UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Saturday 31 May 2014 9.00 am to 11.00 am

PHYSICS (7)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (7)
QUANTUM CONDENSED MATTER PHYSICS - **ANSWERS**

Candidates offering this paper should attempt a total of **three** *questions. The questions to be attempted are* **1**, **2** *and* **one** *other question.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains **four** *sides, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
2×20 Page Answer Book	Mathematical Formulae handbook
Rough workpad	Approved calculator allowed
Yellow master coversheet	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

QUANTUM CONDENSED MATTER PHYSICS

1 *Attempt* **all** *parts of this question. Answers should be concise and relevant formulae may be assumed without proof.*

(a) Say $n = (3\text{Å})^{-3} \approx 3 \times 10^{28}$ m^{-3}.

Plugging in $\omega^2 \approx 10^{32}$ (rad/s)^2, and $\omega \approx 2\pi \times 10^{15}$ Hz.

(b) $\Psi_0 = \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle + |3\rangle)$

$\varepsilon_0 = E_0 + 2t$

(which agrees with what one gets from a more general tight-binding calculation)

(c) No prefactors are needed for the general scaling, so simply:

Total interactions energy $E_I \sim NU$

Total kinetic energy $E_K \sim NE_F \sim N(N/V)^{2/3}$

And from the competition between these two energies $U_c \sim E_F \sim (N/V)^{2/3}$

(They will get the same scaling if they remember and quote from the lectures a more quantitative derivation of the Stoner instability.)

2 All bookwork, marking scheme to be optimised.

3 ... Show that the energy dispersion in this material has two bands, which for $k_z = 0$ take the form

$$E_\pm = \frac{\hbar^2}{2m}\left(k^2 \pm q_R\,|\boldsymbol{k}|\right),$$

and express q_R in terms of λ, $\hbar$ and m. [5]

3(a)

$$\begin{vmatrix} \hbar^2k^2/2m - E & -\lambda(k_y+ik_x) \\ -\lambda(k_y-ik_x) & \hbar^2k^2/2m - E \end{vmatrix} = 0$$

$$\left(\frac{\hbar^2k^2}{2m}-E\right)^2 - \lambda^2\underbrace{(k_y^2+k_x^2)}_{=k^2 \text{ for } k_z=0} = 0$$

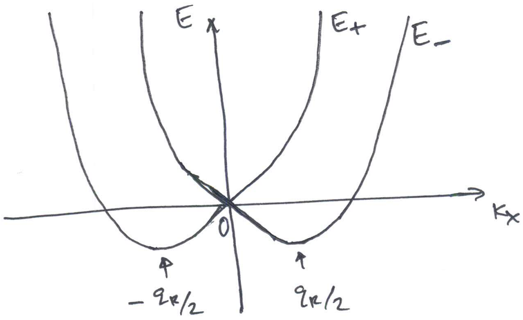
$$E^2 - 2E\frac{\hbar^2k^2}{2m} + \left(\frac{\hbar^2k^2}{2m}\right)^2 - \lambda^2k^2 = 0$$

$$E_\pm = \frac{\hbar^2k^2}{2m} \pm \frac{1}{2}\sqrt{\cancel{4\left(\frac{\hbar^2k^2}{2m}\right)^2} - \cancel{4\left(\frac{\hbar^2k^2}{2m}\right)^2} + 4\lambda^2k^2}$$

$$\therefore \boxed{E_\pm = \frac{\hbar^2k^2}{2m} \pm \lambda|\vec{k}| = \frac{\hbar^2}{2m}\left(k^2 \pm q_R|\vec{k}|\right) \quad \text{w/ } q_R = \frac{2m}{\hbar^2}\lambda}$$

Sketch the dispersion of the spin-split bands $E_+(\boldsymbol{k})$ and $E_-(\boldsymbol{k})$ along $\boldsymbol{k} = (k_x, 0, 0)$, indicating the minimal value of E_- and the value(s) of k_x for which it occurs. [4]

3(b)



$$\frac{\partial E_-}{\partial|k_x|} = 0 \Rightarrow k_x = \pm q_R/2$$

$$\min(E_-) = \frac{\hbar^2}{2m}\left(\frac{q_R^2}{4} - \frac{q_R^2}{2}\right) = -\frac{\hbar^2 q_R^2}{8m}$$

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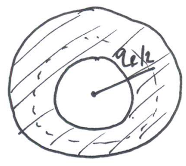
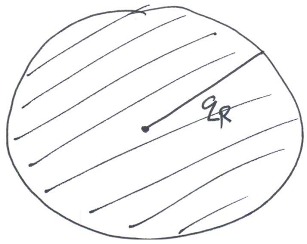
For $\boldsymbol{k}$ in the $k_z = 0$ plane, sketch the Fermi surfaces for three different values of the Fermi energy: (i) $-\hbar^2 q_R^2/(8m) < E_F < 0$, (ii) $E_F = 0$, and (iii) $E_F > 0$. In each case clearly indicate which $\boldsymbol{k}$ states are unoccupied, singly occupied, and doubly occupied. [6]

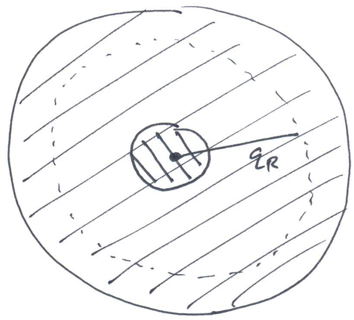
3(c) (i) $\min(E_-) < E_F < 0$:

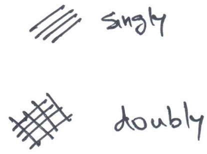


(ii) $E_F = 0$:



(iii) $E_F > 0$





Measurements of the Fermi surface cross-section in this material, for $k_z = 0$ and $E_F > 0$, reveal two Fermi wavevectors: $k_1 = 0.11\text{Å}^{-1}$ and $k_2 = 0.01\text{Å}^{-1}$. Find the value of q_R. [4]

3(d) $k_1 = 0.11\,\text{Å}^{-1}$ $\quad k_2 = 0.01\,\text{Å}^{-1}$

$$E_F = E_-(k_1) = E_+(k_2)$$

$$= \frac{\hbar^2}{2m}\left(k_1^2 - q_R k_1\right) = \frac{\hbar^2}{2m}\left(k_2^2 + q_R k_2\right)$$

$$\Rightarrow k_1^2 - q_R k_1 = k_2^2 + q_R k_2$$

$$q_R(k_1 + k_2) = k_1^2 - k_2^2 \quad \Rightarrow \boxed{q_R = k_1 - k_2 = 0.1\,\text{Å}^{-1}}$$

For $k_z = 0$, the spin eigenstates of H in the $E_-(\boldsymbol{k})$ band are

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} (k_y + ik_x)/\,|\boldsymbol{k}| \\ 1 \end{pmatrix}.$$

Deduce the spin eigenstates in the $E_+(\boldsymbol{k})$ band and sketch how the spin direction varies with the direction of $\boldsymbol{k}$ on the two Fermi surfaces, $|\boldsymbol{k}| = k_1$ and $|\boldsymbol{k}| = k_2$. [6]

[*You may use use the following eigenstates and eigenvalues of the relevant Pauli matrices:*

$$\sigma_x \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} = \pm \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} \quad , \quad \sigma_y \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \end{pmatrix} = \pm \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \end{pmatrix}$$

]

3(e) E_+ eigenstates must be orthogonal:

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} (k_y + ik_x)/|\vec{k}| \\ -1 \end{pmatrix}$$

Can evalute $\langle \sigma_x \rangle$ and $\langle \sigma_y \rangle$, but can also be done by inspection...

E.g.: for $k_x = 0$, $k_y > 0$: $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\rightarrow$

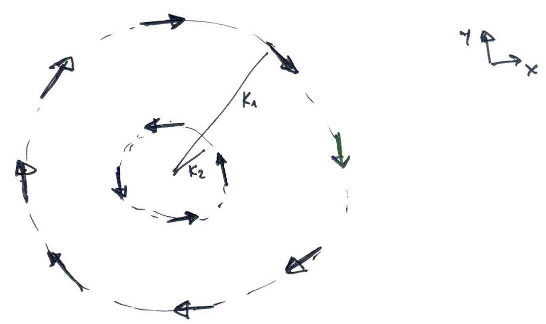
$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\leftarrow$

$k_x > 0$, $k_y = 0$: $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}_- = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}$ $\downarrow$

$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix}$ $\uparrow$

etc

$\therefore$



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4 *Attempt* **either** *this question* **or** *question 3.*

Show that the intrinsic carrier concentration in a non-degenerate semiconductor is given by

$$n_i^2 = np = \frac{1}{2}(m_e m_h)^{3/2}\left(\frac{k_B T}{\pi\hbar^2}\right)^3 \exp\left(-E_g/k_B T\right)$$

where m_e and m_h are the electron and hole effective masses, E_g is the band gap energy, and n and p are the electron and hole concentrations. [8]

4(a)

$$n = \underbrace{2}_{\text{spin}} \frac{1}{(2\pi)^3} \int 4\pi k^2 \, dk \; e^{-\left(\frac{\hbar^2 k^2}{2m_e} + E_c - \mu\right)/kT}$$

$$= \frac{1}{\pi^2} e^{-(E_c-\mu)/kT} \int k^2 \, dk \; e^{-\frac{\hbar^2 k^2}{2m_e kT}}$$

$$= \frac{1}{\pi^2} e^{-(E_c-\mu)/kT} \left(\frac{2m_e kT}{\hbar^2}\right)^{3/2} \underbrace{\int x^2 e^{-x^2} dx}_{\sqrt{\pi}/4}$$

$$= \frac{1}{\sqrt{2}} \left(\frac{m_e kT}{\pi\hbar^2}\right)^{3/2} e^{-(E_c-\mu)/kT}$$

OK to deduce/quote:

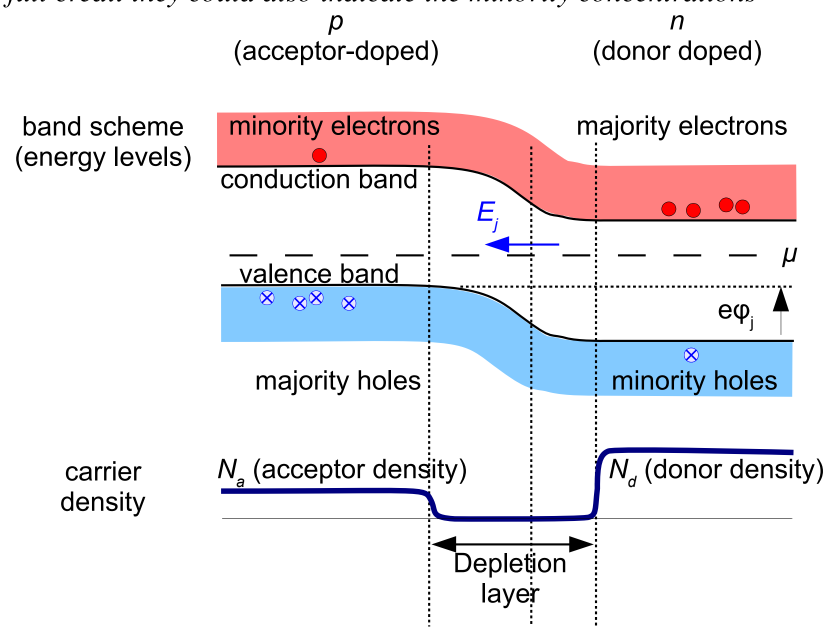
$$p = \frac{1}{\sqrt{2}} \left(\frac{m_h kT}{\pi\hbar^2}\right)^{3/2} e^{(E_v-\mu)/kT}$$

& $E_c - E_v = E_g$

$$\therefore \; n_i^2 = np = \frac{1}{2}(m_e m_h)^{3/2}\left(\frac{kT}{\pi\hbar^2}\right)^3 e^{-E_g/kT}$$ (bookwork)

Make an annotated sketch of the bending of the conduction and valence bands, and of the electron and hole concentrations, across an unbiased $p-n$ junction. [5]

4 (b) This picture is from the course notes... however, given what they just derived above, for full credit they could also indicate the minority concentrations



The current in a reverse-biased $p-n$ junction diode with an applied voltage V follows the diode equation:

$$I = I_{sat}\left[1 - \exp\left(-eV/k_BT\right)\right] ,$$

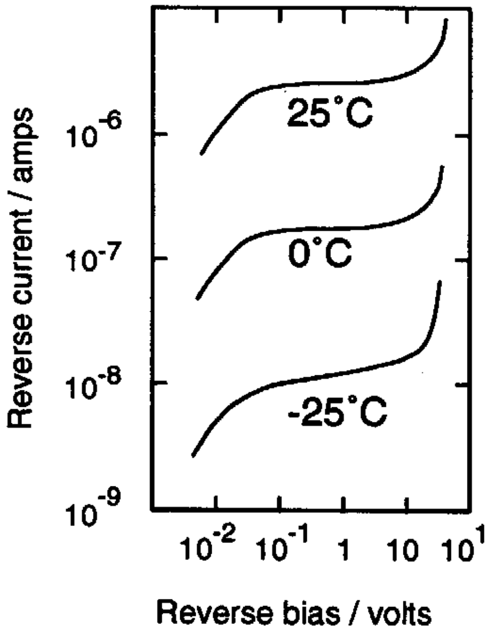
where the saturation current I_{sat} is proportional to n_i^2. Identify the origins of the two contributions to I. [4]

(c) $I = I_{sat}\left[1 - e^{-eV/kT}\right]$

generation / drift current doesn't depend on V (minority carriers swept by the in-built field in the depletion region)

recombination / diffusion current (majority carriers climbing the potential across the junction)

The diagram below shows measurements of the current across a germanium $p - n$ diode under reverse bias. Explain the form of the curves and their temperature dependence. [4]



4(d)

- For all T, $I \to 0$ for $V \to 0$
- The plateau $I \approx I_{sat}$ always reached for $eV \gtrsim kT$ (300K $\approx$ 25 meV)
- Plateau $I \approx I_{sat} \propto n_i^2 \propto e^{-\mathcal{E}_g/kT}$
- See breakdown (of the diode & mostly) for $V \gtrsim 10V$

Deduce the value of the band gap in germanium. [4]

4(e) $I_{sat} \propto e^{-E_g/k_BT}$

Take $T_1 = 300K$ $T_2 = 250K$

$$\frac{I_{sat}(T_1)}{I_{sat}(T_2)} \approx 200 \quad \text{(from graph)}$$

$$-\frac{E_g}{k_BT_1} + \frac{E_g}{k_BT_2} = \ln 200$$

$$E_g \frac{T_1 - T_2}{k_B T_1 T_2} = \ln 200$$

$$E_g = \frac{k_B T_1 T_2}{T_1 - T_2} \ln 200 = k_B \times 1500K \cdot \ln 200$$

$$\approx k_B \times 8000K$$

$$\approx 1eV$$

END OF PAPER