NST II ETP (1) MAY 2010

Relativity

1(a) The 4-momenta of the two particles are $(\gamma mc, \gamma mv, 0, 0)$ and $(2mc, 0, 0, 0)$. The composite system has 4-momentum $((\gamma+2)mc, \gamma mv, 0, 0)$, and invariant $m^2c^2((\gamma+2)^2 - \gamma^2 v^2) = m^2c^2(4\gamma+5)$, which has to equal $4\alpha^2c^2$ to produce two particles of mass αm at rest in their joint CoM frame. Hence $\gamma = \alpha^2 - 5/4$

1(b) The gravitational bending of light is $\Delta\phi = 4GM/Rc^2$, which evaluates to 204 arcsec. You get [2] for GM/Rc^2, [1] for the 4 and [1] for working it out.

1(c) The formulae are on pages 7 and 8 of the formula book. $\Gamma^{\beta}_{\beta\alpha} = \Gamma^{\beta}_{\alpha\beta} = \coth\alpha$, $\Gamma^{\alpha}_{\beta\beta} = -\sin\alpha\cosh\alpha$, $R^{\alpha}_{\beta\alpha\beta} = \sinh^2\alpha$, $R^{\beta}_{\alpha\beta\alpha} = 1$.

2(a) This is one of the 17 handouts for the course, though the lecturer varied the content a bit.

- Collapsed objects are the theoretically predicted end-points of stellar evolution as there is a well-defined limit to the size of object (white dwarf) that can be supported by electron degeneracy pressure (1.4 $M_\odot$, the Chandrasekhar limit).
- Even neutron degeneracy pressure only gets you to about 3 $M_\odot$ in neutron stars.
- There are many X-ray emitting objects known to have masses 5-10 $M_\odot$.
- Some objects (e.g. MGC-6-30) show spectral features of extreme Kerr black holes, presumably due to spin-up following accretion from binary companion.
- Rotation curve of Galaxy clearly indicates presence of 3.6×10^6 $M_\odot$ black hole.
- Extragalactic radio sources typically have black holes in range 10^8 to 10^9 $M_\odot$.
- Jets of radio sources show relativistic beaming effects and superluminal motions. These have to arise from potential wells that are very deep and can only exist in black holes, rather than other compact objects.
- We've actually no direct observational evidence at all...

2(b) This is another complete handout. The answers will be very mathematical.

- The derivative of a vector (here contravariant) $A^i_{,j} \equiv \dfrac{\partial A^i}{\partial x^j}$ is not a tensor, as it transforms incorrectly under a change of coordinates $x \longrightarrow \overline{x}$.
- We can correct this by making the covariant derivative

$$A^i_{;j} = \frac{\partial A^i}{\partial x^j} + \Gamma^i_{jk}A^k$$

where Γ^i_{jk} is the connection. All relevant formulae are in the formula book provided in the Exam.

- Parallel transport of the vectors A^i and A_i uses the covariant derivative and is given by

$$\frac{\mathrm{d}A^i}{\mathrm{d}\tau} + \Gamma^i_{jk} A^j \dot{x}^k = 0 \ ; \quad \frac{\mathrm{d}A_i}{\mathrm{d}\tau} + \Gamma^j_{ik} A_j \dot{x}^k = 0 \ .$$

- Example of transport of vector around a sphere — rotates by amount equal to solid angle of loop.

2(c) This section is a bit shorter in the notes, but obviously central to the course!

- Gravity is not like electromagnetism, where there is a distinct coupling constant q, quite separate from the mass m.
- Gravitation supposes that inertial and gravitational mass are identical (Principle of Equivalence).
- PoE implies that is always possible to find a locally Minkowskian frame for which $\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$, where $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.
- Equivalence of gravitational and inertial mass tested to very hight accuracy by null experiments, improving the Eotvos experiment.
- Could possibly mention inertial forces (Coriolis and centrifugal) from other courses.

3 Proof of Newtonian orbit equation not required, but here it is anyway... From Newton's law in polars $\ddot{r} - r\dot{\phi}^2 = -GM/r^2$ and $h = r^2\dot{\phi}$ (conservation of angular momentum) gives the energy equation

$$\tfrac{1}{2}m\dot{r}^2 + \frac{h^2}{2mr^2} - \frac{GMm}{r} = E$$

Substituting $u \equiv 1/r$ gets $\dot{r} = -h\mathrm{d}u/\mathrm{d}\phi$, so rearranging

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 + u^2 = \frac{2GMu}{h^2} + \frac{2mE}{h^2}$$

Differentiating and dividing by $\mathrm{d}u/\mathrm{d}\phi$ gives the Newtonian equation given Newton thus has $u = GM/h^2$, so $r = h^2/GM$ and the orbital period is $T = 2\pi/\dot{\phi} = 2\pi(r^3/GM)^{1/2}$. Works for all $0 < h < \infty$.

Einstein has $u'' = 0 \Rightarrow 3GMu^2/c^2 - u + GM/h^2 = 0$, so there are two solutions:

$$u = \frac{c^2}{6GM}\left(1 \pm \sqrt{1 - \frac{12G^2M^2}{h^2c^2}}\right) . \tag{1}$$

The outer solution (minus sign, i.e. smaller u) for $h^2 > 12G^2M^2/c^2$ is a stable orbit that approaches the Newtonian case for $h \to \infty$. The last stable orbit is this at $R = 6GM/c^2$. The plus sign is an unstable interior orbit that I would not advise attempting even if you are equipped with the starship Voyager and anxious to clip a few years journey-time off your trip back to the Alpha Quadrant. They'll probably draw the effective potential, which neatly illustrates the stability of the orbits.

For $h \gg GMc$ we can expand the square root. The inner orbit approaches $r = 1/u = 3GM/c^2$ as required, and the outer one has $u = GM/h^2$, which is the Newtonian result above.

Rearranging (1), we get

$$\left(\frac{6GM}{Rc^2} - 1\right)^2 = 1 - \frac{12G^2M^2}{h^2c^2} \Rightarrow h^2 = \frac{GMR^2}{R - 3GM/c^2}$$

With the numerical values given it's a close call: $\sqrt{12}GM/c^2$=1.543 × 10^{10} m and h/c=1.508 × 10^{10} m, so there's no minimum, and no stable orbit. The correction factor given in the hint only amounts to about 0.5%, but is given to make a definitive conversion between v and h.

Using the above formula for h, the value of h/R is 0.1802 c for the circular orbit at this R. The circular velocity follows from $vR = h/\sqrt{1 + h^2/R^2c^2} = 0.1777\ c$, quite a bit more of a correction...

4 (I set $c = G = 1$ in this answer.) A particle falling in radially has $\dot{t}(1 - 2M/r) = \gamma = 1$ if it is to be at rest at infinity.

From the metric, setting $\mathrm{d}s^2 = \mathrm{d}\tau^2$ for a material particle we have

$$1 = \frac{\gamma^2 - \dot{r}^2}{1 - 2M/r} \Rightarrow \dot{r} = -\sqrt{2M/r}$$

which already proves the second part, but we'll do that later. Dividing, we get

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\sqrt{\frac{2M}{r}}\left(1 - \frac{2M}{r}\right)$$

as required. To the stationary observer in frame S' we get (from the metric) $\mathrm{d}s = \mathrm{d}r' = \mathrm{d}r(1 - 2M/r)^{-1/2}$ and $\mathrm{d}s = \mathrm{d}t' = \mathrm{d}t(1 - 2M/r)^{1/2}$, so that $\mathrm{d}r'/\mathrm{d}t' = -\sqrt{2M/r}$, confirming our earlier result. This way of doing it is explicitly in their notes. A better way of getter the relative velocity v between 4-velocities v_1 and v_2 (works in Special or General Relativity) is $v^2 = 1 - 1/(v_1 \cdot v_2)^2$, but they didn't cover this.

A photon still has $\dot{t}(1 - 2M/r) =$ constant. From the metric $\mathrm{d}s^2 = 0$ we get $\dot{r} = (1 - 2M/r)\dot{t}$, so setting $\dot{t}(1 - 2M/r) = p_0$ (the frequency at infinity) the photon has 4-vector $p = (\dot{t}, \dot{r}, 0, 0) = (p_0/(1 - 2M/r), p_0, 0, 0)$. The observed frequency is $p \cdot v = g_{\mu\nu}p^\mu v^\nu$, where v is the 4-velocity of the observer. The observer at infinity ($v = (1, 0, 0,)$) sees $\nu = p_0$, whereas the emitter ($v = ((1 - 2M/r)^{-1}, -\sqrt{2M/r}, 0, 0)$) sees $p_0 \dfrac{1 + \sqrt{2M/r}}{1 - 2M/r} = \dfrac{1}{1 - \sqrt{2M/r}}$. The received frequency is $\nu_0(1 - \sqrt{2M/r})$. It's all compatible with special relativity and Newtonian gravity. I've done this by a method that always works for problems involving redshift in GR.

There are other ways of doing this, which they'll probably use in preference. The first notes that the stationary observer sees a Doppler shift of $\nu_0(1 - 2M/r)^{1/2}/(1 - \sqrt{2M/r})$ immediately the spacecraft goes past him, and the we use $\mathrm{d}t' = \mathrm{d}t(1 - 2M/r)^{1/2}$ again to confirm the result.

The more direct way notes that the spacecraft moves $\mathrm{d}t = \mathrm{d}\tau(1 - 2M/r)^{-1}$ and $\mathrm{d}r = -\mathrm{d}\tau\sqrt{2M/r}$, and that, from the metric, it takes $\mathrm{d}t = |\mathrm{d}r|(1 - 2M/r)^{-1} = \mathrm{d}\tau\sqrt{2M/r}(1 - 2M/r)^{-1}$ for the signal to travel back, so that the total $\mathrm{d}t$ (which is what the observer at infinity sees) is $\mathrm{d}t = \mathrm{d}\tau(1 + \sqrt{2M/r})/(1 - 2M/r)^{-1} = \mathrm{d}\tau(1 - \sqrt{2M/r})^{-1}$. Hence result.