Lecture-24 & 25

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Thermal sensors, Temperature ICs, PTAT
Thermo couples
written in the lecture 23

** Now, Quantum Mechanics & Quantum Computation **

Early Experiments, Rutherford's Model

14 mag

- Electron Charge to Mass Ratio by JJ Thomson in 1897
- Electron charge measured independently by Millikan in 1909
- Nuclear Theory of the atom (measurement of r_a) proposed by E. Rutherford after a series of experiments in 1909–1914, broadening of beams of finely collimated α-particles (He^+) passing through thin metal foils.

* Quantum Mechanics had to be developed for physics of small particles

* Special Relativity for physics of High veloci

T. 8.3: Prime evidenze sperimentali della struttura atomica.

Early Experiments — Rutherford's Model

- Electron charge-to-mass ratio measured by J. J. Thomson in 1897.
- Electron charge measured by R. Millikan in 1909.
- Nuclear theory of the atom (measurement of r_a) proposed by E. Rutherford after a series of experiments in 1909–1914: broadening of beams of finely collimated α-particles (He^+) passing through thin metal foils (several thousand atomic layers).

Rutherford's experiments led to the "planetary" picture of the atom: a small nucleus of atomic charge Zq is surrounded by Z electrons, where Z indicates the position of the element in the periodic table.
As the α-particles are rather heavy, they are deflected mainly by the nuclei. In a thin foil, this typically happens only once. It is found that the nucleus can be considered as a geometrical point down to a distance $r_n \simeq 2 \times 10^{-13} \sqrt{Z}$ cm, $Z = 1, 2, \ldots$, whence $r_n \ll r_a$. The electron radius can be determined in a similar way using x-ray diffusion.

Planetary Model of the Atom — I

The simplest atom is that of hydrogen. The planetary model depicts it as an electron moving nearby the proton under the effect of a Coulombic potential V. As the proton is much more massive than the electron, its position can be approximated with that of the atom's center of mass, and placed in the origin. Choosing the energy reference such that $V(\infty) = 0$, it is

$$V(\boldsymbol{r}) = V(r) = -\frac{q^2}{4\pi\varepsilon_0 r}, \quad \boldsymbol{F} = -\operatorname{grad} V = -\frac{q^2}{4\pi\varepsilon_0 r^2}\frac{\boldsymbol{r}}{r}.$$

$$T + V = \frac{1}{2}mu^2 - \frac{q^2}{4\pi\varepsilon_0 r} = E = \text{const},$$

where $\boldsymbol{r}$ is the electron position and u the velocity's module. The model is extended to more complicated atoms by considering an outer electron moving nearby a core of net charge q embedding Z protons and $Z-1$ electrons, or to even more complicated cases (*hydrogenic-like* systems).

As $T = E - V = E + |V| \geq 0$, two cases are possible:

$$E < 0 \implies V \leq E < 0, \; r_{\max} < \infty \text{ (bound electron)},$$

$$E \geq 0 \implies V \leq 0 \leq E, \; r_{\max} = \infty \text{ (free electron)}.$$

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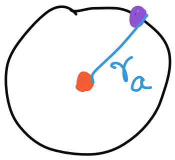
→ "planetary Model of an Atom" (I) was developed firstly



Here the force is of coulombic type

in conserved s/s = T.E = P.E + K.E

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$$V(r) = \frac{-q^2}{4\pi\varepsilon_0 r}, \quad F = -\text{grad } V$$

$$= \frac{-q^2}{4\pi\varepsilon_0 r^2}\frac{\vec{r}}{r}$$

$$K.E + P.E = \frac{1}{2}mu^2 - \frac{q^2}{4\pi\varepsilon_0 r}$$

(Hydrogenic like systems)

$u \Rightarrow$ velocity's module

$$\underbrace{T}_{K.E} + \underbrace{V}_{P.E} = \underbrace{E}_{T.E}$$

* $E < 0 \Rightarrow V \leq E < 0$, $r_{max} < \infty$ (Bound electron)

$E > 0 \Rightarrow V \leq 0 \leq E$, $r_{max} = \infty$ (free electron)

Planetary Model of the Atom — II

By way of example, for a bound electron in a uniform circular motion it is

$$F = |\boldsymbol{F}| = m|\boldsymbol{a}| = m\frac{u^2}{r} = \frac{2T}{r}\,.$$

On the other hand, $F = q^2/(4\pi\varepsilon_0 r^2) = -V/r$, whence

$$T = -\frac{V}{2}\,, \qquad E = T + V = \frac{V}{2} = -\frac{q^2}{8\pi\varepsilon_0 r} = \text{const} < 0\,,$$

$$\frac{\mathrm{d}E}{\mathrm{d}r} = \frac{|E|}{r} > 0\,,$$

that is, the total energy is larger at larger orbits. The model is thus able to explain phenomena like the excitation and ionization of atoms, and the inverse. It also explains some properties of crystals:

- If all electrons are bound, the crystal is an insulator.
- If some electrons are free inside the crystal, the latter is a conductor or a semiconductor.
- The existence of the minimum extraction energy E_W is also explained by the model.

planetary Model of the Atom-II

$$F = |F| = m|a| = \frac{mu^2}{r} = \frac{2T}{r} \quad -①$$

centripetal force

&

$$F = \frac{q^2}{4\pi\varepsilon_0 r^2} = -\frac{V}{r} \quad -②$$

from ① & ②

$$T = -\frac{V}{2}$$

$$\Rightarrow \quad T + V = E \Rightarrow E = \frac{V}{2} = \frac{-q^2}{8\pi\varepsilon_0 r}$$

$$= \text{constant} < 0$$

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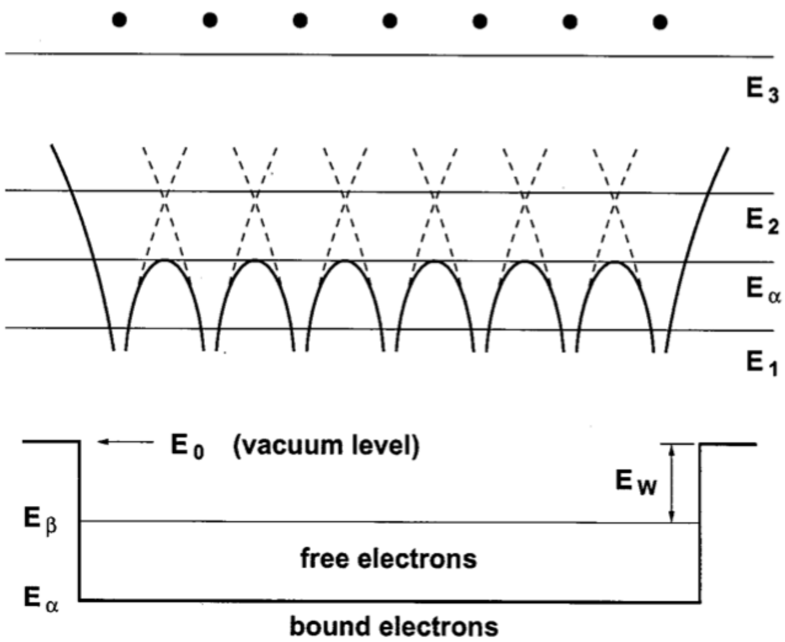

T. 8.7: Andamento schematico dell'energia potenziale in un cristallo.

T. 8.8: Esempi di risultati sperimentali che contraddicono leggi "classiche".

Phenomena Contradicting "Classical" Laws

About 1900, experimental evidence was found for a number of phenomena that contradict the calculations based on the known physical laws, that is, the laws of Analytical Mechanics, Electromagnetism, and Statistical Mechanics. Among such phenomena are:

1. Stability of the atom. (decay of electron could not be explained by planetary model)
2. Spectral lines of excited atoms.
3. Some aspects of the photoelectric effect.
4. Spectrum of the blackbody radiation.
5. Compton effect.
6. Hall effect.
7. Diffraction of electrons.

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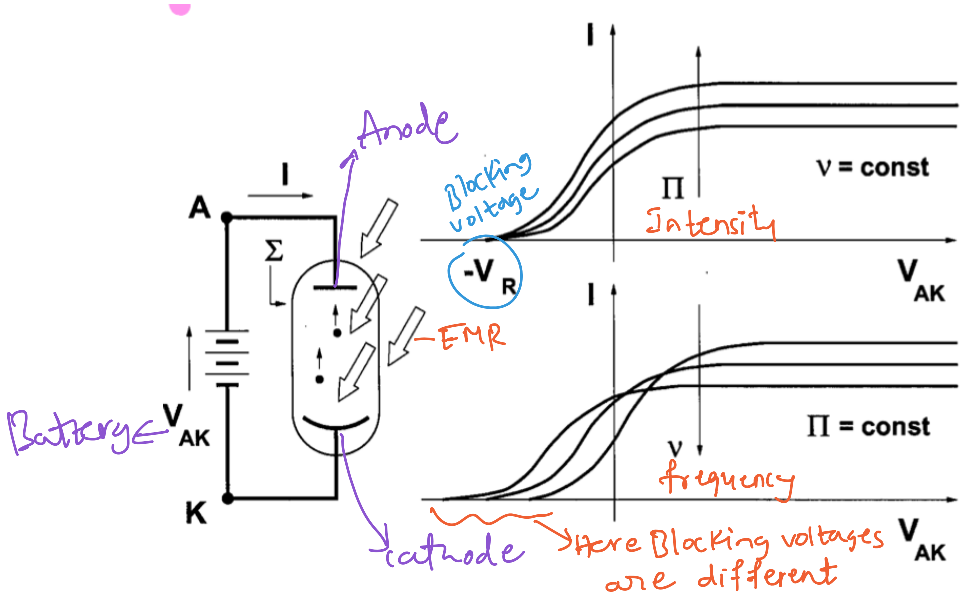
photo electric effect



T. 8.11: Misura dell'effetto fotoelettrico.

EMR: Electro magnetic Radiation
which we vary by (Frequency & Intensity)

Note:

T. 8.12: Tensione di spegnimento nell'effetto fotoelettrico.

Photoelectric Effect — I

It is found that the impinging e.m. radiation extracts charges from a metal (H. Hertz) and that these charges are electrons (J. J. Thomson, 1899). This is interpreted as an increase of the electron energy by an amount E_L due to absorption from the e.m. field, such that the electron overcomes the barrier E_W and is extracted from the metal (the cathode in this case). While the electron travels from the cathode to the anode, energy conservation yields

$$\frac{1}{2}mu_A^2 - \frac{1}{2}mu_K^2 = qV_{AK}\,.$$

Letting $V_{AK} = -V_R$ such that $u_A = 0$ provides

$$\frac{1}{2}mu_K^2 = qV_R \geq 0\,,$$

that is, the energy of the electrons that inside the cathode have the maximum energy and do not suffer energy losses while they are extracted. As

$$\frac{1}{2}mu_K^2 = E_L - E_W \quad \Longrightarrow \quad qV_R = E_L - E_W\,,$$

the above provides a method for measuring E_L.

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T. 8.13: Relazione fra l'energia assorbita da un elettrone e la frequenza della radiazione.

Photoelectric Effect — II

The photoelectric effect is used to determine the dependence of E_L on the parameters characterizing the e.m. field. Taking a monochromatic case at frequency ν, important quantities are the intensity per unit frequency, $\eta = \mathrm{d}E/(\mathrm{d}\Sigma\, \mathrm{d}t\, \mathrm{d}\nu)$ ($[\eta] = \mathrm{J/cm^2}$), and the power per unit frequency

$$\Pi \doteq \int_\Sigma \eta \, \mathrm{d}\Sigma = \frac{\mathrm{d}E}{\mathrm{d}t\, \mathrm{d}\nu} = \frac{\mathrm{d}P}{\mathrm{d}\nu}.$$

The planetary atomic model predicts that, if Π increases, the current I should increase (which happens), and the blocking voltage V_R should also increase (which *does not* happen).
In addition, it is unexpectedly found that both I and V_R depend on the frequency ν. In particular, the energy E_L that the electron absorbs from the e.m. field is found to be proportional to the frequency through the Planck constant:

$$E_L = h\nu\,, \qquad h \simeq 6.626 \times 10^{-34}\ \mathrm{J\,s}\,.$$

If ν is such that $h\nu < E_W$, no current is measured.

T. 8.14: Definizioni e proprietà generali del corpo nero.

Studied using law of Thermodynamics

Blackbody Radiation — I

$$\text{Def:}\quad \begin{cases} \eta & \textit{emissivity}\ (\mathrm{J/cm^2}) \\ \alpha & \textit{absorption power} \\ \textit{blackbody} & \text{a body such that } \alpha = 1\ \forall\nu \end{cases}$$

η is the intensity emitted by the body per unit frequency, α the fraction of the incident intensity absorbed by the body at a frequency ν. By Kirchhoff's law (1859), for any body in thermal equilibrium with radiation it is

$$\frac{\eta}{\alpha} = K(\nu, T)\,.$$

For a blackbody at equilibrium: $\eta = K$. By Stefan's law (1879)

$$\int_0^\infty \eta(\nu, T)\,\mathrm{d}\nu = \sigma\,T^4\,,$$

where $\sigma = 5.67 \times 10^{-12}\ \mathrm{W\,cm^{-2}\,K^{-4}}$ is the Stefan-Boltzmann constant. The relation between emissivity and spectral energy density u is found from:

$$\boxed{\eta = \frac{\mathrm{d}E}{\mathrm{d}x\mathrm{d}y\mathrm{d}t\mathrm{d}\nu} = c\,\frac{\mathrm{d}E}{\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}\nu} = c\,u\,.}$$

The energy density, in turn, is $w_{\mathrm{em}}^{\mathrm{eq}}(T) = \int_0^\infty u(\nu, T)\,\mathrm{d}\nu\,.$

For any body that is in thermal equilibrium with the radiation then

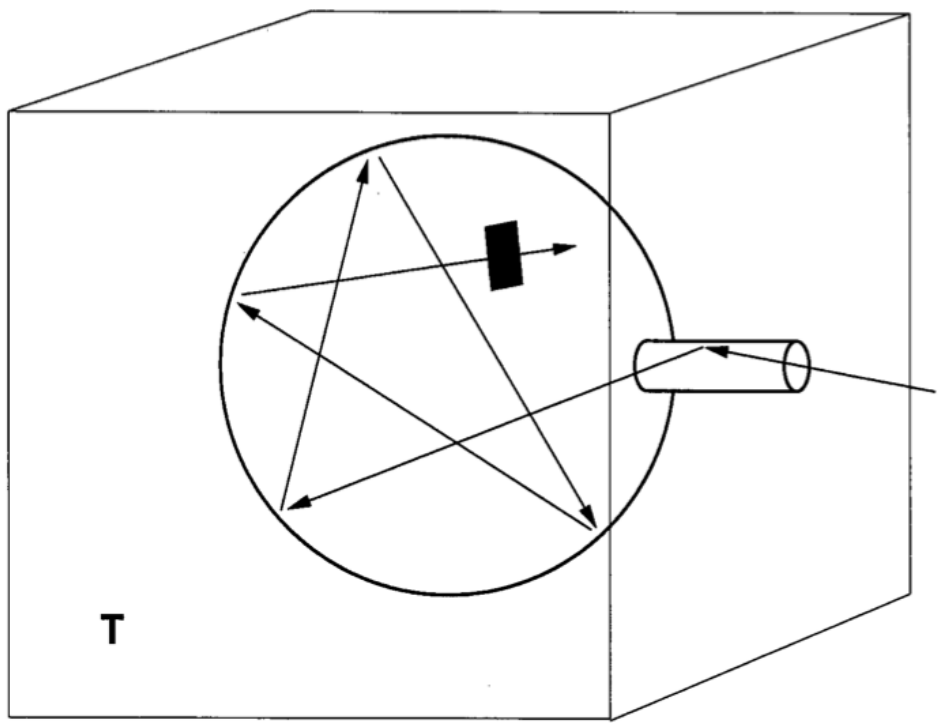
$$\boxed{\frac{\eta}{\alpha} = K(\nu, T)}$$

- Spectral energy density (u) : Energy/volume . frequency
- Emissivity : (η) EMR emitted by a body
- Stefan's law

$$\int_0^{\infty} \eta(\nu, T)\, d\nu = \sigma T^4$$

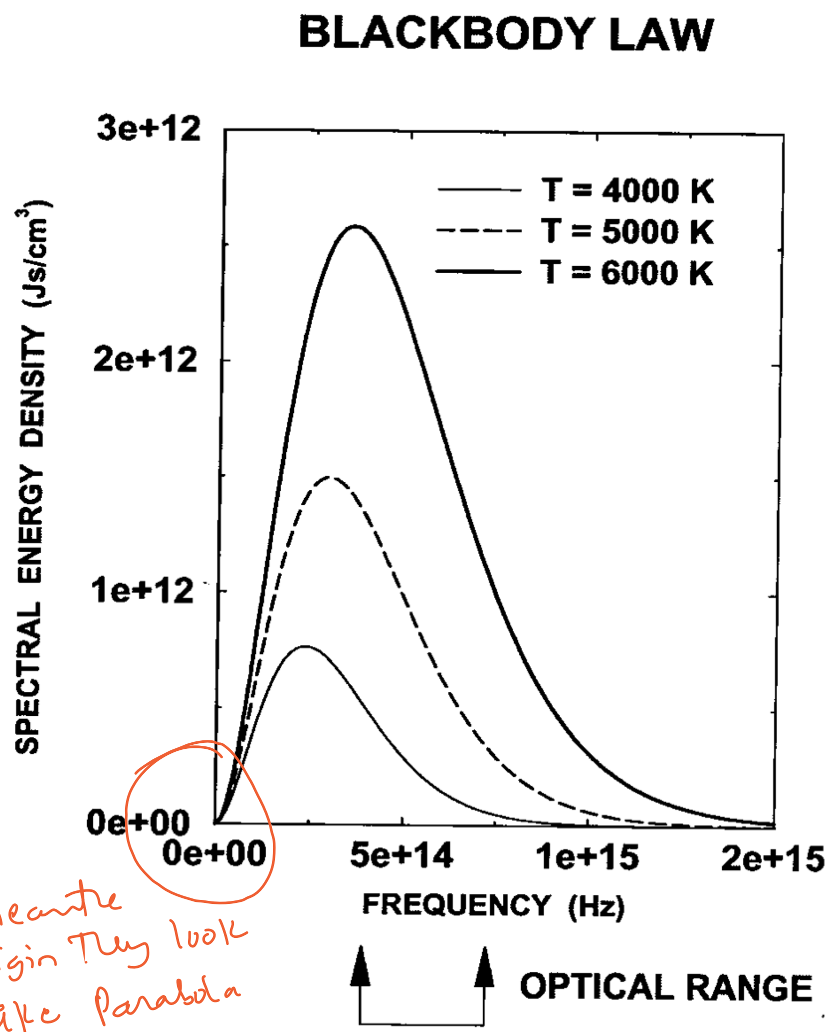
$$\eta = \frac{dE}{dx\,dy\,dt\,d\nu} = c\,\frac{dE}{dx\,dy\,dz\,d\nu}$$

$$\Rightarrow \quad c \quad u$$



T. 8.15: Rappresentazione schematica del corpo nero.

Q) How to calculate Spectral Energy density of a Black Body?



near the origin They look like Parabola

Ans

The energy density is

$$W_m^{eq}(T) = \int_0^\infty u(T,\nu)\, d\nu$$

The calculation of Spectral Energy Density is with reasoning as follows, we want to calculate energy per unit volume & frequency.

We try to split the problem into Two parts

1) We try to describe EMF as superposition monochromatic waves

2) Then we take infinitisimal interval $d\nu$ and calculate how many frequencies are there in the elementary interval.

** So, we have no. of frequencies & no. of EM modes per unit frequency.

** Then we calculate seperately the energy of each EM mode

So we have two ingredients

(i) Energy of each mode

(ii) No. of Modes per unit frequency

Then, if we take the product we obtain energy per unit frequency. Then we divide by volume & frequency $\Rightarrow$ Spectral energy density.

(i) Calculation of no. of EM Modes:

- According to Thermodynamics, The SED of The Cavity does not depend on the Shape of The Cavity.

Now, We take The Cavity to be a Box.

T. 8.17: **Calcolo della densità dei modi del campo elettromagnetico.**

Electromagnetic-Mode Density

Let an e.m. field be confined within a box of sides d_1, d_2, d_3 and volume $V = d_1 d_2 d_3$. By expanding into a Fourier series the field's components, the frequency ν of each mode of the e.m. field is related to the wave vector $\boldsymbol{k}$ by the following:

Wave Vector associated with individual Frequency.

$$\boldsymbol{k} = 2\pi \left(\frac{n_1}{d_1}\boldsymbol{i}_1 + \frac{n_2}{d_2}\boldsymbol{i}_2 + \frac{n_3}{d_3}\boldsymbol{i}_3 \right) , \qquad n_i = 0, \pm 1, \pm 2, \ldots ,$$

Integer

$$k^2 = (2\pi)^2 \left(\frac{n_1^2}{d_1^2} + \frac{n_2^2}{d_2^2} + \frac{n_3^2}{d_3^2} \right) , \qquad \boxed{\omega = 2\pi\,\nu = ck} .$$

Square Modulus

As the volume in $\boldsymbol{k}$ space associated to each $\boldsymbol{k}$ vector is

$$(2\pi/d_1)\,(2\pi/d_2)\,(2\pi/d_3) = (2\pi)^3/V ,$$

the number of modes in a sphere of radius k is, accounting for polarization,

$$N_\nu = \frac{2\,(4/3)\pi\, k^3}{(2\pi)^3/V} = \frac{8\pi\,(2\pi)^3\nu^3}{3c^3}\,\frac{V}{(2\pi)^3} = \frac{8}{3}\pi\, V\,\frac{\nu^3}{c^3} .$$

The number of modes per unit frequency and volume is then

$$\frac{1}{V}\,\frac{\mathrm{d}N_\nu}{\mathrm{d}\nu} = 8\pi\,\frac{\nu^2}{c^3} .$$

→ For any vector 'k', if you change any of the indices n_1, n_2, n_3 by one unit, you go to next vector 'k'

(This reasoning is idendical to the discussion of the possible 'k' vectors in the Brillouin Zone)

- So 'k' vectors are equally spaced in 'k' space!

i.e we can associate an elementary volume to each 'k' vector.

$$\text{volume in 'k' space} = \left(\frac{2\pi}{d_1}\right)\left(\frac{2\pi}{d_2}\right)\left(\frac{2\pi}{d_3}\right)$$

$$\Rightarrow \frac{(2\pi)^3}{d_1 d_2 d_3} = \frac{(2\pi)^3}{V}$$

$$\omega = 2\pi\nu = ck$$

$$\omega \propto \nu \propto k$$

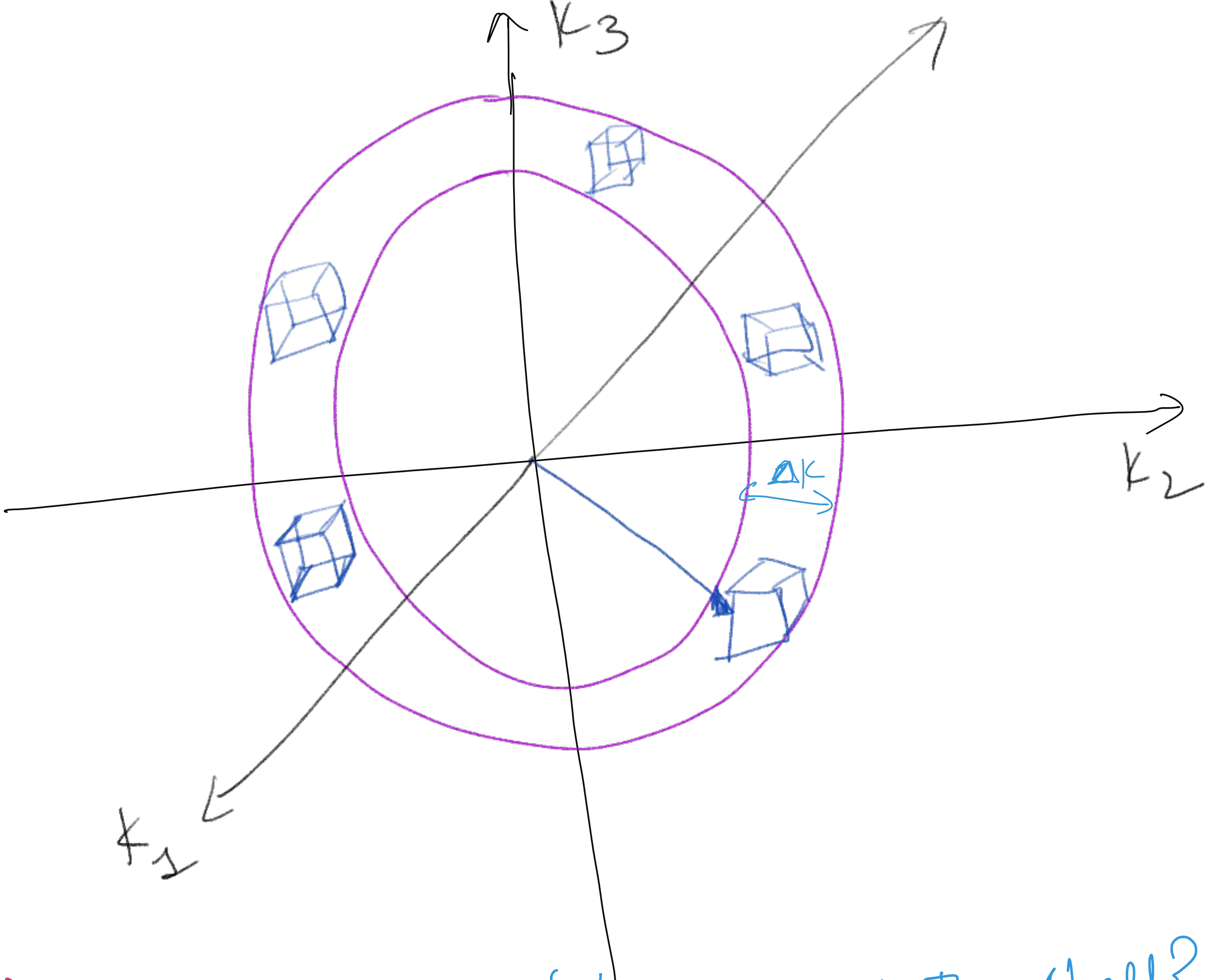
However 'k' is a vector, but the 'k' that appears above is the modulus of a vector.

∴ The goal is to find no. of EM modes per frequency

$\Delta \because \omega \propto \nu \propto k$

$\Rightarrow$ $\boxed{\text{elementary interval of } '\nu' \simeq \text{elementary interval of } 'k'}$

This 'k' vector can be oriented anywhere in space so it defines a sphere of radius 'k' in k-space.



Q) How many vectors 'k' has inside the shell?

$\Rightarrow$ $\dfrac{\text{Volume of the shell}}{\text{Volume of the little cubes that are at the tips of the shell}}$

$\Downarrow$

Total no. of cubes associated whose tip is inside the shell

→ EMF have a polarisation of '2'

$$N_v = \frac{2\left(\frac{4}{3}\right)\pi k^3}{(2\pi)^3 / V}$$

$$\Rightarrow \frac{8}{3}\pi V \frac{\nu^3}{c^3}$$

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The no. of Modes / frequency & volume

$$\frac{1}{V}\frac{dN_\nu}{d\nu} = 8\pi\frac{\nu^2}{c^3}$$

2nd step: Now we have to calculate the energy associated with each Mode of each frequency

* We try to calculate the Energy of Modes whose Frequency is ν

* According to Thermodynamics the distribution of energies in the system under Equilibrium follows the Maxwell Boltzmann distribution law.

treating each mode as a classical particle.

T. 8.18: Statistica di Maxwell-Boltzmann e calcolo dell'energia media.

each EM modes is treated as a linear Harmonic oscillators.
i.e (classical particle)

Maxwell-Boltzmann Statistics

In a system of classical particles in equilibrium at temperature T, the average occupation number at energy ϵ is

$$\boxed{P(\epsilon) = P_0 \exp(-\beta\epsilon)\,, \qquad \beta = \frac{1}{k_B T}}$$

where $k_B = 1.38\times10^{-23}$ J/K is Boltzmann's constant. The average over P of a function $\alpha(\epsilon)$ is defined as

$$\langle\alpha\rangle \doteq \frac{\int_0^\infty \alpha P\,\mathrm{d}\epsilon}{\int_0^\infty P\,\mathrm{d}\epsilon} = \beta \int_0^\infty \alpha\, \exp(-\beta\epsilon)\,\mathrm{d}\epsilon\,,$$

as $\int_0^\infty \exp(-\beta\epsilon)\,\mathrm{d}\epsilon = (1/\beta)\,[\exp(-\beta\epsilon)]_\infty^0 = 1/\beta$. In particular, the average energy reads

$$\langle\epsilon\rangle = \beta \int_0^\infty \epsilon\, \exp(-\beta\epsilon)\,\mathrm{d}\epsilon = \beta \int_0^\infty -\frac{\partial \exp(-\beta\epsilon)}{\partial\beta}\,\mathrm{d}\epsilon =$$

$$-\beta\frac{\mathrm{d}}{\mathrm{d}\beta}\int_0^\infty \exp(-\beta\epsilon)\,\mathrm{d}\epsilon = -\beta\frac{\mathrm{d}(1/\beta)}{\mathrm{d}\beta} = \frac{1}{\beta} = k_B T\,.$$

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Blackbody Radiation — II

Combining the expression of the number of modes per unit frequency and volume, $8\pi\nu^2/c^3$, with that of the average energy of the particles in a system in equilibrium, k_BT, an expression for the spectral energy density of the blackbody is found:

only true for low frequencies

$$u(\nu, T) = 8\pi \frac{k_BT}{c^3}\nu^2 \qquad \text{(Rayleigh-Jeans law).}$$

The above reproduces the experimental results only at low frequencies. Moreover, it makes the energy density $w^{\text{em}}_{\text{eq}}(T) = \int_0^\infty u\,\mathrm{d}\nu$ to diverge. The correct expression of u is found by making the average energy to depend on the mode frequency:

$$\boxed{k_BT \quad \Longleftarrow \quad h\nu/\left\{\exp[h\nu/(k_BT)] - 1\right\}},$$

where $h \simeq 6.626 \times 10^{-34}$ J s is the Planck constant. This yields

$$u(\nu, T) = 8\pi \frac{h\nu^3/c^3}{\exp[h\nu/(k_BT)] - 1} \qquad \text{(Planck law, 1900),}$$

which for $h\nu \ll k_BT$ renders the Rayleigh-Jeans law. However, this result is based on the *ad hoc* hypothesis that the e.m. energy is exchanged only in integer multiples of $h\nu$.

$h\nu \Rightarrow$ elementary energy
(plank's Hypothesis)

T. 8.20: Ipotesi di Planck.

Planck's Hypothesis

In order to explain the features of the blackbody radiation, Planck made in 1900 the hypothesis that a monochromatic e.m. energy is absorbed or emitted only in quantities that are integer multiples of a fixed quantity $h\nu$, where h is a suitable constant. The occupation number then becomes

$$P(\epsilon) \quad \Leftarrow \quad P_n \doteq P_0 \exp(-n\beta h\nu), \qquad \beta = \frac{1}{k_B T}.$$

Letting $\theta = \beta h\nu$, the average $\langle nh\nu \rangle$ is found from

$$\sum_{n=0}^{\infty} P_n = P_0 \left(1 + e^{-\theta} + e^{-2\theta} + \ldots\right) = \frac{P_0}{1 - \exp(-\theta)},$$

$$\sum_{n=0}^{\infty} nh\nu\, P_n = h\nu\, P_0 \left(e^{-\theta} + 2e^{-2\theta} + 3e^{-3\theta} + \ldots\right).$$

As $n \exp(-n\theta) = -\mathrm{d}\exp(-n\theta)/\mathrm{d}\theta$,

$$\sum_{n=0}^{\infty} nh\nu\, P_n = -h\nu \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sum_{n=0}^{\infty} P_n - P_0\right) = h\nu \frac{P_0 \exp(-\theta)}{[1 - \exp(-\theta)]^2},$$

$$\langle nh\nu \rangle = \frac{\sum_{n=0}^{\infty} nh\nu\, P_n}{\sum_{n=0}^{\infty} P_n} = \frac{h\nu}{\exp(\theta) - 1}.$$

T. 8.21: Ipotesi di Einstein.

Einstein's Theory of the Photoelectric Effect

In 1905 Einstein proposed the following explanation of the photoelectric effect: *the transport of the e.m. energy is "quantized"*, specifically, a monochromatic e.m. wave of frequency ν is made of the flux of identical objects (*photons*), each carrying the energy $h\nu$.
In the interaction with a photon, an electron may absorb an energy up to $h\nu$. If the absorbed amount is $h\nu$, the photon is annihilated. This theory provides a correct explanation of the photoelectric effect, as detailed below.

- The photoelectric current increases as the spectral power Π increases at constant ν, because the number of photons is larger.
- The blocking voltage V_R increases as ν increases at constant Π, because the photons are more energetic; however, they are fewer, which explains why the curves intersect each other: $\Pi = \mathrm{d}E/(\mathrm{d}t\,\mathrm{d}\nu) = h\nu\,\mathrm{d}N/(\mathrm{d}t\,\mathrm{d}\nu) = \text{const}$.

Compton effect

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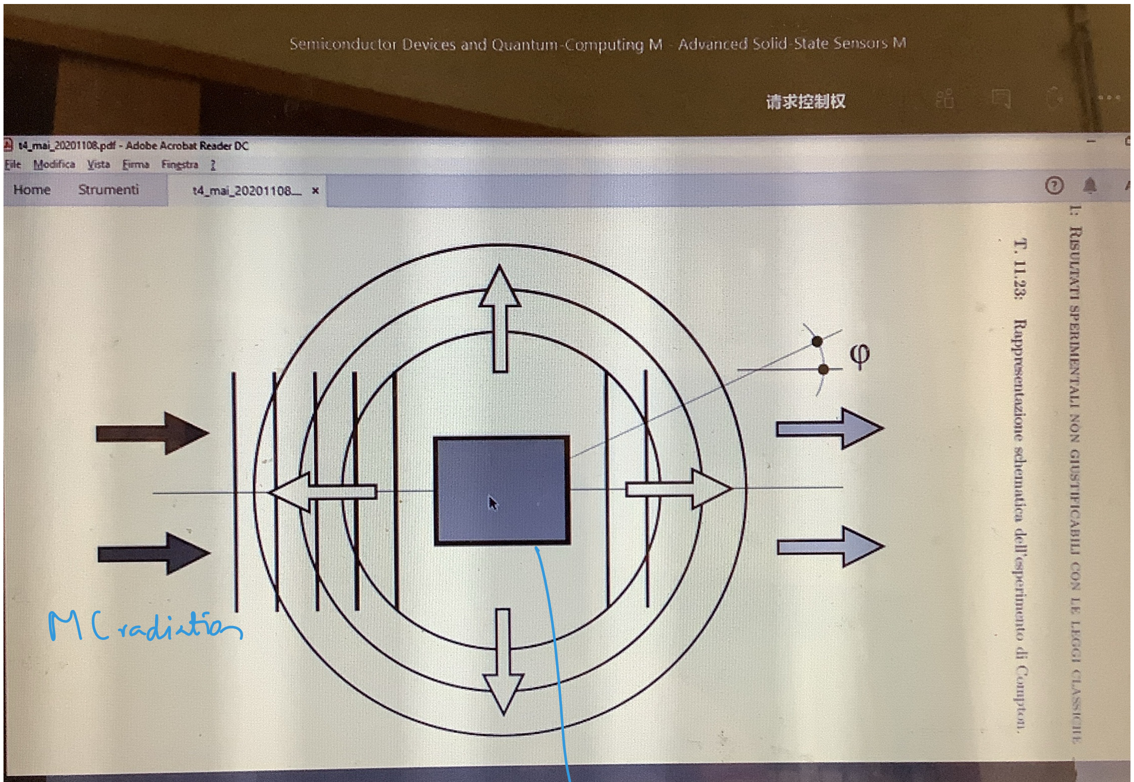
Experiment



MC radiation

dielectric Material

T. 8.22: Relazioni dinamiche relativistiche.

Dynamic Relations of Special Relativity

The relativistic relations for the dynamics of a free particle of rest mass m_0 and velocity $\boldsymbol{u}$ are

$$\boldsymbol{p} = \frac{m_0\, \boldsymbol{u}}{\sqrt{1 - u^2/c^2}}, \qquad E = mc^2,$$

where $m \doteq m_0/\sqrt{1 - u^2/c^2}$ is the relativistic mass. From the expressions of m and E it follows

$$m^2c^2 - m^2u^2 = m_0^2c^2, \qquad m^2c^2 = E^2/c^2,$$

whence the energy can be expressed in terms of momentum as

$$E^2/c^2 - p^2 = m_0^2c^2, \qquad p = \sqrt{E^2/c^2 - m_0^2c^2}.$$

The above shows that for a particle with null rest mass ($m_0 = 0$) it is $p = E/c$. At low velocities $u^2/c^2 \ll 1$ whence

$$m \simeq \frac{m_0}{1 - u^2/(2c^2)} \simeq m_0 \left(1 + \frac{u^2}{2c^2}\right), \quad E \simeq m_0c^2 + \frac{1}{2}m_0u^2,$$

namely $E - E_0 = m_0u^2/2 = T$, where $E_0 \doteq m_0c^2$ is the rest energy.

T. 8.23: Collisione elettrone-fotone.

Electron-Photon Collision

The concept of photon can also be used to explain the Compton effect. As the photon's velocity is c it must be $m_0 = 0$; from Special Relativity it follows $p = E/c$ for the photon's momentum, which is consistent with classical electromagnetism.
If the interaction between an electron and the e.m. field is described as a two-particle collision (specifically, assuming an isolated system made of an electron and a photon) the calculation can be based upon the energy- and momentum-conservation equations.
The dynamical quantities for the photon are then expressed as

$$E = h\nu\,, \qquad p = \frac{E}{c} = \frac{h\nu}{c} = \frac{h\nu}{\lambda\nu} = \frac{h}{\lambda}\,.$$

Using the reduced Planck constant $\hbar \doteq h/(2\pi) \simeq 1.055 \times 10^{-34}$ J s, the above become

$$E = \hbar\, 2\pi\nu = \hbar\omega\,, \qquad p = \frac{\hbar}{\lambda/(2\pi)} = \hbar k \;\Rightarrow\; \boldsymbol{p} = \hbar\boldsymbol{k}\,.$$

For consistency, the dynamical quantities for the electron involved in the collision are expressed by the relativistic expressions $\boldsymbol{p} = m\boldsymbol{u}$, $E = mc^2$, $m = m_0/\sqrt{1 - u^2/c^2}$.

T. 8.24: Calcolo della variazione di lunghezza d'onda per effetto Compton.

Compton Effect

$$\left\{\begin{array}{l} h\nu + m_0c^2 = h\nu' + mc^2 \\ \left.\begin{array}{l} h\nu/c = (h\nu'/c)\cos\varphi + mu\cos\psi \\ 0 = (h\nu'/c)\sin\varphi - mu\sin\psi \end{array}\right\} \end{array}\right. \quad \begin{array}{l}\text{Energy conservation} \\ \text{Momentum conservation}\end{array}$$

From the energy-cons. equation: $m^2c^4 = [h(\nu - \nu') + m_0c^2]^2$

$$\Longrightarrow \quad h^2(\nu - \nu')^2 = -2m_0c^2h(\nu - \nu') + \underbrace{m^2c^4 - m_0^2c^4}_{m^2u^2c^2}$$

From the momentum-cons. eqs., after multiplication by c:

$$h^2(\nu - \nu'\cos\varphi)^2 + h^2(\nu'\sin\varphi)^2 = m^2u^2c^2$$

$$m^2u^2c^2 = h^2[\nu^2 + \nu'^2 - 2\nu\nu'\underbrace{(1 - 2\sin^2(\varphi/2))}_{\cos\varphi}]$$

Inserting $m^2u^2c^2 - h^2(\nu - \nu')^2 = 2m_0c^2h(\nu - \nu')$ in the above yields $2m_0c^2h(\nu - \nu') = 4h^2\nu\nu'\sin^2(\varphi/2)$, whence

$$c\,(1/\nu' - 1/\nu) = \frac{2h}{m_0c}\sin^2(\varphi/2)\,, \qquad \lambda' - \lambda = 2\lambda_c\sin^2(\varphi/2)\,.$$

Here, $c = \lambda\nu = \lambda'\nu'$ has been used, and $\lambda_c \doteq h/(m_0c) \simeq 2.43 \times 10^{-10}$ cm is the Compton wavelength (1923), corresponding to $\nu_c \simeq 1.2 \times 10^{20}$ Hz. As $\Delta\lambda/\lambda \leq 2\nu/\nu_c$, the experiment is made using x-rays ($\nu \sim 10^{18}$ Hz).