UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Friday 28 May 2021 11.00 am to 13.00 pm

PHYSICS (4)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (4)

OPTICS AND ELECTRODYNAMICS

Candidates offering this paper should attempt a total of ***five*** *questions:* ***three*** *questions from Section A and* ***two*** *questions from Section B.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains ***five*** *sides, including this coversheet, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
2 × 20 Page Answer Book	Mathematical Formulae handbook
Metric graph paper	Approved calculator allowed
Rough workpad	
Yellow master coversheet	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

*Attempt **all** questions in this Section. Answers should be concise and relevant formulae may be assumed without proof.*

A1 An unpolarised plane electromagnetic wave, travelling in the $\hat{\mathbf{z}}$ direction of a Cartesian coordinate system, is incident on a diffuse collection of small, randomly positioned, neutral particles. Two different kinds of particles are present: N_1 of them can only be polarised in the $\hat{\mathbf{x}}$ direction, and N_2 of them can only be polarised in the $\hat{\mathbf{y}}$ direction; but both have the same polarisability α.

(a) A far-field observer is located in the $\hat{\mathbf{x}} - \hat{\mathbf{z}}$ plane, at angle ψ with respect to the $\hat{\mathbf{z}}$ axis, derive an expression for the degree of polarisation measured.

(b) If the observer is located in the $\hat{\mathbf{y}} - \hat{\mathbf{z}}$ plane, what is the degree of polarisation measured? Justify your reasoning. [4]

A2 An input transmission line having characteristic impedance Z_i carries microwave power P_i towards a matched antenna having a gain G_a. A distance d away an identical antenna, having effective area $A_e = \lambda^2 G_a / 4\pi$, acts as a receiver. It is connected to an output transmission line having characteristic impedance Z_t. Assuming that Z_i and Z_t are real valued, derive expressions for the following:

(a) The power travelling along the output transmission line, P_t. How does it vary with wavelength?

(b) The power reflected back along the input transmission line, P_r. How does it vary with wavelength? [4]

A3 The terminals of a parallel-plate capacitor are held at a constant potential difference, V, creating a uniform internal field. Field emission causes an electron to be released from one of the plates. Derive an expression for the power radiated during flight. Base your derivation on the non-relativistic Lienart-Wiechert field

$$\mathbf{E}(\mathbf{r}, t) = \frac{e}{4\pi\epsilon_0 c} \left[\frac{\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \dot{\boldsymbol{\beta}})}{R} \right]_{\text{ret}},$$

where the symbols have their usual meanings. [4]

SECTION B

Attempt two questions from this section

B4 A certain optical component is described by the Jones matrix J,

$$\mathsf{J} = \begin{pmatrix} a & ib \\ -ib & a \end{pmatrix},$$

where a and b are real constants.

(a) Find the eigenvalues and eigenvectors of J. [4]

(b) A linearly polarised field is incident on this component. Draw a vector diagram to illustrate how the eigenvalues and eigenvectors of the Jones matrix combine to determine the polarisation of the resultant field. [2]

(c) Write out the Jones matrix J entirely in terms of its eigenvalues. [2]

Suppose that a new Jones matrix $\mathsf{J}(z)$ describes the evolution of a field as it propagates through some continuous medium, and that the largest and smallest eigenvalues evolve with distance z according to $\lambda^+ = e^{-\gamma^+ z}$ and $\gamma^- = e^{-\gamma^- z}$ respectively, where $\gamma^\pm = \alpha^\pm - i\beta^\pm$ are complex propagation coefficients.

(d) show that the new Jones matrix can be written

$$\mathsf{J}(z) = e^{-\gamma z}\,\mathrm{sech}(\delta z)\begin{pmatrix} 1 & -i\tanh(\delta z) \\ i\tanh(\delta z) & 1 \end{pmatrix},$$

where $\gamma = (\gamma^+ + \gamma^-)/2$ and $\delta = (\gamma^+ - \gamma^-)/2$. [4]

(e) If both propagation constants are real, i.e. $\beta^+ = \beta^- = 0$, and a linearly polarised field is applied to the input, describe how the various terms in the equation for $\mathsf{J}(z)$ influence the evolution of the field with z. [2]

(f) If both propagation constants are imaginary, i.e. $\alpha^+ = \alpha^- = 0$, and a linearly polarised field is applied to the input, explain, using the imaginary forms of the hyperbolic functions listed below, how $\mathsf{J}(z)$ describes the way in which the field evolves with z. [2]

$\big[$ Hint: $\cosh(ix) = \cos(x)$ and $\sinh(ix) = i\sin(x)$ $\big]$.

(g) Show that the Jones matrix of this distributed system, having complex propagation constants $\gamma^\pm$, can be written

$$\mathsf{J}'(z) = e^{-\mathsf{K}z},$$

where the characteristics of the medium are contained in the matrix K. [3]

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B5 An optical source produces a quasi-monochromatic plane wave having the form

$$\mathbf{E}(\mathbf{r},t) = a(t - \mathbf{k}\cdot\mathbf{r}/\omega)\, e^{i\mathbf{k}\cdot\mathbf{r}} e^{-i\omega t}\hat{\mathbf{z}},$$

where $a(t - \mathbf{k}\cdot\mathbf{r}/\omega)$ is a complex-valued random variable.

(a) Use $\mathbf{E}(\mathbf{r},t)$ to describe what is meant by 'temporal coherence function', giving your answer in terms of τ. It is often said that certain temporal coherence functions do not depend on absolute time; what does this phrase mean? [3]

Suppose that the wave from the source, $\mathbf{E}(\mathbf{r},t)$, is divided into two parts, and then a system of mirrors causes the parts to cross at 90°, so that the field at crossover becomes

$$\mathbf{E}'(\mathbf{r},t) = \frac{1}{\sqrt{2}}\left[a(t - \mathbf{k}_1\cdot\mathbf{r}/\omega)\, e^{i\mathbf{k}_1\cdot\mathbf{r}} + a(t - \mathbf{k}_2\cdot\mathbf{r}/\omega)\, e^{i\mathbf{k}_2\cdot\mathbf{r}}\right] e^{-i\omega t}\hat{\mathbf{z}},$$

where

$$\mathbf{k}_1 = k\left(\frac{\hat{\mathbf{x}}+\hat{\mathbf{y}}}{\sqrt{2}}\right) \qquad \mathbf{k}_2 = k\left(\frac{\hat{\mathbf{x}}-\hat{\mathbf{y}}}{\sqrt{2}}\right).$$

(b) If the temporal coherence function does not depend on absolute time, show that the intensity of the light along the y axis has the form

$$I(y) = I_0 + \mathrm{Re}\left[\Gamma(\tau(y))\right],$$

where $\Gamma(\tau)$ is the temporal coherence function of the source field. Give an expression for $\tau(y)$. [4]

(c) If the source emits a lifetime-broadened spectral line, sketch the functional form of $I(y)$. [3]

(d) By considering the direction of the Poynting vector, comment on whether a camera could be used to record $I(y)$. [3]

(e) Write down an expression for the intensity of the light, $I(x)$, along the x axis. [2]

(f) Why would it be difficult for a camera to record $I(x)$? [2]

(g) One of the crossing fields is delayed by $\lambda/(4\sqrt{2})$ prior to superposition. Derive an expression for the new intensity along the y axis in terms of the temporal coherence function of the source. [2]

B6 (a) Assuming Minkowski space-time, state whether the following are invariant, covariant or contravariant under the Lorentz transform:

(i) the 4-momentum $P^\alpha \equiv \{E/c, p_x, p_y, p_z\}$,

(ii) the transformed 4-current density $\frac{\partial x^\beta}{\partial x'^\delta} g_{\beta\alpha} J^\alpha$, where $g_{\beta\alpha}$ is the metric tensor,

(iii) differentiation with respect to contravariant space-time $\frac{\partial}{\partial x^\alpha}$, and

(iv) $\partial_\alpha A^\alpha$. [4]

(b) The antisymmetric field-strength tensor is given by

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{bmatrix}.$$

$F^{\alpha\beta}$, when viewed from an inertial frame moving in the x-direction at constant velocity v, gives rise to a new field-strength tensor $F'^{\gamma\delta}$. The expression

$$F'^{\gamma\delta} = \frac{\partial x'^\gamma}{\partial x^\alpha} F^{\alpha\beta} \frac{\partial x'^\delta}{\partial x^\beta}$$

transforms a second-order contravariant tensor between two inertial frames. Use this expression to relate $\mathbf{E}'$ and $\mathbf{B}'$ to $\mathbf{E}$ and $\mathbf{B}$. [6]

(c) A plane wave has the following form

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \exp\left[\mathrm{i}(\mathbf{k} \cdot \mathbf{r} - \omega t)\right].$$

What is the form of the field in an inertial frame that is travelling in some general direction $\mathbf{v}$? You should use the notation $\mathbf{k}_\perp$ and $\mathbf{k}_{||}$ for vector components that are perpendicular and parallel to the direction of $\mathbf{v}$. [1]

(d) Derive expressions that relate the observed frequencies, wave vectors, and electric-field vectors in the moving frame to those in the rest frame. [3]

(e) Using these relationships, show that $K^\alpha \equiv \{\omega/c, k_x, k_y, k_z\}$ does indeed transform as a contravariant vector, and describe the nature of the contraction $R_\alpha K^\alpha$, where R^α is the position 4-vector. [3]

(f) An optical resonator comprises two plane mirrors, with an optical beam bouncing backwards and forwards between them. If the resonator is at resonance in its rest frame, will it be at resonance in all other inertial frames? Justify your reasoning. [2]

END OF PAPER