NATURAL SCIENCES TRIPOS Part II

May–June 2020 **1 hour 15 minutes**

PHYSICS (1)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (1)

THERMAL AND STATISTICAL PHYSICS

Candidates offering this paper should attempt a total of ***four*** *questions:* ***three*** *questions from Section A and* ***one*** *question from Section B.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains ***four*** *sides, including this coversheet. You may use the formula handbook for values of constants and mathematical formulae, which you may quote without proof.*

You have 75 minutes (plus any pre-agreed individual adjustment) to answer this paper. Do not start to read the questions on the subsequent pages of this question paper until the start of the time period.

Please treat this as a closed-book exam and write your answers within the time period. Downloading and uploading times should not be included in the allocated exam time. If you wish to print out the paper, do so in advance. You can pause your work on the exam in case of an external distraction, or delay uploading your work in case of technical problems.

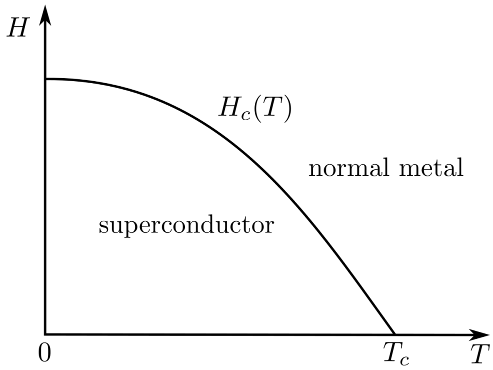
Section A and the chosen section B question should be uploaded as separate pdfs. Please name the files 1234X_Qi.pdf, where 1234X is your examination code and i is the number of the question/section (A or 4 or 5).

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Master coversheet	Mathematical Formulae handbook
	Approved calculator allowed

SECTION A

*Attempt **all** questions in this Section. Answers should be concise and relevant formulae may be assumed without proof.*

1 Type-I superconductors display the phase diagram shown in the figure below as a function of temperature T and magnetic field H (taken along the z-direction).



To a good approximation, the magnetic moment m for a system of volume V satisfies

$$m = \begin{cases} 0 & \text{normal metal,} \\ -VH & \text{superconductor.} \end{cases}$$

The particle number is constant for these systems, hence the change in the Gibbs free energy is $\mathrm{d}G = -S\mathrm{d}T - \mu_0 m \mathrm{d}H$, where S is the entropy and μ_0 is the vacuum permeability. Show that the phase boundary $H_\mathrm{c}(T)$ satisfies

$$L = -\mu_0 V T H_\mathrm{c} \frac{\mathrm{d}H_\mathrm{c}}{\mathrm{d}T},$$

where L is the latent heat of the superconductor–normal-metal transition. [4]

2 Consider N independent, distinguishable, two-level systems, each with energies $\pm\varepsilon_0$ where $\varepsilon_0 > 0$. Write down the number $\Omega(E)$ of microstates with energy $E = (N_+ - N_-)\varepsilon_0$ and the corresponding Boltzmann entropy $S(E)$. (Here $N_\pm$ is the number of systems with energy $\pm\varepsilon_0$.) Show that the temperature, as defined via $S(E)$, is positive for $E < 0$ but is negative for $E > 0$. [4]

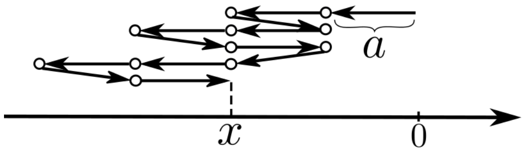
3 In certain three-dimensional systems, electrons obey a linear dispersion relation $\varepsilon_{\boldsymbol{k}} = \hbar v|\boldsymbol{k}|$ with momentum $\boldsymbol{k}$ and velocity v. By comparing the grand potential and the internal energy, or otherwise, show that the pressure p, volume V and the internal energy U are related as $pV = U/3$ for this electronic system. [4]

SECTION B

Attempt one question from this section

4 (a) Consider a system where particles may occupy a set of energy levels $\{\varepsilon_j\}$ with corresponding occupation numbers $\{n_j\}$. Write down the canonical partition function $Z(T, V, N)$ (with temperature T, volume V, and particle number N) and use it to express the grand partition function $\Xi(T, V, \mu)$ with chemical potential μ. Show that $\Xi(T, V, \mu) = \prod_j \Xi_j$ where the product is over energy levels. Hence comment on simplifications in using Ξ instead of Z. [5]

Consider a chain of N elements, each of which can point either to the left or to the right. Each of the elements has length a. The joints of the chain turn freely, but the elements tend to point to the left: pointing the j-th element to the right costs energy $\varepsilon_j > 0$. The figure shows an example of a chain with $N = 12$ elements and endpoint displacement $x = -2a$.



[*All the physics happens in one dimension (horizontally); the vertical separation of the elements only aids visualisation and is to be neglected.*]

(b) Introducing the variable $n_j = 1$ ($n_j = 0$) if the j-th element points to the right (left), write down the partition function $Z(T, x)$ for the chain in equilibrium at temperature T and with endpoint displacement x. (You may leave the sums over n_j unevaluated.) [3]

(c) Now consider the partition function $Y(T, f) = \sum_x Z(T, x)\mathrm{e}^{\beta f x}$ where $\beta = (k_\mathrm{B}T)^{-1}$. Show that

$$Y(T, f) = 2^N \prod_{j=1}^{N} \mathrm{e}^{-\beta\varepsilon_j/2} \cosh\left[\beta\left(\frac{\varepsilon_j}{2} - fa\right)\right].$$

[3]

(d) What is the physical meaning of the parameter f? [2]

(e) Show that the mean displacement $\langle x\rangle$ and the variance $\langle x^2\rangle - \langle x\rangle^2$ are related by

$$\langle x^2\rangle - \langle x\rangle^2 = k_\mathrm{B}T\frac{\partial\langle x\rangle}{\partial f}.$$

[3]

(f) Calculate $\langle x\rangle$ for $f \neq 0$ and discuss the case when $|f|a \ll \varepsilon_j$. [3]

(TURN OVER

5 (a) Describe the Debye model of lattice vibrations. Express its key parameters in terms of the speed of transverse and longitudinal waves as well as the number N of atoms and the volume V of the crystal. [5]

The modes of lattice vibrations are harmonic oscillators of frequency $\omega_{\boldsymbol{k}}$. As such, they have zero-point vibrations responsible for the second term in the energy $\varepsilon_{\boldsymbol{k}} = \hbar\omega_{\boldsymbol{k}}(n_{\boldsymbol{k}} + 1/2)$, where $n_{\boldsymbol{k}}$ is the number of phonons (i.e., oscillator quanta) of momentum $\boldsymbol{k}$ and frequency $\omega_{\boldsymbol{k}}$. The rest of the question discusses the effects of these zero-point vibrations.

(b) Consider first a single harmonic oscillator of frequency ω in equilibrium at temperature T. Show that the average number of oscillator quanta satisfies

$$\langle n \rangle = \frac{1}{\exp(\beta\hbar\omega) - 1}, \text{ where } \beta = \frac{1}{k_{\mathrm{B}}T}.$$ [3]

(c) How do zero-point vibrations contribute to the internal energy and the specific heat? [2]

(d) Consider again the harmonic oscillator in part (b). Using the fact that the expectation values of kinetic and potential energies are equal, show that the mean square displacement satisfies

$$\langle x^2 \rangle = \frac{\hbar}{m\omega}\left[\frac{1}{\exp(\beta\hbar\omega) - 1} + \frac{1}{2}\right],$$

where m is the mass. Identify the contribution due to zero-point vibrations. [3]

(e) The mean square displacement of an atom in a vibrating crystal (of N atoms, each of mass m) satisfies the analogous relation

$$\langle \boldsymbol{x}^2 \rangle = \frac{\hbar}{Nm}\int_{0^+}^{\omega_{\mathrm{D}}} \mathrm{d}\omega\, g(\omega)\,\frac{1}{\omega}\left[\frac{1}{\exp(\beta\hbar\omega) - 1} + \frac{1}{2}\right],$$

where ω_{D} is the Debye frequency and 0^+ is a positive infinitesimal introduced to exclude homogeneous translations of the crystal. Working with the d-dimensional Debye model, find the $T = 0$ and $T > 0$ values of d above which $\langle \boldsymbol{x}^2 \rangle$ is finite. [4]

(f) What do your results in (e) imply about the existence of crystalline order in the ground state and at non-zero temperatures? [2]

END OF PAPER