UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

May–June 2020 **1 hour 15 minutes**

PHYSICS (2)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (2)

RELATIVITY

Candidates offering this paper should attempt a total of ***four*** *questions:* ***three*** *questions from Section A and* ***one*** *question from Section B.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains ***four*** *sides, including this coversheet. You may use the formula handbook for values of constants and mathematical formulae, which you may quote without proof.*

You have 75 minutes (plus any pre-agreed individual adjustment) to answer this paper. Do not start to read the questions on the subsequent pages of this question paper until the start of the time period.

Please treat this as a closed-book exam and write your answers within the time period. Downloading and uploading times should not be included in the allocated exam time. If you wish to print out the paper, do so in advance. You can pause your work on the exam in case of an external distraction, or delay uploading your work in case of technical problems.

Section A and the chosen section B question should be uploaded as separate pdfs. Please name the files 1234X_Qi.pdf, where 1234X is your examination code and i is the number of the question/section (A or 4 or 5).

STATIONERY REQUIREMENTS
Master coversheet

SPECIAL REQUIREMENTS
Mathematical Formulae handbook
Approved calculator allowed

SECTION A

*Attempt **all** questions in this Section. Answers should be concise and relevant formulae may be assumed without proof.*

1 An observer detects light from a source moving at speed v. In the frame of the source, there is an angle θ between the direction of motion and rays reaching the observer. What angle does the observer measure between the direction of motion and the light rays? [4]

In the frame of the source the path of the light ray is (taking motion to be along the x axis)

$$x = ct\cos\theta, \qquad y = ct\sin\theta \tag{1}$$

Performing a Lorentz transformation to the observer's frame moving with $v_x = v$ (observer moving *away* from source)

$$x' = \gamma(ct\cos\theta - vt) \qquad y' = ct\sin\theta \tag{2}$$

This means that the angle measured by the observer is (two equivalent forms)

$$\tan\theta' = \frac{c\sin\theta}{\gamma(c\cos\theta - v)} \tag{3}$$

$$\cos\theta' = \frac{c\cos\theta - v}{c - v\cos\theta}. \tag{4}$$

The angle is bigger when the source is moving away, smaller when the source approaches ('beaming').

Can also be done by relativistic velocity addition.

---

2 General relativity predicts that the centre of the Earth is younger than its surface. Assuming a uniform distribution of mass, estimate the size of this effect. The radius of the Earth is $R = 6371$ km and its age is $T = 4.54 \times 10^9$ years (ignore rotation of the Earth). [4]

$\big[$*In the weak-field limit,* $g_{00} \approx 1 + \frac{2\Phi}{c^2}$*, where* Φ *is the gravitational potential.*$\big]$

I'm not sure what's been done on gravitational dilation, but with the given formula we can easily relate the proper time τ to the coordinate time t

$$\tau = \sqrt{1 + \frac{2\Phi}{c^2}}t \approx \left(1 + \frac{\Phi}{c^2}\right)t,$$

showing that in a negative gravitational potential $\tau < t$: time runs slower in a gravitational field. The difference in the proper time in two regions is then

$$\Delta\tau \approx \frac{\Delta\Phi}{c^2}t.$$

To apply this to the Earth, one needs the standard form of the potential:

$$\Phi(r) = \begin{cases} -\frac{gR^2}{r} & r \geq R \\ \frac{gr^2}{2R} - \frac{3gR}{2} & r \leq R, \end{cases} \tag{5}$$

and we find

$$\Delta\tau = -\frac{gR}{2c^2}\tau.$$

Putting in the given data and using $g = 9.82\ \mathrm{ms}^{-2}$ gives $\Delta\tau \approx 1.58$ years. To 2 s.f.: 1.6 years.

---

3 The Schwarzschild metric is

$$\mathrm{d}s^2 = c^2\left(1 - \frac{2\mu}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2\mu}{r}\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2$$

where $\mu = GM/c^2$ and $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2$. Show that the geodesic equation for the radial coordinate implies that, for circular orbits, the square of the period is proportional to the cube of the radius.

The radial geodesic equation is

$$g'_{rr}(r)\dot{r}^2 - 2r\dot{\phi}^2 + g'_{tt}(r)\dot{t}^2 = 2g_{rr}(r)\dot{r}^2 + 2g_{rr}\ddot{r}.$$

For a circular orbit

$$-2r\dot{\phi}^2 + g'_{tt}(r)\dot{t}^2 = 0,$$

and $d\phi/dt = 2\pi/T$, where T is the period. Rearranging

$$\frac{4\pi^2}{T^2} = \frac{g'_{tt}(r)}{r} = \frac{2GM}{r^3},$$

which is Kepler III.

(TURN OVER

SECTION B
Attempt one question from this section

4 (a) State the symmetries of the Riemann curvature tensor R_{abcd} and show that there are six independent components in three dimensions. [6]

(b) In three dimensions,

$$R_{abcd} = 2\left(g_{b[c}R_{d]a} - g_{a[c}R_{d]b}\right) + Rg_{a[c}g_{d]b}\ ,$$

where $R_{ab} = R_{cab}{}^{c}$ is the Ricci tensor and $R = R_{a}{}^{a}$ is the scalar curvature. Show that this R_{abcd} satisfies the required symmetries, and how this fixes the form uniquely. [4]

(c) Find the metric of a conical surface with opening angle Θ in terms of the angular coordinate ϕ and distance from the tip r. Show that this metric coincides with that of the plane written in terms of polar coordinates (r, ϕ'), where ϕ' has period different from 2π. Find the period in terms of Θ. [3]

(d) Because $R_{abcd} = 0$ in free space in two space and one time dimensions, the most general static metric describing a point mass at the origin has the form

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \{\text{conical metric}\}$$

parameterised by Θ. Describe the orbits of a test particle. [3]

(e) The Einstein equations are

$$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R = -\kappa T_{ab}$$

for some constant κ. The static spherically symmetric metric

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - r_0^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)$$

has $R_{\theta\phi\theta\phi} = r_0^2\sin^2\theta$, with the remaining components not related by symmetry equal to zero. Find the form of T_{ab} that gives rise to this metric. [3]

(a) The argument is lengthy but it's in the notes. Symmetries are:

1. $R_{abc}{}^{d} = -R_{bac}{}^{d}$.
2. $R_{abc}{}^{d} + R_{bca}{}^{d} + R_{cab}{}^{d} = 0$
3. $R_{abcd} = -R_{abdc}$
4. $R_{abcd} = R_{cdab}$

There are six independent components. The argument goes:

1. Antisymmetry in the first two indices (property 1) gives 3 components for each assignment of the second two. Similarly for antisymmetry in the second two indices (property 3). This brings us to $3 \times 3 = 9$ components.

2. Think of this as a 3×3 matrix. Property 4 (symmetry between first two and second two indices) means this is a symmetric matrix, so only 6 components.
3. Finally, property 2 adds no further conditions. Since it is equivalent to $R_{[abc]}{}^d = 0$. $R_{[abc]}{}^d$ has only one component and one of the indices will be equal to d. But $R_{abcc} = 0$ by antisymmetry in the last two indices.

(b) After checking properties, the argument goes:

1. R_{ab} has only six components since $R_{ab} = R_{ba}$.
2. Since R_{abcd} and R_{ab} are linearly related it is possible to express R_{abcd} in terms of R_{ab} uniquely.
3. It remains to verify that $R_{cab}{}^c = R_{ab}$

$$R_{abc}{}^a = 2\left(g_{b[c}R_{d]a} - g_{a[c}R_{d]b}\right)g^{da} + Rg_{a[c}g_{d]b}g^{da} \tag{6}$$

$$= g_{bc}R - R_{bc} - R_{bc} + 3R_{bc} + \frac{R}{2}(g_{bc} - 3g_{bc}) \tag{7}$$

$$= R_{bc} \tag{8}$$

(c) From elementary geometry the metric is

$$ds^2 = dr^2 + r^2 \sin^2 \Theta d\phi^2$$

We can turn this into the usual planar metric

$$ds^2 = dr^2 + r^2 d\phi'^2$$

by choosing $\phi' = \sin\Theta\phi$. The period of ϕ' is $2\pi\sin\Theta$, so the angle deficit is $D = 2\pi(1 - \sin\Theta)$.

(d) The geodesics are straight lines in the (r, ϕ') plane described by the equation $r\cos\phi' = c$ (or rotations thereof). This corresponds to $r\cos[\sin\Theta)\phi] = c$. In particular $r \to \infty$ at $\phi = \pm\frac{\pi}{2\sin\Theta}$.

(e) Finding the Ricci tensor

$$R_{\theta\theta} = R_{\phi\theta\theta}{}^\phi = -R_{\theta\phi\theta}{}^\phi = 1 \tag{9}$$

$$R_{\phi\phi} = \sin^2\theta R_{\theta\theta} \tag{10}$$

The scalar curvature is $R = R_a{}^a = 2/r_0^2$. This gives

$$G_{ab} = -\delta_{a,0}\delta_{b,0}/r_0^2$$

so $T_{00} = 1/(\kappa r_0^2)$ is the only non-zero component, describing a universe of dust.

(TURN OVER

5 (a) By considering the geodesic equation for the tangent vector t^a in the form

$$\frac{\mathrm{d}t_a}{\mathrm{d}u} = \frac{1}{2}(\partial_a g_{bc})t^b t^c\,,$$

show that g_{ab} and $\tilde{g}_{ab} = g_{ab}/A$ have the same null geodesics for any function of spacetime $A(x)$. [4]

(b) Find the differential equations obeyed by geodesics of a metric of the form

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - n(\boldsymbol{x})^2\mathrm{d}\ell^2\,, \qquad (\star)$$

where $\mathrm{d}\ell^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$. [5]

(c) As a function of arc length ℓ, the curvature $k(\ell)$ of a planar curve $\boldsymbol{x}(\ell)$ is defined in terms of the unit tangent vector $\boldsymbol{T}(\ell) = \mathrm{d}\boldsymbol{x}/\mathrm{d}\ell$ and unit normal vector $\boldsymbol{N}(\ell)$ by

$$\frac{\mathrm{d}\boldsymbol{T}}{\mathrm{d}\ell} = k(\ell)\boldsymbol{N}(\ell)\,.$$

(Arc length and unit vectors are defined with respect to the Euclidean metric.) Show, for a light ray, that $k(\ell) = \boldsymbol{N}\cdot\nabla\log n$ for a metric of the form $\star$. [6]

$\left[\mathrm{d}s^2 = 0 \text{ so } \mathrm{d}\ell/\mathrm{d}t = c/n(\boldsymbol{x})\,.\right]$

There is a coordinate system in which the Schwarzschild metric takes the form

$$\mathrm{d}s^2 = c^2\left(\frac{1-\frac{\mu}{2r}}{1+\frac{\mu}{2r}}\right)^2\mathrm{d}t^2 - \left(1+\frac{\mu}{2r}\right)^4\mathrm{d}\ell^2$$

where $\mu = GM/c^2$. By the result of part (a), this has the same null geodesics as a metric of the form $(\star)$.

(d) The curvature of a curve can also be expressed as $k(\ell) = \mathrm{d}\phi/\mathrm{d}\ell$, where ϕ is the angle to a fixed axis. By integrating $k(\ell)$ along a straight light ray that passes at a distance R from a point mass M, obtain an approximate expression for the angle of deflection $\Delta\phi$ of the ray. Work to first order in M. [4]

(a) Consider the geodesic equation in the form

$$\frac{dt_a}{du} = \frac{1}{2}(\partial_a g_{bc})t^b t^c.$$

Note that $\partial_a\tilde{g}_{bc} = \left(\partial_a g_{bc} - g_{bc}\partial_a \log A\right)/A$. Since the geodesic is null by assumption $g_{bc}t^b t^c = 0$ and the equation is unchanged.

(b) Since the metric is time independent we have $\dot{t} = 0$, so $t = \text{constant} \times t$. Evaluating the remaining EL equations

$$\frac{\partial L}{\partial \boldsymbol{x}} = -2n\dot{\boldsymbol{x}}^2\nabla n \qquad (11)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{x}}} = -2n^2\ddot{\boldsymbol{x}} - 4n(\dot{\boldsymbol{x}}\cdot\nabla n)\dot{\boldsymbol{x}}. \qquad (12)$$

Putting it together

$$\ddot{\boldsymbol{x}} + 2\dot{\boldsymbol{x}}(\dot{\boldsymbol{x}} \cdot \nabla \log n) - \dot{\boldsymbol{x}}^2 \log n = 0.$$

(Alternatively one could use the geodesic equation and the formula given for the Christoffel symbols)

(c) Rewriting the definition of curvature given

$$\boldsymbol{x}'' = k(s)\boldsymbol{n}.$$

To see the relation to previous part, we use that t and the arc length ℓ are related by $t = n\ell/c$. This means

$$\dot{\boldsymbol{x}} = c\boldsymbol{x}'/n \tag{13}$$

$$\ddot{\boldsymbol{x}} = c^2\boldsymbol{x}''/n^2 - c\boldsymbol{x}'(\boldsymbol{x}' \cdot \nabla n)/n^2, \tag{14}$$

where ' denotes differentiation with respect to ℓ. The geodesic equation becomes

$$\boldsymbol{x}'' = -\boldsymbol{x}'(\boldsymbol{x}' \cdot \nabla) \log n + \nabla \log n = 0,$$

where we used $\boldsymbol{x}'^2 = 1$. Noticing that the RHS is normal to the curve we identify

$$k(s)\boldsymbol{n} = \nabla \log n - \boldsymbol{x}'(\boldsymbol{x}' \cdot \nabla) \log n,$$

which gives the stated result.

(d) The first step is to put the metric in the right form. We do this by dividing through by $A = \left(\frac{1-\frac{\mu}{2r}}{1+\frac{\mu}{2r}}\right)^2$ which leaves the null geodesics unchanged (part (a) of the question).

To lowest order in M we have

$$n = \frac{B}{A} \sim 1 + \frac{2\mu}{r},$$

so $\log n \sim \frac{2\mu}{r}$. For a light ray parallel to the x-axis we have

$$k = \partial_y \frac{2\mu}{\sqrt{x^2+y^2}} = -\frac{2\mu y}{(x^2+y^2)^{3/2}}.$$

So the total deflection is

$$\mathit{\Delta\Phi} = \int_{-\infty}^{\infty} k(x)dx = -\int_{-\infty}^{\infty} \frac{2\mu R}{(x^2+R^2)^{3/2}}dx = -\frac{4\mu}{R}.$$

I'm not going to be fussy about the sign as it depends on the convention for the angle.

END OF PAPER