Math 353 Week 7

Ricardo Hernandez and Symbolab Calculator

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1 3.2.1

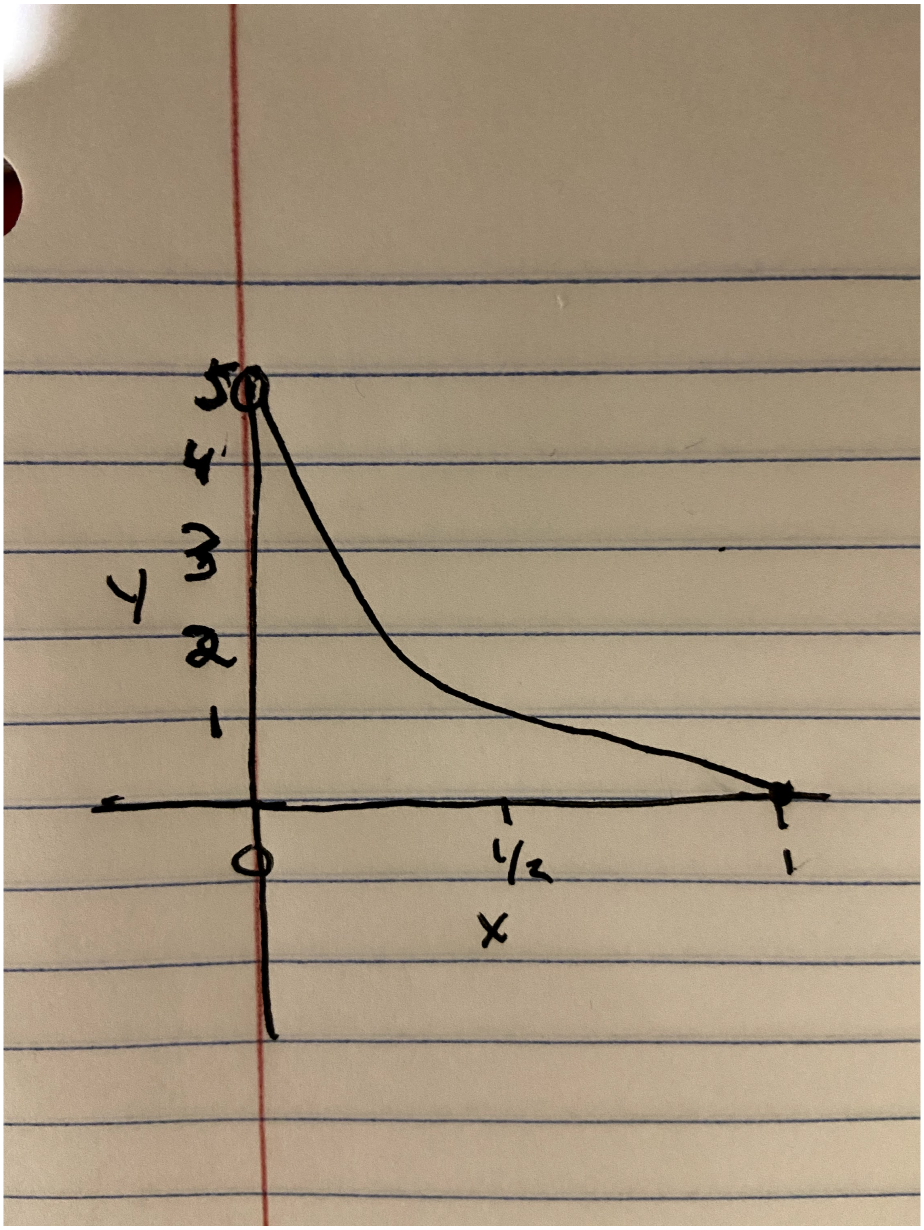
1.1 a.)

1. $f(x)$ meets the first requirement because $f(x) \geq 0$ for all x since the output will always be positive because of the exponent in the equation $f(x) = 5(1-x)^4$.

2.$f(x)$ satisfies the second requirement because

$$\int_0^1 5(1-x)^4\,dx = 1.$$

1.2 b.)



1.3 c.)

The cdf can be solved using u substitution.

$$\int_0^x 5(1-t)^4\, dt$$

where $u = 1 - t$ and $du = -1$.

$$\int_0^x -5u^4\, du$$

$$-5\int_0^x u^4\, du$$

Now do the integral and then plug in u into the equation.

$$\int_0^x u^4\, du = \frac{u^5}{5}$$

$$-5\int_0^x u^4\, du = -u^5$$

$$= -(1-t)^5 + C$$

Solving for C

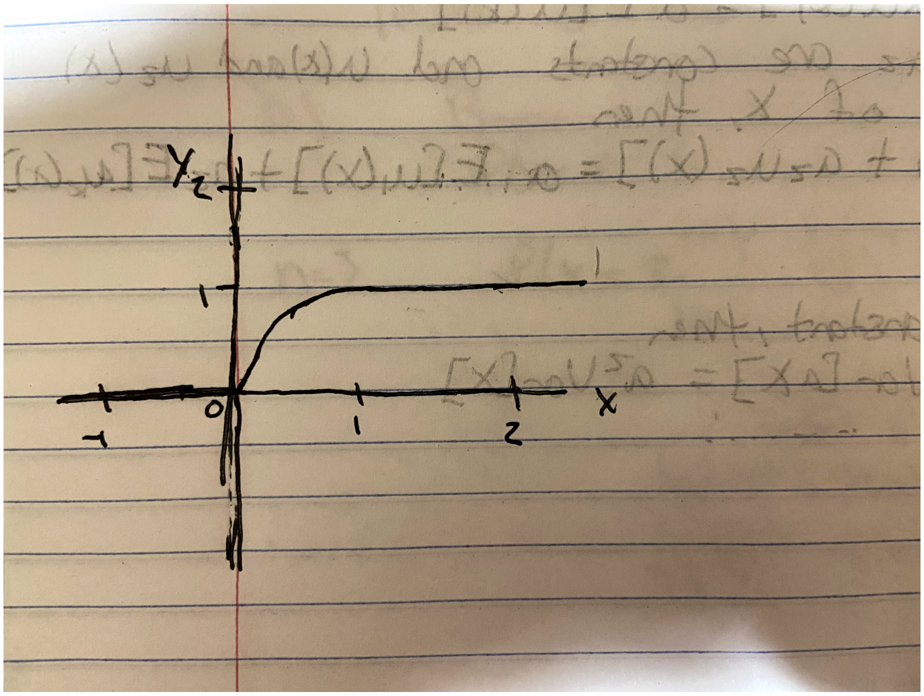
$$(t-1)^5 + C\Big|_0^1 = (0+C) - (-1+C) = 1$$

$C = 1$

$$= (t-1)^5\Big|_0^x = (x-1)^5 + 1$$

thus the cdf is,

$$F(X) = P(X \le x) = \int_{-\infty}^x f(t) = \begin{cases} 0 & \text{if } x \le 0 \\ \int_0^x 5(1-t)^4\, dt = (x-1)^5 + 1 & \text{if } 0 < x \le 1 \\ 1 & \text{if } x > 1 \end{cases}.$$



1.4 d.)

To calculate $P(0 \le X \le 0.5)$, we do the following

$$\int_0^{0.5} 5(1-x)^4\,dx = (.5-1)^5 + 1 \approx 0.969$$

To calculate $P(X \le 0)$, we do the following

$$P(X \le 0) = \int_0^0 5(1-x)^4\,dx = 0$$

To calculate $P(X > 0.25)$ we need to find $1 - P(X \le 0.25)$

$$1 - P(X \le 0.25) = 1 - \int_0^{.25} 5(1-x)^4\,dx = (.25-1)^5 + 1 \approx 1 - 0.762 \approx 0.237.$$

2 3.2.6

2.1 a.)

$$\int_{-1}^{x} \frac{3}{4}(1-t^2)\,dt = \frac{3}{4} - \frac{3}{4}t^2\,dt$$

$$= \frac{3}{4}x - \frac{x^3}{4} + C\Big|_{-1}^{x} = (\frac{3}{4}x - \frac{x^3}{4} + C) - (-\frac{3}{4} + \frac{1}{4} + C)$$

$$= \frac{3}{4}x - \frac{x^3}{4} + \frac{1}{2}$$

Thus, the cdf is.

$$F(X) = P(X \leq x) = \int_{-\infty}^{x} f(t) = \begin{cases} 0 & \text{if } x \leq -1 \\ \int_{-1}^{x} \frac{3}{4}(1-t^2)\,dt = \frac{3}{4}x - \frac{x^3}{4} + \frac{1}{2} & \text{if } -1 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}.$$

2.2 b.)

Finding the first quartile

$$0.25 = F(\pi_{0.25}) = \frac{3}{4}(\pi_{0.25}) - \frac{(\pi_{0.25})^3}{4} + \frac{1}{2}$$

Using the Symbolab Quadratic Equation Calculator to find $(\pi_{0.25})$ we get the following roots.

$$(\pi_{0.25}) = -0.35, (\pi_{0.25}) = -1.53,\ and\ (\pi_{0.25}) = 1.87$$

Out of these three, only -0.35 Lies between - 1 and 1. So,

$$F(\pi_{0.25}) \approx -0.35.$$

Finding the second quartile

$$0.50 = F(\pi_{0.50}) = \frac{3}{4}(\pi_{0.50}) - \frac{(\pi_{0.50})^3}{4} + \frac{1}{2}$$

Using the Symbolab Quadratic Equation Calculator to find $(\pi_{0.50})$ we get the following roots.

$$\pi_{0.50} = 0, \pi_{0.50} = 1.73,\ and\ \pi_{0.50} = -1.73$$

Out of these three roots, only 0 lies between -1 and 1. So,

$$F(\pi_{0.50}) = 0.$$

Finding the third quartile

$$0.75 = F(\pi_{0.75}) = \frac{3}{4}(\pi_{0.75}) - \frac{(\pi_{0.75})^3}{4} + \frac{1}{2}$$

Using the Symbolab Quadratic Equation Calculator to find ($\pi_{0.75}$) we get the following roots.

$$\pi_{0.75} = 0.35, \pi_{0.75} = 1.53, \ and \ \pi_{0.75} = -1.87$$

Out of these three roots, only 0.35 lies between -1 and 1. So,

$$F(\pi_{0.75}) \approx 0.35.$$

□

3 3.2.10

Given that $F(X) = 1 - e^{-x^2}$ for $x \geq 0$ and the following relationship

$$F'(X) = f(x)$$

Taking the derivative of $F(X)$, we get

$$F'(X) = 2xe^{-x^2}$$

Thus, the pdf is

$$f(x) = 2xe^{-x^2}, \ x \geq 0.$$

□

4 3.3.9

Question. Commuter trains arrive at a certain train station every 20 min, starting at 6:00 AM. If a passenger arrives at a time that is uniformally distributed between 6:00 and 6:40 AM, find the probability the passenger has to wait// If X denotes the wait time by a person, we assume that X is $U(0, 40)$. Thus the pdf is,

$$f(x) = \frac{1}{40}, 0 \leq x \leq 40.$$

4.1 a. less than 5 min for a train, and)

Since the train runs every 20 minutes, there are 2 times in our 40 minute period that the train arrives. 6:20 and 6:40. We have two times in which a person should arrive to wait less than 5 minutes.
6:15 and 6:35, so we do the following.

$$P(15 < x < 20) + P(35 < x < 40) = \int_{15}^{20} \frac{1}{40} dx + \int_{35}^{40} \frac{1}{40} dx = 0.25.$$

□

4.2 b. more than 10 min for a train.)

Since the train runs every 20 minutes, there are 2 times in our 40 minute period that the train arrives. 6:20 and 6:40. We have two times in which a person should arrive latest to wait more than 10 minutes.
6:10 and 6:30, so we do the following.

$$P(0 < x < 10) + P(20 < x < 30) = \int_0^{10} \frac{1}{40} dx + \int_{20}^{30} \frac{1}{40} dx = 0.5.$$

□

5 3.3.13

$$\lambda = 2$$

The pdf is,

$$f(x) = 2e^{-2x}$$

The cdf is,

$$F(x) = P(X \leq x) = F(x) = \int_{-\infty}^{x} f(t)dt = \int_0^x f(t)dt = 1 - e^{-2x},\ x > 0. \quad (1)$$

and

$$F(x) = P(X > x) = 1 - P(X \leq x) = 1 - (1 - e^{-2x}) = e^{-2x},\ x > 0. \quad (2)$$

5.1 a.)

Show that $P(X > 2 + 1|X > 1) = P(X > 2)$

Proof. Lets try to find the probabilities first of $P(X > 2 + 1)$ and $P(X > 1)$. Lets first try to find $P(X > 2 + 1)$.

$$P(X > 2 + 1) = P(X > 3)$$

Using Equation 2, find $F(3)$,

$$P(X > x) = e^{-2(3)} = e^{-6} = 0.0025.$$

Using Equation 2, find $F(1)$,

$$P(X > x) = e^{-2(1)} = e^{-2} = 0.135.$$

Using the definition of conditional probability, such that A = $P(X > 2+1)$ and B = $P(X > 1)$.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

$P(A \cap B) = P(A)$ since the intersection is $P(X > 3)$.

$$P(A|B) = \frac{0.0025}{0.135} =\approx 0.018.$$

Now we need to find $P(X > 2)$

$$P(X > x) = e^{-2(2)} = e^{-4} = 0.018.$$

Thus $P(X > 2 + 1|X > 1) = P(X > 2)$, as desired. $\square$

5.2 b.)

Generalizing part a.) Show that $P(X > a + b|X > b) = P(X > a)$

Proof. Lets first try to find $P(X > a + b)$. Using Equation 2, find $F(a + b)$,

$$P(X > x) = e^{-2(a+b)} = e^{-2a-2b}.$$

Using Equation 2, find $F(b)$,

$$P(X > x) = e^{-2(b)} = e^{-2b}.$$

Using the definition of conditional probability, such that A = $P(X > a+b)$ and B = $P(X > b)$.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

$P(A \cap B) = P(A)$ since the intersection is $P(X > a + b)$.

$$P(A|B) = \frac{e^{-2a-2b}}{e^{-2b}} = \frac{e^{-2a} \cdot e^{-2b}}{e^{-2b}} = e^{-2a}$$

Now we need to find $P(X > a)$

$$P(X > x) = e^{-2(a)} = e^{-2a}.$$

Thus $P(X > a + b|X > 1) = P(X > a)$, as desired. $\square$

6 3.3.14

$$\lambda = 0.001$$

The pdf is,

$$f(x) = 0.001e^{-0.001x}$$

The cdf is,

$$F(x) = P(X \leq x) = F(x) = \int_{-\infty}^{x} f(t)dt = \int_{0}^{x} f(t)dt = 1 - e^{-0.001x}, \ x > 0. \tag{3}$$

and

$$F(x) = P(X > x) = 1 - P(X \leq x) = 1 - (1 - e^{-0.001x}) = e^{-0.001x}, \ x > 0. \tag{4}$$

6.1 a.)

Find $P(X > 200)$ using equation (4.),

$$P(X > 200) = e^{-0.001(200)} \approx 0.818.$$

6.2 b.)

Evaluate $P(X > 1100|X > 900)$. Lets find $P(X > 1100)$ first,

$$P(X > 1100) = e^{-0.001(1100)} = 0.33$$

Now lets find $P(X > 900)$ first,

$$P(X > 900) = e^{-0.001(900)} = 0.406$$

Using the definition of conditional probability, such that A = $P(X > 1100)$ and B = $P(X > 900)$.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

$P(A \cap B) = P(A)$ since the intersection is $P(X > 1100)$.

$$P(A|B) = \frac{0.33}{0.406} = .812$$

6.3 c.)

It does make me question because in an exponential distribution you would expect that as the hours increase, the probability that the bulb will still work also increase. In our case, the probability decreases as we increase the number of hours.

7 3.5.2

Let Y = X^2 where Xhas the pdf $fx(x) = 2x, 0 < x < 1$. Show that Y is $U(0, 1)$. Lets find the cdf of X,

$$F'(x) = f(x).$$

$$\int_0^x 2x dx$$

$$= x^2\Big|_0^x$$

$$= x^2$$

Y has the range 0 to 1 an given that $Y = X^2$

$$F_y(y) = P(Y \leq y) = P(X^2 \leq y) = P(X \leq \sqrt{y}) = F(\sqrt{y}) = y$$

Therefore using the first rule below,

$$F'(x) = f(x).$$

$$f_Y(y) = 1, 0 < y < 1.$$

Thus, Y is $U(0,1)$. $\square$

8 3.5.6

Question: Let $Y = (1-2X)^3$ where X has the pdf $f_X(x) = 3(1-2x)^2$, $0 < x < 1$. Show that Y is $U(-1,1)$. Lets find the cdf of Y,

$$F_y(y) = P(Y \leq y) = P(1-2X)^3 \leq y)$$

$$= P((1-2X) \leq y^{\frac{1}{3}})$$

$$= P(-2X \leq y^{\frac{1}{3}} - 1)$$

$$= P(X \geq \frac{1-y^{\frac{1}{3}}}{2})$$

$$= 1 - P(X \geq \frac{1-y^{\frac{1}{3}}}{2}) = P(X < \frac{1-y^{\frac{1}{3}}}{2}) = 1 - F(\frac{1-y^{\frac{1}{3}}}{2})$$

Now that we have the cdf, we can find the pdf using

$$F'(Y) = f(y).$$

$$-\frac{d}{dy}\int_0^{\frac{1-y^{\frac{1}{3}}}{2}} 3(1-2x)^2 dx.$$

$$= -3(1 - 2 \cdot \frac{1-y^{\frac{1}{3}}}{2})^2 \cdot (-\frac{y^{\frac{-2}{3}}}{6})$$

$$= -3(1 - \cdot 1 + y^{\frac{1}{3}})^2 \cdot (-\frac{y^{\frac{-2}{3}}}{6})$$

$$= -3y^{\frac{2}{3}} \cdot -\frac{y^{\frac{-2}{3}}}{6}$$

$$\frac{1}{2}.$$

Showing that Y is $U(-1,1)$. $\square$