UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Tuesday 28 May 2013 9.00 am to 11.00 am

EXPERIMENTAL AND THEORETICAL PHYSICS (2)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (2)

Candidates offering this paper should attempt a total of **three** *questions. The questions to be attempted are* **1**, **2** *and* **one** *other question.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains **four** *sides, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
2×20 Page Answer Book	Mathematical Formulae handbook
Rough workpad	Approved calculator allowed
Yellow master coversheet	

RELATIVITY

1 *Attempt* **all** *parts of this question. Answers should be concise, and relevant formulae may be assumed without proof.*

(a) In Minkowski spacetime, two momentarily coincident observers travel towards a small and distant planet. To one observer the planet appears to have twice the angular diameter as to the other observer. What is the relative speed of the observers? [4]

(b) For a pressureless perfect fluid (e.g. dust), show that the components u^μ of its 4-velocity must satisfy $u^\mu \nabla_\mu u^\nu = 0$, where ∇_μ denotes a covariant derivative, and show that the worldline of each dust particle is a geodesic. [4]

(c) A spacetime has the line-element

$$ds^2 = c^2\,dt^2 - a^2(t)(dx^2 + dy^2 + dz^2).$$

By considering the effective Lagrangian for a free particle, or otherwise, show that the only non-zero connection coefficients are

$$\Gamma^0{}_{11} = \Gamma^0{}_{22} = \Gamma^0{}_{33} = \frac{aa'}{c^2} \qquad \text{and} \qquad \Gamma^1{}_{10} = \Gamma^2{}_{20} = \Gamma^3{}_{30} = \frac{a'}{a},$$

where a prime denotes d/dt. [4]

2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) the equivalence principle and local inertial coordinates;

(b) the use of accretion discs around compact objects as a test of general relativity;

(c) advanced Eddington–Finkelstein coordinates in the Schwarzschild spacetime.

3 *Attempt* **either** *this question* **or** *question 4.*

[*Note that throughout this question the choice of units is such that the speed of light* $c = 1$.]

The coordinates (t, x) define an inertial frame S in a $(1+1)$-dimensional Minkowski spacetime with line-element $ds^2 = dt^2 - dx^2$.

The lightcone coordinates (p, q) in S are defined by $p \equiv t - x$ and $q \equiv t + x$. Find the form of the line-element and the components of the metric in these new coordinates. [2]

Suppose that the coordinates (t', x') define the inertial frame S' moving along the x-axis of S at a constant speed v and that the axes of S and S' coincide at $t = t' = 0$. Show that the lightcone coordinates in S' are related to those in S by

$$p' = \alpha p, \qquad q' = \frac{q}{\alpha},$$

where α is a constant, and find an expression for α in terms of v. [4]

A spaceship is moving along the x-axis of S and an astronaut onboard measures the spaceship to have a constant acceleration a. Show that the lightcone coordinates of the spaceship must satisfy

$$\dot{p}\dot{q} = 1 \qquad \text{and} \qquad \ddot{p}\ddot{q} = -a^2,$$

where a dot denotes differentiation with respect to the astronaut's proper time τ. [4]

If the spaceship has coordinates $t = 0$ and $x = 1/a$ at $\tau = 0$, obtain expressions for the lightcone coordinates $p(\tau)$ and $q(\tau)$ along the spaceship's worldline. [6]

Hence show that the worldline in (t, x) coordinates is given by

$$t(\tau) = \frac{1}{a}\sinh a\tau, \qquad x(\tau) = \frac{1}{a}\cosh a\tau,$$

and sketch the form of the worldline on a spacetime diagram. What is the physical significance of the line $x = t$ for the astronaut? [3]

A photon of frequency ν_0 is emitted by a stationary star located at $x = L$, where $0 < L < 1/a$, and travels in the positive x-direction. If the photon is observed by the astronaut at proper time τ, obtain an expression for the measured frequency. What is the latest time t at which the photon could be emitted if it is to be observed by the astronaut? [3]

Rindler coordinates (ξ, ζ) are related to (t, x) coordinates by

$$t = \frac{1}{a}e^{a\zeta}\sinh a\xi, \qquad x = \frac{1}{a}e^{a\zeta}\cosh a\xi.$$

Find expressions for the Rindler coordinates $\xi(\tau)$ and $\zeta(\tau)$ along the spaceship's worldline, and hence interpret these coordinates physically. Sketch the Rindler coordinate curves on a spacetime diagram. [3]

4 *Attempt* **either** *this question* **or** *question 3.*

The Friedmann–Robertson–Walker (FRW) line-element for a spatially homogeneous and isotropic universe may be written as

$$ds^2 = c^2 dt^2 - R^2(t)\left[d\chi^2 + S^2(\chi)\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right],$$

where t is cosmic time, (χ, θ, ϕ) are comoving spherical polar coordinates, $R(t)$ is the scale factor, and $S(\chi) = \sin\chi, \chi$ or $\sinh\chi$ depending on whether the universe is spatially closed, flat or open, respectively.

A photon is emitted at cosmic time t_e by a comoving observer $\mathcal{E}$ with coordinates (χ, θ, ϕ) and is received at the current cosmic time t_0 by another comoving observer $\mathcal{O}$ at the origin $\chi = 0$. Show that the redshift z of the photon as measured by $\mathcal{O}$ satisfies

$$1 + z = \frac{R_0}{R(t_e)},$$

where $R_0 \equiv R(t_0)$, and interpret this result physically. [5]

Show that the χ-coordinate of $\mathcal{E}$ is given by

$$\chi(z) = \frac{c}{R_0}\int_0^z \frac{d\bar{z}}{H(\bar{z})},$$

where $H(z)$ is the Hubble parameter as a function of redshift, and obtain an analogous expression for the look-back time $t_0 - t_e$. [4]

The proper-motion distance to an object is defined by $d_M \equiv v/\dot{\theta}$, where v is the proper transverse velocity of (some part of) the object, which is assumed known from astrophysics, and $\dot{\theta}$ is the corresponding observed angular velocity. Show that the proper-motion distance of $\mathcal{E}$ from $\mathcal{O}$ is

$$d_M = R_0 S(\chi(z)).$$ [6]

Show that the present-day proper volume of the region of space lying in the infinitesimal redshift range $z \to z + dz$ and subtending an infinitesimal solid angle $d\Omega$ at the observer $\mathcal{O}$ is

$$dV_0 = \frac{cR_0^2 S^2(\chi(z))}{H(z)}\, dz\, d\Omega.$$ [4]

In a spatially-flat Einstein–de-Sitter universe, the Hubble parameter varies with redshift as $H(z) = H_0(1+z)^{3/2}$. In such a universe, galaxies of a certain type formed at redshift $z = z_f$ and thereafter have a constant comoving space density. If their present-day proper space density is n_0, show that the number of such galaxies per unit solid angle on the sky is

$$N = \frac{4c^3 n_0}{H_0^3}\int_0^{z_f} \frac{\left[1 - (1+z)^{-1/2}\right]^2}{(1+z)^{3/2}}\, dz.$$ [6]

END OF PAPER