UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Friday 31 May 2013 9.00 am to 11.00 am

EXPERIMENTAL AND THEORETICAL PHYSICS (5)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (5)

Candidates offering this paper should attempt a total of **three** *questions. The questions to be attempted are* **1**, **2** *and* **one** *other question.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains **four** *sides, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
2×20 Page Answer Book	Mathematical Formulae handbook
Rough workpad	Approved calculator allowed
Yellow master coversheet	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

ASTROPHYSICAL FLUID DYNAMICS

1 *Answer* **all** *parts of this question. Answers should be concise and relevant formulae may be assumed without proof.*

(a) An incompressible fluid of viscosity η and density ρ flows steadily under gravity down an inclined plane which makes an angle α to the horizontal. The depth of the flow perpendicular to the plane is h. The flow velocity $\boldsymbol{u}$ obeys the equation

$$\boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho}\nabla p + \boldsymbol{g} + \frac{\eta}{\rho}\nabla^2 \boldsymbol{u}$$

where $\boldsymbol{g}$ is the gravitational acceleration and p the pressure. Explain why this reduces to

$$0 = \boldsymbol{g} + \frac{\eta}{\rho}\nabla^2 \boldsymbol{u}$$

and hence find an expression for the maximum speed in the flow. [4]

(b) The Galactic disc in the direction perpendicular to the disc can be approximated as a static slab of gas of total thickness $2a$ and uniform density of gas ρ. Neglecting viscosity, how long does it take a star falling from rest from a height a above the mid-plane of the disc to pass through the mid-plane? [4]

(c) For a strong adiabatic shock the density ρ and speed u in the downstream gas (2) relative to their values in the upstream gas (1) are

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{\gamma + 1}{\gamma - 1},$$

where γ is the adiabatic index of the gas. Show that the downstream temperature of the gas in a strong shock is given by

$$\frac{k_{\mathrm{B}} T}{m} = \frac{2(\gamma - 1)}{(\gamma + 1)^2} u_1^2,$$

where k_{B} is the Boltzmann constant and m is the mass per particle. [4]

2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) Alfvén waves;

(b) viscous accretion discs;

(c) shocks with strong cooling.

3 *Attempt* **either** *this question* **or** *question 4.*

Euler's equation for the motion of a fluid in a gravitational potential ψ is

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho}\nabla p - \nabla\psi,$$

where ρ is the density, p the pressure and $\boldsymbol{u}$ the gas velocity. Show that the quantity

$$H = \frac{1}{2}u^2 + \int \frac{\mathrm{d}p}{\rho} + \psi$$

is constant along a streamline for steady flow. [4]

Consider steady flow of a gas in the z-direction in a tube of variable cross-section $A(z)$. Show that, when gravity can be neglected,

$$u^2 \nabla \ln u = -c_\mathrm{s}^2 \nabla \ln \rho,$$

where c_s^2 is the local sound speed in the gas, and hence

$$(u^2 - c_\mathrm{s}^2)\nabla \ln u = c_\mathrm{s}^2 \nabla \ln A. \qquad [4]$$

With the aid of a sketch, discuss the flow through a de Laval nozzle. [4]

A model for the formation of a collimated flow (jet) from an Active Galactic Nucleus (AGN) ignores gravity and assumes that the pressure distribution in the gas surrounding the AGN produces, in effect, a de Laval nozzle in the out-flowing gas from the AGN. If the flow rate of gas is $\dot{M}$ and the narrowest point of the flow has a cross-sectional area A_m, find an expression for the density at the sonic transition if the gas has an adiabatic equation of state $p = K\rho^\gamma$. [4]

Show that the speed of the gas satisfies

$$\frac{1}{2}u^2 + \frac{K\gamma}{\gamma - 1}\left(\frac{\dot{M}}{Au}\right)^{\gamma-1} = \frac{1}{2}\left(\frac{\gamma+1}{\gamma-1}\right)c_{\mathrm{s,m}}^2,$$

where $c_{\mathrm{s,m}}$ is the sound speed at the narrowest point of the flow. [4]

At large distances, r, from the AGN, A increases as r^2. Find the gas velocity at very large r and also how ρ depends on r. [5]

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4 *Attempt* **either** *this question* **or** *question 3.*

Hydrostatic equilibrium of a self-gravitating gaseous object is described by the equations

$$\nabla p = -\rho \nabla \psi; \qquad \nabla^2 \psi = 4\pi G \rho$$

where p is the pressure, ρ is the density and ψ is the gravitational potential. Explain why for a spherically-symmetric system $p = p(\psi)$ and $\rho = \rho(\psi)$ and hence we must have a barotropic equation of state. [3]

For a barotropic equation of state of the form $p = K\rho^{1+1/n}$, and assuming suitable boundary conditions, show that

$$\rho = \left(\frac{\psi_{\mathrm{T}} - \psi}{(n+1)K} \right)^n$$

and identify ψ_{T}. [4]

Hence show that Poisson's equation can be cast in dimensionless form using the dimensionless variables $\theta = (\rho/\rho_{\mathrm{c}})^{1/n}$ and $\xi = \alpha r$, where ρ_{c} is the central density, r the radial coordinate and $\alpha^2 = 4\pi G \rho_{\mathrm{c}}^{1-1/n}/[(n+1)K]$. [7]

Show that the mass of the object scales as

$$M \propto \rho_{\mathrm{c}}^{\frac{3}{2n} - \frac{1}{2}} K^{\frac{3}{2}},$$

and find a similar scaling relationship for the radius of the object R. [3]

Under what conditions is the observed scaling relationship for a star $M \propto R$ recovered? [2]

Clouds of cold monatomic hydrogen in the Galaxy radiate very inefficiently and are well-modelled as adiabatic systems. What mass–radius scaling relationship would these clouds be expected to follow and how does the mean density vary with mass? [4]

The most massive gas clouds are formed from molecular gas. Suggest why this is so. Assuming they radiate efficiently and can therefore be considered as isothermal systems, what mass–radius relationship is now expected? [2]

END OF PAPER