UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Friday 1 June 2012 09.00 to 11.00

EXPERIMENTAL AND THEORETICAL PHYSICS (3)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (3)

*Candidates offering this paper should attempt a total of* **three** *questions. The questions to be attempted are* **1**, **2** *and* **one** *other question.*

*The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains* **four** *sides, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS
$2 \times 20$ Page Answer Book
Metric graph paper
Rough workpad
Yellow master coversheet

SPECIAL REQUIREMENTS
Mathematical Formulae handbook
Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

ADVANCED QUANTUM PHYSICS

1 *Attempt* **all** *parts of this question. Answers should be concise and relevant formulae may be assumed without proof.*

(a) Carbon has electronic structure $(1s)^2(2s)^2(2p)^2$. Derive all possible spectroscopic terms $({}^{2S+1}L_J)$ for the partially filled shell within the LS coupling scheme. [4]

(b) Estimate the Stark shift of the ground state of a KRb molecule in a weak electric field of strength $\mathcal{E} = 1\,\mathrm{kV\,cm^{-1}}$. The molecule may be viewed as a rigid rotor with moment of inertia $I = 7.52 \times 10^{-45}\,\mathrm{kg\,m^2}$ and electric dipole moment $d = 1.89 \times 10^{-30}$ Cm. [4]

[*You may assume that the z-component of the dipole moment has matrix element* $\langle \ell = 1, m_\ell = 0 \,|\, \hat{d}_z \,|\, \ell = 0, m_\ell = 0\rangle = d/\sqrt{3}$, *with* $\ell, m_\ell$ *labelling the molecule's angular momentum and its projection along z.*]

(c) The electric field in a single mode cavity is described by the operator $\hat{\mathcal{E}} = \mathcal{E}_0(\hat{a} + \hat{a}^\dagger)$ where $\hat{a}^{(\dagger)}$ are the usual ladder operators, with $[\hat{a}, \hat{a}^\dagger] = 1$. The cavity is in a (normalised) coherent state $|\alpha\rangle$, such that $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ with $\alpha$ a complex number. Show that the uncertainty in the electric field is $\Delta\mathcal{E} = \mathcal{E}_0$. [4]

2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) the JWKB approximation;

(b) the fine structure of the Hydrogen atom;

(c) the operating principles of a three-level laser.

3 *Attempt* **either** *this question* **or** *question 4.*

A particle of mass $m$ and charge $q$ moves in the $xy$-plane subject to a magnetic field $B$ normal to this plane. The field has a gradient in the $x$-direction, with $B(x) = \alpha x$.

Starting from the general expression for the Hamiltonian in the presence of a vector potential, show that, with an appropriate choice of gauge, the Hamiltonian for the particle can be written

$$\hat{H} = \frac{1}{2m}\left[\hat{p}_x^2 + \left(\hat{p}_y - \frac{1}{2}q\alpha\hat{x}^2\right)^2\right],$$

where $\hat{p}_{x,y}$ are the momentum operators and $\hat{x}$ the position operator in the $x$-direction. [8]

For periodic boundary conditions in the $y$-direction over a length $L_y$, explain why the energy eigenstates in position representation can be written as $\Psi(x, y) = e^{iky}\Phi(x)$ and state the allowed values of $k$. [2]

Hence show that the energy eigenvalues follow from the solution of the one-dimensional Schrödinger equation with an effective potential [2]

$$V(x) = \frac{1}{2m}\left(\hbar k - \frac{1}{2}q\alpha x^2\right)^2 .$$

Sketch the form of the potential in the cases $k/(q\alpha) > 0$ and $k/(q\alpha) < 0$. [2]

Sketch the wave functions $\Phi(x)$ for the *two* lowest energy states, when $k/(q\alpha)$ is large and negative, and when $k/(q\alpha)$ is large and positive. [3]

Explain why the energies of the two lowest energy states are nearly degenerate for large positive $k/(q\alpha)$. By expanding $V(x)$ close to one of its minima, show that each of these two energies is approximately

$$E_k = \frac{1}{m}\left(\frac{\hbar^3 q\alpha k}{2}\right)^{1/2} .$$

[6]

Calculate the group velocity in the $y$-direction of wave packets formed from these states. [2]

4 *Attempt* **either** *this question* **or** *question 3.*

A quantum mechanical system has eigenstates $|\psi_n\rangle$ with energies $\hbar\omega_n$ $(\omega_0 \leq \omega_1 \leq \omega_2 \ldots)$. At time $t = -\infty$ the system is in the ground state $|\psi_0\rangle$. A weak perturbation $\hat{V}(t)$ is applied. Representing the state in terms of the unperturbed eigenstates as

$$|\Psi(t)\rangle = \sum_n c_n(t) e^{-i\omega_n t} |\psi_n\rangle,$$

show that the coefficients $c_n$, for $n \neq 0$, are approximately

$$c_n(t) = \frac{1}{i\hbar} \int_{-\infty}^{t} e^{i(\omega_n - \omega_0)t'} \langle\psi_n|\hat{V}(t')|\psi_0\rangle \, \mathrm{d}t',$$

making clear any assumptions that you make. [12]

An electron in a semiconductor quantum well can be represented as a particle confined to a box by infinite potential barriers, with position $x$ in the range $0 < x < w$, within which the potential is initially zero. Initially the electron is in its ground state $|\psi_0\rangle$. At $t = 0$, a weak electric field pulse is applied, with

$$\mathcal{E}(t) = \mathcal{E}_0 e^{-t/\tau} \qquad \text{for } t \geq 0$$

and with the electric field uniform and pointing in the $x$-direction.

Explain why, in first-order perturbation theory, the electric field pulse cannot excite the electron into the second excited state $|\psi_2\rangle$. [2]

Denoting the electron mass by $m_\mathrm{e}$ and charge by $-e$, find an expression for the probability that at $t = \infty$ the electron is in the first excited state, $|\psi_1\rangle$. [6]

If, instead, the electron had started in the state $|\psi_1\rangle$ at $t = -\infty$, describe without detailed calculation the effects of the above weak electric field pulse. [5]

$$\left[ \textit{You may use the result } \int_0^w x \sin\frac{2\pi x}{w} \sin\frac{(m+1)\pi x}{w} \mathrm{d}x = \frac{-8w^2(1+m)}{\pi^2(m-1)^2(3+m)^2} \textit{ for even } m. \right]$$

END OF PAPER