AFD 2010

1)a) Supernova injects $10^{44}$ J into interstellar medium of density $10^6$ hydrogen atoms $m^{-3}$
Assuming initial mass of explosion can be ignored, estimate radius of blast wave ~~one~~ 1000 yr later

Dimensional analysis $R = R(t, E, \rho)$

$[m] = [kg\, m^2 s^{-2}]^a [s]^b [kg\, m^{-3}]^c$

$a + c = 0$, $2a - 3c = 1$, $b - 2a = 0$

$a = 1/5$, $b = 2/5$, $c = -1/5$

$R \propto E^{1/5} t^{2/5} \rho^{-1/5}$

b) uniform region of $1.5M$, radius $R$
find value of $R$ for which sound speed crossing time = collapse time

$t_s = \frac{2R}{c_s}$, $c_s^2 = \frac{\gamma p}{\rho}$ $\Rightarrow$ $t_s = 2R\sqrt{\frac{\rho}{\gamma p}}$

Jeans length $\lambda_J = \sqrt{\frac{\pi c_s^2}{G\rho_0}}$ $\Rightarrow$ collapse time $t_c = \sqrt{\frac{\pi}{G\rho_0}}$

$$2R\sqrt{\frac{\rho_0}{\gamma p}} = \sqrt{\frac{\pi}{G\rho_0}}$$

$$R = \frac{1}{2}\sqrt{\frac{\pi\gamma p}{G\rho_0^2}}$$

c) $-\omega^2 u_1 + (c_s^2 + v_A^2)(k\cdot u_1)k + (v_A\cdot k)[(v_A\cdot k)u_1 - (v_A\cdot u_1)k - (k\cdot u_1)v_A] = 0$
- dispersion relation for a magnetised ionised medium

$k \parallel v_A$: $k \perp u_1$ gives $-\omega^2 u_1 + v_A^2 k^2 u_1 = 0$

$v^2 = v_A^2$

- Alfven waves

$k \parallel u_1$ gives $-\omega^2 u_1 + c_s^2 k^2 u_1 = 0$

$v^2 = c_s^2$

- sound waves

$K \perp V_A$ : $V_A \cdot K = 0 \Rightarrow -W^2 u_1 + (c_s^2 + V_A^2)(K \cdot u_1)K = 0$

magnetosonic waves (longitudinal)

3) star in hydrostatic equilibrium
show that density $\rho(r)$ satisfies
$dP = -\rho d\phi$ , $\phi$ = gravitational potential

hydrostatic equilibrium has $u, \partial/\partial t = 0$
momentum equation

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\frac{1}{\rho}\nabla p + g \qquad (g = -\nabla\phi)$$

$$\frac{1}{\rho}\nabla p = -\nabla\phi$$

assume barotropic EOS $P = P(\rho)$ only

$$\nabla P = \frac{\partial P}{\partial \rho}\nabla\rho$$

$$\frac{1}{\rho}\frac{dP}{d\rho}\nabla\rho = -\nabla\phi$$

~~sy~~ spherically symmetric $\Rightarrow \nabla = d/dr$

integrate :

$$\frac{1}{\rho}\frac{dP}{d\rho}\rho = -\phi$$

~~d P/d r~~ $\frac{1}{\rho}\frac{dP}{dr} = -\frac{d\phi}{dr}$

integrate : $dP = -\rho d\phi$

spherically symmetric polytrope , $P = K\rho^{1+1/n}$
show how this leads to the Lane-Emden equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^n = 0$$

barotropic $P = P(\rho)$, polytropic $P = K\rho^{1+1/n}$

$dP = K(1+1/n)\rho^{1/n} d\rho = -\rho d\phi$

$K(1+1/n)\rho^{1/n-1} d\rho = -d\phi$

$$\phi = -\frac{K(n+1)}{n}\int \rho^{1/n-1} d\rho = -K(n+1)\rho^{1/n} + c$$

$\phi - \phi_T = -K(n+1)\rho^{1/n}$ ($\phi_T$ at surface)

in centre of star $\rho = \rho_c$, $\phi = \phi_c$

$\phi_c - \phi_T = -K(n+1)\rho_c^{1/n}$

$$\frac{\rho}{\rho_c} = \left(\frac{\phi - \phi_T}{\phi_c - \phi_T}\right)^n = \theta^n$$

$\nabla^2 \phi = 4\pi G\rho$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) = 4\pi G\rho \text{ in spherical polars}$$

$$= 4\pi G\rho_c\theta^n$$

$$\theta^n = -\frac{1}{4\pi G\rho_c r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \phi}{\partial r}\right)$$

$$\theta = \frac{\phi - \phi_T}{\phi_c - \phi_T} \Rightarrow \nabla^2\theta = \frac{\nabla^2\phi}{\phi_c - \phi_T} = \frac{4\pi G\rho_c\theta^n}{\phi_c - \phi_T}$$

$$\xi = \sqrt{\frac{4\pi G\rho_c}{-(\phi_c - \phi_T)}}\, r = \alpha r$$

$$\nabla^2\theta = -\frac{\xi^2}{r^2}\theta^n$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = -\frac{\xi^2}{r^2}\theta^n$$

$$\frac{\partial}{\partial r} = \alpha\frac{\partial}{\partial \xi} \Rightarrow \alpha\frac{\partial}{\partial\xi}\left(\frac{\xi^2}{\alpha^2}\cdot\alpha\frac{\partial}{\partial\xi}(\theta)\right) = -\xi^2\theta^n$$

$$\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial\theta}{\partial\xi}\right) + \theta^n = 0$$

$$\alpha^2 = \frac{4\pi G\rho_c}{\phi_T - \phi_c}, \quad \phi_T - \phi_c = K(n+1)\rho_c^{1/n}$$

$$\alpha^2 = \frac{4\pi G\rho_c^{1-1/n}}{K(n+1)}$$

$P_c = K\rho_c^{1+1/n}$

$\alpha^2 = \dfrac{4\pi G\rho_c^2}{(n+1)P_c}$

for $n=5$ show that $\dfrac{\rho}{\rho_c} = \left(1+\dfrac{\xi^2}{3}\right)^{-5/2}$ is a solution

$\rho/\rho_c = \theta^n$

show that $\theta = (1+\xi^2/3)^{-1/2}$ is a solution

by $\theta = (1+B\xi^2)^{-1/2}$ .

$\dfrac{1}{\xi^2}\dfrac{\partial}{\partial\xi}\left(\xi^2\dfrac{\partial}{\partial\xi}(1+B\xi^2)^{-1/2}\right)$

$= \dfrac{1}{\xi^2}\dfrac{\partial}{\partial\xi}\left(-B\xi^3(1+B\xi^2)^{-3/2}\right)$

$= -\dfrac{1}{\xi^2}\left(-3B\xi^2(1+B\xi^2)^{-3/2} + 3B\xi^4(1+B\xi^2)^{-5/2}\right)$

$= \left[3B(1+\cancel{B\xi^2}) - 3\cancel{B\xi^2}\right](1+B\xi^2)^{-5/2}$

$= 3B\theta^5$

Lane-Emden $\nabla_\xi^2\theta = \theta^5$

$\hookrightarrow 3B\theta^5 = \theta^5$

$B = 1/3$

$\theta = (1+\xi^2/3)^{-1/2} \Rightarrow \rho/\rho_c = (1+\xi^2/3)^{-5/2}$

show that total mass of star is given by

$M = \dfrac{18}{\sqrt{2\pi}}\left(\dfrac{P_c}{G}\right)^{3/2}\rho_c^{-2}$

$M = \int 4\pi r^2\rho\, dr$, with $r = \xi/\alpha$, $\rho = \rho_c(1+\xi^2/3)^{-5/2}$

$dr = 1/\alpha\, d\xi$

$M = 4\pi\int_0^\infty \dfrac{1}{\alpha^2}\xi^2\rho_c(1+\xi^2/3)^{-5/2}\dfrac{1}{\alpha}d\xi$

$= \dfrac{4\pi\rho_c}{\alpha^3}\int_0^\infty \xi^2(1+\xi^2/3)^{-5/2}\, d\xi$

by parts, change variables with $x = 1/\sqrt{3}\,\xi$, $d\xi = \sqrt{3}\,dx$

$$M = \frac{4\pi\rho_c}{\alpha^3}\int_0^\infty 3x^2(1+x^2)^{-5/2}\sqrt{3}\,dx$$

use result $\int_0^\infty x^2(1+x^2)^{-5/2}\,dx = 1/3$

$$M = \frac{4\pi\rho_c}{\alpha^3}\,3\sqrt{3}\,\frac{1}{3} = \frac{4\sqrt{3}\pi\rho_c}{\alpha^3}$$

$$\alpha^3 = \left(\frac{4\pi G\rho_c^2}{(n+1)P_c}\right)^{3/2}$$

$$M = 4\sqrt{3}\pi\rho_c\left(\frac{(n+1)P_c}{4\pi G\rho_c^2}\right)^{3/2} = 4\sqrt{3}\pi\rho_c\left(\frac{3P_c}{2\pi G\rho_c^2}\right)^{3/2} \quad (n=5)$$

$$M = \frac{18}{\sqrt{2\pi}}\,\frac{1}{\rho_c^2}\left(\frac{P_c}{G}\right)^{3/2}$$

let $R$ be the radius at which $\rho/\rho_c = 10^{-3}$
assuming stellar material satisfies perfect gas law, show that central temp $T_c \propto \frac{GM}{R}$

$$P = \frac{R_* \rho T}{\mu}$$

at $\xi = \alpha R$, $\rho/\rho_c = 10^{-3} = (1 + \alpha^2R^2/3)^{-5/2}$

$$M \propto \rho_c^{-2}(P_c/G)^{3/2}$$

$$R \propto 1/\alpha \propto \left(\frac{P_c}{G}\right)^{1/2}\frac{1}{\rho_c}$$

$$\frac{GM}{R} \propto G\rho_c^{-2}\left(\frac{P_c}{G}\right)^{3/2}\rho_c\left(\frac{G}{P_c}\right)^{1/2} \propto \frac{P_c}{\rho_c}$$

$$\frac{P_c}{\rho_c} = \frac{R_* T_c}{\mu}$$

$$T_c \propto \frac{GM}{R}$$

4) Star produces spherically symmetric wind
write down continuity equation and Bernoulli's equation
Mass conservation gives $\dot{M} = \rho u A = \text{const.}$
$\dot{M} = 4\pi r^2 \rho(r) v(r) = \text{const.}$
in steady state, mass is not accumulating at any radius
so mass flux through all radii is the same

Bernoulli : $\frac{1}{2}v^2 + \int \frac{dP}{\rho} + \Psi = \text{const.}$

$$\int \frac{dP}{\rho} = \int \frac{d(K\rho^\gamma)}{\rho} = \gamma K \int \rho^{\gamma-2} d\rho = \frac{K\gamma\rho^{\gamma-1}}{\gamma-1}$$

$$\int \frac{dP}{\rho} = \frac{\gamma P}{\rho(1+\gamma)}$$

$$\Psi = -\frac{GM}{r}$$

$$\frac{1}{2}v^2 + \frac{\gamma P}{\rho(1+\gamma)} - \frac{GM}{r} = \text{const.}$$

Bernoulli gives conservation of energy:
$H = KE + PE + \text{enthalpy} = \text{const.}$ (p u mass)
$KE = \frac{1}{2}v^2$, $PE = \Psi$, enthalpy $= \int \frac{dP}{\rho}$

wind satisfies $P = K\rho^{3/2}$
Show that Mach number $M$ and adiabatic sound speed $c$
are related by
$c^5 M \xi^2 = c_0 K^2 \gamma^2 = A$
$\frac{1}{2}c^2 M^2 + 2c^2 - c_0^2 \xi^{-1} = E$

mass conservation:
$4\pi r^2 \rho(r) c M = \dot{M}$
$\dot{M} = 4\pi \xi^2 R^2 \rho M c$
$c^2 = \frac{\gamma P}{\rho} = K\gamma\rho^{1/2} \Rightarrow \rho \propto c^4$

$\dot{M} \propto \xi^2 M c^5 \Rightarrow c^5 M \xi^2 = A$

Bernoulli $\frac{1}{2}v^2 + \frac{\gamma}{\gamma-1}\frac{p}{\rho} - \frac{GM}{r} = \text{const}$

$$\frac{1}{2}M^2c^2 + \frac{\gamma}{\gamma-1}K\rho^{1/2} - \frac{GM}{\xi R} = \text{const}$$

$$\frac{\gamma p}{\rho} = c^2, \quad \gamma = 3/2 \quad \Rightarrow \quad \gamma/\gamma-1 \; p/\rho = 2c^2$$

$$c_0^2 = GM/R$$

$$\frac{1}{2}c^2M^2 + 2c^2 - c_0^2\xi^{-1} = \text{const.} = E$$

eliminate $c$: $c^2\left(\frac{1}{2}M^2 + 2\right) = E + c_0^2\xi^{-1}$

$$c^5 = A/M\xi^2$$

$$c^2 = \frac{2(E + c_0^2\xi^{-1})}{M^2+4} = \left(\frac{A}{M\xi^2}\right)^{2/5}$$

$$2(E + c_0^2\xi^{-1})\frac{\xi^{4/5}}{A^{2/5}} = \frac{1}{(M^2)^{1/5}}(M^2+4) = f(M^2)$$

$$f(M^2) = \frac{2}{A^{2/5}}\frac{1}{\xi^{1/5}}(E\xi + c_0^2) = \frac{1}{2A^{2/5}}\frac{1}{\xi^{1/5}}\left(4 + \frac{4E}{c_0^2}\xi\right)c_0^2$$

$$\xi_0 = c_0^2/4E$$

$$f(M^2) = \frac{1}{2A^{2/5}}c_0^2 f(\xi/\xi_0)\left(\frac{4E}{c_0^2}\right)^{1/5} = Bf(\xi/\xi_0)$$

$$B = \frac{c_0^2}{2A^{2/5}}\left(\frac{4E}{c_0^2}\right)^{1/5} = (c_0^2)^{4/5}\, 2^{-3/5}\, A^{-2/5} E^{1/5}$$

$$B = \left(\frac{c_0^8 E}{8A^2}\right)^{1/5}$$

$f(M^2) = B\mathcal{E}^{-1/5}\mathcal{E}_0^{1/5}(\mathcal{E}/\mathcal{E}_0 + 4) = Bf(\mathcal{E}/\mathcal{E}_0)$

considering shape of curve $f(\mathcal{E})$, show that wind can only make a transition from subsonic to supersonic flow if $B=1$, $M=1$ at $\mathcal{E}=\mathcal{E}_0$

$f(x) = \frac{x+4}{x^{1/5}}$, $\frac{\partial f}{\partial x} = \frac{1}{x^{1/5}} - \frac{1}{5}\frac{(x+4)}{x^{6/5}}$

min at $x = \frac{1}{5}(x+4) \Rightarrow x = 1$

at $x \to 0$, $f(x) \to \infty$

at large $x$, $f(x) \to x^{4/5}$

$f(\mathcal{E}/\mathcal{E}_0)$ has min at $\mathcal{E}=\mathcal{E}_0$, $f(\mathcal{E}/\mathcal{E}_0) = 5$

$f(M^2) = Bf(\mathcal{E}/\mathcal{E}_0)$

$f(M^2)$ has min at $M^2=1$, $f(M^2) = 5$

at minimum, $5 = B \cdot 5 \Rightarrow$ must have $B=1$ at minimum

so at $\mathcal{E}=\mathcal{E}_0$, $B = M = 1$

$(\mathcal{E}/\mathcal{E}_0)^{-1/5}(\mathcal{E}/\mathcal{E}_0 + 4) = B(M^2+4)M^{-2/5}$

as $\mathcal{E}/\mathcal{E}_0 \to 0$, $BM^{-2/5}(M^2+4) \to 4(\mathcal{E}/\mathcal{E}_0)^{-1/5}$

for $\mathcal{E}/\mathcal{E}_0 \to 1$, $M^{-2/5}(M^2+4) = (\mathcal{E}/\mathcal{E}_0 + 4)(\mathcal{E}/\mathcal{E}_0)^{-1/5}$

as $\mathcal{E}/\mathcal{E}_0 \to 0$, $M \to 0$

$4(\mathcal{E}/\mathcal{E}_0)^{-1/5} = 4BM^{-2/5}$

$B \sim 1 \Rightarrow M \sim (\mathcal{E}/\mathcal{E}_0)^{1/2}$