lorentz transformation between a stationary frame S and another frame S′ moving at velocity v in the x direction. A particle has velocity components (u'_x, u'_y, u'_z) in S′. Find an expression for the components (u_x, u_y, u_z) of its velocity in S. [5]

A particle moving along the x-axis at velocity v has a rest-frame acceleration a. Show that $\dfrac{dv}{d\tau} = a\gamma^{-2}$, where τ is the particle's proper time. [3]

Hence show that a particle accelerating uniformly from rest at $(t = 0, x = 0)$ in S has a velocity $v = c\tanh(a\tau/c)$ and follows the spacetime trajectory

$$t = \frac{c}{a}\sinh(a\tau/c), \quad x = \frac{c^2}{a}(\cosh(a\tau/c) - 1) \ .$$ [4]

Suppose that a spacecraft is equipped with a photon drive that operates by combining matter and anti-matter and using the energy released to make a collimated beam of electromagnetic radiation. If the initial mass of the spacecraft is M_i show that, when it has reached a velocity v, the mass is given by

$$M = M_\mathrm{i}\left(\frac{c - v}{c + v}\right)^{1/2} \ .$$ [4]

The spacecraft is to travel to and come to rest at a stellar system 100 light years from the Earth at a speed such that the time taken to get there (as recorded by the travellers) is 10 years. Assuming that the time taken to accelerate and decelerate can be neglected, give an estimate of the fraction of the spacecraft's mass that must be anti-matter. [5]

After take-off it is decided to increase the travellers' comfort by limiting acceleration to $a = 10\ \mathrm{m\ s^{-2}}$, but to retain the same cruising speed. How much extra time will the travellers say they spend on the journey? [4]

(TURN OVER

4 *Attempt* **either** *this question* **or** *question 3.*
The Schwarzschild metric has the form

$$\mathrm{d}s^2 = \left(1 - \frac{2\mu}{r}\right) c^2 \mathrm{d}t^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} \mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2)$$

where $\mu = GM/c^2$.

Show that, for geodesic motion in the equatorial plane $\theta = \pi/2$, the equations of motion for a massive particle are

$$\begin{aligned} \left(1 - \frac{2\mu}{r}\right)\dot{t} &= k, \\ r^2\dot{\phi} &= h, \\ \dot{r}^2 - \frac{2\mu h^2}{r^3} + \frac{h^2}{r^2} - \frac{2\mu c^2}{r} &= (k^2 - 1)c^2, \end{aligned}$$

where k and h are constants and dots denote differentiation with respect to some affine parameter σ. [6]

For geodesic motion of a massive particle in the circle $r = R$, show that

$$h^2 = \frac{\mu c^2 R^2}{(R - 3\mu)} \quad \text{and} \quad k^2 = \frac{(R - 2\mu)^2}{R(R - 3\mu)}.$$ [5]

Bob is an astronaut in a space capsule with no engine in a circular orbit at $r = R$ (where $R > 3\mu$). His colleague Alice uses her rocket-pack to maintain a hovering position at a fixed point of space at the same radius R. Show that their relative velocity as they pass each other is

$$v_{\mathrm{AB}} = c\sqrt{\frac{\mu}{R - 2\mu}}.$$ [5]

Another colleague Carol is falling from rest at infinity on a radial geodesic. She arrives at the same point in space just as Bob and Alice are passing each other. Show that their relative velocities are

$$v_{\mathrm{AC}} = c\sqrt{\frac{2\mu}{R}} \quad \text{and} \quad v_{\mathrm{BC}} = c\sqrt{\frac{3\mu}{R}}.$$ [6]

Show that these relative three velocities are consistent with the formulæ for the addition of velocities in Special Relativity. [3]

[*You may assume without proof the formula* $\gamma_{\mathrm{AB}} = v_{\mathrm{A}} \cdot v_{\mathrm{B}}/c^2$, *where* $v_{\mathrm{A,B}}$ *are 4-velocities,* $\gamma_{\mathrm{AB}} = (1 - \beta_{\mathrm{AB}}^2)^{-1/2}$ *and* $c\beta_{\mathrm{AB}}$ *is the relative velocity of frames A and B.*]

END OF PAPER