TSP 2012

1) a) temp at which thermal de Broglie wavelength is of the order of interparticle spacing

- $n\lambda^3 \sim 1$

electrons in solids - $m \sim m_e$, $n \sim 10^{28}$

$T \sim 10^4$ K

atomic gas - $m \sim 10^{-26}$, $n \sim 10^{25}$

$T \sim 10^{-2}$ K

neutron star - $m \sim 10^{-27}$, $n \sim 10^{40}$

$T \sim 10^9$ K

b) $C = (220\ \mathrm{J\,K^{-5/2}\,m^{-3}})\, T^{3/2}$

$\epsilon = \alpha p^n$

$$g(\epsilon)\,d\epsilon = \frac{4\pi k^2 dk}{(2\pi)^3} \qquad \text{p u vol}$$

$$= \frac{p^2 dp}{2\pi^2 \hbar^3}$$

$$U = \int \epsilon g(\epsilon) n(\epsilon)\, d\epsilon \propto \int \frac{\alpha p^n \cdot p^2 dp}{e^{\beta\alpha p^n} - 1}$$

$x = \beta\alpha p^n$

$$p = \frac{1}{(\beta\alpha)^{1/n}} x^{1/n} \quad , \quad dp = \frac{1}{n}\frac{1}{(\beta\alpha)^{1/n}} x^{1/n - 1}\, dx$$

$$U = \int \frac{x/\beta \; 1/(\beta\alpha)^{2/n}\, x^{2/n}\, \frac{1}{n}\, 1/(\beta\alpha)^{1/n}\, x^{1/n-1}\, dx}{e^x - 1} \qquad \propto \quad \frac{1}{\beta^{3/n+1}}$$

need $U \propto T^{5/2}$ $\Rightarrow$ $\frac{3}{n} + 1 = \frac{5}{2}$, $n = 2$

c) entropies of 2 phases equal on coexistence line in p-T plane

$$dS = \frac{\partial S}{\partial p} dp + \frac{\partial S}{\partial T} dT$$

$$\frac{\partial S}{\partial p} = - \frac{\partial V}{\partial T} = - \alpha_T V$$

$$\frac{\partial S}{\partial T} = \frac{C_p}{T}$$

$$dS = - \alpha_T V dp + \frac{C_p}{T} dT = 0$$

$$\frac{dT}{dp} = \frac{TV \Delta \alpha_T}{\Delta C_p}$$

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3) Grand potential for non-interacting fermions

$$\Xi_k = \sum_{n=0}^{1} (e^{-\beta(\epsilon-\mu)})^n = 1 + e^{-\beta(\epsilon_k-\mu)}$$

$$\phi_k = -k_B T \ln \Xi_k = -k_B T \ln(1 - e^{-\beta(\epsilon_k-\mu)})$$

$$n_F(\epsilon, \mu) = -\frac{\partial \phi_k}{\partial \mu} = \frac{1}{e^{\beta(\epsilon-\mu)}+1}$$

$$1 - n_F(\epsilon, \mu) = \frac{1}{e^{-\beta(\epsilon-\mu)}+1}$$

$$\phi_k = k_B T \ln(1 - n_F(\epsilon, \mu))$$

$$\Xi = \prod_k \Xi_k$$

$$\phi = -k_B T \ln \Xi = -k_B T \ln \prod_k \Xi_k = -k_B T \sum_k \ln \Xi_k = \sum_k \phi_k$$

need density of states to convert to integral:

$$\phi(T, \mu) = \int_0^\infty \phi(\epsilon) g(\epsilon) d\epsilon$$

$$\phi(\epsilon) = k_B T \ln(1 - n_F(\epsilon, \mu))$$

$$\phi(T, \mu) = k_B T \int_0^\infty d\epsilon \, g(\epsilon) \ln[1 - n_F(\epsilon, \mu)]$$

$$\phi(T, \mu) = k_B T \left[\int_0^\epsilon d\epsilon' g(\epsilon') \ln[1 - n_F(\epsilon, \mu)] \right]_0^\infty - k_B T \int_0^\infty \int_0^\epsilon d\epsilon' g(\epsilon') \frac{\partial}{\partial \epsilon} [\ln(1 - n_F(\epsilon, \mu$$

$$= \int_0^\infty \int_0^\epsilon d\epsilon' g(\epsilon') n_F(\epsilon, \mu) d\epsilon$$

using $-\frac{\partial}{\partial \epsilon}(k_B T \ln(1 - n_F(\epsilon, \mu))) = n_F(\epsilon, \mu)$

$$\phi(T, \mu) = \left[\int_0^\epsilon d\epsilon' \int_0^{\epsilon'} d\epsilon'' g(\epsilon'') \, n_F(\epsilon, \mu) \right]_0^\infty + \int_0^\infty \int_0^\epsilon d\epsilon' \int_0^{\epsilon'} d\epsilon'' g(\epsilon'') \left[-\frac{\partial n_F(\epsilon, \mu)}{\partial \epsilon} \right] d\epsilon$$

$$= \int_0^\infty \phi(0, \epsilon) \left[-\frac{\partial n_F(\epsilon, \mu)}{\partial \epsilon} \right] d\epsilon$$

$$\phi(0, \epsilon) = \int_0^\epsilon d\epsilon' \int_0^{\epsilon'} d\epsilon'' g(\epsilon'')$$

$$g(\epsilon) = 1 + \alpha\cos\left(\frac{2\pi\epsilon}{\Delta}\right)$$

$$\phi(0,\epsilon) = -\int_0^{\epsilon} d\epsilon' \int_0^{\epsilon'} d\epsilon''\left(1+\alpha\cos\left(\frac{2\pi\epsilon''}{\Delta}\right)\right)$$

$$= -\int_0^{\epsilon} d\epsilon' \left(\epsilon' + \frac{\Delta\alpha}{2\pi}\sin\left(\frac{2\pi\epsilon'}{\Delta}\right)\right)$$

$$= -\left[\frac{1}{2}\epsilon^2 - \frac{\alpha\Delta^2}{4\pi^2}\cos\left(\frac{2\pi\epsilon}{\Delta}\right)\right]$$

$$= \frac{\alpha\Delta^2}{4\pi^2}\cos\left(\frac{2\pi\epsilon}{\Delta}\right) - \frac{1}{2}\epsilon^2$$

$$\phi(T,\mu) = \underbrace{\int_0^{\infty}\frac{\alpha\Delta^2}{4\pi^2}\cos\left(\frac{2\pi\epsilon}{\Delta}\right)\left[-\frac{\partial n_F(\epsilon,\mu)}{\partial\epsilon}\right]d\epsilon}_{①} + \underbrace{\frac{1}{2}\int_0^{\infty}\epsilon^2\frac{\partial n_F(\epsilon,\mu)}{\partial\epsilon}d\epsilon}_{②}$$

① $$\mathrm{Re}\left[e^{2\pi i(\epsilon-\mu)/\Delta}\right] = \mathrm{Re}\left[\frac{e^{2\pi i\epsilon/\Delta}}{e^{2\pi i\mu/\Delta}}\right] = \frac{\cos\left(\frac{2\pi\epsilon}{\Delta}\right)}{\cos\left(\frac{2\pi\mu}{\Delta}\right)}$$

$$\cos\left(\frac{2\pi\epsilon}{\Delta}\right) = e^{2\pi i(\epsilon-\mu)/\Delta}\cos\left(\frac{2\pi\mu}{\Delta}\right)$$

$$\int_0^{\infty}\frac{\alpha\Delta^2}{4\pi^2}\cos\left(\frac{2\pi\epsilon}{\Delta}\right)\left[-\frac{\partial n_F(\epsilon,\mu)}{\partial\epsilon}\right]d\epsilon = \int_0^{\infty}\frac{\alpha\Delta^2}{4\pi^2}e^{2\pi i(\epsilon-\mu)/\Delta}\cos\left(\frac{2\pi\mu}{\Delta}\right)\left[-\frac{\partial n_F(\epsilon,\mu)}{\partial\epsilon}\right]d\epsilon$$

$$= \frac{\alpha\Delta^2}{4\pi^2}\cos\left(\frac{2\pi\mu}{\Delta}\right)\frac{x}{\sinh x}$$

② $$\frac{1}{2}\int_0^{\infty}\epsilon^2\frac{\partial n_F(\epsilon,\mu)}{\partial\epsilon}d\epsilon = \frac{1}{2}\left[\epsilon^2 n_F(\epsilon,\mu)\right]_0^{\infty} - \int_0^{\infty}\epsilon\, n_F(\epsilon,\mu)\,d\epsilon$$

$$= -\int_0^{\epsilon_F}\epsilon\,d\epsilon - \frac{\pi^2}{6}(k_BT)^2 \qquad \text{(Sommerfeld)}$$

$$= -\frac{1}{2}\mu^2 - \frac{\pi^2}{6}(k_BT)^2$$

$$\phi(0,\mu) = \frac{\alpha\Delta^2}{4\pi^2}\cos\frac{2\pi\mu}{\Delta} - \frac{1}{2}\mu^2$$

$$② = \phi(0,\mu) - \frac{\pi^2}{6}(k_BT)^2 - \frac{\alpha\Delta^2}{4\pi^2}\cos\frac{2\pi\mu}{\Delta}$$

$$\phi(T,\mu) = \phi(0,\mu) - \frac{\pi^2}{6}k_B^2T^2 + \frac{\alpha\Delta^2}{4\pi^2}\cos\frac{2\pi\mu}{\Delta}\left(\frac{x}{\sinh x} - 1\right)$$

4) energy of state with angular momentum J has energy $g m_J \mu_B B$

m_J can take values $-J, -J+1 \dots J-1, J$

Sum over all m_J states : $Z = \sum_{m=-J}^{J} \exp\left(\frac{m g \mu_B B}{k_B T}\right)$

$$Z = \sum_{m=-J}^{J} \exp\left(m \frac{T_B}{T}\right)$$

entropy:

$$F = -k_B T \ln Z = -k_B T \ln\left(\sum \exp\left(m \frac{T_B}{T}\right)\right)$$

$$S = -\frac{\partial F}{\partial T} = k_B \ln\left(\sum \exp\left(m \frac{T_B}{T}\right)\right) + k_B T\left(-\frac{T_B}{T^2}\right) \frac{\sum m \exp\left(m \frac{T_B}{T}\right)}{\sum \exp\left(m \frac{T_B}{T}\right)}$$

$T_B \propto B$ - S only depends on ratio $\frac{B}{T}$

high T limit

$$Z = \sum_{m=-J}^{J} \exp\left(m \frac{T_B}{T}\right) \sim \sum_{m=-J}^{J} \left(1 + m\frac{T_B}{T} + \frac{1}{2}\left(\frac{T_B}{T}\right)^2 m^2\right)$$

$$Z = 2J+1 + \frac{1}{2}\left(\frac{T_B}{T}\right)^2 \frac{1}{3} J(J+1)(2J+1)$$

$$F = -k_B T \ln Z = -k_B T\left[\ln(2J+1) + \ln\left(1 + \frac{1}{6}\left(\frac{T_B}{T}\right)^2 J(J+1)\right)\right]$$

$$\sim -k_B T\left[\ln(2J+1) + \frac{1}{6}\left(\frac{T_B}{T}\right)^2 J(J+1)\right]$$

$$S = -\frac{\partial F}{\partial T} = k_B \ln(2J+1) - \frac{k_B}{6}\left(\frac{T_B}{T}\right)^2 J(J+1)$$

$$S = k_B\left[\ln(2J+1) - J(J+1)\frac{T_B^2}{6T^2}\right]$$

heat capacity $M k_B T/\theta$ for $T \ll \theta$

at start of demagnetisation $s = k_B\left[\ln(2J+1) - J(J+1)\frac{T_{B_i}^2}{6T_i^2}\right]$

total entropy $S = N s(T_i, B_i)$

$$C = T\frac{\partial S}{\partial T} = \frac{M k_B T}{\theta}$$

$$S = \int_{T_i}^{T_{min}} \frac{C}{T}\, dT = \int_{T_i}^{T_{min}} \frac{M k_B}{\theta}\, dT = \frac{M k_B}{\theta}(T_{min} - T_i)$$

$$N s(T_i, B_i) = M k_B (T_i - T_{min})/\theta$$

$$N k_B \left[\ln(2J+1) - J(J+1) \frac{g^2 \mu_B^2 B_i^2}{6 k_B^2 T_i^2} \right] = \frac{M k_B}{\theta} (T_i - T_{min})$$

$$T_{min} = T_i - \frac{N\theta}{M} \left[\ln(2J+1) - J(J+1) \frac{g^2 \mu_B^2 B_i^2}{6 k_B^2 T_i^2} \right]$$

$\frac{\theta}{T} \gg 1$ - neglect $\ln(2J+1)$ term.

$$T_{min} = T_i - \frac{N \theta J(J+1) g^2 \mu_B^2 B_i^2}{6 M k_B^2 T_i^2}$$